ISSN 2518-7929 (Print)
ISSN 2663-5011 (Online)

BULLETIN
OF THE KARAGANDA UNIVERSITY

MATHEMATICS
SERIES

No. 1(113)/2024

ISSN 2518-7929 (Print)
ISSN 2663–5011(Online)
Индексі 74618
Индекс 74618

ҚАРАҒАНДЫ УНИВЕРСИТЕТІНІҢ **ХАБАРШЫСЫ**

ВЕСТНИК КАРАГАНДИНСКОГО УНИВЕРСИТЕТА

BULLETIN OF THE KARAGANDA UNIVERSITY

МАТЕМАТИКА СЕРИЯСЫ

СЕРИЯ МАТЕМАТИКА

MATHEMATICS SERIES

No. 1(113)/2024

1996 жылдан бастап шығады
Издается с 1996 года
Founded in 1996

Жылына 4 рет шығады
Выходит 4 раза в год
Published 4 times a year

Қарағанды / Караганда / Karaganda
2024

Postal address: 28, University Str., Karaganda, 100024, Kazakhstan
E-mail: math.bku@gmail.com. Web-site: mathematics-vestnik.ksu.kz

Bulletin of the Karaganda University. Mathematics series.
ISSN 2518-7929 (Print). ISSN 2663–5011 (Online).

Proprietary: NLC «Karagandy University of the name of academician E.A. Buketov».

Registered by the Ministry of Information and Social Development of the Republic of Kazakhstan.
Rediscount certificate No. KZ43VPY00027385 dated 30.09.2020.

Signed in print 29.03.2024. Format 60×84 1/8. Offset paper. Volume 27.75 p.sh. Circulation 200 copies.
Price upon request. Order № 23.

Printed in the Publishing house of NLC «Karagandy University of the name of academician E.A. Buketov».
28, University Str., Karaganda, 100024, Kazakhstan. Tel.: (7212) 35–63–16. E-mail: izd_kargu@mail.ru

CONTENT

MATHEMATICS

ERRATA

ANNIVERSARIES

EVENTS

Bulletin of the Karaganda University. Mathematics series, No. 1(113), 2024, pp. 5–20

MATHEMATICS

https://doi.org/10.31489/2024M1/5-20 *Research article*

On the class of pointwise and integrally loaded differential equations

K.R. Aida-zade[1,2], V.M. Abdullayev[1,3,*]

[1] *Institute of Control Systems of Ministry of Science and Education of Republic of Azerbaijan, Baku, Azerbaijan;*
[2] *Institute of Mathematics and Mechanics of Ministry of Science and Education of Republic of Azerbaijan, Baku, Azerbaijan;*
[3] *Azerbaijan State Oil and Industry University, Baku, Azerbaijan*
(E-mail: kamil_aydazade@rambler.ru, vaqif_ab@rambler.ru)

We investigate a system of linear ordinary differential equations containing point and integral loadings with nonlocal boundary conditions. Boundary conditions include integral and point values of the unknown function. An essential feature of the problem is that the kernels of the integral terms in the differential equations depend only on the integration variable. It is shown that similar problems arise during feedback control of objects with both lumped and distributed parameters during point and integral measurements of the current state for the controllable object. The problem statement considered in the paper generalizes a lot of previously studied problems regarding loaded differential equations with nonlocal boundary conditions. By introducing auxiliary parameters, we obtain necessary conditions for the existence and uniqueness of a solution to the problem under consideration. To solve the problem numerically, we propose to use a representation of the solution to the original problem, which includes four matrix functions that are solutions to four auxiliary Cauchy problems. Using solutions to the auxiliary problems in boundary conditions, we obtain the values of the unknown function at the loading points. This is enough to get the desired solution. The paper describes the application of the method using the example of solving a test model problem.

Keywords: integro-differential equation, system of loaded equations, integral conditions, conditions of existence and uniqueness.

2020 Mathematics Subject Classification: 34A12, 34B10, 45J05.

Introduction

The paper studies the existence and uniqueness of the solution of nonlocal problems with respect to systems of linear ordinary differential equations, which are pointwise and integrally loaded, and the kernels of integral terms depend on one variable of integration. The nonlocal conditions are linear and contain point and integral values of the unknown function. Such problems are also called pointwise and integrally loaded and they arise in many practical applications [1–4]. The specific feature of the integral terms in the equations is important for the proposed approach to obtaining the existence and uniqueness conditions for the solution of the problem and for its both analytical and numerical solutions.

*Corresponding author. *E-mail: vaqif_ab@rambler.ru*
Received: 19 July 2023; *Accepted:* 06 December 2023.

The paper describes an example of an optimal feedback control problem for a heating process, which leads to the class of nonlocal problems considered in the paper. Feedback is carried out due to point and integral measurements of the rod's temperature, the results of which are used to form the current values of the control action [5, 6].

In the paper, it is shown that the considered class of nonlocal problems, by introducing new variables, can be reduced to well-studied pointwise loaded problems with separated boundary conditions [7–9]. But taking into account the significant increase in the dimension of the problem, such an approach to the study of the original problem is not recommended.

The approach to obtaining the existence and uniqueness conditions of a solution to the problem is used to a certain extent for the proposed method for solving the problem both in an analytical form in case of a constant matrix of a dynamical system, and for a numerical solution with a variable matrix of the system. We present a study and an analytical method for solving one illustrative problem using the proposed approach.

1 Problem statement and its analysis

We consider the following system of pointwise and integrally loaded differential equations:

$$\frac{du(x)}{dx} = A(x)u(x) + \sum_{i=1}^{l_1} B_i^1(x)u(x_i) +$$

$$+ \sum_{j=1}^{l_2} B_j^2(x) \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\xi)u(\xi)d\xi + D(x),\ x \in [x_0, x_f] \tag{1}$$

with non-local conditions

$$\sum_{i=1}^{l_3} \alpha_i u(x_{L_2+i}) + \sum_{j=1}^{l_4} \int\limits_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\xi)u(\xi)d\xi = \gamma. \tag{2}$$

Here $u(\cdot) \in \mathbb{R}^n$ is an unknown continuously differentiable function. There are: non-negative integers l_1, l_2, l_3, l_4; continuous n-dimensional square matrix functions $A(x)$, $B_i^1(x)$, $B_j^2(x)$, $i = 1, 2, \dots, l_1$, $j = 1, 2, \dots, l_2$, at $x \in [x_0, x_f]$, $C_j(x)$ – at $x \in [x_{L_1+2j-1}, x_{L_1+2j}]$, $j = 1, 2, \dots, l_2$, $\beta_j(x)$ – at $x \in [x_{L_3+2j-1}, x_{L_3+2j}]$; continuous n-dimensional vector function $D(x)$; n-dimensional vector γ; points x_i, $i = 1, 2, \dots, L_4$, $L_1 = l_1$, $L_2 = L_1 + 2l_2$, $L_3 = L_2 + l_3$, $L_4 = L_3 + 2l_4$ from segment $[x_0, x_f]$ (some of the indicated points may coincide), and it is assumed that, without loss of generality, the following conditions are satisfied:

$$x_{L_1+2j} \geq x_{L_1+2j-1}, \quad j = 1, 2, \dots, l_2, \quad x_{L_3+2j} \geq x_{L_3+2j-1}, \quad j = 1, 2, \dots, l_4.$$

In the problem, it is required to find a continuously differentiable vector function $u(\cdot) \in \mathbb{R}^n$ for $x \in [x_0, x_f]$, satisfying the system of pointwise and integrally loaded differential equations (1) and nonlocal conditions (2), containing point and integral values of the unknown function.

An essential feature of problem (1) and (2) is the dependence of the integrands in equation (1) on one variable of integration. For example, optimal feedback control problems lead to such a problem [5,6]. In particular, the control synthesis problem for the heating process of a rod with the length d in the furnace, which can be described by the boundary value problem for the parabolic equation:

$$u_t'(x,t) = a^2 u_{x^2}''(x,t) + \mu(x)\left[\vartheta(t) - u(x,t)\right] \tag{3}$$

with some initial and boundary conditions

$$u(x,0) = \varphi(x),\ x \in [0,\ d]\,, \tag{4}$$

$$\frac{\partial u(0,t)}{\partial x} = \mu_1\left[\vartheta(t) - u(0,t)\right],\ t \in [0,T]\,, \tag{5}$$

$$\frac{\partial u(d,t)}{\partial x} = -\mu_1\left[\vartheta(t) - u(d,t)\right],\ t \in [0,T]\,. \tag{6}$$

Here $u(x,t)$ is the temperature of the rod at the point x at the moment t, $x \in [0,d]$, $t \in [0,T]$; $\mu(x)$; $\varphi(x)$, a, μ_1 are the specified functions and process parameters; $\vartheta(t)$ is a control function that determines the temperature inside the furnace. There is a certain optimality criterion characterizing the choice of control $\vartheta(t)$. Assume that the given points x_i and segments $[\tilde{x}_{2j-1}, \tilde{x}_{2j}]$ of the rod, we take the point $u(x_i,t)$, $i = 1,2,\ldots,l_1$ and integral $u(x,t), x \in [\tilde{x}_{2j-1}, \tilde{x}_{2j}], j = 1,2,\ldots,l_2$ measurements of the temperature. The measurement results are used to form the current temperature value in the furnace (feedback control) in the form of the following relationship

$$\vartheta(t) = \sum_{i=1}^{l_1} k_{1i} u(x_i,t) + \sum_{j=1}^{l_2} k_{2j} \int\limits_{\tilde{x}_{2j-1}}^{\tilde{x}_{2j}} \beta_j(\xi) u(\xi,t) d\xi. \tag{7}$$

Here the given functions $\beta_i(x), x \in [x_{2j-1}, x_{2j}]$ are weighted, the constant coefficients $k_{1i},\ k_{2j}$, $i = 1,2,\ldots,l_1$, are the optimizable feedback parameters [5, 6].

Substituting expression (7) into equation (3) and using the difference approximation of the derivatives with respect to t

$$\frac{\partial u(x,t_s)}{\partial t} = \frac{u(x,t_s) - u(x,t_{s-1})}{h_t} + O(h_t),$$

we obtain the following system of loaded differential equations:

$$a_0 \frac{d^2 u_s(x)}{dx^2} = a_1 u_s(x) + a_2(x)\ \left[\sum_{i=1}^{l_1} k_{1i} u_s(x_i) + \right.$$

$$\left. + \sum_{j=1}^{l_2} k_{2j} \int\limits_{\tilde{x}_{2j-1}}^{\tilde{x}_{2j}} \beta_j(\xi) u_s(\xi) d\xi \right] + f_s(x)\,, \quad x \in [0; d]\ \ ,\ \ s = 1,2,...,N_t. \tag{8}$$

Conditions (4)–(6) can be written in the form

$$\frac{du_s(0)}{dx} = \mu_1 \left[\sum_{i=1}^{l_1} k_{1i} u_s(x_i) + \sum_{j=1}^{l_2} k_{2j} \int\limits_{\tilde{x}_{2j-1}}^{\tilde{x}_{2j}} \beta_j(\xi) u_s(\xi) d\xi - u_s(0)\right], \tag{9}$$

$$\frac{du_s(d)}{dx} = -\mu_1 \left[\sum_{i=1}^{l_1} k_{1i} u_s(x_i) + \sum_{j=1}^{l_2} k_{2j} \int\limits_{\tilde{x}_{2j-1}}^{\tilde{x}_{2j}} \beta_j(\xi) u_s(\xi) d\xi - u_s(d)\right]. \tag{10}$$

In (8)–(10) the following notations are used: h_t is the discretization step, $u_0(x) = u(x,0) = \varphi(x)$, $x \in [0,d]$, $u_s(x) = u(x,t_s), t_s = sh_t$, $s = 1,2,\ldots,N_t$, $h_t = {}^{T}\!/_{N_t}$, $a_0 = a^2 h_t$, $a_1(x) = 1 + a_2(x)$, $a_2(x) = h_t \mu(x)$, $f_s(x) = u_{s-1}(x)$.

To determine the feedback parameters $k_{1i}, k_{2j}, i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$ using any first-order numerical iterative optimization methods, first, it is required to construct formulas for the components of the gradient of the objective functional in terms of the optimizable parameters, and second, to numerically solve problem (8)–(10) for given current values of these parameters. It is clear that system (8) can be easily reduced to the considered system of first-order differential equations (1) and (2).

Note that some special cases of problem (1) and (2) were studied earlier. When $B_j^2(x) \equiv 0$, $j = 1, 2, \ldots, l_2$, we have a point-loaded system of differential equations, investigated in many papers, particularly, in [2–4, 7]. When $B_j^1(x) \equiv 0$, $j = 1, 2, \ldots, l_1$, we obtain an integro-differential system of equations whose kernels $C_j(x)$ depend only on the variable of integration [10–18]. Conditions (2) also generalize many other local and nonlocal conditions. Their particular cases are Cauchy conditions, two-point and multipoint conditions, conditions of an integral type [4, 19].

Problem (1) and (2), by introducing new unknowns, can be reduced to a two-point boundary value problem for a system of point-loaded differential equations. Let's show how it is done. We introduce new n-dimensional variables $\vartheta^j(x), j = 1, 2, \ldots, l_2$, satisfying the system of differential equations:

$$\begin{aligned} \frac{d\vartheta^j(x)}{dx} &= C_j(x)u(x), \quad x_{L_1+2j-1} < x \le x_{L_1+2j},\ j = 1, 2, \ldots, l_2, \\ \vartheta^j(x) &= 0_n, \qquad\qquad x \le x_{L_1+2j-1},\ \ j = 1, 2, \ldots, l_2, \end{aligned} \tag{11}$$

where 0_n is the n–dimensional zero vector. System (1) will have only point loading:

$$\frac{du(x)}{dt} = A(x)u(x) + \sum_{i=1}^{l_1} B_i^1(x)u(x_i) + \sum_{j=1}^{l_2} B_j^2(x)\vartheta^j(x_{L_1+2j}) + D(x), x \in [x_0, x_f]. \tag{12}$$

By introducing new n-dimensional vectors $w^j(x)$, $j = 1, 2, \ldots, l_4$, satisfying the system of differential equations

$$\begin{aligned} \frac{dw^j(x)}{dx} &= \beta_j(x)u(x), \quad x_{L_3+2j-1} < x \le x_{L_3+2j},\ j = 1, 2, \ldots, l_4, \\ w^j(x) &= 0_n, \qquad\qquad x \le x_{L_3+2j-1},\ \ j = 1, 2, \ldots, l_4, \end{aligned} \tag{13}$$

conditions (2) are reduced to the multipoint conditions

$$\sum_{i=1}^{l_3} \alpha_i u(x_{L_2+i}) + \sum_{j=1}^{l_4} w^j(x_{L_3+2j}) = \gamma. \tag{14}$$

The order of the resulting linear system of loaded differential equations (11)–(13) is $(l_2 + l_4 + 1)n$. Using the approach proposed in [8,9], multipoint conditions (14) can be reduced to separated boundary conditions. To do this, each of the $2(l_3+l_4+1)$ segments between all the points x_0, x_{L_2+i}, $i = 1, 2, \ldots, l_3$, x_{L_3+2j}, $j = 1, 2, \ldots, l_4$, x_f after their ordering, is divided into two parts. For each of the halves of these segments between the points, systems of differential equations are introduced for new variables corresponding to $u(x)$, $\vartheta^i(x)$, $w^j(x)$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$, but in different directions of change of the argument x. As a result, we obtain a system of differential equations of the order $2(l_3 + l_4 + 1)(l_2 + l_4 + 1)n$ with two-point boundary conditions of the form (after individual scaling for each segment and reducing them to segments of a unit length):

$$A_1 w(0) = A_2, \qquad A_3 w(1) = A_4,$$

where A_1, A_3 are the square matrices of size $2(l_3 + l_4 + 1)(l_2 + l_4 + 1)n$; A_2, A_4 are the vectors of the corresponding dimension.

Point-loaded equations with two-point and multipoint conditions have been studied well enough, necessary and sufficient existence and uniqueness conditions of a solution were obtained for them in [11] and [17], approaches to their numerical solution were proposed in [2, 7, 20]. Optimization and optimal control problems, inverse problems in various formulations described by point-loaded equations we investigated in [21–23], numerical methods for their solution are described in [21, 22].

Considering a significant increase in the dimension of the original problem (1) and (2), when it is reduced to a problem with point-loaded differential equations with separated boundary conditions, the use of the previously proposed methods both for study and their numerical solution is inappropriate. This is especially true for optimization and optimal control problems that require multiple solutions of problems of the kind (1) and (2).

Therefore, this paper studies the existence and uniqueness of solutions to problem (1), (2), and also proposes an approach to solving that does not require an increase in the dimension of the original problem.

2 Existence and uniqueness conditions for the solution of problem (1) and (2)

Consider the following auxiliary system of differential equations:

$$\frac{du(x)}{dx} = A(x)u(x) + \sum_{i=1}^{l_1} B_i^1(x)\breve{\lambda}^i + \sum_{j=1}^{l_2} B_j^2(x)\widehat{\lambda}^j + D(x),\, x \in [x_0, x_f] \tag{15}$$

with conditions (2). Here, $\breve{\lambda}^i$, $\widehat{\lambda}^j$, $i = 1, 2, \ldots, l_1$, $\quad j = 1, 2, \ldots, l_2$ are arbitrary n-dimensional vectors, the functions and parameters are the same as in equation (1).

Under the accepted assumptions on the functions involved in the problem, the solution to system (15) for an arbitrarily given initial condition

$$u(x_0) = u_0 \tag{16}$$

according to the Cauchy formula can be written as:

$$u(x) = F(x)u_0 + F(x)\int_{x_0}^{x} F^{-1}(\xi)R(\xi)d\xi, \quad x \in [x_0, x_f], \tag{17}$$

$$R(\xi) = \sum_{i=1}^{l_1} B_i^1(\xi)\breve{\lambda}^i + \sum_{j=1}^{l_2} B_j^2(\xi)\widehat{\lambda}^j + D(\xi). \tag{18}$$

Here, the n-dimensional square fundamental matrix $F(x)$ is a solution to the Cauchy problem

$$\frac{dF(x)}{dx} = A(x)F(x), \quad F(x_0) = I_n, \quad x \in [x_0, x_f], \tag{19}$$

where I_n is the n-dimensional identity matrix.

Let us introduce the notation:

$$\breve{F}^i(x) = F(x)\int_{x_0}^{x} F^{-1}(\xi)B_i^1(\xi)d\xi, \quad i = 1, 2, \ldots, l_1, \tag{20}$$

$$\widehat{F}^j(x) = F(x)\int_{x_0}^{x} F^{-1}(\xi)B_j^2(\xi)d\xi, \quad j = 1, 2, \ldots, l_2, \tag{21}$$

$$F^1(x) = F(x) \int_{x_0}^{x} F^{-1}(\xi) D(\xi) d\xi. \tag{22}$$

Then solution (17)–(18) of the system of differential equations (15) with an arbitrary given initial condition u_0 and the vectors $\overset{\smile}{\lambda}{}^{i}$, $\overset{\frown}{\lambda}{}^{j}$, $i = 1, 2, \ldots, l_1$, $\quad j = 1, 2, \ldots, l_2$ can be written as:

$$u(x) = F(x)u_0 + \sum_{i=1}^{l_1} \overset{\smile}{F}{}^{i}(x) \overset{\smile}{\lambda}{}^{i} + \sum_{j=1}^{l_2} \overset{\frown}{F}{}^{j}(x) \overset{\frown}{\lambda}{}^{j} + F^1(x). \tag{23}$$

Considering an arbitrariness of the n-dimensional vectors u_0, $\overset{\smile}{\lambda}{}^{i}$, $\overset{\frown}{\lambda}{}^{j}$, $i = 1, 2, \ldots, l_1$, $\quad j = 1, 2, \ldots, l_2$, we require that they fulfill the following conditions:

$$\overset{\smile}{\lambda}{}^{\nu} = u(x_\nu), \quad \nu = 1, 2, \ldots, l_1, \tag{24}$$

$$\overset{\frown}{\lambda}{}^{\mu} = \int_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\xi) u(\xi) d\xi, \quad \mu = 1, 2, \ldots, l_2, \tag{25}$$

and conditions (2). It is clear that the total number of conditions in (2), (24) and (25) is equal and coincides with the total dimension of an arbitrary vector u_0, $\overset{\smile}{\lambda}{}^{i}$, $\overset{\frown}{\lambda}{}^{j}$, $i = 1, 2, \ldots, l_1$, $\quad j = 1, 2, \ldots, l_2$. Let us introduce the notation for vectors: $\overset{\smile}{\Lambda} = (\overset{\smile}{\lambda}{}^{1}, \overset{\smile}{\lambda}{}^{2}, ..., \overset{\smile}{\lambda}{}^{l_1})^T \in \mathbb{R}^{l_1 n}$, $\overset{\frown}{\Lambda} = (\overset{\frown}{\lambda}{}^{1}, \overset{\frown}{\lambda}{}^{2}, ..., \overset{\frown}{\lambda}{}^{l_2})^T \in \mathbb{R}^{l_2 n}$, $\Lambda = (\overset{\smile}{\Lambda}, \overset{\frown}{\Lambda}) \in \mathbb{R}^{(l_1+l_2)n}$.

"T" is the transposition sign.

From (24), taking into account (23), we obtain:

$$\overset{\smile}{\lambda}{}^{\nu} = F(x_\nu)u_0 + \sum_{i=1}^{l_1} \overset{\smile}{F}{}^{i}(x_\nu) \overset{\smile}{\lambda}{}^{i} + \sum_{j=1}^{l_2} \overset{\frown}{F}{}^{j}(x_\nu)\, \overset{\frown}{\lambda}{}^{j} + F^1(x_\nu). \quad \nu = 1, 2, ..., l_1. \tag{26}$$

From (25), taking into account (23), for $\mu = 1, 2, \ldots, l_2$, we obtain:

$$\overset{\frown}{\lambda}{}^{\mu} = \int_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\eta) \left[F(\eta)u_0 + \sum_{i=1}^{l_1} \overset{\smile}{F}{}^{i}(\eta)\, \overset{\smile}{\lambda}{}^{i} + \sum_{j=1}^{l_2} \overset{\frown}{F}{}^{j}(\eta) \overset{\frown}{\lambda}{}^{j} + F^1(\eta) \right] d\eta. \tag{27}$$

From conditions (2), taking into account (23), we obtain:

$$\sum_{i=1}^{l_3} \alpha_i \left[F(x_{L_2+i})u_0 + \sum_{s=1}^{l_1} \overset{\smile}{F}{}^{s}(x_{L_2+i}) \overset{\smile}{\lambda}{}^{s} + \sum_{j=1}^{l_2} \overset{\frown}{F}{}^{j}(x_{L_2+i})\, \overset{\frown}{\lambda}{}^{j} + F^1(x_{L_2+i}) \right] +$$

$$+ \sum_{j=1}^{l_4} \int_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\eta) \left[F(\eta)u_0 + \sum_{i=1}^{l_1} \overset{\smile}{F}{}^{i}(\eta)\, \overset{\smile}{\lambda}{}^{i} + \sum_{s=1}^{l_2} \overset{\frown}{F}{}^{s}(\eta) \overset{\frown}{\lambda}{}^{s} + F^1(\eta) \right] d\eta = \gamma. \tag{28}$$

Relations (26)–(28) are systems of linear algebraic equations of an $l_1 n$, $l_2 n$ and n-th order, respectively, with respect to the unknown n-dimensional vectors u_0, $\overset{\smile}{\lambda}{}^{i}$, $\overset{\frown}{\lambda}{}^{j}$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$. The total number of equations in these systems corresponds to the total number of unknowns: $(u_0, \overset{\smile}{\Lambda}, \overset{\frown}{\Lambda}) \in$

$\mathbb{R}^{(l_1+l_2+1)n}$. After simple transformations and grouping, the resulting algebraic system can be reduced to the form:

$$\begin{cases} G_{11}^i u_0 + G_{12}^i \breve{\Lambda} + G_{13}^i \widehat{\Lambda} = G_{10}^i, \ i = 1, 2, ..., l_1, \\ G_{21}^j u_0 + G_{22}^j \breve{\Lambda} + G_{23}^j \widehat{\Lambda} = G_{20}^j, \ j = 1, 2, ..., l_2, \\ G_{31} u_0 + G_{32} \breve{\Lambda} + G_{33} \widehat{\Lambda} = G_{30}. \end{cases} \tag{29}$$

The matrix coefficients participating in (29) are determined from (26)–(28):

$$G_{11}^i = F(x_i) \in \mathbb{R}^{n\times n},$$

$$G_{12}^i = \left(\breve{F}^1(x_i), ..., \breve{F}^{i-1}(x_i), \breve{F}^i(x_i) - \mathrm{I}_n, \breve{F}^{i+1}(x_i), ..., \breve{F}^{l_1}(x_i)\right) \in \mathbb{R}^{n\times l_1 n},$$

$$G_{13}^i = \left(\widehat{F}^1(x_i), \widehat{F}^2(x_i)..., \widehat{F}^{l_2}(x_i)\right) \in \mathbb{R}^{n\times l_2 n},$$

$$G_{10}^i = -F^1(x_i) \in \mathbb{R}^{l_1 n}, \quad i = 1, 2, \ldots, l_1,$$

$$C_{21}^j = \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta) F(\eta) d\eta \in \mathbb{R}^{n\times n},$$

$$G_{22}^j = \left(\int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\breve{F}^1(\eta)d\eta, ..., \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\breve{F}^{l_1}(\eta_i)d\eta \right) \in \mathbb{R}^{n\times l_1 n},$$

$$G_{23}^j = \left(\int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\widehat{F}^1(\eta)d\eta, ..., \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\widehat{F}^{j-1}(\eta)d\eta, \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\widehat{F}^j(\eta)d\eta - \mathrm{I}_n, \right.$$
$$\left. \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\widehat{F}^{j+1}(\eta)d\eta, ..., \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\widehat{F}^{l_2}(\eta)d\eta \right) \in \mathbb{R}^{n\times l_2 n},$$

$$G_{20}^j = -\int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j-1}} C_j(\eta)F^1(\eta)d\eta \in \mathbb{R}^{l_2 n}, \quad j = 1, 2, \ldots, l_2,$$

$$G_{31} = \sum_{i=1}^{l_3} \alpha_i F(x_{L_2+i}) + \sum_{j=1}^{l_4} \int\limits_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\eta) F(\eta) d\eta \in \mathbb{R}^{n\times n},$$

$$G_{32} = \sum_{i=1}^{l_3} \alpha_i \sum_{s=1}^{l_1} \breve{F}^s(x_{L_2+i}) + \sum_{j=1}^{l_4} \int\limits_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\eta) \sum_{i=1}^{l_1} \breve{F}^i(\eta) d\eta \in \mathbb{R}^{n\times l_1 n},$$

$$G_{33} = \sum_{i=1}^{l_3} \alpha_i \sum_{j=1}^{l_2} \widehat{F}^j(x_{L_2+i}) + \sum_{j=1}^{l_4} \int\limits_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\eta) \sum_{s=1}^{l_2} \widehat{F}^s(\eta) d\eta \in \mathbb{R}^{n\times l_2 n},$$

$$G_{30} = \gamma - \sum_{i=1}^{l_3} \alpha_i F^1(x_{L_2+i}) - \sum_{j=1}^{l_4} \int\limits_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\eta) F^1(\eta) d\eta \in \mathbb{R}^n.$$

From the solution of algebraic system (29), we determine the initial value of the unknown function $u_0 = u(x_0)$, the point values

$$\overset{\smile}{\lambda}^{i} = u(x_i), i = 1, 2, \ldots, l_1,$$

and the integral values $\overset{\frown}{\lambda}^{j} = \int\limits_{L_1+2j-1}^{L_1+2j} C(\xi)u(\xi)d\xi, \quad j = 1, 2, \ldots, l_2$. This allows us to solve the Cauchy problem for the system of differential equations (15) instead of loaded system (1) with initial conditions (16) without using nonlocal conditions (2).

Thus, the existence and uniqueness of a solution to problem (1) and (2) depends on the existence and uniqueness of the vectors $u_0, \overset{\smile}{\lambda}^{i}, \overset{\frown}{\lambda}^{j}$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$, which are solutions of algebraic system (29). This implies the following theorem.

Theorem 1. For the existence and uniqueness of a solution to problem (1) and (2), the rank of the $(l_1 + l_2 + 1)n$ -dimensional square matrix of algebraic system (29) must satisfy the condition:

$$rank \begin{bmatrix} G_{11}^1 & ... & G_{11}^{l_1} & G_{21}^1 & ... & G_{21}^{l_2} & G_{31} \\ G_{12}^1 & ... & G_{12}^{l_1} & G_{22}^1 & ... & G_{22}^{l_2} & G_{32} \\ G_{13}^1 & ... & G_{13}^{l_1} & G_{23}^1 & ... & G_{23}^{l_2} & G_{33} \end{bmatrix}^T = (l_1 + l_2 + 1)n. \quad (30)$$

It is clear that if the rank of the matrix in (30) is less than n, then algebraic system (29) may have no solutions or have an infinite number of solutions depending on the rank of augmented matrix (29). Consequently, original problem (1) and (2) may have an infinite number of solutions or not have them at all, respectively.

The above approach to studying the existence and uniqueness of a solution to problem (1) and (2) can also be used to solve the problem. But, as can be seen from the above formulas, to find the coefficients of the system of algebraic equations (29), it is necessary to have a fundamental matrix of solutions $F(x)$ and its inverse matrix $F^{-1}(x)$, $x \in [x_0, x_f]$. If the condition $A(x) \neq const$, $x \in [x_0, x_f]$ is met, the construction of these matrices in an analytical form is not possible in practice, and using numerical methods requires a large amount of computation and memory.

In the next section, we present an approach to the numerical solution of problem (1) and (2) is presented that does not require knowledge of the matrix $F^{-1}(x)$, $x \in [x_0, x_f]$.

3 Approach to the solution of the problem

Below, we propose an approach to solving problem (1) and (2) using auxiliary Cauchy problems for linear systems of differential equations. For the numerical solution of the auxiliary Cauchy problems, known methods and software packages can be used.

The proposed approach is based on the representation of solution (23) to auxiliary problem (15), (16) and the Cauchy problems given in the following theorem.

Theorem 2. The solution of the system of differential equations (15) for arbitrarily given independent initial condition (16) and n-dimensional vectors $\overset{\smile}{\lambda}^{i}, \overset{\frown}{\lambda}^{j}$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$, can be uniquely represented as (23), if n-dimensional square matrix functions $F(x)$, $\overset{\smile}{F}^{i}(x)$, $\overset{\frown}{F}^{j}(x)$ and vector function

$F^1(x)$, at $x \in [x_0, x_f]$, are solutions of the corresponding Cauchy problems (19) and

$$\frac{d\breve{F}^{i}(x)}{dx} = A(x)\breve{F}^{i}(x) + B_i^1(x), \quad \breve{F}^{i}(x_0) = 0, \quad i = 1, 2, ..., l_1, \tag{31}$$

$$\frac{d\widehat{F}^{j}(x)}{dx} = A(x)\widehat{F}^{j}(x) + B_j^2(x), \ \widehat{F}^{j}(x_0) = 0, \ \ j = 1, 2, ..., l_2, \tag{32}$$

$$\frac{dF^1(x)}{dx} = A(x)F^1(x) + D(x), \ F^1(x_0) = 0. \tag{33}$$

Proof. According to Cauchy formula, the unique solutions to problems (31)–(33) are the functions $\breve{F}^{i}(x)$, $\widehat{F}^{j}(x)$, $F^1(x)$, respectively, defined by formulas (20)–(22). These formulas involve the matrix function $F(x)$, which is a fundamental solution to homogeneous systems with respect to (31)–(33) and the unique solution to Cauchy problem (19). It is clear that the functions $F(x)$, $\breve{F}^{i}(x)$, $\widehat{F}^{j}(x)$, $F^1(x)$ are independent of the initial condition u_0 and parameters $\breve{\lambda}^{i}$, $\widehat{\lambda}^{j}$. But the representation of the solution to the system of differential equations (15) in the form (23), by virtue of Cauchy formula (18), is unique for arbitrarily and independently given vectors u_0, $\breve{\lambda}^{i}$, $\widehat{\lambda}^{j}$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$.

From the above, we can formulate the following approach to solving original problem (1) and (2).

First, we solve auxiliary Cauchy problems (19), (31)–(33). After finding the functions $F(x)$, $\breve{F}^{i}(x)$, $\widehat{F}^{j}(x)$, $F^1(x)$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$, further, taking into account the arbitrariness of the parameters $\breve{\lambda}^{i}$, $\widehat{\lambda}^{j}$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$, $u_0 \in \mathbb{R}^n$ in problem (15) and (2), we require that they fulfill conditions (24), (25) and (2). Then, from representation (23), we have:

$$\breve{\lambda}^{\nu} = u(x_\nu) = F(x_\nu)u_0 + \sum_{i=1}^{l_1} \breve{F}^{i}(x_\nu)\breve{\lambda}^{i} + \sum_{j=1}^{l_2} \widehat{F}^{j}(x_\nu)\widehat{\lambda}^{j} + F^1(x_\nu), \quad \nu = 1, 2, ..., l_1, \tag{34}$$

$$\widehat{\lambda}^{\mu} = \int\limits_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\xi)u(\xi)d\xi = \int\limits_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\xi)F(\xi)\, u_0 d\xi + \int\limits_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\xi) \sum_{i=1}^{l_1} \breve{F}^{i}(\xi)\, \breve{\lambda}^{i} d\xi +$$

$$+ \int\limits_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\xi) \sum_{j=1}^{l_2} \widehat{F}^{j}(\xi)\, \widehat{\lambda}^{j} d\xi + \int\limits_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\xi)F^1(\xi)d\xi\,, \ \mu = 1, 2, ..., l_2, \tag{35}$$

$$\sum_{i=1}^{l_3} \alpha_i \left[F(x_{L_2+i})u_0 + \sum_{s=1}^{l_1} \breve{F}^{s}(x_{L_2+i})\breve{\lambda}^{s} + \sum_{j=1}^{l_2} \widehat{F}^{j}(x_{L_2+i})\widehat{\lambda}^{j} + F^1(x_{L_2+i}) \right] +$$

$$+ \sum_{j=1}^{l_4} \int\limits_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\xi) \left[F(\xi)u_0 + \sum_{i=1}^{l_1} \breve{F}^{i}(\xi)\, \breve{\lambda}^{i} + \sum_{s=1}^{l_2} \widehat{F}^{s}(\xi)\widehat{\lambda}^{s} + F^1(\xi) \right] d\eta = \gamma. \tag{36}$$

From (34)–(36) we get the system of $(l_1 + l_2 + 1)n$ linear equations with respect to unknowns n-dimensional vectors u_0, $\breve{\lambda}^{\nu}$, $\widehat{\lambda}^{\mu}$, $\nu = 1, 2, \ldots, l_1$, $\mu = 1, 2, \ldots, l_2$. Having determined these vectors, from representation (23), we find the desired solution to problem (1) and (2).

If among $B_i^1(x)$, $i = 1, 2, \ldots, l_1$, or $B_j^2(x)$, $j = 1, 2, \ldots, l_2$, there are functions having the same or different constant coefficients, then the number of auxiliary problems (31)–(33) can be reduced by the number of coinciding functions. For example, if $k_1 B_{i_1}^1(x) = k_2 B_{i_2}^1(x) = ... = k_s B_{i_s}^1(x)$, then instead of vectors $\breve{\lambda}^{i_1}, ..., \breve{\lambda}^{i_s}$ it suffices to introduce into (24) one vector $\breve{\lambda}^{i_*} = \sum\limits_{q=1}^{s} k_q u(x_{i_q})$.

Similarly, if $k_1 B_{j_1}^1(x) = k_2 B_{j_2}^2(x) = ... = k_s B_{j_s}^2(x)$, then instead of vectors $\frown{\lambda}^{j_1}, ..., \frown{\lambda}^{j_s}$ in (25) we introduce one vector $\frown{\lambda}^{j_*} = \sum\limits_{q=1}^{s} k_q \int\limits_{x_{L_1+2j_q-1}}^{x_{L_1+2j_q}} C_{j_q}(\xi) u(\xi) d\xi$. One of such cases will be demonstrated by the example of an illustrative problem given in the next section.

4 Illustrative problem

Consider the following problem:

$$\frac{du(x)}{dx} = 3u(x) + 2u(1) + 3u(2) + 6 \int\limits_2^3 u(\tau) d\tau - 6x^2 + 4x - 118, \; x \in [0, 4], \tag{37}$$

$$u(0) - 2u(3) + u(4) + 3 \int\limits_1^2 u(\tau) d\tau = 13. \tag{38}$$

In equation (37) $A(x) = const = 3$, $x \in [0, 4]$; the functions $B_1^1(x) = 2$ and $B_2^1(x) = 3$ differ in constant coefficients; $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 2$, $x_4 = 3$, $x_5 = 0$, $x_6 = 3$, $x_7 = 4$, $x_8 = 1$, $x_9 = 2$, $l_1 = 2$, $l_2 = l_4 = 1$, $l_3 = 3$, $L_1 = 2$, $L_2 = 4$, $L_3 = 7$, $L_4 = 9$; $D(x) = -6x^2 + 4x - 118$; $C_1(x) = 1$; $\alpha_1 = 1$, $\alpha_2 = -2$, $\alpha_3 = 1$, $\beta_1 = 3$, $\gamma = 13$.

It is easy to verify that the solution to problem (37) and (38) is the function: $u(x) = 2x^2 + 1$.

Let us introduce the notation

$$\breve{\lambda}^1 = 2u(1) + 3u(2), \quad \frown{\lambda}^1 = \int\limits_2^3 u(\xi) d\xi. \tag{39}$$

Let us construct auxiliary problems (31)–(33):

$$\frac{dF(x)}{dx} = 3F(x), \quad F(0) = 1, \tag{40}$$

$$\frac{d\breve{F}^1(x)}{dx} = 3\breve{F}^1(x) + 1, \quad \breve{F}^1(0) = 0, \tag{41}$$

$$\frac{d\frown{F}^1(x)}{dx} = 3\frown{F}^1(x) + 6, \quad \frown{F}^1(0) = 0. \tag{42}$$

$$\frac{dF^1(x)}{dx} = 3F^1(x) - 6x^2 + 4x - 118, \quad F^1(0) = 0. \tag{43}$$

It is not difficult to determine solutions to Cauchy problems (40)–(43):

$$F(x) = e^{3x}, \quad \breve{F}^1(x) = \frac{1}{3}e^{3x} - \frac{1}{3},$$

$$\overset{\frown}{F}^{1}(x) = 2e^{3x} - 2, \quad F^1(x) = 2x^2 - \frac{118}{3}e^{3x} + \frac{118}{3}.$$

Using representation (23) and notation (39), we obtain:

$$\overset{\smile}{\lambda}^{1} = \left(2F(1) + 3F^0(2)\right)\, u_0 + \left(2F^1(1) + 3F^1(2)\right) +$$

$$+ \left(2\overset{\smile}{F}^{1}(1) + 3\overset{\smile}{F}^{1}(2)\right) \overset{\smile}{\lambda}^{1} + \left(2\overset{\frown}{F}^{1}(1) + 3\overset{\frown}{F}^{1}(2)\right) \lambda^1, \tag{44}$$

$$\overset{\frown}{\lambda}^{1} = \int_2^3 \left[F(\xi)\, u_0 + F^1(\xi) + \overset{\smile}{F}^{1}(\xi)\, \overset{\smile}{\lambda}^{1} + \overset{\frown}{F}^{1}(\xi)\, \overset{\frown}{\lambda}^{1}\right] d\xi. \tag{45}$$

Substituting the found functions $F(x)$, $\overset{\smile}{F}^{1}(x)$, $\overset{\frown}{F}^{1}(x)$, $F^1(x)$ into (44), (45) and adding condition (38), we obtain the algebraic system (29):

$$\begin{cases} 3\left(3e^6 + 2e^3\right)\, u_0 + \left((3e^6 + 2e^3) - 8\right) \overset{\smile}{\lambda}^{1} + \left(6(3e^6 + 2e^3) - 30\right) \overset{\frown}{\lambda}^{1} = \\ \qquad = 118(3e^6 + 2e^3) - 674\,, \\ 3\left(e^9 - e^6\right)\, u_0 + \left((e^9 - e^6) - 3\right) \overset{\smile}{\lambda}^{1} + \left(6(e^9 - e^6) - 27\right) \overset{\frown}{\lambda}^{1} = \\ \qquad = 118(e^9 - e^6) - 468\,, \\ 3\left((e^{12} - 2e^9 + e^6 - e^3) + 1\right)\, u_0 + \left((e^{12} - 2e^9 + e^6 - e^3) - 2\right) \overset{\smile}{\lambda}^{1} + \\ + \left(6(e^{12} - 2e^9 + e^6 - e^3) - 12\right) \overset{\frown}{\lambda}^{1} = 118(e^{12} - 2e^9 + e^6 - e^3) - 227. \end{cases}$$

Direct computation shows that the rank of the matrix of this system is equal to 3, and its only solution is:

$$u_0 = 1, \quad \overset{\smile}{\lambda}^{1} = 33, \quad \overset{\frown}{\lambda}^{1} = \frac{41}{3}.$$

Then from representation (23) we obtain the required solution:

$$u(x) = F(x)u_0 + \overset{\smile}{F}^{1}(x) \cdot \overset{\smile}{\lambda}^{1} + \overset{\frown}{F}^{1}(x) \cdot \overset{\frown}{\lambda}^{1} + F^1(x) = 2x^2 + 1, \quad x \in [0,\, 4].$$

Conclusion

We have proposed an approach to the study and solving a class of nonlocal problems with respect to linear ordinary pointwise and integrally loaded differential equations. The main specificity of integral loadings is that the kernels of the integral terms depend on only one variable of integration. This made it possible to reduce solving the original problem to solving auxiliary Cauchy problems with respect to ordinary differential equations.

The considered problem is of independent interest. But as shown in the paper, the optimal control problems for objects with feedback are reduced to it, in which the current state measurements of an object can be of a point and interval nature.

We have obtained existence and uniqueness conditions of the solution for the considered class of problems, and provided study and solution of one illustrative problem.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Нахушев А.М. Нагруженные уравнения и их применение / А.М. Нахушев. — М.: Наука, 2012. — 232 с.

2 Assanova A.T. Numerical solution of systems of loaded ordinary differential equations with multipoint conditions / A.T. Assanova, A.E. Imanchiyev, Z.M. Kadirbayeva // Comput. Math. and Math. Phys. — 2018. — 58. — No. 4. — P. 508–516. https://doi.org/10.1134/S096554251804005X

3 Дженалиев М.Т. К теории линейных краевых задач для нагруженных дифференциальных уравнений / М.Т. Дженалиев. — Алматы: Комп. центр ИТПМ, 1995. — 270 с.

4 Abdullayev V.M. Identification of the functions of response to loading for stationary systems / V.M. Abdullayev // Cybern. Syst. Anal. — 2017. — 53. — No. 3. — P. 417–425. https://doi.org/10.1007/s10559-017-9942-6

5 Abdullayev V.M. Numerical solution of the problem of determining the number and locations of state observation points in feedback control of a heating process / V.M. Abdullayev, K.R. Aida-zade // Comput. Math. and Math. Phys. — 2018. — 58. — No. 1. — P. 78–89. https://doi.org/10.1134/S0965542518010025

6 Aida-zade K.R. Synthesis of locally lumped controls for membrane stabilization with optimization of sensor and vibration suppressor locations / K.R. Aida-zade, V.A. Hashimov // Comput. Math. and Math. Phys. — 2020. — 60. — No. 7. — P. 1126–1142. https://doi.org/10.1134/S0965542520050024

7 Abdullaev V.M. On the numerical solution of loaded systems of ordinary differential equations / V.M. Abdullaev, K.R. Aida-zade // Comput. Math. and Math. Phys. — 2004. — 44. — No. 9. — P. 1505–1515.

8 Абрамов А.А. Пакет прикладных программ для решения линейных двухточечных краевых задач / А.А. Абрамов, Н.Г. Бураго, В.В. Диткин и др. // Сообщения по программному обеспечению ЭВМ. — М.: ВЦ АН СССР, 1982. — 63 с.

9 Moszynski K. A method of solving the boundary value problem for a system of linear ordinary differential equation / K. Moszynski // Algorytmy. — 1964. — 2. — No. 3. — P. 25–43.

10 Aida-zade K.R. To the solution of integro-differential equations with nonlocal conditions / K.R. Aida-zade, V.M. Abdullayev // Turkish Journal of Mathematics. — 2022. — 46. — No. 1. — P. 177–188. https://doi.org/10.3906/mat-2108-89

11 Яковлев М.Н. Оценки решений систем нагруженных интегро-дифференциальных уравнений, подчиненных многоточечным и интегральным краевым условиям / М.Н. Яковлев // Зап. науч. семинаров ЛОМИ. — 1983. — 124. — С. 131–139.

12 Baiburin M.M. Exact solution to systems of linear first-order integro-differential equations with multipoint and integral conditions / M.M. Baiburin, E. Providas // Mathematical Analysis and Applications. Springer Optimization and Its Applications — Berlin: Springer-Cham, 2019. — 154. — P. 1–16. https://doi.org/10.1007/978-3-030-31339-5_1

13 Bakirova E.A. A problem with parameter for the integro-differential equations / E.A. Bakirova, A.T. Assanova, Zh.M. Kadirbayeva // Mathematical Modelling and Analysis. — 2021. — 26. — No. 1. — P. 34–54. https://doi.org/10.3846/mma.2021.11977

14 Assanova A.T. On solvability of boundary value problem for a nonlinear Fredholm integro-differential equation / S.S. Zhumatov, S.T. Mynbayeva, S.G. Karakenova // Bulletin of the

Karaganda University. Mathematics series. — 2022. — No. 1(105). — P. 25–34. https://doi.org/10.31489/2022m1/25-34

15 Mardanov M.J. Existence and uniqueness of solutions for the system of integro-differential equations with three-point and nonlinear integral boundary conditions / M.J. Mardanov, Y.A. Sharifov, K.E. Ismayilova // Bulletin of the Karaganda University. Mathematics series. — 2020. — No. 3(99). — P. 26–37. https://doi.org/10.31489/2020m3/26-37

16 Mardanov M.J. Existence and uniqueness results for the first-order non-linear impulsive integro-differential equations with two-point boundary conditions / M.J. Mardanov, R.S. Mammadov, S.Yu. Gasimov, Y.A. Sharifov // Bulletin of the Karaganda University. Mathematics series. — 2021. — No. 2(102). — P. 74–83. https://doi.org/10.31489/2021m2/74-83

17 Dzhumabaev D.S. Criteria for the unique solvability of a linear two-point boundary value problem for systems of integro-differential equations / D.S. Dzhumabaev, E.A. Bakirova // Differ. Equations. — 2013. — 49. — No. 9. — P. 1087–1102. https://doi.org/10.1134/S0012266113090048

18 Parasidis I.N. On the exact solution of nonlinear integro-differential equations / I.N. Parasidis, E. Providas // Applications of Nonlinear Analysis. Springer Optimization and Its Applications — Berlin: Springer-Cham, 2018. — 134. — P. 591–609. https://doi.org/10.1007/978-3-319-89815-5_21

19 Aida-zade K.R. On the solution of boundary value problems with nonseparated multipoint and integral conditions / K.R. Aida-zade, V.M. Abdullaev // Differ. Equations. — 2013. — 49. — No. 9. — P. 1114–1125. https://doi.org/10.1134/S0012266113090061

20 Alikhanov A.A. Boundary value problems for certain classes of loaded differential equations and solving them by finite difference methods / A.A. Alikhanov, A.M. Berezgov, M.X. Shkhanukov-Lafishev // Comput. Math. and Math. Phys. — 2008. — 48. — No. 9. — P. 1581–1590. https://doi.org/10.1134/S096554250809008X

21 Abdullayev V.M. Approach to the numerical solution of optimal control problems for loaded differential equations with nonlocal conditions / V.M. Abdullayev, K.R. Aida-zade // Comput. Math. and Math. Phys. — 2019. — 59. — No. 5. — P. 696–707. https://doi.org/10.1134/S0965542519050026

22 Aida-Zade K.R. Numerical method for solving the parametric identification problem for loaded differential equations / K.R. Aida-Zade, V.M. Abdullayev // Bull. Iran. Math. Soc. — 2019. — 45. — No. 6. — P. 1725–1742. https://doi.org/10.1007/s41980-019-00225-3

23 Dzhenaliev M.T. Optimal control of linear loaded parabolic equations / M.T. Dzhenaliev // Differ. Equations. — 1989. — 25. — No. 4. — P. 437–445.

Нүктелік және интегралдық жүктелген дифференциалдық теңдеулер класы жайында

К.Р. Айда-заде[1,2], В.М. Абдуллаев[1,3]

[1] *Әзірбайжан Республикасы Ғылым және білім министрлігінің Басқару жүйелері институты, Баку, Әзірбайжан;*
[2] *Әзірбайжан Республикасы Ғылым және білім министрлігінің Математика және механика институты, Баку, Әзірбайжан;*
[3] *Әзірбайжан мемлекеттік мұнай және өнеркәсіп университеті, Баку, Әзірбайжан*

Бейлокал шекаралық шарттары бар нүктелік және интегралдық жүктемелерден тұратын сызықтық қарапайым дифференциалдық теңдеулер жүйесі зерттелген. Шекаралық шарттарға белгісіз функцияның интегралдық және нүктелік мәндері жатады. Есептің маңызды шарты дифференциалдық теңдеулердегі интегралдық мүшелердің ядролары тек интеграциялық айнымалыға тәуелділігінде. Ұқсас есептердің басқарылатын объектінің ағымдағы күйін нүктелік және интегралдық өлшеулер кезінде біріктірілген және бөлінген параметрлері бар екі объектінің де кері байланысын бақылау кезінде туындайтыны көрсетілген. Мақалада қарастырылған есептің қойылуы бейлокал шекаралық шарттармен жүктелген дифференциалдық теңдеулер бойынша бұрын зерттелген көптеген есептерді жалпылайды. Көмекші параметрлерді енгізу арқылы қарастырылатын есептің шешімінің бар және жалғыз болуының қажетті шарттары алынды. Есепті сандық түрде шешу үшін төрт көмекші Коши есебінің шешімі болып табылатын төрт матрицалық функцияны қамтитын бастапқы есептің шешімін пайдалану ұсынылады. Шекаралық жағдайларда көмекші есептердің шешімдерін пайдалана отырып, жүктеу нүктелеріндегі белгісіз функцияның мәндері алынды. Бұл қажетті шешімді алу үшін жеткілікті. Мақалада модельдік есепті шешудің мысалы арқылы әдісті қолдану көрсетілген.

Кілт сөздер: интегралдық-дифференциалдық теңдеу, жүктелген теңдеулер жүйесі, интегралдық шарттар, бейлокал шарттар, бар және жалғыз болу шарттары.

О классе точечно и интегрально нагруженных дифференциальных уравнений

К.Р. Айда-заде[1,2], В.М. Абдуллаев[1,3]

[1] *Институт систем управления Министерства науки и образования Азербайджанской Республики, Баку, Азербайджан;*
[2] *Институт математики и механики Министерства науки и образования Азербайджанской Республики, Баку, Азербайджан;*
[3] *Азербайджанский государственный университет нефти и промышленности, Баку, Азербайджан*

Исследована система линейных обыкновенных дифференциальных уравнений, содержащая точечные и интегральные нагружения, с нелокальными краевыми условиями. Краевые условия включают интегральные и точечные значения неизвестной функции. Существенным условием в задаче является то, что ядра интегральных слагаемых в дифференциальных уравнениях зависят лишь от переменной интегрирования. Показано, что подобные задачи возникают при управлении с обратной связью как объектами с сосредоточенными, так и распределенными параметрами при точечных и интегральных замерах текущего состояния управляемого объекта. Указанная в статье постановка задачи обобщает многие исследованные ранее задачи относительно нагруженных дифференциальных уравнений с нелокальными краевыми условиями. Введением вспомогательных параметров получены необходимые условия существования и единственности решения рассматриваемой задачи. Для численного решения задачи предложено использовать представление решения исходной задачи, включающее четыре матричные функции, являющиеся решениями четырех вспомогательных задач Коши. Используя решения вспомогательных задач в краевых условиях, получены значения неизвестной функции в точках нагружения. Это достаточно, чтобы получить искомое решение. В статье приведено изложение применения метода на примере решения модельной задачи.

Ключевые слова: интегро-дифференциальное уравнение, система нагруженных уравнений, интегральные условия, нелокальные условия, условия существования и единственности.

References

1 Nakhushev, A.M. (2012). *Nagruzhennye uravneniia i ikh primenenie [Loaded Equations and their Applications]*. Moscow: Nauka [in Russian].

2 Assanova, A.T., Imanchiyev, A.E., & Kadirbayeva, Z.M. (2018). Numerical solution of systems of loaded ordinary differential equations with multipoint conditions. *Comput. Math. and Math. Phys., 58*(4), 508–516. https://doi.org/10.1134/S096554251804005X

3 Dzhenaliev, M.T. (1995). *K teorii lineinykh kraevykh zadach dlia nagruzhennykh differentsialnykh uravnenii [On the Theory of Linear Boundary Value Problems for Loaded Differential Equations].* Almaty: Kompiuternyi tsentr ITPM [in Russian].

4 Abdullayev, V.M. (2017). Identification of the functions of response to loading for stationary systems. *Cybern. Syst. Anal., 53*(3), 417–425. https://doi.org/10.1007/s10559-017-9942-6

5 Abdullayev, V.M., & Aida-zade, K.R. (2018). Numerical solution of the problem of determining the number and locations of state observation points in feedback control of a heating process. *Comput. Math. and Math. Phys., 58*(1), 78–89. https://doi.org/10.1134/S0965542518010025

6 Aida-zade, K.R., & Hashimov, V.A. (2020). Synthesis of locally lumped controls for membrane stabilization with optimization of sensor and vibration suppressor locations. *Comput. Math. and Math. Phys., 60*(7), 1126–1142. https://doi.org/10.1134/S0965542520050024

7 Abdullaev, V.M., & Aida-zade, K.R. (2004). On the numerical solution of loaded systems of ordinary differential equations. *Comput. Math. and Math. Phys., 44*(9), 1505–1515.

8 Abramov, A.A., Burago, N.G., & Dyshlo, A.L. et al. (1982). Paket prikladnykh programm dlia resheniia lineinykh dvukhtochechnykh kraevykh zadach. Soobshcheniia po programmnomu obespe-cheniiu EVM [Software package for solving linear two-point boundary value problems]. *Reports on Computer Software.* Moscow: VTs AN SSSR [in Russian].

9 Moszynski, K. (1964) A method of solving the boundary value problem for a system of linear ordinary differential equation. *Algorytmy, 2*(3), 25–43.

10 Aida-zade, K.R., & Abdullayev, V.M. (2022). To the solution of integro-differential equations with nonlocal conditions. *Turkish Journal of Mathematics, 46*(1), 177–188. https://doi.org/10.3906/mat-2108-89

11 Yakovlev, M.N. (1983). Otsenki reshenii sistem nagruzhennykh integro-differentsialnykh uravnenii, podchinennykh mnogotochechnym i integralnym kraevym usloviiam [Estimates for solutions to systems of loaded integro-differential equations subject to multi-point and integral boundary conditions]. *Zapiski nauchnykh seminarov Leningradskogo otdeleniia Matematicheskogo instituta — Notes of scientific seminars of Leningrad branch of the Mathematical Institute, 124*, 131–139 [in Russian].

12 Baiburin, M.M., & Providas, E. (2019). Exact solution to systems of linear first-order integro-differential equations with multipoint and integral conditions. *Mathematical Analysis and Applications. Springer Optimization and Its Applications.* Berlin: Springer-Cham, 154, 1–16. https://doi.org/10.1007/978-3-030-31339-5_1

13 Bakirova, E.A., Assanova, A.T., & Kadirbayeva, Zh.M. (2021). A problem with parameter for the integro-differential equations. *Mathematical Modelling and Analysis, 26*(1), 34–54. https://doi.org/10.3846/mma.2021.11977

14 Assanova, A.T., Zhumatov, S.S., Mynbayeva, S.T., & Karakenova, S.G. (2022). On solvability of boundary value problem for a nonlinear Fredholm integro-differential equation. *Bulletin of the Karaganda University. Mathematics series,* 1(105), 25–34. https://doi.org/10.31489/2022m1/25-34

15 Mardanov, M.J., Sharifov, Y.A., & Ismayilova, K.E. (2020). Existence and uniqueness of solutions for the system of integro-differential equations with three-point and nonlinear integral boundary conditions. *Bulletin of the Karaganda University. Mathematics series,* 3(99), 26–37. https://doi.org/10.31489/2020m3/26-37

16 Mardanov, M.J., Mammadov, R.S., Gasimov, S.Yu., & Sharifov, Y.A. (2021). Existence and uniqueness results for the first-order non-linear impulsive integro-differential equations with two-point boundary conditions. *Bulletin of the Karaganda University. Mathematics series,* 2(102),

74–83. https://doi.org/10.31489/2021m2/74-83

17 Dzhumabaev, D.S., & Bakirova, E.A. (2013). Criteria for the unique solvability of a linear two-point boundary value problem for systems of integro-differential equations. *Differ. Equations, 49*(9), 1087–1102. https://doi.org/10.1134/S0012266113090048

18 Parasidis, I.N., & Providas E. (2018). On the exact solution of nonlinear integro-differential equations. *Applications of Nonlinear Analysis. Springer Optimization and Its Applications.* Berlin: Springer-Cham, 134, 591–609. https://doi.org/10.1007/978-3-319-89815-5_21

19 Aida-zade, K.R., & Abdullaev, V.M. (2013). On the solution of boundary value problems with nonseparated multipoint and integral conditions. *Differ. Equations, 49*(9), 1114–1125. https://doi.org/10.1134/S0012266113090061

20 Alikhanov, A.A., Berezgov A.M., & Shkhanukov-Lafishev M.X. (2008). Boundary value problems for certain classes of loaded differential equations and solving them by finite difference methods. *Comput. Math. and Math. Phys., 48*(9), 1581-1590. https://doi.org/10.1134/S096554250809008X

21 Abdullayev,V.M., & Aida-zade, K.R. (2019). Approach to the numerical solution of optimal control problems for loaded differential equations with nonlocal conditions. *Comput. Math. and Math. Phys., 59*(5), 696–707. https://doi.org/10.1134/S0965542519050026

22 Aida-Zade, K.R., & Abdullayev, V.M. (2019). Numerical method for solving the parametric identification problem for loaded differential equations. *Bull. Iran. Math. Soc., 45*(6), 1725–1742. https://doi.org/10.1007/s41980-019-00225-3

23 Dzhenaliev, M.T. (1989). Optimal control of linear loaded parabolic equations. *Differ. Equations, 25*(4), 437–445.

*Author Information**

Kamil Aida-zade — Corresponding Member of the National Academy of Sciences of Azerbaijan, Doctor of physical and mathematical sciences, Professor, Head of the Department Institute of Control Systems of Ministry of Science and Education of Republic of Azerbaijan; Senior Researcher of Institute of Mathematics and Mechanics of Ministry of Science and Education of Republic of Azerbaijan, 68 Bakhtiyar Vahabzadeh street, Baku, AZ1141, Azerbaijan; e-mail: *kamil_aydazade@rambler.ru*; https://orcid.org/0000-0002-8439-5585

Vagif Abdullayev (*corresponding author*) — Doctor of physical and mathematical sciences, Professor, Professor of Azerbaijan State Oil and Industry University; Senior Researcher of Institute of Control Systems of Ministry of Science and Education of Republic of Azerbaijan; Azerbaijan State Oil and Industry University, 20 Azadlyg avenue, Baku, AZ1010, Azerbaijan; e-mail: *vaqif_ab@rambler.ru*; https://orcid.org/0000-0001-7772-1226

*The author's name is presented in the order: First, Middle and Last Names.

https://doi.org/10.31489/2024M1/21-27 *Research article*

On some estimations of deviations between real solution and numerical solution of dynamical equations with regard for Baumgarte constraint stabilization

M.I. Akylbayev[1,*], I.E. Kaspirovich[2]

[1] *Peoples Friendship University named after academician A.Kuatbekov, Shymkent, Kazakhstan;*
[2] *Patrice Lumumba Peoples' Friendship University of Russia, Moscow, Russian Federation*
(E-mail: musabek_kz@mail.ru, kaspirovich_ie@rudn.ru)

The numerical solution of a system of differential equations with constraints can be unstable due to the accumulation of rounding errors during the implementation of the difference scheme of numerical integration. To limit the amount of accumulation, the Baumgarte constraint stabilization method is used. In order to estimate the deviation of real solution from the numerical one the method of constraint stabilization can be used to derive required formulas. The well-known technique of expansion the deviation function to Taylor series is being used. The paper considers the estimation of the error of the numerical solution obtained by the first-order Euler method.

Keywords: constraint stabilization, numerical integration, stability, dynamics, system of differential levels, numerical methods, numerical solution, difference scheme, rounding.

2020 Mathematics Subject Classification: 65D30.

Introduction

The description of the dynamics of the system using Hamilton or Lagrange formalisms assumes the solution of differential equations or a qualitative study of their properties [1]. It is not always possible to obtain analytically the solution of systems of differential equations. Therefore, it is necessary to resort to numerical integration methods [2] or to methods of investigating the properties of solutions using methods of the qualitative theory of differential equations [3].

The use of numerical methods for solving differential equations is associated with the inevitable accumulation of numerical integration errors. Therefore, the result of the numerical solution reflects the real picture only with some degree of accuracy. The fact is that the implementation of one or another difference scheme of numerical integration is accompanied by the accumulation of numerous errors, in particular rounding errors.

Baumgarte showed [4] that the classical method of determining the reactions of contact constraints used in mechanics leads to an inevitable accumulation of numerical integration errors associated with an increase in the values of deviations from the constraints equations caused by errors in setting the initial conditions. To reduce these deviations, Baumgarte proposed using linear combinations of constraints equations together with their derivatives. The equations that establish the relationship between linear combinations of constraints and their derivatives are called the equations of perturbations of constraints. In essence, the Baumgarte method boils down to replacing the constraints equations with servo constraints equations. The method of bond stabilization proposed by Baumgarte proved popular and

*Corresponding author. *E-mail: musabek_kz@mail.ru*

The research was funded by the Russian Science Foundation and Moscow city project No. 23-21-10065, https://rscf.ru/project/23-21-10065/.

Received: 22 November 2023; *Accepted:* 13 December 2023.

caused the emergence of various modifications. Thus, Yu. Ascher proposed a method for stabilizing systems of higher-order differential algebraic equations with constraints [5].

The conditions imposed on the behavior of solving a system of dynamic equations with deviations from the constraints equations leads to additional requirements for determining the constraints reactions. For these purposes, the concept of program constraints was introduced.

The first-order Euler difference scheme is the simplest scheme for numerical integration of systems of first-order differential equations. When integrating, the area under the curve is searched for as an area collections of rectangles. Any set of rectangles of finite length will not be able to completely cover the area of a curved trapezoid, so the numerical solution of the integrable equation does not coincide with the real one. The estimate of the maximum value of this error can be calculated by considering the deviation of the numerical solution from the real one.

In theoretical mechanics there is a specific set of problems that define their goal as constructing a system of ordinary differential equations based on the given properties. Such problems are being called inverse ones. In some cases we need to find the specific constraints equations that provide the system with its requested properties. These constraints are defined as program constraints. Methods for solving systems of differential algebraic equations were investigated in work [6]. If a system contains some ambiguity by its internal random parameters than it can be considered as stochastic. Some inverse dynamical problems for the system with stochastic parameters are considered in papers [7–9]. In some cases the system of motion equations is required to be constructed with regard for Baumgarte constraint stabilization method implemented in it. In papers [10, 11] it was shown that perturbation parameters are connected with dissipative function that pumps energy out of the system. New advanced numerical methods were investigated for inverse-like problems in works [12, 13].

1 Problem Statement

Let the state of a mechanical system be given by the set of generalized coordinates $q = \left(q^1, ..., q^n\right)$. The change in the position of the mechanical system in time implies the dependence of the vector q on time t: $q = q(t)$. The rate of change in the position of the system is determined by the velocity vector: $v(t) = dq(t)/dt = \dot{q}(t) = (\dot{q}^1, ..., \dot{q}^n)$. Let's consider that the system of motion equations is presented in form:

$$\begin{aligned} \dot{q} &= \nu; \\ \dot{\nu} &= a(q, t), \end{aligned} \tag{1}$$

where $a(q, t)$ is a given function. Let's introduce a vector state $x = (q, \nu)$ and rewrite (1) in a matrix form:

$$\dot{x} = F(x, t). \tag{2}$$

Suppose that the motion is restricted and the kinematic state vector $x(t)$ is limited by a set of mechanical constraints described by the equations:

$$h_i(q, t) = 0, \ \ i = 1, ..., m, \ \ m < n. \tag{3}$$

Here and in the future, corresponding to Einstein's notation, repeating indices imply summation by the same indices.

In order to solve system (2) with constraints (3) the method of Lagrange multipliers is used. But during the numerical integration with Euler first order scheme we will inevitably face with solution's instability. To solve this problem J. Baumgarte [1] suggested to consider an arbitrary linear of constraints and its full time derivatives while solving the system of differential algebraic equations with constraints. According to this stabilization method our system will take form:

$$\begin{cases} \dot{x} = F(x, t), \\ \ddot{h} + A\dot{h} + Bh = 0, \end{cases}$$

where A and B are matrices with arbitrary components that are called perturbation parameters. By manipulating the values of these components, we can achieve a stable numerical solution. We can algebraically solve obtained system and derive $\dot{x}$:

$$\dot{x} = X(x + \aleph, t). \tag{4}$$

Symbol $\aleph$ here stands for the terms with perturbation parameters and provides numerical stability.

2 Numerical Integration

In numerical integration, it is assumed that the differentials of the function and the independent argument are represented in finite differences $dx(t) \approx \Delta x(t)$, $dt \approx \Delta t$. Thus, the functions become functions of a discrete argument.

Let equation (4) be determined on a set $[t_1, t_2]$. In the simplest difference schemes, this set can be divided by points $t_1 = t_{(1)}, t_{(2)}, ..., t_{(2)} = t_{(N)}$ and $(N-1)$ equal length segments $\tau = t_{(\alpha+1)} - t_{(\alpha)}$, corresponding to the integration step. Finite increment of a state vector $x(t)$ can be represented as a difference:

$$\Delta x \left(t_{(\alpha)}\right) = x \left(t_{(\alpha+1)}\right) - x \left(t_{(\alpha)}\right), \;\; \alpha = 1, ..., N-1.$$

We use Euler first order difference scheme to solve equation (4) numerically:

$$x_{(\alpha+1)} = x_{(\alpha)} + \tau X_{(\alpha)}, \qquad \alpha = 1, ..., N-1. \tag{5}$$

To estimate the deviation error, the real solution will be denoted $\tilde{x}(t)$. It satisfies the system (2) with constraints (3). It does not include stabilization terms. Consider the deviation of a real solution from a numerical one (5) at the moment $t_{(\alpha)} : \tilde{x}(t_{(\alpha)}) - x_{(\alpha)}$. Let's expend $\tilde{x}(t_{(\alpha)})$ at $t_{(\alpha)}$ to Taylor series:

$$\tilde{x}(t_{(\alpha)}) = \tilde{x}(t_{(\alpha-1)}) + \tau\dot{\tilde{x}}(t_{(\alpha-1)}) + \frac{\tau^2}{2}\ddot{\tilde{x}}(\zeta),$$

where $\zeta : \zeta \in \left[t_{(\alpha-1)}, t_{(\alpha)}\right]$. Taking into account (2) deviation $\tilde{x}(t_{(\alpha)}) - x_{(\alpha)}$ will be written in the form:

$$\tilde{x}(t_{(\alpha)}) - x_{(\alpha)} = \tilde{x}(t_{(\alpha-1)}) - x_{(\alpha-1)} + \tau \left(\tilde{X}\left(t_{(\alpha-1)}\right) - X_{(\alpha-1)}\right) + \frac{\tau^2}{2}\ddot{\tilde{x}}(\zeta). \tag{6}$$

If we apply mean value theorem to the term $\tilde{X}\left(t_{(\alpha-1)}\right) - X_{(\alpha-1)}$ we will obtain the following relation:

$$\tilde{X}\left(\tilde{x}(t_{(\alpha-1)}), t_{(\alpha-1)}\right) - X\left(x_{(\alpha-1)} + \aleph, t_{(\alpha-1)}\right) = \frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right)\left(\tilde{x}(t_{(\alpha-1)}) - x_{(\alpha-1)} - \aleph\right),$$

where $x_\zeta \in \left[\tilde{x}(t_{(\alpha-1)}), x_{(\alpha-1)}\right]$ or $x_\zeta \in \left[x_{(\alpha-1)}, \tilde{x}(t_{(\alpha-1)})\right]$ depending on which value is greater. $\frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right)$ – matrix $[2n \times 2n]$.

Let's denote the deviation $\tilde{x}(t_{(\alpha)}) - x_{(\alpha)} = \Delta_{(\alpha)}$, then the ratio (6) can be rewritten as:

$$\Delta_{(\alpha)} = \left(I_{2n} + \tau\frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right)\right)\Delta_{(\alpha-1)} - \tau\aleph\frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right) + \frac{\tau^2}{2}\ddot{\tilde{x}}(\zeta), \tag{7}$$

where I_{2n} is a unit matrix.

Denote $\Im = \max\limits_{t\in[t_0,t_k]}\left(\frac{\tau}{2}\ddot{\tilde{x}}(\zeta) - \aleph\frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right)\right)$. Taking into account the triangle inequality, the ratio (7) will take form:

$$\left|\Delta_{(\alpha)}\right| \leq \left|\Delta_{(\alpha-1)}\right|\left|I_{2n} + \tau\frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right)\right| + \tau\Im.$$

The norm of a vector or matrix by components is understood as the maximum value of the modulus of its components: $\left|\Delta_{(\alpha)}\right| = \max\limits_{k=1,\dots,2n} \left|\Delta_{k(\alpha)}\right|$. Solving this recursive inequality with respect to the first element, we obtain:

$$\left|\Delta_{(\alpha)}\right| \leq \left|\Delta_{(1)}\right| \left|I_{2n} + \tau \frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right)\right|^{\alpha-1} + \tau \Im \sum\nolimits_{l=1}^{\alpha-2} \left|I_{2n} + \tau \frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right)\right|^{l}.$$

Let's assume that the integration step is small enough, so the expression $1 + \tau \frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right)$ is positive $\forall t \in [t_0, t_k]$ even, if the derivative $\frac{\partial X}{\partial x}$ is negative. Also apply to the second term the formula of the sum of a finite number of elements of geometric series, we get:

$$\left|\Delta_{(\alpha)}\right| \leq \left|\Delta_{(1)}\right| \left|I_{2n} + \tau \frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right)\right|^{\alpha-1} + \Im \frac{\left|I_{2n} + \tau \frac{\partial X}{\partial x}(x_\zeta, t_{(\alpha-1)})\right|^{\alpha-1} - 1}{\left|\frac{\partial X}{\partial x}(x_\zeta, t_{(\alpha-1)})\right|}. \tag{8}$$

As

$$I_{2n} + \tau \frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right) \leq \exp\left(\tau \frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right)\right),$$

then:

$$\left|I_{2n} + \tau \frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right)\right|^{\alpha-1} \leq \left|\exp\left(\tau(\alpha-1) \frac{\partial X}{\partial x}\left(x_\zeta, t_{(\alpha-1)}\right)\right)\right|$$

and $t_k = t_0 + (N-1)\tau$, $\alpha \leq N$, $\tau(\alpha-1) \leq t_k - t_0$. Then the ratio (8) will be written in the form:

$$\left|\Delta_{(\alpha)}\right| \leq \left|\Delta_{(1)}\right| \left|\exp\left(t_k - t_0\right) \frac{\partial X}{\partial x}\left(x_\zeta, t_\zeta\right)\right| + \Im \frac{\left|\exp\left(t_k - t_0\right) \frac{\partial X}{\partial x}\left(x_\zeta, t_\zeta\right)\right| - 1}{\left|\frac{\partial X}{\partial x}\left(x_\zeta, t_\zeta\right)\right|}. \tag{9}$$

The right side of this ratio does not include the node number, so you can also enter the norm for nodes: $\alpha \left|\left|\Delta\right|\right| = \max\limits_{\alpha=1,\dots,N-1} \left|\Delta_{(\alpha)}\right|$. Then the relation (9) allows you to set the ratio for the maximum possible error in numerical integration using the Euler difference scheme:

$$\left|\left|\Delta\right|\right| \leq \left|\Delta_{(1)}\right| \left|\exp\left(t_k - t_0\right) \frac{\partial X}{\partial x}\left(x_\zeta, t_\zeta\right)\right| + \Im \frac{\left|\exp\left(t_k - t_0\right) \frac{\partial X}{\partial x}\left(x_\zeta, t_\zeta\right)\right| - 1}{\left|\frac{\partial X}{\partial x}\left(x_\zeta, t_\zeta\right)\right|}.$$

Conclusion

It follows from this relation that the maximum possible error exponentially depends on the length of the segment on which the integration takes place. Also, the second term of this relation contains the stabilization term $\aleph$ associated with the equations of perturbed constraints. Therefore, a change in the values of the perturbation parameters affects the maximum deviation error during numerical integration. However, due to the arbitrariness of the type of functions $X(x,t)$, it is extremely difficult to draw a conclusion about the direct relationship between the perturbation parameters and the maximum deviation value. Only in some cases, discussed below, the estimates of the perturbation parameters can be determined by the formula, while ensuring the stability of the numerical solution.

Acknowledgments

The research was funded by the Russian Science Foundation and Moscow city project No. 23-21-10065, https://rscf.ru/ project/23-21-10065/.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Степанов В.В. Курс дифференциальных уравнений / В.В. Степанов // URSS. — 2016. — 512 с.

2 Atkinson K. Numerical solutions of ordinary differential equations / K. Atkinson. — John Wiley & Sons, Inc., Hoboken, New Jersey, 2009. — 261 p.

3 Немыцкий В.В. Качественная теория дифференциальных уравнений / В.В. Немыцкий, В. Степанов. — 4-е изд. — Ленанд, 2017. — 552 с.

4 Baumgarte J. Stabilization of constraints and integrals of motion in dynamical systems / J. Baumgartee // Computer Methods in Applied Mechanics and Engineering. — 1972. — Vol. 1. — No. 1. — P. 1–16.

5 Ascher U.M. Stabilization of DAEs and invariant manifolds / U.M. Ascher, H. Chin, S. Reich // Numerische Mathematik. — 1994. — Vol. 67. — No. 2. — P. 131–149.

6 Mukharlyamov R.G. Differential-algebraic equations of programmed motions of Lagrangian dynamical systems / R.G. Mukharlyamov // Mechanics of Solids. — 2011. — Vol. 46. — No. 4. — P. 534–543. https://doi.org/10.3103/S0025654411040042

7 Tleubergenov M.I. Construction of the differential equations system of the program motion in Lagrangian variables in the presence of random perturbations / M.I. Tleubergenov, G.K. Vassilina, D.T. Azhymbaev // Bulletin of the Karaganda University. Mathematics series. — 2022. — No. 1(105). — P. 118–126. https://doi.org/10.31489/2022M1/118-126

8 Tleubergenov M.I. Construction of stochastic differential equations of motion in canonical variables / M.I. Tleubergenov, G.K. Vassilina, S.R. Seisenbayeva // Bulletin of the Karaganda University. Mathematics series. — 2022. — No. 3(107). — P. 152–162. https://doi.org/10.31489/2022M3/152-162

9 Tleubergenov M.I. On construction of a field of forces along given trajectories in the presence of random perturbations // M.I. Tleubergenov, G.K. Vassilina, G.A. Tuzelbaeva // Bulletin of the Karaganda University. Mathematics series. — 2021. — No. 1(101). — P. 98–103. https://doi.org/10.31489/2021m1/98-103

10 Kaspirovich I.E. On Constructing Dynamic Equations Methods with Allowance for Atabilization of Constraints / I.E. Kaspirovich, R.G. Mukharlyamov // Mechanics of Solids. — 2019. — Vol. 54. — No. 4. — P. 589–597.

11 Kaspirovich I.E. Possible solutions of inverse dynamical problems with regards for nonlinear constraint stabilization function / I.E. Kaspirovich, R.G. Mukharlyamov // Journal of Physics: Conference Series. — 2020. — Vol. 1705. — P. 012013.

12 Omkar R. Numerical solution of differential-difference equations having an interior layer using nonstandard finite differences / R. Omkar, M. Lalu, K. Phaneendra // Bulletin of the Karaganda University. Mathematics series. — 2023. — No. 2(110). — P. 104–115. https://doi.org/10.31489/2023M2/104-115

13 Sinsoysal B. Numerical method to solution of generalized model Buckley-Leverett in a class of discontinuous functions / B. Sinsoysal, M. Rasulov, R. Iskenderova // Bulletin of the Karaganda University. Mathematics series. — 2023. — No. 1(109). — P. 131–140. https://doi.org/10.31489/2023M1/131-140

Баумгарт байланысының тұрақтануын ескере отырып, динамикалық теңдеулердің нақты және сандық шешімі арасындағы ауытқулардың кейбір бағалаулары туралы

М.И. Ақылбаев[1], И.Е. Каспирович[2]

[1] *Академик Ә. Қуатбеков атындағы Халықтар достығы университеті, Шымкент, Қазақстан;*
[2] *П. Лумумба атындағы Ресей халықтар достығы университеті, Мәскеу, Ресей*

Сандық интегралдауда айырымдық схемасын жүзеге асыру кезінде дөңгелектеу қателерінің жиналуына байланысты байланысы бар дифференциалдық теңдеулер жүйесінің сандық шешімі тұрақсыз болуы мүмкін. Жиналу мөлшерін шектеу үшін Баумгарт байланысын тұрақтандыру әдісі қолданылады. Нақты шешімнің сандық шешімнен ауытқуын бағалауда қажетті формулаларды алу үшін тұрақтандыру әдісін пайдалануға болады. Ауытқу функциясын Тейлор қатарына жіктеудің белгілі әдісі қолданылған. Мақалада бірінші ретті Эйлер әдісімен алынған сандық шешімнің қателігін бағалау қарастырылды.

Кілт сөздер: байланыстарды тұрақтандыру, сандық интегралдау, тұрақтылық, динамика, дифференциалдық теңдеулер жүйесі, сандық әдістер, сандық шешім, айырымдық схемасы, дөңгелектеу.

О некоторых оценках отклонений между реальным и численным решениями динамических уравнений с учетом стабилизации связи Баумгарта

М.И. Акылбаев[1], И.Е. Каспирович[2]

[1] *Университет дружбы народов имени академика А. Куатбекова, Шымкент, Казахстан;*
[2] *Российский университет дружбы народов имени П. Лумумбы, Москва, Россия*

Численное решение систем дифференциальных уравнений со связями может быть нестабильным из-за накопления ошибок округления при реализации разностной схемы численного интегрирования. Для ограничения величины накопления использован метод стабилизации связей Баумгарта. Для оценки отклонения реального решения от численного может быть применен метод стабилизации для получения требуемых формул. Использован хорошо известный метод разложения функции отклонения в ряд Тейлора. В статье рассмотрена оценка погрешности численного решения, полученного методом Эйлера первого порядка.

Ключевые слова: стабилизация связей, численное интегрирование, устойчивость, динамика, система дифференциальных уровней, численные методы, численное решение, разностная схема, округление.

References

1 Stepanov, V.V. (2016). *Kurs differentsialnykh uravnenii [Course of differential equations]*. URSS [in Russian].

2 Atkinson, K. (2009). *Numerical solutions of ordinary differential equations.* John Wiley & Sons, Inc., Hoboken, New Jersey.

3 Nemytskii, V. (2017). *Kachestvennaia teoriia differentsialnykh uravnenii [Qualitative theory of differential equations]*. Lenand [in Russian].

4 Baumgarte, J.(1972). Stabilization of constraints and integrals of motion in dynamical systems. *Computer Methods in Applied Mechanics and Engineering, 1*(1), 1–16.

5 Ascher, U.M., Chin, H., & Reich, S. (1994). Stabilization of DAEs and invariant manifolds. *Numerische Mathematik, 67*(2), 131–149.

6 Mukharlyamov, R. (2011). Differential-algebraic equations of programmed motions of Lagrangian dynamical systems. *Mechanics of Solids, 46*(4), 534–543. https://doi.org/10.3103/S0025654411040042

7 Tleubergenov, M., Vassilina, G.K., & Azhymbaev, D.T. (2022). Construction of the differential equations system of the program motion in Lagrangian variables in the presence of random perturbations. *Bulletin of the Karaganda University. Mathematics series*, 1(105), 118–126. https://doi.org/10.31489/2022M1/118-126

8 Tleubergenov, M. (2022). Construction of stochastic differential equations of motion in canonical variables. *Bulletin of the Karaganda University. Mathematics series*, 3(107), 152–162. https://doi.org/10.31489/2022M3/152-162

9 Tleubergenov, M. (2021). On construction of a field of forces along given trajectories in the presence of random perturbations. *Bulletin of the Karaganda University. Mathematics series*, 1(101), 98–103. https://doi.org/10.31489/2021m1/98-103

10 Kaspirovich, I., & Mukharlyamov, R. (2019). On Constructing Dynamic Equations Methods with Allowance for Atabilization of Constraints. *Mechanics of Solids, 54*(4), 589–597.

11 Kaspirovich, I., & Mukharlyamov, R. (2020). Possible solutions of inverse dynamical problems with regards for nonlinear constraint stabilization function. *Journal of Physics: Conference Series, 1705*, 012013.

12 Omkar, R., Lalu, M., & Phaneendra, K. (2023). Numerical solution of differential-difference equations having an interior layer using nonstandard finite differences. *Bulletin of the Karaganda University. Mathematics series*, 2(110), 104–115. https://doi.org/10.31489/2023M2/104-115

13 Sinsoysal, B. (2023). Numerical method to solution of generalized model Buckley-Leverett in a class of discontinuous functions. *Bulletin of the Karaganda University. Mathematics series*, 1(109), 131–140. https://doi.org/10.31489/2023M1/131-140

Author Information[*]

Musabek Islamovich Akylbayev (*corresponding author*) — Candidate (PhD) of Technical Sciences, Associate Professor of Mathematics, Peoples' Friendship University named after Academician A. Kuatbekov, 14 Tereshkova street, Shymkent, 160000, Kazakhstan; e-mail: *musabek_kz@mail.ru*; https://orcid.org/0000-0003-1383-4592

Ivan Evgenyevich Kaspirovich — Doctor of physical and mathematical sciences, Assistant at Institute of Physical Research and Technologies at Patrice Lumumba Peoples' Friendship University of Russia, 6 Miklukho-Maklaya street, Moscow, 117198, Russian Federation; e-mail: *kaspirovich_ie@rudn.ru*; https://orcid.org/0000-0002-6583-6621

[*]The author's name is presented in the order: First, Middle and Last Names.

https://doi.org/10.31489/2024M1/28-45 *Research article*

On the behaviors of solutions of a nonlinear diffusion system with a source and nonlinear boundary conditions

M.M. Aripov[1], Z.R. Rakhmonov[1], A.A. Alimov[1,2,*]

[1] *National University of Uzbekistan named after Mirzo Ulugbek, Tashkent, Uzbekistan;*
[2] *Tashkent branch of the G.V. Plekhanov Russian University of Economics, Tashkent, Uzbekistan*
(E-mail: mirsaidaripov@mail.ru, zraxmonov@inbox.ru, akram.alimov18@gmail.com)

We study the global solvability and unsolvability of a nonlinear diffusion system with nonlinear boundary conditions in the case of slow diffusion. We obtain the critical exponent of the Fujita type and the critical global existence exponent, which plays a significant part in analyzing the qualitative characteristics of nonlinear models of reaction-diffusion, heat transfer, filtration, and other physical, chemical, and biological processes. In the global solvability case, the key components of the asymptotic solutions are obtained. Iterative methods, which quickly converge to the exact solution while maintaining the qualitative characteristics of the nonlinear processes under study, are known to require the presence of an appropriate initial approximation. This presents a significant challenge for the numerical solution of nonlinear problems. A successful selection of initial approximations allows for the resolution of this challenge, which depends on the value of the numerical parameters of the equation, which are primarily in the computations recommended using an asymptotic formula. Using the asymptotics of self-similar solutions as the initial approximation for the iterative process, numerical calculations and analysis of the results are carried out. The outcomes of numerical experiments demonstrate that the results are in excellent accord with the physics of the process under consideration of the nonlinear diffusion system.

Keywords: blow-up, nonlinear boundary condition, critical global existence curve, degenerate parabolic systems, critical exponents of Fujita type.

2020 Mathematics Subject Classification: 35A01, 35B44, 35K57, 35K65.

Introduction

The source for this article's discussion of the doubly degenerate parabolic equations is as follows:

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u_i^k}{\partial x}\right|^{m-1}\frac{\partial u_i^k}{\partial x}\right) + u_i^{p_i}, \quad x \in R_+, \quad t > 0, \quad i = 1, 2, \tag{1}$$

coupled through nonlinear boundary conditions:

$$-\left|\frac{\partial u_i^k}{\partial x}\right|^{m-1}\frac{\partial u_i^k}{\partial x}\bigg|_{x=0} = u_{3-i}^{q_i}(0,t), \quad t > 0, \quad i = 1, 2, \tag{2}$$

where $m > 1$, $k \geq 1$, and q_i, $p_i > 0$ are numerical parameters. The following preliminary information should be considered:

$$u_i|_{t=0} = u_{i0}(x), \quad i = 1, 2. \tag{3}$$

It is expected that the function and its corresponding first- and second-order derivatives conform to a set of criteria. Specifically, these derivatives should exhibit a degree of continuity, non-negativity, and compactness within the domain of R_+.

*Corresponding author. *E-mail: akram.alimov18@gmail.com*
Received: 25 July 2023; *Accepted:* 11 December 2023.

Population dynamics, heat transfer, chemical processes, and other phenomena all use parabolic equations with nonlinearity (1).

The functions $u_1(t,x)$, $u_2(t,x)$ represent the biological two populations' densities during migration, the thickness of two types of chemical reagents during a chemical reaction, and the temperatures of two various sorts of materials during propagation. By incorporating the dependent on the power-law of shear stress and displacement velocity, equation (1) becomes an invaluable tool for analyzing a liquid medium with inconsistent fluxes. This allows for a comprehensive understanding of the complex dynamics and behavior exhibited under polytropic conditions, providing specialized professionals and enthusiasts with the means to look into the details of these systems.

Parabolic equations (1) with nonlinearity have a significant role in several scientific fields, such as population dynamics, heat transfer, chemical reactions, and many others. They are widely employed to investigate a variety of phenomena, such as the biological densities of two populations during migration and the thickness of two different kinds of chemical reagents during a chemical reaction. These equations are also used to determine the temperature of two different types of materials during propagation. In population dynamics, the functions $u_1(t,x)$, $u_2(t,x)$ describe the growth or decline of animal or plant populations. Similarly, in heat transfer, they help to determine the heat flux in a material with varying temperatures. Furthermore, they are used to describe unsteady flows in a liquid media, especially when shear stress and displacement velocity exhibit a power-law relationship.

The local presence of ineffective solutions to problem (1)–(3) in the problem-solving domain has been a topic of much discussion and analysis. The strict testing and experimentation conducted in this field have consistently shown that the usual integration method is a reliable approach for determining this specific phenomenon. This widely acknowledged fact within the community of experts demonstrates the thorough knowledge and expertise that underpins our understanding of complex systems. Moreover, it is worth mentioning that such a local existence can be easily established and understood by applying the comparison principle, which has been extensively reviewed in several studies ([1; 316], [2; 26], [3–11]). Therefore, it is safe to say that the determination of the local existence in this particular problem can be achieved with a high level of accuracy and precision, using the appropriate tools and methods at hand.

The study of nonlinear parabolic systems has piqued the interest of researchers all around the world. With the aim of understanding the global existence and blow-up conditions of such systems, researchers have employed diverse techniques and strategies to investigate this phenomenon. The existing literature in this area is extensive, with several noteworthy contributions from experts in the field (see [1; 176], [2,3,7–9,12] and references therein). The essential for several nonlinear parabolic equations in mathematical physics, the Fujita exponent is one of the major topics of research, which has drawn significant attention from mathematicians. Researchers have delved deep into this area, studying various aspects of critical Fujita exponents in great detail (see [2,10,11,13–16] and references therein). Overall, the understanding of nonlinear parabolic systems' global existence and blow-up circumstances, as well as the critical Fujita exponent, continues to be an area of active research. With further study and investigation, researchers hope to gain deeper insights into these systems, leading to a better understanding of the complex phenomena that underlie them.

Let us now consider and revisit some well-known results. In the research conducted by V.A. Galaktionov, and H.A. Levine mentioned in reference [4], they extensively investigated the situation using a single equation

$$\begin{cases} u_t = \left(u^k\right)_{xx}, & x>0,\ \ 0<t<T, \\ -(u^k)_x(0,t) = u^q(0,t), & 0<t<T, \\ u(x,0)=u_0(x), & x>0, \end{cases} \tag{4}$$

and the gradient diffusion heat conduction equation

$$\begin{cases} u_t = \left(|u_x|^{k-1} u_x\right)_x, & x > 0,\ 0 < t < T, \\ -|u_x|^{k-1} u_x(0,t) = u^q(0,t), & 0 < t < T, \\ u(x,0) = u_0(x), & x > 0, \end{cases} \tag{5}$$

with $k \geq 1$, $q > 0$, and u_0 has compact support. It has been established that for the problem (4), $q_0 = \frac{1}{2}(k+1)$ is the critical global exponent, where $q_c = k+1$ is the crucial Fujita exponent, as opposed to (5), the critical Fujita exponent is $q_c = 2k$ as well as the critical global exponent being $q_0 = \frac{2k}{k+1}$.

In [5] authors analyzed the following issue with single equation and gradient diffusion:

$$\begin{cases} \rho(x)u_t = \left(|u_x|^{k-2} u_x\right)_x + \rho(x)u^{\beta}, & (x,t) \in R_+ \times (0,+\infty), \\ -|u_x|^{k-2} u_x(0,t) = u^m(0,t), & t > 0, \\ u(x,0) = u_0(x) > 0, & x \in R_+, \end{cases} \tag{6}$$

with $k > 2$, β, $m > 0$, $\rho(x) = x^{-n}$, $n \in R$, $u_0(x)$ is a bounded, continuous, nonnegative, and nontrivial initial value. They determined that the problem (6):

– in case of $0 < \beta \leq 1$, and $0 < m \leq \frac{(2-n)(k-1)}{k-n}$ the issue can be resolved globally;

– in case of $\beta < 1$, and $m > \frac{(2-n)(k-1)}{k-n}$ the issue has a blow-up solution.

Consideration of the following problem is the focus of the research conducted by Zhaoyin Xiang, Chunlai Mu, and Yulan Wang in their study published in [12]. The problem under scrutiny has been given thorough attention and analysis by the researchers.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial u^{m_1}}{\partial x}\right|^{p_1-2} \dfrac{\partial u^{m_1}}{\partial x}\right) \\ \dfrac{\partial v}{\partial t} = \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial v^{m_2}}{\partial x}\right|^{p_2-2} \dfrac{\partial v^{m_2}}{\partial x}\right) \end{cases}, \quad (x,t) \in R_+ \times (0,T), \tag{7}$$

$$\begin{cases} -\left|\dfrac{\partial u^{m_1}}{\partial x}\right|^{p_1-2} \dfrac{\partial u^{m_1}}{\partial x}\Bigg|_{x=0} = v^{q_1}(0,t) \\ -\left|\dfrac{\partial v^{m_2}}{\partial x}\right|^{p_2-2} \dfrac{\partial v^{m_2}}{\partial x}\Bigg|_{x=0} = u^{q_2}(0,t) \end{cases}, \quad t \in (0,T), \tag{8}$$

$$\begin{cases} u(x,0) = u_0(x) \\ v(x,0) = v_0(x) \end{cases}, \quad x \in R_+, \tag{9}$$

where $m_i > 1$, $p_i > 2$, $q_i > 0$, $i = 1,2$. They determined that:

(i) in case of $q_1 q_2 \leq ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/p_1 p_2$ the problem's every nonnegative solutions (7)–(9) are all global in time;

(ii) in case of $q_1 q_2 > ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/p_1 p_2$, then the problem (7)–(9) has solutions that blow-up in a limited length of time.

If $q_1 q_2 > ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/p_1 p_2$:

(i) in case of $\min\{\alpha_1+\beta_1, \alpha_2+\beta_2\} > 0$, then solution of the problem (7)–(9) is global in time;

(ii) in case of $\max\{\alpha_1+\beta_1, \alpha_2+\beta_2\} < 0$, then the solution of problem (7)–(9) is blow-up.

Many mathematical models of nonlinear cross-diffusion in [17, 18] are described using nonlinearly linked partial differential equation systems. Finding explicit analytical solutions for these systems is difficult, though. To tackle the complexities of these systems, researchers have delved into the realm of numerical methods, employing them to derive approximations. In their pursuit, they have turned to the use of nonlinear parabolic equations, coupled with nonlinear boundary conditions, as a means to accurately describe and analyze these intricate systems. By harnessing the power of these mathematical tools, deeper investigation of the intricacies is possible for researchers and intricacies of these complex phenomena, providing valuable insights that pave the way for advancements in their respective fields. To investigate the qualitative properties of a cross-diffusion system with nonlocal boundary conditions and nonlinearity, self-similar analysis and the standard equation approach have been employed. The results of these studies have helped researchers understand the behavior of these systems under different conditions. Despite the challenges posed by the nonlinearly coupled systems of partial differential equations, and improvements in numerical techniques have paved the way for obtaining accurate approximations, thus making significant contributions to the field of nonlinear cross-diffusion. The quest for further exploration and understanding of these systems continues to fuel research in this area. The situation of slow diffusion, researchers have devised several self-similar solutions to tackle the cross-diffusion problem. The intricate nature of a nonlinear cross-diffusion system, comprised of interconnected parabolic equations, poses a significant challenge in the realm of mathematical analysis. These complex systems often exhibit behavior that defies traditional methods of solution due to the presence of nonlinear boundary conditions. As a result, finding global solutions becomes an arduous task requiring advanced computational techniques and deep understanding of the underlying dynamics at play. Self-similar analysis and the comparison principle were used to identify the critical exponents, namely the global solvability and Fujita type critical exponents. The comparison theorem has further enabled researchers to establish upper and lower limits for global solutions and blow-up solutions, respectively. These findings underscore the importance of carefully considering numerical parameters when dealing with slow-diffusion scenarios.

This article, influenced by the works we have mentioned earlier, serves a twofold purpose. First, it aims to identify the (1)–(3) system's essential global existence curve, and in order to achieve that, the article emphasizes the importance of constructing self-similar super-solution and sub-solution. Second, the essay presents a theory regarding the critical curve of the Fujita type supported by certain recent findings. As opposed to dealing with a single equation, we are dealing with a system, we need to devise some innovative strategies to tackle the challenges that come with it. In conclusion, this article is a valuable addition to the literature on critical global existence curves, self-similar super- and subsolutions, and the critical curve of the Fujita type.

It is widely accepted in the field of mathematics that degenerate equations often lack classical solutions. When confronted with such equations, mathematicians have to find other solutions that are more general in nature. In conclusion, while degenerate equations may present unique challenges, there are still various ways to approach them and derive meaningful solutions.

Definition. The function $u(x,t)$ is viewed as an insufficient solution to problems (1)–(3) in $\Omega = \{(0,+\infty)\times(0,T)\}$, if $0 \le u_i(x,t) \in C(\Omega)$, $\left|\dfrac{\partial u_i^k}{\partial x}\right|^{m-1}\dfrac{\partial u_i^k}{\partial x} \in C(\Omega)$, $i=1,2$, if it complies with (1)–(3) with regard to distribution in Ω, where the longest time period that can be allowed is $T>0$, see [5].

1 Main results

Solutions to the global existence and nonexistence theorems play a crucial role in understanding complex systems. To further explore this topic, it is necessary to discuss the creation of self-similar sub-and super-solutions to equations (1)–(3). These solutions provide valuable insights into the behavior of these equations under various conditions. By examining the properties of sub-solutions, we can gain a deeper understanding of how certain factors contribute to the existence of global solutions. As opposed to that, studying super-solutions allows us to analyze situations where nonexistence solutions arise. This comprehensive approach enables researchers and professionals to make informed decisions when dealing with complex systems in their respective fields.

We will use the comparison principle to prove our first theorem, which focuses on determining the conditions necessary for the global solution of problem (1)–(3). By establishing a framework for analyzing self-similar sub-solutions and super-solutions, we gain valuable insights into the intricacies of global existence and nonexistence solutions. This theorem represents a significant advancement in our understanding of complex systems, as it showcases the interplay between comparison principles and the concept of self-similarity. The comprehensive examination of these factors allows us to delve deeper into the realm of global solutions, providing a solid foundation for further research and analysis in this field. Our findings highlight the importance of considering self-similar sub- and super-solutions when studying problems with global implications.

Theorem 1. If $r_1 r_2 \le \left(\dfrac{m}{m+1}\right)^2 (k+1-s_1)(k+1-s_2)$, then every nonnegative solution of the problem (1)–(3) is global in time.

Proof. By emphasizing the construction of a self-similar super-solution, one can gain an additional understanding of the theorem and its intricate nuances. This particular super-solution serves as a powerful demonstration of the theorem's validity and its ability to address complex problems. Through meticulous analysis, it becomes evident that this super-solution possesses certain limitations for any given $t > 0$. As researchers strive towards achieving their objective, their attention has been directed towards the identification and analysis of strict super-solutions that conform to the self-similar form. These endeavors pave the way for a more comprehensive understanding of the intricacies involved in this intricate realm of study,

$$\bar{u}_i(t,x) = e^{h_{2i-1}} \left(N + e^{-K_i x e^{-h_{2i} t}}\right)^{\frac{1}{k}}, \tag{10}$$

where $K_i > 0$, $h_{2i-1,\,2i} > 0$, $N = \max\left\{\|\bar{u}_i\|_\infty^k + 1\right\}$; $i = 1, 2$.

Using comparison principles and the substitution of (10) into (1)-(2), it has been determined:

$$\frac{\partial \bar{u}_i}{\partial t} = h_{2i-1} \cdot e^{h_{2i-1} t} \cdot \left(N + e^{-K_i x e^{-h_{2i} t}}\right)^{\frac{1}{k}} + e^{(h_{2i-1} - h_{2i})t} \cdot \frac{1}{k} \cdot K_i \cdot x \cdot h_{2i} \left(N + e^{-K_i x e^{-h_{2i} t}}\right)^{\frac{1}{k}-1} \ge$$

$$\ge h_{2i-1} e^{h_{2i-1} t} \left(N + e^{-K_i x e^{-h_{2i} t}}\right)^{\frac{1}{k}} \ge h_{2i-1} e^{h_{2i-1} t} N^{\frac{1}{k}},$$

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial \bar{u}_i^k}{\partial x}\right|^{m-1} \frac{\partial \bar{u}_i^k}{\partial x}\right) = m K_i^{m+1} e^{[h_{2i-1} k m - (m+1) h_{2i}]t} \times e^{-m K_i x e^{-h_{2i} t}} \le m K_i^{m+1} e^{[h_{2i-1} k m - (m+1) h_{2i}]t},$$

$$\bar{u}_i^{s_i} = e^{s_i h_{2i-1} t} \left(N + e^{-K_i x e^{-h_{2i} t}}\right)^{\frac{s_i}{k}} \le e^{s_i h_{2i-1} t} (N+1)^{\frac{s_i}{k}},$$

$$-\left|\frac{\partial \bar{u}_i^k}{\partial x}\right|^{m-1} \frac{\partial \bar{u}_i^k}{\partial x}\bigg|_{x=0} = K_i^m e^{(h_{2i-1} k - h_{2i}) m t}, \; \bar{u}_{3-i}^{r_i}\big|_{x=0} = e^{r_i h_{5-2i} t} (N+1)^{\frac{r_i}{k}}.$$

The solution $\bar{u}_i$ is regarded as global, if inequalities:

$$\frac{\partial \bar{u}_i}{\partial t} \ge \frac{\partial}{\partial x}\left(\left|\frac{\partial \bar{u}_i^k}{\partial x}\right|^{m-1} \frac{\partial \bar{u}_i^k}{\partial x}\right) + \bar{u}_i^{s_i}, \qquad i = 1, 2, \tag{11}$$

hold for any $x \in R_+, t > 0$. Eventually, the following expressions have been achieved using the computations above in (11):

$$
\begin{aligned}
& h_{2i-1}e^{h_{2i-1}t}N^{\frac{1}{k}} \geq mK_i^{m+1}e^{[h_{2i-1}km-(m+1)h_{2i}]t} + e^{s_i h_{2i-1}t}\left(N+1\right)^{\frac{s_i}{k}}, \\
& K_i^m e^{(h_{2i-1}k-h_{2i})mt} = e^{r_i h_{5-2i}t}\left(N+1\right)^{\frac{r_i}{k}}, \\
& K_i = (N+1)^{\frac{r_i}{km}}, \;\; r_i h_{5-2i} = \left(h_{2i-1}k - h_{2i}\right)m, \;\; i=1,2, \\
& h_{2i-1} \geq h_{2i-1}km - (m+1)h_{2i} + s_i h_{2i-1}, h_1 k - h_2 = \frac{r_1}{m}h_3, \\
& h_3 k - h_4 = \frac{r_2}{m}h_1, \\
& h_{2i} \geq \frac{\left(km + s_i - 1\right)h_{2i-1}}{m+1}, \\
& h_1 k - \frac{r_1}{m}h_3 \geq \frac{\left(km + s_1 - 1\right)h_1}{m+1}, \\
& h_3 k - \frac{r_1}{m}h_1 \geq \frac{\left(km + s_2 - 1\right)h_3}{m+1}.
\end{aligned}
$$

Thus, it is evident that for the solution of the problem (1)–(3) to be global in time, the last inequality should always hold for any $m > 1, \; k \geq 1$, as the theorem proves.

Remark. Theorem 1 demonstrates that $r_1 r_2 = \left(\dfrac{m}{m+1}\right)^2 \left(k+1-s_1\right)\left(k+1-s_2\right)$ is critical global existence of the problem (1)–(3).

Theorem 2. If $0 < p_i \leq 1$, and $q_i \geq \dfrac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ or $p_i > 1$, and $r_i \leq \dfrac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ then, each of the solutions to (1)–(3) blows up.

Proof. To prove the theorem, it was necessary to search for sub-solutions of the problem (1)–(3), and this was achieved by looking for them in the next form:

$$
\underline{u}_i(t,x) = t^{\alpha_i} f_i(\xi_i), \qquad \xi_i = x t^{-\beta_i}, \tag{12}
$$

where $\alpha_i = \dfrac{1}{1-p_i}, \;\; \beta_i = \dfrac{p_i - km}{(p_i-1)(m+1)}, \;\; i=1,2.$

By analyzing the super-solutions obtained from equation (12), we can observe the emergence of a self-similar form in the resulting equations (1)–(3). These self-similar inequalities and boundary conditions play a pivotal role in determining whether a solution is deemed as a blow-up solution or not. It is imperative to adhere to these self-similar inequalities and boundary conditions in order to accurately classify and understand the behavior of the system under study. The presence of such intricate relationships highlights the complexity of the problem at hand, requiring a comprehensive and meticulous approach for its exploration. To fully comprehend the underlying dynamics, further research and analysis are warranted to delve deeper into these self-similar forms and their implications on the overall system:

$$
\frac{d}{d\xi_i}\left(\left|\frac{df_i^k}{d\xi_i}\right|^{m-1}\frac{df_i^k}{d\xi_i}\right) + \beta_i \xi_i \frac{df}{d\xi_i} - \alpha_i f_i + f_i^{p_i} \geq 0, \tag{13}
$$

$$
-\left|\frac{\partial \underline{u}_i^k}{\partial x}\right|^{m-1}\frac{\partial \underline{u}_i^k}{\partial x}\bigg|_{x=0} \leq \underline{u}_{3-i}^{q_i}(0,t). \tag{14}
$$

Let

$$f_i(\xi_i) = A_i \left(a_i^{\frac{m+1}{m}} - \xi_i^{\frac{m+1}{m}} \right)^{\frac{m}{mk-1}}. \tag{15}$$

By substituting equation (15) into inequalities (13) and (14), we can derive the necessary conditions that unequivocally illustrate the occurrence of equation (14) under all circumstances. This crucial step not only solidifies our understanding of the underlying principles, but also provides a robust framework for further analysis and exploration within this complex system:

$$\left(\frac{k(m+1)}{mk-1}\right)^m \left(\frac{m+1}{mk-1}\right) A_i^{mk} \geq \beta_i \frac{m+1}{mk-1} A_i,$$

$$A_i \geq \left[\beta_i \left(\frac{mk-1}{k(m+1)}\right)^m\right]^{\frac{1}{mk-1}},$$

$$f_i^{p_i} = A_i^{p_i} \left(a_i^{\frac{m+1}{m}} - \xi_i^{\frac{m+1}{m}} \right)_+^{\frac{m}{mk-1}} \left(a_i^{\frac{m+1}{m}} - \xi_i^{\frac{m+1}{m}} \right)_+^{\frac{m}{mk-1}(p_i-1)} \leq$$

$$\leq A_i^{p_i} a_i^{\frac{(m+1)(p_i-1)}{mk-1}} \left(a_i^{\frac{m+1}{m}} - \xi_i^{\frac{m+1}{m}} \right)_+^{\frac{m}{mk-1}},$$

$$A_i^{p_i} a_i^{\frac{(m+1)(p_i-1)}{mk-1}} \geq \alpha_i A_i + A_i^{mk} \left(\frac{k(m+1)}{mk-1}\right)^m.$$

By taking

$$a_i^{\frac{(m+1)(p_i-1)}{mk-1}} \geq \alpha_i A_i^{1-p_i} + A_i^{mk-p_i} \left(\frac{k(m+1)}{mk-1}\right)^m,$$

$0 < p_i \leq 1$, and $q_i \geq \dfrac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ can be easily checked and ensure that A_1, and A_2 can be taken sufficient to prevent inequalities (13) and (14) are valid. Because of this, if the initial data $u_1(x,0)$, $u_2(x,0)$ are large enough that $u_{10}(x) \geq \underline{u}_1(x,0)$, $u_{20}(x) \geq \underline{u}_2(x,0)$, then $\underline{u}_i(t,x)$, $i = 1,2$ is a subsolution to (1)–(3). In accordance with the comparison principle, it is established that when dealing with a substantial amount of initial data, the solutions provided in (1)–(3) will eventually blow up within a finite time frame. The comprehensive proof has been successfully concluded, cementing this understanding.

Theorem 3. If $q_1 q_2 < \left(\dfrac{m(k+1)}{m+1}\right)^2$, and $p_i > \left(1 + \dfrac{1}{k}\right) m + \dfrac{1}{k}$, then every solution of problem (1)–(3) blows up in finite time.

Proof. It is vital to comprehend that the delineated by (1)–(3) can be convincingly shown for equations that lack a source. The necessary conditions for this to occur can be satisfied entirely through internal mechanisms. As such, we proceed to build our targeted solution in a subsequent manner.

$$u_{ib}(t,x) = t^{\mu_i} g_i(\xi_i), \qquad \xi_i = x t^{-\gamma_i}, \tag{16}$$

where g_i are two compactly supported functions,

$$\mu_i = \frac{m[m(k+1) + (m+1)q_i]}{(m(k+1))^2 - (m+1)^2 q_i q_{3-i}},$$

$$\gamma_i = \frac{m[mk(k+1) + (mk-1)q_i] - (m+1)q_1 q_2}{(m(k+1))^2 - (m+1)^2 q_i q_{3-i}}.$$

We now insert (16) into (1)–(3) and derive the following result:

$$\frac{d}{d\xi_i}\left(\left|\frac{dg_i^k}{d\xi_i}\right|^{m-1}\frac{dg_i^k}{d\xi_i}\right)+\gamma_i\xi_i\frac{dg_i}{d\xi_i}-\mu_i g_i\geq 0, \tag{17}$$

$$-\left|\frac{dg_i^k}{d\xi_i}\right|^{m-1}\frac{dg_i^k}{d\xi_i}\Bigg|_{\xi_i=0}\leq g_{3-i}^{q_i}(0). \tag{18}$$

Finding self-similar solutions to the issue (17), (18) is now necessary.

Let

$$\bar{g}_i(\xi_i)=B_i(b_i-\xi_i)^{\frac{m}{mk-1}}, \tag{19}$$

then by inserting (19) into (17), and (18), we obtain

$$\begin{aligned}
&\frac{d\bar{g}_i}{d\xi_i}=-\frac{B_i m}{mk-1}(b_i-\xi_i)^{\frac{m}{mk-1}-1},\\
&\gamma_i\xi_i\frac{dg_i}{d\xi_i}-\mu_i g_i=-\frac{B_i m}{mk-1}\xi_i\left(b_i-\xi_i\right)^{\frac{m}{mk-1}-1}-\mu_i B_i\left(b_i-\xi_i\right)^{\frac{m}{mk-1}}=\\
&=-\frac{B_i m}{mk-1}\xi_i\left(b_i-\xi_i\right)^{\frac{m}{mk-1}-1}-\mu_i B_i\left(b_i-\xi_i\right)^{\frac{m}{mk-1}-1}\left(b_i-\xi_i\right)\geq\\
&\geq-\left(\frac{b_i B_i m}{mk-1}-\mu_i b_i B_i\right)(b_i-\xi_i)_+^{\frac{m}{mk-1}-1},\\
&\frac{d}{d\xi_i}\left(\left|\frac{dg_i^k}{d\xi_i}\right|^{m-1}\frac{dg_i^k}{d\xi_i}\right)=B_i^{mk}\left(\frac{m}{mk-1}\right)^{m+1}k^m\left(b_i-\xi_i\right)_+^{\frac{m}{mk-1}-1}\geq\\
&\geq b_i B_i\left(\mu_i+\frac{m}{mk-1}\right)(b_i-\xi_i)_+^{\frac{m}{mk-1}-1},\\
&B_i^{mk-1}\geq\frac{b_i}{k^m}\left(\frac{mk-1}{m}\right)^{m+1}\left(\mu_i+\frac{m}{mk-1}\right),\\
&-\left|\frac{dg_i^k}{d\xi_i}\right|^{m-1}\frac{dg_i^k}{d\xi_i}\Bigg|_{\xi_i=0}\leq\bar{g}_{3-i}^{q_i}(0).
\end{aligned}$$

The following benefits result from applying comparison principles to the aforementioned expressions:

$$\begin{aligned}
&-\left|\frac{dg_i^k}{d\xi_i}\right|^{m-1}\frac{dg_i^k}{d\xi_i}\Bigg|_{\xi_i=0}=\left|B_i^k\left(b_i-\xi_i\right)_+^{\frac{m}{mk-1}-1}\right|^{m-1}\cdot\left(B_i^k\left(b_i-\xi_i\right)_+^{\frac{m}{mk-1}-1}\right)\Bigg|_{\xi_i=0}=\\
&=B_i^{mk}\left(b_i-\xi_i\right)_+^{\left(\frac{m}{mk-1}-1\right)m}\Bigg|_{\xi_i=0}=B_i^{mk}b_i^{\frac{m}{mk-1}}\leq B_{3-i}^{q_i}b_{3-i}^{\frac{q_i m}{mk-1}}.
\end{aligned}$$

And this illustrates unequivocally that when $p_i>\left(1+\frac{1}{k}\right)m+\frac{1}{k}$, equations (17) and (18) hold true. The concept of comparison leads us to conclude that (1)–(3) have solutions that invariably end in blow-up in a finite amount of time.

Theorem 4. If $q_1q_2\leq\left(m\left(k+1\right)\right)^2$, and $p_i>1$, then every solution of the problem (1)–(3) is blow-up in finite time.

Proof. The same approach used in [11, 18] can be used to establish Theorem 4.

Let us demonstrate how self-similar solutions asymptotically behave.

The case $q_1 q_2 > \left(\dfrac{m(k+1)}{m+1}\right)^2$, and $\dfrac{1}{m} < p_i \leq 1$. Take into account the following self-similar solution of (1)–(3).

Auxiliary systems of equations are a fundamental aspect of mathematical problem-solving in various fields. The intricacy of these systems can often be overwhelming, but with the right methods and techniques, they can be simplified. Through the application of specific transformations, such as substitution or elimination, the complex nature of these systems can be broken down into more manageable components. These methods have been extensively studied and proven effective in numerous academic research papers. By implementing these strategies, professionals and enthusiasts alike can confidently approach and solve even the most intricate auxiliary systems of equations:

$$u_i(x,t) = (T+t)^{\alpha_i}\,\varphi_i(\xi_i), \xi_i = x(T+t)^{-\beta_i},$$

where α_i and β_i parameters defined above.

$$\frac{d}{d\xi_i}\left(\left|\frac{d\varphi_i^k}{d\xi_i}\right|^{m-1}\frac{d\varphi_i^k}{d\xi_i}\right) + \beta_i\xi_i\frac{d\varphi_i}{d\xi_i} - \alpha_i\varphi_i + \varphi_i^{\beta_i} = 0, \tag{20}$$

$$-\left|\frac{d\varphi_i^k}{d\xi_i}\right|^{m-1}\frac{d\varphi_i^k}{d\xi_i}\Bigg|_{\xi_i=0} = \varphi_{3-i}^{q_i}(0). \tag{21}$$

Let us consider the function

$$\bar{\varphi}_i(\xi_i) = \left(d_i - D_i\xi_i^{\frac{m+1}{m}}\right)^{\frac{m}{mk-1}},\ \ d_i > 0,\ \ D_i = \frac{\beta_i^{\frac{1}{m}}(mk-1)}{k(m+1)}.$$

Theorem 5. The compactly supported solution of problem (20)-(21) has the asymptotic

$$\varphi_i(\xi_i) = \bar{\varphi}_i(\xi_i)(1+o(1)),$$

when $\xi_i \to \left(\dfrac{d_i}{D_i}\right)^{\frac{m}{m+1}} = \xi_{i0}$.

Proof. The function φ_i is looked for in the following form

$$\varphi_i(\xi_i) = \bar{\varphi}_i(\xi_i)\omega_i(\eta_i).$$

It is enough to show that $\omega_i \approx 1$. Let

$$\eta_i = -\ln\left(d_i - D_i\xi_i^{\frac{m+1}{m}}\right), \quad \text{and} \quad \eta_i \overset{\xi_i\to\xi_{i0}}{\longrightarrow} +\infty. \tag{22}$$

Upon substituting (22) into (20)-(21) we get the next expressions:

$$\bar{\varphi}_i(\xi_i) = e^{-\frac{m}{mk-1}\eta_i}, \varphi_i(\xi_i) = e^{-\frac{m}{mk-1}\eta_i}\omega_i, \quad \xi_i = \left(d_i - e^{-\eta_i}\right)^{\frac{m}{m+1}} D_i^{-\frac{m}{m+1}},$$

$$\frac{d\xi_i}{d\eta_i} = \left(\frac{m+1}{m}\right) D_i^{\frac{m}{m+1}} e^{\eta_i} \left(d_i - e^{-\eta_i}\right)^{\frac{1}{m+1}},$$

$$\beta_i \xi_i \frac{d\varphi_i}{d\xi_i} - \alpha_i \varphi_i + \varphi_i^{p_i} = \beta_i \left(\frac{m+1}{m}\right) e^{(1-\frac{m}{mk-1})\eta_i} \left(d_i - e^{-\eta_i}\right) \times$$

$$\times \left(\omega_i^{'} - \frac{m}{mk-1}\omega_i\right) - \alpha_i e^{-\frac{m}{mk-1}\eta_i}\omega_i + e^{-\frac{mp_i}{mk-1}\eta_i}\omega_i^{p_i},$$

$$\frac{d}{d\xi_i}\left(\left|\frac{d\varphi_i^k}{d\xi_i}\right|^{m-1} \frac{d\varphi_i^k}{d\xi_i}\right) = \left(\frac{m+1}{m}\right)^{m+1} D_i^m e^{(1-\frac{m}{mk-1})\eta_i} \times$$

$$\times \left(d_i - e^{-\eta_i}\right) \left[\left((L_i\omega)^m\right)' + \left(\frac{e^{-\eta_i}}{d_i - e^{-\eta_i}} - \frac{m}{mk-1}\right) \cdot (L_i\omega)^m \right],$$

where $L_i\omega = \left(\omega_i^k\right)' - \dfrac{mk}{mk-1}\omega_i^k$.

Now (20) takes a next look:

$$\left((L_i\omega)^m \right)' + \left(a_1(\eta_i) - \frac{m}{mk-1}\right)(L_i\omega)^m + a_2(\eta_i)\omega_i^{1-k} L_i\omega - a_3(\eta_i)\omega_i + a_4(\eta_i)\omega_i^{p_i} = 0,$$

where $a_1(\eta_i) = \dfrac{e^{-\eta_i}}{d_i - e^{-\eta_i}}$, $a_2(\eta_i) = \dfrac{\beta_i}{k}\left(\dfrac{m}{D_i(m+1)}\right)^m$,

$$a_3(\eta_i) = d_i \left(\frac{m}{m+1}\right)^{m+1} D_i^{-m} a_1(\eta_i) e^{-\frac{m(p_i-1)}{mk-1}\eta_i}, \eta_i \in [\eta_0; +\infty).$$

In a specific region around $+\infty$, the solutions to the last system fulfill the following inequalities:

$$\omega_i > 0, \ \left(\omega_i^k\right)' - \frac{mk}{mk-1}\omega_i^k \neq 0.$$

Assuming that $\nu_i(\eta_i) = (L_i\omega)^m$, then

$$\nu_i^{'}(\eta_i) = -\left(a_1(\eta_i) - \frac{m}{mk-1}\right)\nu_i - a_2(\eta_i)\omega_i^{1-k} L_i\omega + a_3(\eta_i) - \omega_i a_4(\eta_i)\omega_i^{p_i}. \tag{23}$$

Furthermore, we consider the functions:

$$\theta_i(\eta_i, \mu_i) = -\left(a_1(\eta_i) - \frac{m}{mk-1}\right)\mu_i - a_2(\eta_i)\omega_i^{1-k} L_i\omega + a_3(\eta_i) - \omega_i a_4(\eta_i)\omega_i^{p_i}, \tag{24}$$

where $\mu_i \in R$.

The functions $\theta_i(\eta_i, \mu_i)$ keep the sign for interval $[\eta_{1i}; +\infty) \subset [\eta_{i0}; +\infty)$ regarding each fixed value μ_i. Therefore, the functions $\theta_i(\eta_i, \mu_i)$ satisfies one of the following inequalities, for all $\eta_i \in [\eta_{1i}; +\infty)$,

$$\nu_i^{'} > 0, \quad \text{or} \quad \nu_i^{'} < 0, \tag{25}$$

from what one can conclude that when $\eta_i \in [\eta_{1i}; +\infty)$:

$$\lim_{\eta_i \to +\infty} a_1(\eta_i) = \lim_{\eta_i \to +\infty} a_3(\eta_i) = 0,$$

$$\overset{0}{a}_2 = \lim_{\eta_i \to +\infty} a_2(\eta_i) = \frac{\beta_i}{k}\left(\frac{m}{\nu_i(m+1)}\right)^m,$$

$$\overset{0}{a}_4 = \lim_{\eta_i \to +\infty} a_4(\eta_i) = \begin{cases} 0, & \text{if} \quad p_i > 1-k+\frac{1}{m}, \\ \left(\frac{m}{m+1}\right)^m \frac{D_i^{-m}}{d_i}, & \text{if } p_i = 1-k+\frac{1}{m}, \\ +\infty, & \text{if} \quad p_i < 1-k+\frac{1}{m}. \end{cases}$$

Suppose now that for the functions $\nu_i(\eta_i)$, a limit $\eta_i \to +\infty$ does not exist. It should be taken into account the situation where one of the inequalities (25) holds. As $\nu_i(\eta_i)$ are oscillating functions around $\overline{\nu_i} = \mu_i$, and in $[\eta_{1i}; +\infty)$, the intersection of this straight line's graph with itself is infinite.

But given that in the interval $[\eta_{1i}; +\infty)$, this is not possible. Since there is only one real inequality (25), it follows from (24) that the graph of the function $\nu_i(\eta_i)$ only crosses the straight line $\overline{\nu_i} = \mu_i$, once over the interval $[\eta_{1i}; +\infty)$. The function $\nu_i(\eta_i)$ therefore has a limit at $\eta \to +\infty$.

The functions $\nu_i(\eta_i)$ are assumed they have a limit at $\eta \to +\infty$. Then, $w_i^{'}(\eta_i)$ has a limit at $\eta \to +\infty$, and this limit is zero. Then

$$\nu_i(\eta_i) = \left(\frac{mk}{mk-1}\right)^m \left(\overset{0}{\omega_i}\right)^{km} + o(1),$$

at $\eta \to +\infty$.

Furthermore, by (23) functions $\nu_i(\eta_i)$ derivatives have limits at $\eta \to +\infty$, which are plainly equal to zero.

As a result, it is required

$$\lim_{\eta_i \to \infty}\left[\left(a_1(\eta_i) - \frac{m}{mk-1}\right)\nu_i + a_2(\eta_i)\omega_i^{1-k} L_i\omega - a_3(\eta_i) + \omega_i a_4(\eta_i)\omega_i^{p_i}\right] = 0.$$

And the following algebraic equations can be obtained

$$\frac{mk}{mk-1}\left(\frac{mk}{mk-1}\right)^m k^m \left(\overset{0}{\omega}\right)^{mk} - \overset{0}{a}_2 \frac{mk}{mk-1}\overset{0}{\omega}_i = 0,$$

or

$$\overset{0}{\omega}_i = \left(\frac{\beta_i}{k}\left(\frac{mk-1}{kD_i(m+1)}\right)^m\right)^{\frac{1}{mk-1}}. \tag{26}$$

The best case: $\overset{0}{\omega}_i = 1$. From the last equation (26), it has been achieved that $\overset{0}{\omega}_i \approx 1$, and thus $\varphi_i(\xi_i) = \bar{\varphi}_i(\xi_i)\omega_i(\eta_i)$.

Theorem 6. If $p_i > 1-k+\dfrac{1}{m}$, and $\quad q_1 q_2 < \dfrac{m^2(k+1-p_1)(k+1-p_2)}{(m+1)^2}$, then

$$u_i(t,x) = c_i(t+T)^{\alpha_i}\bar{g}_i(\xi_i)(1+o(1)),$$

where $c_i = \left(\dfrac{mk-1}{m} b_i\gamma_i\right)^{\frac{1}{mk-1}} \left(\dfrac{mk-1}{B_i m}\right)$.

Proof. Theorem 6 is demonstrated in a manner similar to that of Theorem 5.

2 Numerical solution of the problem

Drawing upon the extensive knowledge in the field of numerical analysis, experts have established that the process of selecting an initial approximation is of utmost importance in maintaining the nonlinear characteristics of a system of equations. Through rigorous research and analysis, it has been determined that an ill-suited initial approximation can lead to significant distortions in the accuracy and efficiency of the numerical solution. As such, professionals in this domain are constantly exploring innovative techniques and methodologies to ensure optimal selection of initial approximations for complex systems. Recognizing this significance, a computer experiment was recently undertaken to investigate the qualitative properties of solutions in relation to the global solvability of the system. To ensure utmost accuracy in our calculations, we employed equation (1) as our primary tool. This equation, which takes into account the second order with respect to x and the first order with respect to t, allows us to accurately model complex systems. By leveraging this approximation method, we can gain a deeper understanding of intricate phenomena and make informed decisions based on highly accurate data. The construction of the iterative process for numerical modeling involved employing the Thomas algorithm to calculate the node values during each step of the iteration. This meticulous approach guarantees the precision and reliability of the numerical analysis for the given system of nonlinear equations.

To shed some light on the effectiveness of different approaches, we conducted a series of numerical experiments. Through these numerical experiments, we were able to gain valuable insights into the influence of different initial approximations on both the convergence of the solution and the preservation of the qualitative properties of the intricate nonlinear processes under study. Our findings revealed that even slight variations in the initial approximations could have a significant impact on the final outcome, highlighting the importance of careful consideration and precise initialization in computational simulations. These results underscore the necessity for thorough numerical analysis and further emphasize the intricate nature of these nonlinear systems. Through our experiments, we were able to gather valuable insights into the behavior of the system of nonlinear equations under different numerical parameters and boundary conditions.

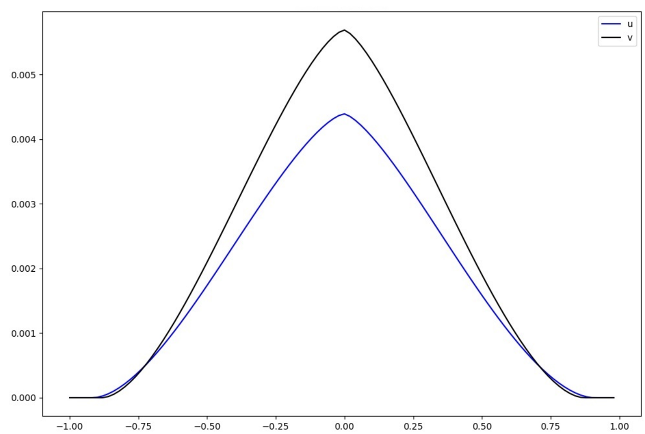

Figure 1. $k = 1.0,\ m = 2.3,\ p_1 = 2.1,\ p_2 = 2.0,\ a_1 = 1,\ a_2 = 1$

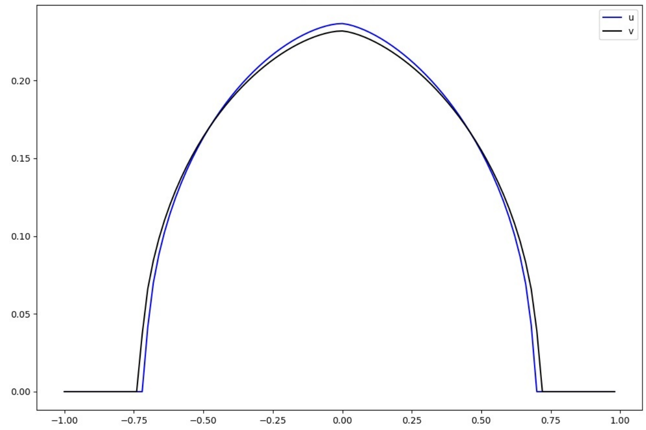

Figure 2. $k = 1.8,\ m = 1.7,\ p_1 = 2.5,\ p_2 = 2.4,\ a_1 = 1,\ a_2 = 1$

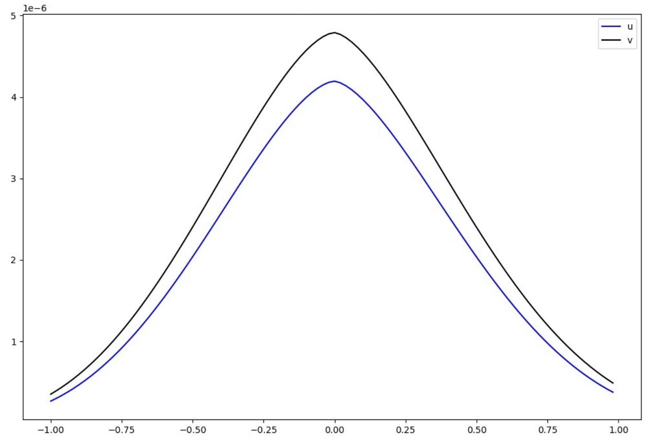

Figure 3. $k = 0.8,\ m = 3.7,\ p_1 = 1.4,\ p_2 = 1.5,\ a_1 = 1,\ a_2 = 1$

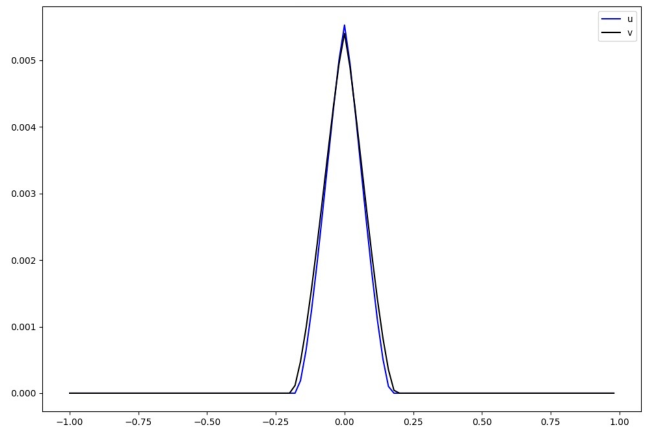

Figure 4. $k = 1.7,\ m = 1.6,\ p_1 = 2.8,\ p_2 = 2.4,\ a_1 = 1,\ a_2 = 1$

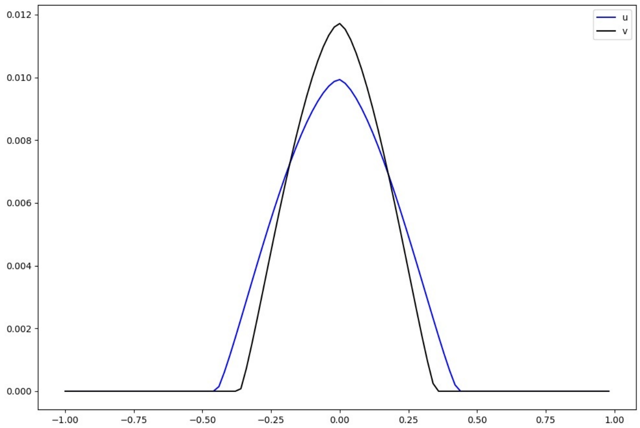

Figure 5. $k = 1.5,\ m = 1.7,\ p_1 = 2.6,\ p_2 = 3.4,\ a_1 = 1,\ a_2 = 1$

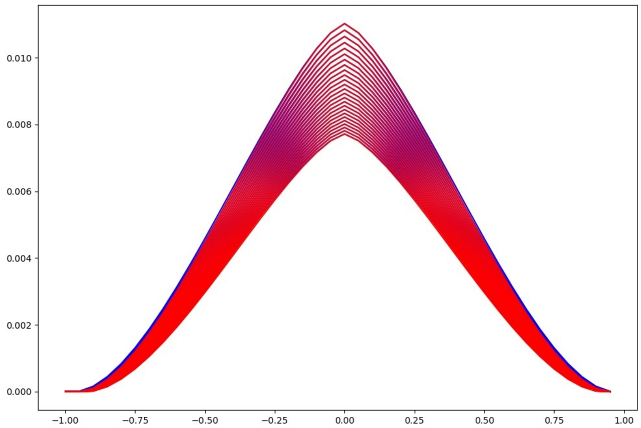

Figure 6. $k = 1.4,\ m = 1.7,\ p_1 = 1.6,\ p_2 = 1.4,\ a_1 = 1,\ a_2 = 1$

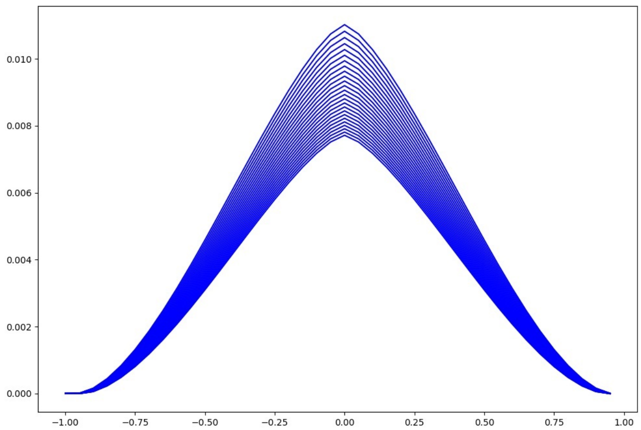

Figure 7. $k = 1.4,\ m = 1.7,\ p_1 = 1.6,\ p_2 = 1.4,\ a_1 = 1,\ a_2 = 1$

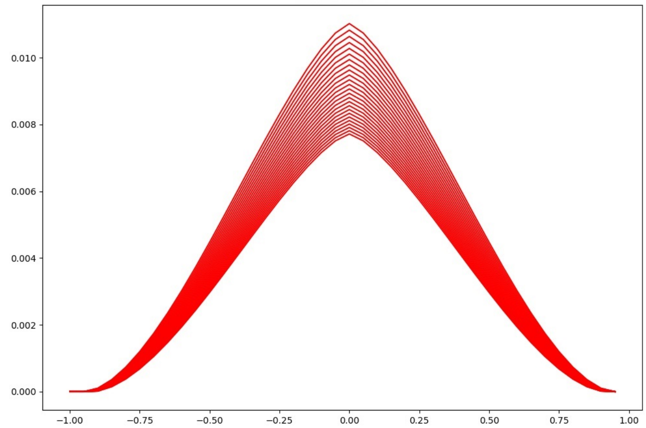

Figure 8. $k = 1.4,\ m = 1.7,\ p_1 = 1.6,\ p_2 = 1.4,\ a_1 = 1,\ a_2 = 1$

Conclusion

It has been established upper and lower estimates for global and unbounded generalized solutions and also Fujita-type critical exponents are obtained for a nonlinear mathematical model of the system of parabolic equations with sources and nonlinear boundary conditions. In the study of a mathematical model of a nonlinear diffusion equation with a double nonlinearity and a source, it has been confirmed that perturbations propagate with finite velocity. This finding sheds light on the behavior of solutions within this complex system, revealing the intricacies of spatial localization. By understanding these properties, researchers can delve deeper into the dynamics of nonlinear diffusion equations, advancing our knowledge in this specialized field of study.

An asymptotic behavior of compactly supported generalized solutions of the nonlinear diffusion problem with a source and with nonlinear damping is proved.

In Figures 1–8, we are presented with a visual representation of the numerical solution to the boundary value problem (1)–(3). These graphs not only provide a comprehensive view of the solution, but also showcase the intricate nature of the problem at hand. By examining these figures, one can discern the complex patterns and behaviors that emerge from this system, further reaffirming the need for rigorous analysis and research in this field. In this case, the process has the property of a finite perturbation propagation velocity. The size of the perturbation propagation region increases with time. The results of numerical experiments provide compelling evidence of the rapid convergence observed in the iterative process. This phenomenon can be attributed to the meticulous selection of the initial approximation, a crucial step that sets the foundation for subsequent computations. Through careful analysis and validation, it becomes evident that this method yields accurate and efficient solutions, making it a valuable tool for tackling complex problems in various domains. All the figures show that the increase in the propagation of a disturbance depends on the numerical parameters of the medium. The numerical experiments conducted in this study have demonstrated the remarkable convergence rate of the iterative process towards the precise solution. This notable result can be attributed to the careful selection of an appropriate initial approximation. Notably, regardless of the variation in numerical parameters, the number of iterations required does not surpass a mere five. Such findings emphasize the efficiency and reliability of our computational methods in solving complex problems.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Samarskii, A.A., Galaktionov, V.A., Kurdyumov, S.P., & Mikhailov, A.P. (1995). *Blow-up in Quasilinear Parabolic Equations*. Berlin: Walter de Gruyter.

2 Kalashnikov, A.S. (1987). Some problems of the qualitative theory of non-linear degenerate parabolic equations of second-order. *Russian Mathematical Surveys, 42*(2), 169–222. https://doi.org/10.1070/RM1987v042n02ABEH001309

3 Deng, K., & Levine, H.A. (2000). The role of critical exponents in blow-up theorems. *The sequel J Math Anal Appl., 243*, 85–126.

4 Galaktionov, V.A., & Levine, H.A. (1996). On critical Fujita exponents for heat equations with nonlinear flux conditions on the boundary. *Israel J Math, 94*, 125–146.

5 Rakhmonov, Z.R., & Tillaev, A.I. (2018). On the behavior of the solution of a nonlinear polytropic filtration problem with a source and multiple nonlinearities. *Nanosystems: physics, chemistry, mathematics, 9*(3), 323–329. https://doi.org/10.17586/2220-8054-2018-9-3-323-329

6 Wu, Z.Q., Zhao, J.N., Yin, J.X., & Li, H.L. (2001). *Nonlinear Diffusion Equations*. River Edge, NJ: World Scientific Publishing Co Inc.

7 Levine, H.A. (1990). The role of critical exponents in blow up theorems. *SIAM Rev, 32*, 262–288.

8 Li, Z.P., Mu, C.L., & Cui, Z.J. (2009). Critical curves for a fast diffusive polytropic filtration system coupled via nonlinear boundary flux. *Math Phys., 60*, 284–296.

9 Chen, B., Mi. Y., & Mu Ch. (2011). Global existence and nonexistence for a doubly degenerate parabolic system coupled via nonlinear boundary flux. *Acta Mathematica Scientia, 31*(2), 681–693.

10 Aripov, M.M., Utebaev, D., Kazimbetova, M.M., & Yarlashov, R.Sh. (2023). On convergence of difference schemes of high accuracy for one pseudo-parabolic Sobolev type equation. *Bulletin of the Karaganda University. Mathematics series*, 1(109), 24–37. https://doi.org/10.31489/2023M1/24-37

11 Aripov, M.M., Utebaev, D., & Nurullaev, Zh.A. (2022). On the convergence of difference schemes of high accuracy for the equation of ion-acoustic waves in a magnetized plasma. *Bulletin of the Karaganda University. Mathematics series*, 4(108), 4–19. https://doi.org/10.31489/2022M4/4-19

12 Zhaoyin, X., Chunlai, M., & Yulan, W. (2009). Critical curve of the non-Newtonian polytropic filtration equations coupled via nonlinear boundary flux. *Rocky mountain journal of mathematics, 39*(2), 76–89. https://www.jstor.org/stable/44239573

13 Aripov, M.M., & Matyakubov, A.S. (2017). To the qualitative properties of solution of system equations not in divergence form of polytrophic filtration in variable density. *Nanosystems: Physics, Chemistry, Mathematics, 8*(3), 317–322. https://doi.org/10.17586/2220-8054-2017-8-1-5-12

14 Rakhmonov, Z., Urunbaev, J., & Alimov, A. (2022). Properties of solutions of a system of nonlinear parabolic equations with nonlinear boundary conditions. *AIP Conference Proceedings, 2637*(1), 040008. https://doi.org/10.1063/5.0119747

15 Rakhmonov, Z., Parovik, R., & Alimov, A. (2021). Global existence and nonexistence for a multidimensional system of parabolic equations with nonlinear boundary conditions. *AIP Conference Proceedings, 2365*(1), 060022. https://doi.org/10.1063/5.0057139

16 Aripov, M., Matyakubov, A., & Bobokandov, M. (2023). Cauchy problem for the heat dissipation equation in non-homogeneous medium. *AIP Conference Proceedings, 2781*(1), 020027. https://doi.org/10.1063/5.0144807

17 Rakhmonov, Z.R., Khaydarov, A.T., & Urunbaev, J.E. (2020). Global Existence and Nonexistence of Solutions to a Cross Diffusion System with Nonlocal Boundary Conditions. *Mathematics and Statistics, 8*(4), 404–409. https://doi.org/10.13189/ms.2020.080405

18 Rakhmonov, Z. (2016). On the properties of solutions of multidimensional nonlinear filtration problem with variable density and nonlocal boundary condition in the case of fast diffusion. *Journal of Siberian Federal University. Mathematics & Physics, 9*, 236–245. https://doi.org/10.17516/1997-1397-2016-9-2-225-234

Бейсызықты шекаралық шарттары және дереккөзі бар бейсызықты диффузиялық жүйе шешімдерінің өзгеруі туралы

М.М. Арипов[1], З.Р. Рахмонов[1], А.А. Алимов[1,2]

[1]*Мырза Ұлықбек атындағы Өзбекстан ұлттық университеті, Ташкент, Өзбекстан;*
[2]*В.Г. Плеханов атындағы Ресей экономикалық университетінің Ташкент филиалы, Ташкент, Өзбекстан*

Мақалада баяу диффузия жағдайындағы бейсызықтық шекаралық шарттары бар бейсызықты диффузиялық жүйенің глобалды шешілетіндігі және шешілмейтіндігі зерттелген. Бейсызықты модельдерінің реакция-диффузия, жылу алмасу, сүзу және басқа да физикалық, химиялық және биологиялық процестердің сапалық сипаттамаларын талдауда маңызды рөл атқаратын критикалық Фуджита типті көрсеткіші алынды. Глобалды шешімділік жағдайында асимптотикалық шешімдердің негізгі компоненттері алынады. Зерттелетін бейсызықты процестердің сапалық сипаттамаларын сақтай отырып, нақты шешімге тез жақындайтын итерациялық әдістер сәйкес бастапқы жуықтаудың болуын талап ететіні белгілі. Бұл бейсызықтық есептерді сандық шешу үшін күрделі мәселе болып табылады. Бастапқы жуықтауларды сәтті таңдау есепті шешуге мүмкіндік береді, ол теңдеудің сандық параметрлерінің мәніне байланысты, олар бірінші кезекте асимптотикалық формуланы қолданатын есептеулерде ұсынылады. Итерациялық процестің бастапқы жуықтауы ретінде өзіндік ұқсас шешімдердің асимптотикасын пайдаланып, сандық есептеулер жүргізілген және нәтижелер талдауы берілген. Сандық тәжірибелерден алынған нәтижелер бейсызық диффузиялық жүйеде қарастырылатын процестің физикасымен тамаша сәйкес келетінін көрсетеді.

Кілт сөздер: күрделену режимі, бейсызықты шекаралық шарт, шешімінің бар болуының критикалық глобалды қисығы, өзгешеленген параболалық жүйелер, Фуджита типті критикалық көрсеткіш.

О поведении решений нелинейной диффузионной системы с источником и нелинейными граничными условиями

М.М. Арипов[1], З.Р. Рахмонов[1], А.А. Алимов[1,2]

[1]*Национальный университет Узбекистана имени Мирзо Улугбека, Ташкент, Узбекистан;*
[2]*Ташкентский филиал Российского экономического университета имени В.Г. Плеханова, Ташкент, Узбекистан*

Изучены глобальная разрешимость и неразрешимость нелинейной диффузионной системы с нелинейными граничными условиями в случае медленной диффузии. Получены критические показатели типа Фуджиты и существования, которые играют существенную роль при анализе качественных характеристик нелинейных моделей реакций—диффузии, теплопереноса, фильтрации и других физических, химических и биологических процессов. В случае глобальной разрешимости получены ключевые компоненты асимптотических решений. Известно, что итерационные методы, быстро сходящиеся к точному решению при сохранении качественных характеристик изучаемых нелинейных процессов, требуют наличия соответствующего начального приближения. Это представляет собой серьезную проблему для численного решения нелинейных задач. Успешный выбор начальных приближений позволяет решить эту задачу, которая зависит от значения числовых параметров уравнения, которые, в первую очередь, в расчетах рекомендуются с использованием асимптотической формулы. Применяя асимптотику автомодельных решений в качестве начального приближения итерационного процесса, проведены численные расчеты и приведен анализ результатов. Результаты численных экспериментов показывают, что полученные результаты прекрасно согласуются с физикой рассматриваемого процесса в нелинейной диффузионной системе.

Ключевые слова: режим с обострением, нелинейное граничное условие, критическая глобальная кривая существования, вырожденные параболические системы, критические показатели типа Фуджиты.

*Author Information**

Mirsaid Mirsiddiqovich Aripov — Doctor of physical-mathematical sciences, Professor at the Department of Applied Mathematics and Computer Analysis of the National University of Uzbekistan, 4 Universitetskaya street, Tashkent, 100174, Uzbekistan; e-mail: *aripovmirsaid@mail.ru*; https://orcid.org/0000-0001-5207-8852

Zafar Ravshanovich Rakhmonov — Doctor of physical-mathematical sciences, Dean of the Faculty of Applied Mathematics and Intellectual Technologies of the National University of Uzbekistan, 4 Universitetskaya street, Tashkent, 100174, Uzbekistan; e-mail: *zraxmonov@inbox.ru*; https://orcid.org/0000-0002-4190-7069

Akram Abdirashidovich Alimov (*corresponding author*) — Senior Lecturer at the Department of Information Systems and Math Disciplines of Tashkent branch of the G.V. Plekhanov Russian University of Economics, 3 Shakhriobod street, Tashkent, 100164, Uzbekistan; e-mail: *akram.alimov18@gmail.com*; https://orcid.org/0009-0009-8518-4366

*The author's name is presented in the order: First, Middle and Last Names.

https://doi.org/10.31489/2024M1/46-59 *Research article*

Problem for differential-algebraic equations with a significant loads

A.T. Assanova[1,*], Zh.M. Kadirbayeva[1,2], R.A. Medetbekova[3], S.T. Mynbayeva[1,4]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *International Information Technology University, Almaty, Kazakhstan;*
[3] *Shymkent University, Shymkent, Kazakhstan;*
[4] *K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan*
(E-mail: assanova@math.kz, zhkadirbayeva@gmail.com, ryskulmedetbekova@mail.ru, mynbaevast80@gmail.com)

In this article, the problem for a differential-algebraic equation with a significant loads is studied. Unlike previously studied problems for differential equations with a significant loads, in the considered equation, there is a matrix in the left part with a derivative that is not invertible. Therefore, the system of equations includes both differential and algebraic equations. To solve the problem, we propose a modification of the Dzhumabaev's parametrization method. The considered problem is reduced to a parametric problem for the differential-algebraic equation with significant loads. We apply the Weierstrass canonical form to this problem. We obtain parametric initial value problem for a differential equations and an algebraic equations with a significant loads. The solvability conditions for the considered problem are established.

Keywords: differential-algebraic equations, equations with significant loads, parameter, parametric initial value problem, solution.

2020 Mathematics Subject Classification: 34A09; 34A36; 34B08; 34K10.

Introduction

Differential equations with significant loads are equations that describe how a system changes over time, taking into account significant external influences or forces, known as "loads". These external influences could represent various factors such as external forces, environmental conditions, or other external factors that affect the behavior of the system.

In the context of scientific and engineering applications, these equations are often used to model dynamic systems where the behavior is influenced by external factors. For example, in the study of diffusion processes, soil moisture dynamics, or the spread of infections, the differential equations with significant loads would mathematically represent how the system evolves over time, considering the impact of external loads on the system's dynamics.

The solutions to these differential equations provide insights into the behavior of the system under the influence of these significant loads, helping researchers and scientists understand and predict the system's evolution over time. The study of such equations is essential in various fields, including physics, biology, engineering, and environmental science [1–17].

Solving differential equations with significant loads can be challenging, and the specific methods you choose depend on the characteristics of the problem. Here are some general approaches: 1) Analytical methods — they include Separation of variables, Integrating factors, Exact equations; 2) Numerical methods — they include Euler's method, Runge-Kutta methods, Finite difference methods; 3) Series solutions — they include Power series. This method is useful for solving linear differential equations with variable coefficients; 4) Transform methods — they include Laplace transform. The Laplace transform

*Corresponding author. *E-mail: assanova@math.kz*

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19675193).

Received: 16 November 2023; *Accepted:* 08 December 2023.

can simplify the differential equation into an algebraic equation, making it easier to solve; 5) Numerical simulation — they are Finite element method, Boundary element method; 6) Special functions — they include Bessel Functions, Legendre Polynomials, etc.; 7) Computer algebra systems and software — they include Mathematica, MATLAB, or Python with libraries like SciPy to numerically solve differential equations or perform symbolic computations.

When dealing with significant loads, it's crucial to consider the nature of the load (constant, time-dependent, etc.) and the type of differential equation (ordinary or partial). In many real-world situations, a combination of analytical and numerical methods, possibly with the aid of computational tools, is necessary for obtaining solutions.

Differential-algebraic equations with significant loads refer to a class of mathematical equations that involve a combination of differential equations and algebraic equations, where the system is subjected to significant external forces or loads. These equations are common in various scientific and engineering applications, especially when modeling complex dynamic systems [18–33].

The presence of significant loads implies that external forces or influences play a substantial role in the behavior of the system. These loads can be time-dependent, leading to a more intricate mathematical formulation.

Solving differential-algebraic equations with significant loads may require specialized numerical methods or a combination of analytical and numerical techniques. The choice of method depends on the specific characteristics of the problem, such as the nature of the loads and the structure of the equations. Some common methods for solving differential-algebraic equations include implicit and explicit numerical methods, index reduction techniques, and advanced numerical solvers.

Researchers often study and develop methods tailored to the specific challenges posed by differential-algebraic equations with significant loads to accurately model and simulate the behavior of dynamic systems in various fields, including physics, engineering, and biology.

The present article considers a problem for the differential-algebraic equation with significant loads, where the left-hand side of the equation involves a non-invertible matrix. To study and solve this problem, a modification of the Dzhumabaev's parametrization method [34] is proposed. Considered problem is reduced to a parametric initial-boundary value problem for the differential-algebraic equations with significant loads.

1 Statement of problem and reduction to a parametric problem

On $[0,T]$ the following problem for differential-algebraic equations with significant loads is considered:

$$E\dot{x}(t) = Ax(t) + E_0\dot{x}(\theta) + A_0x(\theta) + f(t), \qquad t \in [0,T], \tag{1}$$

$$Bx(0) + Cx(T) = d, \tag{2}$$

where the matrices $E, A \in \mathbb{C}^{n,n}$, $E_0, A_0 \in \mathbb{C}^{n,n}$, and the function $f(t) \in C([0,T],\mathbb{C}^n)$, $0 < \theta < T$, the matrices $B, C \in \mathbb{C}^{n,n}$, the vector $d \in \mathbb{C}^n$.

We suppose that the matrix pair (E, A) is regular.

A solution to problem (1), (2) is called a function $x(t) \in C([0,T],\mathbb{C}^n)$ having derivative $\dot{x}(t) \in C([0,T],\mathbb{C}^n)$, satisfies to differential-algebraic equations with significant loads (1) and two-point condition (2).

The aim of the paper is to propose a constructive method for solving problem (1), (2).

For solving the problem for differential-algebraic equations with significant loads (1), (2) Dzhumabaev's parametrization method is applied [34].

We introduce a parameter ξ in the following form: $E\xi = Ex(0)$, a.e. as a value of the unknown function at the left endpoint. Then, in the problem (1), (2) we replace $x(t)$ by a new function in the form $x(t) = y(t) + \xi$. The two-point problem for differential-algebraic equations with significant loads (1), (2) transfers to the parametric problem

$$E\dot{y}(t) = Ay(t) + E_0\dot{y}(\theta) + A_0 y(\theta) + [A + A_0]\xi + f(t), \qquad t \in [0, T], \tag{3}$$

$$Ey(0) = 0, \tag{4}$$

$$[B + C]\xi + Cy(T) = d. \tag{5}$$

We obtain the parametric for differential-algebraic equations with significant loads and initial condition (3)–(5). Relation (5) can be interpreted as an algebraic equation, containing unknown parameter ξ and value of the unknown function $y(t)$ at the point $t = T$.

A solution to the parametric problem for differential-algebraic equations with significant loads and initial condition (3)–(5) is called a pair $(y(t), \xi)$ with elements $y(t) \in C([0, T], \mathbb{C}^n)$ and $\xi \in \mathbb{C}^n$, satisfies to system (3), initial condition (4) and system of algebraic equations (5).

Subsequently, based on the properties of the obtained parametric problem (3)–(5), we give the solvability conditions to the considered problem (1), (2). For this purpose, in next Section the Weierstrass canonical form is applied to the parametric problem (3)–(5).

2 Weierstrass canonical form and solution to parametric problem

Further, we apply Weierstrass canonical form [18], it is a specific representation of differential-algebraic equations. The Weierstrass canonical form simplifies the analysis and numerical solution of differential-algebraic equations by separating the differential and algebraic components of the system. It provides a structured representation that is easier to work with when applying numerical integration techniques or performing stability analysis.

Let P and Q be nonsingular matrices on dimension n which transform (3) to the Weierstrass canonical form

$$PEQ = \begin{bmatrix} I_{n_1} & O_{n_2} \\ O_{n_1} & N \end{bmatrix}, \qquad PAQ = \begin{bmatrix} J & O_{n_2} \\ O_{n_1} & I_{n_2} \end{bmatrix}, \qquad Pf = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix}, \tag{6}$$

where I_{n_1} and I_{n_2} is a identity matrices on dimension n_1, n_2, respectively, O_{n_1} and O_{n_2} is a null matrices on dimension n_1, n_2, respectively, N is a nilpotent matrix on dimension n_2, J is a matrix in Jordan canonical form on dimension n_1, $n_1 + n_2 = n$. Following [18], we call the index of nilpotency of N in (6) the index of the matrix pair (E, A), denoted by $\nu = \text{ind}(E, A)$.

We suppose that the matrices E_0 and A_0 have the forms

$$PE_0Q = \begin{bmatrix} L_{n_1} & O_{n_2} \\ O_{n_1} & L_{n_2} \end{bmatrix}, \qquad PA_0Q = \begin{bmatrix} M_{n_1} & O_{n_2} \\ O_{n_1} & M_{n_2} \end{bmatrix}, \tag{7}$$

where L_{n_1}, M_{n_1} and L_{n_2}, M_{n_2} are a constant matrices on dimension n_1, n_2, respectively.

Using (6), (7) we reduce parametric problem (3)–(5) to the next form:

$$\dot{\tilde{y}}_1(t) = J(\tilde{y}_1(t) + \tilde{\xi}_1) + L_{n_1}\dot{\tilde{y}}_1(\theta) + M_{n_1}\tilde{y}_1(\theta) + M_{n_1}\tilde{\xi}_1 + \tilde{f}_1(t), \tag{8}$$

$$\tilde{y}_1(0) = 0, \tag{9}$$

$$N\dot{\tilde{y}}_2(t) = \tilde{y}_2(t) + \tilde{\xi}_2 + L_{n_2}\dot{\tilde{y}}_2(\theta) + M_{n_2}\tilde{y}_2(\theta) + M_{n_2}\tilde{\xi}_2 + \tilde{f}_2(t), \tag{10}$$

$$N\tilde{y}_2(0) = 0, \tag{11}$$

$$[\tilde{B} + \tilde{C}]\tilde{\xi} = d - \tilde{C}\tilde{y}(T), \tag{12}$$

where $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)^T = Q^{-1}y$, $\tilde{y}_1(t) \in C([0,T], \mathbb{C}^{n_1})$, $\tilde{y}_2(t) \in C([0,T], \mathbb{C}^{n_2})$, $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2)^T = Q^{-1}\xi$, $\tilde{\xi}_1 \in \mathbb{C}^{n_1}$, $\tilde{\xi}_2 \in \mathbb{C}^{n_2}$, $\tilde{B} = BQ, \tilde{C} = CQ$.

Problems (8), (9) and (10), (11) are initial value problems with parameter for differential equations with significant loads.

A pair $(\tilde{y}(t), \tilde{\xi})$ with $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)^T$ and $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2)^T$ is called a solution to problem (8)–(12), if it satisfies the initial value problems (8), (9) and (10), (11), and the relation (12).

From equations (8) and (10) we determine the values $\dot{\tilde{y}}_1(\theta)$ and $\dot{\tilde{y}}_2(\theta)$. We have

$$\dot{\tilde{y}}_1(\theta) = J(\tilde{y}_1(\theta) + \tilde{\xi}_1) + L_{n_1}\dot{\tilde{y}}_1(\theta) + M_{n_1}\tilde{y}_1(\theta) + M_{n_1}\tilde{\xi}_1 + \tilde{f}_1(\theta), \tag{13}$$

$$N\dot{\tilde{y}}_2(\theta) = \tilde{y}_2(\theta) + \tilde{\xi}_2 + L_{n_2}\dot{\tilde{y}}_2(\theta) + M_{n_2}\tilde{y}_2(\theta) + M_{n_2}\tilde{\xi}_2 + \tilde{f}_2(\theta). \tag{14}$$

From (13) and (14) we obtain

$$[I_{n_1} - L_{n_1}]\dot{\tilde{y}}_1(\theta) = [J + M_{n_1}]\tilde{y}_1(\theta) + [J + M_{n_1}]\tilde{\xi}_1 + \tilde{f}_1(\theta), \tag{15}$$

$$[N - L_{n_2}]\dot{\tilde{y}}_2(\theta) = [I_{n_2} + M_{n_2}]\tilde{y}_2(\theta) + [I_{n_2} + M_{n_2}]\tilde{\xi}_2 + \tilde{f}_2(\theta). \tag{16}$$

Assuming that the matrices $I_{n_1} - L_{n_1}$ and $N - L_{n_2}$ are non-singular in (15), (16), we have the following presentations for $\dot{\tilde{y}}_1(\theta)$ and $\dot{\tilde{y}}_2(\theta)$:

$$\dot{\tilde{y}}_1(\theta) = [I_{n_1} - L_{n_1}]^{-1}[J + M_{n_1}]\tilde{y}_1(\theta) + [I_{n_1} - L_{n_1}]^{-1}[J + M_{n_1}]\tilde{\xi}_1 + [I_{n_1} - L_{n_1}]^{-1}\tilde{f}_1(\theta), \tag{17}$$

$$\dot{\tilde{y}}_2(\theta) = [N - L_{n_2}]^{-1}[I_{n_2} + M_{n_2}]\tilde{y}_2(\theta) + [N - L_{n_2}]^{-1}[I_{n_2} + M_{n_2}]\tilde{\xi}_2 + [N - L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{18}$$

Substituting (17), (18) into the equations (8), (10) instead of the values $\dot{\tilde{y}}_1(\theta)$ and $\dot{\tilde{y}}_2(\theta)$, we obtain

$$\dot{\tilde{y}}_1(t) = J\tilde{y}_1(t) + J\tilde{\xi}_1 + \tilde{L}_{n_1}\tilde{y}_1(\theta) + \tilde{L}_{n_1}\tilde{\xi}_1 + \tilde{f}_1(t) + L_{n_1}[I_{n_1} - L_{n_1}]^{-1}\tilde{f}_1(\theta), \tag{19}$$

$$N\dot{\tilde{y}}_2(t) = \tilde{y}_2(t) + \tilde{\xi}_2 + \tilde{L}_{n_2}\tilde{y}_2(\theta) + \tilde{L}_{n_2}\tilde{\xi}_2 + \tilde{f}_2(t) + L_{n_2}[N - L_{n_2}]^{-1}\tilde{f}_2(\theta), \tag{20}$$

where $\tilde{L}_{n_1} = L_{n_1}[I_{n_1} - L_{n_1}]^{-1}[J + M_{n_1}] + M_{n_1}$, $\tilde{L}_{n_2} = L_{n_2}[N - L_{n_2}]^{-1}[I_{n_2} + M_{n_2}] + M_{n_2}$.

For fixed $\tilde{\xi}_1$ solution to initial value problem (19), (9) has the next representation:

$$\tilde{y}_1(t) = \int_0^t e^{(t-s)J}dsJ\tilde{\xi}_1 + \int_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}\tilde{y}_1(\theta) + \int_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}\tilde{\xi}_1 +$$

$$+ \int_0^t e^{(t-s)J}\tilde{f}_1(s)ds + \int_0^t e^{(t-s)J}dsJL_{n_1}[I_{n_1} - L_{n_1}]^{-1}\tilde{f}_1(\theta), \qquad t \in [0,T]. \tag{21}$$

By Lemma 2.8 [18] and property of matrix N, for fixed $\tilde{\xi}_2$ equation (20) has the unique solution in the form:

$$\tilde{y}_2(t) = -\sum_{j=0}^{\nu-1} N^j\Big[\tilde{\xi}_2 + \tilde{L}_{n_2}\tilde{y}_2(\theta) + \tilde{L}_{n_2}\tilde{\xi}_2 + \tilde{f}_2(t) + L_{n_2}[N - L_{n_2}]^{-1}\tilde{f}_2(\theta)\Big]^{(j)} =$$

$$= -\sum_{j=0}^{\nu-1} N^j\tilde{f}_2^{(j)}(t) - \tilde{L}_{n_2}\tilde{y}_2(\theta) - \tilde{\xi}_2 - \tilde{L}_{n_2}\tilde{\xi}_2 - L_{n_2}[N - L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{22}$$

From expressions (21) and (22) we determine values of functions $\tilde{y}_1(t)$ and $\tilde{y}_2(t)$ at the point $t = \theta$:

$$\tilde{y}_1(\theta) = \int_0^\theta e^{(\theta-s)J}dsJ\tilde{\xi}_1 + \int_0^\theta e^{(\theta-s)J}dsJ\tilde{L}_{n_1}\tilde{y}_1(\theta) + \int_0^\theta e^{(\theta-s)J}dsJ\tilde{L}_{n_1}\tilde{\xi}_1 +$$

$$+\int\limits_0^\theta e^{(\theta-s)J}\tilde{f}_1(s)ds+\int\limits_0^\theta e^{(\theta-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta), \tag{23}$$

$$\tilde{y}_2(\theta)=-\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(\theta)-\tilde{L}_{n_2}\tilde{y}_2(\theta)-\tilde{\xi}_2-\tilde{L}_{n_2}\tilde{\xi}_2-L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{24}$$

From equations (23) and (24) we have

$$\Big[I_{n_1}-\int\limits_0^\theta e^{(\theta-s)J}dsJ\tilde{L}_{n_1}\Big]\tilde{y}_1(\theta)=\int\limits_0^\theta e^{(\theta-s)J}dsJ\tilde{\xi}_1+\int\limits_0^\theta e^{(\theta-s)J}dsJ\tilde{L}_{n_1}\tilde{\xi}_1+$$

$$+\int\limits_0^\theta e^{(\theta-s)J}\tilde{f}_1(s)ds+\int\limits_0^\theta e^{(\theta-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta), \tag{25}$$

$$[I_{n_2}+\tilde{L}_{n_2}]\tilde{y}_2(\theta)=-\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(\theta)-\tilde{\xi}_2-\tilde{L}_{n_2}\tilde{\xi}_2-L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{26}$$

We suppose that the matrices $D_{n_1}=I_{n_1}-\int\limits_0^\theta e^{(\theta-s)J}dsJ\tilde{L}_{n_1}$ and $D_{n_2}=I_{n_2}+\tilde{L}_{n_2}$ are invertible, a.e. non-singular. Then, from algebraic equations (25), (26), we obtain the expressions for $\tilde{y}_1(\theta)$ and $\tilde{y}_2(\theta)$:

$$\tilde{y}_1(\theta)=D_{n_1}^{-1}\int\limits_0^\theta e^{(\theta-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]\tilde{\xi}_1+$$

$$+D_{n_1}^{-1}\int\limits_0^\theta e^{(\theta-s)J}\tilde{f}_1(s)ds+D_{n_1}^{-1}\int\limits_0^\theta e^{(\theta-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta), \tag{27}$$

$$\tilde{y}_2(\theta)=-D_{n_2}^{-1}\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(\theta)-\tilde{\xi}_2-D_{n_2}^{-1}L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{28}$$

Substituting (27) and (28) into the expressions (21), (22) instead of the values $\tilde{y}_1(\theta)$ and $\tilde{y}_2(\theta)$, we obtain

$$\tilde{y}_1(t)=\int\limits_0^t e^{(t-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]\tilde{\xi}_1+\int\limits_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}D_{n_1}^{-1}\int\limits_0^\theta e^{(\theta-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]\tilde{\xi}_1+$$

$$+\int\limits_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}\Big\{D_{n_1}^{-1}\int\limits_0^\theta e^{(\theta-s)J}\tilde{f}_1(s)ds+D_{n_1}^{-1}\int\limits_0^\theta e^{(\theta-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta)\Big\}+$$

$$+\int\limits_0^t e^{(t-s)J}\tilde{f}_1(s)ds+\int\limits_0^t e^{(t-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta),\qquad t\in[0,T], \tag{29}$$

$$\tilde{y}_2(t)=-\tilde{\xi}_2-\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(t)+\tilde{L}_{n_2}D_{n_2}^{-1}\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(\theta)+$$

$$+\tilde{L}_{n_2}D_{n_2}^{-1}L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta)-L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta). \qquad (30)$$

Hence, taking into account initial condition (11), we obtain that the second component of the parameter $\tilde{\xi}$ is uniquely determined and the vector $\tilde{\xi}_2$ has the next form

$$\tilde{\xi}_2=-\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(0)+\tilde{L}_{n_2}D_{n_2}^{-1}\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(\theta)+$$

$$+\tilde{L}_{n_2}D_{n_2}^{-1}L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta)-L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta). \qquad (31)$$

As can be seen from (30) and (31) the second components of $\tilde{y}$ and $\tilde{\xi}$ became known.

Now, we are interested in finding only the first components $\tilde{y}_1$ and $\tilde{\xi}_1$ which are interrelated by (29). Therefore, the appropriate number of imposed boundary conditions must match the number of n_1 differential equations in (1).

A natural question arises: what should be the structure of boundary matrices B and C?

3 The solvability of problem (1), (2)

We assume that the $n\times n$ matrices and the right-hand side n-vector of the relation (2) are of the form

$$BQ=\tilde{B}=\begin{bmatrix}\tilde{B}_1 & O_{n_2}\\ O_{n_1} & O_{n_2}\end{bmatrix},\qquad CQ=\tilde{C}=\begin{bmatrix}\tilde{C}_1 & O_{n_2}\\ O_{n_1} & O_{n_2}\end{bmatrix},\qquad d=\begin{bmatrix}d_1\\ O\end{bmatrix}, \qquad (32)$$

where $\tilde{B}_1,\tilde{C}_1\in\mathbb{C}^{n_1,n_1}$ and $d\in\mathbb{C}^{n_1}$.

Now, by substituting (29) into (2), we get the following algebraic equation with respect to $\tilde{\xi}_1$:

$$\Phi\tilde{\xi}_1=\tilde{d}, \qquad (33)$$

where

$$\Phi=\tilde{B}_1+\tilde{C}_1+\tilde{C}_1\int_0^T e^{(T-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]+\tilde{C}_1\int_0^T e^{(T-s)J}dsJ\tilde{L}_{n_1}D_{n_1}^{-1}\int_0^\theta e^{(\theta-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]$$

and

$$\tilde{d}=d_1-\tilde{C}_1\int_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}D_{n_1}^{-1}\Big\{\int_0^\theta e^{(\theta-s)J}\tilde{f}_1(s)ds+\int_0^\theta e^{(\theta-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta)\Big\}-$$

$$-\tilde{C}_1\int_0^t e^{(t-s)J}\tilde{f}_1(s)ds-\tilde{C}_1\int_0^t e^{(t-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta).$$

If the matrix Φ is nonsingular, a.e. is invertible, then system of algebraic equations (33) has the unique solution $\tilde{\xi}_1^*=\Phi^{-1}\tilde{d}$. Substituting $\tilde{\xi}_1^*$ into (29), we find $\tilde{y}_1^*$ and hence the first components of the unique solution $(\tilde{y}^*,\tilde{\xi}^*)$ of the parametric problem (8)–(12):

$$\tilde{y}_1^*(t)=\int_0^t e^{(t-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]\Phi^{-1}\tilde{d}+\int_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}D_{n_1}^{-1}\int_0^\theta e^{(\theta-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]\Phi^{-1}\tilde{d}+$$

$$+\int\limits_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}\Big\{D_{n_1}^{-1}\int\limits_0^\theta e^{(\theta-s)J}\tilde{f}_1(s)ds + D_{n_1}^{-1}\int\limits_0^\theta e^{(\theta-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta)\Big\}+$$

$$+\int\limits_0^t e^{(t-s)J}\tilde{f}_1(s)ds + \int\limits_0^t e^{(t-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta), \qquad t\in[0,T], \tag{34}$$

$$\tilde{\xi}_1^* = \Phi^{-1}\tilde{d}. \tag{35}$$

As shown earlier, the second components of $(\tilde{y}^*, \tilde{\xi}^*)$ are determined by:

$$\tilde{y}_2^*(t) = \sum_{j=0}^{\nu-1} N^j \tilde{f}_2^{(j)}(0) - \sum_{j=0}^{\nu-1} N^j \tilde{f}_2^{(j)}(t), \qquad t\in[0,T], \tag{36}$$

$$\tilde{\xi}_2^* = -\sum_{j=0}^{\nu-1} N^j \tilde{f}_2^{(j)}(0) + \tilde{L}_{n_2}D_{n_2}^{-1}\sum_{j=0}^{\nu-1} N^j \tilde{f}_2^{(j)}(\theta)+$$

$$+\tilde{L}_{n_2}D_{n_2}^{-1}L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta) - L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{37}$$

Therefore, taking into account the interrelation between the parametric problem (3)–(5) and the initial value problem with parameter (8)-(12), we can summarize our result for this case.

Theorem 1. Let (E, A) be a regular pair of square matrices and let P and Q be nonsingular matrices which transform (3) to Weierstrass canonical form (6). Furthermore, let $\nu = \mathrm{ind}(E, A)$ and $f \in C^\nu([0, T], \mathbb{C}^n)$. Assume that:

i) the matrices E_0 and A_0 have the forms (7) with constant matrices L_{n_1}, M_{n_1} and L_{n_2}, M_{n_2} on dimension n_1, n_2, respectively;

ii) the matrices $I_{n_1} - L_{n_1}$ and $N - L_{n_2}$ are non-singular;

iii) the matrices $D_{n_1} = I_{n_1} - \int\limits_0^\theta e^{(\theta-s)J}dsJ\tilde{L}_{n_1}$ and $D_{n_2} = I_{n_2} + \tilde{L}_{n_2}$ are non-singular, where

$\tilde{L}_{n_1} = L_{n_1}[I_{n_1} - L_{n_1}]^{-1}[J + M_{n_1}] + M_{n_1}, \qquad \tilde{L}_{n_2} = L_{n_2}[N - L_{n_2}]^{-1}[I_{n_2} + M_{n_2}] + M_{n_2}.$

Then the initial value problem with parameter (8)–(12) with the matrices B, C, and vector d of the form (32) has a unique solution $(\tilde{y}^*, \tilde{\xi}^*)$ if and only if the matrix

$$\Phi = \tilde{B}_1 + \tilde{C}_1 + \tilde{C}_1\int\limits_0^T e^{(T-s)J}dsJ[I_{n_1} + \tilde{L}_{n_1}] + \tilde{C}_1\int\limits_0^T e^{(T-s)J}dsJ\tilde{L}_{n_1}D_{n_1}^{-1}\int\limits_0^\theta e^{(\theta-s)J}dsJ[I_{n_1} + \tilde{L}_{n_1}]$$

is nonsingular.

Taking into account the interrelation between of the parametric problem (3)–(5) and the initial value problem with parameter (8)–(12), we write the unique solution $(y^*(t), \xi^*)$ of the parametric problem (3)–(5) in the following form

$$(y^*(t), \xi^*) = Q\begin{bmatrix}(\tilde{y}_1^*(t), \tilde{\xi}_1^*)\\(\tilde{y}_2^*(t), \tilde{\xi}_2^*)\end{bmatrix}, \tag{38}$$

where the functions $\tilde{y}_1^*(t)$, $\tilde{y}_2^*(t)$ and the vectors $\tilde{\xi}_1^*$, $\tilde{\xi}_2^*$ are determined by (34), (36) and (35), (37), respectively,

$$Pf(t) = \begin{bmatrix}\tilde{f}_1(t)\\\tilde{f}_2(t)\end{bmatrix},$$

$$\tilde{d} = d_1 - \tilde{C}_1 \int\limits_0^t e^{(t-s)J} ds J \tilde{L}_{n_1} D_{n_1}^{-1} \Big\{ \int\limits_0^\theta e^{(\theta-s)J} \tilde{f}_1(s) ds + \int\limits_0^\theta e^{(\theta-s)J} ds J L_{n_1} [I_{n_1} - L_{n_1}]^{-1} \tilde{f}_1(\theta) \Big\} -$$

$$-\tilde{C}_1 \int\limits_0^t e^{(t-s)J} \tilde{f}_1(s) ds - \tilde{C}_1 \int\limits_0^t e^{(t-s)J} ds J L_{n_1} [I_{n_1} - L_{n_1}]^{-1} \tilde{f}_1(\theta).$$

From the equivalence problems (3)–(5) and (1), (2) it follows that

Theorem 2. Let (E, A) be a regular pair of square matrices and let P and Q be nonsingular matrices which transform (3) to Weierstrass canonical form (6). Furthermore, let $\nu = \text{ind}(E, A)$ and $f \in C^\nu([0, T], \mathbb{C}^n)$. Assume that:

i) the matrices E_0 and A_0 have the forms (7) with constant matrices L_{n_1}, M_{n_1} and L_{n_2}, M_{n_2} on dimension n_1, n_2, respectively;

ii) the matrices $I_{n_1} - L_{n_1}$ and $N - L_{n_2}$ are non-singular;

iii) the matrices $D_{n_1} = I_{n_1} - \int\limits_0^\theta e^{(\theta-s)J} ds J \tilde{L}_{n_1}$ and $D_{n_2} = I_{n_2} + \tilde{L}_{n_2}$ are non-singular, where

$\tilde{L}_{n_1} = L_{n_1}[I_{n_1} - L_{n_1}]^{-1}[J + M_{n_1}] + M_{n_1}, \qquad \tilde{L}_{n_2} = L_{n_2}[N - L_{n_2}]^{-1}[I_{n_2} + M_{n_2}] + M_{n_2}.$

Then the problem (1), (2) with the matrices B, C, and vector d of the form (32) has a unique solution if and only if the matrix

$$\Phi = \tilde{B}_1 + \tilde{C}_1 + \tilde{C}_1 \int\limits_0^T e^{(T-s)J} ds J [I_{n_1} + \tilde{L}_{n_1}] + \tilde{C}_1 \int\limits_0^T e^{(T-s)J} ds J \tilde{L}_{n_1} D_{n_1}^{-1} \int\limits_0^\theta e^{(\theta-s)J} ds J [I_{n_1} + \tilde{L}_{n_1}]$$

is nonsingular. And, the solution $x^*(t)$ of problem (1), (2) is determined by equality

$$x^*(t) = y^*(t) + \xi^*, \qquad t \in [0, T],$$

where the function $y^*(t)$ and the vector ξ^* are defined from (38) with components $\tilde{y}_1^*(t)$, $\tilde{y}_2^*(t)$, $\tilde{\xi}_1^*$, $\tilde{\xi}_2^*$ give the expressions (34)–(37).

Acknowledgments

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19675193).

Author Contributions

Z.M. Kadirbayeva and S.T.Mynbayeva collected and analyzed data, and led manuscript preparation. R.A. Medetbekova assisted in data collection and analysis. A.T. Assanova served as the principal investigator of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Нахушев А.М. Уравнения математической биологии / А.М. Нахушев., Нальчик: Высш. шк. — 1995. — 301 с.

2 Дженалиев М.Т. Нагруженные уравнения как возмущения дифференциальных уравнений / М.Т. Дженалиев, М.И. Рамазанов. — Алматы: Ғылым, 2010. — 334 с.

3 Нахушев А.М. Нагруженные уравнения и их применение / А.М. Нахушев. — М.: Наука, 2012. — 232 с.

4 Yesbayev A.N. Investigation of the model for the essentially loaded heat equation / A.N. Yesbayev, G.A. Yessenbayeva, M.I. Ramazanov // Eurasian Physical Technical Journal. — 2019. — 16. — No. 1. — P. 121–128. https://doi.org/10.31489/2019no1/121-128

5 Yesbayev A.N. On the kernel properties of the integral equation for the model with the essentially loaded heat equation / A.N. Yesbayev, G.A. Yessenbayeva, G.A. Yessenbayeva, M.I. Ramazanov // Eurasian Physical Technical Journal. — 2019. — 16. — No. 2. — P. 105–112. https://doi.org/10.31489/2019No2/105-112

6 Akhmanova D.M. On strongly loaded heat equations / D.M. Akhmanova, M.T. Kosmakova, B.A. Shaldykova // Bulletin of the Karaganda University. Mathematics series. — 2019. — No. 4(96). — P. 8–14. https://doi.org/10.31489/2019M4/8-14

7 Assanova A.T. Numerical solution of systems of loaded ordinary differential equations with multipoint conditions / A.T. Assanova, A.E. Imanchiyev, Z.M. Kadirbayeva // Comput. Math. Math. Phys. — 2018. — 58. — No. 4. — P. 508–516. https://doi.org/10.1134/S096554251804005X

8 Assanova A.T. On the numerical algorithms of parametrization method for solving a two-point boundary-value problem for impulsive systems of loaded differential equations / A.T. Assanova, Z.M. Kadirbayeva // Comput. and Appl. Math. — 2018. — 37. — No. 4. — P. 4966–4976. https://doi.org/10.1007/s40314-018-0611-9

9 Kadirbayeva Zh.M. Numerical implementation of solving a control problem for loaded differential equations with multi-point condition / Zh.M. Kadirbayeva, A.D. Dzhumabaev // Bulletin of the Karaganda University. Mathematics series. — 2020. — No. 4(100). — P. 81–91. https://doi.org/10.31489/2020M4/81-91

10 Kadirbayeva Z.M. A numerical method for solving boundary value problem for essentially loaded differential equations / Z.M. Kadirbayeva // Lobachevskii J. Math. — 2021. — 42. — No. 3. — P. 551–559. https://doi.org/10.1134/S1995080221030112

11 Kadirbayeva Z.M. A computational method for solving the boundary value problem for impulsive systems of essentially loaded differential equations / Z.M. Kadirbayeva, S.S. Kabdrakhova, S.T. Mynbayeva // Lobachevskii J. Math. — 2021. — 42. — No. 15. — P. 3675–3683. https://doi.org/10.1134/S1995080222030131

12 Kadirbayeva Z.M. A numerical solution of problem for essentially loaded differential equations with an integro-multipoint condition / Z.M. Kadirbayeva, S.S. Kabdrakhova // Open Mathematics. — 2022. — 20. — No. 1. — P. 1173–1183. https://doi.org/10.1515/math-2022-0496

13 Abildayeva A. A multi-point problem for a system of differential equations with piecewise-constant argument of generalized type as a neural network model / A. Abildayeva, A. Assanova, A. Imanchiyev // Eurasian Math. J. — 2022. — 13. — No. 2. — P. 8–17. https://doi.org/10.32523/2077-9879-2022-13-2-08-17

14 Uteshova R.E. A novel numerical implementation for solving problem for loaded depcag / R.E. Uteshova, Zh.M. Kadirbayeva, G. Marat, B.B. Minglibayeva // International J. Math. Phys. — 2022. — 13. — No. 2. — P. 50–57. https://doi.org/10.26577/ijmph.2022.v13.i2.07

15 Kadirbayeva Z.M. On an algorithm for solving a problem with parameter for the essentially

loaded differential equations / Z.M. Kadirbayeva // Lobachevskii J. Math. — 2022. — 43. — No. 11. — P. 3183–3191. https://doi.org/10.1134/S1995080222140177

16 Bakirova E.A. A computational method for solving a boundary-value problem for differential equations with piecewise constant argument of generalized type / E.A. Bakirova, Z.M. Kadirbayeva, G.I. Salgarayeva // Lobachevskii J. Math. — 2022. — 43. — No. 11. — P. 3057–3064. https://doi.org/10.1134/S1995080222140050

17 Kadirbayeva Z. A new computational approach for solving a boundary-value problem for DEPCAG / Z. Kadirbayeva, A. Assanova, E. Bakirova // Advances in the Theory of Nonlinear Analysis and its Applications. — 2023. — 7. — No. 2. — P. 362–376. https://doi.org/10.31197/atnaa.1202501

18 Kunkel P. Differential-Algebraic Equations. Analysis and Numerical Solution / P. Kunkel, V. Mehrmann. — Zurich: European Mathematical Society Publishing House, 2006. — 377 p.

19 Campbell S.L. Direct transcription solution of high index optimal control problems and regular Euler-Lagrange equations / S.L. Campbell, R. Marz // J. Comp. Appl. Math. — 2007. — 202. — P. 186–202. https://doi.org/10.1016/j.cam.2006.02.024

20 Berger T. Zero dynamics and funnel control of linear differential-algebraic systems / T. Berger, A. Ilchmann, T. Reis // Math. Control Signals Syst. — 2012. — 24. — P. 219–263. https://doi.org/10.1016/10.1007/s00498-012-0085-z

21 Lamour R. Differential-Algebraic Equations: A Projector Based Analysis / R. Lamour, R. Marz, C. Tischendorf. — Heidelberg: Springer Berlin, 2012. — 659 p.

22 Ilchmann A. Surveys in Differential-Algebraic Equations I / A. Ilchmann, T. Reis. — Berlin Heidelberg: Springer-Verlag, 2013. — 231 p.

23 Berger T. On the standard canonical form of time-varying linear DAEs / T. Berger, A. Ilchmann // Quarterly Appl. Math. — 2013. — 61. — No. 1. — P. 69–87. https://doi.org/10.1090/S0033-569X-2012-01285-1

24 Linh V.H. Asymptotic integration of linear differential-algebraic equations / V.H. Linh, N.N. Tuan // Electr. J. Qual. Theory Differ. Equ. — 2014. — 12. — P. 1–17.

25 Chuiko S.M. A generalized matrix differential-algebraic equation / S.M. Chuiko // J. Math. Sci. — 2015. — 210. — No. 1. — P. 9–21.

26 Lamour R. Boundary-Value Problems for Differential-Algebraic Equations: A Survey / R. Lamour, R. März, E. Weinmüller // In book: Surveys in Differential-Algebraic Equations III. — Berlin Heidelberg: Springer-Verlag, 2015. — P. 177–309.

27 Campbell S.L. Solving higher index DAE optimal control problems / S.L. Campbell, P. Kunkel // Numerical Algebra, Control and Optimization. — 2016. — 6. — No. 4. — P. 447–472. https://doi.org/10.3934/naco.2016020

28 Berger T. Linear relations and the Kronecker canonical form / T. Berger, C. Trunk, H. Winkler // Linear Algebra and its Applications. — 2016. — 488. — No. 1. — P. 13–44. https://doi.org/10.1016/j.laa.2015.09.033

29 Iwata S. On the Kronecker canonical form of singular mixed matrix pencils / S. Iwata, M. Takamatsu // SIAM J. Control Optim. — 2017. — 55. — No. 3. — P. 2134–2150. DOI: https://doi.org/10.1137/16M1057516

30 Chuiko S.M. On a reduction of the order in a differential-algebraic system / S.M. Chuiko // J. Math. Sci. — 2018. — 235. — No. 1. — P. 2–14.

31 Ilchmann A. Optimal control of differential-algebraic equations from an ordinary differential equation perspective / A. Ilchmann, L. Leben, J. Witschel, K. Worthmann // Optim. Control Appl. Meth. — 2019. — 40. — P. 351–366. https://doi.org/10.1002/oca.2481

32 Leben L. Finite rank perturbations of linear relations and matrix pencils / L. Leben, F.M. Perla,

F. Philipp, C. Trunk, H. Winkler // Complex Anal. Operator Theory. — 2021. — 15. — P. 15–37. https://doi.org/10.1007/s11785-021-01082-x.

33 Samusenko P.F. Asymptotic solutions of boundary value problem for singularly perturbed system of differential-algebraic equations / P.F. Samusenko, M.B. Vira // Carpathian Math. Publ. — 2022. — 14. — No. 1. — P. 49–60. https://doi.org/10.15330/cmp.14.1.49-60

34 Dzhumabaev D.S. Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation / D.S. Dzhumabaev // Comput. Math. Math. Phys. — 1989. — No. 29. — P. 34–46.

Елеулі жүктемелері бар дифференциалдық-алгебралық теңдеулер үшін есеп

А.Т. Асанова[1], Ж.М. Кадирбаева[1,2], Р.А. Медетбекова[3], С.Т. Мынбаева[1,4]

[1] *Математика және математикалық модельдеу институты, Алматы, Қазақстан;*
[2] *Халықаралық ақпараттық технологиялар университеті, Алматы, Қазақстан;*
[3] *Шымкент университеті, Шымкент, Қазақстан;*
[4] *Қ. Жұбанов атындағы Ақтөбе өңірлік университеті, Ақтөбе, Қазақстан*

Мақалада елеулі жүктемелері бар дифференциалдық-алгебралық теңдеулер үшін есеп зерттелген. Елеулі жүктемелері бар дифференциалдық теңдеулер үшін бұрын зерттелген есептерден айырмашылығы, қарастырылып отырған теңдеудің сол жағындағы туындының алдында қайтымсыз матрица бар. Ендеше, теңдеулер жүйесі дифференциалдық теңдеулермен қоса, алгебралық теңдеулерді де қамтиды. Қойылған есепті шешу үшін Жұмабаевтың параметрлеу әдісінің модификациясы ұсынылады. Қарастырылып отырған есеп елеулі жүктемелері бар дифференциалдық-алгебралық теңдеулер үшін параметрлік есепке келтірілген. Осы есепке Вейерштрасс канондық формасы қолданылады. Елеулі жүктемелері бар дифференциалдық және алгебралық теңдеулер үшін параметрлік бастапқы есеп алынған. Зерттеліп отырған есептің шешілімділік шарттары анықталған.

Кілт сөздер: дифференциалдық-алгебралық теңдеулер, елеулі жүктемелері бар теңдеулер, параметр, параметрлік бастапқы есеп, шешім.

Задача для дифференциально-алгебраических уравнений с существенными нагрузками

А.Т. Асанова[1], Ж.М. Кадирбаева[1,2], Р.А. Медетбекова[3], С.Т. Мынбаева[1,4]

[1] *Институт математики и математического моделирования, Алматы, Казахстан;*
[2] *Международный университет информационных технологий, Алматы, Казахстан;*
[3] *Шымкентский университет, Шымкент, Казахстан;*
[4] *Актюбинский региональный университет имени К. Жубанова, Актобе, Казахстан*

В статье исследована задача для дифференциально-алгебраического уравнения с существенными нагрузками. В отличие от ранее изученных задач для дифференциальных уравнений с существенной нагрузкой, в рассматриваемом уравнении в левой части при производной имеется необратимая матрица. Следовательно, система уравнений включает в себя как дифференциальные, так и алгебраические уравнения. Для решения поставленной задачи предложена модификация метода параметризации Джумабаева, и задача сведена к параметрической задаче для дифференциально-алгебраического уравнения с существенными нагрузками. К этой задаче применяется каноническая форма Вейерштрасса. Получена параметрическая начальная задача для дифференциальных и алгебраических уравнений с существенными нагрузками. Установлены условия разрешимости исследуемой задачи.

Ключевые слова: дифференциально-алгебраические уравнения, уравнения с существенной нагрузкой, параметр, параметрическая начальная задача, решение.

References

1 Nakhushev, A.M. (1995). Uravneniia matematicheskoi biologii *[Equations of mathematical biology.]* Nalchik: Vysshaia shkola [in Russian].

2 Jenaliyev, M.T., & Ramazanov, M.I. (2010). Nagruzhennye uravneniia kak vozmushcheniia differentsialnykh uravnenii *[Loaded equations are as perturbations of differential equations.]* Almaty: Gylym [in Russian].

3 Nakhushev, A.M. (2012). Nagruzhennye uravneniia i ikh primenenie *[Loaded equations and their applications.]* Moscow: Nauka [in Russian].

4 Yesbayev, A.N., Yessenbayeva, G.A., & Ramazanov, M.I. (2019). Investigation of the model for the essentially loaded heat equation. *Eurasian Physical Technical Journal, 16*(1), 121–128. https://doi.org/10.31489/2019no1/121-128

5 Yesbayev, A.N., Yessenbayeva, G.A., Yessenbayeva, G.A., & Ramazanov, M.I. (2019). On the kernel properties of the integral equation for the model with the essentially loaded heat equation. *Eurasian Physical Technical Journal, 16*(2), 105–112. https://doi.org/10.31489/2019No2/105-112

6 Akhmanova, D.M., Kosmakova, M.T., & Shaldykova, B.A. (2019). On strongly loaded heat equations. *Bulletin of the Karaganda University. Mathematics series*, 4(96), 8–14. https://doi.org/10.31489/2019M4/8-14

7 Assanova, A.T., Imanchiyev, A.E., & Kadirbayeva, Z.M. (2018). Numerical solution of systems of loaded ordinary differential equations with multipoint conditions. *Comput. Math. Math. Phys., 58*(4), 508–516. https://doi.org/10.1134/S096554251804005X

8 Assanova, A.T., & Kadirbayeva, Z.M. (2018). On the numerical algorithms of parametrization method for solving a two-point boundary-value problem for impulsive systems of loaded differential equations. *Comput. and Appl. Math., 37*(4), 4966–4976. https://doi.org/10.1007/s40314-018-0611-9

9 Kadirbayeva, Zh.M., & Dzhumabaev, A.D. (2020). Numerical implementation of solving a control problem for loaded differential equations with multi-point condition. *Bulletin of the Karaganda University. Mathematics series*, 4(100), 81–91. https://doi.org/10.31489/2020M4/81-91

10 Kadirbayeva, Z.M. (2021). A numerical method for solving boundary value problem for essentially loaded differential equations. *Lobachevskii J. Math., 42*(3), 551–559. https://doi.org/10.1134/S1995080221030112

11 Kadirbayeva, Z.M., Kabdrakhova, S.S., & Mynbayeva, S.T. (2021). A computational method for solving the boundary value problem for impulsive systems of essentially loaded differential equations. *Lobachevskii J. Math., 42*(15), 3675–3683. https://doi.org/10.1134/S1995080222030131

12 Kadirbayeva, Z.M., & Kabdrakhova, S.S. (2022). A numerical solution of problem for essentially loaded differential equations with an integro-multipoint condition. *Open Mathematics, 20*(1), 1173–1183. https://doi.org/10.1515/math-2022-0496.

13 Abildayeva, A., Assanova, A., & Imanchiyev, A. (2022). A multi-point problem for a system of differential equations with piecewise-constant argument of generalized type as a neural network model. *Eurasian Math. J., 13*(2), 8–17. https://doi.org/10.32523/2077-9879-2022-13-2-08-17

14 Uteshova, R.E., Kadirbayeva, Zh.M., Marat, G., & Minglibayeva, B.B. (2022). A novel numerical implementation for solving problem for loaded depcag. *International J. Math. Phys., 13*(2), 50–57. https://doi.org/10.26577/ijmph.2022.v13.i2.07

15 Kadirbayeva, Z.M. (2022). On an algorithm for solving a problem with parameter for the essentially loaded differential equations. *Lobachevskii J. Math., 43*(11), 3183–3191. https://doi.org/10.1134/S1995080222140177

16 Bakirova, E.A., Kadirbayeva, Z.M., & Salgarayeva, G.I. (2022). A computational method for solving a boundary-value problem for differential equations with piecewise constant argument of generalized type. *Lobachevskii J. Math., 43*(11), 3057–3064. https://doi.org/10.1134/S1995080222140050.

17 Kadirbayeva, Z., Assanova, A., & Bakirova, E. (2023). A new computational approach for solving a boundary-value problem for DEPCAG. *Advances in the Theory of Nonlinear Analysis and its Applications, 7*(2), 362–376. https://doi.org/10.31197/atnaa.1202501

18 Kunkel, P., & Mehrmann, V. (2006). *Differential-Algebraic Equations. Analysis and Numerical Solution.* Zurich: European Mathematical Society Publishing House.

19 Campbell, S.L., & Marz, R. (2007). Direct transcription solution of high index optimal control problems and regular Euler-Lagrange equations. *J. Comp. Appl. Math., 202,* 186–202. https://doi.org/10.1016/j.cam.2006.02.024

20 Berger, T., Ilchmann, A., & Reis, T. (2012). Zero dynamics and funnel control of linear differential-algebraic systems. *Math. Control Signals Syst., 24*, 219–263. https://doi.org/10.1016/10.1007/s00498-012-0085-z

21 Lamour, R., Marz, R., & Tischendorf, C. (2012). *Differential-Algebraic Equations: A Projector Based Analysis.* Springer-Verlag, Berlin Heidelberg.

22 Ilchmann, A. & Reis, T. (2013). *Surveys in Differential-Algebraic Equations I.* Springer-Verlag, Berlin Heidelberg.

23 Berger, T., & Ilchmann, A. (2013). On the standard canonical form of time-varying linear DAEs. *Quarterly Appl. Math., 61*(1), 69–87. https://doi.org/10.1090/S0033-569X-2012-01285-1.

24 Linh, V.H., & Tuan, N.N. (2014). Asymptotic integration of linear differential-algebraic equations. *Electr. J. Qual. Theory Differ. Equ., 12*, 1–17.

25 Chuiko, S.M. (2015). A generalized matrix differential-algebraic equation. *J. Math. Sci., 210*(1), 9–21.

26 Lamour, R., März, R., & Weinmüller, E. (2015). *Boundary-Value Problems for Differential-Algebraic Equations: A Survey.* In book: Surveys in Differential-Algebraic Equations III. Springer-Verlag, Berlin Heidelberg, 177–309.

27 Campbell, S.L., & Kunkel, P. (2016). Solving higher index DAE optimal control problems. *Numerical Algebra, Control and Optimization, 6*(4), 447–472. https://doi.org/10.3934/naco.2016020.

28 Berger, T., Trunk, C., & Winkler, H. (2016). Linear relations and the Kronecker canonical form. *Linear Algebra and its Applications, 488*(1), 13–44. https://doi.org/10.1016/j.laa.2015.09.033

29 Iwata, S., & Takamatsu, M. (2017). On the Kronecker canonical form of singular mixed matrix pencils. *SIAM J. Control Optim., 55*(3), 2134–2150. https://doi.org/10.1137/16M1057516

30 Chuiko, S.M. (2018). On a reduction of the order in a differential-algebraic system. *J. Math. Sci., 235*(1), 2–14.

31 Ilchmann, A., Leben, L., Witschel, J., & Worthmann, K. (2019). Optimal control of differential-algebraic equations from an ordinary differential equation perspective. *Optim. Control Appl. Meth., 40*, 351–366. https://doi.org/10.1002/oca.2481

32 Leben, L., Perla, F.M., Philipp, F., Trunk, C., & Winkler, H. (2021). Finite rank perturbations of linear relations and matrix pencils. *Complex Anal. Operator Theory, 15*, 15–37. https://doi.org/10.1007/s11785-021-01082-x

33 Samusenko, P.F., & Vira, M.B. (2022). Asymptotic solutions of boundary value problem for singularly perturbed system of differential-algebraic equations. *Carpathian Math. Publ., 14*(1), 49–60. https://doi.org/10.15330/cmp.14.1.49-60

34 Dzhumabaev, D.S. (1989). Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation. *Comput. Math. Math. Phys,* (29), 34–46.

Author Information[*]

Anar Turmaganbetkyzy Assanova (*corresponding author*) — Doctor of physical and mathematical sciences, Professor, Head of the Department of Differential Equations and Dynamical Systems, Institute of Mathematics and Mathematical Modeling, 28 Shevshenko street, Almaty, 050010, Kazakhstan; e-mail: *assanova@math.kz*; https://orcid.org/0000-0001-8697-8920

Zhazira Muratbekovna Kadirbayeva — Candidate of physical and mathematical sciences, Associate Professor, Chief Researcher of the Department of Differential Equations and Dynamical Systems, Institute of Mathematics and Mathematical Modeling, 28 Shevshenko street, 050010, Associate Professor of International Information Technology University, Almaty, Kazakhstan, e-mail: *zhkadirbayeva@gmail.com*; https://orcid.org/0000-0001-8861-4100

Ryskul Ashimaliyevna Medetbekova — Candidate of physical and mathematical sciences, Associate Professor, Associate Professor of Shymkent University, mcr Nurtas, 101 Taibekov street, Shymkent, 160000, Kazakhstan, e-mail: *ryskulmedetbekova@mail.ru*; https://orcid.org/0000-0002-0216-1479

Sandugash Tabyldyevna Mynbayeva — PhD, Chief Researcher of the Department of Differential Equations and Dynamical Systems, Institute of Mathematics and Mathematical Modeling, 28 Shevshenko street, 050010, Almaty, Associate Professor of K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan, e-mail: *mynbaevast80@gmail.com*; https://orcid.org/0000-0001-6266-9357

[*]The author's name is presented in the order: First, Middle and Last Names.

https://doi.org/10.31489/2024M1/60-72 *Research article*

On solvability of the inverse problem for a fourth-order parabolic equation with a complex-valued coefficient

A.B. Imanbetova[1], A.A. Sarsenbi[2,3], B. Seilbekov[2,4,*]

[1] *South Kazakhstan University of the name of M. Auezov, Shymkent, Kazakhstan;*
[2] *Scientific Institute "Theoretical and Applied Mathematics", South Kazakhstan University of the name of M. Auezov, Shymkent, Kazakhstan;*
[3] *Tashenev University, Shymkent, Kazakhstan;*
[4] *South Kazakhstan State Pedagogical University, Shymkent, Kazakhstan*
(E-mail: aselek_enu@mail.ru, abdisalam.sarsenbi@auezov.edu.kz, bolat_3084@mail.ru)

In this paper, the inverse problem for a fourth-order parabolic equation with a variable complex-valued coefficient is studied by the method of separation of variables. The properties of the eigenvalues of the Dirichlet and Neumann boundary value problems for a non-self-conjugate fourth-order ordinary differential equation with a complex-valued coefficient are established. Known results on the Riesz basis property of eigenfunctions of boundary value problems for ordinary differential equations with strongly regular boundary conditions in the space $L_2(-1,1)$ are used. On the basis of the Riesz basis property of eigenfunctions, formal solutions of the problems under study are constructed and theorems on the existence and uniqueness of solutions are proved. When proving theorems on the existence and uniqueness of solutions, the Bessel inequality for the Fourier coefficients of expansions of functions from space $L_2(-1,1)$ into a Fourier series in the Riesz basis is widely used. The representations of solutions in the form of Fourier series in terms of eigenfunctions of boundary value problems for a fourth-order equation with involution are derived. The convergence of the obtained solutions is discussed.

Keywords: parabolic equation, inverse problem, classical solution, Fourier method, strongly regular boundary conditions, Riesz basis.

2020 Mathematics Subject Classification: 34L10, 35D35, 35K35, 35Q70.

Introduction

With the further development of the theory of solvability of differential equations, with the advent of new mathematical models in various fields of natural sciences, it becomes more and more important to formulate new mathematical problems and to study more general cases of classical differential equations. These are direct and inverse problems for the fourth-order partial differential equations. A lot of papers are devoted to the study of boundary value problems for the fourth-order partial differential equations (see, for example, [1, 2], and references therein).

It should be noted that boundary value problems with complex-valued coefficients are of particular interest. The existence and uniqueness of the solution of mixed problems for the heat equation with a complex-valued coefficient was established in [3]. The solvability of mixed problems for a perturbed wave equation with involution and with a variable complex-valued coefficient was studied in [4, 5]. The solvability of inverse problems for the perturbed heat equation with involution and with a variable complex-valued coefficient was considered in [6–8].

The results on the existence of a unique solution to inverse problems for a fourth-order partial differential equation with real coefficients depending on x and t can be found in [9, 10].

*Corresponding author. *E-mail: bolat_3084@mail.ru*

This research is funded by the Committee of Science of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP13068539).

Received: 19 June 2023; *Accepted:* 10 November 2023.

This article presents the results of studies of inverse problems for a fourth-order parabolic equation with a variable complex-valued coefficient. The existence and uniqueness of the solution of mixed inverse problems for a one-dimensional fourth-order equation is established

$$u_t(x,t) + \frac{\partial^4}{\partial x^4} u(x,t) + q(x)\, u(x,t) = f(x), \tag{1}$$

where $q(x) = q_1(x) + iq_2(x)$. We will use $\Omega = \{-1 < x < 1,\ 0 < t < T\}$ to denote an open domain, and $\bar{\Omega} = \{-1 \le x \le 1,\ 0 \le t \le T\}$ to denote a closed domain.

The space $C^{k,l}_{x,t}(\Omega)$ consists of all functions $u(x,t)$ having continuous derivatives with respect to t and x of the order l, k respectively, in the domain Ω.

1 Problem Statement

Let us introduce a non-self-conjugate fourth-order differential operator $L_q : D(L_q) \subset L_2(-1,1) \to L_2(-1,1)$ by the formula

$$L_q y = y^{IV}(x) + q(x)\, y(x), \ \ -1 \le x \le 1,$$

with the domain of definition

$$D(L_q) = \left\{ y(x) \in W_2^4[-1,1] : \ U_i(y) = 0, \ \ i = 1,2,3,4, \right\}, \tag{2}$$

where the linear forms $U_i(y)$ are written as $U_i(y)$

$$U_i(y) = a_{i1}y'''(-1) + a_{i2}y'''(1) + a_{i3}y''(-1) + a_{i4}y''(1) + a_{i5}y'(-1) + a_{i6}y'(1) + a_{i7}y(-1) + a_{i8}y(1),$$

with given complex coefficients a_{ij}, $W_2^4[-1,1] = \left\{ y(x) \in C^3[-1,1] : y^{IV}(x) \in L_2(-1,1) \right\}$ is the Sobolev space. Assume that the linear forms $U_1(y)$, $U_2(y)$, $U_3(y)$, $U_4(y)$ are linearly independent. The order of the highest derivative of the form will be called the order of the form. Then the maximum number of forms of order 3 will be not more than two. Boundary conditions (2) can easily be reduced to the form

$$a_{11}y'''(-1) + a_{12}y'''(1) + a_{13}y''(-1) + a_{14}y''(1) + a_{15}y'(-1) + a_{16}y'(1) + a_{17}y(-1) + a_{18}y(1) = 0,$$

$$a_{21}y'''(-1) + a_{22}y'''(1) + a_{23}y''(-1) + a_{24}y''(1) + a_{25}y'(-1) + a_{26}y'(1) + a_{27}y(-1) + a_{28}y(1) = 0,$$

$$a_{33}y''(-1) + a_{34}y''(1) + a_{35}y'(-1) + a_{36}y'(1) + a_{37}y(-1) + a_{38}y(1) = 0,$$

$$a_{43}y''(-1) + a_{44}y''(1) + a_{45}y'(-1) + a_{46}y'(1) + a_{47}y(-1) + a_{48}y(1) = 0, \tag{3}$$

called the normalized boundary conditions [11; 66]. For the sake of simplicity, we have not changed the notation of the coefficients. We proceed similarly if the order of the highest derivative of the forms is less than 3.

Let us rewrite equation (1) in the form

$$u_t(x,t) + L_q u(x,t) = f(x), \ \ (x,t) \in \Omega, \tag{4}$$

and then consider a differential operator L_q with domain generated by one of the following two boundary conditions:

D: Dirichlet boundary conditions

$$U_1(u) = u(-1,t) = 0, \ U_2(u) = u(1,t) = 0, \ U_3(u) = u_{xx}(-1,t) = 0,$$
$$U_4(u) = u_{xx}(1,t) = 0, \ t \in (0,T). \tag{5}$$

N: Neumann boundary conditions

$$U_1(u) = u_x(-1,t) = 0,\ U_2(u) = u_x(1,t) = 0,\ U_3(u) = u_{xxx}(-1,t) = 0,$$

$$U_4(u) = u_{xxx}(1,t) = 0,\ \ t \in (0,T)\,. \tag{6}$$

We have to find a pair of functions $u(x,t)$ and $f(x)$ satisfying equation (4) in the domain Ω and conditions

$$u(x,0) = \varphi(x)\,,\ u(x,T) = \psi(x)\,,\ x \in (-1,1)\,, \tag{7}$$

where $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions.

Definition 1. A pair of functions $u(x,t)$ and $f(x)$ is called a solution to inverse problem (4), (5), and (7) if the following three conditions are satisfied:

1) the function $u(x,t) \in C\left(\bar{\Omega}\right) \bigcap C^{2,0}_{x,t}\left(\bar{\Omega}\right)$;

2) there are derivatives $u_t(x,t)$, $u_{xxx}(x,t)$ and $u_{xxxx}(x,t)$ continuous in the open domain Ω, $f(x) \in C[-1,1]$;

3) functions $u(x,t)$ and $f(x)$ satisfy equation (4), and the function$u(x,t)$ satisfies conditions (5), (7) in the usual sense.

The notion of a solution to inverse problem (4), (6) with boundary conditions (7) is defined similarly.

To prove the existence and uniqueness of a solution to the inverse problem posed, we use the Fourier method. The advantage of this method is that we will have a representation of the solution to the inverse problem in the form of Fourier series. A disadvantage of the Fourier method may be increased requirements for initial data. However, the aim of this work is not to reduce the smoothness of the initial data.

In this regard, it is necessary to solve the inverse problem of convergence of expansions of functions from a certain class in terms of eigenfunctions of the following spectral problem:

$$L_q X(x) = \lambda X(x)\,,\ \ -1 \le x \le 1. \tag{8}$$

2 Properties of eigenfunctions of spectral problems

It is easier to prove the convergence of expansions of operator L_q in eigenfunctions if the system of eigenfunctions $\{X_k(x)\}$ forms a Riesz basis in the class $L_2(-1,1)$. Therefore, in this section, we study the basis property of the eigenfunctions of a differential operator L_q. The differential operator L_q is not a self-conjugate operator. The conjugate spectral problem is written as

$$L_q^* Z(x) = \overline{\lambda} Z(x)\,, \tag{9}$$

where $L_q^* Z(x) = Z^{IV}(x) + \overline{q}(x) Z(x)$ is the operator conjugate to the operator L_q. The domain of definition of the conjugate operator L_q^* is given by one of the boundary conditions (D) or (N) so that $D(L_q) = D\left(L_q^*\right)$. Suppose that all eigenvalues of the operators L_q are simple and zero is not an eigenvalue. The systems of eigenfunctions $\{X_k(x)\}$ and $\{Z_k(x)\}$ satisfy the biorthogonality condition [11; 30]

$$(X_k, Z_n) = \int_{-1}^{1} X_k(x)\,\bar{Z}_n(x)\,dx = \delta_{kn},$$

where δ_{kn} is the Kronecker symbol. In the case of positive self-conjugate operators, the eigenvalues are real and positive. In the case of nonself-conjugate operators, the eigenvalues can be complex numbers. Therefore, it is necessary to study the condition of non-negativity of their real parts.

Lemma 1. Let $q(x) \in C[-1,1]$. Then the inequality $|\mathrm{Im}\ \lambda_k| \le \max|q_2(x)|$ holds for all eigenvalues λ_k of the operator L_q. Under an additional condition $\mathrm{Re}\ q(x) = q_1(x) \ge 0$ in the interval $-1 \le x \le 1$, all eigenvalues λ_k of the operator L_q satisfy the inequality $\mathrm{Re}\ \lambda_k > 0$.

Proof. Consider equation (8) with boundary conditions (5) or (6). We multiply both parts of equation (8) by the complex conjugate function $\bar{X}_k(x)$ and integrate the resulting equality twice by parts over the interval $(-1,1)$. After this, the non-integral terms that arise disappear, and we obtain the equality

$$\int_{-1}^{1} \left|X''_k(x)\right|^2 dx + \int_{-1}^{1} q(x)\left|X_k(x)\right|^2 dx = \lambda_k \int_{-1}^{1} \left|X_k(x)\right|^2 dx.$$

Writing out the real and imaginary parts of the last equality separately, we get the following two relations:

$$\int_{-1}^{1} q_2(x)\left|X_k(x)\right|^2 dx = \mathrm{Im}\,\lambda_k \int_{-1}^{1} \left|X_k(x)\right|^2 dx,$$

$$\int_{-1}^{1} \left|X''_k(x)\right|^2 dx + \int_{-1}^{1} q_1(x)\left|X_k(x)\right|^2 dx = \mathrm{Re}\,\lambda_k \int_{-1}^{1} \left|X_k(x)\right|^2 dx.$$

From the first equality we obtain the first assertion of the lemma

$$\max_{x\in[-1,1]} |q_2(x)| \ge |\mathrm{Im}\,\lambda_k|\,,\ \ k \in N.$$

To prove the second assertion of the lemma, we assume the contrary. Let there be a subsequence $\{\lambda_{n_k}\}$ satisfying the condition $\mathrm{Re}\,\lambda_{n_k} < 0$. Then the second relation implies the inequality

$$\int_{-1}^{1} \left|X''_{n_k}(x)\right|^2 dx + \int_{-1}^{1} q_1(x)\left|X_{n_k}(x)\right|^2 dx = \mathrm{Re}\,\lambda_{n_k} \int_{-1}^{1} \left|X_{n_k}(x)\right|^2 dx < 0,$$

whence, by virtue of $q_1(x) \ge 0$, we get a contradiction, which proves the lemma.

Note that this lemma is valid for continuous $q(x) \in C[-1,1]$. In this case $\mathrm{Re}\,\lambda_k > 0$, starting from some number k_0, as $\mathrm{Re}\,\lambda_k \ge |\min q_1(x)|$ for $k \ge k_0$, if $\min q_1(x) < 0$.

For further presentation, let us dwell on some well-known facts. Let $\lambda = \rho^4$. In the complex ρ-plane, consider a fixed region S_ν, $\nu = 0,1,2,...,7$, defined by the inequality $\frac{\nu\pi}{4} \le \arg\rho \le \frac{(\nu+1)\pi}{4}$. We enumerate $\omega_1, \omega_2, \omega_3, \omega_4$ different roots of the number $\sqrt[4]{-1}$ so that for $\rho \in S_\nu$, $\mathrm{Re}(\rho\omega_1) \le \mathrm{Re}(\rho\omega_2) \le \mathrm{Re}(\rho\omega_3) \le \mathrm{Re}(\rho\omega_4)$.

It is well known that the normalized boundary conditions (3) are called regular (see, for example, [11; 67] if the numbers θ_{-1}, θ_1 defined by the equality

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = \begin{vmatrix} a_{11}\omega_1^3 & (a_{11} + sa_{12})\,\omega_2^3 & \left(a_{11} + \frac{1}{s}a_{12}\right)\omega_3^3 & a_{12}\omega_4^3 \\ a_{21}\omega_1^3 & (a_{21} + sa_{22})\,\omega_2^3 & \left(a_{21} + \frac{1}{s}a_{22}\right)\omega_3^3 & a_{22}\omega_4^3 \\ a_{33}\omega_1^2 & (a_{33} + sa_{34})\,\omega_2^2 & \left(a_{33} + \frac{1}{s}a_{34}\right)\omega_3^2 & a_{34}\omega_4^2 \\ a_{43}\omega_1^2 & (a_{43} + sa_{44})\,\omega_2^2 & \left(a_{43} + \frac{1}{s}a_{44}\right)\omega_3^2 & a_{44}\omega_4^2 \end{vmatrix}$$

are different from zero. Here the power of the number ω_j is equal to the order of the highest derivative of the corresponding boundary condition. We proceed similarly if the order of the highest derivative of the forms is less than 3.

If the additional condition $\theta_0^2 - 4\theta_{-1}\theta_1 \ne 0$ is satisfied, then the boundary conditions (3) are called strongly regular.

Note that the differential operator L_q generated by strongly regular boundary conditions can have only a finite number of multiple eigenvalues.

The papers [12], [13] imply the following important theorem.

Theorem 1. [12], [13]. If the operator L_q is generated by strongly regular boundary conditions, then the eigenfunctions and associated functions of this operator form a Riesz basis in the space $L_2(-1,1)$.

It is easy to check that the boundary conditions (5) (and (6)) are strongly regular, so the system of eigenfunctions $\{X_k(x)\}$ of the operator L_q forms a Riesz basis in the space $L_2(-1,1)$. This is also valid for the system of eigenfunctions $\{Z_k(x)\}$ of the operator L_q^*.

Everywhere below we will assume that all eigenvalues of the operator L_q are single.

Lemma 2. For any function $\varphi \in D(L_q)$ each of the Fourier series

$$\varphi(x) = \sum_{k=1}^{\infty} (\varphi, Z_k) X_k(x), \quad \varphi(x) = \sum_{k=1}^{\infty} (\varphi, X_k) Z_k(x), \tag{10}$$

by eigenfunctions $\{X_k(x)\}$, $\{Z_k(x)\}$ converges uniformly for $-1 \le x \le 1$.

Proof. Let us rewrite equation (8) in the form (the number $\lambda = 0$ is not an eigenvalue)

$$X_k(x) = \frac{X_k^{IV}(x) + q(x) X_k(x)}{\lambda_k}.$$

Then

$$(\varphi, X_k) = \int_{-1}^{1} \varphi(x) \bar{X}_k(x)\, dx = \int_{-1}^{1} \varphi(x) \frac{\bar{X}_k^{IV}(x) + \bar{q}(x) \bar{X}_k(x)}{\bar{\lambda}_k} dx =$$

$$= \frac{1}{\bar{\lambda}_k} \int_{-1}^{1} \left[\varphi^{IV}(x) + q(x)\varphi(x)\right] \bar{X}_k(x)\, dx = \frac{1}{\bar{\lambda}_k} \left(L_q\varphi, \bar{X}_k\right).$$

Using this relation, the second series in (10) can be written as

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{A_k}{\bar{\lambda}_k} Z_k(x), \tag{11}$$

where

$$A_k = \int_{-1}^{1} \left[\varphi^{IV}(x) + q(x)\varphi(x)\right] \bar{X}_k(x)\, dx.$$

On the other hand, it is well known that the conjugate spectral problem is equivalent to the integral equation

$$Z_k(x) = \overline{\lambda_k} \int_{-1}^{1} G^*(x,t) \bar{Z}_k(t)\, dt,$$

where $G^*(x,t)$ is the Green's function of the conjugate boundary value problem for $\lambda = 0$. By definition [11; 45], the Green's function $G^*(x,t)$ is continuous for $x \in [-1,1]$ and $t \in [-1,1]$ and therefore it is bounded. Let's denote $C_k(x) = \int_{-1}^{1} G^*(x,t) \bar{Z}_k(t)\, dt$. Then equality (11) takes the form

$$\sum_{k=1}^{\infty} \frac{A_k}{\bar{\lambda}_k} Z_k(x) = \sum_{k=1}^{\infty} A_k C_k(x).$$

Further, using the inequality $ab \leq \frac{1}{2}\left(a^2+b^2\right)$, we obtain the following estimate

$$\sum_{k=1}^{\infty}\left|\frac{A_k}{\overline{\lambda_k}}Z_k\left(x\right)\right| = \sum_{k=1}^{\infty}\left|A_k C_k\left(x\right)\right| \leq \sum_{k=1}^{\infty}\left|A_k\right|^2 + \sum_{k=1}^{\infty}\left|C_k\left(x\right)\right|^2. \tag{12}$$

As the quantities A_k are the Fourier coefficients of the expansion in the Riesz basis $Z_k\left(x\right)$, $k = 1,2,3,...$, and $C_k\left(x\right)$ are the Fourier coefficients of the expansion of the Green's function $G\left(x,t\right)$ in the Riesz basis $\{X_k\left(x\right)\}$, due to the Bessel inequality for the Riesz bases, both series on the right side of inequality (12) converge and

$$\sum_{k=1}^{\infty}\left|C_k\left(x\right)\right|^2 \leq \int_{-1}^{1}\left|G^*\left(x,t\right)\right|^2 dt \leq M_0, \ \forall x \in \left[-1,1\right].$$

This implies absolute and uniform convergence of the second series (10) . The absolute and uniform convergence of the first series (10) is proved similarly. The lemma is proved.

3 Formal solution to the inverse problem

In this section, we construct a formal solution to the inverse problem for equation (4) with Dirichlet boundary conditions (5) and conditions (8). Recall that if the domain $D\left(L_q\right)$ of the operator L_q is generated by one of the boundary conditions (D), (N), then each of the systems $\{X_k\left(x\right)\}$ and $\{Z_k\left(x\right)\}$, consisting of the eigenfunctions of the operators L_q and L_q^*, respectively, forms a Riesz basis in the space $L_2\left(-1,1\right)$. The functions $u\left(x,t\right)$ and $f\left(x\right)$ can be represented as Fourier series

$$u(x,t) = \sum_{k=0}^{\infty} C_k\left(t\right) X_k\left(x\right), \tag{13}$$

$$f(x) = \sum_{k=0}^{\infty} f_k X_k\left(x\right), \tag{14}$$

$$C_k\left(t\right) = \int_{-1}^{1} u\left(x,t\right)\bar{Z}_k\left(x\right)dx, \ \ f_k = \int_{-1}^{1} f\left(x\right)\bar{Z}_k\left(x\right)dx, \tag{15}$$

where $C_k\left(t\right)$ are unknown functions and f_k are unknown constants. Substituting (13) and (14) into equation (4), we obtain the equation

$$C'_k\left(t\right) + \lambda_k C_k\left(t\right) = f_k,$$

whose solution will be written in the form

$$C_k\left(t\right) = D_k \cdot e^{-\lambda_k t} + \frac{f_k}{\lambda_k}. \tag{16}$$

As, according to condition (7) and formula (15),

$$C_k\left(0\right) = \int_{-1}^{1} u\left(x,0\right)\bar{Z}_k\left(x\right)dx = \int_{-1}^{1} \varphi\left(x\right)\bar{Z}_k\left(x\right)dx = \varphi_k,$$

$$C_k(T) = \int_{-1}^{1} u(x,T)\,\bar{Z}_k(x)\,dx = \int_{-1}^{1} \psi(x)\,\bar{Z}_k(x)\,dx = \psi_k,$$

from equality (16) we get

$$\begin{cases} D_k + \frac{f_k}{\lambda_k} = \varphi_k, \\ D_k e^{-\lambda_k T} + \frac{f_k}{\lambda_k} = \psi_k. \end{cases}$$

Solving this system of equations, we find the unknown quantities

$$D_k = \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}},\ \ f_k = (\varphi_k - D_k)\,\lambda_k,$$

using which from relation (16) we find

$$C_k(t) = \varphi_k - \frac{1 - e^{-\lambda_k t}}{1 - e^{-\lambda_k T}}\,[\varphi_k - \psi_k]\,.$$

Substituting the found values of the unknowns $C_k(t)$ and f_k into (13) and (14), we find the formal solution to the inverse problem in the form of the following series

$$u(x,t) = \varphi(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \left(e^{-\lambda_k t} - 1\right) X_k(x), \tag{17}$$

and

$$f(x) = L_q\varphi(x) - \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \lambda_k X_k(x)\,. \tag{18}$$

Now we have to prove that the functions (17) and (18) will be the classical solution to the studied inverse problems.

4 Main results

In [9], the authors of this work proposed a new approach to prove the uniform convergence of formally differentiated series, which represent a formal solution to the inverse problem for the equation of a fourth-order hyperbolic equation with complex-valued coefficients. This approach has two advantages: 1) the first advantage is the use of estimates of the norms of eigenfunctions derivatives through the norm of eigenfunctions [14]; the second advantage is the use of the properties of uniform boundedness of Riesz bases consisting of eigenfunctions of the differential operator [15]. In this section, this approach is developed for the case of inverse problems for a fourth-order parabolic equation with complex-valued coefficients. It is clear that the formal solutions to hyperbolic and parabolic equations have completely different structures. The conditions for the existence of solutions are also different.

Let us formulate the main result of the present work. The solvability of the inverse problem (4), (7) with the Dirichlet boundary conditions (5) is formulated as the following theorem.

Theorem 2. Let $q(x) \in C^4[-1,1]$, and functions $\varphi,\ \psi$ are such that $\varphi,\ \psi,\ L_q\varphi,\ L_q\psi \in D(L_q)$. Then inverse problem (4), (5), (7) has a unique solution, which can be represented as Fourier series (17), (18).

Proof. We have to show that the resulting formal solution in the form of series (17), (18) satisfies equation (4) and conditions (5), (7). Let us first show that series (17), (18), as well as the formal derivative with respect to the variable t and formal derivatives up to the fourth order with respect to the variable x of series (17), converge uniformly in the open domain Ω, i.e. let us prove the uniform convergence of the series (17), (18) and the uniform convergence of the formally differentiated series

$$u_t(x,t) = -\sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \lambda_k e^{-\lambda_k t} X_k(x), \tag{19}$$

$$u_x(x,t) = \varphi'(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \left(e^{-\lambda_k t} - 1\right) X'_k(x), \tag{20}$$

$$u_{xx}(x,t) = \varphi''(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \left(e^{-\lambda_k t} - 1\right) X''_k(x), \tag{21}$$

$$u_{xxx}(x,t) = \varphi'''(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \left(e^{-\lambda_k t} - 1\right) X'''_k(x), \tag{22}$$

$$u_{xxxx}(x,t) = \varphi^{IV}(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \left(e^{-\lambda_k t} - 1\right) X_k^{IV}(x). \tag{23}$$

The uniform convergence of series (17) follows from the obvious inequality

$$|u(x,t)| \le |\varphi(x)| + \left|\sum_{k=0}^{\infty} (\varphi, Z_k) X_k(x)\right| + \left|\sum_{k=0}^{\infty} (\psi, Z_k) X_k(x)\right|,$$

and Lemma 2, taking into account Lemma 1 ($\operatorname{Re}\lambda_k > 0$).

To prove series (18) in the expressions $\varphi_k = (\varphi, Z_k)$, $\psi_k = (\psi, Z_k)$, the function $Z_k(x)$ is replaced by the conjugate equation (9). Then

$$\lambda_k \varphi_k = \lambda_k (\varphi, Z_k) = \left(\varphi, L_q^* Z_k\right) = (L_q \varphi, Z_k), \ \lambda_k \psi_k = (L_q \psi, Z_k). \tag{24}$$

Substituting them into (18), we obtain

$$f(x) = L_q \varphi(x) - \sum_{k=0}^{\infty} \frac{(L_q \varphi, Z_k) - (L_q \psi, Z_k)}{1 - e^{-\lambda_k T}} \cdot X_k(x).$$

Hence we get the inequality

$$|f(x)| \le |L_q \varphi(x)| + \left|\sum_{k=0}^{\infty} (L_q \varphi, Z_k) X_k(x)\right| + \left|\sum_{k=0}^{\infty} (L_q \psi, Z_k) X_k(x)\right|.$$

As, by the condition of the theorem $L_q\varphi$, $L_q\psi \in D(L_q)$, then, by virtue of Lemma 2, both series on the right-hand side of the last inequality converge uniformly. The uniform convergence of the series (17), (18) is proved. The uniform convergence of the series (19) is proved as well as the convergence of the series (18), taking into account the boundedness of the quantities $\lambda_k e^{-\lambda_k \tau} \to 0$, $k \to \infty$.

Let us prove the uniform convergence of series (20)–(23). Applying (24) to the series (20) we obtain the relation

$$|u_x(x,t)| \le \left|\varphi'(x)\right| + \left|\sum_{k=0}^{\infty} \frac{(L_q \varphi, Z_k) - (L_q \psi, Z_k)}{\lambda_k \left(1 - e^{-\lambda_k T}\right)} \left(e^{-\lambda_k t} - 1\right) X'_k(x)\right|.$$

In [14] the validity of the estimates

$$\max\left|X_k^{(s)}(x)\right| \le c_1\left(\sqrt[4]{|\lambda_k|}\right)^s \max|X_k(x)|, \; s=1,2,3, \tag{25}$$

for the eigenfunctions of the fourth-order differential operator is shown. Using estimates (25), from the last inequality we obtain the estimate

$$|u_x(x,t)| \le \left|\varphi'(x)\right| + c_1\sum_{k=0}^{\infty}\frac{|(L_q\varphi, Z_k)-(L_q\psi, Z_k)|}{\left(\sqrt[4]{|\lambda_k|}\right)^3}\max|X_k(x)|.$$

It follows from [15] that only uniformly bounded systems of eigenfunctions of ordinary differential operators can be Riesz bases. Therefore, due to the conditions of the theorem $L_q\varphi$, $L_q\psi \in D(L_q)$, the Bessel inequality for the Riesz bases, and the asymptotics of the eigenvalues [11; 99], the series on the right-hand side of the following inequality

$$|u_x(x,t)| \le \left|\varphi'(x)\right| + c_1\sum_{k=0}^{\infty}\left[(L_q\varphi, Z_k)^2 + (L_q\psi, Z_k)^2 + \frac{2}{\left(\sqrt{\lambda_k}\right)^3}\right]$$

converges. The uniform convergence of series (20) is proved.

Using the estimates (25), the convergence of series (21), (22) in the open domain Ω is similarly proved. Consider the uniform convergence of the series

$$u_{xxxx}(x,t) = \varphi^{IV}(x) + \sum_{k=0}^{\infty}\frac{(L_q\varphi, Z_k)-(L_q\psi, Z_k)}{\lambda_k\left(1-e^{-\lambda_k T}\right)}\left(e^{-\lambda_k t}-1\right)X_k^{IV}(x).$$

Replacing the fourth derivative with the help of equation (8), we obtain the estimate

$$|u_{xxxx}(x,t)| \le \left|\varphi^{IV}(x)\right| + \left|\sum_{k=0}^{\infty}\frac{q(x)}{\lambda_k}\left[(L_q\varphi, Z_k)X_k(x) - (L_q\psi, Z_k)X_k(x)\right]\right|$$

$$+\left|\sum_{k=0}^{\infty}\left[(L_q\varphi, Z_k)X_k(x) - (L_q\psi, Z_k)X_k(x)\right]\right|. \tag{26}$$

The second series on the right-hand side of (26) converges by virtue of the conditions of the theorem $L_q\varphi$, $L_q\psi \in D(L_q)$ and Lemma 2. The convergence of the first series in (26) follows from the uniform boundedness of the system $\{X_k(x)\}$ [15], the Bessel inequality for the Riesz bases, the asymptotics of the eigenvalues [11; 99], and the boundedness of the function $q(x)$. This proves the uniform convergence of the series $u_{xxxx}(x,t)$ in the open domain Ω. Thus, we have shown that series (17), (18) satisfy equation (4).

Obviously, the formal solution (17) satisfies conditions (7):

$$\lim_{t\to 0+0} u(x,t) = \lim_{t\to 0+0}\left[\varphi(x) + \sum_{k=0}^{\infty}\frac{\varphi_k-\psi_k}{1-e^{-\lambda_k T}}\left(e^{-\lambda_k t}-1\right)X_k(x)\right] = \varphi(x),$$

$$\lim_{t\to T-0} u(x,t) = \lim_{t\to T-0}\left[\varphi(x) + \sum_{k=0}^{\infty}\frac{\varphi_k-\psi_k}{1-e^{-\lambda_k T}}\left(e^{-\lambda_k t}-1\right)X_k(x)\right] = \psi(x).$$

The boundary conditions are satisfied as each term of the series (17) satisfies them. The existence of a classical solution to problem (4), (5), (7) has been completely proved. To prove the uniqueness of the solution, we assume the contrary. Suppose that there are two sets of solutions $\{u_1(x,t),\ f_1(x)\}$ and $\{u_2(x,t),\ f_2(x)\}$ to the inverse problem (4), (5), (7). Denote

$$u(x,t) = u_1(x,t) - u_2(x,t)$$

and

$$f(x) = f_1(x) - f_2(x).$$

Then the functions $u(x,t)$ and $f(x)$ satisfy equation (4), boundary conditions (5), and homogeneous conditions

$$u(x,0) = 0,\ u(x,T) = 0,\ x \in [-1,1]. \tag{27}$$

Consider the Fourier coefficients:

$$u_k(t) = \int_{-1}^{1} u(x,t)\,\bar{X}_k(x)\,dx,\ k \in N, \tag{28}$$

$$f_k = \int_{-1}^{1} f(x)\,\bar{X}_k(x)\,dx,\ k \in N, \tag{29}$$

and note that the homogeneous conditions (27) lead to equalities

$$u_k(x,0) = u_k(x,T) = 0.$$

Differentiating equality (28) with respect to the variable t, we obtain

$$u'_k(t) = \int_{-1}^{1} u'_t(x,t)\,\bar{X}_k(x)\,dx,$$

where the derivative $u_t(x,t)$ will be replaced using equation (4)

$$u'_k(t) = \int_{-1}^{1} \left[-u_{xxxx}(x,t) - q(x)\,u(x,t)\right]\bar{X}_k(x)\,dx + \int_{-1}^{1} f(x)\,\bar{X}_k(x)\,dx,$$

or

$$u'_k(t) = \int_{-1}^{1} \left[-u_{xxxx}(x,t) - q(x)\,u(x,t)\right]\bar{X}_k(x)\,dx + f_k.$$

After integrating by parts four times, we get

$$u'_k(t) = \int_{-1}^{1} \left[-\bar{X}_k^{IV}(x) - \overline{q}(x)\,\bar{X}_k(x)\right] u(x,t)\,dx + f_k,$$

or

$$u'_k(t) = \int_{-1}^{1} -\overline{\lambda_k}\bar{X}_k(x)\,u(x,t)\,dx + f_k.$$

The last equality can be rewritten as

$$u'_k(t) + \overline{\lambda}_k u_k(t) = f_k.$$

As (29) is satisfied, i.e., $u_k(0) = u_k(T) = 0$, the last relation implies

$$f_k = 0,\ u_k(t) \equiv 0.$$

The basis property of the system $\{X_k(x)\}$ implies the equality

$$f(x) \equiv 0,\ u(x,t) \equiv 0,\ (x,t) \in \Omega.$$

The uniqueness of the solution is proved. The theorem is completely proved. The assertion of the theorem is fully applicable to the case of inverse problem (4), (6), (7).

Conclusion

The inverse problem of determining the right side for a fourth-order parabolic equation with a complex-valued variable coefficient is studied. The existence of a unique solution to the inverse problem with Dirichlet and Neumann boundary conditions is established

Acknowledgments

This research is funded by the Committee of Science of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP13068539). The authors thank the anonymous reviewers for the careful reading of the manuscript, their suggestions, and additional references that have improved the article.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Janno, J., & Seletski, A. (2015). Reconstruction of coefficients of higher order nonlinear wave equations by measuring solitary waves. *Wave Motion, 52,* 15–25. https://doi.org/10.1016/j.wavemoti.2014.08.005

2 Kabanikhin, S.I. (2011). Inverse and Ill-posed Problems: Theory and Applications. Berlin, Germany: De Gruyter. https://doi.org/10.1515/9783110224016

3 Ionkin, N.I., & Moiseev, E.I. (1979). A problem for a heat equation with two-point boundary conditions. *Differ. Uravn., 15*(7), 1284–1295. https://www.mathnet.ru/eng/de3763

4 Mussirepova, E., Sarsenbi, A.A., & Sarsenbi, A.M. (2022). Solvability of mixed problems for the wave equation with reflection of the argument. *Math Meth Appl Sci., 45*(17), 11262–11271. https://doi.org/10.1002/mma.8448

5 Ashyralyev, A., Ashyralyyev, A., & Abdalmohammed, B. (2023). On the hyperbolic type differential equation with time involution. *Bulletin of the Karaganda University. Mathematics series,* 1(109), 38–47. https://doi.org/10.31489/2023m1/38-47

6 Mussirepova, E., Sarsenbi, A., & Sarsenbi, A. (2022). The inverse problem for the heat equation with reflection of the argument and with a complex coefficient. *Bound Value Probl.,* 99. https://doi.org/10.1186/s13661-022-01675-1

7 Ashyralyev, A., Erdogan, A.S., & Sarsenbi, A. (2020). A note on the parabolic identification problem with involution and Dirichlet condition. *Bulletin of the Karaganda University. Mathematics series,* 3(99), 130–139. https://doi.org/10.31489/2020m3/130-139

8 Sarsenbi, A.A. (2018). A solvability conditions of mixed problems for equations of parabolic type with involution. *Bulletin of the Karaganda University. Mathematics series,* 4(92), 87–93. https://doi.org/10.31489/2018M4/87-93

9 Imanbetova, A., Sarsenbi, A., & Seilbekov, B. (2023). Inverse problem for a fourth-order hyperbolic equation with a complex-valued coefficient. *MDPI Mathematics, 11*(15), 3432. https://doi.org/10.3390/math11153432

10 Mehraliyev, Y., Allahverdiyeva, S., & Ramazanova, A. (2023). On one coefficient inverse boundary value problem for a linear pseudoparabolic equation of the fourth order. *AIMS Mathematics, 8*(2), 2622–2633. https://doi.org/10.3934/math.2023136

11 Naimark, M.A. (1968). *Linear Differential Operators.* New York: Ungar.

12 Kesel'man, G.M. (1964). On the unconditional convergence of eigenfunction expansions of certain differential operators. *Izv. Vyssh. Uchebn. Zaved. Mat.,* (2), 82–93. https://www.mathnet.ru/eng/ivm2384

13 Mikhailov, V.P. (1962). On Riesz bases in $L_2(0,1)$. *Dokl. Akad. Nauk SSSR, 144*(5), 981–984. https://www.mathnet.ru/eng/dan26607

14 Kerimov, N.B. (1996). On the basis property and uniform minimality of systems of root functions of differential operators. I. *Differential Equations, 32*(3), 321–327.

15 Sarsenbi, A.M. (2008). Necessary and sufficient conditions for the Riesz basis property of the eigen- and associated functions of high-order differential operators on an interval. *Doklady Mathematics, 77*(2), 290–292. https://doi.org/10.1134/S1064562408020336

Комплекс мәнді коэффициенті бар төртінші ретті параболалық теңдеу үшін кері есептің шешімділігі туралы

А.Б. Иманбетова[1], А.А. Сәрсенбі[2,3], Б.Н. Сейлбеков[2,4]

[1]*М. Әуезов атындағы Оңтүстік Қазақстан университеті, Шымкент, Қазақстан;*
[2]*М. Әуезов атындағы Оңтүстік Қазақстан университеті, Теориялық және қолданбалы математика ғылыми институты, Қазақстан;*
[3]*Ж.А. Тәшенов атындағы университет, Шымкент, Қазақстан;*
[4]*Оңтүстік Қазақстан мемлекеттік педагогикалық университеті, Шымкент, Қазақстан*

Мақалада айнымалыларды ажырату әдісімен айнымалы комплексті коэффициенті бар төртінші ретті параболалық теңдеу үшін кері есеп зерттелген. Комплекс мәнді коэффиценті бар өзіне-өзі түйіндес емес төртінші ретті біртекті дифференциалдық теңдеу үшін Дирихле және Нейман шекаралық есептерінің меншікті мәндерінің қасиеттері белгіленген. Күшті регулярлы шекаралық шарттары бар біртекті дифференциалдық теңдеулер үшін шекаралық есептердің меншікті функцияларының $L_2(-1,1)$ кеңістігіндегі Рис базистік қасиеті бойынша белгілі нәтижелер пайдаланылады. Меншікті функциялардың Рис базистік қасиетінің негізінде зерттелетін есептердің формальды шешімдері құрастырылып, шешімдердің бар болуы мен жалғыздығы туралы теоремалар дәлелденген. Шешімдердің бар екендігі мен жалғыздығы туралы теореманы дәлелдеу кезінде Бессель теңсіздігі Фурье коэффициенттері $L_2(-1,1)$ кеңістігінен Рис базисі бойынша Фурье қатарына функциялардың жіктелінуі үшін кеңінен қолданылады. Инволюциялы төртінші ретті теңдеу үшін шеттік есептердің меншікті функциялары бойынша Фурье қатарлары түрінде шешімдердің түр сипаты жазылды. Алынған шешімдердің жинақтылығы да талқыланған.

Кілт сөздер: параболалық теңдеу, кері есеп, классикалық шешім, Фурье әдісі, күшті регулярлы шекаралық шарттар, Рис базисі.

О разрешимости обратной задачи для параболического уравнения четвертого порядка с комплекснозначным коэффициентом

А.Б. Иманбетова[1], А.А. Сарсенби[2,3], Б.Н. Сейлбеков[2,4]

[1] *Южно-Казахстанский университет имени М. Ауэзова, Шымкент, Казахстан;*
[2] *Научный институт «Теоретическая и прикладная математика» Южно-Казахстанского университета имени М. Ауэзова, Шымкент, Казахстан;*
[3] *Университет имени Ж.А. Ташенова, Шымкент, Казахстан;*
[4] *Южно-Казахстанский государственный педагогический университет, Шымкент, Казахстан*

В настоящей работе методом разделения переменных изучена обратная задача для параболического уравнения четвертого порядка с переменным комплекснозначным коэффициентом. Установлены свойства собственных значений краевых задач Дирихле и Неймана для несамосопряженного обыкновенного дифференциального уравнения четвертого порядка с комплекснозначным коэффициентом. Использованы известные результаты о базисности Рисса в пространстве $L_2(-1,1)$ собственных функций краевых задач для обыкновенных дифференциальных уравнений с усиленно регулярными краевыми условиями. На основании базисности Рисса собственных функций построены формальные решения изучаемых задач и доказаны теоремы о существовании и единственности решения. При доказательстве теорем о существовании и единственности решений применено неравенство Бесселя для коэффициентов Фурье разложений функций из пространства $L_2(-1,1)$ в ряд Фурье по базису Рисса. Выписаны представления решений в виде рядов Фурье по собственным функциям краевых задач для уравнения четвертого порядка с инволюцией. Также обсуждена сходимость полученных решений.

Ключевые слова: параболическое уравнение, обратная задача, классическое решение, метод Фурье, усиленно регулярные краевые условия, базис Рисса.

Author Information[*]

Asselkhan Bostandykovna Imanbetova — Doctoral student, Department of Mathematics, South Kazakhstan University of the name of M. Auezov, 68 A. Baitursynov street, Shymkent, 160012, Kazakhstan; e-mail: *aselek_enu@mail.ru*; https://orcid.org/0000-0002-0757-9540

Abdissalam Sarsenbi — Doctor of Philosophy (PhD), Scientific Institute "Theoretical and Applied Mathematics", South Kazakhstan University of the name of M. Auezov, Tauke Khana street, 5; Department of Mathematics and Informatics, Tashenev University, 21 Kunaev street, Shymkent, 160000, Kazakhstan; e-mail: *abdisalam.sarsenbi@auezov.edu.kz*; https://orcid.org/0000-0002-1667-3010

Bolat Seilbekov (*corresponding author*) — Doctor of Philosophy (PhD), Scientific Institute "Theoretical and Applied Mathematics", South Kazakhstan University of the name of M. Auezov, 5 Tauke Khana street; Department of Mathematics, South Kazakhstan State Pedagogical University, 13 A. Baitursynov street, Shymkent, 160000, Kazakhstan; e-mail: *bolat_3084@mail.ru*; https://orcid.org/0000-0002-6325-9891

[*] The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics series, No. 1(113), 2024, pp. 73–83

https://doi.org/10.31489/2024M1/73-83 *Research article*

Embeddings of a Multi-Weighted Anisotropic Sobolev Type Space

G.Sh. Iskakova[1,*], M.S. Aitenova[2,3], A.K. Sexenbayeva[4]

[1]*Karaganda Buketov University, Karaganda, Kazakhstan;*
[2]*Karaganda University of Kazpotrebsoyuz, Karaganda, Kazakhstan;*
[3]*Kazakhstan Branch of Lomonosov Moscow State University, Astana, Kazakhstan;*
[4]*L.N. Gumilyov Eurasian National University, Astana, Kazakhstan*
(E-mail: iskakova.1975@mail.ru, abibekove@mail.ru, aiisha08@mail.ru)

Parameters such as various integral and differential characteristics of functions, smoothness properties of regions and their boundaries, as well as many classes of weight functions cause complex relationships and embedding conditions for multi-weighted anisotropic Sobolev type spaces. The desire not to restrict these parameters leads to the development of new approaches based on the introduction of alternative definitions of spaces and norms in them or on special localization methods. This article examines the embeddings of multi-weighted anisotropic Sobolev type spaces with anisotropy in all the defining characteristics of the norm of space, including differential indices, summability indices, as well as weight coefficients. The applied localization method made it possible to obtain an embedding for the case of an arbitrary domain and weights of a general type, which is important in applications in differential operators' theory, numerical analysis.

Keywords: Anisotropic Sobolev Spaces, Multi-Weighted Spaces, Embedding Theorems, Localization Methods, Weighted Functions.

2020 Mathematics Subject Classification: 46E35.

Introduction

In the article embeddings of multi-weighted anisotropic Sobolev type spaces $W^{\bar{l}}_{\bar{p},p}(G,\bar{\rho},\upsilon)$ described by a finite norm

$$|u; W^{\bar{l}}_{\bar{p},p}(G,\bar{\rho},\upsilon)| = \sum_{i=1}^{n} \Big(\int\limits_G |D_i^{l_i} u|_i^p \rho_i dx \Big)^{1/p_i} + \Big(\int\limits_G |u|^p \upsilon dx \Big)^{1/p}$$

are investigated in the case when $\rho_1, ..., \rho_n$ and υ are connected by certain relations on average on parallelepipeds in G with an adjustable edge length.

The article extensively utilizes approaches developed in the works [1–5]. Nonetheless, they have enabled a slight expansion of the class of weights for which the considered embedding is valid. As before [5], the spaces are anisotropic in terms of derivative orders, integrability indices, and weight factors for these derivatives. Also, for the introduced class of weights, the localization method [1–4] allows not imposing conditions on the domains, in which the spaces are considered and embeddings are set. So, the conditions under which the embeddings (1) occur for a sufficiently broad class of weights, restricted by a special condition $\Pi_{(\delta,\,\varepsilon)}$ introduced in Definition 1 are studied. The localization method (Lemma 3) with the introduction of so-called "characteristic parallelepipeds" (3) allows considering the domain G, with no additional conditions imposed.

---

*Corresponding author. *E-mail: iskakova.1975@mail.ru*

This work is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP15473253).

Received: 21 September 2023; *Accepted:* 11 December 2023.

For domains with conditions, fundamental embedding results have been obtained for anisotropic Sobolev spaces [6, 7]; as for weighted cases, in [8–10] the weights are functions of the distance to the boundary, and inequalities in [8] are considered in cylindrical domains, in [9] — on a domain with cusp, and in [10] an open connected domain is considered with various conditions. In [11], along with other findings, an anisotropic Sobolev inequality is obtained for smooth bounded domains and the class of p-admissible weights. In the work [12], Sobolev spaces are considered in open domains with certain conditions of smoothness imposed on the functions introduced in them, anisotropy in terms of derivative orders, and integrable classes of weights; alternative descriptions of these spaces are presented, including norms and properties of the density of smooth functions within them, and the relationships between the weights and anisotropic properties of Sobolev spaces are described. In [13], alternative definitions of spaces are introduced, through which embeddings of weighted spaces are obtained. Here, we also present studies of anisotropic Sobolev type spaces [14, 15], and their embeddings [16–18].

1 Set up

Let us introduce the notation. Let R^n be an n-dimensional arithmetic space with a norm

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \quad x = (x_1, ..., x_n).$$

Denote by

$$\bar{l} = (l_1, \dots, l_n), \quad \bar{a} = (a_1, \dots, a_n), \quad \text{and } \bar{b} = (b_1, \dots, b_n)$$

fixed vectors with coordinates $l_i, a_i > 0, b_i \geq 0, \ \ i = 1, \dots, n$. Set for $\lambda > 0$:

$$\bar{a} \pm \lambda = (a_1 \pm \lambda, \dots, a_n \pm \lambda), \lambda\bar{a} = (\lambda_1 a_1, \dots, \lambda_n a_n),$$

$$\bar{b} \ : \ \bar{a} = (b_1/a_1, \dots, b_n/a_n), \bar{b}\bar{a} = (b_1 a_1, \dots, b_n a_n),$$

$$\lambda^{\bar{b}} = (\lambda^{b_1}, \dots, \lambda^{b_n}), \bar{a}^{\lambda} = (a_1^{\lambda}, \dots, a_n^{\lambda_n}), \bar{a}^{\bar{b}} = (a_1^{b_1}, \dots, a_n^{b_n}),$$

$$|b| = \sum_{i=1}^{n} b_i, \ \prod \bar{b} = \prod_{i=1}^{n} b_i, \ \sum \bar{b} = \sum_{i=1}^{n} b_i.$$

For a multi-index $\alpha = (\alpha_1, ..., \alpha_n)$ $D^\alpha = \frac{\partial^{\alpha_1 + ... + \alpha_n}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$, for integers l_i $D_i^{l_i} u = \frac{\partial^{l_i}}{\partial x_i^{l_i}} u$.

Let $L(G; \, loc)$ be the space of locally-summable functions, $L_q^\alpha(G; \, \omega)$ is the Lebesgue weighted space of functions $u(\cdot)$, with a finite semi-norm

$$\left|u; L_q^\alpha(G;\omega)\right| = |D^\alpha u; L_q \ (G;\omega)| = \left(\int\limits_G |D^\alpha u|^q \omega(x) dx\right)^{1/q},$$

where $1 \leq q < \infty$,

$$D^\alpha u = \frac{\partial^\alpha u}{\partial x^\alpha} = \frac{\partial^{\alpha_1 + ... + \alpha_n} u}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

are mixed derivatives corresponding to the multi-index $\alpha = \alpha_1, ..., \alpha_n$. We denote the class

$$C^\infty W = \{u \in C^\infty \ \ : |u; W_{\bar{p}, p}^{\bar{l}}(G; \, \bar{\rho}, \upsilon)| < \infty\}$$

by $C^\infty W$.

We consider issues related to the description of the conditions under which the embedding takes place:

$$W^{\bar{l}}_{\bar{p},\,p}(G;\,\bar{\rho},\upsilon)\to L^{\alpha}_{q}(G;\omega). \tag{1}$$

The embedding (1) is revealed through the embedding inequality as follows:

$$\left(\int_G |D^{\alpha}u|^q\omega(x)dx\right)^{1/q}\le C|u;W^{\bar{l}}_{\bar{p},\,p}(G;\,\bar{\rho},\upsilon)|, u\in C^{\infty}W.$$

This article discusses the embeddings of spaces $W^{\bar{l}}_{\bar{p},\,p}\,(G;\,\bar{\rho},\,\upsilon)$ in the case when $\kappa_{\alpha}=\left|(1+\alpha):\bar{l}\right|>1.$

Further, by introducing weights satisfying multiplicative boundedness conditions on average on parallelepipeds (Definition 1), we obtain embeddings of multiweighted anisotropic Sobolev type spaces $W^{\bar{l}}_{\bar{p},p}(G,\bar{\rho},\upsilon)$ into the weighted space $L^{\alpha}_{q}(G;\,\omega)$ on a domain G with irregular geometry, and anisotropy is present in the orders of derivatives, in terms of summability and in weight multipliers for these derivatives.

2 Preliminaries

Let $a=(a_1,...,a_n)$, $a_i>0$. We denote the parallelepiped

$$Q_a=Q_a(x)\{y\in R^n\quad:|y_i-x_i|<\frac{a_i}{2},\ i=1,2,\ldots,n\},$$

by $Q_a=Q_a(x)$. For $\lambda>0$ let $\lambda Q=\lambda Q_a=Q_{\lambda a.}$

We will consider a positive vector function $\bar{d}(x)=(d_1(x),...,\,d_n(x))$, satisfying the conditions:

1) $Q(x)=Q_{\bar{d}(x)}(x)\subset G;$

2) $\sup\limits_G d_i(x)<\infty$, $i=1,...,n;$

3) There exist the constants $0<\varepsilon<1$ and $b_0>1$:

$$\forall i=1,...,n\quad b_0^{-1}\le\frac{d_i(y)}{d_i(x)}\le b_0$$

as soon as

$$Q_{(\varepsilon)}(y)\cap Q_{(\varepsilon)}(x)\ne\emptyset,$$

where $Q_{(\varepsilon)}(x)=(1-\varepsilon)Q(x)$. We call the function $\bar{d}(x)$ by the edge length function. Let $\{Q(x),x\in G\}$ be the parallelepiped family

$$Q(x)=\{y\in R^n\quad:|y_i-x_i|<\frac{d_i(x)}{2},\ \ i=1,2,\ldots,n\},$$

satisfying the conditions 1)–3). Let $\rho_i(x)$, $\omega(x)$, $\upsilon(x)$ be the weights in G, and

$$\sigma_i(x)=\tilde{\rho}_i(x)d_i^{l_ip_i'}(x)\in L(G;loc),$$

$$\tilde{\rho}_i=\rho_i^{1-p_i'}(p_i'=\frac{p_i}{p_i-1}),1<p_i<\infty\ (i=1,...,n).$$

Suppose

$$B(x)=\left[\prod_{i=1}^{n}\left(\int\limits_{Q_{(\varepsilon)}(x)}\sigma_i\right)^{1/p_i'}\right]^{1/n}|Q_{\varepsilon}(x)|^{-1},$$

$$N_{(\delta)}(Q)\{e\subset Q\ :\ |e|\leq\delta|Q|\},\ 0\leq\delta<1.$$

Let

$$M_{(\varepsilon,\delta)}(x)=B(x)\inf_{e\in N_{(\delta)}(Q_{(\varepsilon)}(x))}\left(\int\limits_{Q_{(\varepsilon)}(x)\setminus e}\upsilon\right)^{1/p},\ \varepsilon\in[0,1).$$

Definition 1. [5] We say that the weight pair $(\bar{\rho},\upsilon)$ satisfies the condition $\Pi_{(\delta,\,\varepsilon)}$ with respect to $\bar{d}(x)=(d_1(x),...,\,d_n(x))$, if there are numbers $\delta\in[0,1)$ and $\varepsilon\in[0,1)$ such that

$$M_{(\varepsilon,\delta)}(x)\geq 1 \text{ for a.a. } x\in G.$$

Short notation:

$$(\bar{\rho},\,\upsilon)\in\Pi_{(\delta,\,\varepsilon)}.$$

At the same time $0<\varepsilon<1$, if $G\subset R^n$, $G\neq R^n$.

Lemma 1. [3] Let $1<p_i,p<q<\infty$, $(i=1,...,n)$, $r=\min\{p_1,...,p_n,p\}$, $\omega\in L_1(\nabla)$, where $\xi\in\nabla=\left(-\frac{1}{2},\frac{1}{2}\right)^n$, $\omega\geq 0$ and

$$M^{1/q}=\sup_{\substack{t>0\\ Q_{(t,\bar{l})}\subset\nabla}}t^{1-\kappa}|Q_{(t,\bar{l})}|_{\omega}^{1/q}<\infty.$$

Then

$$\left(\int\limits_{\nabla}|D^{\alpha}f|^q\omega(\xi)d\xi\right)^{1/q}\leq cM^{1/q}\left\|f\right\|_{W_r^{\bar{l}}(\nabla)},$$

where c does not depend on $f\in C^{\infty}(\bar{\nabla})$.

Let $\left\{\bar{P}(x),\,x\in E\right\}$ be the closed parallelepiped family

$$\bar{P}(x)=\{y\in R^n\ :\ |y_i-x_i|\leq a_i(x)/2,\ \ i=1,...,n\}\,,\tag{2}$$

where $a(x)=(a_1(x),\ldots,a_n(x))$ is a a positive vector function defined on a bounded set E in R^n.

Theorem 1. Let E be a bounded set in R^n, $\left\{\bar{P}(x),x\in E\right\}$ is the closed parallelepiped family (2), satisfying the conditions:

1) $\sup\limits_{x\in E}a_i(x)<\infty,\ (i=1,\ldots n)$;

2) there is a number $c>0$ such that

$$c^{-1}\leq a_i(y)/a_i(x)\leq c\ (i=1,\ldots n),$$

as soon as

$$\bar{Q}(x)\cap\bar{P}(y)\neq\emptyset.$$

Then it is possible to distinguish from $\{\bar{P}(x),\ x\in E\}$ no more than a countable subfamily $\{\bar{P}^j\}_{j\in J}$, $\bar{P}^j=\bar{P}(x^j)$, such that:

a) $E\subset\bigcup_{j\in J}\bar{P}^j$;

b) $\sum_{j\in J}X_{\bar{P}^j}(x)\leq\kappa_1=\kappa_1(c,n)<\infty$ for any x in R^n;

c) $\{\bar{P}^j\}_{j\in J}$ is represented as a union of no more than $\kappa_2=\kappa_1+1$ subfamilies $\{\bar{P}^j\}_{j\in J_v}$ of pairwise disjoint parallelepipeds.

The cover $\{\bar{P}^j\}$ of the set E, which has the properties of finite multiplicity and finite separability (respectively, properties b) and c)), in Theorem 1, we will call B-covering.

Let $X_i = X_i(G)$ $(i = 1, 2)$ be spaces of functions defined in G, with the seminorms $\|\cdot\|_{X_i(G)}$, $X_i(G)$ is the space of functions $X_i(G)/G$ with an induced seminorm $\|\cdot\|_{X_i(G)}$.

Lemma 2. [5] Let the spaces X_i $(i = 1, 2)$ meet the following conditions:

i1) $C^\infty(\bar{Q}) \subset X_i$, $Q \in I^n$ such that $\bar{Q} \subset G$;

i2) $\|f\|_{X_i(G_1)} \leq \|f\|_{X_i(G_2)}$, if $G_1 \subset G_2 \subset G$;

$$\|f\|_{X_i(G_k)} = \lim_{k\to\infty} \|f\|_{X_i(G_k)},$$

if

$$G_k \subset G_{k+1} \ (k \geq 1), \ G = \bigcup_1^\infty G_k \subset G;$$

i3) $\|f\|_{X_i(G\setminus e)} = \|f\|_{X_i(G)}$, if $|e| = 0$;

i4) There are numbers $s_i \geq 1, \ c_i \geq 1$ $(i = 1, 2)$ such that for any family

$$G = \bigcup_j G_j \subset G$$

$\{G_i\}$ of open sets such that

$$\|f\|^{s_1}_{X_i(G)} \leq c_1^{s_1} \sum_j \|f\|^{s_1}_{X_1(G_j)}, \ \ f \in X_1(G) \, \|f\|^{s_2}_{X_2(G_j)} \leq c_2^{s_2} \|f\|^{s_2}_{X_2(G)},$$

if G_j do not intersect in pairs with $f \in X_2(G)$;

i5) There are such a parallelepiped family $\{P(x), \, x \in G\}$ and a positive function $K(x)$ on G, that

$$\|f\|_{X_1(Q(x))} \leq cK(x) \, \|f\|_{X(Q(x))} \ \ \forall f \in C^\infty(\bar{Q}(x));$$

i6) From the family $\{P(x), \, x \in G\}$, we can distinguish B-covers $E = G \cap B(x, r)$, multiplicity κ_1 and coefficients of finite separability κ_2, which do not depend on r. Then we have a true inequality

$$\|f\|_{X_1} \leq cK \, \|f\|_{X_2}, \ f \in C^\infty(G) \cap X_2,$$

where

$$K = K_{s_1 s_2} = \begin{cases} \sup\limits_{x\in G} K(x), & \text{where } s_2 \leq s_1 \\ \sup \left(\sum_{j\in J} \left(K(x^j)\right)^{s_1 s_2/(s_2-s_1)}\right)^{(s_2-s_1)/(s_1 s_2)}, & \text{where } s_2 > s_1 \end{cases}$$

and sup is taken over all at most countable families of $\{Q^j\}_{j\in J}$ pairwise non-intersecting parallelepipeds $Q^j = Q(x^j)$.

3 Localization and embedding theorems

Below we will consider "characteristic parallelepipeds" of the form

$$Q(x) = Q^\tau_{\bar{\rho}}(x) = \left\{ y \in R^n : |y_i - x_i| < \frac{(\tau(x)\,\rho_i(x))^{1/l_i}}{2}, \ i = 1, 2, ..., n \right\}, \tag{3}$$

where $\tau(x)$ such a positive function in G that the functions

$$d_i(x) = (\tau(x) \ \rho_i(x))^{1/l_i} \leq 1$$

and satisfy the conditions 1)–3). With respect to the functions ρ_i $(i = 1, 2, ..., n)$ we will assume that

$$c_1 \leq \frac{\rho_i(y)}{\rho_i(x)} \leq c_2, \quad \text{if} \quad y \in Q^{\tau}_{(\varepsilon)}(x) \;\; (i = 1, ..., n)$$

and we introduce the following values connecting to the "characteristic parallelepipeds" $Q(x) = Q^{\tau}_{\bar{\rho}}(x)$ of the weight of ρ_i, ω and numeric parameters p_i, p, q, $\alpha = (\alpha_1, ..., \alpha_n)$ and $\bar{l} = (l_1, ..., l_n)$; namely, let us put

$$A^{\tau}_{\bar{p},p,q}(x|\bar{\rho}, \omega) = \sup_{t>0} \left\{ t^{1-\kappa} \left(\int_{\bar{p}} \omega \right)^{1/q}, p_t = ((1-\varepsilon)\, t)^{1/\bar{l}} Q_{\bar{\rho}^{1/\bar{l}}} \subset (1-\varepsilon)^{1/\bar{l}} Q(x) \right\},$$

$$K^{\tau}_{\bar{p},p,q}(x|\bar{\rho}, \omega) = \left[\prod_{i=1}^{n} \left(\tau(x)^{1-\frac{1}{p}-\alpha_i-\frac{1}{r}} \right)^{1/l_i} \right] A^{\tau}_{\bar{p},p,q}(x|\bar{\rho}, \omega).$$

Lemma 3. [3–5] Let $1 < p_i, p < q < \infty$ $(i = 1, ..., n)$, $r = \min\limits_{1 \leq i \leq n}(p_i, p)$, and the conditions are met:

$$M^{r}_{(\delta,\varepsilon)}(x|\bar{\rho}, \tau(\cdot)) \geq 1$$

and

$$\mathbf{K}_{\bar{p},p,q}(x|\bar{\rho}, \omega) < \infty.$$

Then

$$\left(\int_Q |D^{\alpha}u|^q \omega \right)^{1/q} \leq c\mathbf{K}^{\alpha}_{(\varepsilon),\bar{p},q}(x|\bar{\rho}, \omega) \left[\sum_{i=1}^{n} \left(\int_Q \left| \rho_i D^{l_i} u \right|^{p_i} \right)^{\frac{1}{p_i}} + \left(\int_Q |\upsilon u|^p \right)^{\frac{1}{p}} \right],$$

where $Q = Q_{(\varepsilon)}(x) = (1-\varepsilon)^{1/\bar{l}} Q^{\tau}_{\bar{\rho}}(x)$.

Proof. Suppose $f(\zeta) = u\left(x + (1-\varepsilon)^{1/\bar{l}} d(x) \cdot \zeta\right)$, where

$$\bar{d}(x) = (d_1(x), ..., d_n(x)) \subset d_i(x) = (\tau(x)\rho_i(x))^{1/l_i}, d_i = (1-\varepsilon)^{1/l_i} d_i(x).$$

Let

$$\hat{\omega}(\zeta) = \omega\left(x + \bar{d} \cdot \zeta\right), \bar{d} = (d_1, ..., d_n), Q = Q_{(\varepsilon)}(x).$$

Then by virtue of Lemma 1

$$\left(\int_{Q_{(\varepsilon)}(x)} |D^{\alpha}u|^q \omega \right)^{1/q} = |Q|^{1/q} \left(\prod_{i=1}^{n} d_i^{-\alpha_i} \right) \left(\int_{\nabla} |D^{\alpha} f|^q \hat{\omega}(\xi)\, d\xi \right)^{1/q} \leq$$

$$\leq \left(M \frac{|\alpha|}{\prod_{i=1}^{n} d_i^{q\alpha_i}} \right)^{\frac{1}{q}} \left[\sum_{i=1}^{n} \left(\int_{\nabla} \left| D^{l_i} f \right|^r d\xi \right)^{1/r} + \left(\int_{\nabla} |f|^r d\xi \right)^{1/r} \right] \leq$$

$$\leq \left(\frac{M}{\prod_{i=1}^{n} d_i^{q\alpha_i}} \right)^{\frac{1}{q}} |Q|^{\frac{1}{q}-\frac{1}{r}} \left\{ \sum_{i=1}^{n} |Q|^{\frac{1}{r}} \left(\int_Q \left| D^{l_i} f \right|^{p_i} \right)^{\frac{1}{p_i}} + \left(\int_Q |\omega|^r dy \right)^{1/r} \right\} \leq$$

$$\leq \left(\frac{M}{\prod_{i=1}^{n} d_i^{q\alpha_i}} \right)^{\frac{1}{q}} |Q|^{\frac{1}{q}-\frac{1}{r}} \tau(x) \left[\sum_{i=1}^{n} |Q|^{\frac{1}{r}-\frac{1}{p_i}} \zeta_i(x) \left\| D^{l_i} u \right\|_{p_i Q} + \tau(x)^{-1} \left(\int_Q |u|^r \right)^{1/r} \right] \leq$$

$$\leq \left(\frac{M}{\prod_{i=1}^{n} d_i^{q\alpha_i}}\right)^{\frac{1}{q}} |Q|^{\frac{1}{q}-\frac{1}{r}} \left\| u; W_{\hat{p},p}^{\hat{l}}\left(Q;\bar{\rho},\upsilon\right)\right\|, \text{where} \bar{\rho} = \left(\rho_1^{p_1},...,\rho_n^{p_n}\right).$$

It remains to note that

$$\left(\frac{M}{\prod\limits_{i=1}^{n} d_i^{q\alpha_i}}\right)^{\frac{1}{q}} |Q|^{\frac{1}{q}-\frac{1}{r}} \tau\left(x\right) = \tau\left(x\right)\prod_{i=1}^{n} d_i^{-\alpha_i} |Q|^{\frac{1}{q}-\frac{1}{r}} \sup_{Q_{(t,\bar{l})}\subset ?} t^{1-\kappa} \left|Q_{(t,\bar{l})}\right|_{\hat{\omega}}^{1/q} =$$

$$= \tau\left(x\right)\prod_{i=1}^{n} d_i^{-\alpha_i} |Q|^{-\frac{1}{r}} \left(\tau\left(x\right)^{x-1} \sup_{t>0}\left\{t^{1-\kappa}\left(\int_{\bar{p}}\omega\right)^{1/q}, p_t = \left(\left(1-\varepsilon\right)t\right)^{1/\bar{l}} Q_{\bar{\rho}^{1/\bar{l}}} \subset Q\right\}\right) =$$

$$= \prod_{i=1}^{n}\left(\tau\left(x\right)^{1-\frac{1}{r}} \rho_i\left(x\right)^{-\alpha_i-\frac{1}{r}}\right)^{1/l_i} A_{\bar{p},p,q}^{\tau}\left(x|\bar{\rho},\upsilon\right) = \mathbf{K}_{\bar{p},p,q,\alpha}^{\tau}(x|\bar{\rho},\omega).$$

Theorem 2. Let $1 < p_i, p < q < \infty\,(i = 1,...,n)$, $\alpha \in Z_+^n$, $\rho^{\bar{p}} = \left(\rho_1^{\bar{p}_1},...,\rho_n^{\bar{p}_n}\right)$. Let $(\rho^{\bar{p}},\upsilon) \in \prod_{(\delta,\varepsilon)}^{r}$ and

$$K = \sup_{x\in G} K_{\bar{p},p,q,\alpha}^{\tau}(x|\bar{\rho},r) < \infty.$$

Then on the class $C^{\infty}W$ the inequality

$$\left(\int_G |D^{\alpha}u|^q\omega(x)dx\right)^{1/q} \leq C\left|u; W_{\hat{p},p}^{\hat{l}}(G;\bar{\rho},\upsilon)\right|, u \in C^{\infty}W$$

is valid with an exact constant $C \leq cK$.

Proof. It follows from Lemma 2 and Theorem 1 that a pair of spaces $X_1 = L_q^{\alpha}(G;\omega)$, $X_2 = (W_{\hat{p},p}^{\hat{l}}(G;\bar{\rho}\upsilon))$ satisfies all the requirements of Lemma 2, from which the statement of the theorem follows.

Corollary 1. Let $1 < p_i, p < q < \infty\,(i = 1,...,n)$, $\kappa = \left|1:\bar{l}\right| \leq 1$, ω is the weight on R^n, which satisfies the conditions of uniform boundedness on unit cubes $Q = Q_1(x) = x + \nabla$:

$$K = \sup_x\left(\int_{Q_1(x)}\omega\right)^{1/q} < \infty.$$

Then

$$\left(\int_G |u|^q\omega(x)dx\right)^{1/q} \leq cK\|u; W_{\bar{p},p}^{\bar{i}}(R^n)\|, u \in C^{\infty}W.$$

Let $\overline{\mu} = (m_1,...,\mu_n) - \infty < m_i, v < \infty(i = 1,...n)$,

$$W_{\bar{p},p}^{\bar{i}}(\overline{\mu},v) = W_{\bar{p},p}^{\bar{i}}(R^n;\overline{\rho},v)$$

with

$$\rho_i(x) = (1+|x|)^{\mu}, v(x) = (1+|x|)^{v}.$$

Corollary 2. Let $1 < p_i, p < q < \infty \, (i = 1, ..., n)$,$\kappa = \left|1 : \bar{l}\right| \leq 1$, $-\infty < \mu_i, v, \gamma < \infty \, (i = 1, ..., n)$, and let the conditions be met:

$$\frac{v}{p} - \frac{1}{n} \sum_{i=n}^{n} \frac{\mu_i}{p_i} \leq 0, \frac{\gamma}{q} \leq \min_{i<i<n} \frac{\mu_i}{p_i}.$$

Then the inequality is true

$$\left(\int_G |u|^q (1 + |x|)^\gamma dx \right)^{1/q} \leq C \|u; W^{\bar{l}}_{\overline{p},p}(\overline{\mu}, v)\|, \; u \in C^\infty W.$$

Embeddings of weighted anisotropic Sobolev type spaces are relevant in applications where it is necessary to consider the heterogeneity of the medium or the complex geometry of the domain, such as in numerical methods for solving differential equations and in the theory of differential operators. Localization methods, in particular, the norms of embeddings on cubes with variable edge length considered in the work, can be applied for embeddings of more complex spaces, including fractional ones. Considering weights that satisfy multiplicative conditions of boundedness on average in parallelepipeds is particularly important for analyzing functions in multidimensional spaces with irregular geometries.

Acknowledgments

This work is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP15473253).

Author Contributions

G.Sh. Iskakova collected and analyzed data, led manuscript preparation, conducted the main research process. M.S. Aitenova and A.K. Sexenbayeva assisted in data collection and analysis. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Отелбаев М. Теоремы вложения пространств с весом и их применения к изучению спектра оператора Шредингера / М. Отелбаев // Тр. МИАН СССР. — 1979. — 150. — С. 265–305.

2 Лизоркин П.И. Теоремы вложения и компактности для пространств соболевского типа с весами / П.И. Лизоркин, М. Отелбаев // Мат. сб. — 1979. — 108. — № 3. — С. 358–377.

3 Кусаинова Л.К. Теоремы вложения анизотропных весовых пространств типа Соболева. I / Л.К. Кусаинова // Изв. вузов. Математика. — 1987. — № 11. — С. 20–31.

4 Кусаинова Л.К. Теоремы вложения анизотропных весовых пространств типа Соболева. II / Л.К. Кусаинова // Изв. вузов. Математика. — 1988. — № 2. — С. 36–45.

5 Искакова Г.Ш. Теоремы вложения анизотропных многовесовых пространств Соболева: канд. дис. ... физ.-мат. наук / Г.Ш. Искакова. — Астана, 2010.

6 Besov O.V. Integral representations of functions, and imbedding theorems for a region with flexible horn condition / O.V. Besov // Proc. Steklov Inst. Math. — 1987. — 170. — P. 11–31.

7 Besov O.V. Sobolev's embedding theorem for anisotropically irregular domains / O.V. Besov // Eurasian mathematical journal. — 2011. — 2. — No. 1. — P. 32–51.

8 Filippas S. On a class of weighted anisotropic Sobolev inequalities / S. Filippas, L. Moschini, A. Tertikas // Journal of Functional Analysis. — 2008. — 255. — No. 1. — P. 90–119. https://doi.org/10.1016/j.jfa.2008.03.014

9 Vasil'eva A.A. Estimates for the Kolmogorov widths of weighted Sobolev classes on a domain with cusp: case of weights that are functions of the distance from the boundary / A.A. Vasil'eva // Eurasian mathematical journal. — 2017. — 8. — No. 4. — P. 102–106.

10 di Blasio G. A weighted anisotropic Sobolev type inequality and its applications to Hardy inequalities / G. di Blasio, G. Pisante, G. Psaradakis // Math. Ann. — 2021. — 379. — P. 1343–1362. https://doi.org/10.1007/s00208-019-01930-4

11 Bal K. Weighted anisotropic Sobolev inequality with extremal and associated singular problems / K. Bal, G. Prashanta // Differential Integral Equations. — 2023. — 36. — No. 1-2. — P. 59–92.

12 Metafune G. Anisotropic Sobolev Spaces with Weights / G. Metafune, L. Negro, Ch. Spina // Tokyo J. Math. Advance Publication. — 2023. — 25 p. https://doi.org/10.3836/tjm/1502179386

13 Li H. Embedding theorem of the weighted Sobolev–Lorentz spaces / H. Li, J. Ruan, Q. Sun // Glasgow Mathematical Journal. — 2022. — 64. — No. 2. — P. 358–375. https://doi.org/10.1017/S0017089521000161

14 Mukherjee S. On a scale of anisotropic Sobolev spaces / S. Mukherjee, I. Tice // arXiv:2312.06044. — 2023. https://arxiv.org/abs/2312.06044

15 Hoai-Minh N. On anisotropic Sobolev spaces / N. Hoai-Minh, S. Marco // Communications in Contemporary Mathematics. — 2019. — 21. — No. 01. — P. 1850017. https://doi.org/10.1142/S0219199718500177

16 Xu Ch. An embedding theorem for anisotropic fractional Sobolev spaces / Ch. Xu, W. Sun // Banach Journal of Mathematical Analysis. — 2021. — 15. — Article number 63. https://doi.org/10.1007/s43037-021-00146-6

17 Bartusel D. Embeddings of anisotropic Besov spaces into Sobolev spaces / D. Bartusel, H. Fuehr // Mathematische Nachrichten. — 2023. — 296. — No. 4. — P. 1380–1393. https://doi.org/10.1002/mana.202100512

18 Bekmaganbetov K.A. The embedding theorems for anisotropic Nikol'skii-Besov spaces with generalized mixed smoothness / K.A. Bekmaganbetov, K.Ye. Kervenev, Ye. Toleugazy // Bulletin of the Karaganda University. Mathematics series. — 2021. — No. 4(104). — P. 28–34. https://doi.org/10.31489/2021m4/28-34

Соболев типті көпсалмақты анизотроптық кеңістіктің енгізулері

Г.Ш. Искакова[1], М.С. Айтенова[2,3], А.К. Сексенбаева[4]

[1]*Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан;*
[2]*Қарағанды Қазтұтынуодағы университеті, Қарағанды, Қазақстан;*
[3]*М.В.Ломоносов атындағы Мәскеу мемлеттік университетінің Қазақстан филиалы, Астана, Қазақстан;*
[4]*Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Астана, Қазақстан*

Функциялардың әртүрлі интегралды-дифференциалдық сипаттамалары, облыстар мен олардың шекараларының тегістігінің қасиеттері, сондай-ақ салмақтық функцияларының көптеген кластары сияқты параметрлер Соболев типіндегі көп салмақтық анизотропты кеңістіктердің күрделі өзара байланыстары мен енгізу шарттарын анықтайды. Бұл параметрлерді шектемеуге деген ұмтылыс жаңа тәсілдердің дамуына әкеледі, олардағы кеңістіктер мен нормаларды анықтаудың балама нұсқаларын енгізуге немесе оқшаулаудың арнайы әдістеріне негізделген. Мақалада көп салмақтық дифференциалдық индекстерді, қосындылану индекстерін қоса алғанда, кеңістік нормасының барлық айқындаушы сипаттамаларында анизотропиясы бар, сондай-ақ салмақ коэффициенттері бар Соболев типті анизотропты кеңістіктердің енгізілуі зерттелген. Қолданылған локализация әдісі дифференциалдық операторлар теориясында және сандық талдаудың қосымшаларында маңызды болып табылатын кез келген облыс пен жалпы типтегі салмақ үшін енгізгенді алуға мүмкіндік береді.

Кілт сөздер: анизотроптық Соболев кеңістіктері, көп салмақтық кеңістіктер, енгізу теоремалары, локализация әдістері, салмақтық функциялар.

Вложения многовесового анизотропного пространства типа Соболева

Г.Ш. Искакова[1], М.С. Айтенова[2,3], А.К. Сексенбаева[4]

[1] *Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан;*
[2] *Карагандинский университет Казпотребсоюза, Караганда, Казахстан;*
[3] *Казахстанский филиал Московского государственного университета им. М.В. Ломоносова, Астана, Казахстан;*
[4] *Евразийский национальный университет имени Л.Н. Гумилева, Астана, Казахстан*

Такие параметры, как различные интегро-дифференциальные характеристики функций, свойства гладкости областей и их границ, а также множества классов весовых функций, обусловливают сложные взаимосвязи и условия вложений многовесовых анизотропных пространств типа Соболева. Стремление не ограничивать эти параметры приводит к развитию новых подходов, основанных на введении альтернативных вариантов определений пространств и норм в них либо на специальных методах локализации. В этой статье исследованы вложения многовесовых анизотропных пространств типа Соболева с анизотропией во всех определяющих характеристиках нормы пространства, включая дифференциальные индексы, индексы суммируемости, а также весовые коэффициенты. Примененный метод локализации позволил получить вложение для случая произвольной области и весов общего типа, что важно в приложениях в теории дифференциальных операторов и численном анализе.

Ключевые слова: анизотропные пространства Соболева, многовесовые пространства, теоремы вложения, методы локализации, весовые функции.

References

1 Otelbaev, M. (1979). Teoremy vlozheniia prostranstv s vesom i ikh primeneniia k izucheniiu spektra operatora Shredingera [Imbedding theorems for spaces with weight and their application to the study of the spectrum of the Schredinger operator]. *Trudy MIAN SSSR — Proceedings of the Steklov Mathematical Institute of the USSR, 150*, 265–305 [in Russian].

2 Lizorkin, P.I., & Otelbaev, M. (1979). Teoremy vlozheniia i kompaktnosti dlia prostranstv sobolevskogo tipa s vesami [Embedding theorems and compactness for spaces of Sobolev type with weights]. *Matematicheskii sbornik — Mathematical Collection, 108*(3), 358–377 [in Russian].

3 Kussainova, L.K. (1987). Teoremy vlozheniia anizotropnykh vesovykh prostranstv tipa Soboleva. I [Embedding theorems of anisotropic weighted Sobolev type spaces. I]. *Izvestiia vuzov. Matematika — News of universities. Mathematics*, 11, 20–31 [in Russian].

4 Kussainova, L.K. (1988). Teoremy vlozheniia anizotropnykh vesovykh prostranstv tipa Soboleva. II [Embedding theorems of anisotropic weighted Sobolev type spaces. II]. *Izvestiia vuzov. Matematika — News of universities. Mathematics*, 2, 36–45 [in Russian].

5 Iskakova, G.Sh. (2010). *Teoremy vlozheniia anizotropnykh mnogovesovykh prostranstv Soboleva [Embedding theorems of anisotropic multi-weighted Sobolev spaces]. Candidate's thesis.* Astana [in Russian].

6 Besov, O.V. (1987). Integral representations of functions, and imbedding theorems for a region with flexible horn condition. *Proc. Steklov Inst. Math., 170,* 11–31.

7 Besov, O.V. (2011). Sobolev's embedding theorem for anisotropically irregular domains. *Eurasian mathematical journal, 2*(1), 32–51.

8 Filippas, S., Moschini, L., & Tertikas, A. (2008). On a class of weighted anisotropic Sobolev inequalities. *Journal of Functional Analysis, 255*(1), 90–119. https://doi.org/10.1016/j.jfa.2008.03.014

9 Vasil'eva, A.A. (2017). Estimates for the Kolmogorov widths of weighted Sobolev classes on a domain with cusp: case of weights that are functions of the distance from the boundary. *Eurasian mathematical journal, 8*(4), 102–106.

10 di Blasio, G., Pisante, G., & Psaradakis, G. (2021). A weighted anisotropic Sobolev type inequality and its applications to Hardy inequalities. *Math. Ann., 379,* 1343–1362. https://doi.org/10.1007/s00208-019-01930-4

11 Bal, K., & Prashanta, G. (2023). Weighted anisotropic Sobolev inequality with extremal and associated singular problems. *Differential Integral Equations, 36*(1-2), 59–92.

12 Metafune, G., Negro L., & Spina, Ch. (2023). Anisotropic Sobolev Spaces with Weights. *Tokyo J. Math. Advance Publication.* https://doi.org/10.3836/tjm/1502179386

13 Li, H., Ruan, J., & Sun, Q. (2022). Embedding theorem of the weighted Sobolev–Lorentz spaces. *Glasgow Mathematical Journal, 64*(2), 358–375. https://doi.org/10.1017/ S0017089521000161

14 Mukherjee, S., & Tice, I. (2023). *On a scale of anisotropic Sobolev spaces.* arXiv:2312.06044. https://arxiv.org/abs/2312.06044

15 Hoai-Minh, N., & Marco, S. (2019). On anisotropic Sobolev spaces. *Communications in Contemporary Mathematics, 21*(01), 1850017. https://doi.org/10.1142/ S0219199718500177

16 Xu, Ch., & Sun, W. (2021). An embedding theorem for anisotropic fractional Sobolev spaces. *Banach Journal of Mathematical Analysis, 15*, 63. https://doi.org/ 10.1007/s43037-021-00146-6

17 Bartusel, D., & Fuehr, H. (2023). Embeddings of anisotropic Besov spaces into Sobolev spaces. *Mathematische Nachrichten, 296*(4), 1380–1393. https://doi.org/10.1002/ mana.202100512

18 Bekmaganbetov, K.A., Kervenev, K.Ye., & Toleugazy, Ye. (2021). The embedding theorems for anisotropic Nikol'skii-Besov spaces with generalized mixed smoothness. *Bulletin of the Karaganda University. Mathematics series,* 4(104), 28–34. https://doi.org/10.31489/2021m4/28-34

Author Information[*]

Gaukhar Shamugutovna Iskakova (*corresponding author*) — Candidate of physical and mathematical sciences, Associate Professor of the Department of Mathematical Analysis and Differential Equations, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; Kazakhstan Branch of Lomonosov Moscow State University, Astana, Kazakhstan; e-mail: *iskakova.1975@mail.ru*; https://orcid.org/0000-0003-4953-6853

Mansia Saparovna Aitenova — Candidate of physical and mathematical sciences, Associate Professor of the Department of Higher Mathematics, Karaganda University of Kazpotrebsoyuz, 9 Akademicheskaya street, Karaganda, 100009, Kazakhstan; e-mail: *abibekove@mail.ru*; https://orcid.org/0009-0007-0884-3327

Aisha Kurmanbekovna Sexenbayeva — Candidate of physical and mathematical sciences, Professor of the Department of Applied Mathematics, L.N.Gumilyov Eurasian National University, 2 Mirzoyan street, Astana, 010000, Kazakhstan; e-mail: *aiisha08@mail.ru*; https://orcid.org/0009-0001-5646-4328

[*]The author's name is presented in the order: First, Middle and Last Names.

https://doi.org/10.31489/2024M1/84-100 Research article

On the solvability of a boundary value problem for a two-dimensional system of Navier-Stokes equations in a cone

M.T. Jenaliyev, A.M. Serik, M.G. Yergaliyev*

Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
(E-mail: muvasharkhan@gmail.com, serikakerke00@gmail.com, ergaliev@math.kz)

Due to the fact that the Navier-Stokes equations are involved in the formulation of a large number of interesting problems that are important from an applied point of view, these equations have been the object of attention of mechanics, mathematicians and other scientists for several decades in a row. But despite this, many problems for the Navier-Stokes equation remain unexplored to this day. In this work, we are exploring the solvability of a boundary value problem for a two-dimensional Navier-Stokes system in a non-cylindrical degenerating domain, namely, in a cone with its vertex at the origin. Previously, we studied cases of the linearized Navier-Stokes system or non-degenerating cylindrical domains, so this work is a logical continuation of our previous research in this direction. To the above-mentioned degenerate domain we associate a family of non-degenerate truncated cones, which, in turn, are formed by a one-to-one transformation into cylindrical domains, where for the problem under consideration we established uniform a priori estimates with respect to changes in the index of the domains. Further, using a priori estimates and the Faedo-Galerkin method, we established the existence, uniqueness of solution in Sobolev classes, and its regularity as the smoothness of the given functions increases.

Keywords: Navier-Stokes system, degenerating domain, Galerkin method.

2020 Mathematics Subject Classification: 35Q30, 35K40, 35K55.

Introduction

As mentioned above, the Navier-Stokes equations have been the object of research by many scientists due to their applied importance ([1–5], and others). A significant number of practical problems have not been solved to this day.

Boundary value problems for parabolic equations in domains with moving boundaries are often models for ecological and medical processes [6], thermal processes in electrical contacts [7], thermo-mechanics processes [8, 9], and so on.

Among the works in this direction, we would like to mention the works [10] and [11], where the solvability of boundary value problems for the Burgers equation (the so-called one-dimensional version of the Navier-Stokes system) in domains with moving boundaries was researched. The results of these works were continued in [12], where by using the Faedo-Galerkin method and a priori estimates, the existence, uniqueness of the regular solution of the researched boundary value problems in Sobolev spaces is established.

Previously, in [13–15], it was shown by the authors that homogeneous boundary value problems for the Burgers equation and the nonlinear heat equation in an angular domain that degenerates at the initial time, along with the trivial solution, have nontrivial solutions. For boundary value problems with different inhomogeneities along the boundary, both unique and non-unique solvability were established

*Corresponding author. *E-mail: ergaliev@math.kz*

This research is supported by the grant project AP19674862 (2023–2025) from the Ministry of Science and Higher Education of the Republic of Kazakhstan.

Received: 31 August 2023; *Accepted:* 13 November 2023.

in the work [16]. Also, the following works [15, 17, 18] devoted to problems in degenerating domains may be of interest to the readers.

In this work, we research the solvability of a boundary value problem with Dirichlet conditions for a two-dimensional Navier-Stokes system in a cone with its vertex at the origin. In Section 1, we present the formulation of the main boundary value problem and a sequence of auxiliary boundary value problems in the truncated cones. Then, in Section 2, these problems are transformed into boundary value problems in cylindrical domains by a change of independent variables. In Section 3, using the previously obtained results from the work [19], we obtain unique solvability of each of the above sequence of problems. In Section 4, auxiliary lemmas and the theorem on uniform a priori estimates are given. Section 5 is devoted to the main result.

1 Preliminary statement of the problem

Let us consider the next cone $Q_{xt_1} = \{x, t_1 : \ |x| < t_1, \ 0 < t_1 < T_1 < \infty\}$, which has its vertex at the origin. Let Ω_{xt_1} be the section of the cone Q_{xt_1} for a given $t_1 \in (0, T_1)$.

In the cone Q_{xt_1}, which degenerates into a point at $t_1 \in (0, T_1)$, we will consider the following boundary value problem (BVP) for a system of Navier-Stokes equations with respect to a two-dimensional vector-function of the fluid velocity $u(x, t_1) = \{u_1(x, t_1), \ u_2(x, t_1)\}$ and the fluid pressure function $p(x, t_1)$:

$$\frac{\partial u}{\partial t_1} - \nu \Delta u + \sum_{i=1}^{2} u_i \frac{\partial u}{\partial x_i} = f - \nabla p, \tag{1}$$

$$\operatorname{div} u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \tag{2}$$

$$u = 0, \ \ \{x, \, t_1\} \in \Sigma_{xt_1} \text{ is the lateral surface of the cone.} \tag{3}$$

Remark 1. Since at the initial moment of time the considering domain degenerates into a point, in the formulation of problem (1)–(3) we do not set the initial condition.

To the problem (1)–(3) we will set a sequence of BVPs, each of which will be considered in the corresponding truncated cone.

Let $n \in \mathbb{N}^* \equiv \{n \in \mathbb{N} : n \geq n_1, 1/n_1 < T_1\}$, $x = \{x_1, x_2\}$, and consider the domain $Q^n_{xt_1} = \{x, t_1 : |x| < t_1, \ 1/n < t_1 < T_1 < \infty\}$ which is an inverted truncated cone and let Ω_{xt_1} be the section of the cone $Q^n_{xt_1}$ for a given $t_1 \in (1/n, T_1)$. As we can see, now the domain $Q^n_{xt_1}$ does not degenerate into a point at the initial moment of time $t_1 = 1/n$. For domains Q_{xt_1} and $Q^n_{xt_1}$, the following inclusions are also true: $Q^{n_1}_{xt_1} \subset Q^{n_1+1}_{xt_1} \subset ... \subset Q_{xt_1}$, moreover, $\lim\limits_{n\to\infty} Q^n_{xt_1} = Q_{xt_1}$.

Now in the non-degenerating domain $Q^n_{xt_1}$ (for each finite $n \in \mathbb{N}^*$) we consider the following BVP:

$$\frac{\partial u_n}{\partial t_1} - \nu \Delta u_n + \sum_{i=1}^{2} u_{in} \frac{\partial u_n}{\partial x_i} = f_n - \nabla p_n, \tag{4}$$

$$\operatorname{div} u_n = \frac{\partial u_{1n}}{\partial x_1} + \frac{\partial u_{2n}}{\partial x_2} = 0, \tag{5}$$

$$u_n = 0 \ \ \{x, \, t_1\} \in \Sigma^n_{xt_1} \text{ is the lateral surface of the cone } \ Q^n_{xt_1}. \tag{6}$$

$$u_n(x, 1/n) = 0, \ \ x \in \Omega_{x1/n} \text{ is the section of the cone at } t_1 = 1/n. \tag{7}$$

A BVP of form (4)–(7) (for each fixed finite $n \in \mathbb{N}^*$) was studied by us in [19], in which we established theorems on unique solvability in Sobolev spaces.

2 Transformation of the problem (4)–(7) and its meaningful statement

Now we transform BVP (4)–(7) so that it would be posed in a cylindrical domain. For this purpose we use the transformation of independent variables and pass from the variables $\{x, t_1\}$ to variables $\{y, t\}$. Then we obtain

$$x_i = \frac{1}{n-t} y_i,\ t_1 = \frac{t}{n-t},\ y_i = \frac{x_i}{t_1},\ t = n - \frac{1}{t_1},\ i = 1, 2;$$

$Q^n_{yt} = \{y, t:\ |y| < 1,\ 0 < t < T\}$ is a cylindrical domain, and Ω is a section of the cylinder Q^n_{yt} for any fixed $t \in [0, T]$,

$$t_1 = 1/n \Leftrightarrow t = 0,\ t_1 = T_1 \Leftrightarrow t = T = n - \frac{1}{T_1}.$$

Since

$$\tilde{u}_{in}(y,t) \triangleq u_{in}\left(\frac{y}{n-t}, \frac{1}{n-t}\right),\ \ \tilde{p}_n(y,t) \triangleq p_n\left(\frac{y}{n-t}, \frac{1}{n-t}\right), \tag{8}$$

we obtain the next derivatives with respect to t_1 of function $u_{in}(x, t_1)$ (8)

$$\frac{\partial u_{in}}{\partial t_1} = \frac{\partial \tilde{u}_{in}(y,t)}{\partial t}(n-t)^2 - \sum_{k=1}^{2} \frac{\partial \tilde{u}_{in}(y,t)}{\partial y_k}(n-t) y_k.$$

As for the derivatives with respect to x_k of function $u_{in}(x, t_1)$ (8), we have:

$$\frac{\partial u_{in}}{\partial x_k} = \frac{\partial \tilde{u}_{in}}{\partial y_k}(n-t), \quad \frac{\partial^2 u_{in}}{\partial x_k^2} = \frac{\partial^2 \tilde{u}_{in}}{\partial y_k^2}(n-t)^2.$$

Using the above we write down the BVP (4)–(7) in the cylindrical domain Q^n_{yt}:

$$\frac{\partial \tilde{u}_n}{\partial t} - \nu \Delta \tilde{u}_n + \frac{1}{n-t} \sum_{i=1}^{2} (\tilde{u}_{in} - y_i) \frac{\partial \tilde{u}_n}{\partial y_i} = \frac{1}{(n-t)^2} \tilde{f}_n - \frac{1}{n-t} \nabla p_n, \tag{9}$$

$$\operatorname{div} \tilde{u}_n = 0,\ \ \{y, t\} \in Q^n_{yt}, \tag{10}$$

$$\tilde{u}_n(y,t) = 0,\ \ \{y,t\} \in \Sigma^n_{yt} = \{y, t:\ |y| = 1,\ 0 < t < T\}, \tag{11}$$

$$\tilde{u}_n(y, 0) = 0,\ \ y \in \Omega = \{y:\ |y| < 1\}. \tag{12}$$

Now instead of BVP (9)–(12) we will consider a more general BVP:

$$\frac{\partial \tilde{u}_n}{\partial t} - \nu \Delta \tilde{u}_n + \alpha(t) \sum_{i=1}^{2} \tilde{u}_{in} \frac{\partial \tilde{u}_n}{\partial y_i} = \sum_{i=1}^{2} \gamma_i(y_i, t) \frac{\partial \tilde{u}_n}{\partial y_i} + \beta(t) \tilde{f}_n - \delta(t) \nabla \tilde{p}_n, \tag{13}$$

$$\operatorname{div} \tilde{u}_n = 0,\ \ \{y, t\} \in Q^n_{yt} = \{y, t:\ |y| < 1,\ \ 0 < t < T\}, \tag{14}$$

$$\tilde{u}_n(y,t) = 0,\ \ \{y,t\} \in \Sigma^n_{yt} = \{y, t:\ |y| = 1,\ 0 < t < T\}, \tag{15}$$

$$\tilde{u}_n(y, 0) = 0,\ \ y \in \Omega = \{y:\ |y| < 1\}, \tag{16}$$

where the given functions $\alpha(t), \beta(t), \gamma_i(y_i, t),\ i = 1, 2$, and $\delta(t)$, satisfy the following conditions

$$\begin{aligned} &\alpha_1 \le \alpha(t),\, \alpha'(t) \le \alpha_2,\ |\beta(t)| \le \beta_1,\ |\beta'(t)| \le \beta_1,\ |\delta(t)| \le \delta_1, \quad \forall\, t \in [0, T], \\ &|\gamma_i(y_i, t)| \le \gamma_1, \quad \left|\frac{\partial \gamma_i(y_i, t)}{\partial t}\right| \le \gamma_1,\ i = 1, 2, \qquad \forall\, \{y, t\} \in Q^n_{yt}, \end{aligned} \tag{17}$$

where $\alpha_1,\ \alpha_2,\ \gamma_1,\ \beta_1,\ \delta_1$, are given positive constants.

It is easy to see that for the coefficients of equations (9) conditions (17) are aslo met.

Let us give a definition of a weak solution to problem (13)–(16). For this purpose we use the following notation [3, 4, 20–22] (here and further the designation $U^2 = U \times U$ is accepted):

$$V = \{\varphi |\, \varphi \in (D(\Omega))^2,\ \mathrm{div}\varphi = 0\},$$

$$H = \text{the closure of } V \text{ in } (L_2(\Omega))^2,$$

$$V = \text{the closure of } V \text{ in } (H^1(\Omega))^2.$$

For $\tilde{f}, \tilde{g} \in H$ we set

$$(\tilde{f}, \tilde{g}) = \int\limits_{\Omega} \tilde{f}(y)\tilde{g}(y)d\,y,\ \ |\tilde{f}| = (\tilde{f}, \tilde{f})^{1/2},$$

and for $\tilde{u}, \tilde{v} \in V$ we set

$$((\tilde{u}, \tilde{v})) = \sum_{i,j=1}^{2} \int\limits_{\Omega} \frac{\partial \tilde{u}_j(y)}{\partial y_i} \frac{\partial \tilde{v}_j(y)}{\partial y_i} d\,y,\ \ \|\tilde{u}\| = ((\tilde{u}, \tilde{u}))^{1/2}.$$

Then, identifying H with its conjugate: $H = H'$, we obtain the following inclusions

$$V \subset H = H' \subset V',$$

and each of these spaces is dense in the subsequent with completely continuous embedding operators. We can understand conditions (15) as conditions of belonging the function $\tilde{u}(y, t)$ to space V for almost all t.

Now we assume that

$$a(\tilde{u}, \tilde{v}) = \sum_{i,j=1}^{2} \int\limits_{\Omega} \frac{\partial \tilde{u}_j}{\partial y_i} \frac{\partial \tilde{v}_j}{\partial y_i} d\,y,\ \ \tilde{u},\ \tilde{v} \in V,\ \forall\, t \in (0, T),$$

$$b(\tilde{u}, \tilde{v}, \tilde{w}) = \sum_{i,k=1}^{2} \int\limits_{\Omega} \tilde{u}_k \frac{\partial \tilde{v}_i}{\partial y_k} \tilde{w}_i\, d\,y,\ \forall\, t \in (0, T),$$

for a triple of such two-dimensional vectors $\tilde{u}, \tilde{v}, \tilde{w}$, for which the corresponding integrals converge.

Problem 1. Let

$$\tilde{f}_n \in L_2(0, T; (H^{-1}(\Omega))^2),$$

be given and functions $\alpha(t), \beta(t), \gamma_i(y_i, t),\ i = 1, 2$, and $\delta(t)$ satisfy conditions (17).

It is required to find such $\tilde{u}_n$ and $\tilde{p}_n$, $\tilde{p}_n \in D'(Q^n_{yt})$, that

$$\tilde{u}_n \in L_2(0, T; V) \cap L_\infty(0, T; H),$$

$$\frac{\partial \tilde{u}_n}{\partial t} - \nu \Delta \tilde{u}_n + \alpha(t) \sum_{i=1}^{2} \tilde{u}_{in} \frac{\partial \tilde{u}_n}{\partial y_i} = \sum_{i=1}^{2} \gamma_i(y_i, t) \frac{\partial \tilde{u}_n}{\partial y_i} + \beta(t) \tilde{f}_n - \delta(t) \nabla \tilde{p}_n, \tag{18}$$

$$\tilde{u}_n(y, 0) = 0. \tag{19}$$

Despite the apparent accuracy, in the formulation of Problem 1 we have one ambiguity: there is no information regarding the derivative $\frac{\partial \tilde{u}_n}{\partial t}(y,t)$ and $\tilde{p}_n(y,t)$, there is only the following relation

$$\frac{\partial \tilde{u}_n}{\partial t} + \delta(t)\nabla \tilde{p}_n = \nu\Delta\tilde{u}_n - \alpha(t)\sum_{i=1}^{2} \tilde{u}_{in}\frac{\partial \tilde{u}_n}{\partial y_i} + \sum_{i=1}^{2}\gamma_i(y_i,t)\frac{\partial \tilde{u}_n}{\partial y_i} + \beta(t)\tilde{f}_n \text{ on } Q^n_{yt},$$

therefore the meaning of condition (19) is not obvious.

If we take $\varphi(y) \in V$, then $(\nabla\tilde{p}_n, \varphi) = 0$ in $(D'(0,T))^2$, and (18) leads to equality

$$\left(\frac{\partial \tilde{u}_n}{\partial t}, \varphi\right) = -\nu a(\tilde{u}_n, \varphi) - \alpha(t) b(\tilde{u}_n, \tilde{u}_n, \varphi) + \sum_{i=1}^{2}\left(\gamma_i(y_i,t)\frac{\partial \tilde{u}_n}{\partial y_i}, \varphi\right) + \beta(t)(\tilde{f}_n, \varphi) \text{ for any } \varphi \in V. \quad (20)$$

Using the following equality

$$b(\tilde{u}_n, \tilde{u}_n, \varphi) = -b(\tilde{u}_n, \varphi, \tilde{u}_n),$$

we get that (20) is equivalent to

$$\left(\frac{\partial \tilde{u}_n}{\partial t}, \varphi\right) = -\nu a(\tilde{u}_n, \varphi) + \alpha(t) b(\tilde{u}_n, \varphi, \tilde{u}_n) + \sum_{i=1}^{2}\left(\gamma_i(y_i,t)\frac{\partial \tilde{u}_n}{\partial y_i}, \varphi\right) + \beta(t)(\tilde{f}_n, \varphi) \text{ for any } \varphi \in V. \quad (21)$$

Let

$$X = \text{the closure of } V \text{ in } (W_1^1(\Omega))^2,$$

we have

$$|b(\tilde{u}_n, \varphi, \tilde{u}_n)| \le C_1\|\tilde{u}_n\|^2_{(L_\infty(\Omega))^2}\sum_{i,j=1}^{2}\left\|\frac{\partial \varphi_j}{\partial y_i}\right\|_{L_1(\Omega)} \le C\|\tilde{u}_n\|^2\|\varphi\|_X,$$

since $V \subset (L_\infty(\Omega))^2$, and therefore

$$b(\tilde{u}_n, \varphi, \tilde{u}_n) = (g, \varphi), \;\; \|g\|_{X'} \le C\|\tilde{u}_n\|^2_{L_\infty(\Omega))^2},$$

whence it follows that $g \in L_1(0,T;X')$.

From (21) we obtain that

$$\frac{\partial \tilde{u}_n}{\partial t} \in L_2(0,T;V') + L_1(0,T;X'),$$

so that (19) makes sense (for example, in X').

Thus, we obtain a different formulation of Problem 1.

Problem 2. Let

$$\tilde{f}_n \in L_2(0,T;V') \quad (22)$$

be given and functions $\alpha(t), \beta(t), \gamma_i(y_i,t),\; i = 1,2,$ and $\delta(t)$, satisfy conditions (17).

It is required to find such $\tilde{u}_n$, that

$$\tilde{u}_n \in L_2(0,T;V) \cap L_\infty(0,T;H), \quad (23)$$

$$\left(\frac{\partial \tilde{u}_n}{\partial t}, \tilde{v}\right) + \nu a(\tilde{u}_n, \tilde{v}) + \alpha(t) b(\tilde{u}_n, \tilde{u}_n, \tilde{v}) - \sum_{i=1}^{2}\left(\gamma_i(y_i,t)\frac{\partial \tilde{u}_n}{\partial y_i}, \tilde{v}\right) = \beta(t)(\tilde{f}_n, \tilde{v}) \;\forall \tilde{v} \in V, \quad (24)$$

$$\tilde{u}_n(y,0) = 0, \;\; y \in \Omega. \quad (25)$$

Next we want to formulate Problem 2 in relation to the BVP (4)–(7). To do this, first, we need the following correspondence of function spaces in terms of independent variables $\{y,t\} \in Q_{yt}^n$ and $\{x,t_1\} \in Q_{xt_1}^n$:

$$\tilde{f}_n(y,t) \in L_2(0,T;V') \Leftrightarrow f_n(x,t_1) \in L_2(1/n,T_1;V'_{t_1}),$$

$$\tilde{u}_n(y,t) \in L_2(0,T;V) \cap L_\infty(0,T;H) \Leftrightarrow u_n(x,t_1) \in L_2(1/n,T_1;V_{t_1}) \cap L_\infty(1/n,T_1;H_{t_1}),$$

etc., where for almost all $t_1 \in [1/n, T_1]$,

$$V_{t_1} = \{\varphi | \, \varphi \in (D(\Omega_{xt_1}))^2, \ \mathrm{div}\varphi = 0\},$$

$$H_{t_1} = \text{the closure of } V_{t_1} \text{ in } (L_2(\Omega_{xt_1}))^2,$$

$$V_{t_1} = \text{the closure of } V_{t_1} \text{ in } (W_2^1(\Omega_{xt_1}))^2.$$

Problem 3. Let

$$f_n(x,t_1) \in L_2(1/n,T_1;V'_{t_1}) \tag{26}$$

be given. It is required to find such $u(x,t_1)$, that

$$u_n(x,t_1) \in L_2(1/n,T_1;V_{t_1}) \cap L_\infty(1/n,T_1;H_{t_1}), \tag{27}$$

$$\left(\frac{\partial u_n}{\partial t_1}, v\right) + \nu a(u_n,v) + b(u_n,u_n,v) = (f,v) \ \forall \, v \in V_{t_1}, \tag{28}$$

$$u_n(x,1/n) = 0, \ x \in \Omega_{x1/n}. \tag{29}$$

Finally, we formulate Problem 4 in relation to the original BVP (1)–(3), which is given in a degenerating cone.

Problem 4. Let

$$f(x,t_1) \in L_2(0,T_1;V'_{t_1}) \tag{30}$$

be given. It is required to find such $u(x,t_1)$, that

$$u(x,t_1) \in L_2(0,T_1;V_{t_1}) \cap L_\infty(0,T_1;H_{t_1}), \tag{31}$$

$$\left(\frac{\partial u}{\partial t_1}, v\right) + \nu a(u,v) + b(u,u,v) = (f,v) \ \forall \, v \in V_{t_1}. \tag{32}$$

Further, we will use the following Lemma ([3], Lemma I.6.1); [4], Lemma II.1.1):

Lemma 1. The trilinear form $\{\tilde{u},\tilde{v},\tilde{w}\} \to b(\tilde{u},\tilde{v},\tilde{w})$ is continuous on $V \times V \times V$, $\forall \, t \in (0,T)$, and the following estimate is valid

$$\left| \int\limits_\Omega \tilde{u}_i \frac{\partial \tilde{v}_k}{\partial y_i} \tilde{w}_k \, d\,y \right| \le \|\tilde{u}_i\|_{L_4(\Omega)} \left\| \frac{\partial \tilde{v}_k}{\partial y_i} \right\|_{L_2(\Omega)} \|\tilde{w}_k\|_{L_4(\Omega)}, \ i,k = 1,2.$$

3 Solvability theorems for problems (22)–(25), (26)–(29) and (30)–(32)

According to the results of [19] we have:

(a) in the case of each of the domains represented by the cylinders Q^n_{yt}, $n \in \mathbb{N}^*$, Theorems 1–3 and Corollary 1 are valid;

(b) and in the case of each of the domains represented by truncated cones $Q^n_{xt_1}$, $n \in \mathbb{N}^*$, Theorems 4–6 and Corollary 2 are valid.

Theorem 1. Let for the functions $\alpha(t), \beta(t), \gamma_i(y_i, t)$ and $\delta(t)$ conditions (17) met. Then Problem 2 (22)–(25) has a unique (weak) solution

$$\tilde{u}_n(y,t) \in \widetilde{W}(0,T) = \{v| \, v \in L_2(0,T;V), \ \frac{\partial v}{\partial t} \in L_2(0,T;V')\}.$$

Theorem 2. Let the following be true along with the conditions of Theorem 1:

$$\frac{\partial \tilde{f}_n}{\partial t} \in L_2(0,T;V'), \ \tilde{f}_n(y,0) \in H.$$

Then for the solution $\tilde{u}_n(y,t)$ to Problem 2 (22)–(25) we have the following inclusion

$$\frac{\partial \tilde{u}_n}{\partial t} \in L_2(0,T;V) \cap L_\infty(0,T;H).$$

Theorem 3. Let the following be true along with the conditions of Theorem 2:

$$\tilde{f}_n \in L_\infty(0,T;H).$$

Then for the solution $\tilde{u}_n(y,t)$ to Problem 2 (22)–(25) we have the following inclusion

$$\tilde{u}_n \in L_\infty(0,T;(W^2_2(\Omega))^2).$$

Corollary 1. Let the following be true along with the conditions of Theorem 3:

$$\tilde{f}_n \in L_2(0,T;H).$$

Then for the solution $\tilde{u}_n(y,t)$ to Problem 2 (22)–(25) we have the following inclusion

$$\tilde{u}_n \in L_2(0,T;(W^2_2(\Omega))^2).$$

Theorem 4. Let $f_n(x,t_1) \in L_2(1/n,T_1;V'_{t_1})$. Then Problem 3 (26)–(29) has a unique (weak) solution

$$u_n(x,t_1) \in W(1/n,T_1) = \{v| \, v \in L_2(1/n,T_1;V_{t_1}), \ \frac{\partial v}{\partial t_1} \in L_2(1/n,T_1;V'_{t_1})\}.$$

Theorem 5. Let the following be true along with the conditions of Theorem 4:

$$\frac{\partial f_n}{\partial t_1} \in L_2(1/n,T_1;V'_{t_1}), \ f_n(x,1/n) \in H_{1/n}.$$

Then for the solution $u_n(x,t_1)$ to Problem 3 (26)–(29) we have the following inclusion

$$\frac{\partial u_n}{\partial t_1} \in L_2(1/n,T_1;V_{t_1}) \cap L_\infty(1/n,T_1;H_{t_1}).$$

Theorem 6. Let the following be true along with the conditions of Theorem 5

$$f_n \in L_\infty(1/n, T_1; H_{t_1}).$$

Then for the solution $u_n(x,t_1)$ to Problem 3 (26)–(29) we have the following inclusion

$$u_n \in L_\infty(1/n, T_1; (W_2^2(\Omega_{xt_1}^n))^2).$$

Corollary 2. Let the following be true along with the conditions of Theorem 6

$$f_n \in L_2(1/n, T_1; H_{t_1}).$$

Then for the solution $u_n(x,t_1)$ to Problem 3 (26)–(29) we have the following inclusion

$$u_n \in L_2(1/n, T_1; (W_2^2(\Omega_{xt_1}^n))^2).$$

Remark 2. Note that Problem 3 (26)–(29) corresponds to BVP (4)–(7).

Further, by using the results of Theorems 4–6 and Corollaries 2, for BVP 3 (26)–(29) we will show the validity of the following theorem.

Theorem 7. Let the conditions of Theorem 6 and Corollary 2 be satisfied. Then there exists a positive constant K independent of n, such that for the solution $u_n(x,t_1)$ to BVP 3 (26)–(29) we have the following estimate

$$\|u_n(x,t_1)\|^2_{(W_2^{2,1}(Q_{xt_1}^n))^2} + \|\nabla p_n(x,t_1)\|^2_{(L_2(Q_{xt_1}^n))^2} \le K \cdot F_n \le K \cdot F,$$

where

$$F_n = |f_n(x,1/n)|^2 + \|f_n(x,t_1)\|^2_{W_2^1(1/n,T_1;V'_{t_1})} + \|f_n(x,t_1)\|^2_{(L_2(Q_{xt_1}^n))^2},$$

$$F = |f(x,0)|^2 + \|f(x,t_1)\|^2_{W_2^1(0,T_1;V'_{t_1})} + \|f(x,t_1)\|^2_{(L_2(Q_{xt_1}))^2}.$$

$$Q_{xt_1}^{n_1} \subset Q_{xt_1}^{n_1+1} \subset ... \subset Q_{xt_1} \text{ and obviously } \lim_{n\to\infty} Q_{xt_1}^n = Q_{xt_1}.$$

The proof of this theorem will be given in the next section.

Now we can formulate the main result of the paper, which will be proved in Section 5 on the basis of the assertion of Theorem 7.

Theorem 8. Let the conditions of Theorem 7 be met. Then in the degenerating domain Q_{xt_1} the two-dimensional BVP 4 for the system of Navier-Stokes equations (30)–(32) has a unique solution $\{u(x,t_1), p(x,t_1)\}$ in space

$$(W_2^{2,1}(Q_{xt_1}))^2 \times L_2(0,T_1; W_2^1(\Omega_{xt_1})/X_{xt_1}),$$

where $W_2^1(\Omega_{xt_1})/X_{xt_1}$ and $\|\psi(x)\|_{W_2^1(\Omega_{xt_1})/X_{xt_1}} \equiv \inf_{k\in X_{xt_1}} \|\psi(x)+k\|_{W_2^1(\Omega_{xt_1})}$ are, respectively, a quotient space and a quotient norm in the subspace X_{xt_1} consisting of all possible constants $k = \text{const}$ defined on the set Ω_{xt_1}.

Remark 3. Problem 4 (30)–(32) corresponds to BVP (1)–(3).

4 Auxiliary lemmas. Proof of Theorem 7

To prove Theorem 7, we need to establish the following lemmas.

Lemma 2. Let the conditions of Theorem 4 be met. Then there exists a positive constant K_1 independent of n, such that for the solution $u_n(x,t_1)$ to BVP (4)–(7) we have the following estimate

$$\|u_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2} + \|\nabla u_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2} \leq K_1 \|f_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2}, \tag{33}$$

where

$$\|u_n(x,t_1)\|^2_{L_2(Q^n_{xt_1})} \equiv \int\limits_{1/n}^{T_1} |u_n(x,t_1)|^2\, d\,t_1,$$

$$|u_n(x,t_1)|^2 = \int\limits_{\Omega_{xt_1}} \left\{[u_{1n}(x,t_1)]^2 + [u_{2n}(x,t_1)]^2\right\}\, d\,x.$$

Proof. By multiplying equation (4) scalarly by the function $u_n(x,t_1)$ in space $L_2(\Omega_{xt_1})$, we obtain

$$\frac{1}{2}\frac{d}{d\,t}|u_n(x,t_1)|^2 + a(u_n(x,t_1),u_n(x,t_1)) = (f_n(x,t_1),u_n(x,t_1)),$$

since $b(u_n(x,t_1),u_n(x,t_1),u_n(x,t_1)) = 0$. From here, according to the Cauchy ε-inequality, using the Poincare inequality ([21], 6.30) and integrating the result from $1/n$ to T_1, we obtain the required inequality (33).

Lemma 3. Let the conditions of Theorem 5 be met. Then there exists a positive constant K_2 independent of n, such that at all $t_1 \in [1/n, T_1]$ for the solution $u_n(x,t_1)$ to BVP (4)–(7) we have the following estimate

$$\left|\frac{\partial u_n(x,t_1)}{\partial t_1}\right|^2 + \int\limits_{1/n}^{t_1} \left\|\frac{\partial u_n(x,t_1)}{\partial t_1}\right\|^2 d\,t_1 \leq K_2\left[|f_n(x,1/n)|^2 + \|f_n(x,t_1)\|^2_{W^1_2(1/n,T_1;V'_{t_1})}\right]. \tag{34}$$

Proof. By multiplying equation (4) scalarly by the function $\frac{\partial u_n(x,t_1)}{\partial t_1}$ in space $L_2(\Omega_{xt_1})$, for $t_1 = 1/n$ we will obtain:

$$\left|\frac{\partial u_n(x,1/n)}{\partial t_1}\right|^2 = \left(f(x,1/n), \frac{\partial u_n(x,1/n)}{\partial t}\right) \leq |f(x,1/n)|\left|\frac{\partial u_n(x,1/n)}{\partial t}\right|,$$

i.e., we get

$$\left|\frac{\partial u_n(x,1/n)}{\partial t_1}\right|^2 \leq C_0 |f(x,1/n)|^2. \tag{35}$$

Now we differentiate equation (4) with respect to t_1, then by multiplying the equation scalarly by the function $\frac{\partial u_n(x,t_1)}{\partial t_1}$ in space $L_2(\Omega^n_{xt_1})$, and considering (by virtue of Lemma II.1.3 from [4]) the following equality

$$b\left(u_n, \frac{\partial u_n}{\partial t_1}, \frac{\partial u_n}{\partial t_1}\right) = 0,$$

we get

$$\frac{1}{2}\frac{d}{d\,t_1}\left|\frac{\partial u_n}{\partial t_1}\right|^2 + \nu\left\|\frac{\partial u_n}{\partial t_1}\right\|^2 + b\left(\frac{\partial u_n}{\partial t_1}, u_n, \frac{\partial u_n}{\partial t_1}\right) = \left(\frac{\partial f_n}{\partial t_1}, \frac{\partial u_n}{\partial t_1}\right). \tag{36}$$

We have

$$\left|b\left(\frac{\partial u_n}{\partial t_1}, u_n, \frac{\partial u_n}{\partial t_1}\right)\right| = \left|-b\left(\frac{\partial u_n}{\partial t_1}, \frac{\partial u_n}{\partial t_1}, u_n\right)\right| \leq C_5 \left\|\frac{\partial u_n}{\partial t_1}\right\|_{\left(L_4(\Omega^n_{xt_1})\right)^2} \left\|\frac{\partial u_n}{\partial t_1}\right\| \|u_n\|_{\left(L_4(\Omega^n_{xt_1})\right)^2} \leq$$

$$\leq C_6 \left\|\frac{\partial u_n}{\partial t_1}\right\|^{3/2} \left|\frac{\partial u_n}{\partial t_1}\right|^{1/2} \|u_n\|_{\left(L_4(\Omega^n_{xt_1})\right)^2} \leq \frac{\nu}{2}\left\|\frac{\partial u_n}{\partial t_1}\right\|^2 + C_7 \left|\frac{\partial u_n}{\partial t_1}\right|^2 \|u_n\|^4_{\left(L_4(\Omega^n_{xt_1})\right)^2}.$$

Here we have used Lemma I.6.2 from [3] and Young's inequality ($p^{-1}+q^{-1}=1$):

$$|AB| = \left|\left(a^{1/p}A\right)\left(a^{1/q}\frac{B}{a}\right)\right| \leq \frac{a}{p}|A|^p + \frac{a}{qa^q}|B|^q,$$

where

$$A = \left\|\frac{\partial u_n}{\partial t_1}\right\|^{3/2},\ B = C_6\,|u_n|^{1/2}\left\|\frac{\partial u_n}{\partial t_1}\right\|_{(L_4(\Omega^n_{xt_1}))^2},\ a=\frac{2\nu}{3},\ p=\frac{4}{3},\ q=4,$$

$$\left(\frac{\partial f_n}{\partial t_1}, \frac{\partial u_n}{\partial t_1}\right) \leq C_8\left|\frac{\partial f_n}{\partial t_1}\right|\left|\frac{\partial u_n}{\partial t_1}\right| \leq \frac{C_8^2}{2}\left|\frac{\partial f_n}{\partial t_1}\right|^2 + \frac{1}{2}\left|\frac{\partial u_n}{\partial t_1}\right|^2.$$

By using these inequalities and relations (35)–(36), we get uniform in t_1 and n required estimate (34). The statement of Lemma 3 is proved.

Lemma 4. Let the conditions of Theorem 6 and Corollary 2 be met. Then there exists a positive constant K_3 independent of n, such that at all $t_1 \in [1/n, T_1]$ for the solution $u_n(x,t_1)$ to BVP (4)–(7) we have the following estimate

$$|\nabla u_n(x,t_1)|^2 + \int_{1/n}^{t_1} |\Delta u_n(x,t_1)|^2\,dt_1 \leq K_3 \cdot F_{1n},$$

where

$$F_{1n} = |f_n(x,1/n)|^2 + \|f_n(x,t_1)\|^2_{W^1_2(1/n,T_1;V'_{t_1})} + \|f_n(x,t_1)\|^2_{L_\infty(1/n,T_1;H_{t_1})}.$$

Proof. First, note that, by Lemma III.3.1 from [4] function $B\,u_n(x,t_1)$, defined by equality

$$< B\,u_n, v >= b(u_n,u_n,v)\ \forall\, v \in V_{t_1}\ \text{ almost everywhere on }\ [1/n, T_1],$$

belongs to space $L_1(1/n,T_1;V'_{t_1})$.

We write equation (24) in the form

$$\nu a(u_n(x,t_1), v(x)) = (g_n(x,t_1), v(x))\ \ \forall\, v \in V_{t_1}, \tag{37}$$

where

$$g_n(x,t_1) = -\frac{\partial u_n}{\partial t_1} - B\,u_n + f_n. \tag{38}$$

Since $u_n \in L_\infty(1/n,T_1;V_{t_1})$ and according to Lemmas I.6.1–I.6.2 from [3]

$$|b(u_n(x,t_1), u_n(x,t_1), v(x)| \leq$$

$$\leq C_0\|u_n(x,t_1)\|_{(L_4(\Omega^n_{xt_1}))^2}\|u_n(x,t_1)\|\|v\|_{(L_4(\Omega^n_{xt_1}))^2} \leq C_1\|u_n(x,t_1)\|^2\|\|v\|_{(L_4(\Omega^n_{xt_1}))^2}, \tag{39}$$

then $B\,u_n \in L_\infty(1/n, T_1; (L_{4/3}(\Omega^n_{xt_1}))^2)$. From (38) and the inclusion

$$-\frac{\partial u_n}{\partial t_1} + f_n \in L_\infty(1/n, T_1; H_{t_1})$$

(here we have used the statement of Lemma 3) we have

$$g_n(x,t_1) \in L_\infty(1/n, T_1; (L_{4/3}(\Omega^n_{xt_1}))^2). \tag{40}$$

Further, applying the theorem from ([23], 309–311) and ([4], I.2.5) for the elliptic BVP (37), we get $u_n \in L_\infty(1/n, T_1; (W^2_{4/3}(\Omega^n_{xt_1}))^2)$, and the following estimates

$$\|u_n(x,t_1)\|^2_{L_\infty(1/n,T_1;(W^2_{4/3}(\Omega^n_{xt_1}))^2)} + \|p_n(x,t_1)\|^2_{L_\infty(1/n,T_1;W^1_{4/3}(\Omega^n_{xt_1})/X^n_{xt_1})} \leq$$
$$\leq K\,\|g_n(x,t_1)\|^2_{L_\infty(1/n,T_1;(L_{4/3}(\Omega^n_{xt_1}))^2)}\,, \tag{41}$$

where $W^1_{4/3}(\Omega^n_{xt_1})/X^n_{xt_1}$ is a quotient space in the subspace $X^n_{xt_1}$ consisting of all possible constants $k =$ const defined on the set $\Omega^n_{xt_1}$. But according to Sobolev embedding theorem $W^2_{4/3}(\Omega^n_{xt_1}) \subset L_\infty(\Omega^n_{xt_1})$, then $u_n \in (L_\infty(Q^n_{xt_1}))^2$.

Now we can improve the inclusion (40). We replace inequality (39) with the following

$$|b(u_n(x,t_1), u_n(x,t_1), v(x))| \leq C_2\|u_n\|_{(L_\infty(Q^n_{xt_1}))^2}\|u_n(x,t_1)\|\|v\|,$$

from which it follows that $B\,u_n \in L_\infty(1/n, T_1; H_{t_1})$. Thus, we obtain that $g_n \in L_\infty(1/n, T_1; H_{t_1})$.

Again, applying the theorem from ([23], 309–311) and ([4], I.2.5) for the elliptic BVP (37), we get that $u_n \in L_\infty(1/n, T_1; (W^2_2(\Omega^n_{xt_1}))^2) \subset L_2(1/n, T_1; (W^2_2(\Omega^n_{xt_1}))^2)$, and estimates for the case of Theorem 6:

$$\|u_n(x,t_1)\|^2_{L_\infty(1/n,T_1;(W^2_2(\Omega^n_{xt_1}))^2)} + \|p_n(x,t_1)\|^2_{L_\infty(1/n,T_1;W^1_2(\Omega^n_{xt_1})/X^n_{xt_1})} \leq$$
$$\leq K\,\|g_n(x,t_1)\|^2_{L_\infty(1/n,T_1;H_{t_1})}\,, \tag{42}$$

and for the case of Corollary 2:

$$\|u_n(x,t_1)\|^2_{L_2(1/n,T_1;(W^2_2(\Omega^n_{xt_1}))^2)} + \|p_n(x,t_1)\|^2_{L_2(1/n,T_1;W^1_2(\Omega^n_{xt_1})/X^n_{xt_1})} \leq$$
$$\leq K\,\|g_n(x,t_1)\|^2_{L_2(1/n,T_1;H_{t_1})}\,, \tag{43}$$

where $W^1_2(\Omega^n_{xt_1})/X^n_{xt_1}$ is a quotient space in the subspace $X^n_{xt_1}$ consisting of all possible constants $k =$ const defined on the set $\Omega^n_{xt_1}$. From here we also get that $\nabla u_n \in L_\infty(1/n, T_1; (W^1_2(\Omega^n_{xt_1}))^2) \subset L_2(1/n, T_1; (W^1_2(\Omega^n_{xt_1}))^2)$.

It remains to estimate the right-hand side in (41)–(43) with respect to the function $f_n(x,t_1)$. According to (38), it remains to estimate only the summand $B\,u_n$. We have

$$\|B\,u_n\|_{L_\infty(1/n,T_1;H_{t_1})} \leq C_3\|u_n(x,t_1)\|,$$

the right-hand side of which is estimated in Lemma 3. This completes the proof of Lemma 4.

Lemma 5. Let the conditions of Theorem 6 and Corollary 2 be satisfied. Then there exists a positive constant K_4 independent of n, such that for a solution to the boundary value problem (4)–(7) the following estimate takes place

$$\left\|\frac{\partial u_n(x,t_1)}{\partial t_1}\right\|^2_{(L_2(Q^n_{xt_1}))^2} + \|\Delta u_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2} + \|\nabla p_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2} \leq K_4 \cdot F_2, \tag{44}$$

where

$$F_2 = |f(x,0)|^2 + \|f(x,t_1)\|^2_{W^1_2(0,T_1;V'_{t_1})} + \|f(x,t_1)\|^2_{(L_2(Q_{xt_1}))^2}.$$

Proof. The proof of Lemma 5 directly follows from the statements of Lemmas 3 and 4.

Thus, the statement of Theorem 7 follows from Lemmas 2, 5 and inequalities

$$\|f_n\|^2_{(L_2(Q^n_{xt_1}))^2} \leq \|f\|^2_{(L_2(Q_{xt_1}))^2},$$

i.e., we obtain the required estimate (44):

$$\|u_n(x,t_1)\|^2_{(W_2^{2,1}(Q^n_{xt_1}))^2} + \|\nabla p_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2} \leq K_4 \cdot F_{2n} \leq K_4 \cdot F_2,$$

where

$$F_{2n} = |f_n(x,1/n)|^2 + \|f_n(x,t_1)\|^2_{W_2^1(1/n,T_1;V'_{t_1})} + \|f_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2}.$$

5 *Proof of Theorem 8: the existence and uniqueness of a solution to boundary value problem (1)–(3)*

Let $\{u_n(x,t_1), p_n(x,t_1)\}$ be a solution to boundary value problem (4)–(7), which exists and is unique according to Theorems 4–6, Corollary 2 and Theorem 7. Denote by $\{\widetilde{u_n}(x,t_1), \widetilde{p_n}(x,t_1)\}$ the continuation of solutions $\{u_n(x,t_1), p_n(x,t_1)\}$ by zero to the entire cone Q_{xt_1}. Theorem 7 implies the following inequality

$$\|\widetilde{u_n}(x,t_1)\|^2_{(W_2^{2,1}(Q_{xt_1}))^2} + \|\nabla\widetilde{p_n}(x,t_1)\|^2_{(L_2(Q_{xt_1}))^2} \leq K \cdot F,$$

that is uniform over n, where

$$F = |f(x,0)|^2 + \|f(x,t_1)\|^2_{W_2^1(0,T_1;V'_{t_1})} + \|f(x,t_1)\|^2_{(L_2(Q_{xt_1}))^2}.$$

It follows that from the bounded sequence $\{\widetilde{u_n}(x,t_1), \nabla\widetilde{p_n}(x,t_1)\}_{n=1}^{\infty}$ it is possible to extract a subsequence (to denote the index of which we keep the letter n), such that the following limit relations take place:

$$\frac{\partial\widetilde{u_n}(x,t_1)}{\partial t_1} \to \frac{\partial u(x,t_1)}{\partial t_1} \text{ weakly in } (L_2(Q_{xt_1}))^2,$$

$$\Delta\widetilde{u_n}(x,t_1) \to \Delta u(x,t_1) \text{ weakly in } (L_2(Q_{xt_1}))^2,$$

$$\widetilde{u_n}(x,t_1) \to u(x,t_1) \text{ strongly in } (L_2(Q_{xt_1}))^2,$$

$$\widetilde{u_{in}}(x,t_1)\frac{\partial\widetilde{u_{kn}}(x,t_1)}{\partial x_i}\widetilde{u_{kn}}(x,t_1) \to u_i(x,t_1)\frac{\partial u_k(x,t_1)}{\partial x_i}u_k(x,t_1) \text{ weakly in } L_2(Q_{xt_1}),\ i,k=1,2,$$

$$\nabla\widetilde{p_n}(x,t_1) \to \nabla p(x,t_1) \text{ weakly in } (L_2(Q_{xt_1}))^2.$$

Further, in a standard way, it is easy to show that

$$\{u(x,t_1), p(x,t_1)\} \in \left\{(W_2^{2,1}(Q_{xt_1}))^2 \times L_2(0,T_1;W_2^1(\Omega_{xt_1})/X_{xt_1})\right\}$$

is the solution of the boundary value problem (1)–(3), where $W_2^1(\Omega_{xt_1})/X_{xt_1}$ is a quotient space in the subspace X_{xt_1} consisting of all possible constants $k = \text{const}$ defined on the set Ω_{xt_1}.

We pass to the proof of uniqueness in problem (1)–(3). Let $\{\bar{u}(x,t_1), \bar{p}(x,t_1)\}$ and $\{u^*(x,t_1), p^*(x,t_1)\}$ be two solutions of the boundary value problem (1)–(3), and let

$$u(x,t_1) = \bar{u}(x,t_1) - u^*(x,t_1),\ p(x,t_1) = \bar{p}(x,t_1) - p^*(x,t_1),$$

which according to (1)–(3) satisfy the following equation:

$$\left(\frac{\partial u}{\partial t_1}, w\right) + \nu a(u,w) + b(u,\bar{u},w) + b(\bar{u},u,w) - b(u,u,w) = 0.$$

If we take as the test function $w = u$, then we will have the equality

$$\frac{1}{2}\frac{d}{d\,t_1}\|u\|^2_{H_{t_1}} + \nu\|\nabla u\|^2_{H_{t_1}} = b(u,u,\bar{u}), \tag{45}$$

since $b(u,u,\bar{u}) = -b(u,\bar{u},u),\ b(\bar{u},u,u) = 0,\ b(u,u,u) = 0.$

Further, proceeding in the same way as in the proof of Lemma 3, from (45) we obtain

$$\frac{d}{d\,t_1}\|u\|^2_{H_{t_1}} \le K\|u\|^2_{H_{t_1}},$$

where K is a positive constant, and by Gronwall's lemma it follows that $u \equiv 0$, and thus the property of uniqueness is proved.

This completes the proof of the main result of the work formulated in the following theorem.

Conclusion

The results of the work can be generalized to the case when the section of the cone for each fixed t_1 can change according to the rule $r = \sqrt{x_1^2 + x_2^2} \le \varphi(t_1),\ t_1 \in [0, T_1],\ \varphi(0) = 0$, under some natural requirements for the function $\varphi(t_1)$. For example, the function $\varphi(t_1)$ must satisfy the following two conditions: 1^0. in a sufficiently short period of time $(0, t_1^*)$ the function $\varphi(t_1)$ could have the representation $\varphi(t_1) = \mu\, t_1$, where μ is the given positive constant (in our work it was equal to one); 2^0. on the interval $[t_1^*, T_1]$ the function $\varphi(t_1)$ would be continuously differentiable and possess the property of monotonicity, providing a one-to-one transformation from independent variables $\{x, t_1\}$ to variables $\{y, t\}$.

Acknowledgments

Supported by the grant project AP19674862 (2023–2025) from the Ministry of Science and Higher Education of the Republic of Kazakhstan.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Burgers J.M. The nonlinear diffusion equation / J.M. Burgers. — Dordrecht: Reidel, 1974. — 174 p.

2 Batchelor G.K. An intronduction to fluid dynamics / G.K. Batchelor. — Cambridge: CUP, 2000. — 615 p.

3 Lions J.-L. Quelques methodes de resolution des problemes aux limites non lineaires / J.-L. Lions. — Paris: Dunod, 1969. — 554 p.

4 Temam R. Navier-Stokes equations. Theory and numerical analysis / R. Temam. — Amsterdam: North Holland Publishing Company, 1979. — 530 p.

5 Vishik M.I. Mathematical problems of statistical hydromechanics / M.I. Vishik, A.B. Fursikov. — Dordrecht: Springer, 1988. — 576 p.

6 Mitropol'ski Yu.A. Free boundary problems for nonlinear evolution equations in metallurgy, medicine, and ecology / Yu.A. Mitropol'ski, A.A. Berezovskii, T.A. Plotnizkii // Ukrainian mathematical journal. — 1992. — 44. — No. 1. — P. 67–75. https://doi.org/10.1007/BF01062627

7 Ким Е.И. Математические модели тепловых процессов в электрических контактах / Е.И. Ким, В.Т. Омельченко, С.Н. Харин. — Алма-Ата: Наука, 1977. — 236 с.

8 Веригин Н.Н. Об одном классе гидромеханических задач для областей с подвижными границами / Н.Н. Веригин // Динамика жидкости со свободными границами. Динамика сплошной среды. — 1980. — 46. — С. 23–32.

9 Карташов Э.М. Проблема теплового удара в областях с движущейся границей на основе новых интегральных соотношений / Э.М. Карташов // Изв. АН РФ. Энергетика. — 1997. — № 4. – С. 122–137.

10 Benia Y. Existence of solutions to Burgers equations in domains that can be transformed into rectangles / Y. Benia, B.-K. Sadallah // Electronic Journal of Differential Equations. — 2016. — 157. — P. 1–13.

11 Benia Y. Existence of solutions to Burgers equations in a non-parabolic domain / Y. Benia, B.-K. Sadallah // Electronic Journal of Differential Equations. — 2018. — 20. — P. 1–13.

12 Sadallah B.-K. Etude d'un probleme 2m-parabolique dans des domaines plan non rectangulaires / B.-K. Sadallah // Bollettino dell'Unione Matematica Italiana. — 1983. — 6. — No. 2-B. — P. 51–112.

13 Amangaliyeva M.M. On the existence of non-trivial solution to homogeneous Burgers equation in a corner domain / M.M. Amangaliyeva, M.T. Jenaliyev, M.I. Ramazanov // Abstracts of Int. Conf. "Mathematics in the Modern World". — Novosibirsk, 2017. — P. 187.

14 Jenaliyev M. On linear and nonlinear heat equations in degenerating domains / M. Jenaliyev, M. Ramazanov, M. Yergaliyev // AIP Conference Proceedings. — 2017. — 1910. — 040001-1–040001-10. https://doi.org/10.1063/1.5013968

15 Jenaliyev M.T. On Solonnikov-Fasano problem for the Burgers equation / M.T. Jenaliyev, M.I. Ramazanov, A.A. Assetov // Bulletin of the Karaganda University. Mathematics series. — 2020. — No. 2(98). — P. 69–83. https://doi.org/10.31489/2020m2/69-83

16 Amangaliyeva A.A. On the solvability of nonhomogeneous boundary value problem for the Burgers equation in the angular domain and related integral equations / M.M. Amangaliyeva, M.T. Jenaliyev, M.T. Kosmakova, M.I. Ramazanov // Springer Proceedings in Mathematics and Statistics. — 2017. — 216. — P. 123–141.

17 Kosmakova M.I. On the solution to a two-dimensional boundary value problem of heat conduction in a degenerating domain / M.T. Kosmakova, V.G. Romanovski, D.M. Akhmanova, Zh.M. Tuleutaeva, A.Yu. Bartashevich // Bulletin of the Karaganda University. Mathematics series. — 2020. — No. 2(98). — P. 100–109. https://doi.org/10.31489/2020m2/100-109

18 Kosmakova M.T. A fractionally loaded boundary value problem two-dimensional in the spatial variable / M.T. Kosmakova, K.A. Izhanova, L.Z. Kasymova // Bulletin of the Karaganda University. Mathematics series. — 2023. — No. 2(110). — P. 72–83. https://doi.org/10.31489/2023m2/72-83

19 Jenaliyev M.T. On the solvability of a boundary value problem for a two-dimensional system of

Navier-Stokes equations in a truncated cone / M.T. Jenaliyev, M.G. Yergaliyev // Lobachevskii Journal of Mathematics. — 2023. — 44. — No. 8. — P. 3307–3320. https://doi.org/10.1134/S199508022308022X

20 Lions J.-L. Problemes aux limites non homogenes et applications, Vol. 1 / J.-L. Lions, E. Magenes. — Dunod: Paris, 1968. — 372 p.

21 Adams R.A. Sobolev spaces (Second Ed.) / R.A. Adams, J.J.F. Fournier. — New York: Academic Press, 2003. — 320 p.

22 Brezis H. Functional analysis, Sobolev spaces and partial differential equations / H. Brezis. — New York: Springer, 2011. — 600 p.

23 Cattabriga L. Su un problema al contorno relativo al sistema di equazioni di Stokes / L. Cattabriga // Rendiconti del Seminario Matematico della Universita di Padova. — 1961. — 31. — P. 308–340.

Конустағы екіөлшемді Навье-Стокс теңдеулерінің жүйесі үшін қойылған шекаралық есептің шешімділігі туралы

М.Т. Жиенәлиев, А.М. Серік, М.Ғ. Ерғалиев

Математика және математикалық модельдеу институты, Алматы, Қазақстан

Навье-Стокс теңдеулері қолданбалы тұрғыдан маңызды көптеген қызықты есептердің қойылуында кездесетіндіктен, бұл теңдеулер бірнеше ондаған жылдар бойы механиктердің, математиктердің және басқа да ғалымдардың назарында болды. Бірақ бұған қарамастан Навье-Стокс теңдеуіне арналған көптеген есептер осы күнге дейін әлі де зерттелмеген. Жұмыста цилиндрлік емес өзгешеленетін облыстағы, атап айтқанда төбесі координаталардың басында орналасқан конуста екіөлшемді Навье-Стокс жүйесі үшін шекаралық есептің шешімділігі зерттелген. Бұған дейін осы есептің сызықты Навье-Стокс жүйесі үшін қойылуы немесе өзгешеленетін емес цилиндрлік облыстардағы қойылуы зерттелген, сондықтан бұл жұмыс осы бағыттағы алдыңғы зерттеулердің логикалық жалғасы болып табылады. Жоғарыда аталған өзгешеленетін облысқа өзгешеленетін емес кесілген конустар жиыны сәйкестікке қойылады. Бұл облыстар өз кезегінде цилиндрлік облыстарға өзара бірмәнді түрлендіру арқылы келтіріледі. Бұдан кейін қарастырылып отырған есеп үшін облыс индексінің өзгеруіне қатысты біртекті априорлық бағалаулары айқындалып, әрі қарай, априорлық бағалаулар мен Фаэдо-Галеркин әдісін қолдана отырып, Соболев кластарындағы шешімнің бар және жалғыз екендігін дәлелдеп, берілген функциялардың тегістігі артқан сайын оның регулярлығын анықтаған.

Кілт сөздер: Навье-Стокс жүйесі, өзгешеленетін облыс, Галеркин әдісі.

О разрешимости одной граничной задачи для двумерной системы уравнений Навье-Стокса в конусе

М.Т. Дженалиев, А.М. Серик, М.Г. Ергалиев

Институт математики и математического моделирования, Алматы, Казахстан

В силу того, что уравнения Навье-Стокса участвуют в постановках большого количества интересных задач, важных с прикладной точки зрения, данные уравнения в течение нескольких десятилетий подряд были объектом внимания механиков, математиков и других ученых. Но, несмотря на это, множество задач для уравнения Навье-Стокса остаются неисследованными и по сей день. В этой работе мы исследуем разрешимость граничной задачи для двумерной системы Навье-Стокса в нецилиндрической вырождающейся области, а именно в конусе с вершиной в начале координат. Ранее мы изучали случаи линеаризованной системы Навье-Стокса, или невырождающихся цилиндрических областей, поэтому данная работа является логическим продолжением наших предыдущих исследований в этом направлении. Вышеупомянутой вырождающейся области мы сопоставляем семейство невырождающихся усеченных конусов, которые, в свою очередь, формируются путем взаимооднозначного преобразования в цилиндрические области, где для рассматриваемой задачи устанавливаются априорные оценки, однородные относительно изменения индекса областей. Далее, используя априорные оценки и метод Фаэдо-Галеркина, мы установили существование, единственность решения в классах Соболева и его регулярность по мере увеличения гладкости заданных функций.

Ключевые слова: система Навье-Стокса, вырождающаяся область, метод Галеркина.

References

1 Burgers, J.M. (1974). *The nonlinear diffusion equation.* Dordrecht: Reidel.

2 Batchelor, G.K. (2000). *An intronduction to fluid dynamics.* Cambridge: CUP.

3 Lions, J.-L. (1969). *Quelques methodes de resolution des problemes aux limites non lineaires.* Paris: Dunod.

4 Temam, R. (1979). *Navier-Stokes equations. Theory and numerical analysis.* Amsterdam: North Holland Publishing Company.

5 Vishik, M.I., & Fursikov, A.B. (1988). *Mathematical problems of statistical hydromechanics.* Dordrecht: Springer.

6 Mitropol'ski, Yu.A., Berezovskii, A.A., & Plotnizkii, T.A. (1992). Free boundary problems for nonlinear evolution equations in metallurgy, medicine, and ecology. *Ukrainian mathematical journal, 44*(1), 67–75.

7 Kim, E.I., Omelchenko, V.T., & Kharin, S.N. (1977). *Matematicheskie modeli teplovykh protsessov v elektricheskikh kontaktakh [Mathematical models of thermal processes in electrical contacts].* Alma-Ata: Nauka [in Russian].

8 Verigin, N.N. (1980). Ob odnom klasse gidromekhanicheskikh zadach dlia oblastei s podvizhnymi granitsami [On a class of hydromechanical problems for areas with moving boundaries]. *Dinamika zhidkosti so svobodnymi granitsami. Dinamika sploshnoi sredy — Dynamics of a liquid with free boundaries. Dynamics of a solid medium, 46*, 23–32 [in Russian].

9 Kartashov, E.M. (1997). Problema teplovogo udara v oblastiakh s dvizhushcheisia granitsei na osnove novykh integralnykh sootnoshenii [The problem of heat stroke in domains with a moving boundary based on new integral relations]. *Izvestiia Akademii nauk Rossiiskoi Federatsii. Energetika — Proceedings of the Academy of Sciences of Russian Federation. Energetics, 4*, 122–137 [in Russian].

10 Benia, Y., & Sadallah, B.-K. (2016). Existence of solutions to Burgers equations in domains that can be transformed into rectangles. *Electronic Journal of Differential Equations, 157*, 1–13.

11 Benia, Y., & Sadallah, B.-K. (2018). Existence of solutions to Burgers equations in a non-parabolic domain. *Electronic Journal of Differential Equations, 20*, 1–13.

12 Sadallah, B.-K. (1983). Etude d'un probleme 2m-parabolique dans des domaines plan non rectangulaires. *Bollettino dell'Unione Matematica Italiana, 6*(2-B), 51–112.

13 Amangaliyeva, M.M., Jenaliyev, M.T., & Ramazanov, M.I. (2017). On the existence of nontrivial solution to homogeneous Burgers equation in a corner domain. *Abstracts of Int. Conf. "Mathematics in the Modern World"*, 187.

14 Jenaliyev, M., Ramazanov, M., & Yergaliyev, M. (2017). On linear and nonlinear heat equations in degenerating domains. *AIP Conference Proceedings,* 1910.

15 Jenaliyev, M.T., Ramazanov, M.I., & Assetov, A.A. (2020). On Solonnikov-Fasano problem for the Burgers equation. *Bulletin of the Karaganda University. Mathematics series,* 2(98), 69–83. https://doi.org/10.31489/2020m2/69-83

16 Amangaliyeva, A.A., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2017). On the solvability of nonhomogeneous boundary value problem for the Burgers equation in the angular domain and related integral equations. *Springer Proceedings in Mathematics and Statistics, 216*, 123–141.

17 Kosmakova, M.I., Romanovski, V.G., Akhmanova, D.M., Tuleutaeva, Zh.M. & Bartashevich A.Yu. (2020). On the solution to a two-dimensional boundary value problem of heat conduction in a degenerating domain. *Bulletin of the Karaganda University. Mathematics series,* 2(98), 100–109. https://doi.org/10.31489/2020m2/100-109

18 Kosmakova, M.T., Izhanova, K.A., & Kasymova, L.Z. (2023). A fractionally loaded boundary value problem two-dimensional in the spatial variable. *Bulletin of the Karaganda University. Mathematics series,* 2(110), 72–83. https://doi.org/10.31489/2023m2/72-83

19 Jenaliyev, M.T., & Yergaliyev, M.G. (2023). On the solvability of a boundary value problem for a two-dimensional system of Navier-Stokes equations in a truncated cone. *Lobachevskii Journal of Mathematics, 44*(8), 3307–3320. https://doi.org/10.1134/S199508022308022X

20 Lions, J.-L., & Magenes, E. (1968). *Problemes aux limites non homogenes et applications.* Paris: Dunod.

21 Adams, R.A., & Fournier, J.J.F. (2003). *Sobolev spaces (Second Ed.).* New York: Academic Press.

22 Brezis, H. (2011). *Functional analysis, Sobolev spaces and partial differential equations.* New York: Springer.

23 Cattabriga, L. (1961). Su un problema al contorno relativo al sistema di equazioni di Stokes. *Rendiconti del Seminario Matematico della Universita di Padova, 31*, 308–340.

*Author Information**

Muvasharkhan Tanabayevich Jenaliyev — Doctor of physical and mathematical sciences, Professor, Cheif Researcher, Institute of Mathematics and Mathematical Modeling, 28 Shevchenko street, Almaty, 050010, Kazakhstan; e-mail: *muvasharkhan@gmail.com*; https://orcid.org/0000-0001-8743-7026

Akerke Mazhitkyzy Serik — PhD student, Mathematician, Institute of Mathematics and Mathematical Modeling, 28 Shevchenko street, Almaty, 050010, Kazakhstan; e-mail: *serikakerke00@gmail.com*; https://orcid.org/0000-0003-3837-4668

Madi Gabidenovich Yergaliyev (*corresponding author*) — PhD, Senior Researcher, Institute of Mathematics and Mathematical Modeling, 28 Shevchenko street, Almaty, 050010, Kazakhstan; e-mail: *ergaliev@math.kz*; https://orcid.org/0000-0001-8638-4647

*The author's name is presented in the order: First, Middle and Last Names.

https://doi.org/10.31489/2024M1/101-111 *Research article*

Some properties of the one-dimensional potentials

T.Sh. Kalmenov, A. Kadirbek*, A. Kydyrbaikyzy

Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
(E-mail: kalmenov.t@mail.ru, royale.ayan@gmail.com, atirgul.kydyrbaykyzy@bk.ru)

The main aim of this paper is to study the properties of the one-dimensional potentials. In this paper, we have studied the connection between the one-dimensional potentials and the self-adjoint part of the operator L_K^{-1}, which L_K^{-1} is the solution to the one-dimensional Cauchy problem. Moreover, a new method is used that allows us to reduce the spectral problem for the Helmholtz potential to the equivalent problem.

Keywords: one-dimensional Helmholtz potential, spectral problem, Fredholm operator.

2020 Mathematics Subject Classification: 34B05.

Introduction

One-dimensional potentials are important in the field of mathematical physics, offering insight into the behavior and characteristics of physical systems in a simplified form. The study of one-dimensional potentials also involves the analysis of eigenvalues and eigenfunctions. These concepts provide valuable information about the energy levels and corresponding wavefunctions associated with the potential.

In the study of elliptic equations, the Laplace and Helmholtz equations hold significant importance due to their wide-ranging applications and deep implications. The solutions to these equations take the form of Newton and Helmholtz potentials, respectively, which offer fundamental insights into the behavior and properties of these equations.

The Newton potential's properties have important applications in various fields, including physics and engineering. Similarly, the Helmholtz potential finds extensive utilization in electromagnetic radiation, seismology, and acoustics due to its inherent connection with the wave equation. In recent years, new methods have been discovered for investigating the potentials of the elliptic equations in multidimensional cases [1–4].

Let Ω be a bounded simply-connected domain in $\mathbb{R}^n$. In multidimensional case, a Newton potential is defined as follows:

$$u(x) = \int_\Omega \varepsilon_n(x-\xi) f(\xi) d\xi, \tag{1}$$

where

$$\varepsilon_n(x-\xi) = \begin{cases} -\frac{1}{2}\ln|x-\xi|, & n=2, \\ \frac{1}{(n-2)\sigma_n}|x-\xi|^{2-n}, & n \geq 3, \end{cases}$$

σ_n is the surface domain of the sphere in $\mathbb{R}^n$, and $\varepsilon_n(x-\xi)$ is a fundamental solution of the Laplace equation, such that

$$\Delta\varepsilon_n(x-\xi) = \delta(x-\xi),$$

here $\delta(x)$ is the Dirac delta function.

*Corresponding author. *E-mail: royale.ayan@gmail.com*

This work is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP14871460).

Received: 31 August 2023; *Accepted:* 11 October 2023.

In [1], the general form of the boundary condition for the Newton potential (1) was discovered by T.Sh. Kalmenov and D. Suragan used their method

$$N[u] = -\frac{u(x)}{2} + \int\limits_{\partial\Omega} \left(\varepsilon_n(x-\xi)\frac{\partial u(\xi)}{\partial n_\xi} - \frac{\partial \varepsilon_n(x-\xi)}{\partial n_\xi} u(\xi) \right) d\xi = 0, \quad x \in \partial\Omega, \tag{2}$$

where $\frac{\partial}{\partial n_\xi}$ is a normal derivative. In the aforementioned study, the eigenvalues and eigenfunctions of the volume potential were discovered for both the 2-disk and the 3-ball. In [2], the eigenfunctions of two-dimensional Newton potential was studied.

In the work [3], for the n-dimensional Helmholtz equation, after transforming the entire space into a finite domain in $\mathbb{R}^n$, the Sommerfeld radiation condition is transferred into a general boundary value condition with the same form as boundary value condition (2). Additional and comprehensive references on this study can be found in [5–7].

In the present paper, we will study the connections between potentials in the Cauchy problem and investigates the eigenvalue problem of the one-dimensional Helmholtz potential as a Fredholm operator, employing a novel methodology. Furthermore, we will analyze the relationship between one-dimensional Newton potential and the solution of the Cauchy problem.

1 *Main functional relations*

It is well known that the one-dimensional Newton potential is defined as follows

$$u(x) = \int\limits_a^b \frac{|x-\xi|}{2} f(\xi)d\xi, \quad x \in (a,b) \subset \mathbb{R},$$

and satisfies the following Poisson equation

$$Lu = \frac{d^2}{dx^2}u(x) = f(x).$$

Let $f \in L_2(a,b)$, then we can find the solution of the one-dimensional Cauchy problem in the following form

$$u_K(x) = L_K^{-1} f := \int\limits_a^x (x-\xi) f(\xi) d\xi.$$

An adjoint operator to L_K^{-1} is

$$\left(L_K^{-1}\right)^* f := \int\limits_x^b (\xi - x) f(\xi) d\xi. \tag{3}$$

By using the Cartesian theorem for operators, we can rewrite operator (3) as follows

$$L_K^{-1} f = \mathfrak{Re}\left(L_K^{-1}\right) f + i\cdot \mathfrak{Im}\left(L_K^{-1}\right) f = \frac{L_K^{-1} + (L_K^{-1})^*}{2} f + i\frac{L_K^{-1} - (L_K^{-1})^*}{2i} f.$$

The operators $\mathfrak{Re}\left(L_K^{-1}\right)$ and $\mathfrak{Im}\left(L_K^{-1}\right)$ are respectively the real part and image part of the operator L_K^{-1}, and they are self-adjoint operators.

It is easily seen that the real part of the operator L_K^{-1} coincides with the one-dimensional Newton potential such that

$$\frac{L_K^{-1} + (L_K^{-1})^*}{2} f = \int\limits_a^b \frac{|x-\xi|}{2} f(\xi) d\xi.$$

Note that we have related the one-dimensional Newton potential to the classical Cauchy problem.

The proof above gives more, namely, we can generalize this fact to high-order differential equations.

Let us consider the following self-adjoint linear differential equation

$$Lu := \frac{d^{(2m)}}{dx^{(2m)}}u(x) + a_1\frac{d^{(2m-2)}}{dx^{(2m-2)}}u(x) + ... + a_m u(x) = f(x), \quad m \in \mathbb{N}, \quad x \in (a,b), \tag{4}$$

where coefficients $a_1, ..., a_m \in \mathbb{R}$ are constants.

By the Malgrange-Ehrenpreis theorem [8] we know that equation (4) has a fundamental solution. Now, we will construct the fundamental solution of the operator (4).

Lemma 1. Let $z(x)$ is a solution of the following homogeneous equation

$$Lz = 0,$$

which satisfies the conditions

$$\begin{aligned} &z(a) = z'(a) = ... = z^{(2m-2)}(a) = 0; \\ &z^{(2m-1)}(a) = 1. \end{aligned} \tag{5}$$

Then the function

$$\varepsilon(x) = \frac{1}{2}\mathrm{sgn}(x) \cdot z(x) \tag{6}$$

is a fundamental solution of the operator L.

Proof. By using the weak derivatives and properties of distributions, and taking into account conditions (5), we have

$$\begin{aligned} \varepsilon'(x) &= \delta(x) \cdot z(x) + \frac{1}{2}\mathrm{sgn}(x) \cdot z'(x) = \frac{1}{2}\mathrm{sgn}(x) \cdot z'(x), \\ \varepsilon''(x) &= \delta(x) \cdot z'(x) + \frac{1}{2}\mathrm{sgn}(x) \cdot z''(x) = \frac{1}{2}\mathrm{sgn}(x) \cdot z''(x), \\ &\cdots \\ \varepsilon^{(2m-2)}(x) &= \frac{1}{2}\mathrm{sgn}(x) \cdot z^{(2m-2)}(x), \\ \varepsilon^{(2m)}(x) &= \delta(x) \cdot z^{(2m-1)}(x) + \frac{1}{2}\mathrm{sgn}(x) \cdot z^{(2m)}(x) \\ &= \delta(x) + \frac{1}{2}\mathrm{sgn}(x) \cdot z^{(2m)}(x). \end{aligned} \tag{7}$$

We see at once that

$$\begin{aligned} L\varepsilon &= \frac{d^{(2m)}}{dx^{(2m)}}\varepsilon(x) + a_1\frac{d^{(2m-2)}}{dx^{(2m-2)}}\varepsilon(x) + ... + a_m\varepsilon(x) \\ &= \frac{1}{2}\mathrm{sgn}(x) \cdot Lz + \delta(x) \\ &= \delta(x), \end{aligned}$$

which is clear from (7).

Using fundamental solution (6) we can write the potential of equation (4) as follows

$$u(x) = L^{-1}f = \varepsilon * f = \frac{1}{2}\int_a^b \operatorname{sgn}(x-\xi)\cdot z(x-\xi)f(\xi)d\xi. \tag{8}$$

On the other hand, the solution of the Cauchy problem with zero-conditions of equation (4) is

$$u_K(x) = L_K^{-1}f = \int_a^x z(x-\xi)f(\xi)d\xi,$$

and

$$\left(L_K^{-1}\right)^* f = \int_x^b z(\xi-x)f(\xi)d\xi.$$

By the Cartesian theorem, the real part of the operator L_K^{-1} is

$$\begin{aligned}\mathfrak{Re}\left(L_K^{-1}\right) f =& \frac{L_K^{-1} + \left(L_K^{-1}\right)^*}{2} f = \\ =& \frac{1}{2}\left[\int_a^x z(x-\xi)f(\xi)d\xi + \int_x^b z(\xi-x)f(\xi)d\xi\right].\end{aligned} \tag{9}$$

Obviously $z(x)$ is an odd function, i.e. $z(-x) = -z(x)$, therefore we can rewrite (9) as

$$\begin{aligned}\mathfrak{Re}\left(L_K^{-1}\right) f =& \frac{1}{2}\left[\int_a^x z(x-\xi)f(\xi)d\xi + \int_x^b z(\xi-x)f(\xi)d\xi\right] = \\ =& \frac{1}{2}\left[\int_a^x \operatorname{sgn}(x-\xi)\cdot z(x-\xi)f(\xi)d\xi + \int_x^b \operatorname{sgn}(x-\xi)\cdot z(x-\xi)f(\xi)d\xi\right] = \\ =& \frac{1}{2}\int_a^b \operatorname{sgn}(x-\xi)\cdot z(x-\xi)f(\xi)d\xi,\end{aligned}$$

which proves the following theorem.

Theorem 1. Potential (8) and the real part of the solution of the Cauchy problem for the equation (4) are equal:

$$\mathfrak{Re}\left(L_K^{-1}\right) f = \frac{L_K^{-1} + \left(L_K^{-1}\right)^*}{2} f = L^{-1}f = \frac{1}{2}\int_a^b \operatorname{sgn}(x-\xi)\cdot z(x-\xi)f(\xi)d\xi.$$

Example. If $a_1 = a_2 = \ldots = a_m = 0$ in (4), which we may assume, then we have the following polyharmonic equation

$$Lu = \Delta^m u(x) = \frac{d^{2m}}{dx^{2m}}u(x) = f(x), \quad x \in (a,b) \subset \mathbb{R}. \tag{10}$$

By using (6) and (8) the polyharmonic Newton potential is given by

$$u(x) = L^{-1}f = \frac{1}{2}\int_a^b \frac{1}{(2m-1)!}|x-\xi|^{2m-1}f(\xi)d\xi.$$

We find the solution of the Cauchy problem for equation (10) as

$$u_K(x) = L_K^{-1}f = \frac{1}{(2m-1)!}\int_a^x (x-\xi)^{2m-1}f(\xi)d\xi,$$

and

$$\left(L_K^{-1}\right)^* f = \frac{1}{(2m-1)!}\int_x^b (\xi-x)^{2m-1}f(\xi)d\xi,$$

where $\left(L_K^{-1}\right)^*$ is an adjoint operator to the operator L_K^{-1}. By direct calculation we obtain

$$\begin{aligned}\mathfrak{Re}\left(L_K^{-1}\right)f =& \frac{L_K^{-1}+\left(L_K^{-1}\right)^*}{2}f = \\ =& \frac{1}{2}\frac{1}{(2m-1)!}\left[\int_a^x (x-\xi)^{2m-1}f(\xi)d\xi + \int_x^b (\xi-x)^{2m-1}f(\xi)d\xi\right] = \\ =& \frac{1}{2}\int_a^b \frac{1}{(2m-1)!}|x-\xi|^{2m-1}f(\xi)d\xi = L^{-1}f,\end{aligned}$$

and $\mathfrak{Re}\left(L_K^{-1}\right)f = L^{-1}f$ as claimed.

2 Spectral problem for Helmholtz potential

Let us consider a one-dimensional Helmholtz equation in $(a,b)\subset\mathbb{R}$

$$Lu = -\frac{d^2}{dx^2}u(x) - k^2u(x) = f(x), \quad x\in(a,b). \tag{11}$$

It is easy to check that a particular solution to the Helmholtz equation is defined as a one-dimensional Helmholtz potential [9]

$$u(x) = -\frac{1}{2}\int_a^b \frac{\sin\left(k|x-\xi|\right)}{k}f(\xi)d\xi, \tag{12}$$

here $\varepsilon_1(x-\xi) := -\frac{1}{2}\frac{\sin(k|x-\xi|)}{k}$ is a fundamental solution of the Helmholtz equation, i.e.

$$-\frac{d^2}{dx^2}\left(\varepsilon_1(x-\xi)\right) - k^2\varepsilon_1(x-\xi) = \delta(x-\xi).$$

In [10] there are the considered boundary conditions of operator (12) with this fundamental solution and with $\frac{e^{ik|x|}}{2ik}$. In this work, we will study the integral operator with $\varepsilon_1(x-\xi) := -\frac{1}{2}\frac{\sin(k|x-\xi|)}{k}$.

Lemma 2. Let $f \in C[a,b]$. Then there is a unique solution to equation (11), defined by the Helmholtz potential (12), and satisfies the following boundary conditions:

$$\begin{aligned} N_1[u] =& \frac{1}{k}\cos(bk)u'(b) + \frac{1}{k}\cos(ak)u'(a) + \sin(bk)u(b) + \sin(ak)u(a) = 0; \\ N_2[u] =& \frac{1}{k}\sin(bk)u'(b) + \frac{1}{k}\sin(ak)u'(a) - \cos(bk)u(b) - \cos(ak)u(a) = 0. \end{aligned} \tag{13}$$

Proof. Replacing $f(\xi)$ by $-\frac{d^2}{d\xi^2}u(\xi) - k^2u(\xi)$ in (12) we can rewrite (12) as

$$\begin{aligned} u(x) =& -\frac{1}{2}\int_a^b \frac{\sin(k|x-\xi|)}{k} f(\xi)d\xi = \\ =& \frac{1}{2}\int_a^b \frac{\sin(k|x-\xi|)}{k} \left(\frac{d^2}{d\xi^2}u(\xi) + k^2u(\xi)\right)d\xi, \end{aligned}$$

hence using integration by parts, we have

$$\begin{aligned} u(x) =& \int_a^b \frac{\sin(k|x-\xi|)}{2k}\left(\frac{d^2}{d\xi^2}u(\xi)\right)d\xi + k^2\int_a^b \frac{\sin(k|x-\xi|)}{2\lambda}u(\xi)d\xi = \\ =& \frac{\sin(k|x-\xi|)}{2k}\frac{d}{d\xi}u(\xi)\Big|_a^b - u(\xi)\frac{d}{d\xi}\left(\frac{\sin(k|x-\xi|)}{2k}\right)\Big|_a^b + \int_a^b \frac{d^2}{d\xi^2}\left(\frac{\sin(k|x-\xi|)}{2k}\right)u(\xi)d\xi + \\ &+k^2\int_a^b \frac{\sin(k|x-\xi|)}{2k}u(\xi)d\xi, \end{aligned}$$

since

$$\frac{d}{d\xi}\left(\frac{\sin(k|x-\xi|)}{2k}\right) = -\frac{\cos(k|x-\xi|)}{2}\cdot \mathrm{sgn}(x-\xi),$$

we obtain

$$\begin{aligned} u(x) =& \frac{\sin(k|x-\xi|)}{2k}\frac{d}{d\xi}u(\xi)\Big|_a^b + u(\xi)\frac{\cos(k|x-\xi|)}{2}\mathrm{sgn}(x-\xi)\Big|_a^b + \int_a^b \delta(x-\xi)u(\xi)d\xi = \\ =& \frac{\sin(k|x-\xi|)}{2k}\frac{d}{d\xi}u(\xi)\Big|_a^b + u(\xi)\frac{\cos(k|x-\xi|)}{2}\mathrm{sgn}(x-\xi)\Big|_a^b + u(x), \end{aligned}$$

we see that the same terms in the equality cancel out, and it follows that

$$\begin{aligned} &\frac{\sin(k|x-\xi|)}{2k}\frac{d}{d\xi}u(\xi)\Big|_a^b + u(\xi)\frac{\cos(k|x-\xi|)}{2}\mathrm{sgn}(x-\xi)\Big|_a^b = \\ =& \frac{\sin(k|x-b|)}{2k}\frac{d}{d\xi}u(b) - \frac{\sin(k|x-a|)}{2k}\frac{d}{d\xi}u(a) + \\ &+u(b)\frac{\cos(k|x-b|)}{2}\mathrm{sgn}(x-b) - u(a)\frac{\cos(k|x-a|)}{2}\mathrm{sgn}(x-a) = 0. \end{aligned}$$

Since $a < x < b$, we have

$$-\frac{\sin(k(x-b))}{2k}\frac{d}{d\xi}u(b) - \frac{\sin(k(x-a))}{2k}\frac{d}{d\xi}u(a) - \frac{\cos(k(x-b))}{2}u(b) - \frac{\cos(k(x-a))}{2}u(a) =$$
$$=\frac{1}{2}\sin(kx)\left(-\frac{1}{k}\cos(kb)u'(b) - \frac{1}{k}\cos(ka)u'(a) - \sin(kb)u(b) - \sin(ka)u(a)\right) +$$
$$+\frac{1}{2}\cos(kx)\left(\frac{1}{k}\sin(kb)u'(b) + \frac{1}{k}\sin(ka)u'(a) - \cos(kb)u(b) - \cos(ka)u(a)\right) = 0.$$

Since $\sin(kx)$ and $\cos(kx)$ are linearly independent, then we obtain the general conditions for the one-dimensional Newton potential as follows:

$$N_1[u] = \frac{1}{k}\cos(kb)u'(b) + \frac{1}{k}\cos(ka)u'(a) + \sin(kb)u(b) + \sin(ka)u(a) = 0;$$
$$N_2[u] = \frac{1}{k}\sin(kb)u'(b) + \frac{1}{k}\sin(ka)u'(a) - \cos(kb)u(b) - \cos(ka)u(a) = 0.$$

On the other hand, the general solution to equation (11) is given by

$$u(x) = -\int_a^x \frac{\sin(k(x-\xi))}{k}f(\xi)d\xi + c_1e^{ikx} + c_2e^{-ikx}, \tag{14}$$

boundary conditions (13) determine c_1 and c_2:

$$c_1 = \frac{1}{4}\int_a^b \frac{e^{-ik\xi}}{ik}f(\xi)d\xi, \quad c_2 = -\frac{1}{4}\int_a^b \frac{e^{ik\xi}}{ik}f(\xi)d\xi. \tag{15}$$

Substituting (15) into (14) we can assert that

$$u(x) = -\int_a^x \frac{\sin(k(x-\xi))}{k}f(\xi)d\xi + \frac{1}{4}\int_a^b \frac{e^{ik(x-\xi)}}{ik}f(\xi)d\xi - \frac{1}{4}\int_a^b \frac{e^{-ik(x-\xi)}}{ik}f(\xi)d\xi =$$
$$= -\int_a^x \frac{\sin(k(x-\xi))}{k}f(\xi)d\xi + \frac{1}{2}\int_a^b \frac{\sin(k(x-\xi))}{k}f(\xi)d\xi =$$
$$= -\frac{1}{2}\int_a^b \frac{\sin(k|x-\xi|)}{k}f(\xi)d\xi.$$

The proof is completed.

Now, we will consider the spectral problem for potential (12) as Fredholm integral operator

$$-\frac{1}{2}\int_0^1 \frac{\sin(k|x-\xi|)}{k}u(\xi)d\xi = \frac{u(x)}{\lambda}, \quad x \in (0,1). \tag{16}$$

Applying lemma 2 we conclude that operator (11) with conditions (13) is inverse operator to potential (12). Therefore, eigenvalue problem (16) and the following problem is equivalent

$$-\frac{d^2}{dx^2}u(x) - k^2u(x) = \lambda u(x) \tag{17}$$

that satisfies the conditions [10]

$$\begin{aligned} N_1[u] &= \frac{1}{k}\cos(k)u'(1) + \frac{1}{k}u'(0) + \sin(k)u(1) = 0, \\ N_2[u] &= \frac{1}{k}\sin(k)u'(1) - \cos(k)u(1) - u(0) = 0. \end{aligned} \tag{18}$$

Since operator (12) is a self-adjoint operator, then it always has real eigenvalues. It follows the problem (17)-(18) also has the real eigenvalue.

A solution to equation (17) is

$$u(x) = C_1 e^{ix\sqrt{\lambda+k^2}} + C_2 e^{-ix\sqrt{\lambda+k^2}}.$$

The conditions (18) implies that

$$\begin{bmatrix} \frac{1}{k}i\sqrt{\lambda+k^2}\cos(k)e^{i\sqrt{\lambda+k^2}} + \frac{1}{k}i\sqrt{\lambda+k^2} + \sin(k)e^{i\sqrt{\lambda+k^2}} \\ \frac{1}{k}i\sqrt{\lambda+k^2}\sin(k)e^{i\sqrt{\lambda+k^2}} - \cos(k)e^{i\sqrt{k^2+\lambda}} - 1 \\ -\frac{1}{k}i\sqrt{\lambda+k^2}\cos(k)e^{-i\sqrt{\lambda+k^2}} - \frac{1}{k}i\sqrt{\lambda+k^2} + \sin(k)e^{-i\sqrt{\lambda+k^2}} \\ -\frac{1}{k}i\sqrt{\lambda+k^2}\sin(k)e^{-i\sqrt{\lambda+k^2}} - \cos(k)e^{-i\sqrt{\lambda+k^2}} - 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0.$$

To have non-trivial solutions of this system, the determinant of the left-hand side matrix must be zero, then

$$\begin{aligned} &-4i\frac{1}{k}\sqrt{\lambda+k^2} - \left(\frac{1}{k^2}(\lambda+k^2)+1\right)\sin(k)\left(e^{i\sqrt{\lambda+k^2}} - e^{-i\sqrt{\lambda+k^2}}\right) - \\ &-\frac{2}{k}i\sqrt{\lambda+k^2}\cos(k)\left(e^{i\sqrt{\lambda+k^2}} + e^{-i\sqrt{\lambda+k^2}}\right) = 0, \end{aligned} \tag{19}$$

equation (19) is a transcendental equation for eigenvalues of problem (17)-(18).

1) If $\lambda + k^2 > 0$, by Euler's formula, we get

$$\begin{aligned} &2\sqrt{\lambda+k^2} + \frac{1}{k}(\lambda+k^2)\sin(k)\sin\left(\sqrt{\lambda+k^2}\right) + \\ &+2\sqrt{\lambda+k^2}\cos(k)\cos\left(\sqrt{\lambda+k^2}\right) + k\sin(k)\sin\left(\sqrt{\lambda+k^2}\right) = 0. \end{aligned} \tag{20}$$

From (20) let $\sqrt{\lambda+k^2} = \pi n + \alpha_n$, here $\alpha_n \to 0$ as $n \to \infty$, then we obtain

$$\frac{1}{2}\left(\frac{k}{\pi n+\alpha_n} + \frac{\pi n+\alpha_n}{k}\right)(-1)^n \sin\alpha_n = -\frac{1+(-1)^n\cos k\cos\alpha_n}{\sin k},$$

it follows that as $n \to \infty$

$$n\cdot\sin\alpha_n = -2k\frac{1+(-1)^n\cos k}{(-1)^n\pi\sin k},$$

thus we have α_n as

$$\alpha_n = -\frac{1+(-1)^n\cos k}{(-1)^n\pi\sin k}\cdot\frac{2k}{n}.$$

Then we can obtain asymptotic behavior of the eigenvalue λ as $n \to \infty$

$$\lambda_n = (\pi n+\alpha_n)^2 - k^2 = \pi^2 n^2 - k^2 - \frac{4k}{(-1)^n}\left(\frac{1+(-1)^n\cos(k)}{\sin(k)}\right) + O\left(\frac{1}{n^2}\right). \tag{21}$$

2) Now we turn to the case $\lambda + k^2 < 0$. Denote by $\mu = i\sqrt{\lambda + k^2} = \sqrt{-\lambda - k^2}$, of course $\mu > 0$. In this case, we can rewrite (19) as

$$-4\mu + \sin(k)\left(\frac{1}{k}\mu^2 - k\right)\left(e^{\mu} - e^{-\mu}\right) - 2\mu\cos(k)\left(e^{\mu} + e^{-\mu}\right) = 0. \tag{22}$$

It is easily seen that equation (22) has only one root

$$\lambda = -\mu^2 - k^2.$$

However, this equation does not have a simple analytical solution, and graphical methods may be needed to approximate the root.

Thus, we described the eigenvalues of the Helmholtz potential.

Theorem 2. The eigenvalues of problem (16) for $\lambda + k^2 > 0$ are the roots of transcendental equation (20) and asymptotic behavior as $n \to \infty$ has the form (21), for $\lambda + k^2 < 0$ has only one eigenvalue as

$$\lambda = -\mu - k^2,$$

where μ is a root of equation (22).

Acknowledgments

This work is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP14871460).

Author Contributions

T.Sh. Kalmenov served as the principal investigator of the research grant and supervised the research process. A. Kadirbek and A. Kydyrbaikyzy contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Kal'menov T.Sh. To spectral problems for the volume potential / T.Sh. Kal'menov, D. Suragan // Doklady Mathematics. — 2009. — 80. — No. 2. — P. 646–649. https://doi.org/10.1134/S1064562409050032

2 Saito N. Data analysis and representation on a general domain using eigenfunctions of Laplacian / N. Saito // Applied and Computational Harmonic Analysis. — 2008. — 25. — No. 1. — P. 68–97. https://doi.org/10.1016/j.acha.2007.09.005

3 Kalmenov T.S. The Sommerfeld problem and inverse problem for the Helmholtz equation / T.S. Kalmenov, S.I. Kabanikhin, A. Les // Journal of Inverse and Ill-posed Problems. — 2021. — 29. — No. 1. — P. 49–64. https://doi.org/10.1515/jiip-2020-0033

4 Kal'menov T.Sh. Boundary conditions for the volume potential for the polyharmonic equation / T.Sh. Kal'menov, D. Suragan // Differential Equations. — 2012. — 48. — No. 4. — P. 604–608.

5 Kal'menov T.Sh. On a nonlocal boundary value problem for the multidimensional heat equation in a noncylindrical domain / T.Sh. Kal'menov, N.E. Tokmagambetov // Siberian Mathematical Journal. — 2013. — 54. — No. 6. — P. 1023–1028. https://doi.org/10.1134/S0037446613060086

6 Kal'menov T.Sh. A criterion for the existence of soliton solutions of telegraph equation / T.Sh. Kal'menov, G.D. Arepova // Bulletin of the Karaganda University. Mathematics series. Spec. Issue. — 2018. — No. 3(91). — P. 45–52. https://doi.org/10.31489/2018M3/45-52

7 Kal'menov T.Sh. On permeable potential boundary conditions for the Laplace–Beltrami operator. / T.Sh. Kal'menov, D. Suragan // Siberian Mathematical Journal. — 2015. — 56. — Iss. 6. — P. 1060–1064. https://doi.org/10.1134/S0037446615060099

8 Hormander L. The analysis of linear partial differential operators I: Distribution theory and Fourier analysis / L. Hormander. — Springer, 2015.

9 Владимиров В.С. Уравнения математической физики. — 4-е изд. / В.С. Владимиров. — М.: Наука. Глав. ред. физ.-мат. литературы, 1981. — 512 с.

10 Kakharman N. On one-dimensional Helmholtz equation / N. Kakharman // Journal of Mathematics, Mechanics and Computer Science. — 2023. — 118. — No. 2. — P. 21–29. https://doi.org/10.26577/JMMCS.2023.v118.i2.03

Бірөлшемді потенциалдардың кейбір қасиеттері

Т.Ш. Кальменов, А. Кәдірбек, Ә. Қыдырбайқызы

Математика және математикалық модельдеу институты, Алматы, Қазақстан

Мақаланың негізгі мақсаты бірөлшемді потенциалдардың қасиеттерін зерттеу. Авторлар бірөлшемді потенциалдар мен бірөлшемді Коши есебінің шешімі болатын L_K^{-1} операторының өзіне-өзі түйіндес бөлігінің арасындағы байланысты зерттеген. Сонымен қатар, жаңа әдіс арқылы Гельмгольц потенциалының спектральды мәселесі эквикалентті есепке келтірілді.

Кілт сөздер: бірөлшемді Гельмгольц потенциалы, спектральды мәселе, Фредгольм операторы.

Некоторые свойства одномерных потенциалов

Т.Ш. Кальменов, А. Кадирбек, А. Кыдырбайкызы

Институт математики и математического моделирования, Алматы, Казахстан

Основной целью данной работы является изучение свойств одномерных потенциалов. В статье мы исследовали связь одномерных потенциалов с самосопряженной частью оператора L_K^{-1}, который является решением одномерной задачи Коши. Более того, использован новый метод, позволяющий свести спектральную задачу для потенциала Гельмгольца к эквивалентной задаче.

Ключевые слова: одномерный потенциал Гельмгольца, спектральная проблема, оператор Фредгольма.

References

1 Kal'menov, T.Sh., & Suragan, D. (2009). To spectral problems for the volume potential. *Doklady Mathematics, 80*(2), 646–649. https://doi.org/10.1134/S1064562409050032

2 Saito, N. (2008). Data analysis and representation on a general domain using eigenfunctions of Laplacian. *Applied and Computational Harmonic Analysis, 25*(1), 68–97. https://doi.org/10.1016/j.acha.2007.09.005

3 Kalmenov, T.S., Kabanikhin, S.I., & Les, A. (2021). The Sommerfeld problem and inverse problem for the Helmholtz equation. *Journal of Inverse and Ill-posed Problems, 29*(1), 49–64. https://doi.org/10.1515/jiip-2020-0033

4 Kal'menov, T.Sh., & Suragan, D. (2012). Boundary conditions for the volume potential for the polyharmonic equation. *Differential Equations, 48*(4), 604–608.

5 Kal'menov, T.Sh., & Tokmagambetov, N.E. (2013). On a nonlocal boundary value problem for the multidimensional heat equation in a noncylindrical domain. *Siberian Mathematical Journal, 54*(6), 1023–1028. https://doi.org/10.1134/S0037446613060086

6 Kal'menov, T.Sh., & Arepova, G.D. (2018). A criterion for the existence of soliton solutions of telegraph equation. *Bulletin of the Karaganda University. Mathematics series. Spec. Issue,* 3(91), 45–52. https://doi.org/10.31489/2018M3/45-52

7 Kal'menov, T.Sh., & Suragan, D. (2015). On permeable potential boundary conditions for the Laplace–Beltrami operator. *Siberian Mathematical Journal, 56*(6), 1060–1064. https://doi.org/10.1134/S0037446615060099

8 Hormander, L. (2015). *The analysis of linear partial differential operators I: Distribution theory and Fourier analysis.* Springer.

9 Vladimirov, V.S. (1981). *Uravneniia matematicheskoi fiziki. — 4-e izdanie [Equations of mathematical physics].* Moscow: Nauka. Glavnaia redaktsiia fiziko-matematicheskoi literatury [in Russian].

10 Kakharman, N. (2023). On one-dimensional Helmholtz equation. *Journal of Mathematics, Mechanics and Computer Science, 118*(2), 21–29. https://doi.org/10.26577/JMMCS.2023.v118.i2.03

*Author Information**

Tynysbek Sharipovich Kalmenov — Doctor of physical and mathematical sciences, Academic of National Academy of Sciences of the Republic of Kazakhstan, Chief Researcher, Institute of Mathematics and Mathematical Modeling, 125 Pushkin street, Almaty, 050010, Kazakhstan; e-mail: *kalmenov.t@mail.ru*; https://orcid.org/0000-0002-1821-2015

Ayan Kadirbek (*corresponding author*) — Bachelor, Specialist, Institute of Mathematics and Mathematical Modeling, 125 Pushkin street, Almaty, 050010, Kazakhstan; email: *royale.ayan@gmail.com.*

Atirgul Kydyrbaikyzy — Bachelor, Specialist, Institute of Mathematics and Mathematical Modeling, 125 Pushkin street, Almaty, 050010, Kazakhstan; email: *atirgul.kydyrbaykyzy@bk.ru.*

*The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics series, No. 1(113), 2024, pp. 112–127

https://doi.org/10.31489/2024M1/112-127 *Research article*

Algebras of binary formulas for $\aleph_0$-categorical weakly circularly minimal theories: monotonic case

B.Sh. Kulpeshov[1,2,*], S.V. Sudoplatov[3,4]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *Kazakh British Technical University, Almaty, Kazakhstan;*
[3] *Novosibirsk State Technical University, Novosibirsk, Russia;*
[4] *Sobolev Institute of Mathematics, Novosibirsk, Russia*
(E-mail: b.kulpeshov@kbtu.kz, sudoplat@math.nsc.ru)

This article concerns the notion of weak circular minimality being a variant of o-minimality for circularly ordered structures. Algebras of binary isolating formulas are studied for countably categorical weakly circularly minimal theories of convexity rank greater than 1 having both a 1-transitive non-primitive automorphism group and a non-trivial strictly monotonic function acting on the universe of a structure. On the basis of the study, the authors present a description of these algebras. It is shown that there exist both commutative and non-commutative algebras among these ones. A strict m-deterministicity of such algebras for some natural number m is also established.

Keywords: circularly ordered structure, binary formula, isolating formula, algebra of formulas, $\aleph_0$-categorical theory, weak circular minimality, convexity rank, automorphism group, transitivity, primitiveness, m-deterministicity.

2020 Mathematics Subject Classification: 03C64.

1 Preliminaries

Algebras of binary formulas are a tool for describing relationships between elements of the sets of realizations of a type at the binary level with respect to the superposition of binary definable sets. A *binary isolating formula* is a formula of the form $\varphi(x, y)$ such that for some parameter a the formula $\varphi(a, y)$ isolates a complete type in $S(\{a\})$. The concepts and notations related to these algebras can be found in the papers [1, 2]. In recent years, algebras of binary formulas have been studied intensively and have been continued in the works [3–7].

Let L be a countable first-order language. Throughout we consider L-structures and assume that L contains a ternary relational symbol K, interpreted as a circular order in these structures (unless otherwise stated).

Let $\mathcal{M} = \langle M, \leq\rangle$ be a linearly ordered set. If we connect two endpoints of $\mathcal{M}$ (possibly, $-\infty$ and $+\infty$), then we obtain a circular order. More formally, the *circular order* is described by a ternary relation K satisfying the following conditions:

(co1) $\forall x\forall y\forall z(K(x, y, z) \to K(y, z, x))$;

(co2) $\forall x\forall y\forall z(K(x, y, z) \land K(y, x, z) \Leftrightarrow x = y \lor y = z \lor z = x)$;

(co3) $\forall x\forall y\forall z(K(x, y, z) \to \forall t[K(x, y, t) \lor K(t, y, z)])$;

(co4) $\forall x\forall y\forall z(K(x, y, z) \lor K(y, x, z))$.

Sometimes we will identify $\mathcal{M}$ and the universe M if a linear/circular order is fixed.

*Corresponding author. *E-mail: b.kulpeshov@kbtu.kz*

The work was supported by Science Committee of Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19674850), and in the framework of the State Contract of the Sobolev Institute of Mathematics, Project No. FWNF-2022-0012.

Received: 17 September 2023; *Accepted:* 04 December 2023.

The notion of *weak circular minimality* was studied initially in [8]. Let $A \subseteq M$, where $\mathcal{M}$ is a circularly ordered structure. The set A is called *convex* if for any $a, b \in A$ the following property is satisfied: for any $c \in M$ with $K(a,c,b)$, $c \in A$ holds, or for any $c \in M$ with $K(b,c,a)$, $c \in A$ holds. A *weakly circularly minimal structure* is a circularly ordered structure $\mathcal{M} = \langle M, K, \ldots \rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in $\mathcal{M}$. Recall [9] that such a structure $\mathcal{M}$ is called *circularly minimal* if any definable (with parameters) of M is a union of finitely many intervals and points in $\mathcal{M}$. Clearly, the weak circular minimality is a generalization of circular minimality. Notice also that any weakly o-minimal structure is weakly circular minimal. The converse, in general, fails. The study of weakly circularly minimal structures was continued in the papers [10–16].

Let $\mathcal{M}$ be an $\aleph_0$-categorical weakly circularly minimal structure, $G := \mathrm{Aut}(\mathcal{M})$. Following the standard group theory terminology, the group G is called *k-transitive* if for any pairwise distinct $a_1, a_2, \ldots, a_k \in M$ and pairwise distinct $b_1, b_2, \ldots, b_k \in M$ there exists $g \in G$ such that $g(a_1) = b_1$, $g(a_2) = b_2, \ldots, g(a_k) = b_k$. A *congruence* on $\mathcal{M}$ is an arbitrary G-invariant equivalence relation on $\mathcal{M}$. The group G is called *primitive* if G is 1-transitive and there are no non-trivial proper congruences on $\mathcal{M}$.

Let $\mathcal{M}$, $\mathcal{N}$ be circularly ordered structures. The 2-*reduct* of $\mathcal{M}$ is a circularly ordered structure with the same universe of $\mathcal{M}$ and consisting of predicates for each $\emptyset$-definable relation on $\mathcal{M}$ of arity ≤ 2 as well as of the ternary predicate K for the circular order, but does not have other predicates of arities more than two. We say that the structure $\mathcal{M}$ is *isomorphic* to $\mathcal{N}$ *up to binarity* or *binarily isomorphic to* $\mathcal{N}$ if the 2-reduct of $\mathcal{M}$ is isomorphic to the 2-reduct of $\mathcal{N}$.

Notation. (1) $K_0(x,y,z) := K(x,y,z) \wedge y \neq x \wedge y \neq z \wedge x \neq z$.

(2) $K(u_1, \ldots, u_n)$ denotes a formula saying that all subtuples of the tuple $\langle u_1, \ldots, u_n \rangle$ having the length 3 (in ascending order) satisfy K; similar notations are used for K_0.

(3) Let A, B, C be disjoint convex subsets of a circularly ordered structure $\mathcal{M}$. We write $K(A,B,C)$ if for any $a, b, c \in M$ with $a \in A$, $b \in B$, $c \in C$ we have $K(a,b,c)$. We extend naturally that notation using, for instance, the notation $K_0(A,d,B,C)$ if $d \notin A \cup B \cup C$ and $K_0(A,d,B) \wedge K_0(d,B,C)$ holds.

Let $f : M \to M$ be an $\emptyset$-definable function with $\mathrm{Dom}(f) = I \subseteq M$, where I is an open convex set. We say that f is *monotonic-to-right (left) on I* if it preserves (reverses) the relation K_0, i.e. for any $a, b, c \in I$ such that $K_0(a,b,c)$ we have $K_0(f(a), f(b), f(c))$ $(K_0(f(c), f(b), f(a)))$.

The following definition can be used in a circular ordered structure as well.

Definition 1. [17, 18] Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T, $A \subseteq M$. *The rank of convexity of the set A* ($RC(A)$) is defined as follows:

1) $RC(A) = -1$ if $A = \emptyset$.

2) $RC(A) = 0$ if A is finite and non-empty.

3) $RC(A) \geq 1$ if A is infinite.

4) $RC(A) \geq \alpha + 1$ if there exists a parametrically definable equivalence relation $E(x,y)$ and an infinite sequence of elements $b_i \in A$, $i \in \omega$ such that:

- For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
- For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.

5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A)) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

The rank of convexity of an 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

In particular, a theory has convexity rank 1 if there are no definable (with parameters) equivalence relations with infinitely many infinite convex classes.

The following theorem characterizes $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structures of convexity rank greater than 1 having a non-trivial strictly monotonic function up to binarity:

Theorem 1. [11] (monotonic case) Let M be an $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater than 1 having a non-trivial strictly monotonic function so that $dcl(a) \neq \{a\}$ for some $a \in M$. Then M is isomorphic up to binarity to $M_{s,m,k} := \langle M,$ K, f^1, $E_1^2, \ldots, E_s^2, E_{s+1}^2\rangle$, where

- M is a circularly ordered structure, M is densely ordered, $s \geq 1$, $k \geq 2$, $m = 1$ or k divides m;
- E_{s+1} is an equivalence relation partitioning M into m infinite convex classes without endpoints, for every $1 \leq i \leq s$ the relation E_i is an equivalence relation partitioning every E_{i+1}–class into infinitely many infinite convex E_i–subclasses without endpoints so that the induced order on E_i–subclasses is dense without endpoints;
- f is a bijection on M so that $f^k(a) = a$ for any $a \in M$, for every $1 \leq i \leq s+1$ $f(E_i(M, a)) = E_i(M, f(a))$ and $\neg E_i(a, f(a))$, and either f is monotonic-to-right on M or f is monotonic-to-left on M (and in this case $k = m = 2$).

In [19] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories with a primitive automorphism group. In [20] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. Here we describe algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and having a non-trivial strictly monotonic function.

2 Results

Example 1. Consider the structure $M_{1,1,2} := \langle M, K^3, f^1, E^2\rangle$ from Theorem 1, where f is monotonic-to-right on M, E is an equivalence relation partitioning M into infinitely many infinite convex classes. We assert that $Th(M_{1,1,2})$ has eight binary isolating formulas:

$$\theta_0(x,y) := x = y, \theta_1(x,y) := K_0(x,y,f(x)) \wedge E(x,y),$$

$$\theta_2(x,y) := K_0(x,y,f(x)) \wedge \neg E(x,y) \wedge \neg E(f(x),y),$$

$$\theta_3(x,y) := K_0(x,y,f(x)) \wedge E(f(x),y),$$

$$\theta_4(x,y) := f(x) = y, \theta_5(x,y) := K_0(f(x),y,x) \wedge E(f(x),y),$$

$$\theta_6(x,y) := K_0(f(x),y,x) \wedge \neg E(x,y) \wedge \neg E(f(x),y),$$

$$\theta_7(x,y) := K_0(f(x),y,x) \wedge E(x,y),$$

and

$$K_0(\theta_0(a,M),\theta_1(a,M),\theta_2(a,M),\theta_3(a,M),\theta_4(a,M),\theta_5(a,M),\theta_6(a,M),\theta_7(a,M))$$

holds for any $a \in M$.

Define labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y) \text{ where } 0 \leq k \leq 7.$$

It is easy to check that for the algebra $\mathfrak{P}_{M_{1,1,2}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5	6	7
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3,4,5\}$	$\{5\}$	$\{5\}$	$\{6\}$	$\{7,0,1\}$
2	$\{2\}$	$\{2\}$	$\{2,3,4,5\}$	$\{6\}$	$\{6\}$	$\{6\}$	$\{6,7,0,1,2\}$	$\{2\}$
3	$\{3\}$	$\{3,4,5\}$	$\{6\}$	$\{7\}$	$\{7\}$	$\{7,0,1\}$	$\{2\}$	$\{3\}$
4	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
5	$\{5\}$	$\{5\}$	$\{6\}$	$\{7,0,1\}$	$\{1\}$	$\{1\}$	$\{2\}$	$\{3,4,5\}$
6	$\{6\}$	$\{6\}$	$\{6,7,0,1,2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2,3,4,5,6\}$	$\{6\}$
7	$\{7\}$	$\{7,0,1\}$	$\{2\}$	$\{3\}$	$\{3\}$	$\{3,4,5\}$	$\{6\}$	$\{7\}$

By the Cayley table the algebra $\mathfrak{P}_{M_{1,1,2}}$ is commutative.

Theorem 2. The algebra $\mathfrak{P}_{M_{s,1,k}}$ of binary isolating formulas having a monotonic-to-right function on M has $2k(s+1)$ labels and is commutative.

Proof of Theorem 2. Indeed, since $f^k(a)=a$, we have the following isolating formulas:

$$f^l(x)=y \ \text{ for every } \ 0\leq l\leq k-1.$$

Since for every $1\leq i\leq s$ the relation E_i is an equivalence relation partitioning every E_{i+1}–class into infinitely many infinite convex E_i–subclasses without endpoints so that the induced order on E_i–subclasses is dense without endpoints, we obtain the following binary isolating formulas:

$$K_0(f^l(x),y,f^{l+1}(x))\wedge E_1(f^l(x),y),\ \text{where } 0\leq l\leq k-1,$$

$$K_0(f^l(x),y,f^{l+1}(x))\wedge\neg E_j(f^l(x),y)\wedge E_{j+1}(f^l(x),y),\ \text{where } 0\leq l\leq k-1, 1\leq j\leq s-1,$$

$$K_0(f^l(x),y,f^{l+1}(x))\wedge\neg E_s(f^l(x),y)\wedge\neg E_s(f^{l+1}(x),y),\ \text{where } 0\leq l\leq k-1,$$

$$K_0(f^l(x),y,f^{l+1}(x))\wedge\neg E_j(f^{l+1}(x),y)\wedge E_{j+1}(f^{l+1}(x),y),\ \text{where } 0\leq l\leq k-1, 1\leq j\leq s-1,$$

$$K_0(f^l(x),y,f^{l+1}(x))\wedge E_1(f^{l+1}(x),y),\ \text{where } 0\leq l\leq k-1.$$

Thus, we obtain $4k+2k(s-1)=2k(s+1)$ binary isolating formulas.

Now we establish commutativity of this algebra. Since for any binary isolating formula $\theta(x,y)$ the following holds:

$$\exists t[x=t\wedge\theta(t,y)]\equiv\theta(x,y) \text{ and } \exists t[\theta(x,t)\wedge t=y]\equiv\theta(x,y),$$

we obtain that $0\cdot l=l\cdot 0=\{l\}$ for any label l with the condition $0\leq l\leq 2k(s+1)-1$.

Obviously,

$$\text{both } \exists t[f^{l_1}(x)=t\wedge f^{l_2}(t)=y], \text{ and } \exists t[f^{l_2}(x)=t\wedge f^{l_1}(t)=y],$$

uniquely determine the formula $f^{l_1+l_2(\text{mod } k)}(x)=y$.

Further, since $K_0(a,f(a),f^2(a),\ldots,f^{k-1}(a))$ holds for any $a\in M$,

$$\exists t[f^{l_1}(x)=t\wedge E_i(f^{l_2}(t),y)\wedge K_0(f^{l_2}(t),y,f^{l_2+1}(t))],$$

where $1\leq i\leq s$, uniquely determines the formula

$$E_i(f^{l_1+l_2(\text{mod } k)}(x),y)\wedge K_0(f^{l_1+l_2(\text{mod } k)}(x),y,f^{l_1+l_2+1(\text{mod } k)}(x))$$

independently from behaviour of the function f. On the other hand, since f is monotonic-to-right on M,

$$\exists t[E_i(f^{l_2}(x),y)\wedge K_0(f^{l_2}(x),y,f^{l_2+1}(x))\wedge f^{l_1}(t)=y]$$

also uniquely determines the formula

$$E_i(f^{l_2+l_1 (\mathrm{mod}\ k)}(x), y) \wedge K_0(f^{l_2+l_1 (\mathrm{mod}\ k)}(x), y, f^{l_2+l_1+1 (\mathrm{mod}\ k)}(x)).$$

Further it is also easy to understand that the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t)) \wedge E_1(f^{l_2}(t), y)] \text{ and}$$

$$\exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge E_1(f^{l_2}(x), t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t), y)]$$

uniquely determine the formula

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge E_1(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y).$$

Now if we consider the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t)) \wedge E_1(f^{l_2+1}(t), y)] \text{ and}$$

$$\exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge E_1(f^{l_2+1}(x), t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t), y)],$$

there is no uniqueness, but both these formulas are compatible with the formulas

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge E_1(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y),$$

$$f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x) = y,$$

$$K_0(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+2 (\mathrm{mod}\ k)}(x)) \wedge E_1(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y).$$

Further, we consider the following formulas:

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$
$$\wedge \neg E_j(f^{l_2}(t), y) \wedge E_{j+1}(f^{l_2}(t), y)] \qquad (*)$$
$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_j(f^{l_2}(x), t) \wedge E_{j+1}(f^{l_2}(x), t)$$
$$\wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t), y)]. \qquad (**)$$

Since $E_1(f^{l_1}(x), t)$ implies $E_{j+1}(f^{l_1}(x), t)$, the formula

$$\exists t[E_1(f^{l_1}(x), t) \wedge \neg E_j(f^{l_2}(t), y) \wedge E_{j+1}(f^{l_2}(t), y)]$$

is compatible with the formula $E_{j+1}(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y) \wedge \neg E_j(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y)$. Consequently, the formulas $(*)$ and $(**)$ uniquely determine the formula

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge \neg E_j(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y)$$
$$\wedge E_{j+1}(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y).$$

Similarly we can show that

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$
$$\wedge \neg E_s(f^{l_2}(t), y) \wedge \neg E_s(f^{l_2+1}(t), y)]$$
$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_s(f^{l_2}(x), t) \wedge \neg E_s(f^{l_2+1}(x), t)$$

$$\wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t), y)]$$

uniquely determine the formula

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge \neg E_s(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y)$$

$$\wedge \neg E_s(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y).$$

Further, considering the following formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge \neg E_j(f^{l_1}(x), t) \wedge E_{j+1}(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$

$$\wedge \neg E_s(f^{l_2}(t), y) \wedge \neg E_s(f^{l_2+1}(t), y)]$$

$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_s(f^{l_2}(x), t) \wedge \neg E_s(f^{l_2+1}(x), t)$$

$$\wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge \neg E_j(f^{l_1}(t), y) \wedge E_{j+1}(f^{l_1}(t), y)],$$

we also obtain that they uniquely determine the formula

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge \neg E_s(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y)$$

$$\wedge \neg E_s(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y).$$

Consider now the following formulas:

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge \neg E_{j_1}(f^{l_1}(x), t) \wedge E_{j_1+1}(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$

$$\wedge \neg E_{j_2}(f^{l_2}(t), y) \wedge E_{j_2+1}(f^{l_2}(t), y)]$$

$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_{j_2}(f^{l_2}(x), t) \wedge E_{j_2+1}(f^{l_2}(x), t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t))$$

$$\wedge \neg E_{j_1}(f^{l_1}(t), y) \wedge E_{j_1+1}(f^{l_1}(t), y)].$$

Let $j = \max\{j_1, j_2\}$. Then it is easy to establish that these formulas uniquely determine the formula

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge \neg E_j(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y)$$

$$\wedge E_{j+1}(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y).$$

At last, consider the following formulas:

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge \neg E_s(f^{l_1}(x), t) \wedge \neg E_s(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$

$$\wedge \neg E_s(f^{l_2}(t), y) \wedge \neg E_s(f^{l_2}(t), y)]$$

$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_s(f^{l_2}(x), t) \wedge \neg E_s(f^{l_2}(x), t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t))$$

$$\wedge \neg E_s(f^{l_1}(t), y) \wedge \neg E_s(f^{l_1}(t), y)].$$

Here we loss the uniqueness: these formulas are compatible with the formula $K_0(f^{l_1+l_2}(x), y, f^{l_1+l_2+2}(x))$ which is in its turn compatible with the following $2s + 3$ formulas:

$$f^{l_1+l_2+1}(x) = y,$$

$$K_0(f^{l_1+l_2}(x), y, f^{l_1+l_2+1}(x)) \wedge E_1(f^{l_1+l_2+1}(x), y),$$

$$K_0(f^{l_1+l_2}(x), y, f^{l_1+l_2+1}(x)) \wedge \neg E_1(f^{l_1+l_2+1}(x), y) \wedge E_2(f^{l_1+l_2+1}(x), y),$$

$$\dots \quad \dots \quad \dots$$

$$K_0(f^{l_1+l_2}(x), y, f^{l_1+l_2+1}(x)) \wedge \neg E_{s-1}(f^{l_1+l_2+1}(x), y) \wedge E_s(f^{l_1+l_2+1}(x), y),$$

$$K_0(f^{l_1+l_2+1}(x), y, f^{l_1+l_2+2}(x)) \wedge \neg E_{s-1}(f^{l_1+l_2+1}(x), y) \wedge E_s(f^{l_1+l_2+1}(x), y),$$

$$\dots \quad \dots \quad \dots$$

$$K_0(f^{l_1+l_2+1}(x), y, f^{l_1+l_2+2}(x)) \wedge \neg E_1(f^{l_1+l_2+1}(x), y) \wedge E_2(f^{l_1+l_2+1}(x), y),$$

$$K_0(f^{l_1+l_2+1}(x), y, f^{l_1+l_2+1+2}(x)) \wedge E_1(f^{l_1+l_2+1}(x), y).$$

Definition 2. [1] Let $p \in S_1(\emptyset)$ be non-algebraic. The algebra $\mathcal{P}_{\nu(p)}$ is said to be *deterministic* if $u_1 \cdot u_2$ is a singleton for any labels $u_1, u_2 \in \rho_{\nu(p)}$.

Generalizing the last definition, we say that the algebra $\mathcal{P}_{\nu(p)}$ is *m-deterministic* if the product $u_1 \cdot u_2$ consists of at most m elements for any labels $u_1, u_2 \in \rho_{\nu(p)}$. We also say that an m-deterministic algebra $\mathcal{P}_{\nu(p)}$ is *strictly m-deterministic* if it is not $(m-1)$-deterministic.

Corollary. The algebra $\mathfrak{P}_{M_{s,1,k}}$ of binary isolating formulas having a monotonic-to-right function on M is strictly $(2s+3)$-deterministic.

Example 2. Consider the structure $M_{1,2,2} := \langle M, K^3, f^1, E_1^2, E_2^2 \rangle$ from Theorem 1, where f is monotonic-to-right on M, E_1 is an equivalence relation partitioning M into infinitely many infinite convex classes, E_2 is an equivalence relation partitioning M into two infinite convex classes. We assert that $Th(M_{1,2,2})$ has ten binary isolating formulas:

$$\theta_0(x,y) := x = y, \theta_1(x,y) := K_0(x, y, f(x)) \wedge E_1(x,y),$$

$$\theta_2(x,y) := K_0(x, y, f(x)) \wedge \neg E_1(x,y) \wedge E_2(x,y),$$

$$\theta_3(x,y) := K_0(x, y, f(x)) \wedge \neg E_2(x,y) \wedge \neg E_1(f(x), y),$$

$$\theta_4(x,y) := K_0(x, y, f(x)) \wedge E_1(f(x), y),$$

$$\theta_5(x,y) := f(x) = y, \theta_6(x,y) := K_0(f(x), y, x) \wedge E_1(f(x), y),$$

$$\theta_7(x,y) := K_0(f(x), y, x) \wedge \neg E_1(f(x), y) \wedge E_2(f(x), y),$$

$$\theta_8(x,y) := K_0(f(x), y, x) \wedge \neg E_2(f(x), y) \wedge \neg E_1(x, y),$$

$$\theta_9(x,y) := K_0(f(x), y, x) \wedge E_1(x,y),$$

and

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \theta_3(a,M), \theta_4(a,M), \theta_5(a,M), \theta_6(a,M), \theta_7(a,M))$$
$$\text{and } K_0(\theta_7(a,M), \theta_8(a,M), \theta_9(a,M), \theta_0(a,M))$$

holds for any $a \in M$.

Define labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y), \text{ where } 0 \le k \le 9.$$

It is easy to check that for the algebra $\mathfrak{P}_{M_{1,2,2}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5	6	7	8	9
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4,5,6\}$	$\{6\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9,0,1\}$
2	$\{2\}$	$\{2\}$	$\{2\}$	$\{3,4,5,6,7\}$	$\{7\}$	$\{7\}$	$\{7\}$	$\{7\}$	$\{8,9,0,1,2\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3,4,5,6,7\}$	$\{8\}$	$\{8\}$	$\{8\}$	$\{8\}$	$\{8,9,0,1,2\}$	$\{3\}$	$\{3\}$
4	$\{4\}$	$\{4,5,6\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{9\}$	$\{9,0,1\}$	$\{2\}$	$\{3\}$	$\{4\}$
5	$\{5\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
6	$\{6\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9,0,1\}$	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4,5,6\}$
7	$\{7\}$	$\{7\}$	$\{7\}$	$\{8,9,0,1,2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{3,4,5,6,7\}$	$\{7\}$
8	$\{8\}$	$\{8\}$	$\{8,9,0,1,2\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3,4,5,6,7\}$	$\{8\}$	$\{8\}$
9	$\{9\}$	$\{9,0,1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{4\}$	$\{4,5,6\}$	$\{7\}$	$\{8\}$	$\{9\}$

By the Cayley table the algebra $\mathfrak{P}_{M_{1,2,2}}$ is commutative.

Theorem 3. The algebra $\mathfrak{P}_{M_{s,m,k}}$ of binary isolating formulas having a monotonic-to-right function on M for $m \neq 1$ has $2k(s+1)+m$ labels, is commutative and strictly $(2s+3)$-deterministic.

Proof of Theorem 3. Similarly as in Theorem 2 we have the following binary isolating formulas:

$$f^l(x) = y \text{ for every } 0 \leq l \leq k-1,$$

$$K_0(f^l(x), y, f^{l+1}(x)) \wedge E_1(f^l(x), y), \text{ where } 0 \leq l \leq k-1,$$

$$K_0(f^l(x), y, f^{l+1}(x)) \wedge \neg E_j(f^l(x), y) \wedge E_{j+1}(f^l(x), y), \text{ where } 0 \leq l \leq k-1, 1 \leq j \leq s-1,$$

$$K_0(f^l(x), y, f^{l+1}(x)) \wedge \neg E_j(f^{l+1}(x), y) \wedge E_{j+1}(f^{l+1}(x), y), \text{ where } 0 \leq l \leq k-1, 1 \leq j \leq s-1,$$

$$K_0(f^l(x), y, f^{l+1}(x)) \wedge E_1(f^{l+1}(x), y), \text{ where } 0 \leq l \leq k-1.$$

Since in this structure there exists additionally the equivalence relation $E_{s+1}(x, y)$ partitioning M into m infinite convex classes, instead of the formulas

$$K_0(f^l(x), y, f^{l+1}(x)) \wedge \neg E_s(f^l(x), y) \wedge \neg E_s(f^{l+1}(x), y), \text{ where } 0 \leq l \leq k-1,$$

additionally the following binary isolating formulas appear:

$$K_0(f^l(x), y, f^{l+1}(x)) \wedge \neg E_s(f^l(x), y) \wedge E_{s+1}(f^l(x), y), \text{ where } 0 \leq l \leq k-1,$$

$$K_0(f^l(x), y, f^{l+1}(x)) \wedge \neg E_s(f^{l+1}(x), y) \wedge E_{s+1}(f^{l+1}(x), y), \text{ where } 0 \leq l \leq k-1.$$

Also, the formulas $\theta^{l,i}(x, y)$ containing the conjunctive term $K_0(f^l(x), y, f^{l+1}(x))$ and extracting the i-th E_{s+1}-class to the right of E_{s+1}-class containing $f^l(x)$ for some $1 \leq i \leq m/k - 1$ (here also $0 \leq l \leq k-1$) will be binary isolating formulas. For example, the formula $\theta^{l,1}(x, y)$ has the following form:

$$\theta^{l,1}(x, y) := K_0(f^l(x), y, f^{l+1}(x)) \wedge \neg E_{s+1}(f^l(x), y) \wedge$$
$$\forall t[K_0(f^l(x), t, y) \wedge \neg E_{s+1}(t, y) \to E_{s+1}(f^l(x), t)].$$

Thus, we obtain $k + k + 2k(s-1) + k + 2k + k(m/k - 1) = 2k(s+1) + m$ binary isolating formulas.

Take arbitrary labels l_1, l_2 and show that $l_1 \cdot l_2 = l_2 \cdot l_1$.

It is easy to establish that the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$
$$\wedge \neg E_s(f^{l_2}(t), y) \wedge E_{s+1}(f^{l_2}(t), y)]$$
$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_s(f^{l_2}(x), t) \wedge E_{s+1}(f^{l_2}(x), t)$$
$$\wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t), y)]$$

uniquely determine the formula

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge \neg E_s(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y)$$
$$\wedge E_{s+1}(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y).$$

Similarly, the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$
$$\wedge \neg E_s(f^{l_2+1}(t), y) \wedge E_{s+1}(f^{l_2+1}(t), y)]$$

$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_s(f^{l_2+1}(x), t) \wedge E_{s+1}(f^{l_2+1}(x), t)$$
$$\wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t), y)]$$

uniquely determine the formula

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge \neg E_s(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y)$$
$$\wedge E_{s+1}(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y).$$

Further, considering the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x), t) \wedge \theta^{l_2,j}(t, y)]$$
$$\text{and } \exists t[\theta^{l_2,j}(x, t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t), y)],$$

we establish that they uniquely determine the formula $\theta^{l_1+l_2 (\mathrm{mod}\ k), j}(x, y)$ for arbitrary $1 \le j \le m/k - 1$.

Similarly, the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge \neg E_s(f^{l_1}(x), t) \wedge E_{s+1}(f^{l_1}(x), t) \wedge \theta^{l_2,j}(t, y)]$$
$$\text{and } \exists t[\theta^{l_2,j}(x, t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge \neg E_s(f^{l_1}(t), y) \wedge E_{s+1}(f^{l_1}(t), y)]$$

uniquely determine the formula $\theta^{l_1+l_2+1 (\mathrm{mod}\ k), j}(x, y)$ for every $1 \le j \le m/k - 1$.

On the other hand, the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge \neg E_s(f^{l_1+1}(x), t) \wedge E_{s+1}(f^{l_1+1}(x), t) \wedge \theta^{l_2,j}(t, y)]$$
$$\text{and } \exists t[\theta^{l_2,j}(x, t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge \neg E_s(f^{l_1+1}(t), y) \wedge E_{s+1}(f^{l_1+1}(t), y)]$$

uniquely determine the formula $\theta^{l_1+l_2 (\mathrm{mod}\ k), j}(x, y)$ for arbitrary $1 \le j \le m/k - 1$.

Also observe that the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge \neg E_j(f^{l_1}(x), t) \wedge E_{j+1}(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$
$$\wedge \neg E_s(f^{l_2}(t), y) \wedge E_{s+1}(f^{l_2}(t), y)]$$
$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_s(f^{l_2}(x), t) \wedge E_{s+1}(f^{l_2}(x), t)$$
$$\wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge \neg E_j(f^{l_1}(t), y) \wedge E_{j+1}(f^{l_1}(t), y)],$$

uniquely determine the formula

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge \neg E_s(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y)$$
$$\wedge E_{s+1}(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y).$$

If we consider the following formulas:

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t)) \wedge E_1(f^{l_2+1}(t), y)] \text{ and}$$
$$\exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge E_1(f^{l_2+1}(x), t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t), y)],$$

there is no uniqueness, but both these formulas are compatible with the formulas

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge E_1(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y),$$

$$f^{l_1+l_2+1(\text{mod } k)}(x) = y,$$

$$K_0(f^{l_1+l_2+1(\text{mod } k)}(x), y, f^{l_1+l_2+2(\text{mod } k)}(x)) \wedge E_1(f^{l_1+l_2+1(\text{mod } k)}(x), y).$$

Consider now the formulas $\theta^{l_1,i}(x,y)$ and $\theta^{l_2,j}(x,y)$ for arbitrary $1 \leq i,j \leq m/k-1$. If $i+j$ (mod $m/k) \neq 0$, it is easy to check that the formulas

$$\exists t[\theta^{l_1,i}(x,t) \wedge \theta^{l_2,j}(t,y)] \text{ and } \exists t[\theta^{l_2,j}(x,t) \wedge \theta^{l_1,i}(t,y)]$$

uniquely determine the formula $\theta^{l_1+l_2(\text{mod } k), i+j(\text{mod } m/k)}(x,y)$.

If $i+j$ (mod $m/k) = 0$, these formulas are compatible with the following $2s+3$ formulas:

$$f^{l_1+l_2+1(\text{mod } k)}(x) = y,$$

$$K_0(f^{l_1+l_2(\text{mod } k)}(x), y, f^{l_1+l_2+1(\text{mod } k)}(x)) \wedge E_1(f^{l_1+l_2+1(\text{mod } k)}(x), y),$$

$$K_0(f^{l_1+l_2(\text{mod } k)}(x), y, f^{l_1+l_2+1(\text{mod } k)}(x)) \wedge \neg E_j(f^{l_1+l_2+1(\text{mod } k)}(x), y)$$

$$\wedge E_{j+1}(f^{l_1+l_2+1(\text{mod } k)}(x), y), \quad 1 \leq j \leq s,$$

$$K_0(f^{l_1+l_2+1(\text{mod } k)}(x), y, f^{l_1+l_2+2(\text{mod } k)}(x)) \wedge E_1(f^{l_1+l_2+1(\text{mod } k)}(x), y),$$

$$K_0(f^{l_1+l_2+1(\text{mod } k)}(x), y, f^{l_1+l_2+2(\text{mod } k)}(x)) \wedge \neg E_j(f^{l_1+l_2+1(\text{mod } k)}(x), y)$$

$$\wedge E_{j+1}(f^{l_1+l_2+1(\text{mod } k)}(x), y), \quad 1 \leq j \leq s.$$

Example 3. Consider the structure $M_{1,2,2} := \langle M, K^3, f^1, E_1^2, E_2^2 \rangle$ from Theorem 1, where f is monotonic-to-left on M, E_1 is an equivalence relation partitioning M into infinitely many infinite convex classes, E_2 is an equivalence relation partitioning M into two infinite convex classes. We assert that $Th(M_{1,2,2})$ has ten binary isolating formulas:

$$\theta_0(x,y) := x = y, \theta_1(x,y) := K_0(x,y,f(x)) \wedge E_1(x,y),$$

$$\theta_2(x,y) := K_0(x,y,f(x)) \wedge \neg E_1(x,y) \wedge E_2(x,y),$$

$$\theta_3(x,y) := K_0(x,y,f(x)) \wedge \neg E_2(x,y) \wedge \neg E_1(f(x),y),$$

$$\theta_4(x,y) := K_0(x,y,f(x)) \wedge E_1(f(x),y),$$

$$\theta_5(x,y) := f(x) = y, \theta_6(x,y) := K_0(f(x),y,x) \wedge E_1(f(x),y),$$

$$\theta_7(x,y) := K_0(f(x),y,x) \wedge \neg E_1(f(x),y) \wedge E_2(f(x),y),$$

$$\theta_8(x,y) := K_0(f(x),y,x) \wedge \neg E_2(f(x),y) \wedge \neg E_1(x,y),$$

$$\theta_9(x,y) := K_0(f(x),y,x) \wedge E_1(x,y),$$

and both

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \theta_3(a,M), \theta_4(a,M), \theta_5(a,M), \theta_6(a,M), \theta_7(a,M))$$

$$\text{and } K_0(\theta_7(a,M), \theta_8(a,M), \theta_9(a,M), \theta_0(a,M))$$

hold for any $a \in M$.

Define labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y), \text{ where } 0 \leq k \leq 9.$$

It is easy to check that for the algebra $\mathfrak{P}_{M_{1,2,2}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5	6	7	8	9
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{4\}$	$\{4,5,6\}$	$\{7\}$	$\{8\}$	$\{9,0,1\}$
2	$\{2\}$	$\{2\}$	$\{2\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3,4,5,6,7\}$	$\{8,9,0,1,2\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3,4,5,6,7\}$	$\{8,9,0,1,2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{3\}$	$\{3\}$
4	$\{4\}$	$\{4,5,6\}$	$\{7\}$	$\{8\}$	$\{9,0,1\}$	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
5	$\{5\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
6	$\{6\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{9\}$	$\{9,0,1\}$	$\{2\}$	$\{3\}$	$\{4,5,6\}$
7	$\{7\}$	$\{7\}$	$\{7\}$	$\{8\}$	$\{8\}$	$\{8\}$	$\{8\}$	$\{8,9,0,1,2\}$	$\{3,4,5,6,7\}$	$\{7\}$
8	$\{8\}$	$\{8\}$	$\{8,9,0,1,2\}$	$\{3,4,5,6,7\}$	$\{7\}$	$\{7\}$	$\{7\}$	$\{7\}$	$\{8\}$	$\{8\}$
9	$\{9\}$	$\{9,0,1\}$	$\{2\}$	$\{3\}$	$\{4,5,6\}$	$\{6\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$

By the Cayley table the algebra $\mathfrak{P}_{M_{1,2,2}}$ is not commutative.

Theorem 4. The algebra $\mathfrak{P}_{M_{s,2,2}}$ of binary isolating formulas having a monotonic-to-left function on M has $4s+6$ labels, is strictly $(2s+3)$-deterministic and is not commutative.

Proof of Theorem 4. In this case we have the following binary isolating formulas:

$$x = y, \quad f(x) = y,$$

$$K_0(x, y, f(x)) \wedge E_1(x, y),$$

$$K_0(x, y, f(x)) \wedge \neg E_j(x, y) \wedge E_{j+1}(x, y), 1 \le j \le s,$$

$$K_0(x, y, f(x)) \wedge \neg E_j(f(x), y) \wedge E_{j+1}(f(x), y), 1 \le j \le s,$$

$$K_0(x, y, f(x)) \wedge E_1(f(x), y),$$

$$K_0(f(x), y, x) \wedge E_1(f(x), y),$$

$$K_0(f(x), y, x) \wedge \neg E_j(f(x), y) \wedge E_{j+1}(f(x), y), 1 \le j \le s,$$

$$K_0(f(x), y, x) \wedge \neg E_j(x, y) \wedge E_{j+1}(x, y), 1 \le j \le s,$$

$$K_0(f(x), y, x) \wedge E_1(x, y).$$

Thus, we obtain $4s+6$ binary isolating formulas.

The formula

$$\exists t[K_0(x, t, f(x)) \wedge E_1(x, t) \wedge K_0(t, y, f(t)) \wedge E_1(t, y)]$$

uniquely determines the formula $K_0(x, y, f(x)) \wedge E_1(x, y)$. Further, the formulas

$$\exists t[K_0(x, t, f(x)) \wedge E_1(x, t) \wedge K_0(t, y, f(t)) \wedge \neg E_j(t, y) \wedge E_{j+1}(t, y)]$$

$$\text{and } \exists t[K_0(x, t, f(x)) \wedge \neg E_j(x, t) \wedge E_{j+1}(x, t) \wedge K_0(t, y, f(t)) \wedge E_1(t, y)]$$

for every $1 \le j \le s$ uniquely determine the formula

$$K_0(x, y, f(x)) \wedge \neg E_j(x, y) \wedge E_{j+1}(x, y).$$

Consider now the formulas

$$\exists t[K_0(x, t, f(x)) \wedge E_1(x, t) \wedge K_0(t, y, f(t)) \wedge \neg E_j(f(t), y) \wedge E_{j+1}(f(t), y)]$$

$$\text{and } \exists t[K_0(x, t, f(x)) \wedge \neg E_j(f(x), t) \wedge E_{j+1}(f(x), t) \wedge K_0(t, y, f(t)) \wedge E_1(t, y)],$$

where $1 \le j \le s$. It is easy to establish that they uniquely determine the formula

$$K_0(x, y, f(x)) \wedge \neg E_j(f(x), y) \wedge E_{j+1}(f(x), y).$$

The formula

$$\exists t[K_0(x, t, f(x)) \wedge E_1(x, t) \wedge K_0(t, y, f(t)) \wedge E_1(f(t), y)]$$

uniquely determines the formula $K_0(x,y,f(x)) \wedge E_1(f(x),y)$. While the formula

$$\exists t[K_0(x,t,f(x)) \wedge E_1(f(x),t) \wedge K_0(t,y,f(t)) \wedge E_1(t,y)]$$

is compatible with the following three formulas:

$$K_0(x,y,f(x)) \wedge E_1(f(x),y), \quad f(x)=y, \quad K_0(f(x),y,x) \wedge E_1(f(x),y).$$

Thus, we established that the algebra $\mathfrak{P}_{M_{s,2,2}}$ is not commutative.

Further, considering the formulas

$$\exists t[K_0(x,t,f(x)) \wedge E_1(x,t) \wedge f(t)=y] \text{ and } \exists t[f(x)=t \wedge K_0(t,y,f(t)) \wedge E_1(t,y)],$$

we obtain that they uniquely determine the formulas

$$K_0(x,y,f(x)) \wedge E_1(f(x),y) \text{ and } K_0(f(x),y,x) \wedge E_1(f(x),y)$$

respectively, also confirming non-commutativity of the algebra $\mathfrak{P}_{M_{s,2,2}}$.

Similarly, the formulas

$$\exists t[K_0(x,t,f(x)) \wedge E_1(f(x),t) \wedge f(t)=y] \text{ and } \exists t[f(x)=t \wedge K_0(t,y,f(t)) \wedge E_1(f(t),y)],$$

uniquely determine the formulas

$$K_0(x,y,f(x)) \wedge E_1(x,y) \text{ and } K_0(f(x),y,x) \wedge E_1(x,y), \text{ respectively.}$$

The formula

$$\exists t[K_0(x,t,f(x)) \wedge E_1((x,t) \wedge K_0(f(t),y,t) \wedge E_1(f(t),y)]$$

is compatible with the following three formulas:

$$K_0(x,y,f(x)) \wedge E_1(f(x),y), \quad f(x)=y, \quad K_0(f(x),y,x) \wedge E_1(f(x),y).$$

While the formula

$$\exists t[K_0(f(x),t,x) \wedge E_1(f(x),t) \wedge K_0(t,y,f(t)) \wedge E_1(t,y)]$$

uniquely determines the formula $K_0(f(x),y,x) \wedge E_1(f(x),y)$.

Further, the formulas

$$\exists t[K_0(x,t,f(x)) \wedge E_1(x,t) \wedge K_0(f(t),y,t) \wedge \neg E_j(f(t),y) \wedge E_{j+1}(f(t),y)]$$

$$\text{and } \exists t[K_0(f(x),t,x) \wedge \neg E_j(f(x),t) \wedge E_{j+1}(f(x),t) \wedge K_0(t,y,f(t)) \wedge E_1(t,y)]$$

for every $1 \leq j \leq s$ uniquely determine the formula

$$K_0(f(x),y,x) \wedge \neg E_j(f(x),y) \wedge E_{j+1}(f(x),y).$$

Similarly, the formulas

$$\exists t[K_0(x,t,f(x)) \wedge E_1(x,t) \wedge K_0(f(t),y,t) \wedge \neg E_j(t,y) \wedge E_{j+1}(t,y)]$$

$$\text{and } \exists t[K_0(f(x),t,x) \wedge \neg E_j(x,t) \wedge E_{j+1}(x,t) \wedge K_0(t,y,f(t)) \wedge E_1(t,y)]$$

for every $1 \leq j \leq s$ uniquely determine the formula

$$K_0(f(x),y,x) \wedge \neg E_j(x,y) \wedge E_{j+1}(x,y).$$

Further, the formulas

$$\exists t[K_0(x,t,f(x)) \wedge E_1(x,t) \wedge K_0(f(t),y,t) \wedge E_1(t,y)]$$

$$\text{and } \exists t[K_0(f(x),t,x) \wedge E_1(x,t) \wedge K_0(t,y,f(t)) \wedge E_1(t,y)]$$

are compatible with the following three formulas:

$$K_0(f(x),y,x) \wedge E_1(x,y), \quad x=y, \quad K_0(x,y,f(x)) \wedge E_1(x,y).$$

The formula

$$\exists t[K_0(x,t,f(x)) \wedge \neg E_s(x,t) \wedge E_{s+1}(x,t) \wedge K_0(f(t),y,t) \wedge \neg E_s(f(t),y) \wedge E_{s+1}(f(t),y)]$$

is compatible with the following $2s+3$ formulas:

$$K_0(x,y,f(x)) \wedge \neg E_j(f(x),y) \wedge E_{j+1}(f(x),y), \quad 1 \leq j \leq s,$$

$$K_0(x,y,f(x)) \wedge E_1(f(x),y),$$

$$f(x)=y,$$

$$K_0(f(x),y,x) \wedge E_1(f(x),y),$$

$$K_0(f(x),y,x) \wedge \neg E_j(f(x),y) \wedge E_{j+1}(f(x),y), \quad 1 \leq j \leq s.$$

While the formula

$$\exists t[K_0(f(x),t,x) \wedge \neg E_s(f(x),t) \wedge E_{s+1}(f(x),t) \wedge K_0(t,y,f(t)) \wedge \neg E_s(t,y) \wedge E_{s+1}(t,y)]$$

uniquely determines the formula

$$K_0(f(x),y,x) \wedge \neg E_s(f(x),y) \wedge E_{s+1}(f(x),y).$$

On the other hand, the formulas

$$\exists t[K_0(x,t,f(x)) \wedge \neg E_s(x,t) \wedge E_{s+1}(x,t) \wedge K_0(f(t),y,t) \wedge \neg E_s(t,y) \wedge E_{s+1}(t,y)]$$

$$\text{and } \exists t[K_0(f(x),t,x) \wedge \neg E_s(x,t) \wedge E_{s+1}(x,t) \wedge K_0(t,y,f(t)) \wedge \neg E_s(t,y) \wedge E_{s+1}(t,y)]$$

are compatible with the same $2s+3$ formulas:

$$K_0(f(x),y,x) \wedge \neg E_j(x,y) \wedge E_{j+1}(x,y), \quad 1 \leq j \leq s,$$

$$K_0(f(x),y,x) \wedge E_1(x,y),$$

$$x=y,$$

$$K_0(x,y,f(x)) \wedge E_1(x,y),$$

$$K_0(x,y,f(x)) \wedge \neg E_j(x,y) \wedge E_{j+1}(x,y), \quad 1 \leq j \leq s.$$

Acknowledgments

The work was supported by Science Committee of Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19674850), and in the framework of the State Contract of the Sobolev Institute of Mathematics, Project No. FWNF-2022-0012.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Sudoplatov, S.V. (2018). *Classification of countable models of complete theories.* Novosibirsk: NSTU.

2 Shulepov, I.V., & Sudoplatov, S.V. (2014). Algebras of distributions for isolating formulas of a complete theory. *Siberian Electronic Mathematical Reports, 11,* 380–407.

3 Sudoplatov, S.V. (2014). Algebras of distributions for semi-isolating formulas of a complete theory. *Siberian Electronic Mathematical Reports, 11,* 408–433.

4 Emelyanov, D.Yu., Kulpeshov, B.Sh., & Sudoplatov, S.V. (2017). Algebras of distributions for binary formulas in countably categorical weakly o-minimal structures. *Algerba and Logic, 56*(1), 13–36. https://doi.org/10.1007/s10469-017-9424-y

5 Baikalova, K.A., Emelyanov, D.Yu., Kulpeshov, B.Sh., Palyutin, E.A., & Sudoplatov, S.V. (2018). On algebras of distributions of binary isolating formulas for theories of abelian groups and their ordered enrichments. *Russian Mathematics, 62*(4), 1–12. https://doi.org/10.3103/S1066369X 18040011

6 Emelyanov, D.Yu., Kulpeshov, B.Sh., & Sudoplatov, S.V. (2019). On algebras of distributions for binary formulas for quite o-minimal theories. *Algebra and Logic, 57*(6), 429–444. https://doi.org/ 10.1007/s10469-019-09515-5

7 Altayeva, A.B., Kulpeshov, B.Sh., & Sudoplatov, S.V. (2021). Algebras of distributions of binary isolating formulas for almost ω-categorical weakly o-minimal theories. *Algebra and Logic, 60*(4), 241–262. https://doi.org/10.1007/s10469-021-09650-y

8 Kulpeshov, B.Sh., & Macpherson, H.D. (2005). Minimality conditions on circularly ordered structures. *Mathematical Logic Quarterly, 51*(4), 377–399. https://doi.org/10.1002/malq.200410040

9 Macpherson, H.D., & Steinhorn, C. (1996). On variants of o-minimality. *Annals of Pure and Applied Logic, 79,* 165–209.

10 Kulpeshov, B.Sh. (2006). On $\aleph_0$-categorical weakly circularly minimal structures. *Mathematical Logic Quarterly, 52*(6), 555–574. https://doi.org/10.1002/malq.200610014

11 Kulpeshov, B.Sh. (2009). Definable functions in the $\aleph_0$-categorical weakly circularly minimal structures. *Siberian Mathematical Journal, 50*(2), 282–301. https://doi.org/10.1007/s11202-009-0034-3

12 Kulpeshov, B.Sh. (2015). On indiscernibility of a set in circularly ordered structures. *Siberian Electronic Mathematical Reports, 12,* 255–266. https://doi.org/10.17377/semi.2015.12.021

13 Kulpeshov, B.Sh., & Verbovskiy, V.V. (2015). On weakly circularly minimal groups. *Mathematical Logic Quarterly, 61*(1-2), 82–90. https://doi.org/10.1002/malq.201300076

14 Kulpeshov, B.Sh., & Altayeva, A.B. (2016). Binary formulas in countably categorical weakly circularly minimal structures. *Algebra and Logic, 55*(3), 226–241. https://doi.org/10.1007/s10469-016-9391-8

15 Kulpeshov, B.Sh. (2016). On almost binarity in weakly circularly minimal structures. *Eurasian Mathematical Journal, 7*(2), 38–49.

16 Altayeva, A.B., & Kulpeshov, B.Sh. (2021). Almost binarity of countably categorical weakly circularly minimal structures. *Mathematical Notes, 110*(6), 813–829. https://doi.org/10.1134/S0001434621110195

17 Kulpeshov, B.Sh. (1998). Weakly o-minimal structures and some of their properties. *The Journal of Symbolic Logic, 63*(4), 1511–1528. https://doi.org/10.2307/2586664

18 Kulpeshov, B.Sh. (2021). A criterion for binarity of almost ω-categorical weakly o-minimal theories. *Siberian Mathematical Journal, 62*(2), 1063–1075. https://doi.org/10.1134/S0037446621060082

19 Emelyanov, D.Yu., Kulpeshov, B.Sh., & Sudoplatov, S.V. (2020). Algebras of binary formulas for compositions of theories. *Algebra and Logic, 59*(4), 295–312. https://doi.org/1007/s10469-020-09602-y

20 Kulpeshov, B.Sh., & Sudoplatov, S.V. (2022). Algebras of binary formulas for weakly circularly minimal theories with non-trivial definable closure. *Lobachevskii Journal of Mathematics, 43*(12), 3532–3540. https://doi.org/10.1134/S199508022215015X

$\aleph_0$-категориялық әлсіз циклдік минималды теориялар үшін бинарлық формулалар алгебралары: монотонды жағдай

Б.Ш. Кулпешов[1,2], С.В. Судоплатов[3,4]

[1] *Математика және математикалық модельдеу институты, Алматы, Қазақстан;*
[2] *Қазақстан-Британ техникалық университеті, Алматы, Қазақстан;*
[3] *Новосібір мемлекеттік техникалық университеті, Новосібір, Ресей;*
[4] *РҒА СБ С.Л. Соболев атындағы Математика институты, Новосібір, Ресей*

Мақалада о-минималдылық тұжырымдамасына қатысты циклдік реттелген құрылымдар үшін нұсқа болып табылатын әлсіз циклдік минималдылық түсінігі қарастырылған. 1-транзитивтілік автоморфизмдердің примитивтілік емес группасы және құрылымның негізгі жиынында әрекет ететін тривиальды емес қатаң монотонды функцияның екеуі де бар дөңестік рангісі бірден үлкен саналымды категориялық әлсіз циклдік минималды теориялары үшін бинарлы оқшаулау формулалар алгебралары зерттелген. Зерттеу нәтижесінде авторлар осы алгебралардың сипаттамасын ұсынған. Олардың арасында коммутативті және коммутативті емес алгебралар бар екені көрсетілген. Мұндай алгебралардың қатаң m-детерминаттылығы кейбір m натурал саны үшін де анықталған.

Кілт сөздер: циклдік реттелген құрылым, бинарлық формула, оқшаулау формуласы, формулалар алгебрасы, саналымды категориялық теория, әлсіз циклдік минималдылық, дөңестік рангісі, автоморфизм группасы, транзитивтілік, примитивтілік, m-детерминаттылық.

Алгебры бинарных формул для $\aleph_0$-категоричных слабо циклически минимальных теорий: монотонный случай

Б.Ш. Кулпешов[1,2], С.В. Судоплатов[3,4]

[1] *Институт математики и математического моделирования, Алматы, Казахстан;*
[2] *Казахстанско-Британский технический университет, Алматы, Казахстан;*
[3] *Новосибирский государственный технический университет, Новосибирск, Россия;*
[4] *Институт математики имени С.Л. Соболева СО РАН, Новосибирск, Россия*

В настоящей статье рассмотрено понятие слабой циклической минимальности, являющейся вариантом o-минимальности для циклически упорядоченных структур. Исследованы алгебры бинарных изолирующих формул для счетно категоричных слабо циклически минимальных теорий ранга выпуклости, большего единицы, имеющих как *1*-транзитивную непримитивную группу автоморфизмов, так и нетривиальную строго монотонную функцию, действующую на основном множестве структуры. В результате исследования авторы представляют описание этих алгебр. Показано, что среди них имеются как коммутативные, так и некоммутативные алгебры. Кроме того, установлена строгая m-детерминированность таких алгебр для некоторого натурального числа m.

Ключевые слова: циклически упорядоченная структура, бинарная формула, изолирующая формула, алгебра формул, счетно категоричная теория, слабая циклическая минимальность, ранг выпуклости, группа автоморфизмов, транзитивность, примитивность, m-детерминированность.

*Author Information**

Beibut Shaiykovich Kulpeshov (*corresponding author*) — Doctor of physical and mathematical sciences, Corresponding Member of National Academy of Sciences of the Repuiblic of Kazakhstan, Professor, Chief Researcher of Institute of Mathematics and Mathematical Modeling, Professor of Kazakh-British Technical University, 28 Shevchenko street, Almaty, 050010, Kazakhstan; e-mail: *b.kulpeshov@kbtu.kz*; https://orcid.org/0000-0002-4242-0463

Sergey Vladimirovich Sudoplatov — Doctor of physical and mathematical sciences, Professor, Deputy Director of Sobolev Institute of Mathematics, Head of Chair of Algebra and Mathematical Logic of Novosibirsk State Technical University, 4 Academician Koptyug avenu, Novosibirsk, 630090, Russia; e-mail: *sudoplat@math.nsc.ru*; https://orcid.org/0000-0002-3268-9389

*The author's name is presented in the order: First, Middle and Last Names.

https://doi.org/10.31489/2024M1/128-139 *Research article*

Isomorphism Theorems of a Series Sum and the Improper Integral

V.P. Malyshev[1], S.Sh. Kazhikenova[2], A.M. Makasheva[1,*], A.Sh. Kazhikenova[3]

[1] *Abishev Chemical and Metallurgical Institute, Karaganda, Kazakhstan;*
[2] *Abylkas Saginov Karaganda Technical University, Karaganda, Kazakhstan;*
[3] *Karaganda Buketov University, Karaganda, Kazakhstan*
(E-mail: eia_hmi@mail.ru, sauleshka555@mail.ru, astra_mun@mail.ru, aigul-kazhikenova@mail.ru)

The discrete and continuous dependencies' relationship question has been investigated. An algorithm for determining the final and total series sums through the equivalence ratio of the series common term a_n and the a_n-model function improper integral mean value within the change unit interval based on the extended integral Cauchy convergence criterion has been developed. Examples of determining for the statistical sum in the Boltzmann distribution, for the first time directly expressed through a_n-model function. This eliminates the need for calculations to accumulate the sum of the series up to a value that is specified by a certain accuracy of this sum. In addition, it allows in this case to vary the energy variation interval with any given accuracy. The conducted studies allow solving both theoretical and practical problems of physics and materials science, directly using the Boltzmann distribution (energy spectrum) to calculate the entropy, which determines the loss of thermal energy in technological processes.

Keywords: isomorphism, series sum, improper integral, Boltzmann distribution, equivalence relation.

2020 Mathematics Subject Classification: 40-02, 40C10, 44A20.

Introduction

The natural sciences are dominated by discrete (nanoparticles, atoms, molecules, genes, etc.), but continuous (electromagnetic fields, wave theory, etc.) quantities are also used to describe real processes. Classical mathematics is built on the continuity and a function limit concepts, which are difficult to apply in practice. The basis of specific calculations are the laws of discrete mathematics, which has developed as a result of only mathematical constructions recognition from a finite number of procedures and finite objects. Evolution models are continuous and discrete dynamical systems. Continuous dynamical systems are described by ordinary differential equations systems or in partial derivatives, discrete dynamical systems are described by difference equations systems. There is a certain relationship between continuous and discrete systems, regardless of the application. It is interesting to compare the continuous model properties and their discrete counterparts. And the functional dependence question arises, when it is possible to carry out the transition from continuous to discrete values and vice [1–3]. As it is known, the discrete dependencies' identification with continuous ones as the argument tends to an infinitely small value dx is the differential and integral calculus. But the relationship between discrete and continuous distributions can turn out to be definite and productive for fixed variation intervals, x, if we applied to them the isomorphism general provisions — one mathematics development the new directions [4]. Prior to this, such a relationship manifested itself when establishing the convergence of a series, i.e. sums of discrete quantities, using the Cauchy, Maclaurin series convergence integral criterion [5], according to which the series $\sum_{n=1}^{\infty} a_n$ converges if for the function $f(x)$, which takes the values a_n at the points n, namely, for $f(n) = a_n$, and for condition of monotone decrease of $f(x)$ in the region $x \geq n_0$ with observance of the inequality $f(x) \geq 0$, the convergence of the improper integral $\int_{n_0}^{\infty} f(x)dx$ is ensured. The integral Cauchy criterion greatly facilitates the series convergence

*Corresponding author. *E-mail: astra_mun@mail.ru*
Received: 11 August 2023; *Accepted:* 17 October 2023.

study, since to reduce this issue to finding out the integral convergence of a well-chosen corresponding function $f(x)$, which is easily done using the integral calculus methods. Thus, this sign establishes a certain equivalence of discrete and a variable continuous distribution. Detailed calculations, but without observing the isomorphism conditions, were presented earlier by the authors [6–10].

It is necessary to verify the proposed version provability for the series sum and the improper integral of the auxiliary a_n-model function isomorphism, the existence of which is entirely determined by the structure and form of the common term of the series and corresponds to its direct purpose for determining its sum. In this regard, the a_n-model function has a peculiarity, the similarity of which has not been found in the literature [11–17]. This originality emphasizes the insufficiency of the integral criterion for the convergence of the series and the improper Cauchy and Maclaurin integral to determine the sum of the series, copying according to the form the series common term. Such insufficiency is a consequence of using only the inequalities of a series sum and the improper integral to prove the series convergence or divergence in terms these integral the convergence or divergence. To determine of a series sum itself, an analytical, quantitative expression of its relationship with the improper integral is required for a given restriction of this relationship. In addition, isomorphism is generally aimed at introducing analytical proofs of mathematical objects similarity instead of their qualitative similarity indications. At the very least, it is not yet possible to determine a series sum based on the area inequalities formed using a_n and $f_n(x)$.

1 *Determination of the equivalence relation based on the Boltzmann distribution*

The Boltzmann distribution for a monatomic ideal gas in a discrete version is expressed as

$$P_i = \frac{N_i}{N} = e^{-\frac{\varepsilon_i}{kT}} \Big/ \sum_{i=1}^{\infty} e^{-\frac{\varepsilon_i}{kT}}, \tag{1}$$

where P_i is the particles fraction with the i-th energy level ε_i, and the first energy level for physical and thermodynamic reasons is equal to zero as the lowest energy value to which the equilibrium system tends. It, like quantum orbitals, is also the most populated, which is mathematically achieved by the value $exp(-\varepsilon_i/kT) = 1$ and following from (1) at $\varepsilon_i = (i-1)\triangle\varepsilon$ by the condition $P_{i+1} \leq P_i$, if $T \geq 0$, and $P_{i+1} = P_i$ at $T \to \infty$.

The series sum $\sum_{i=1}^{\infty} e^{-\frac{\varepsilon_i}{kT}}$ until recently had no limit direct expression and was estimated either from specific spectral distributions or as a continuous function

$$\int_0^{\infty} e^{-\frac{\varepsilon}{kT}} d\varepsilon = -kT \left| e^{-\frac{\varepsilon}{kT}} \right|_0^{\infty} = kT. \tag{2}$$

This result raises questions, first of all, due to its explicit energy dimension, J·particle^{-1}, which belongs not to the function, but to the argument. The function is a continuous sum of populations of infinitesimally different energy levels, and the function must be dimensionless. We will try to solve this question further as a special case of a more general solution. Meanwhile, it is possible to fairly strictly determine of the series sum (statistical sum) using the equivalence relation A, for which the series common term must be expressed in more detail and in the accepted notation

$$a_n = e^{-\frac{(n-1)\triangle\varepsilon}{kT}}, \tag{3}$$

which provide the condition: at $n = 1$ $a_1 = 1$, $\varepsilon = 0$. In this case, $\triangle\varepsilon$ is the energy variation step and kT are constants (isothermal energy distribution is considered). a_n-model function (3) is expressed as

$$f_n(x) = e^{-\frac{(x-1)\triangle\varepsilon}{kT}}, \tag{4}$$

and its improper integral (4) as

$$\int_0^\infty e^{-\frac{(x-1)\triangle\varepsilon}{kT}}dx = -\frac{kT}{\triangle\varepsilon}\left|e^{-\frac{(x-1)\triangle\varepsilon}{kT}}\right|_0^\infty = \frac{kT}{\triangle\varepsilon}e^{\frac{\triangle\varepsilon}{kT}}. \quad (5)$$

The integral (5) does not contain $x = n$ that is, it converges, and hence the series $\sum_{n=1}^{\infty} a_n$ converges according to the integral criterion for the Cauchy and Maclaurin series convergence.

If we consider the integrand (2) as a_n-model for the series $\sum_0^\infty e^{-\frac{n\triangle\varepsilon}{kT}}$, then its improper integral is expressed as

$$\int_0^\infty e^{-\frac{x\triangle\varepsilon}{kT}}dx = -\frac{kT}{\triangle\varepsilon}\left|e^{-\frac{x\triangle\varepsilon}{kT}}\right|_0^\infty = \frac{kT}{\triangle\varepsilon}. \quad (6)$$

This dimensionless result (6) can be numerically equal to kT only at $\triangle\varepsilon = 1$ J·particle^{-1}. This interval is quite acceptable, as well as any others that do not have a fundamental physical justification. It is much more convincing to use $\triangle\varepsilon = kT$. But let us continue the isomorphism possibility analysis on the chosen example in the most general form.

To determine the equivalence relation, it is required to check its independence from n in an arbitrary unit interval

$$A_1 = \frac{\int_{x=n-1}^{x=n} e^{-\frac{(x-1)\triangle\varepsilon}{kT}}dx}{e^{-\frac{(n-1)\triangle\varepsilon}{kT}}} = \frac{-\frac{kT}{\triangle\varepsilon}\left|e^{-\frac{(x-1)\triangle\varepsilon}{kT}}\right|_{x=n-1}^{x=n}}{e^{-\frac{(n-1)\triangle\varepsilon}{kT}}} = \frac{-\frac{kT}{\triangle\varepsilon}(e^{-\frac{(n-1)\triangle\varepsilon}{kT}} - e^{-\frac{(n-2)\triangle\varepsilon}{kT}})}{e^{-\frac{(n-1)\triangle\varepsilon}{kT}}} = \frac{kT}{\triangle\varepsilon}(e^{\frac{\triangle\varepsilon}{kT}}-1) \neq f_n(n). \quad (7)$$

The equivalence relation does not depend on n, so it is possible to use formula (7) to find the series sum. But first need to make sure that get the same result for the first unit interval, $0 \div 1$:

$$A_1 = \frac{\int_0^1 f_n(x)dx}{a_n} = \frac{-\frac{kT}{\triangle\varepsilon}(1-e^{\frac{\triangle\varepsilon}{kT}})}{1} = \frac{kT}{\triangle\varepsilon}(e^{\frac{\triangle\varepsilon}{kT}} - 1).$$

Then, according to (5) and (7), we establish an isomorphism and find the partition function in the direct calculation formula form [6], which can be used to directly calculate the series sum

$$\Sigma_{n=1}^\infty a_n \cong \frac{1}{A}\int_0^\infty f_n(x)dx/\Sigma_{n=1}^\infty e^{-\frac{(n-1)\triangle\varepsilon}{kT}} = \frac{\frac{kT}{\triangle\varepsilon}e^{\frac{\triangle\varepsilon}{kT}}}{\frac{kT}{\triangle\varepsilon}(e^{\frac{\triangle\varepsilon}{kT}} - 1)} = \frac{e^{\frac{\triangle\varepsilon}{kT}}}{e^{\frac{\triangle\varepsilon}{kT}} - 1} = \frac{1}{1 - e^{-\frac{\triangle\varepsilon}{kT}}}. \quad (8)$$

Using (8), we obtain the Boltzmann distribution calculation formula (1) with the usual indexing $i = n$

$$P_i = \frac{N_i}{N} = e^{-\frac{(i-1)\triangle\varepsilon}{kT}}/\Sigma_{i=1}^\infty e^{-\frac{(i-1)\triangle\varepsilon}{kT}} = e^{-\frac{(i-1)\triangle\varepsilon}{kT}} \bigg/ \frac{e^{\frac{\triangle\varepsilon}{kT}}}{e^{\frac{\triangle\varepsilon}{kT}} - 1} = e^{-\frac{i\triangle\varepsilon}{kT}}(e^{\frac{\triangle\varepsilon}{kT}} - 1). \quad (9)$$

So, the entropy calculations have been dramatically simplified and become more accurate and variable [8–10]. The results of the calculation statistical sum members according to the formula (3), their sum accumulation for comparison with the calculation data according to the formula for the sum direct expression (8) are shown in Table. And Boltzmann distribution results in the form (9) to control $P_i \to 0$ and the sum convergence P_i to unity. Arbitrary $1000K$ temperature values and energy variation step $\triangle\varepsilon = \frac{1000k}{2} = 6,9032 \cdot 10^{-21}$ J·particle^{-1} were chosen for calculation.

Direct calculation by (8) gives the value $\Sigma_{i=1}^\infty = 2.5415$, from which the stepwise summation result is only 0.0001 less. The accuracy of calculating the statistical sum, and indeed any convergent series, can be determined or specified if the direct expression of this sum through the equivalence relation A is known, since it is possible to select any interval of the sum in a commensurate set of its members.

In this case, as in the general case, it is of great importance to set the exact limits of the discrete and integral sums in order to avoid discontinuities in the continuous summation by at least one unit interval. The fact is that the series is given by the common term number inclusive that is, by the interval from $(n-1)$ to n.

Therefore, to cover the area $a_n \times 1$, we need to start integrating from a value equal to $n-1$, and end with the series higher term number, $m > n$. Then the partial sums in the continuous coverage of the series, from a_1 to a_∞ can be reflected with the appropriate limits of integration: from a_1 to $a_{n>1}$, that is $\int_{x=0}^{x=n} f_n(x)dx$; from a_{n+1} to $a_{m>n} - \int_{x=n}^{x=m>n} f_n(x)dx$; from a_{m+1} to $a_\infty - \int_{x=m}^{\infty} f_n(x)dx$.

This implies the strict equality of the total sum of the series s to its initial s_n ("partial", "final"), intermediate s_m ("middle") and residual R_m parts

$$s = s_n + s_m + R_m. \tag{10}$$

If any of the parts is missing (there is no need to take it into account), then the remaining ones adjoin one another with the transition of the previous part upper limit to the next one lower limit, as established above. In this case, it is possible to determine one of the three sums by difference with the other two. The same possibility of difference calculation is applicable to the total sum (10).

Determining the equivalence relation of the series and the improper integral and establishing their isomorphism drastically simplifies computational procedures, since it allows one to find calculation formulas for all terms of equality (10)

$$s_n = \frac{1}{A}\int_0^n f_n(x)dx = \frac{1}{\frac{kT}{\triangle\varepsilon}(e^{\frac{\triangle\varepsilon}{kT}}-1)}\int_{x=0}^{x=n} e^{-\frac{(n-1)\triangle\varepsilon}{kT}}dx = \frac{1-e^{-\frac{n\triangle\varepsilon}{kT}}}{1-e^{-\frac{\triangle\varepsilon}{kT}}},$$

$$\Sigma P_n = \frac{s_n}{s} = \frac{1-e^{-\frac{n\triangle\varepsilon}{kT}}}{1-e^{-\frac{\triangle\varepsilon}{kT}}} \Big/ \frac{1}{1-e^{-\frac{\triangle\varepsilon}{kT}}} = 1-e^{-\frac{n\triangle\varepsilon}{kT}},$$

$$s_m = \frac{1}{A}\int_{x=n}^{x=m>n} f_n(x)dx = -\frac{1}{A}\cdot\frac{kT}{\triangle\varepsilon}\Big|e^{-\frac{(x-1)\triangle\varepsilon}{kT}}\Big|_n^m = e^{-\frac{n\triangle\varepsilon}{kT}}\frac{1-e^{-\frac{(n-m)\triangle\varepsilon}{kT}}}{1-e^{-\frac{\triangle\varepsilon}{kT}}},$$

$$\Sigma P_m = \frac{s_m}{s} = e^{-\frac{n\triangle\varepsilon}{kT}}\frac{1-e^{\frac{(n-m)\triangle\varepsilon}{kT}}}{1-e^{-\frac{\triangle\varepsilon}{kT}}} \Big/ \frac{1}{1-e^{-\frac{\triangle\varepsilon}{kT}}} = e^{-\frac{n\triangle\varepsilon}{kT}} - e^{-\frac{m\triangle\varepsilon}{kT}},$$

$$R_m = \frac{1}{A}\int_{x=m}^{\infty} f_n(x)dx = -\frac{1}{A}\cdot\frac{kT}{\triangle\varepsilon}\Big|e^{-\frac{(x-1)\triangle\varepsilon}{kT}}\Big|_m^\infty = \frac{e^{-\frac{m\triangle\varepsilon}{kT}}}{1-e^{-\frac{\triangle\varepsilon}{kT}}}, \tag{11}$$

$$\Sigma P_{R_m} = \frac{R_m}{s} = \frac{e^{-\frac{m\triangle\varepsilon}{kT}}}{1-e^{-\frac{\triangle\varepsilon}{kT}}} \Big/ \frac{1}{1-e^{-\frac{\triangle\varepsilon}{kT}}} = e^{-\frac{m\triangle\varepsilon}{kT}}. \tag{12}$$

In this case, the ΣP_s total value is unity:

$$\Sigma P_s = \Sigma P_n + \Sigma P_m + \Sigma P_{R_m} = 1 - e^{\frac{-n\triangle\varepsilon}{kT}} + e^{\frac{-n\triangle\varepsilon}{kT}} - e^{\frac{-m\triangle\varepsilon}{kT}} + e^{\frac{-m\triangle\varepsilon}{kT}} = 1.$$

This indicates each relative sum probabilistic meaning, as follows from the Boltzmann distribution itself, which is a series of relative P_i values that sum to one. This result indicates the need for the constant joint presence of all series members as a whole.

The same is displayed by expressions for ΣP_n, ΣP_m and ΣP_{Rm}, containing $e^{\frac{-n\triangle\varepsilon}{kT}}$ and $e^{\frac{-m\triangle\varepsilon}{kT}}$ exponents. There are some arbitrary energy quantities $n\triangle\varepsilon$ and $m\triangle\varepsilon$ opposed to the kT system thermal energy state. And the corresponding probabilities of overcoming energy barriers $n\triangle\varepsilon$ or $m\triangle\varepsilon$ by the particles of the system (or not overcoming them at values $1-\Sigma P_n$ and $1-\Sigma P_m$).

This primarily refers to the activation energy, which expresses the probability of exceeding the barrier ε_a by the exponent $exp(-\varepsilon_a/(kT)$ in various chemical, physical, and mechanical processes [18]. The found energy quantity impact probability and its contribution to the process general conditions everything is not specified. In the rate constant, which was developed by Arrhenius in the 19^{th} century, the exponent included in it is treated as a thermodynamically determined quantity, without connection with the Boltzmann distribution, and even more so without analytical justification and expression.

2 Randomized particles concept

Much more modern is the randomized particles concept (RPC) proposed at the beginning of the 21^{st} century by the authors of [19, 20] which is entirely based on overcoming or not overcoming the natural thermal barriers of melting kT_m and boiling kT_b by its fractional (probabilistic) content of three energy particles classes: with energy less than kT_m – crystal-mobile (crm); with energy greater than kT_m, but less than kT_b – liquid-mobile (lqm); with energy more than kT_b – vapor-mobile (vm). The entire spectrum of these particles is present at any temperature and in any state of substance aggregation, in their probabilities sum it is unity with the difference from it to the fraction of *crm*-particles through P_{crm} – the fraction of kT_m superbarrier particles and from the difference with this fraction of vapor-mobile particles P_{vm} – the fraction liquid particles:

$$P_{crm} = 1 - exp(-kT_m) = 1 - exp(-T_m/T),$$

$$P_{lqm} = exp(-T_m/T) - exp(-T_b/T),$$

$$P_{vm} = exp(-T_b/T),$$

$$P_{crm} + P_{lqm} + P_{vm} = 1.$$

Each particle's energy variety plays its role in ensuring the corresponding state of aggregation stability and in preparing for the transition to it from other states, depending on the substance temperature, based on system-wide criteria for limiting stability, in particular, in proximity to the golden ratio.

In this case, the relationship between changes in the content of particles certain types and some substance physicochemical properties, for example, with the metal plasticity, the melts viscosity, and other properties, are revealed. With the advent of the randomized particles concept, finally, a general "zero model" of matter as a whole took place, and it goes back to the universal, and therefore fundamental, Boltzmann distribution according to the primary randomized (thermal) characteristic of matter — its kinetic energy. It is all the more important to strengthen the evidence-based part of RPC in order to eliminate some of its shortcomings.

First, from a physical point of view, the distribution is discrete both in content and form, since the energy is quantized. This quantum of energy should be present at least in a general form in the calculation formulas. Secondly, the possibility of expressing the partition function in a mathematically rigorous isomorphism terms of continuous and discrete distributions must necessarily be taken into account in the Boltzmann distribution, excluding the approximation in determining this sum. Thirdly, each energy class and energy spectrum as a whole must be expressed independently, directly, thereby proving the possibility of their isolation and unity in accordance with the superposition principle as a fundamental property of complex systems.

3 Isomorphism of discrete and continuous mappings of the Boltzmann distribution

The discreteness and continuity isomorphism analysis, which refers to accuracy expression of the partition function terms calculating, and along with this, the particles distribution over energy levels P_i with a further entropy definition, is necessary for a more rigorous Boltzmann distribution justification. In this case, the residual sum R_m (11) and its relative share P_{Rm} (12) play a special role. To determine the calculation accuracy of all discrete quantities, only the residual sum related to series all members a_n matters, so we use the index n' for the obtained absolute R_n and relative P_{Rn} expressions of the residual sum. Since it contains the terms closest to zero, which have a residual, almost negligible value, then it determines the calculating accuracy of the series sum δ (up to the residual sum ones)

$$\delta = \frac{R_n}{s} = P_{Rn} = e^{-\frac{n'\triangle\varepsilon}{kT}}. \qquad (13)$$

We obtain the calculation formula excluding from (13) the required number of the series terms n' to calculate their share with a given accuracy δ,

$$n' = -\frac{kT}{\triangle\varepsilon} ln\delta. \qquad (14)$$

The higher δ , the greater the number of terms of the series must be taken into account.

So, we limited ourselves to the results rounded to the nearest 10^{-4} when calculating the data in the Table. Substitute in (14) the used values $\triangle\varepsilon = 6.9032 \cdot 10^{-21} J \cdot \text{particle}^{-1}$, $T = 1000K$ and $k = 1.38064 \cdot 10^{-23} J \cdot \text{particle}^{-1} K^{-1}$, we obtain the value $n' = 18.4$. Rounding of the table data ends exactly in the interval $n = 1819$. Let's set a rougher accuracy limit $\delta = 0.001$, we get $n' = 13.8$, and this corresponds to the table data, where all values after $n = 13 \div 14$ cease to differ in the third decimal place. For control, we will take an even rougher accuracy $\delta = 0.01(1\%)$ and get $n' = 9.2$. We are convinced that in the interval $n = 9 \div 10$ the data difference ends by more than 1%, abs.

Table

Partition function members a_i, its accumulation $\Sigma_{i=1}^{i} a_i$, Boltzmann distribution P_i and its accumulation $\Sigma_{i=1}^{n} P_i$ at $T =$ 1000 K and $\triangle\varepsilon = \frac{1000k}{2}$, $J \cdot particle^{-1}$

i (n)	a_i	$\sum_{i=1}^{i} a_i$	P_i	$\sum_{i=1}^{i} P_i$
1	1	1	0.3933	0.3935
2	0.6065	1.6065	0.2386	0.6321
3	0.3679	1.9744	0.1448	0.7769
4	0.2231	2.1975	0.0878	0.8647
5	0.1353	2.3329	0.0532	0.9179
6	0.0821	2.4150	0.0323	0.9502
7	0.0498	2.4648	0.0196	0.9698
8	0.0302	2.4949	0.0119	0.9817
9	0.0183	2.5133	0.0072	0.9889
10	0.0111	2.5244	0.0044	0.9933
11	0.0067	2.5311	0.0026	0.9959
12	0.0041	2.5352	0.0016	0.9975
13	0.0025	2.5377	0.0010	0.9985
14	0.0016	2.5392	0.0006	0.9991
15	0.0009	2.5401	0.0004	0.9994
16	0.0006	2.5406	0.0002	0.9997
17	0.0003	2.5410	0.0001	0.9998
18	0.0002	2.5412	0.0001	0.9999
19	0.0001	2.5413	0.0000	0.9999
20	0.0001	2.5414	0.0000	1.0000

It is possible to estimate the requirement for the variation interval $\triangle\varepsilon$ by setting it to the smallest value $\triangle\varepsilon = 1.3806410^{-23} J \cdot \text{particle}^{-1}$. In this case, formula (14) takes the form

$$n' = -T \ln \delta. \tag{15}$$

We obtain the series terms required number $n' = 9210$, having set a practically acceptable calculation accuracy $\delta = 10^{-4}$, which would be difficult to implement without taking into account the discrete and continuous distributions isomorphism. Finally, using formula (15), one can determine the convergence condition, discrete and continuous distributions identification: since the latter of them are characterized by infinitely high accuracy, or vanishingly small error, this can be ensured by the condition $\delta \to 0$, under which the required number of series sum terms tends to infinity

$$\frac{\lim n'}{\delta \to 0} = \infty.$$

From formula (14) we release $\triangle\varepsilon$ and find its limit:

$$\triangle\varepsilon = -\frac{kT}{n'} \ln \delta, \quad \frac{\lim \triangle\varepsilon}{n' \to \infty} = 0.$$

Next, the isomorphism conditions (7) with respect to the equivalence relation A

$$\lim_{\triangle\varepsilon \to 0} A = \lim_{\triangle\varepsilon \to 0} \frac{kT}{\triangle\varepsilon}(e^{\frac{\triangle\varepsilon}{kT}} - 1) = \frac{0}{0}.$$

This uncertainty is revealed by L'Hopital's rule:

$$\lim_{\triangle\varepsilon \to 0} A = \lim_{\triangle\varepsilon \to 0} \frac{kT}{\triangle\varepsilon}(e^{\frac{\triangle\varepsilon}{kT}} - 1) = \lim_{\triangle\varepsilon \to 0} \frac{dkT(e^{\frac{\triangle\varepsilon}{kT}} - 1)d\varepsilon}{d\varepsilon} = \lim_{\triangle\varepsilon \to 0} \frac{kT}{kT} e^{\frac{\triangle\varepsilon}{kT}} = 1.$$

Thus, we can assume that establishing the discrete and continuous sequences isomorphism represents a broader scope of the problem than limiting the close proximity of both. And in any case, it involves both internal and external resources of the analyzed mathematical objects in solving the problem.

Conclusion

Establishing the series sum and an improper integral isomorphism requires knowledge or selection of a function that takes the series common term values at the points $x = n$. As such, a function is recommended that completely repeats the structure and the series common term form - a_n-model function, $f_n(x)$. The equivalence relation between the elements of the a_n-model function improper integral and the series of a common term is defined in an arbitrary unit interval and must satisfy the condition

$$A = \frac{\int_{x=n-1}^{n} f_n(x)dx}{a_n} \neq f(n).$$

In this case, it is possible to transfer A to each unit interval and to their entire set as a whole, ensuring complete mutual invertibility of both the two sets the elements and the series sum with an improper integral

$$\sum_{i=1}^{\infty} a_n \cong \frac{1}{A} \int_0^{\infty} f_n(x)dx.$$

The found isomorphism extends to finite sums of both convergent and divergent series.

Thanks to the established algorithm for identifying the series sum and an improper integral isomorphism, the Cauchy and Maclaurin series convergence integral criterion is developed in terms of proving not only convergence, but also determining the series sum.

The presented partition function is defined in its direct expression for the first time in many years of its discovery. Thanks to this, the concept of randomized particles received a more rigorous justification. This allowed the Boltzmann distribution itself to be expressed not in continuity approxima-tion, but taking into account its discreteness, that is, closer to its physical reality. At the same time, it became possible to single out its final and residual sums in the particle energy spectrum, and to associate the latter with a given accuracy of calculating the statistical sum, taking into account the necessary and this sufficient sum members number.

The decisive role of the a_n-model function is revealed in determining the isomorphism and accuracy of the calculation of compared infinite sequences as the most closely related to both the discrete and continuous sides of their isomorphism, automatically appearing along with the common term of the series and disappearing after the establishment or rejection of their isomorphism. Its modest role as a discreteness reducer and carrier to the improper integral of a continuous mapping is reminiscent of very fundamental isomorphic self-organization procedures in nature and in solving general scientific problems, consisting in maintaining their unity through the analytical connection of arbitrary elements of each of such sets.

The a_n-model function decisive role is to determine the isomorphism and compared infinite sequences calculation accuracy, closely related to both the discrete and continuous sides of their isomorphism, automatically appearing along with the series common term and disappearing after the establishment or rejection of their isomorphism. Its modest role as a discreteness reducer and carrier to the improper integral of a continuous mapping is reminiscent of the fundamental isomorphic self-organization procedures in nature and in solving general scientific problems, which consist in maintaining their unity through the analytical connection of such sets arbitrary each element.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Федотов В.Х. О непрерывных и дискретных моделях эволюции / В.Х. Федотов, Н.И. Кольцов // Вестн. Казан. технолог. ун-та. — 2014. — № 17. — С. 9–13.

2 Парийская Е.Ю. Сравнительный анализ математических моделей и подходов к моделированию и анализу непрерывно-дискретных систем / Е.Ю. Парийская // Дифференциальные уравнения и процессы управления. — 1997. — № 1. — С. 91–120.

3 Kvasnikov V.P. Technology for restoring functional dependencies to determine reliability parameters / V.P. Kvasnikov, S.V. Yehorov, T.Yu. Shkvarnytska // Bulletin of the Karaganda University. Mathematics series. — 2021. — № 1(101). — С. 78–86. https://doi.org/10.31489/2021m1/78-86

4 Битюков В.И. Изоморфизм. Математический энциклопедический словарь / В.И. Битюков, А.Н. Колмогоров. — М.: Сов. энцикл., 1988. — 224 с.

5 Бронштейн И.Н. Справочник по математике для инженеров и учащихся втузов / И.Н. Бронштейн, К.А. Семендяев. — М.: Наука, 1986. — 544 с.

6 Малышев В.П. О взаимосвязи и соразмерности дискретных и непрерывных зависимостей / В.П. Малышев, А.М. Макашева, Ю.С. Зубрина (Красикова) // Докл. НАН РК. — 2016. — № 1. — С. 49–56.

7 Малышев В.П. Распределение Больцмана как бесконечная последовательность и сходящийся ряд / В.П. Малышев, Ю.С. Зубрина, А.М. Макашева // Вестн. НАН РК. — 2017. — № 1. — С. 49–58.

8 Malyshev V.P. Quantization of particle energy in the analysis of the Boltzmann distribution and entropy / V.P. Malyshev, Yu.S. Zubrina, A.M. Makasheva // Journal of Mathematics and System Science. — 2017. — Vol. 7. — P. 278–288. https://doi.org/10.17265/2159-5291/2017.10.002

9 Malyshev V.P. Analytical determination of the statistic Sum and the Boltzmann distribution / V.P. Malyshev, Yu.S. Zubrina, A.M. Makasheva // Abstracts of the XXI International Conference on Chemical Thermodynamics in Russia. — Novosibirsk: Akademgorodok, 2017. — P. 74.

10 Малышев В.П. Распределение и энтропия Больцмана как бесконечные сходящиеся последовательности / В.П. Малышев, А.М. Макашева, Ю.С. Красикова // Вестн. Караганд. ун-та. Сер. Физика. — 2018. — № 3(91). — С. 42–58.

11 Фихтенгольц Г.М. Курс дифференциального и интегрального исчисления / Г.М. Фихтенгольц. — Т. 2. — М.: Физматлит, 2003.

12 Graham R.L. Concrete Mathematics / R.L. Graham, D.E. Knuth, O. Patashnik. — Massachusetts: Addison-Wesley, 2nd ed., 1994. — 604 p.

13 Бермант А.Ф. Краткий курс математического анализа / А.Ф. Бермант. — М.: Наука, 1964. — С. 576–578.

14 Ramnath S. On the isomorphism of expressions / S. Ramnath, P.Y. Zhao // Information Processing Letters. — 2000. — Vol. 74. — No. 3-4. — P. 97–102.

15 Keshvardoost K. Definable Isomorphism Problem / K. Keshvardoost, B. Klin, S. Lasota, J. Ochremiak, S. Torunczyk // Logical methods in Computer Science. — 2019. — Vol. 15. — No. 4. — P. 14–19.

16 Ibraev Sh.Sh. Cohomology of simple modules for algebraic groups / Sh.Sh. Ibraev, L.S. Kainbaeva, S.K. Menlikhozhayeva // Bulletin of the Karaganda University. Mathematics series. — 2020. — № 1(97). — С. 37–43. https://doi.org/10.31489/2020M1/37-43

17 Dummit D.S. Abstract algebra / D.S. Dummit, R.M. Foote. — Wiley: Hoboken, NJ, 2004. — 946 p.

18 Малышев В.П. Распределение Больцмана как основа универсального выражения энергии активации вязкого течения, химических реакций и механического разрушения / В.П. Малышев, А.М. Макашева // Цветные металлы. — 2018. — № 10. — С. 6–12. https://doi.org/10.17580/tsm.2018.10.01

19 Малышев В.П. Концепция хаотизированных частиц как основа единого отображения твердого, жидкого и газообразного состояний вещества / В.П. Малышев, А.М. Нурмагамбетова (Макашева) // Вестн. Казах. нац. ун-та. Сер. Химия. — 2004. — № 3. — С. 53–67.

20 Malyshev V.P. Invariants of ratio of crystal-mobile, liquid-mobile, and vaporized chaotized particles in solid, liquid, and gas states of substance / V.P. Malyshev, A.M. Makasheva, L.A. Bekbaeva // Bulletin of the University of Karaganda – Chemistry — 2021. — No. 4(104). — P. 69–78. https://doi.org/10.31489/2021Ch4/69-78

Қатардың қосындысы мен меншіксіз интегралдың изоморфизмі туралы теоремалар

В.П. Малышев[1], С.Ш. Кажикенова[2], А.М. Мақашева[1], А.Ш. Кажикенова[3]

[1] *Ж. Әбішев атындағы Химия-металлургия институты, Қарағанды, Қазақстан;*
[2] *Ә. Сағынов атындағы Қарағанды техникалық университеті, Қарағанды, Қазақстан;*
[3] *Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан*

Мақалада дискретті және үздіксіз тәуелділіктердің байланысы туралы мәселе зерттелген. Коши қатарының үйлесімділігінің кеңейтілген интегралдық белгісі негізінде a_n қатарының жалпы мүшесінің эквиваленттік қатынасы және олардың өзгеруінің бірлік интервалы аралығындағы a_n-тәрізді функцияның орташа интегралдық шамасы арқылы қатардың соңғы және толық қосындыларын анықтау алгоритмі жасалды. Больцман үлестіріміндегі статистикалық қосынды үшін мұндай сомаларды анықтау мысалдары келтірілген, олар алғаш рет a_n-тәрізді функция арқылы тікелей көрсетілді. Бұл осы соманың белгілі бір дәлдігімен берілген мәнге дейін қатардың қосындысын жинақтау үшін есептеулер жүргізу қажеттілігін жояды. Сонымен қатар, бұл жағдайда энергияның өзгеру аралығын кез келген дәлдікпен өзгертуге мүмкіндік береді. Жүргізілген зерттеулер технологиялық процестердегі жылу энергиясының жоғалуын анықтайтын энтропияны есептеу үшін Больцманның таралуын (энергия спектрін) тікелей қолдана отырып, физика мен материалтану саласындағы теориялық және практикалық мәселелерді шешуге мүмкіндік береді.

Кілт сөздер: изоморфизм, қатар қосындысы, меншіксіз интеграл, бірлік интервал, эквиваленттік қатынас.

Теоремы об изоморфизме суммы ряда и несобственного интеграла

В.П. Малышев[1], С.Ш. Кажикенова[2], А.М. Макашева[1], А.Ш. Кажикенова[3]

[1] *Химико-металлургический институт имени Ж. Абишева, Караганда, Казахстан;*
[2] *Карагандинский технический университет имени Абылкаса Сагинова, Караганда, Казахстан;*
[3] *Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан*

Исследован вопрос о взаимосвязи дискретных и непрерывных зависимостей. На основе расширенного интегрального признака сходимости ряда Коши разработан алгоритм определения конечной и полной сумм ряда через отношение эквивалентности общего члена ряда a_n и среднеинтегральной величины a_n-образной функции в пределах единичного интервала их изменения. Приведены примеры определения таких сумм для статистической суммы в распределении Больцмана, впервые непосредственно выраженной через a_n-образную функцию. Это исключает необходимость проведения расчетов по накоплению суммы ряда до значения, которое задается определенной точностью этой суммы. Кроме того, она позволяет в данном случае варьировать интервал вариации энергии с любой заданной точностью. Проведенные исследования позволяют решать как теоретические, так и практические задачи физики и материаловедения, непосредственно используя распределение (энергетический спектр) Больцмана для расчета энтропии, определяющей потери тепловой энергии в технологических процессах.

Ключевые слова: изоморфизм, сумма ряда, несобственный интеграл, единичный интервал, отношение эквивалентности.

References

1 Fedotov, V.Kh., & Koltsov, N.I. (2014). O nepreryvnykh i diskretnykh modeliakh evoliutsii [On continuous and discrete models of evolution]. *Vestnik Kazanskogo tekhnologicheskogo universiteta — Bulletin of the Kazan Technological University,* 17, 9–13 [in Russian].

2 Pariiskaia, E.Yu. (1997). Sravnitelnyi analiz matematicheskikh modelei i podkhodov k modelirovaniiu i analizu nepreryvno-diskretnykh sistem [Comparative analysis of mathematical models and approaches to modeling and analysis of continuous-discrete systems]. *Differentsialnye uravneniia i protsessy upravleniia — Differential Equations and Control Processes,* 1, 91–120 [in Russian].

3 Kvasnikov, V.P., Yehorov, S.V. & Shkvarnytska, T.Yu. (2021). Technology for restoring functional dependencies to determine reliability parameters. *Bulletin of the Karaganda University. Mathematics series,* 1(101), 78–86. https://doi.org/10.31489/2021M1/78-86

4 Bitiukov, V.I., & Kolmogorov, A.N. (1988). *Izomorfizm. Matematicheskii entsiklopedicheskii slovar [Isomorphism. Mathematical Encyclopedic Dictionary].* Moscow: Sovetskaia entsiklopediia [in Russian].

5 Bronshtein, I.N., & Semendiaev, K.A. (1986). S*pravochnik po matematike dlia inzhenerov i uchashchikhsia vtuzov [Handbook of mathematics for engineers and students of technical universities].* Moscow: Nauka [in Russian].

6 Malyshev, V.P., Makasheva, A.M., & Zubrina, Yu.S. (2016). O vzaimosviazi i sorazmernosti diskretnykh i nepreryvnykh zavisimostei [On the relationship and proportionality of discrete and continuous dependencies]. *Doklady Natsionalnoi akademii nauk Respubliki Kazakhstan — Reports of the National Academy of Sciences of the Republic of Kazakhstan,* 1, 49–56 [in Russian].

7 Malyshev, V.P., Zubrina, Yu.S., & Makasheva, A.M. (2017). Raspredelenie Boltsmana kak beskonechnaia posledovatelnost i skhodiashchiisia riad [The Boltzmann distribution as an infinite sequence and a convergent series]. *Vestnik Natsionalnoi akademii nauk Respubliki Kazakhstan — Bulletin of the National Academy of Sciences of the Republic of Kazakhstan,* 1, 49–58 [in Russian].

8 Malyshev, V.P., Zubrina, Yu.S., & Makasheva, A.M. (2017). Quantization of particle energy in the analysis of the Boltzmann distribution and entropy. *Journal of Mathematics and System Science, 7,* 278–288. https://doi.org/10.17265/2159-5291/2017.10.002

9 Malyshev, V.P., Zubrina, Yu.S., & Makasheva, A.M. (2017). Analytical determination of the statistic Sum and the Boltzmann distribution. *Abstracts of the XXI International Conference on Chemical Thermodynamics in Russia,* 74.

10 Malyshev, V.P., Makasheva, A.M., & Krasikova, Yu.S. (2018). Raspredelenie i entropiia Boltsmana kak beskonechnye skhodiashchiesia posledovatelnosti [Distribution and entropy of Boltzmann as infinite convergent sequences]. *Vestnik Karagandinskogo universiteta. Seriia Fizika — Bulletin of Karaganda University, Physics series,* 3(91), 42–58 [in Russian].

11 Fikhtengolts, G.M. (2003). *Kurs differentsialnogo i integralnogo ischisleniia [Course of differential and integral calculus].* Moscow: Fizmatlit [in Russian].

12 Graham, R.L., Knuth, D.E., & Patashnik, O. (1994). *Concrete Mathematics,* 2nd ed. Massachusetts: Addison-Wesley.

13 Bermant, A.F. (1964). *Kratkii kurs matematicheskogo analiza [A Short Course in mathematical analysis].* Moscow: Nauka [in Russian].

14 Ramnath, S., & Zhao, P.Y. (2000). On the isomorphism of expressions. *Information Processing Letters, 74*(3–4), 97–102.

15 Keshvardoost, K., Klin, B., Lasota, S., Ochremiak, J., & et al. (2019). Definable Isomorphism Problem. *Logical methods in Computer Science, 15*(4), 14.

16 Ibraev, Sh.Sh., Kainbaeva, L.S., & Menlikhozhayeva, S.K. (2020). Cohomology of simple modules for algebraic groups. *Bulletin of the Karaganda University. Mathematics series, 1*(97), 37–43. https://doi.org/10.31489/2020M1/37-43

17 Dummit, D.S., & Foote, R.M. (2004). *Abstract algebra.* Hoboken: NJ, Wiley.

18 Malyshev, V.P., & Makasheva, A.M. (2018). Raspredelenie Boltsmana kak osnova universalnogo vyrazheniia energii aktivatsii viazkogo techeniia, khimicheskikh reaktsii i mekhanicheskogo razrusheniia [Boltzmann distribution as the basis for a universal expression of the activation energy of viscous flow, chemical reactions and mechanical destruction]. *Tsvetnye metally — Non-ferrous metals,* 10, 6–12 [in Russian]. https://doi.org/10.17580/tsm.2018.10.01

19 Malyshev, V.P., & Makasheva, A.M. (2004). Kontseptsiia khaotizirovannykh chastits kak osnova edinogo otobrazheniia tverdogo, zhidkogo i gazoobraznogo sostoianii veshchestva [The concept of randomized particles as the basis for a unified display of solid, liquid and gaseous states of matter]. *Vestnik Kazakhskogo natsionalnogo universiteta. Seriia Khimiia — Bulletin of the Kazakh National University, Chemistry series, 3*(35), 53–67 [in Russian].

20 Malyshev, V.P., Makasheva, A.M., & Bekbaeva, L.A. (2021). Invariants of ratio of crystal-mobile, liquid-mobile, and vaporized chaotized particles in solid, liquid, and gas states of substance. *Bulletin of the University of Karaganda – Chemistry,* 4(104), 69–78. https://doi.org/10.31489/2021Ch4/69-78

*Author Information**

Vitalyi Pavlovich Malyshev — Doctor of technical sciences, Professor, Chief Researcher at the Laboratory of Entropy and Information Analysis of Complex Physicochemical Systems, Abishev Chemical and Metallurgical Institute, 63 Ermekov street, Karaganda, 100009, Kazakhstan; e-mail: *eia_hmi@mail.ru*; https://orcid.org/0000-0002-3996-1533

Saule Sharapatovna Kazhikenova — Doctor of technical sciences, Professor, Head of the Department of Higher Mathematics, Abylkas Saginov Karaganda Technical University, 56 Nursultan Nazarbayev Avenue, Karaganda, 100000, Kazakhstan; e-mail: *sauleshka555@mail.ru*; https://orcid.org/0000-0002-6937-1577

Astra Mundukovna Makasheva (*corresponding author*) — Doctor of technical sciences, Professor, Head of the Laboratory of Chief Researcher at the Laboratory of Entropy and Information Analysis of Complex Physicochemical Systems, Abishev Chemical and Metallurgical Institute, 63 Ermekov street, Karaganda, 100009, Kazakhstan; e-mail: *astra_mun@mail.ru*; https://orcid.org/0000-0003-2249-3435

Aigul Sharapatovna Kazhikenova — Candidate of technical sciences, Associate Professor of the Department of Higher Mathematics, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *aigul-kazhikenova@mail.ru*; https://orcid.org/0000-0003-4053-1546

*The author's name is presented in the order: First, Middle and Last Names.

https://doi.org/10.31489/2024M1/140-148 Research article

A boundary value problem for the fourth-order degenerate equation of the mixed type

J.A. Otarova

Karakalpak State University named after Berdakh, Nukus, Uzbekistan
(E-mail: j.otarova@mail.ru)

Many problems in mechanics, physics, and geophysics lead to solving partial differential equations that are not included in the known classes of elliptic, parabolic or hyperbolic equations. Such equations, as a rule, began to be called non-classical equations of mathematical physics. The theory of degenerate equations is one of the central branches of the modern theory of partial differential equations. This is primarily due to the identification of a variety of applied problems, the mathematical modeling of which serves the study of various types of degenerate equations. The study of boundary value problems for mixed type's equations of the fourth-order with power-law degeneration remains relevant. In this work, a boundary value problem in a rectangular domain for a degenerate equation of the fourth-order mixed-type is posed and investigated. Well-posedness of the boundary value problem for a fourth-order partial differential equation is established by proving the existence and uniqueness of the solution. Under sufficient conditions, a solution to the problem under consideration was explicitly found by the variable separation method.

Keywords: fourth-order mixed type equation, Bessel functions, Fourier series, completeness, regular solution.

2020 Mathematics Subject Classification: 35M12.

Introduction

In the modern theory of partial differential equations, the studies of degenerate equations and equations of a mixed type occupy an important place, which is explained both by the theoretical significance of the results obtained and the presence of their practical applications in the gas dynamics of transonic flows, magnetic hydrodynamics, in the theory of infinitesimal bending of surfaces, in various sections of continuum mechanics and other branches of knowledge.

The fundamental results for second-order degenerate equations of elliptic type were obtained by academician M.V. Keldysh (1951), where first the cases were indicated in which the characteristic part of the domain boundary can be freed from boundary conditions, which are then replaced by the condition of boundedness of solutions. The work of M.V. Keldysh spurred numerous further studies in the direction he indicated. Later, A.V. Bitsadze noted in his work that the boundedness condition can be replaced by a boundary condition with a certain weight function. The results he obtained were then developed and generalized by O.A. Oleinik. It is also worth noting the works of V.P. Glushko, Yu.B. Savchenko [1], S.A. Iskhokov [2], S.N. Sidorov [3]. In particular, the work [2] was focused on the study of the unique solvability of a variational problem for an elliptic equation with a "non-power" degeneration. It is also noteworthy that a "power" degeneration was initiated in the research of M.I. Vishik and V.V. Grushin. Degenerate elliptic equations of high order, including their connection with pseudodifferential operators, were studied in the works of A.D. Baev [4–6].

K.B. Sabitov [7] investigated the Dirichlet problem for the second-order degenerate equation of the mixed type of first kind in a rectangular domain. By the methods of spectral analysis, the criteria of uniqueness of a solution that is constructed in the form of the sum of a Fourier series was established. The question of the correctness of the formulation of the Dirichlet problem depending on

Received: 10 July 2023; *Accepted:* 15 December 2023.

the degree of degeneracy was investigated for a mixed type equation of second kind by K.B. Sabitov and A.Kh. Tregubova (Suleimanova) [8, 9]. A boundary-value problem with nonlocal boundary conditions for a mixed type equation was studied by M.E. Lerner and O.A. Repin in work [10]. The uniqueness of solutions of the problem was proved by using the principle of extremum, the existence of solutions of the problem was proved by methods of integral transformations and equations. Nonlocal boundary value problems of Bitsadze-Samarsky type for a fourth-order mixed type equation were studied by L.R. Rustamova in [11]. Many authors also studied boundary value problems for degenerate equations [12–14]. In the present paper, the boundary value problem is studied for the fourth-order degenerate equation in a rectangular domain.

1 *Problem formulation*

In the domain
$$\Omega = \{(x,t) :\ 0 < x < b,\ -T < t < T\,,\ \ T > 0\}$$
we consider the equation
$$Lu \equiv \operatorname{sgn} t \cdot |t|^m u_{xxxx} - u_{tt} + a^2 \operatorname{sgn} t \cdot |t|^m u = 0, \tag{1}$$
where $m = const > 0,\ a = const \geq 0$. The equation $Lu = 0$ for $t > 0$ has the form
$$|t|^m \left(u_{xxxx} + a^2 u\right) - u_{tt} = 0, \tag{2}$$
for $t < 0$
$$|t|^m \left(u_{xxxx} + a^2 u\right) + u_{tt} = 0. \tag{3}$$
We denote $\Omega^+ = \Omega \cap (t > 0)\,, \Omega^- = \Omega \cap (t < 0)\,.$

Problem A. Find in the domain Ω a bounded function $u\,(x,t)$ satisfying the conditions
$$u\,(x,t) \in C\left(\bar{\Omega}\right) \cap C^{2,1}_{x,t}\,(\Omega) \cap C^{4,2}_{x,t}\left(\Omega^+ \cup \Omega^-\right), \tag{4}$$
$$Lu = 0, \quad (x,t) \in \Omega, \tag{5}$$
$$\frac{\partial^k u}{\partial t^k}\,(x,+0) = \frac{\partial^k u}{\partial t^k}\,(x,-0)\,, \quad k = 0,1 \tag{6}$$
$$\left.\begin{array}{c} u\,(0,t) = u\,(b,t) = 0, -T \leq t \leq T, \\ u_{xx}\,(0,t) = u_{xx}\,(b,t) = 0, -T \leq t \leq T, \end{array}\right\} \tag{7}$$
$$u\,(x,T) = \varphi\,(x)\,, \quad 0 \leq x \leq b, \tag{8}$$
$$u\,(x,-T) = \psi\,(x)\,, \quad 0 \leq x \leq b, \tag{9}$$
$\varphi\,(x)$ and $\psi\,(x)$ are given sufficiently smooth functions, moreover $\varphi\,(0) = \varphi\,(b) = 0$, $\psi\,(0) = \psi\,(b) = 0$.

2 *The existence of a solution*

To prove the existence of a solution to the problem, we use the method of separation of variables, i.e. particular solutions of equation (1) that are not equal to zero in the domain Ω, will be sought in the form of a product $u\,(x,t) = X\,(x) \cdot T\,(t)\,,$ satisfying zero boundary conditions (7). The following theorem holds:

Theorem 1.
$$\left\{\begin{array}{l} \gamma\,(k) = K_{1/(2q)}\ (p_k T^q) \neq 0, \\ \delta\,(k) = \tilde{Y}_{1/(2q)}\ (p_k T^q) \neq 0. \end{array}\right. \tag{10}$$

Proof. By substituting this product into equation (1), we obtain

$$X^{IV}(X) - \lambda^4 X(X) = 0, \quad 0 < x < b, \tag{11}$$

we solve equation (11) with conditions (7), which change to the following

$$X(0) = X(b) = X''(0) = X''(b) = 0, \tag{12}$$

$$T''(t) - \left(\lambda^4 + a^2\right) \operatorname{sgn} \cdot |t|^m T(t) = 0, \quad -T < t < T, \tag{13}$$

where λ is the separation constant.

The solution to problem (11), (12) has the form

$$X_k(x) = \sqrt{\frac{2}{b}} \sin \lambda_k x, \quad \lambda_k = \frac{k\pi}{b}, \quad k = 1, 2, \ldots \tag{14}$$

From equation (13), following [15], (with $\lambda = \lambda_k$) for $t > 0$ we obtain

$$T(t) = W(p_k t^q)\sqrt{t} = \sqrt{t}\, W(z), \tag{15}$$

in which $q = (m+2)/2$, $p_k^2 = \left(a^2 + \lambda_k\right)/q^2$. Then we obtain the modified Bessel equation [16]

$$W''(z) + \frac{1}{z} W'(z) - \left(1 + \frac{\nu^2}{z^2}\right) W(z) = 0,$$

where $z = p_k t^q$, $\nu = 1/(2q) = 1/(m+2) \in (0, 1/2)$, the general solution of which is determined by the formula as follows

$$W(z) = C_1 I_{1/(2q)}(z) + C_2 K_{1/(2q)}(z), \quad z > 0, \tag{16}$$

where $I_{1/(2q)}(z)$ and $K_{1/(2q)}(z)$ are the modified Bessel functions and C_1, C_2 are arbitrary constants. Taking into account (15), (16), the general solution (13) for $t > 0$ can be written as

$$T_k^+(t) = A_k \sqrt{t} I_{1/(2q)}(p_k t^q) + B_k \sqrt{t} K_{1/(2q)}(p_k t^q), \quad t > 0, \tag{17}$$

where A_k, B_k are arbitrary constants.

In the same way, from equation (13) for t<0 we get

$$T(t) = Z(p_k (-t)^q)\sqrt{-t} = \sqrt{-t} Z(z), \tag{18}$$

and the Bessel equation

$$Z''(z) + \frac{1}{z} Z'(z) + \left(1 - \frac{\nu^2}{z^2}\right) Z(z) = 0.$$

The general solution is written as

$$Z(z) = C_1 J_{1/(2q)}(z) + C_2 Y_{1/(2q)}(z), \quad z > 0, \tag{19}$$

where $J_{1/(2q)}(z)$, $Y_{1/(2q)}(z)$ are the Bessel functions. Taking into account (18), (19), the general solution of equation (13) for $t < 0$ can be written as

$$T_k^-(t) = C_k \sqrt{-t} J_{1/(2q)}(p_k (-t)^q) + D_k Y_{1/(2q)}(p_k (-t)^q), \quad t < 0,$$

where C_k, D_k are arbitrary constants.

Therefore, the solutions of equation (13) for $t > 0$ have the form of (17), and for $t < 0$ have the form of (21). To find the unknown constants A_k, B_k, C_k, D_k, we use the gluing conditions (6), that respectively change to the following conditions

$$T_k\left(0+0\right) = T_k\left(0-0\right), \tag{20}$$

$$T_k{}'\left(0+0\right) = T_k{}'\left(0-0\right). \tag{21}$$

Condition (20) is satisfied for any A_k, B_k if $D_k = -\pi B_k/2$, and condition (21) is satisfied for $C_k = \pi B_k ctg\left(\pi/(4q)\right)/2 \ - A_k$ and when $D_k = -\pi B_k/2$. Considering all of these, the solution of equation (13) can be written as

$$T_k\left(t\right) = \left\{ \begin{array}{c} T_k^+\left(t\right) = A_k\sqrt{t}I_{1/(2q)}\ \left(p_k t^q\right) + B_k\sqrt{t}K_{1/(2q)}\ \left(p_k t^q\right), \quad t>0, \\ T_k^-\left(t\right) = -A_k\sqrt{-t}J_{1/(2q)}\ \left(p_k(-t)^q\right) - \frac{1}{2}\pi B_k\tilde{Y}_{1/(2q)}\ \left(p_k(-t)^q\right), \quad t<0, \end{array}\right. \tag{22}$$

where

$$\tilde{Y}_{1/(2q)}\ \left(p_k(-t)^q\right) = \frac{\pi}{2\sin\left(\pi/2q\ \right)}\left[J_{1/(2q)}\ \left(p_k(-t)^q\right) + J_{-1/(2q)}\ \left(p_k(-t)^q\right)\right].$$

For function (22), the equality, $T_k{}''\left(0+0\right) = T_k{}''\left(0-0\right)$, holds i.e. functions (22) belong to the class $C^2\left[-T;T\right]$ and satisfy equation (13). Functions (22) are not limited, because $\sqrt{t}I_{1/(2q)}\ \left(p_k t^q\right) \to \infty$, therefore we assume $A_k = 0, \ \forall\, k \in N$, then

$$T_k\left(t\right) = \left\{ \begin{array}{l} T_k^+\left(t\right) = B_k\sqrt{t}K_{1/(2q)}\ \left(p_k t^q\right), \quad t>0, \\ T_k^-\left(t\right) = B_k\tilde{Y}_{1/(2q)}\ \left(p_k(-t)^q\right), \quad t<0. \end{array}\right. \tag{23}$$

3 The uniqueness of the solution

Theorem 2. If there is a solution to problem A, then it is unique when

$$\lim_{x\to 0+0} u_{xx}\left(x,t\right)\sin\frac{\pi k}{p}x = \lim_{x\to p-0} u_{xx}\left(x,t\right)\sin\frac{\pi k}{p}x = 0, \quad T \le t \le -T \tag{24}$$

and if condition (10) is satisfied for all $k \in N$.

Proof. Let $u\left(x,t\right)$ be the solution of problem (4)–(9). Consider the functions (14) which form into $L_2\left(0,b\right)$ a complete orthonormal system.

We denote

$$u\left(x,t\right) = \left\{ \begin{array}{l} u^+\left(x,t\right), (x,t)\in\Omega^+, \\ u^-\left(x,t\right), (x,t)\in\Omega^-. \end{array}\right. \tag{25}$$

We consider the integral

$$\int_0^b u^+\left(x,t\right)X_k\left(x\right)dx = \alpha_k\left(t\right), \ \ k=1,2,\ldots \tag{26}$$

Suppose that the partial derivative $u_{xx}\left(x,t\right)$ satisfies conditions (24).

Differentiating (26) with respect to t twice, taking into account equation (2) and conditions (7) we have

$$\alpha''{}_k\left(t\right) - |t|^m\left(a^2 + \left(\frac{\pi k}{b}\right)^4\right)\alpha_k\left(t\right) = 0, \quad k=1,2,... \tag{27}$$

For negative values of t we denote the integral

$$\int_0^b u^-(x,t)\, X_k(x)\, dx = \beta_k(t), \qquad k = 1,2,\ldots \tag{28}$$

By a similar transformation from (28) and (3) we obtain

$$\int_0^b u^-(x,t)\, X_k(x)\, dx = \beta_k(t), \qquad k = 1,2,\ldots \tag{29}$$

Equations (27) and (29) for $\lambda = \lambda_k$ coincide with equation (13), i.e. for $t > 0,\ \alpha_k(t) = T_k^+(t)$, and for $t < 0$, there will be $\beta_k(t) = T_k^-(t)$, which means that functions $\alpha_k(t)$ and $\beta_k(t)$ are determined by functions (25). To find the coefficients B_k, we use the boundary conditions (8), (9), i.e. $\alpha_k(T) = \varphi_k,\ \ \beta_k(-T) = \psi_k$ (8), (9) and formulas (26), (28), then:

$$\alpha_k(T) = \int_0^b u(x,T) \sin\frac{\pi k}{b} x dx = \int_0^b \varphi(x) \sin\frac{\pi k}{b} x dx = \varphi_k, \tag{30}$$

$$\beta_k(-T) = \int_0^b u(x,-T) \sin\frac{\pi k}{b} x dx = \int_0^b \psi(x) \sin\frac{\pi k}{b} x dx = \psi_k, \tag{31}$$

then from (23), (30) and (31), taking into account condition (10), we have

$$\begin{cases} B_k = \frac{\varphi_k}{\sqrt{T}\gamma(k)}, & t > 0, \\ B_k = \frac{\psi_k}{\sqrt{T}\delta(k)}, & t < 0. \end{cases} \tag{32}$$

Substituting (32) into (23), we find the functions $T_k(t)$:

$$T_k(t) = \begin{cases} \varphi_k \sqrt{\frac{t}{T}} \frac{K_{1/(2q)}\left(p_k t^q\right)}{\gamma(k)}, & t > 0, \\ \psi_k \sqrt{\frac{-t}{T}}_k \frac{\tilde{Y}_{1/(2q)}\left(p_k(-t)^q\right)}{\delta(k)}, & t < 0. \end{cases} \tag{33}$$

Let now $\varphi(x) \equiv 0$ and $\psi(x) \equiv 0$. Then, from equalities (30), (31) and solution (33), it follows that $T_k(t) = 0, \forall\, k \in N$. Therefore, by virtue of (26) and (28)

$$\int_0^b u^+(x,t)\, X_k(x)\, dx = 0, \qquad k = 1,2,\ldots$$

$$\int_0^b u^-(x,t)\, X_k(x)\, dx = 0, \qquad k = 1,2,\ldots$$

Hence follows that $u(x,t) \equiv 0$ for all $x \in [0,b]$ and $t \in [-T;T]$, due to the completeness of system (24) in $L_2(0,b)$.

Based on the Bessel asymptotic formula [16], for large k, we have estimate

$$\begin{cases} \left|\sqrt{k}\delta(k)\right| \geq C > 0, \\ \left|\sqrt{k}\gamma(k)\right| \geq C_0 > 0. \end{cases} \tag{34}$$

Under conditions (10) and (34), taking into account (25) and (33), the solution to problem (1), (4)–(9) can be written as

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) \cdot \sin\frac{\pi k}{b}x. \quad (35)$$

Given (34) and if the functions $\varphi(x) \in C^{3+\gamma}$, $0 < \gamma < T$; $\varphi''(0) = \varphi''(b) = 0$, $\varphi(0) = \varphi(b) = 0$ and $\psi(x) \in C^{3+\delta}$, $-T < \delta < 0$; $\psi(0) = \psi(b) = \psi''(0) = \psi''(b) = 0$, then for φ_k and ψ_k the estimates $|\varphi_k| \leq C_3/k^{3+\gamma}$; $|\psi_k| \leq C_4/k^{3+\delta}$; C_3, $C_4 = const > 0$ are valid. Then the series (35) converges uniformly in the domain $\bar{\Omega}$ and it can be differentiated term-by-term twice with respect to, t and 4 times with respect to, x. Therefore, for the solution of problem A we have $u(x,t) \in C^{4,2}_{x,t}(\bar{\Omega})$.

Since the constructed solution is $u(x,t) \in C^{4,2}_{x,t}(\bar{\Omega})$, then the condition (24) of Theorem 2 is always satisfied.

Conflict of Interest

The author declares no conflict of interest.

References

1 Glushko V.P. Higher-order degenerate elliptic equations: Spaces, operators, boundary-value problems / V.P. Glushko, Yu.B. Savchenko // Journal of Soviet Mathematics. — 1987. — Vol. 39. — P. 3088–3148. https://doi.org/10.1007/BF01087547

2 Исхоков С.А. О гладкости решения эллиптического уравнения с нестепенным вырождением / С.А. Исхоков // Докл. АН — 2001. — Т. 378, № 3. — С. 306–309.

3 Sidorov S.N. Nonlocal problems for an equation of mixed parabolic-hyperbolic type with power degeneration / S.N. Sidorov // Russian Mathematics. — 2015. — Vol. 59. — P. 46–55. https://doi.org/10.3103/S1066369X15120051

4 Баев А.Д. Об общих краевых задачах в полупространстве для вырождающихся эллиптических уравнений высокого порядка / А.Д. Баев // Докл. АН — 2008. — Т. 422, №6. — С. 727–728.

5 Баев А.Д. Теорема о существовании и единственности решения одной краевой задачи в полосе для вырождающегося эллиптического уравнения высокого порядка / А.Д. Баев, С.С. Бунеев // Изв. Саратов. ун-та. Новая сер. Сер. Механика. Математика. Информатика. — 2012. — Т. 12, Вып. 3. — С. 9–17. https://doi.org/10.18500/1816-9791-2012-12-3-8-17

6 Баев А.Д. О вырождающихся эллиптических уравнениях высокого порядка и псевдодифференциальных операторах с вырождением / А.Д. Баев, Р.А. Ковалевский, П.А. Кобылинский // Докл. АН. — 2016. — Т. 471, № 4. — С. 387–390.

7 Сабитов К.Б. Задача Дирихле для уравнения смешанного типа в прямоугольной области / К.Б. Сабитов // Докл. АН. — 2007. — Т. 413, № 1. — С. 23–26.

8 Sabitov K.B. The dirichlet problem for a mixed-type equation of the second kind in a rectangular domain / K.B. Sabitov, A.Kh. Suleimanova // Russian Mathematics. — 2007. — Vol. 51. — P. 42–50. https://doi.org/10.3103/S1066369X07040068

9 Трегубова (Сулейманова) А.Х. Задача Дирихле и видоизмененные задачи для уравнений смешанного типа с характеристическим вырождением: автореф. дис. ... канд. физ. мат. наук / А.Х. Трегубова (Сулейманова). — Казань, 2009. — 18 с.

10 Lerner M.E. A problem with two nonlocal boundary conditions for a mixed-type equation / M.E. Lerner, O.A. Repin // Siberian Mathematical Journal. — 1999. — Vol. 40. — P. 1064–1078. https://doi.org/10.1007/BF02677530

11 Рустамова Л.Р. Нелокальные краевые задачи типа Бицадзе-Самарского для уравнения смешанного типа четвëртого порядка / Р.Л. Рустамова // Изв. вузов. Сев.-Кавказ. регион. Естественные науки. — 2007. — № 4. — С. 14–16.

12 Асхатов Р.М. Решение основных краевых задач вырождающихся эллиптических уравнений методом потенциалов / Р.М. Асхатов, Р.Н. Абайдуллин // Физико-математические науки. Учен. зап. Казан. ун-та. Сер. Физ.-мат. науки. — 2015. — Т. 157, № 1. — С. 5–14.

13 Энбом Е.А. Задача Коши для вырождающегося уравнения третьего порядка / Е.А. Энбом // Науч. докл. Ежегод. межвуз. 55-ой науч. конф. Самар. гос. пед. ун-та. — Самара, 2001. — С. 80–87.

14 Апаков Ю.П. Краевая задача для вырождающегося уравнения высокого нечетного порядка / Ю.П. Апаков, Б.Ю. Иргашев // Украин. мат. журн. — 2014. — Т. 66, № 10. — С. 1318–1331.

15 Моисеев Е.И. О решении спектральным методом одной нелокальной краевой задачи / Е.И. Моисеев // Дифференциальные уравнения. — 1999. — Т. 35, № 8. — С. 1094–1100.

16 Дунаев А.С. Специальные функции: учеб. пос. / А.С. Дунаев, В.И. Шлычков. — Екатеринбург: УрФУ, 2015. — 938 с.

Төртінші ретті аралас типті өзгешеленетін теңдеу үшін шекаралық есеп

Ж.А. Отарова

Бердах атындағы Қарақалпақ мемлекеттік университеті, Нүкіс, Өзбекстан

Механика, физика және геофизикадағы көптеген есептер эллиптикалық, параболалық немесе гиперболалық теңдеулердің белгілі кластарына кірмейтін дербес туындылы дифференциалдық теңдеулерді шешуге әкеледі. Мұндай теңдеулер әдетте, математикалық физиканың классикалық емес теңдеулері деп аталады. Өзгешеленетін теңдеулер теориясы қазіргі дербес дифференциалдық теңдеулер теориясының негізгі бөлімдерінің бірі. Бұл, ең алдымен, әртүрлі қолданбалы есептерді анықтаумен түсіндіріледі, олардың математикалық модельдеуі өзгешеленетін теңдеулердің әртүрлі түрлерін зерттеуге қолданылады. Осы уақытқа дейін өзгешеленетін теңдеулерге арналған есептер негізінен модельдік теңдеулер мен төменгі мүшелерінің жеткілікті тегіс коэффициенттері бар төменгі мүшелері бар теңдеулер үшін зерттелді. Төртінші ретті аралас типті дәрежелері өзгешеленетін теңдеулер үшін шекаралық есептерді зерттеу өзекті болып қала береді. Мақалада төртінші ретті аралас типті бір өзгешеленетін теңдеу үшін тікбұрышты облыста шекаралық есеп қойылып, зерттелді. Төртінші ретті дербес туындылы дифференциалдық теңдеу үшін шекаралық есептің корректілігі шешімнің бар болуы мен жалғыздығын дәлелдеу арқылы белгіленеді. Жеткілікті шарттар қойылған жағдайда қарастырылған есептің шешімі айнымалыларды бөлу әдісімен табылды.

Кілт сөздер: төртінші ретті аралас теңдеу, Бессель функциялары, Фурье қатары, толықтық, регулярлы шешім.

Краевая задача для вырождающегося уравнения смешанного типа четвёртого порядка

Ж.А. Отарова

Каракалпакский государственный университет имени Бердаха, Нукус, Узбекистан

Многие задачи механики, физики, геофизики приводят к решению уравнений в частных производных, которые не входят в известные классы эллиптических, параболических или гиперболических уравнений. Такие уравнения, как правило, стали называть неклассическими уравнениями математической физики. Теория вырождающихся уравнений является одним из центральных разделов современной теории уравнений с частными производными. Это объясняется, прежде всего, выявлением множества прикладных задач, математическое моделирование которых обслуживает изучение различных типов вырождающихся уравнений. До настоящего времени задачи для вырождающихся уравнений, в основном, исследованы для модельных уравнений и уравнений с младшими членами с достаточно гладкими коэффициентами при младших членах. Исследование краевых задач для уравнений смешанного типа четвертого порядка со степенным вырождением остаётся актуальным. В этой работе поставлена и исследована краевая задача в прямоугольной области для одного вырождающегося уравнения смешанного типа четвертого порядка. Корректность краевой задачи для уравнения с частными производными четвертого порядка устанавливается доказательством существования и единственности решения. При достаточных условиях найдено решение рассматриваемой задачи в явном виде методом разделения переменных.

Ключевые слова: смешанное уравнение четвёртого порядка, функции Бесселя, ряд Фурье, полнота, регулярное решение.

References

1 Glushko, V.P., & Savchenko, Yu.B. (1987). Higher-order degenerate elliptic equations: Spaces, operators, boundary-value problems. *Journal of Soviet Mathematics, 39,* 3088–3148. https://doi.org/10.1007/BF01087547

2 Iskhokov, S.A. (2001). O gladkosti resheniia ellipticheskogo uravneniia s nestepennym vyrozhdeniem [On the smoothness of the solution of an elliptic equation with non-power degeneracy]. *Doklady Akademii nauk — Reports of the USSR Academy of Sciences, 378*(3), 306–309 [in Russian].

3 Sidorov, S.N. (2015). Nonlocal problems for an equation of mixed parabolic-hyperbolic type with power degeneration. *Russian Mathematics, 59,* 46–55 https://doi.org/10.3103/S1066369X15120051

4 Baev, A.D. (2008). Ob obshchikh kraevykh zadachakh v poluprostranstve dlia vyrozhdaiushchikhsia ellipticheskikh uravnenii vysokogo poriadka [On general boundary value problems in a half-space for degenerate elliptic equations of high order]. *Doklady Akademii nauk — Reports of the Academy of Sciences, 422*(6), 727–728 [in Russian].

5 Baev, A.D., & Buneev, S.S. (2012). Teorema o sushchestvovanii i edinstvennosti resheniia odnoi kraevoi zadachi v polose dlia vyrozhdaiushchegosia ellipticheskogo uravneniia vysokogo poriadka [The theorem on the existence and uniqueness of the solution of a single boundary value problem in a strip for a degenerate elliptic equation of high order]. *Izvestiia Saratovskogo universiteta. Novaia seriia. Seriia Mekhanika. Matematika. Informatika — News of Saratov University. New Series. Series: Mechanics. Mathematics. Computer science, 12*(3), 9–17 [in Russian]. https://doi.org/10.18500/1816-9791-2012-12-3-8-17

6 Baev, A.D., Kovalevskii, R.A., & Kobylinskii, P.A. (2016). O vyrozhdaiushchikhsia ellipticheskikh uravneniiakh vysokogo poriadka i psevdodifferentsialnykh operatorakh s vyrozhdeniem [On degenerate high order elliptic equations and pseudodifferential operators with degeneration]. *Doklady Akademii nauk — Reports of the Academy of Sciences, 471*(4), 387– 390 [in Russian].

7 Sabitov, K.B. (2007). Zadacha Dirikhle dlia uravneniia smeshannogo tipa v priamougolnoi oblasti [Dirichle problem for a mixed-type equation in a rectangular domain]. *Doklady Akademii nauk — Reports of the Academy of Sciences, 413*(1), 23–26 [in Russian].

8 Sabitov, K.B., & Suleimanova, A.Kh. (2007). The dirichlet problem for a mixed-type equation of the second kind in a rectangular domain. *Russian Mathematics, 51,* 42–50. https://doi.org/10.3103/S1066369X07040068

9 Tregubova (Suleimanova), A.Kh. (2009). Zadacha Dirikhle i vidoizmenennye zadachi dlia uravnenii smeshannogo tipa s kharakteristicheskim vyrozhdeniem [The Dirichlet problem and modified problems for equations of mixed type with characteristic degeneration]. *Extended abstract of Candidate's thesis.* Kazan [in Russian].

10 Lerner, M.E., & Repin, O.A. (1999). A problem with two nonlocal boundary conditions for a mixed-type equation. *Siberian Mathematical Journal, 40,* 1064–1078. https://doi.org/10.1007/BF02677530

11 Rustamova, L.R. (2007). Nelokalnye kraevye zadachi tipa Bitsadze-Samarskogo dlia uravneniia smeshannogo tipa chetvertogo poriadka [Nonlocal boundary-value problems of Bitsadze - Samarsky type for a fourth-order mixed-type equation]. *Izvestiia vuzov. Severo-Kavkazskii region. Estestvennye nauki — News of Universities. North Caucasus region. Natural Sciences*, (4), 14–16 [in Russian].

12 Askhatov, R.M., & Abaidullin R.N. (2015). Reshenie osnovnykh kraevykh zadach vyrozhdaiushchikhsia ellipticheskikh uravnenii metodom potentsialov [The solution of the main boundary value problems of degenerate elliptic equations by the method of potentials]. *Fiziko-matematicheskie nauki. Uchenye zapiski Kazanskogo universiteta. Seriia Fizikomatematiches-kie nauki — Physics and mathematics, Scientific notes of Kazan University. Series physical and mathematical sciences, 157*(1), 5–14 [in Russian].

13 Enbom, E.A. (2001). Zadacha Koshi dlia vyrozhdaiushchegosia uravneniia tretego poriadka [Cauchy problem for a third order degenerate equation]. *Nauchnye doklady Ezhegodnoi mezhvuzovskoi 55-oi nauchnoi konferentsii Samarskogo gosudarstvennogo pedagogicheskogo universiteta — Scientific reports of the annual interuniversity 55th scientific conference of Samara State Pedagogical University,* 80–87 [in Russian].

14 Apakov, Yu.P., & Irgashev, B.Yu. (2014). Kraevaia zadacha dlia vyrozhdaiushchegosia uravneniia vysokogo nechetnogo poriadka [A boundary value problem for a degenerate equation of high odd order]. *Ukrainskii matematicheskii zhurnal — Ukrainian mathematical journal, 66*(10), 1318–1331 [in Russian].

15 Moiseev, E.I. (1999). O reshenii spektralnym metodom odnoi nelokalnoi kraevoi zadachi [On the solution by a spectral method of a single non-local boundary value problem]. *Differentsialnye uravneniia — Differential equations, 35*(8), 1094–1100 [in Russian].

16 Dunaev, A. S., & Shlychkov, A.S. (2015). *Spetsialnye funktsii [Special functions].* Ekaterinburg: Uralskii federalnyi universitet [in Russian].

*Author Information**

Jamila Amanbayevna Otarova — Candidate of physical and mathematical sciences, Associate Professor, Head of the Department of Differential equations, Karakalpak State University named after Berdakh, 1 Chardzhou Abdirov Street, Nukus, 230112, Uzbekistan / Karakalpakstan; e-mail: *j.otarova@mail.ru*; https://orcid.org/0000-0003-3661-7937

*The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics series, No. 1(113), 2024, pp. 149–161

https://doi.org/10.31489/2024M1/149-161 *Research article*

Factorization of abstract operators into two second degree operators and its applications to integro-differential equations

I.N. Parasidis*, E. Providas

University of Thessaly, Larissa, Greece
(E-mail: paras@uth.gr, providas@uth.gr)

Boundary value problem $\mathbf{B}_1 x = f$ with an abstract linear operator B_1, corresponding to an Fredholm integro-differential equation with ordinary or partial differential operator is researched. An exact solution to $\mathbf{B}_1 x = f$ in the case when a bijective operator $\mathbf{B}_1$ has a factorization of the form $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$ where $\mathbf{B}$ and $\mathbf{B}_0$ are two linear more simple than $\mathbf{B}_1$ second degree abstract operators, received. Conditions for factorization of the operator $\mathbf{B}_1$ and a criterion for bijectivity of $\mathbf{B}_1$ are found.

Keywords: correct operator, bijective operator, factorization (decomposition) of linear operators, Fredholm integro-differential equations, boundary value problems, exact solutions.

2020 Mathematics Subject Classification: 34A30, 34K10, 45J05, 45K05, 47A62, 47A68, 47B01, 47G20.

Introduction

Integro-differential equations (IDEs) are used to model many problems in science, engineering, economics, medicine, control theory, micro-inhomogeneous media and viscoelasticity [1–9]. Very important tools in solving of Boundary Value Problems (BVPs) with IDEs are the Parametrization Method [10] and the Factorization (Decomposition) method, but the applicability of the last method is confined to certain kinds of integro-differential operators, corresponding to BVPs and cannot be universal for all problems. There are several types of decomposition methods for solving BVPs with IDEs. The most popular is Adomyan decomposition method and its modifications, where the Adomyan polynomials are used, and approximate solutions of given BVPs are obtained [11–19]. Other types of decomposition method were considered in [4], [5], [20]. Factorization of tensor integro-differential wave equations of the acoustics of dispersive viscoelastic anisotropic media is performed for the one-dimensional case in [4]. The integro-differential one-dimensional tensor wave equations of the electrodynamics of dispersive anisotropic media are factorized in [5]. The initial first order integro-differential operator with arbitrary nonpositive parameters was decomposed on three factors in [20] and further the sufficient conditions for the existence of a solution are obtained on half line.

We propose in this article another factorization method on two factors which successfully was applied in the articles by the authors [6], [7], [21]–[26] and by another author in [27]. Here we generalize the results of these papers and study a more complicated boundary value problem with an abstract operator equation

$$\mathbf{B}_1 u = \mathcal{A}^2 u - V\Phi(A_0 u) - Y\Phi(A_0^2 u) - S\Psi(\mathcal{A}A_0 u) -$$
$$-G\Psi(\mathcal{A}^2 u) = f, \quad D(\mathbf{B}_1) = D(\mathcal{A}^2)$$

on a Banach space X, where $\mathcal{A}, A_0$ are abstract linear differential operators, the functional vectors Φ, Ψ are defined on X_m and vectors $V, Y, S, G \in X_m$. We obtained the conditions on the vectors

*Corresponding author. *E-mail: paras@uth.gr*

Received: 22 June 2023; *Accepted:* 12 September 2023.

V, Y, S, G under which the operator $\mathbf{B}_1$ can be factorized in a product of two second degree operators, i.e. $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$ with

$$\mathbf{B}_0 u = A_0^2 u - S_0 \Phi(A_0 u) - G_0 \Phi(A_0^2 u) = f, \quad D(\mathbf{B}_0) = D(A_0^2),$$

$$\mathbf{B} u = A^2 u - S \Psi(A u) - G \Psi(A^2 u) = f, \qquad D(\mathbf{B}) = D(A^2)$$

and then found the exact solution in closed form of the given problem, using the exact solutions of the above two simple problems. Using of the obtained formula for the exact solution of the equation $\mathbf{B}_1 u = f$ makes it possible to easily obtain exact solutions of a class of Fredholm IDEs with ordinary or partial differential operators. The decomposition method, applied to abstract operator equation

$$Tu = Au - Ku - G\Psi(Au) = f, \quad D(T) = D(A) = \{u \in X_n : \Phi(u) = \mathbf{0}\}$$

on a Banach space X for solving boundary value problems for n-th order linear Volterra-Fredholm integro-differential equations of convolution type, was used in [6], [7], where were constructed the closed-form solutions to the two-phase integral model of Euler-Bernoulli nanobeams in bending under transverse distributed load and various types of boundary conditions. In [21] the operator B_1 corresponding to the abstract operator equation

$$B_1 u = AA_0 u - S\langle A_0 u, \Phi^t \rangle_{H^m} - G\langle AA_0 u, F^t \rangle_{H^m} = f$$

on a Hilbert space H was factorized in two operators, i.e. $B_1 = B_G B_{G_0}$, where

$$B_{G_0} u = A_0 u - G_0 \langle A_0 u, \Phi^t \rangle_{H^m} = f, \qquad D(B_{G_0}) = D(A_0),$$

$$B_G u = Au - G\langle Au, F^t \rangle_{H^m} = f, \qquad D(B_G) = D(A).$$

Further, using the exact solutions of these two simple equations, the exact solution of $B_1 u = f$ was obtained. An exact solution to the abstract operator equation

$$B_1 u = \mathcal{A}u - S\Phi(A_0 u) - G\Psi(\mathcal{A}u) = f, \quad D(B_1) = D(\mathcal{A})$$

on a Banach space was found in [22] by factorization of B_1 in two simple operators, and then the corresponding theory was applied for solving of Hyperbolic integro-differential equations with integral boundary conditions. The exact solution to the abstract operator equation

$$B_1 u = A^2 u - S\Phi(Au) - G\Psi(A^2 u) = f, \quad D(B_1) = D(A^2)$$

on a Banach space was obtained in [23]. The operator B_1 corresponding to the abstract operator equations

$$B_1 u = A^2 u - SF(Au) - SF(A^2 u) = f,$$

$$D(B_1) = \{u \in D(A^2) : \Phi(u) = N\Psi(Au),\ \Phi(Au) = DF(Au) + N\Psi(A^2 u)\}, \quad \text{and}$$

$$B_1 u = A^2 u - SF(Au) - SF(A^2 u) = f,$$

$$D(B_1) = \{u \in D(A^2) : \Phi(u) = N\Psi(u),\ \Phi(Au) = DF(Au) + N\Psi(Au)\},$$

where D, N are matrices, S, G are vectors, by decomposition method for squared operators is factorized in $B_1 = B^2$ and then the exact solution of $B_1 u = f$ in closed form is easily obtained in [24], [25], respectively. The exact solution to the abstract operator equation

$$B_1 u = \mathcal{A}u - S\Phi(u) - G\Psi(A_0 u) = f, \quad D(B_1) = D(AA_0)$$

on a Banach space by factorization of B_1 in two simple operators B, B_0, was investigated in [26]. The exact solution in closed analytical form to the abstract operator equation

$$B_1u = \mathcal{A}u - S_0F(Au) - G_0\Phi(Au) = f, \quad D(B_1) = D(\mathcal{A})$$

was obtained by decomposition method in [27], and then was applied for solving some ordinary integro-differential and partial integro-differential equations. Our decomposition method is simple to use and can be easily incorporated into any Computer Algebra (CAS). The paper is organized as follows. In Section 1 we give an introduction, terminology and notation. In Section 2 we develop the theory for the solution of the problem $\mathbf{B}_1x = f$ when $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$ with $\mathbf{B}$ and $\mathbf{B}_0$ being two linear second degree abstract operators and give an example of boundary problem with integro-differential equation demonstrating the power and usefulness of the methods presented.

Preliminaries

Throughout this paper by X we denote the complex Banach space and by X^* the adjoint space of X, i.e. the set of all complex-valued linear and bounded functionals f on X. We denote by $f(x)$ the value of f on x. We write $D(A)$ and $R(A)$ for the domain and the range of the operator $A : X \to Y$, respectively. An operator $A : X \to Y$ is said to be *injective or uniquelly solvable* if for all $u_1, u_2 \in D(A)$ such that $Au_1 = Au_2$, follows that $u_1 = u_2$. Remind that a linear operator A is injective if and only if $\ker A = \{0\}$. An operator $A : X \to Y$ is called *surjective* or *everywhere solvable* if $R(A) = Y$. The operator $A : X \to Y$ is said to be *bijective* if A is both injective and surjective. An operator A and the corresponding problem $Au = f$ are called *correct* if A is bijective and its inverse A^{-1} is bounded on Y. Lastly, if for operator $B_1 : X \to X$ there exist two operators B and B_0 such that $B_1 = BB_0$ then we say that BB_0 is *a decomposition (factorization)* of B_1. If $g_i \in X$ and $\psi_i \in X^*, i = 1, \dots, m$, $x \in X$, then we denote by $G = (g_1, \dots, g_m)$, $\Psi =$col$(\psi_1, \dots, \psi_m)$ and $\Psi(x) =$col$(\psi_1(x), \dots, \psi_m(x))$ and write $G \in X_m$, $\Psi \in X_m^*$. If $G = (g_1, ..., g_m)$, $g_1, ..., g_m \in D(A)$, then we write $G \in [D(A)]_m$. We will denote by $\Psi(G)$ the $m \times m$ matrix whose i, j-th entry $\psi_i(g_j)$ is the value of functional ψ_i on element g_j. Note that $\Psi(GC) = \Psi(G)C$, where C is a $m \times k$ constant matrix. We will also denote by I_m the identity $m \times m$ matrix.

We will use the following Theorem, that have been shown in [20] and is recalled here but with a different notation tailored to the needs of the present article.

Theorem 1. Let X be a complex Banach space, the vectors $G_0 = (g_{10}, ..., g_{m0})$, $G = (g_1, ..., g_m)$, $S = (s_1, ..., s_m) \in X_m$, the components of the vectors $\Psi = col(\psi_1, ..., \psi_m)$ and $\Phi = col(\phi_1, ..., \phi_m)$ belong to X^* and the operators $B_0, B, B_1 : X \to X$ defined by

$$B_0u = A_0u - G_0\Phi(A_0u) = f, \qquad D(B_0) = D(A_0),$$

$$Bu = Au - G\Psi(Au) = f, \qquad D(B) = D(A),$$

$$B_1u = AA_0u - S\Phi(A_0u) - G\Psi(AA_0u) = f, \quad D(B_1) = D(AA_0), \tag{1}$$

where $A_0, A : X \to X$ are linear correct operators and $G_0 \in [D(A)]_m$. Then the following statements are fulfilled:

(i) If

$$S \in [R(B)]_m \quad \text{and} \quad S = BG_0 = AG_0 - G\Psi(AG_0), \tag{2}$$

then the operator B_1 can be factorized in $B_1 = BB_0$.

(ii) If the components of the vector Φ are linearly independent elements of X^* and the operator B_1 can be factorized in $B_1 = BB_0$, then (2) is fulfilled.

(iii) If there exists a vector $G_0 \in [D(A)]_m$, satisfying the equation $AG_0 - G\Psi(AG_0) = S$, then B_1 is bijective if and only if the operators B_0 and B are bijective, which means that

$$\det V = \det[I_m - \Phi(G_0)] \neq 0 \quad \text{and} \quad \det L = \det[I_m - \Psi(G)] \neq 0.$$

In this case, the unique solution to the boundary value problem (1) for any $f \in X$, is given by

$$u = B_1^{-1} f = A_0^{-1} v + A_0^{-1} G_0 V^{-1} \Phi(v), \quad \text{where} \quad v = A^{-1} f + A^{-1} G L^{-1} \Psi(f). \tag{3}$$

Lemma 2. Let X be a complex Banach space. Then a linear operator $A : X \to X$ is bijective if and only if A^2 is bijective.

Proof. Let A be bijective and $u \in \ker A^2$. Then $A^2 u = 0$. Applying twise the operator A^{-1} to this equation we obtain $u = 0$ which proves that $\ker A^2 = \{0\}$. Consider the equation $A^2 u = f, \quad f \in X$. Applying twise the operator A^{-1} to this equation, we obtain $u = A^{-1}\left(A^{-1} f\right) = A^{-2} f$ for every $f \in X$, which proves that $R(A^2) = X$. Thus A^2 is a bijective operator.

Conversely, let A^2 be bijective. Then $\ker A^2 = \{0\}$ and $R(A^2) = X$, and from well-known relations

$$\ker A \subset \ker A^2, \quad R(A^2) \subset R(A),$$

for a linear operator $A : X \to X$, follows that $\ker A = \{0\}$ and $R(A) = X$. Hence A is a bijective operator.

Bellow we prove the main theorem.

Theorem 3. Let X be a complex Banach space, $A_0, A_0^2, A, A^2 : X \overset{on}{\to} X$ linear operators and the vectors $V, Y, G, S \in X_m$, $\Phi, \Psi \in [X^*]_m$, $S_0, G_0 \in [D(A^2)]_m$. Then for the operators $\mathbf{B}_0, \mathbf{B}, \mathbf{B}_1 : X \to X$, defined by

$$\mathbf{B}_0 u = A_0^2 u - S_0 \Phi(A_0 u) - G_0 \Phi(A_0^2 u) = f, \quad D(\mathbf{B}_0) = D(A_0^2), \tag{4}$$

$$\mathbf{B} u = A^2 u - S \Psi(A u) - G \Psi(A^2 u) = f, \qquad D(\mathbf{B}) = D(A^2), \tag{5}$$

$$\mathbf{B}_1 u = A^2 A_0^2 u - V \Phi(A_0 u) - Y \Phi(A_0^2 u) - S \Psi(A A_0^2 u) - G \Psi(A^2 A_0^2 u) = f,$$
$$D(\mathbf{B}_1) = D(A^2 A_0^2) = \{u \in D(A_0^2) : A_0^2 u \in D(A^2)\}, \tag{6}$$

hold true the next statements:

(i) If the vectors $G_0 = (g_{10}, ..., g_{m0})$ and $S_0 = (s_{10}, ..., s_{m0})$ belong to $[D(A^2)]_m$ and satisfy the system of equations

$$V = \mathbf{B} S_0 = A^2 S_0 - S \Psi(A S_0) - G \Psi(A^2 S_0), \tag{7}$$

$$Y = \mathbf{B} G_0 = A^2 G_0 - S \Psi(A G_0) - G \Psi(A^2 G_0), \tag{8}$$

then the operator $\mathbf{B}_1$ can be factorized in $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$.

(ii) If $G_0 = (g_{10}, ..., g_{m0})$, $S_0 = (s_{10}, ..., s_{m0}) \in [D(A^2)]_m$ and the operator $\mathbf{B}_1$ is factorized in $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$, where A, A_0 are bijective, and if the functional vectors

$$\hat{\boldsymbol{\Phi}}(f) = \left(\Phi * A_0^{-1} A^{-2}\right)(f) = \Phi(A_0^{-1} A^{-2} f), \quad \check{\boldsymbol{\Phi}}(f) = \left(\Phi * A^{-2}\right)(f) = \Phi(A^{-2} f)$$

are linearly independent on X, then (7), (8) hold true.

(iii) The operators $\mathbf{B}_0, \mathbf{B}$ are bijective if and only if, respectively,

$$\det \mathbf{L}_0 = \det \begin{pmatrix} I_m - \Phi(A^{-1} S_0) & -\Phi(A^{-1} G_0) \\ -\Phi(S_0) & I_m - \Phi(G_0) \end{pmatrix} \neq 0, \tag{9}$$

$$\det \mathbf{L} = \det \begin{pmatrix} I_m - \Psi(A^{-1} S) & -\Psi(A^{-1} G) \\ -\Psi(S) & I_m - \Psi(G) \end{pmatrix} \neq 0, \tag{10}$$

and in this case the unique solutions of (4), (5) for any $f \in X$ are given by

$$u = \mathbf{B}_0^{-1} f = A_0^{-2} f + (A_0^{-2} S_0, A_0^{-2} G_0) \mathbf{L}_0^{-1} \begin{pmatrix} \Phi(A_0^{-1} f) \\ \Phi(f) \end{pmatrix}, \tag{11}$$

$$u = \mathbf{B}^{-1} f = A^{-2} f + (A^{-2} S, A^{-2} G) \mathbf{L}^{-1} \begin{pmatrix} \Psi(A^{-1} f) \\ \Psi(f) \end{pmatrix}, \tag{12}$$

respectively.

(iv) If V, Y are defined by (7), (8) and A, A_0 are bijective operators, then $\mathbf{B}_1$ is bijective if and only if (9) and (10) are fulfilled, and the unique solution of (6) in this case for every $f \in X$ is given by

$$u = A_0^{-2} v + (A_0^{-2} S_0, A_0^{-2} G_0) \mathbf{L}_0^{-1} \begin{pmatrix} \Phi(A_0^{-1} v) \\ \Phi(v) \end{pmatrix}, \text{ where} \tag{13}$$

$$v = A^{-2} f + (A^{-2} S, A^{-2} G) \mathbf{L}^{-1} \begin{pmatrix} \Psi(A^{-1} f) \\ \Psi(f) \end{pmatrix}. \tag{14}$$

Proof (i). Taking into account that $G_0,\, S_0 \in [D(A^2)]_m$ we obtain

$$D(\mathbf{B}\mathbf{B}_0) = \{u \in D(\mathbf{B}_0) : \mathbf{B}_0 u \in D(\mathbf{B})\} =$$

$$= \{u \in D(A_0^2) : A_0^2 u - S_0 \Phi(A_0 u) - G_0 \Phi(A_0^2 u) \in D(A^2)\} =$$
$$= \{u \in D(A_0^2) : A_0^2 u \in D(A^2)\} = D(A^2 A_0^2) = D(\mathbf{B}_1).$$

We put $y = \mathbf{B}_0 u$. Then for each $u \in D(A^2 A_0^2)$ since (5) and (4) we have

$$\mathbf{B}\mathbf{B}_0 u = \mathbf{B} y = A^2 y - S\Psi(Ay) - G\Psi(A^2 y) =$$

$$= A^2 \mathbf{B}_0 u - S\Psi(A\mathbf{B}_0 u) - G\Psi(A^2 \mathbf{B}_0 u) =$$
$$= A^2 [A_0^2 u - S_0 \Phi(A_0 u) - G_0 \Phi(A_0^2 u)] -$$
$$- S\Psi\left(A[A_0^2 u - S_0 \Phi(A_0 u) - G_0 \Phi(A_0^2 u)]\right) -$$
$$- G\Psi\left(A^2[A_0^2 u - S_0 \Phi(A_0 u) - G_0 \Phi(A_0^2 u)]\right) =$$
$$= A^2 A_0^2 u - A^2 S_0 \Phi(A_0 u) - A^2 G_0 \Phi(A_0^2 u) -$$
$$- S\Psi\left(A A_0^2 u - A S_0 \Phi(A_0 u) - A G_0 \Phi(A_0^2 u)\right) -$$
$$- G\Psi\left(A^2 A_0^2 u - A^2 S_0 \Phi(A_0 u) - A^2 G_0 \Phi(A_0^2 u)\right) =$$
$$= A^2 A_0^2 u - A^2 S_0 \Phi(A_0 u) - A^2 G_0 \Phi(A_0^2 u) -$$
$$- S\Psi(A A_0^2 u) + S\Psi(A S_0) \Phi(A_0 u) +$$
$$+ S\Psi(A G_0) \Phi(A_0^2 u) - G\Psi(A^2 A_0^2 u) +$$
$$+ G\Psi(A^2 S_0) \Phi(A_0 u) + G\Psi(A^2 G_0) \Phi(A_0^2 u).$$

So we obtain

$$\mathbf{B}\mathbf{B}_0 u = A^2 A_0^2 u - [A^2 S_0 - S\Psi(A S_0) - G\Psi(A^2 S_0)] \Phi(A_0 u) -$$
$$- [A^2 G_0 - S\Psi(A G_0) - G\Psi(A^2 G_0)] \Phi(A_0^2 u) -$$
$$- S\Psi(A A_0^2 u) - G\Psi(A^2 A_0^2 u), \quad \text{or}$$

$$\mathbf{B}\mathbf{B}_0 u = A^2 A_0^2 u - \mathbf{B} S_0 \Phi(A_0 u) - \mathbf{B} G_0 \Phi(A_0^2 u) - S\Psi(A A_0^2 u) - G\Psi(A^2 A_0^2 u), \tag{15}$$

where the relations

$$\mathbf{B}S_0 = A^2 S_0 - S\Psi(AS_0) - G\Psi(A^2 S_0),$$

$$\mathbf{B}G_0 = A^2 G_0 - S\Psi(AG_0) - G\Psi(A^2 G_0)$$

follow by substituting $u = S_0$ and $u = G_0$ in (5). By comparing (6) with (15), it is easy to verify that $\mathbf{B}\mathbf{B}_0 u = \mathbf{B}_1 u$ for each $u \in D(A^2 A_0^2)$ if (7), (8) hold true.

(ii) Let now $\mathbf{B}\mathbf{B}_0 u = \mathbf{B}_1 u$ for each $u \in D(A^2 A_0^2)$. Then by subtraction for each $u \in D(A^2 A_0^2)$, we get $\mathbf{B}\mathbf{B}_0 u - \mathbf{B}_1 u = 0$, which implies

$$(\mathbf{B}S_0 - V)\Phi(A_0 u) + (\mathbf{B}G_0 - Y)\Phi(A_0^2 u) = 0,$$

or, since the operators A, A_0 are bijective and, by Lemma 2, the operators A^2, A_0^2 are bijective too, we have

$$(\mathbf{B}S_0 - V)\Phi(A_0^{-1}A^{-2}A^2A_0^2 u) + (\mathbf{B}G_0 - Y)\Phi(A^{-2}A^2A_0^2 u) = 0,$$

or denoting $f = A^2 A_0^2 u$, for each $f \in X$ we get

$$(\mathbf{B}S_0 - V)\Phi(A_0^{-1}A^{-2} f) + (\mathbf{B}G_0 - Y)\Phi(A^{-2} f) = 0,$$

which is

$$(\mathbf{B}S_0 - V)\hat{\mathbf{\Phi}}(f) + (\mathbf{B}G_0 - Y)\check{\mathbf{\Phi}}(f) = 0, \quad \forall f \in X.$$

The last equation, because of the vectors $\hat{\mathbf{\Phi}}, \check{\mathbf{\Phi}}$ are linear independent on X, gives $V = \mathbf{B}S_0$, $Y = \mathbf{B}G_0$.

(iii)-(iv) Let the operator $\mathbf{B}_1$ and the vectors V, Y be defined by (6), (7) and (8), respectively. Equation (6) can also be written in matrix notation as

$$\mathbf{B}_1 u = A^2 A_0^2 u - \big(\mathbf{B}S_0, \mathbf{B}G_0\big)\begin{pmatrix}\Phi(A_0 u)\\ \Phi(A_0^2 u)\end{pmatrix} - (S, G)\begin{pmatrix}\Psi(AA_0^2 u)\\ \Psi(A^2 A_0^2 u)\end{pmatrix} = f,$$

or

$$\mathbf{B}_1 u = A^2 A_0^2 u - \big(\mathbf{B}S_0, \mathbf{B}G_0\big)\begin{pmatrix}\Phi(A_0^{-1} A_0^2 u)\\ \Phi(A_0^2 u)\end{pmatrix} - (S, G)\begin{pmatrix}\Psi(A^{-1}A^2 A_0^2 u)\\ \Psi(A^2 A_0^2 u)\end{pmatrix} = f,$$

or

$$\mathbf{B}_1 u = \mathcal{A}\mathcal{A}_0 u - \widetilde{S}\widetilde{\Phi}(\mathcal{A}_0 u) - \widetilde{G}\widetilde{\Psi}(\mathcal{A}\mathcal{A}_0 u) = f, \quad D(\mathbf{B}_1) = D(\mathcal{A}\mathcal{A}_0), \tag{16}$$

where

$$\mathcal{A} = A^2,\ \mathcal{A}_0 = A_0^2, \quad \widetilde{S} = \mathbf{B}\widetilde{G}_0, \quad \widetilde{G} = (S, G),\ \widetilde{G}_0 = (S_0, G_0), \tag{17}$$

$$\widetilde{\Phi} = col\big(\Phi * A_0^{-1}, \Phi\big), \quad \widetilde{\Psi} = col\big(\Psi * A^{-1}, \Psi\big) \tag{18}$$

and $(\Phi * A_0^{-1})(v) = \Phi(A_0^{-1} v)$, $(\Psi * A^{-1})(v) = \Psi(A^{-1} v)$. Remind that by Lemma 2, the operators $\mathcal{A} = A^2$ and $\mathcal{A}_0 = A_0^2$ are bijective, because of A and A_0 are bijective. Notice that the components of the vectors $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are bounded on X, since the operators A_0^{-1}, A^{-1} are bounded, the components of the vectors Φ and Ψ belong to X^* and for any $f \in X$ the elements $A_0^{-1}A^{-2} f$, $A^{-2} f \in X$. It is easy to verify that Equations (4) and (5) can be equivalently represented in matrix form:

$$\mathbf{B}_0 u = \mathcal{A}_0 u - \widetilde{G}_0 \widetilde{\Phi}(\mathcal{A}_0 u) = f, \quad D(\mathbf{B}_0) = D(\mathcal{A}_0),$$

$$\mathbf{B} u = \mathcal{A} u - \widetilde{G}\widetilde{\Psi}(\mathcal{A} u) = f, \qquad D(\mathbf{B}) = D(\mathcal{A}).$$

Now by Theorem 1, where instead of $B, B_0, B_1, S, G, \Phi, \Psi, A, A_0, L, V$ and m we have $\mathbf{B}, \mathbf{B}_0, \mathbf{B}_1, \widetilde{S}, \widetilde{G}, \widetilde{\Phi}, \widetilde{\Psi}, \mathcal{A}, \mathcal{A}_0, \mathbf{L}, \mathbf{V}$ and $2m$, respectively, we conclude that the operator $\mathbf{B}_1$ can be factorized in $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$ if

$$\mathcal{A}\widetilde{G}_0 - \widetilde{G}\widetilde{\Psi}(\mathcal{A}\widetilde{G}_0) = \widetilde{G}. \tag{19}$$

It is easy to verify that Equation (19) is equivalent to System (7), (8). Also by Theorem 1, the operator $\mathbf{B}_1$ is bijective if and only if

$$\det \mathbf{V} = \det[I_{2m} - \widetilde{\Phi}(\widetilde{G}_0)] \neq 0 \quad \text{and} \quad \det \mathbf{L} = \det[I_{2m} - \widetilde{\Psi}(\widetilde{G})] \neq 0,$$

respectively. The last inequalities, since

$$\widetilde{\Phi}(\widetilde{G}_0) = \begin{pmatrix} (\Phi * A_0^{-1})(S_0) & (\Phi * A_0^{-1})(G_0) \\ \Phi(S_0) & \Phi(G_0) \end{pmatrix} = \begin{pmatrix} \Phi(A_0^{-1}S_0) & \Phi(A_0^{-1}G_0) \\ \Phi(S_0) & \Phi(G_0) \end{pmatrix},$$

$$\widetilde{\Psi}(\widetilde{G}) = \begin{pmatrix} (\Psi * A^{-1})(S) & (\Psi * A^{-1})(G) \\ \Psi(S) & \Psi(G) \end{pmatrix} = \begin{pmatrix} \Psi(A^{-1}S) & \Psi(A^{-1}G) \\ \Psi(S) & \Psi(G) \end{pmatrix},$$

give (9) and (10). Let $\mathbf{B}_1 u = \mathbf{B}\mathbf{B}_0 u = f$, $f \in X$. By Theorem 1 using (3), since $\mathbf{B}, \mathbf{B}_0$ are bijective operators, we obtain the unique solution of (16) or (6)

$$u = \mathbf{B}_0^{-1} v = \mathcal{A}_0^{-1} v + \mathcal{A}_0^{-1} \widetilde{G}_0 \mathbf{L}_0^{-1} \widetilde{\Phi}(v), \text{ where}$$

$$v = \mathbf{B}^{-1} f = \mathcal{A}^{-1} f + \mathcal{A}^{-1} \widetilde{G} \mathbf{L}^{-1} \widetilde{\Psi}(f),$$

which since (17), (18) gives

$$u = \mathbf{B}_0^{-1} v = A_0^{-2} v + (A_0^{-2} S_0, A_0^{-2} G_0) \mathbf{L}_0^{-1} \begin{pmatrix} \Phi(A_0^{-1} v) \\ \Phi(v) \end{pmatrix}, \text{ where} \tag{20}$$

$$v = \mathbf{B}^{-1} f = A^{-2} f + (A^{-2} S, A^{-2} G) \mathbf{L}^{-1} \begin{pmatrix} \Psi(A^{-1} f) \\ \Psi(f) \end{pmatrix}. \tag{21}$$

So we proved (13), (14). From (20), (21) we immediately obtain (11), (12). The theorem is proved.

The next theorem follows from Theorem 3 and is useful in applications and gives the decomposition $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$, where $\mathbf{B}, \mathbf{B}_0$ beforehand we do not know. Also this theorem gives a criterion for the bijectivity of $\mathbf{B}_1$ and the solution of $\mathbf{B}_1 u = f$ in an elegant way.

Theorem 4. Let the space X and the vectors V, Y, S, G, Φ, Ψ be defined as in Theorem 3 and the operator $B_1 : X \to X$ by

$$\mathbf{B}_1 u = \mathcal{A}^2 u - V\Phi(A_0 u) - Y\Phi(A_0^2 u) - S\Psi(\mathcal{A} A_0 u) - G\Psi(\mathcal{A}^2 u) = f, \; D(\mathbf{B}_1) = D(\mathcal{A}^2), \tag{22}$$

where $A_0 : X \to X$ is a bijective n_1-order differential operator and $\mathcal{A} : X \to X$ is a n-order differential operator, $n_1 < n$. Suppose that there exists a bijective linear differential $n - n_1$ order operator $A : X \to X$ such that

$$\mathcal{A} = A A_0, \quad D(\mathbf{B}_1) = D(A^2 A_0^2) \tag{23}$$

and

$$\det \mathbf{L} = \det \begin{pmatrix} I_m - \Psi(A^{-1}S) & -\Psi(A^{-1}G) \\ -\Psi(S) & I_m - \Psi(G) \end{pmatrix} \neq 0. \tag{24}$$

Then the operator $\mathbf{B}_1$ is factorized into $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$, where $\mathbf{B}_0$, $\mathbf{B}$ are defined by (4), (5), respectively, and

$$S_0 = A^{-2} V + (A^{-2} S, A^{-2} G) \mathbf{L}^{-1} \begin{pmatrix} \Psi(A^{-1} V) \\ \Psi(V) \end{pmatrix}, \tag{25}$$

$$G_0 = A^{-2} Y + (A^{-2} S, A^{-2} G) \mathbf{L}^{-1} \begin{pmatrix} \Psi(A^{-1} Y) \\ \Psi(Y) \end{pmatrix}. \tag{26}$$

Furtermore the operator $\mathbf{B}_1$ is bijective if (9) is fulfilled, and in this case a unique solution to the boundary value problem (22), (23) for any $f \in X$ is given by (13), (14).

Proof. Substituting $\mathcal{A} = AA_0$ into (22) we obtain the operator $\mathbf{B}_1$ in the form (6). Construct the operators $\mathbf{B}_0$ and $\mathbf{B}$ by using (4) and (5), respectively, where for $\mathbf{B}$ we take the elements G, S, Ψ and A from (22) and (23), and for $\mathbf{B}_0$ the elements A_0, Φ and S_0, G_0 from (22) and (25), (26).

Note that the operator $\mathbf{B}$, by Theorem 3 (iii), since (24) and bijectivity of A, is bijective, and that taking into account (12) the system of equations (25), (26) can be represented as $S_0 = \mathbf{B}^{-1}V$ and $G_0 = \mathbf{B}^{-1}Y$. The last system, because of bijectivity of $\mathbf{B}$, is equivalent to the system $V = \mathbf{B}S_0$ and $Y = \mathbf{B}G_0$, which is the system (7), (8). Then by Theorem 3 (i), the operator $\mathbf{B}_1$ can be factorized in $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$. Furthermore by Theorem 3 (iv), since (24) and bijectivity of A_0, the operator $\mathbf{B}_1$ is bijective if (9) holds. The unique solution to (22), (23), by Theorem 3 (iv), is given by (13), (14). The theorem is proved.

A reader can prove easily by Lemma 2 the next proposition.

Proposition 5. Let the operators $A_0, A : C[0,1] \to C[0,1]$ be defined by

$$A_0 u(t) = u'(t) = f, \quad D(A_0) = \{u(t) \in C^1[0,1] : \, u(0) = 0\}, \tag{27}$$

$$Au(t) = u'(t) = f, \quad D(A) = \{u(t) \in C^1[0,1] : \, u(1) = 0\}. \tag{28}$$

Then:

(i) The operators A_0, A are bijective and the unique solution of the problem (27) and (28) is given by

$$u(t) = A_0^{-1} f(t) = \int_0^t f(x)dx, \tag{29}$$

$$u(t) = A^{-1} f(t) = \int_0^t f(x)dx - \int_0^1 f(x)dx, \tag{30}$$

respectively.

(ii) The operators $A_0^2, A^2 : C[0,1] \to C[0,1]$ are defined by

$$A_0^2 u(t) = u''(t) = f, \quad D(A_0^2) = \{u(t) \in C^2[0,1] : \, u(0) = 0, \, u'(0) = 0\}, \tag{31}$$

$$A^2 u(t) = u''(t) = f, \quad D(A^2) = \{u(t) \in C^2[0,1] : \, u(1) = 0, \, u'(1) = 0\}, \tag{32}$$

and bijective. The unique solution of the problem (31), (32) is given by

$$u(t) = A_0^{-2} f(t) = \int_0^t (t-x) f(x)dx, \tag{33}$$

$$u(t) = A^{-2} f(t) = \int_0^t (t-x) f(x)dx - \int_0^1 (t-x) f(x)dx, \tag{34}$$

respectively.

Example 6. Let the operator $\mathbf{B}_1 : C[0,1] \to C[0,1]$ be defined by

$$\mathbf{B}_1 u = u^{(4)}(t) - (5-2t)\int_0^1 x^2\, u'(x)dx - (6t-3)\int_0^1 x^2\, u''(x)dx -$$

$$-12\int_0^1 x\, u'''(x)dx - (2t+1)\int_0^1 x\, u^{(4)}(x)dx = 2 - 3t, \tag{35}$$

$$D(\mathbf{B}_1) = \{u(x) \in C^4[0,1] : u(0) = u'(0) = u''(1) = u'''(1) = 0\}. \tag{36}$$

Then:

(i) $\mathbf{B}_1$ can be factorized as a product of two operators and is bijective.

(ii) The unique solution of Problem (35)-(36) is given by

$$u(t) = -\frac{t^2(12271t^3 - 46530t^2 + 63410t - 33760)}{531448}. \quad (37)$$

Proof (i). If we compare equation (35) with equation (22), it is natural to take $A_0u = u'(x)$, $A_0^2u = u''(x)$, $\mathcal{A}A_0u = u'''(x)$, $\mathcal{A}^2u = u^{(4)}$, $n_1 = 1$, $V = 5-2t$, $Y = 6t-3$, $S = 12$, $G = 2t+1$, f=2-3t, $\Phi(A_0u) = \int_0^1 x^2\, u'(x)dx$, $\Phi(A_0^2u) = \int_0^1 x^2\, u''(x)dx$, $\Psi(\mathcal{A}A_0u) = \int_0^1 x\, u'''(x)dx$, $\Psi(\mathcal{A}^2u) = \int_0^1 x\, u^{(4)}(x)dx$.

Then

$$\Phi(v) = \int_0^1 x^2\, v(x)dx, \quad \Psi(v) = \int_0^1 x\, v(x)dx. \quad (38)$$

It is evident that $\Phi, \Psi \in X^*$. We chooce the operator A to satisfy (23), namely $\mathcal{A}u = AA_0u$, $D(\mathbf{B}_1) = D(A^2A_0^2)$. From $\mathcal{A}u(x) = AA_0u(x)$, $\mathcal{A}A_0u = u'''(x)$ and $A_0u(x) = u'(x)$ we get $\mathcal{A}A_0u(x) = AA_0^2u(x) = Au''(x) = u'''(x)$. Denote $v(x) = u''(x)$, then $Av(x) = v'(x)$. Let $D(A_0) = \{u(x) \in C^1[0,1] : u(0) = 0\}$, $D(A) = \{v(x) \in C^1[0,1] : v(1) = 0\}$. So we proved that the operators A_0, A are defined as in (27), (28). Then the operators A_0^2, A^2 are defined as in (31), (32), respectively. Further we find

$$D(A^2A_0^2) = \{u(t) \in D(A_0^2) : A_0^2u(t) \in D(A^2)\} =$$

$$= \{u(t) \in C^2[0,1] : u(0) = u'(0) = 0,\ u''(t) \in C^2[0,1],\ u''(1) = u'''(1) = 0\} =$$

$$= \{u(t) \in C^4[0,1] : u(0) = u'(0) = 0,\ u''(1) = u'''(1) = 0\} = D(\mathbf{B}_1).$$

This proves that the conditions (23) are satisfied and so we can apply Theorem 4. Using (30) and (38) by simple calculations we find

$A^{-1}S = \int_0^t Sdx - \int_0^1 Sdx = 12t - 12$,

$A^{-1}G = \int_0^t Gdx - \int_0^1 Gdx = \int_0^t (2x+1)dx - \int_0^1 (2x+1)dx = t^2 + t - 2$,

$\Psi(A^{-1}S) = \int_0^1 x(12x-12)dx = -2$, $\quad \Psi(A^{-1}G) = \int_0^1 x(x^2+x-2)dx = -5/12$,

$\Psi(S) = \int_0^1 12xdx = 6$, $\quad \Psi(G) = \int_0^1 x(2x+1)dx = 7/6$.

By (10), we obtain $\mathbf{L} = \begin{pmatrix} 3 & 5/12 \\ -6 & -1/6 \end{pmatrix}$. Then $\det \mathbf{L} \neq 0$ and $\mathbf{L}^{-1} = \begin{pmatrix} -1/12 & -5/24 \\ 3 & 3/2 \end{pmatrix}$. By Theorem 4, the operator $\mathbf{B}_1$ is factorized in $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$, where $\mathbf{B}_0$, $\mathbf{B}$ and S_0, G_0 are defined by (4), (5) and (25), (26), respectively. Using (34) for $S = 12$, $G = 2x+1$, we obtain

$A^{-2}S = \int_0^t (t-x)Sdx - \int_0^1 (t-x)Sdx = 6(t-1)^2$,

$A^{-2}G = t^3/3 + t^2/2 - 2t + 7/6$.

By (30), (34) for $V = 5-2x$, $Y = 6x-3$ we get

$A^{-1}V = \int_0^t V(x)dx - \int_0^1 V(x)dx = 5t - t^2 - 4$,

$A^{-1}Y = \int_0^t Y(x)dx - \int_0^1 Y(x)dx = 3t^2 - 3t$,

$A^{-2}V = \int_0^t (t-x)V(x)dx - \int_0^1 (t-x)V(x)dx = -t^3/3 + 5t^2/2 - 4t + 11/6$,

$A^{-2}Y = t^3 - 3t^2/2 + 1/2$, and further by (38) we have

$\Psi(A^{-1}V) = \int_0^1 t(5t - t^2 - 4)dt = -7/12$, $\quad \Psi(A^{-1}Y) = -1/4$,

$\Psi(V) = \int_0^1 t(5-2t)dt = 11/6$, $\quad \Psi(Y) = 1/2$.

Applying (25), (26) and the above calculations we get

$S_0 = S_0(t) = (t-1)^2$, $\quad G_0 = G_0(t) = t(t-1)^2$.

By (29) we find $A_0^{-1}S_0 = \int_0^t S_0(x)dx = t(t^2 - 3t + 3)/3$,

$A_0^{-1}G_0 = \int_0^t G_0(x)dx = t^2(3t^2 - 8t + 6)/12$. Then

$\Phi(A_0^{-1}S_0) = \int_0^1 t^2 t(t^2 - 3t + 3)/3dt = 19/180$,

$\Phi(A_0^{-1}G_0) = \int_0^1 t^2 t^2(3t^2 - 8t + 6)/12dt = 31/1260$,

$\Phi(S_0) = \int_0^1 t^2(t-1)^2 dt = 1/30,$
$\Phi(G_0) = \int_0^1 t^3(t-1)^2 dt = 1/60.$
Using (9) we obtain $\mathbf{L}_0 = \begin{pmatrix} 161/180 & -31/1260 \\ -1/30 & 59/60 \end{pmatrix}$. It is easy to verify that

$$\det \mathbf{L}_0 \neq 0, \qquad \mathbf{L}_0^{-1} = \frac{1}{66431}\begin{pmatrix} 74340 & 1860 \\ 2520 & 67620 \end{pmatrix}.$$

Then, by the Theorem 4, the operator $\mathbf{B}_1$ is bijective.

(ii) Now we find the solution of (35)-(36). Using (29), (33), (30), (34), (38) we find

$$A_0^{-2}S_0 = \frac{t^2}{12}(t^2 - 4t + 6), \quad A_0^{-2}G_0 = \frac{t^3}{60}(3t^2 - 10t + 10),$$

and for $f = 2 - 3t$

$$A^{-1}f = \frac{1}{2}(4t - 3t^2 - 1), \quad A^{-2}f = \frac{1}{2}(-t^3 + 2t^2 - t),$$

$$\Psi(f) = \int_0^1 (2 - 3t)t dt = 0, \quad \Psi(A^{-1}f) = 1/24.$$

Using (14) from the above we get

$$v = v(t) = -\frac{11t^3 - 25t^2 + 17t - 3}{24}.$$

Then by (29), (33) and (38) we have

$$A_0^{-1}v(t) = \int_0^t v(x)dx = -\frac{t(33t^3 - 100t^2 + 102t - 36)}{288},$$

$$A_0^{-2}v(t) = \int_0^t (t - x)v(x)dx = -\frac{t^2(33t^3 - 125t^2 + 170t - 90)}{1440},$$

$$A_0^{-2}S_0 = \int_0^t (t - x)S_0(x)dx = \int_0^t (t - x)(x - 1)^2 dx = \frac{t^2(t^2 - 4t + 6)}{12},$$

$$A_0^{-2}G_0 = \int_0^t (t - x)G_0(x)dx = \int_0^t (t - x)x(x - 1)^2 dx = \frac{t^3(3t^2 - 10t + 10)}{60},$$

$$\Phi(v) = \int_0^1 x^2 v(x)dx = -\frac{1}{288},$$

$$\Phi(A_0^{-1}v) = -\int_0^1 x^2 A_0^{-1}v(x)dx = -\int_0^1 x^2 \left[\frac{x(33x^3 - 100x^2 + 102x - 36)}{288}\right] dx = \frac{29}{15120}.$$

Substituting these values into (13) we obtain (37).

Author Contributions

I.N. Parasidis collected and analyzed data, and led manuscript preparation. E. Providas assisted in data collection and analysis.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Minakov, A.A., & Schick, C. (2021). Integro-differential equation for the non-equilibrium thermal response of glass-forming materials: Analytical solutions. *Symmetry, 13*(256), 41–57. https://doi.org/10.3390/sym13020256

2 Gillespie, W.R. (1993). Generalized pharmacokinetic modeling for drugs with nonlinear binding: I. Theoretical framework. *J. Pharmacokinet Biopharm, 21*(1), 99–124. https://doi.org/10.1007/BF01061777

3 Volinsky, I. (2021). Stability Analysis of a Mathematical Model of Hepatitis B Virus with Unbounded Memory Control on the Immune System in the Neighborhood of the Equilibrium Free Point. *Symmetry, 13*(8), 1437. https://doi.org/10.3390/sym13081437

4 Barkovsky, L.M., & Furs, A.N. (2002). Factorization of integro-differential equations of the acoustics of dispersive viscoelastic anisotropic media and the tensor integral operators of wake packet velosities. *Acoustical Physics, 48*(2), 128–132.

5 Barkovsky, L.M., & Furs, A.N. (2001). Factorization of integro-differential equations of optics of dispersive anisotropic media and tensor integral operators of wave packet velocities. *Optics and Spectroscopy, 90*(4), 561–567. https://doi.org/10.1134/1.1366751

6 Providas, E. (2022). On the Exact Solution of Nonlocal Euler-Bernoulli Beam Equations via a Direct Approach for Volterra-Fredholm Integro-Differential Equations. *Applied Math., 2*(2), 269–283. https://doi.org/10.3390/appliedmath2020017

7 Providas, E. (2022). Closed-Form Solution of the Bending Two-Phase Integral Model of Euler-Bernoulli Nanobeams. *Algorithms, 15*, 151. https://doi.org/10.3390/a15050151

8 Dong, S.H. (2007). *Factorization method in quantum mechanics.* Fund. Theories Phys., 150. Springer, Dordrecht.

9 Kalimbetov, B.T., & Etmishev, Kh.F. (2020). Asymptotic solutions of scalar integro-differential equations with partial derivatives and with rapidly oscillating coefficients. *Bulletin of the Karaganda University. Mathematics series*, 1(97), 52–67. https://doi.org/10.31489/2020M1/52-67

10 Dzhumabaev, D.S., & Mynbayeva, S.T. (2020). One approach to solve a nonlinear boundary value problem for the Fredholm integro-differential equation. *Bulletin of the Karaganda University. Mathematics series*, 1(97), 27–36. https://doi.org/10.31489/2020M1/27-36

11 Dawood, L.A., Hamoud, A.A., & Mohammed, N.M. (2020). Laplace discrete decomposition method for solving nonlinear Volterra-Fredholm integro-differential equations. *J. Math. Computer Sci., 21*(2), 158–163. http://dx.doi.org/10.22436/jmcs.021.02.07

12 Bakodah, H.O., Al-Mazmumy, M., & Almuhalbedi, S.O. (2017). An Efficient Modification of the Adomian decomposition Method for Solving Integro-Differential Equations. *Mathematical Sciences Letters, 6*(1), 15–21. https://doi.org/10.18576/msl/060103

13 Hashim, I. (2006). Adomian decomposition method for solving BVPs for fourth-order integro-differential equations. *Journal of Computational and Applied Mathematics, 193*(2), 658–664. https://doi.org/10.1016/j.cam.2005.05.034

14 Adomian, G., & Rach, R. (1986). On linear and nonlinear integro-differential equations. *J. Math. Anal. Appl., 113*, 199–201.

15 Al-Khaled, K., & Allan, F. (2005). Decomposition Method for solving nonlinear Integro-differential Equations. *J. Appl. Math. and Computing, 19*(1-2), 415–425. http://dx.doi.org/10.1007/BF02935815

16 Hamoud, A.A., & Ghadl, K.P. (2017). The Combined Modified Laplace with Adomian Decomposition Method for solving the nonlinear Volterra-Fredholm Integro-differential Equations. *J. KSIAM, 21*(1), 17–28. http://dx.doi.org/10.12941/jksiam.2017.21.017

17 El-Sayed, S.M., & Abdel-Aziz, M.R. (2003). A Comparison of Adomian's Decomposition Method and Wavelet-Galerkin Method for solving Integro-Differential Equations. *Applied mathematics and Computation, 136*, 151–159.

18 El-Sayed, S., Kaya D., & Zarea, S. (2004). The decomposition method applied to solve high-order linear Volterra-Fredholm integro-differential equations. *Int. J. Nonlinear Sci. Numer. Simul., 5*(2), 105–112.

19 Hamoud, A.A., & Ghadle, K.P. (2017). The reliable modified of Laplace Adomian decomposition method to solve nonlinear interval Volterra-Fredholm integral equations. *Korean J. Math., 25*(3), 323–334.

20 Khachatryan, A.Kh., & Khachatryan Kh.A. (2008). Factorization of one convolution-type Integro-Differential Equation on positive half line. *Ukr. mat. Journ., 60*(11), 1823–1839. https://doi.org/10.1007/s11253-009-0172-6

21 Parasidis, I.N., Providas, E., & Tsekrekos, P.C. (2012). Factorization of linear operators and some eigenvalue problems of special operators. *Vestnik of Bashkir University, 17*(2), 830–839.

22 Providas, E., Pulkina l.S., & Parasidis, I.N. (2021). Factorization of ordinary and Hyperbolic integro-differential equations with integral boundary conditions in a Banach space. *Vestnik of Samara University. Natural Science Series, 27*(1), 29–43. https://doi.org/10.18287/2541-7525-2021-27-1-29-43

23 Providas, E., & Parasidis, I.N. (2021). A Procedure for Factoring and Solving Nonlocal Boundary Value Problems for a Type of Linear Integro-Differential Equations. *Algorithms, 14*, 346. https://doi.org/10.3390/a14120346

24 Parasidis, I.N. (2019). Extension and decomposition methods for differential and integro-differential equations. *Eurasian Mathematical Journal, 10*(3), 48–67. https://doi.org/10.32523/2077-9879-2019-10-3-48-67.

25 Vassiliev, N.N., Parasidis, I.N., & Providas, E. (2019). Decomposition-extension method for squared operators. *Information and Control Systems*, 2, 2–9. https://doi.org/10.31799/1684-8853-2019-2-2-9

26 Parasidis, I.N., & Providas, E. (2021). Factorization method for solving nonlocal boundary value problems in Banach space. *Bulletin of the Karaganda University. Mathematics series*, 3(103), 76–86. https://doi.org/10.31489/2021M3/76-86

27 Tsilika, K.D. (2021). Decomposition of abstract linear operators on Banach spaces. *Information and Control Systems*, 2, 2–12. https://doi.org/10.31799/1684-8853-2021-2-2-12

Абстрактілі операторларды екінші дәрежелі екі операторға факторизациялау және оны интегралдық дифференциалдық теңдеулерге қолдану

И.Н. Парасидис, Е. Провидас

Фессалия университеті, Лариса, Греция

Мақалада дербес туындылы дифференциалдық оператормен немесе Фредгольм интегралдық-дифференциалдық теңдеуіне сәйкес келетін қарапайым дифференциалдық операторы бар B_1 абстрактылы сызықтық операторымен $\mathbf{B}_1 x = f$ шекаралық есебі зерттелген. Биективті оператор $\mathbf{B}_1$ түріндегі факторизацияны өткізген жағдайда $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$, $\mathbf{B}_1 x = f$ есебінің дәл аналитикалық шешімі алынды, мұндағы $\mathbf{B}$, $\mathbf{B}_0$ $\mathbf{B}_1$ қарағанда қарапайым, екінші дәрежелі екі сызықтық абстрактылы оператор. $\mathbf{B}_1$ операторының факторизациялау шарттары және биективтіліктің критерийі табылды.

Кілт сөздер: корректілі оператор, биективті оператор, сызықтық операторларды факторизациялау (жіктеу), Фредгольм интегралдық-дифференциалдық теңдеулері, шекаралық есептер, дәл шешімдер.

Факторизация абстрактных операторов на два оператора второй степени и ее приложения к интегро-дифференциальным уравнениям

И.Н. Парасидис, Е. Провидас

Университет Фессалии, Лариса, Греция

Исследована краевая задача $\mathbf{B}_1 x = f$ с абстрактным линейным оператором B_1, соответствующая интегро-дифференциальному уравнению Фредгольма с обыкновенным дифференциальным оператором или дифференциальным оператором в частных производных. Получено точное аналитическое решение задачи $\mathbf{B}_1 x = f$ в случае, когда биективный оператор $\mathbf{B}_1$ допускает факторизацию вида $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$, где $\mathbf{B}$, $\mathbf{B}_0$ — два линейных абстрактных оператора второй степени, более простых, чем $\mathbf{B}_1$. Найдены условия факторизации и критерий биективности оператора $\mathbf{B}_1$.

Ключевые слова: корректный оператор, биективный оператор, факторизация (разложение) линейных операторов, интегро-дифференциальные уравнения Фредгольма, краевые задачи, точные решения.

*Author Information**

Ioannis Nestorios Parasidis (*corresponding author*) — Professor, PhD, University of Thessaly, Larissa, Greece; e-mail: *paras@uth.gr*; https://orcid.org/0000-0002-7900-9256

Efthimios Providas — Professor, PhD, Vice-Rector of the University Thessaly, Larissa, Greece; e-mail: *providas@uth.gr*; https://orcid.org/0000-0002-0675-4351

*The author's name is presented in the order: First, Middle and Last Names.

https://doi.org/10.31489/2024M1/162-173 *Research article*

The first boundary value problem for the fractional diffusion equation in a degenerate angular domain

M.I. Ramazanov[1], A.V. Pskhu[2], M.T. Omarov[1,*]

[1] *Institute of Applied Mathematics, Karaganda Buketov University, Karaganda, Kazakhstan;*
[2] *Institute of Applied Mathematics and Automation of Kabardin-Balkar Scientific Center of RAS, Nalchik, Russia*
(E-mail: ramamur@mail.ru, pskhu@list.ru, madiomarovt@gmail.com)

This article addresses the problems observed in branching fractal structures, where super-slow transport processes can occur, a phenomenon described by diffusion equations with a fractional time derivative. The characteristic feature of these processes is their extremely slow relaxation rate, where a physical quantity changes more gradually than its first derivative. Such phenomena are sometimes categorized as processes with "residual memory". The study presents a solution to the first boundary problem in an angular domain degenerating into a point at the initial moment of time for a fractional diffusion equation with the Riemann-Liouville fractional differentiation operator with respect to time. It establishes the existence theorem of the problem under investigation and constructs a solution representation. The need for understanding these super-slow processes and their impact on fractal structures is identified and justified. The paper demonstrates how these processes contribute to the broader understanding of fractional diffusion equations, proving the theorem's existence and formulating a solution representation.

Keywords: partial differential equation, fractional calculus, angular domain, kernel, weak singularity, parabolic cylinder, Carleman-Vekua equation, general solution, unique solution, Riemann-Lioville fractional operator.

2020 Mathematics Subject Classification: 35R11.

Introduction

The paper discusses an equation of the form:

$$\left(\frac{\partial^\alpha}{\partial t^\alpha} - \frac{\partial^2}{\partial x^2}\right) u(x,t) = f(x,t), \quad (0 < \alpha < 1), \tag{1}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is a fractional derivative of an order α with respect to the variable t, starting from the point $t = 0$. This type of fractional differentiation is defined by the Riemann-Liouville operator:

$${}_a g^{(\nu)}(x) \equiv {}_a\mathrm{D}_x^\nu g(x) = \begin{cases} \frac{1}{\Gamma(-\nu)} \int_a^x (x-\xi)^{-\nu-1} g(\xi)d\xi, & \nu < 0, \\ \frac{1}{\Gamma(1-\nu)} \mathrm{D}_x \int_a^x (x-\xi)^{-\nu} g(\xi)d\xi, & 0 \le \nu < 1, \\ \frac{1}{\Gamma(2-\nu)} \mathrm{D}_x^2 \int_a^x (x-\xi)^{-\nu+1} g(\xi)d\xi, & 1 \le \nu < 2, \\ \dots\dots\dots\dots\dots\dots, & \dots\dots\dots \end{cases}$$

Fractional diffusion equations (where $0 < \alpha \le 1$) have been extensively studied in recent years. This surge in interest is due to their widespread applications in physics and modeling, as referenced in sources [1–5]. The primary methodologies for exploring diffusion-wave equations are detailed in publications [6–24], while monographs [25] and [26] provide a comprehensive bibliography on the subject.

*Corresponding author. *E-mail: madiomarovt@gmail.com*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488729).

Received: 18 October 2023; *Accepted:* 04 December 2023.

Nearly all studies related to equation (1) have focused on initial and boundary problems in both limited and unlimited cylindrical areas. Specifically, the first boundary problem for the fractional diffusion equation in a rectangular area was examined in [17,18]. In publication [27], the first boundary problem for the fractional diffusion-wave equation in a non-cylindrical area was solved. However, the area where the solution is sought does not degenerate into a point at the initial moment in time.

The aim of this study is to solve the first boundary problem for equation (1) in a domain that is not cylindrical, but rather angular, and degenerates into a point at the initial moment in time.

In relation to the boundary value problems for the heat conduction equation with a diffusion coefficient α set to 1:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u(x,t) = f(x,t),$$

these problems have been investigated in non-cylindrical domains by several authors [28–32]. It is important to underline that boundary value problems for the Laplace equation in domains with evolving boundaries are distinct from the classical ones defined in fixed cylindrical domains. The reason is that the dimensions of the domain where the solution is sought are time-dependent, which makes these problems unsuitable for classical variable separation and integral transformation methods.

The potential theory approach allows reformulating the boundary value problem into a Volterra system of second kind integral equations. In such cases, if the domain's boundary does not exist at the initial time, then the corresponding system of integral equations can be solved by the method of successive approximations due to the weak singularity of their kernels. In contrast, if the boundary exists at the initial time, the integral equations of the boundary value problem might admit additional solutions, and the implementation of the Picard method encounters certain mathematical complexities. Similar issues occur for boundary value problems of the Dirichlet problem for the Laplace equation in non-cylindrical domains that originate at the initial moment in time.

1 Problem Statement

To determine a regular solution for the fractional time-derivative heat equation:

$$\left(\frac{\partial^\alpha}{\partial t^\alpha} - \frac{\partial^2}{\partial x^2}\right) u(x,t) = f(x,t), \quad (0 < \alpha < 1),$$

within the domain

$$D = \{(x,t) : 0 < x < t, 0 < t < \infty\},$$

that adheres to the boundary conditions:

$$u(0,t) = 0, \quad u(t,t) = 0, \quad 0 < t < \infty. \tag{2}$$

We denote $u(x,t)$ as a regular solution of equation (1) in domain D such that:

$$t^{1-\gamma} u(x,t) \in C(\overline{D})$$

for some $\gamma > 0$. Additionally, the solution $u(x,t)$ must be continuous within D and possess a continuous partial derivative with respect to x, and its second-order derivative with respect to x, ${}_0D_t^v u(x,y)$, must be continuous in the variable t at fixed x inside the domain D and up to the boundary set $\{0 < x < t\}$, with $u(x,t)$ fulfilling equation (1) at every point in D.

2 Main Result

Definitions are introduced as follows:

$$\beta = \frac{\alpha}{2}, \quad \omega_{\beta,\mu}(x,t) = t^{\mu-1} W\left(-\beta, \mu; -\frac{|x|}{t^{\beta}}\right),$$

$$\omega(x,t) = \omega_{\beta,0}(x,t),$$

in which

$$W(-\beta, \mu; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\mu - \beta k)}$$

represents the Wright function, as discussed in [33].

The following statement holds true:

Theorem 1. Let the conditions be satisfied: $t^{1-\gamma} g_i(t) \in C[0,T]$, $i = 1,2$, for some $\gamma > 0$, and $t^{1-\mu} f(x,t) \in C(\overline{D})$, $\mu \geq 0$, if $f(x,t)$ satisfies the Holder condition with respect to the variable x.

Then the solution to problem (1)-(2) exists and can be expressed as

$$u(x,t) = \int_0^t \psi_1(\tau)\omega(x,t-\tau)d\tau + \int_0^t \psi_2(\tau)\omega(T-x,t-\tau)d\tau + F(x,t), \tag{3}$$

here

$$F(x,t) = \frac{1}{2}\int_0^t \int_0^{\tau} f(s,\tau)\omega_{\beta,\mu}(x-s,t-\tau)dsd\tau,$$

and $\psi_1(t), \psi_2(t)$ from (3) are the solutions to the system of integral equations

$$\begin{cases} \psi_1(t) + \int_0^t \psi_2(\tau)\omega(\tau,t-\tau)d\tau = -F(0,t), \\ \psi_2(t) + \int_0^t \psi_1(\tau)\omega(t,t-\tau)d\tau + \int_0^t \psi_2(\tau)\omega(\tau-\mathrm{t},t-\tau)d\tau = -F(t,t). \end{cases} \tag{4}$$

From the first equation of this system (4), we obtain

$$\psi_1(t) = -\int_0^t \psi_2(\tau)\omega(t,\tau-t)d\tau - F(0,t),$$

and substituting $\psi_1(t)$ into the second equation of the system (4), we get [33]:

$$\begin{aligned} &\psi_2(t) + \int_0^t \left(-\int_0^{\tau} \psi_2(\xi)\omega(\xi,\tau-\xi)d\xi - F(0,\tau)\right)\omega(t,t-\tau)d\tau+ \\ &+\int_0^t \psi_2(\tau)\omega(\tau-t,t-\tau)d\tau = -F(t,t). \end{aligned} \tag{5}$$

Substituting into the repeated integral and changing the order of integration as well as the dummy variables ξ and τ, from equation (5), we arrive at a special integral equation of the second kind in the form of a Volterra equation:

$$\psi_2(t) - \int_0^t \mathcal{K}(t,\tau)\psi_2(\tau)d\tau = \mathcal{F}(t), \tag{6}$$

here

$$\mathcal{K}(t,\tau) = \int_{\tau}^t \omega_{\beta,0}(\tau,\xi-\tau)\omega_{\beta,0}(t,t-\xi)d\xi - \omega_{\beta,0}(\tau-t,t-\tau), \tag{7}$$

and

$$\mathcal{F}(t) = \int_0^t F(0,\tau)\omega(t,\tau - t)d\tau - F(t,t). \tag{8}$$

To perform the calculation of integral (6) having (7) and (8), it is necessary to employ.

$$\int_\tau^t \omega_{\beta,0}(\tau,\xi-\tau)\omega_{\beta,0}(t,t-\xi)d\xi.$$

The convolution formula is applied to the Wright function as referenced in [33], and this is expressed through the function $\omega_{\beta,\mu}(x,t)$:

$$\int_0^y \omega_{\alpha,\delta}\left(x_1,\xi\right)\omega_{\alpha,\mu}\left(x_2,(y-\xi)\right)d\xi = \omega_{\alpha,\delta+\mu}\left(x_1+x_2,y\right).$$

Then we obtain,

$$\begin{aligned}&\int_\tau^t \omega_{\beta,0}(\tau,\xi-\tau)\omega_{\beta,0}(t,t-\xi)d\xi = \|\xi-\tau=\eta\| = \\ &= \int_0^{t-\tau} \omega_{\beta,0}(\tau,\eta)\omega_{\beta,0}(t,(t-\tau)-\eta)d\eta = \\ &= \omega_{\beta,0}(t+\tau,t-\tau).\end{aligned}$$

Therefore, the conclusive kernel $\mathcal{K}_\beta(t,\tau)$ is determined by the following relation:

$$\mathcal{K}_\beta(t,\tau) = \omega_{\beta,0}(t+\tau,t-\tau) - \omega_{\beta,0}(\tau-t,t-\tau). \tag{9}$$

The second term of kernel (9) has a weak singularity, since the following estimate is valid for it:

$$\omega_{\beta,0}(\tau-t,t-\tau) \le \frac{C(\beta)}{(t-\tau)^\beta}. \tag{10}$$

Indeed, by applying the estimate found in [26]:

$$|\omega_\mu(x,y)| \leqslant C(\beta,\mu,\theta)|x|^{-\theta}y^{\beta\theta+\mu-1},$$
$$\theta \geqslant \begin{cases}0, & (-\mu)\notin\mathbb{N}\cup\{0\}\\ -1, & (-\mu)\in\mathbb{N}\cup\{0\}\end{cases},$$

taking into account that $\mu = 0$, and choosing

$$\theta = -\frac{\beta}{1-\beta} > -1.$$

This leads to the confirmation of inequality (10). Next, we aim to demonstrate the special nature of the kernel $\mathcal{K}_\beta(t,\tau)$.

Lemma. If $0 < \beta \le 1/2$, the equality holds true

$$\lim_{t\to 0}\int_0^t \mathcal{K}_\beta(t,\tau)d\tau = 1. \tag{11}$$

Proof. Initially, when t is small, these inequalities are applicable:

$$\omega_{\beta,0}(t,t-\tau) \geq \omega_{\beta,0}(t+\tau,t-\tau) \geq \omega_{\beta,0}(2t,t-\tau).$$

Using equation [33]

$$D^{v}_{0t}\omega_{\beta,\mu}(x,y) = \omega_{\beta,\mu-v}(x,y)$$

these results in

$$\lim_{t\to 0}\int_0^t \omega_{\beta,0}(bt,t-\tau)d\tau = \lim_{t\to 0}\omega_{\beta,1}(bt,t) = 1, \quad b = 1,2.$$

Therefore, considering inequality (10), we establish the validity of equality (11).

The kernel's properties make it unsuitable for solving the corresponding integral equation through the method of successive approximations. This limitation of the integral equation is due to the solution domain for the problem collapsing to a single point at the start. Otherwise, if this collapse didn't occur, the kernel for the integral equation would possess a weak singularity, enabling the use of Picard's method for finding a solution [33].

3 Solution of the special integral equation (6)

To solve the integral equation mentioned in equation (6), we apply the Carleman-Vekua method. This involves using a specific integral equation, which we refer to as the characteristic equation.

$$\psi_2(t) - \int_0^t \mathcal{K}_{\frac{1}{2}}(t,\tau)\psi_2(\tau)d\tau = \mathcal{Q}(t) \tag{12}$$

here

$$\mathcal{K}_{1/2}(t,\tau) = \frac{1}{2a\sqrt{\pi}}\left\{\frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right) + \frac{1}{(t-\tau)^{\frac{1}{2}}}\exp\left(-\frac{t-\tau}{4a^2}\right)\right\}. \tag{13}$$

Relation (13) can be verified directly using the following formula [34; 5.2.10(2)] for $(n = -2)$,

$$\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!\Gamma[1+(n-k)/2]} = \frac{1}{\sqrt{\pi}}2^{(n+1)/2}e^{-x^2/8}D_{-n-1}\left(\frac{x\sqrt{2}}{2}\right),$$

here $D_{-n-1}(z)$ is the function of a parabolic cylinder.

At the same time, the kernel $\mathcal{K}_{\frac{1}{2}}(t,\tau)$ possesses a similar property as described in equation (11):

$$\lim_{t\to 0}\int_0^t \mathcal{K}_{\frac{1}{2}}(t,\tau)d\tau = 1.$$

This means that the kernel difference $\mathcal{K}_{\frac{1}{2}}(t,\tau) - \mathcal{K}_{\beta}(t,\tau) = \widetilde{K}(t,\tau)$ has a weak singularity. We will employ the regularization method to solve the characteristic equation, known as the Carleman-Vekua equation, and to do so, we will express equation (12) in a particular form:

$$\psi_2(t) - \int_0^t \mathcal{K}_{\frac{1}{2}}(t,\tau)\psi_2(\tau)d\tau = \mathcal{Q}(t) - \int_0^t \widetilde{K}(t,\tau)\psi_2(\tau)d\tau. \tag{14}$$

Assuming the right-hand side of this equality is temporarily known and denoting it by

$$\mathcal{Q}(t) = \mathcal{F}(t) - \int_0^t \widetilde{K}(t,\tau)\psi_2(\tau)d\tau.$$

Equation (14) can be represented in the following form:

$$\mathbb{K}_{1/2}\psi_2 \equiv \psi_2(t) - \int_0^t \mathcal{K}_{\frac{1}{2}}(t,\tau)\psi_2(\tau)d\tau = Q(t). \qquad (15)$$

In [35] it is shown that the general solution of equation (15) in the weight class of functions

$$\sqrt{t}\exp\left(-\frac{t}{4a^2}\right)\varphi\left(t\right) \in L_\infty\left(0,\infty\right)$$

has the form:

$$\mathbf{K}\psi_2 \equiv \psi_2(t) - \left[\mathbb{K}_{\frac{1}{2}}\right]^{-1}\mathcal{Q}(t) = c_0\psi_0(t) \qquad (16)$$

and the function

$$\psi_0(t) = \frac{1}{\sqrt{t}}\exp\left(-\frac{t}{4a^2}\right) + \frac{\sqrt{\pi}}{2a}\operatorname{erf}\left(\frac{\sqrt{t}}{2a}\right) + \frac{\sqrt{\pi}}{2a}$$

is the general solution of the corresponding homogeneous integral equation.

The integral equation (16) is already solvable by the method of successive approximations and the solution to the corresponding homogeneous equation will be determined by the equality:

$$\psi_{2,0}(t) = c_0[\mathbf{K}]^{\mathbf{-1}}\left[\psi_0(t)\right].$$

Similarly, as in the work [33], it is proven that function (6) is a solution to equation (1) and satisfies conditions (2), thus proving the validity of Theorem 1.

Conclusion

It is shown that in a non-cylindrical domain that degenerates at the initial moment of time into a point, the first boundary value problem for a fractional diffusion equation with the Riemann-Liouville fractional differentiation operator with respect to a time variable is singular, that is, it may not have a unique solution.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488729).

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Нахушев А.М. Дробное исчисление и его применение / А.М. Нахушев. — М.: Физматлит, 2003. — 272 с.

2 Учайкин В.В. Метод дробных производных / В.В. Учайкин. — Ульяновск: Артишок, 2008. — 512 с.

3 Mainardi F. Fractional calculus and waves in linear viscoelasticity. An introduction to mathematical models / F. Mainardi. — London: Imperial College Press, 2010. — 347 p. https://doi.org/10.1142/p614

4 Тарасов В.Е. Модели теоретической физики с интегро-дифференцированием дробного порядка / В.Е. Тарасов. — М.–Ижевск: Ин-т комп. исслед., 2011. — 568 с.

5 Atanackovic T.M. Fractional calculus with applications in mechanics. Vibrations and diffusion processes / T.M. Atanackovic, S. Pilipovic, B. Stankovic, D. Zorica. — London: John Wiley and Sons, Inc., Hoboken, NJ, 2014. — 315 p. https://doi.org/10.1002/9781118577530

6 Wyss. W. The fractional diffusion equation / W. Wyss // J. Math. Phys. — 1986. — 27. — No. 11. — P. 2782–2785. https://doi.org/10.1063/1.527251

7 Schneider W.R. Fractional diffusion and wave equations / W.R. Schneider, W. Wyss // J. Math. Phys. — 1989. — 30. — No. 1. — P. 134–144. https://doi.org/10.1063/1.528578

8 Kochubei A.N. Diffusion of fractional order / A.N. Kochubei // Differential Equations. — 1990. — 26. — No. 4. — P. 485–492.

9 Fujita Ya. Integrodifferential equation which interpolates the heat equation and the wave equation / Ya. Fujita // Osaka J. Math. — 1990. — 27. — No. 2. — P. 309–321. https://doi.org/10.18910/8094

10 Геккиева С.Х. Краевая задача для обобщенного уравнения переноса с дробной производной по времени / С.Х. Геккиева // Докл. Адыгской (Черкесской) междунар. акад. наук. — 1994. — 1. — № 1. — С. 17,18.

11 Mainardi F. The fundamental solutions for the fractional diffusion-wave equation / F. Mainardi / Appl. Math. Lett. — 1996. — 9. — № 6. — P. 23–28. https://doi.org/10.1016/0893-9659(96)00089-4

12 Шогенов В.Х. Обобщенное уравнение переноса и дробные производные / В.Х. Шогенов, С.К. Кумыкова, М.Х. Шхануков-Лафишев // Докл. НАН Украины. — 1997. — № 12. — С. 47–54.

13 Buckwar E. Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations / E. Buckwar, Yu. Luchko. // J. Math. Anal. Appl. — 1998. — 227. — No. 1. — P. 81–97. https://doi.org/10.1006/jmaa.1998.6078

14 Gorenflo R. Wright functions as scale-invariant solutions of the diffusion-wave equation / R. Gorenflo, Yu. Luchko, F. Mainardi // J. Comput. Appl. Math. — 2000. — 118. — No. 1-2. — P. 175–191. https://doi.org/10.1016/S0377-0427(00)00288-0

15 Gorenflo R. Mapping between solutions of fractional diffusion-wave equations / R. Gorenflo, A. Iskenderov, Yu. Luchko // Fract. Calc. Appl. Anal. — 2000. — 3. — No. 1. — P. 75–86.

16 Agrawal. O.P. Solution for a fractional diffusion-wave equation defined in a bounded domain / O.P. Agrawal // Nonlinear Dynam. — 2002. — 29. — P. 145–155. https://doi.org/10.1023/A:1016539022492

17 Псху А.В. Решение первой краевой задачи для уравнения диффузии дробного порядка / А.В. Псху // Диф. ур. — 2003. — 39. — № 9. — С. 1286–1289.

18 Псху А.В. Решение краевых задач для уравнения диффузии дробного порядка методом функции Грина / А.В. Псху // Диф. ур. — 39. — № 10. — С. 1430—1433. https://doi.org/10.1023/B:DIEQ.0000012703.45373.aa

19 Eidelman S. D. Cauchy problem for fractional diffusion equations / S.D. Eidelman, A.N. Kochubei // J. Differential Equations. — 2004. — 199. — No. 2. — P. 211–255. https://doi.org/10.1016/j.jde.2003.12.002

20 Ворошилов А.А. Задача Коши для диффузионно-волнового уравнения с частной производной Капуто / А.А. Ворошилов, А.А. Килбас // Диф. ур. — 2006. — 42. — № 5. — С. 599–609. https://doi.org/10.1134/S0012266106050041

21 Yang. Y. Spectral collocation method for the time-fractional diffusion-wave equation and convergence analysis / Y. Yang, Y. Chen, Y. Huang, H. Wei // Computers and Mathematics with Applications. — 2017. — 73. — No. 6. — P. 1218–1232. https://doi.org/10.1016/j.camwa.2016.08.017

22 Псху А.В. Фундаментальное решение диффузионно-волнового уравнения дробного порядка / А.В. Псху // Изв. РАН. Сер. матем. — 2009. — 73. — № 2. — С. 141–182. https://doi.org/10.4213/im2429

23 Luchko Yu. Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation // Fract. Calc. Appl. Anal. — 2012. — 15. — No. 1. — P. 141–160. https://doi.org/10.2478/s13540-012-0010-7

24 Мамчуев М.О. Необходимые нелокальные условия для диффузионно-волнового уравнения / М.О. Мамчуев // Вестн. Самар. гос. ун-та. Естественнонауч. сер. — 2014. — 118. — № 7. — С. 45–59. https://doi.org/10.18287/2541-7525-2014-20-7-45-59

25 Kilbas A.A. Theory and applications of fractional differential equations / A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. — Elsevier, 2006. — 523 p. https://doi.org/10.1016/s0304-0208(06)x8001-5

26 Псху А.В. Уравнения в частных производных дробного порядка / А.В. Псху. — М.: Наука, 2005. — 272 с.

27 Gevrey M. Sur les équations aux dérivées partielles du type parabolique / M. Gevrey // J. Math. Pures Appl. — 1913. — 6. — No. 9. — P. 305–476.

28 Mamchuev M.O. Fourier Problem for Fractional Diffusion–Wave Equation / M.O. Mamchuev, A.M. Mamchuev // Lobachevskii Journal of Mathematics. — 2023. — 44. — No. 2. — P. 620–628. https://doi.org/10.1134/S1995080223020257

29 Liu J. A new fractional derivative for solving time fractional diffusion wave equation / J. Liu, X. Yang, Y. Feng, L. Geng // Mathematical Methods in the Applied Sciences. — 2022. — 46. — No. 1. — P. 267–272. https://doi.org/10.1002/mma.8509

30 Kharin S. Mathematical model of electrical contact bouncing / S. Kharin // AIP Conference Proceedings. — 2015. — 1676. — No. 1. — 020019. https://doi.org/10.1063/1.4930445

31 Дженалиев М.Т. Нагруженные уравнения как возмущения дифференциальных уравнений / М.Т. Дженалиев, М.И. Рамазанов. — Алматы: Fылым, 2010. — 334 с.

32 Pskhu A.V. Boundary value problem for fractional diffusion equation in a curvilinear angle domain / A.V. Pskhu, M.I. Ramazanov, N.K. Gulmanov, S.A. Iskakov // Bulletin of the Karaganda university. Mathematics series. — 2022. — № 1(105). — P. 83–95. https://doi.org/10.31489/2022M1/83-95

33 Псху А.В. Первая краевая задача для дробного диффузионно-волнового уравнения в нецилиндрической области / А.В. Псху // Изв. РАН. Сер. матем. — 2017. — 81. — № 6. — С. 158–179. https://doi.org/10.4213/im8520

34 Прудников А.П. Интегралы и ряды: справоч.: [В 3-х т.] — Т. 2. Специальные функции / А.П. Прудников, Ю.А. Брычков, О.И. Маричев. — 2-е изд., испр. — М.: Физматлит, 2003. — 664 с.

35 Jenaliyev M. On a Volterra equation of the second kind with "incompressible" kernel / M. Jenaliyev, M. Amangaliyeva, M. Kosmakova, M. Ramazanov // Advances in Difference Equations. — 2015. — 71. — P. 1–14. https://doi.org/10.1186/s13662-015-0418-6

Бұрыштық жойылмалы аядағы бөлшекті диффузия теңдеуі үшін бірінші шекаралық есеп

М.И. Рамазанов[1], А.В. Псху[2], М.Т. Омаров[1]

[1] *Қолданбалы математика институты, Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан;*
[2] *РҒА Кабардин-Балқар ғылыми орталығының Қолданбалы математика және автоматтандыру институты, Нальчик, Ресей*

Мақала тармақталған фракталды құрылымдарда байқалатын мәселелерді қарастырады, мұнда уақыт бойынша бөлшектік туындылары бар диффузиялық теңдеулермен сипатталатын өте баяу транспорттық процестер болуы мүмкін. Осы процестердің ерекше белгісі — олардың өте баяу релаксация жылдамдығы, мұнда физикалық шама оның бірінші туындысынан гөрі біртіндеп өзгереді. Мұндай құбылыстар кейде «қалдық жады» бар процестер ретінде жіктеледі. Зерттеуде уақыт бойынша Риман-Лиувилль бөлшектік дифференциалдау операторы бар бөлшектік диффузиялық теңдеу үшін бұрыштық облыста, бастапқы уақыт моментінде нүктеге дегенерацияланған бірінші шекаралық есептің шешімі ұсынылған. Онда зерттелетін есептің бар екендігі туралы теорема анықталған және есептің шешімі көрсетілген. Мақалада осындай өте баяу процестерді және олардың фракталды құрылымдарға әсерін түсінудің қажеттілігі атап өтілген. Жұмыс бөлшектік диффузиялық теңдеулердің кеңірек түсінілуіне осы процестердің қалай ықпал ететінін көрсетеді, теореманың бар екендігін дәлелдейді және есептің шешімін тұжырымдайды.

Кілт сөздер: дербес туынды теңдеу, бөлшек есептеу, бұрыштық облыс, ядро, әлсіз ерекшелік, параболикалық цилиндр, Карлеман-Векуа теңдеуі, жалпы шешім, жалғыз шешім, Риман-Лиувилльдің бөлшекті операторы.

Первая краевая задача для дробного диффузионного уравнения в угловой вырождающейся области

М.И. Рамазанов[1], А.В. Псху[2], М.Т. Омаров[1]

[1] *Институт прикладной математики, Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан;*
[2] *Институт прикладной математики и автоматизации, Кабардино-Балкарский научный центр РАН, Нальчик, Россия*

Статья рассматривает проблемы, наблюдаемые в ветвящихся фрактальных структурах, где могут происходить сверхмедленные транспортные процессы; явление, описываемое диффузионными уравнениями с дробной производной по времени. Характерной особенностью этих процессов является их крайне медленная скорость релаксации, при которой физическая величина изменяется более постепенно, чем её первая производная. Такие явления иногда классифицируются как процессы с «остаточной памятью». В исследовании представлено решение первой краевой задачи в угловой области, вырождающейся в точку в начальный момент времени, для дробного диффузионного уравнения с оператором дробного дифференцирования Римана-Лиувилля по времени. В нем устанавливается теорема существования исследуемой задачи и строится представление решения. Авторами подчёркивается необходимость понимания этих сверхмедленных процессов и их влияния на фрактальные структуры. Работа демонстрирует, как эти процессы способствуют более широкому пониманию дробных диффузионных уравнений, доказывая существование теоремы и формулируя представление решения.

Ключевые слова: уравнение в частных производных, дробное исчисление, угловая область, ядро, слабая особенность, параболический цилиндр, уравнение Карлемана-Векуа, общее решение, единственное решение, дробный оператор Римана-Лиувилля.

References

1 Nakhushev, A.M. (2003). *Drobnoe ischislenie i ego primenenie [Fractional calculus and its application].* Moscow: Fizmatlit [in Russian].

2 Uchaikin, V.V. (2008). *Metod drobnykh proizvodnykh [Fractional derivative method].* Ulianovsk: Artishok [in Russian].

3 Mainardi, F. (2010). *Fractional calculus and waves in linear viscoelasticity. An introduction to mathematical models.* London. Imperial College Press. https://doi.org/10.1142/p614

4 Tarasov, V.E. (2011). *Modeli teoreticheskoi fiziki s integro-differentsirovaniem drobnogo poriadka [Models of theoretical physics with fractional order integro-differentiation].* Institut kompiuternykh issledovanii Moscow—Izhevsk: Institut kompiuternykh issledovanii [in Russian].

5 Atanackovic, T.M., Pilipovic, S., Stankovic, B., & Zorica, D. (2014). *Fractional calculus with applications in mechanics. Vibrations and diffusion processes.* London. John Wiley and Sons, Inc., Hoboken, NJ. https://doi.org/10.1002/9781118577530

6 Wyss, W. (1986). The fractional diffusion equation. *J. Math. Phys., 27*(11), 2782–2785. https://doi.org/10.1063/1.527251

7 Schneider, W.R., & Wyss, W. (1989). Fractional diffusion and wave equations. *J. Math. Phys., 30*(1), 134–144. https://doi.org/10.1063/1.528578

8 Kochubei, A.N. (1990). Diffusion of fractional order. *Differential Equations, 26*(4), 485–492.

9 Fujita, Ya. (1990). Integrodifferential equation which interpolates the heat equation and the wave equation. *Osaka J. Math., 27*(2), 309–321. https://doi.org/10.18910/8094

10 Gekkieva, S.Kh. (1994). Kraevaia zadacha dlia obobshchennogo uravneniia perenosa s drobnoi proizvodnoi po vremeni [Boundary value problem for the generalized transport equation with fractional time derivative]. *Doklady Adygskoi (Cherkesskoi) mezhdunarodnoi akademii nauk — Reports of the Adyghe (Circassian) International Academy of Sciences, 1*(1), 17,18 [in Russian].

11 Mainardi, F. (1996). The fundamental solutions for the fractional diffusion-wave equation. *Appl. Math. Lett., 9*(6), 23–28. https://doi.org/10.1016/0893-9659(96)00089-4

12 Shogenov, V.Kh., Kumykova, S.K., & Shkhanukov-Lafishev, M.Kh. (1997). Obobshchennoe uravnenie perenosa i drobnye proizvodnye [Generalized transport equation and fractional derivatives]. *Doklady Natsionalnoi akademii nauk Ukrainy — Reports of the NAS of Ukraine,* (12), 47–54 [in Russian].

13 Buckwar, E., & Luchko, Yu. (1998). Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations. *J. Math. Anal. Appl., 227*(1), 81–97. https://doi.org/10.1006/jmaa.1998.6078

14 Gorenflo, R., Luchko, Yu., & Mainardi, F. (2000). Wright functions as scale-invariant solutions of the diffusion-wave equation. *J. Comput. Appl. Math., 118*(1-2), 175–191. https://doi.org/10.1016/S0377-0427(00)00288-0

15 Gorenflo, R., Iskenderov, A., & Luchko, Yu. (2000). Mapping between solutions of fractional diffusion-wave equations. *Fract. Calc. Appl. Anal., 3*(1), 75–86.

16 Agrawal, O.P. (2002). Solution for a fractional diffusion-wave equation defined in a bounded domain. *Nonlinear Dynam., 29(1)*, 145–155. https://doi.org/10.1023/A:1016539022492

17 Pskhu, A.V. (2003). Reshenie pervoi kraevoi zadachi dlia uravneniia diffuzii drobnogo poriadka [Solution of the first boundary value problem for the fractional order diffusion equation]. *Differentsialnye uravneniia — Differential equations, 39*(9), 1286–1289 [in Russian].

18 Pskhu, A.V. (2003). Reshenie kraevykh zadach dlia uravneniia diffuzii drobnogo poriadka metodom funktsii Grina [Solving boundary value problems for the fractional order diffusion equation using the Green's function method]. *Differentsialnye uravneniia — Differential equations, 39*(10), 1430–1433 [in Russian]. https://doi.org/10.1023/B:DIEQ.0000012703.45373.aa

19 Eidelman, S.D., & Kochubei, A.N. (2004). Cauchy problem for fractional diffusion equations. *J. Differential Equations, 199*(2), 211–255. https://doi.org/10.1016/j.jde.2003.12.002

20 Voroshilov, A.A., & Kilbas, A.A. (2006). Zadacha Koshi dlia diffuzionno-volnovogo uravneniia s chastnoi proizvodnoi Kaputo [Cauchy problem for the diffusion-wave equation with Caputo's partial derivative]. *Differentsialnye uravneniia — Differential equations, 42*(5), 599–609 [in Russian]. https://doi.org/10.1134/S0012266106050041

21 Yang, Y., Chen, Y., Huang, Y., & Wei, H. (2017). Spectral collocation method for the time-fractional diffusion-wave equation and convergence analysis. *Computers and Mathematics with Applications, 73*(6), 1218–1232. https://doi.org/10.1016/j.camwa.2016.08.017

22 Pskhu, A.V. (2009). Fundamentalnoe reshenie diffuzionno-volnovogo uravneniia drobnogo poriadka. [Fundamental solution of the diffusion-wave equation of fractional order] *Izvestiia RAN. Seriia matematicheskaia — News of the Russian Academy of Sciences. Mathematical series, 73*(2), 141–182 [in Russian]. https://doi.org/10.4213/im2429

23 Luchko, Yu. (2012). Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation. *Fract. Calc. Appl. Anal., 15*(1), 141–160. https://doi.org/10.2478/s13540-012-0010-7

24 Mamchuev, M.O. (2014). Neobkhodimye nelokalnye usloviia dlia diffuzionno-volnovogo uravneniia [Necessary nonlocal conditions for the diffusion-wave equation]. *Vestnik Samarskogo gosudarstvennogo universiteta. Estestvennonauchnaia seriia — Vestnik of Samara University. Natural Science Series, 118*(7), 45–59 [in Russian]. https://doi.org/10.18287/2541-7525-2014-20-7-45-59

25 Kilbas, A.A., Srivastava, H.M., & Trujillo, J.J. (2006). *Theory and applications of fractional differential equations.* Elsevier. https://doi.org/10.1016/s0304-0208(06)x8001-5

26 Pskhu, A.V. (2005). *Uravneniia v chastnykh proizvodnykh drobnogo poriadka [Fractional partial differential equations.* Moscow: Nauka [in Russian].

27 Gevrey, M. (1913). Sur les équations aux dérivées partielles du type parabolique [On partial differential equations of the parabolic type]. *J. Math. Pures Appl., 6*(9), 305–476 [in French].

28 Mamchuev, M.O., & Mamchuev, A.M. (2023). Fourier Problem for Fractional Diffusion–Wave Equation. *Lobachevskii Journal of Mathematics, 44*(2), 620–628. https://doi.org/10.1134/S1995080223020257

29 Liu, J., Yang, X., Feng, Y., & Geng, L. (2022). A new fractional derivative for solving time fractional diffusion wave equation. *Mathematical Methods in the Applied Sciences, 46*(1), 267–272. https://doi.org/10.1002/mma.8509

30 Kharin, S. (2015). Mathematical model of electrical contact bouncing. *AIP Conference Proceedings, 1676*(1). https://doi.org/10.1063/1.4930445

31 Dzhenaliev, M.T., & Ramazanov, M.I. (2010). *Nagruzhennye uravneniia kak vozmushcheniia differentsialnykh uravnenii [Loaded equations as perturbations of differential equations].* Almaty: Gylym [in Russian].

32 Pskhu, A.V., Ramazanov, M.I., Gulmanov, N.K., & Iskakov, S.A. (2022). Boundary value problem for fractional diffusion equation in a curvilinear angle domain. *Bulletin of the Karaganda university. Mathematics series,* 1(105), 83–95. https://doi.org/10.31489/2022M1/83-95

33 Pskhu, A.V. (2017). Pervaia kraevaia zadacha dlia drobnogo diffuzionno-volnovogo uravneniia v netsilindricheskoi oblasti [The first boundary value problem for the fractional diffusion-wave equation in a non-cylindrical domain.]. *Izvestiia RAN. Seriia matematicheskaia — News of the Russian Academy of Sciences. Mathematical series, 81* (6), 158–179 [in Russian]. https://doi.org/10.4213/im8520

34 Prudnikov, A.P., Brychkov, Yu.A., & Marichev, O.I. (2003). *Integraly i riady: spravochnik: [v 3-kh tomakh]. Tom 2. Spetsialnye funktsii [Integrals and series: handbook: in volume 2. Special functions.]* 2-e izdanie, ispravlennoe. Moscow: Fizmatlit [in Russian].

35 Jenaliyev, M., Amangaliyeva, M., Kosmakova, M., & Ramazanov, M. (2015). On a Volterra equation of the second kind with "incompressible" kernel. *Advances in Difference Equations,* 71, 1–14. https://doi.org/10.1186/s13662-015-0418-6.

*Author Information**

Murat Ibraevich Ramazanov — Doctor of physical and mathematical sciences, Professor, Institute of Applied Mathematics, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *ramamur@mail.ru*; https://orcid.org/0000-0002-2297-5488

Arsen Vladimirovich Pskhu — Doctor of physical and mathematical sciences, Docent, A. D. of IAMA KBSC RAS, Head of the Department of Fractional Calculus, The Institute of Applied Mathematics and Automation of Kabardin-Balkar Scientific Center of RAS, st. Shortanova, 89 A, Nalchik, 360000, Kabardin-Balkar Republic, Russia; e-mail: *pskhu@list.ru*; https://orcid.org/0000-0002-0506-3516

Madi Tulegenovich Omarov (*corresponding author*) — Master of natural sciences, teacher, Institute of Applied Mathematics, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *madiomarovt@gmail.com*; https://orcid.org/0000-0002-9026-5912

*The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics series, No. 1(113), 2024, pp. 174–193

https://doi.org/10.31489/2024M1/174-193 *Research article*

Existence of Hilfer fractional neutral stochastic differential systems with infinite delay

S. Sivasankar[1], R. Udhayakumar[1,*], V. Muthukumaran[2], G. Gokul[1], S. Al-Omari[3]

[1] *School of Advanced Sciences, Vellore Institute of Technology, Vellore, India;*
[2] *College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, India;*
[3] *Faculty of Science, Al-Balqa Applied University, Amman, Jordan*
(E-mail: sivajisivasankar@gmail.com, udhayaram.v@gmail.com, muthu.v2404@gmail.com, gokulg2110@gmail.com, shridehalomari@bau.edu.jo)

The goal of this study is to propose the existence of mild solutions to delay fractional neutral stochastic differential systems with almost sectorial operators involving the Hilfer fractional (HF) derivative in Hilbert space, which generalized the famous Riemann-Liouville fractional derivative. The main techniques rely on the basic principles and concepts from fractional calculus, semigroup theory, almost sectorial operators, stochastic analysis, and the Mönch fixed point theorem via the measure of noncompactness (MNC). Particularly, the existence result of the equation is obtained under some weakly compactness conditions. An example is given at the end of this article to show the applications of the obtained abstract results.

Keywords: Hilfer fractional evolution system, Neutral system, Measure of noncompactness, Fixed point theorem.

2020 Mathematics Subject Classification: 47H10, 47H08, 34K30, 34K50.

Introduction

Applications for fractional calculus extend from engineering and natural phenomena to financial views and physical accomplishments, and the subject is always growing. Fields like viscoelasticity, electrical engineering circuits, the vibration of seismic movements, biological systems, etc. usually contain an increasing number of fractional frameworks. Numerous good monographs provide the essential scientific methods for the attractiveness of this research topic. It should be possible to compare frameworks with practical systems of fractional power to the framework of ordinary integer order. Regarding fractional order, the derivative of the framework sum in the practical system might be correct. Numerous models in scattering, sensor fusion, automation, and so forth might all be used using this system. Learners can examine the literature [1–3], as well as research articles [4–8] that deal with the concept of fractional evolution systems to gain a thorough understanding of the concepts as well as the specifics of how it is implemented.

Due to the prevalence of neutral differential equations in many applications of applied mathematics, only neutral systems have received substantial attention in recent decades. In most cases, neutral systems with or without delay serve as an optimal configuration of numerous partial neutral systems that emerge in problems related to heat stream in components, viscoelasticity, acoustic waves, and various natural processes. One may mention [9–11] for a very helpful discussion on neutral systems involved in differential equations. Instead of deterministic models, stochastic ones should be studied since both natural and manufactured systems are prone to noise or uncontrolled fluctuations. Differential equations with stochastic components contain unpredictability in their theoretical depiction of a specific event. For a general overview of stochastic differential equations (SDE) and its applications [12–15].

*Corresponding author. *E-mail: udhayaram.v@gmail.com*
Received: 27 July 2023; *Accepted:* 06 November 2023.

The R-L and Caputo fractional derivatives were among the additional fractional order derivatives that Hilfer [16] started. The significance and consequences of the Hilfer fractional derivative (HFD) have also been found through conceptual forecasts of experiments in hard materials, pharmaceutical industries, set architecture design, architecture, and other fields. Gu and Trujillo [17] recently showed that the HFD evolution problem has an integral solution using a fixed point approach and a MNC strategy. In order to identify the derivative's order, he constructed the greatest current variable $\zeta \in [0,1]$ and a fractional variable η so that $\zeta = 0$ generates the R-L derivative and $\zeta = 1$ generates the Caputo derivative. Numerous papers have been written about Hilfer fractional calculus [18, 19]. According to [20–23], researchers discovered a mild solution for HF differential systems employing almost sectorial operators and a fixed point method.

The research articles [24–27] to improve the fractional existence for fractional calculus by utilizing almost sectorial operators. Investigators in the study by [20–22] employed almost sectorial operators to get their results using Schauder's fixed point theorem. Researchers have subsequently constructed nonlocal fractional differential equations with or without delay using non-dense fields, semigroups, cosine families, many fixed point strategy, and the MNC. To the best of our knowledge, the existence of HF neutral stochastic differential systems using the measure of noncompactness mentioned in this study is an exposed area of research that appears to give an extra incentive for completing this research.

The following subject will be looked at in this article: HF stochastic differential systems contain almost sectorial operators with nonlocal condition

$$D_{0^+}^{\eta,\zeta}\big[z(\rho) - \eth(\rho, z_\rho))\big] = \widehat{A}z(\rho) + \mathcal{F}\big(\rho, z_\rho\big) + \mathcal{H}\big(\rho, z_\rho\big)\frac{dW(\rho)}{d\rho}, \qquad \rho \in \mathfrak{D}' = (0,d], \tag{1}$$

$$I_{0^+}^{(1-\eta)(1-\zeta)}z(0) + \aleph(z_\rho) = \alpha \in L^2(\Delta, B_{\rm r}), \quad \rho \in (-\infty, 0], \tag{2}$$

where $\widehat{A}$ denote the almost sectorial operator, which generate an analytic semigroup $\{T(\rho), \rho \geq 0\}$ on $\mathbb{Y}$. Consider $z(\cdot)$ is the value in a Hilbert space $\mathbb{Y}$ with $\|\cdot\|$ and $D_{0^+}^{\eta,\zeta}$ represents the HFD of order $\eta,\ 0 < \eta < 1$ and type $\zeta,\ 0 \leq \zeta \leq 1$. The histories $z_\rho : (-\infty, 0] \to B_{\rm r},\ z_\rho(a) = z(\rho + a),\ a \leq 0$ connected with the abstract phase space $B_{\rm r}$. Fix $\mathfrak{D} = [0,d]$, and let $\mathcal{F} : \mathfrak{D} \times B_{\rm r} \to \mathbb{Y}$, $\mathcal{H} : \mathfrak{D} \times B_{\rm r} \to L_2^0(\mathcal{J}, \mathbb{Y})$ and $\eth : \mathfrak{D} \times B_{\rm r} \to \mathbb{Y}$ are the $\mathbb{Y}$-valued function and non-local term $\aleph : B_{\rm r} \to \mathbb{Y}$.

Now let's break up our content into the following sections. In Section 1 we outline a few crucial ideas and details from our study that are referenced throughout the body of this article. We discussed the existence of a mild solution to the problem in Section 2. We provide an illustration of our main notion in Section 3. Then, a few conclusions are offered.

1 *Preliminaries*

The fundamental concepts, theorems, and lemma that are used throughout the whole work are introduced here.

The notations $(\mathbb{Y}, \|\cdot\|)$ and $(\mathcal{J}, \|\cdot\|)$ signify two real distinct Hilbert spaces. Suppose $(\Delta, \mathscr{F}, P)$ is a full probability area connected with full family of right continuous growing sub σ-algebra $\{\mathscr{F}_\rho : \rho \in \mathfrak{D}\}$ fulfills $\mathscr{F}_\rho \subset \mathscr{F}$. Consider $W = \{W(\rho)\}_{\rho \geq 0}$ is a Q-Wiener strategy identified on $(\Delta, \mathscr{F}, P)$ with the correlation operator Q such that $Tr(Q) < \infty$. We assume there exists a full orthonormal system e_k, $k \geq 1$ in U, a limited series of non-negative real integers χ_k such that $Qe_k = \chi_k e_k,\ k = 1, 2, \cdots$ and $\{\mu_k\}$ of independent Brownian movements such that

$$\big(W(\rho), e\big)_U = \sum_{k=1}^{\infty} \sqrt{\chi_k}(e_k, e)\mu_k(\rho), \quad e \in U\ \rho \geq 0.$$

Assuming that the area of all Q-Hilbert-Schmidt operators $\varphi : Q^{\frac{1}{2}}\mathcal{J} \to \mathbb{Y}$ with the inner product $\|\varphi\|_Q^2 = \langle \varphi, \varphi \rangle = Tr(\varphi Q \varphi)$ is signified by the symbols $L_2^0 = L_2\big(Q^{\frac{1}{2}}\mathcal{J}, \mathbb{Y}\big)$. Let us consider the resolvent operator of $\widehat{A}$, $0 \in \rho(\widehat{A})$, where $\mathsf{S}(\cdot)$ is uniformly bounded, that is, $\|\mathsf{S}(\rho)\| \leq M,\ M \geq 1$, and $\rho \geq 0$. Thus, given $\delta \in (0,1]$, the fractional power operator $\widehat{A}^{\delta}$ on its range $D(\widehat{A}^{\delta})$ may be obtained. Furthermore, $D(\widehat{A}^{\delta})$ is dense in $\mathbb{Y}$.

The succeeding substantial characteristic of $\widehat{A}^{\delta}$ will be discussed.

Theorem 1. [1]

1 If $0 < \delta \leq 1$, then $\mathbb{Y}_{\delta} = D(\widehat{A}^{\delta})$ is a Banach space with $\|z\|_{\delta} = \|\widehat{A}^{\delta} z\|,\ z \in \mathbb{Y}_{\chi}$.

2 Assume $0 < \gamma < \delta \leq 1$, embedding $D(\widehat{A}^{\delta}) \to D(\widehat{A}^{\gamma})$ and the implementation are compact whenever $\widehat{A}$ is compact.

3 For all $0 < \delta \leq 1$, there exists $C_{\delta} > 0$ such that

$$\|\widehat{A}^{\delta}\mathsf{S}(\rho)\| \leq \frac{C_{\delta}}{\rho^{\delta}},\ 0 < \rho \leq d.$$

Consider $\mathsf{C} : \mathfrak{D} \to \mathbb{Y}$ is the family of all continuous functions, where $\mathfrak{D} = [0,d]$ and $\mathfrak{D}' = (0,d]$ with $d > 0$. Choose

$$Y = \big\{z \in \mathsf{C} : \lim_{\rho \to 0} \rho^{1-\zeta+\eta\zeta-\eta\vartheta} z(\rho) \text{ exists and finite}\big\},$$

which is the Banach space and its $\|\cdot\|_Y$, specified as

$$\|z\|_Y = \sup_{\rho \in \mathfrak{D}'} \big\{\rho^{1-\zeta+\eta\zeta-\eta\vartheta}\|z(\rho)\|\big\}.$$

Fix $\mathscr{B}_P(\mathfrak{D}) = \{u \in \mathsf{C}$ such that $\|u\| \leq P\}$. Let $z(\rho) = \rho^{-1+\zeta-\eta\zeta+\eta\vartheta}y(\rho),\ \rho \in (0,d]$ then, $z \in Y$ if and only if $y \in \mathsf{C}$ and $\|z\|_Y = \|y\|$. We produce $\mathcal{H}$ with $\|\mathcal{H}\|_{L^p(\mathfrak{D},\mathbb{R}^+)}$, where $\mathcal{H} \in L^p(\mathfrak{D},\mathbb{R}^+)$ for some p along with $1 \leq p \leq \infty$. Also $L^p(\mathfrak{D},\mathbb{Y})$ represent the Banach space of functions $\mathcal{H} : \mathfrak{D} \times B_{\mathrm{r}} \to \mathbb{Y}$ which are the Bochner integrable normed by $\|\mathcal{H}\|_{L^p(\mathfrak{D},\mathbb{Y})}$.

Definition 1. [16] For the function $\mathcal{H} : [d,+\infty) \to \mathbb{R}$, the HFD of order $0 < \eta < 1$ and type $\zeta \in [0,1]$, presented by

$$D^{\eta,\zeta}_{d^+}\mathcal{F}(\rho) = [I^{(1-\eta)\zeta}_{d^+} D(I^{(1-\eta)(1-\zeta)}_{d^+}\mathcal{F})](\rho).$$

The abstract phase space B_{r} is now specified. Assume that $w : (-\infty,0] \to (0,+\infty)$ is continuous along $l = \int_{-\infty}^{0} w(\rho)d\rho < +\infty$. Now, for all $n > 0$, we obtain

$$B = \big\{\varepsilon : [-n,0] \to \mathbb{Y} \text{ such that } \varepsilon(\rho) \ \text{ is bounded and measurable}\big\},$$

and set the space B with the norm

$$\|\varepsilon\|_{[-n,0]} = \sup_{\tau \in [-n,0]} \|\varepsilon(\tau)\|, \text{ for all } \varepsilon \in \mathscr{B}.$$

We now specify,

$$B_{\mathrm{r}} = \Big\{\varepsilon : (-\infty,0] \to \mathbb{Y} \text{ such that for all } n > 0,\ \varepsilon|_{[-n,0]} \in \mathscr{B} \text{ and } \int_{-\infty}^{0} w(\tau)\|\varepsilon\|_{[\tau,0]}d\tau < +\infty\Big\}.$$

Suppose B_{r} is endowed with

$$\|\varepsilon\|_{B_{\mathrm{r}}} = \int_{-\infty}^{0} w(\tau)\|\varepsilon\|_{[\tau,0]}d\tau, \text{ for all } \varepsilon \in B_{\mathrm{r}},$$

therefore $(B_{\mathrm{r}}, \|\cdot\|)$ is a Banach space.

Now, we specify the space

$$B'_{\mathrm{r}} = \{z : (-\infty, d] \to \mathbb{Y} \text{ such that } z|_{\mathfrak{D}} \in \mathtt{C},\ \alpha \in B_{\mathrm{r}}\}.$$

Let us consider the seminorm $\|\cdot\|_d$ in B'_{r} defined as

$$\|z\|_d = \|\alpha\|_{B_{\mathrm{r}}} + \sup\{\|z(\tau)\| : \tau \in [0, d]\},\ z \in B'_{\mathrm{r}}.$$

Lemma 1. If $z \in B'_{\mathrm{r}}$, then for all $\rho \in \mathfrak{D},\ z_\rho \in B_{\mathrm{r}}$. Furthermore,

$$l|z(\rho)| \leq \|z_\rho\|_{B_{\mathrm{r}}} \leq \|\alpha\|_{B_{\mathrm{r}}} + l \sup_{r\in[0,\rho]} |z(r)|,$$

where $l = \int_{-\infty}^{0} w(\rho)d\rho < \infty$.

Definition 2. [25] We explain the family of closed linear operators Θ_ω^ϑ, for $0 < \vartheta < 1,\ 0 < \omega < \frac{\pi}{2}$, the sector $S_\omega = \{\theta \in \mathbb{C}\backslash\{0\} \text{ with } |arg\ \theta| \leq \omega\}$ and $\widehat{A} : D(\widehat{A}) \subset \mathbb{Y} \to \mathbb{Y}$ that fulfills
(i) $\sigma(\widehat{A}) \subseteq S_\omega$;
(ii) $\|(\theta - \widehat{A})^{-1}\| \leq \mathcal{K}_\varepsilon |\iota|^{-\vartheta}$, for all $\omega < \varepsilon < \pi$ and there exists $\mathcal{K}_\varepsilon$ as a constant,
afterward $\widehat{A} \in \Theta_\omega^{-\vartheta}$ is specified like almost sectorial operator on $\mathbb{Y}$.

Theorem 2. [3] $\mathsf{S}_\eta(\rho)$ and $\mathsf{Q}_\eta(\rho)$ are continuous in the uniform operator topology, for $\rho > 0$, for all $d > 0$, the continuity is uniform on $[d, \infty)$.

Lemma 2. [28] Suppose $\{T_\eta(\rho)\}_{\rho>0}$ is a compact operator, then $\{\mathsf{S}_{\eta,\zeta}(\rho)\}_{\rho>0}$ and $\{\mathsf{Q}_\eta(\rho)\}_{\rho>0}$ are also compact linear operators.

Lemma 3. [17] System (1)-(2) is unique to an integral equation offered by

$$\begin{aligned} z(\rho) =& \frac{\alpha(0) - \aleph(z_\rho) - \eth(0, \alpha(0))}{\Gamma(\zeta(1-\eta)+\eta)} \rho^{-(1-\eta)(\zeta-1)} + \eth(\rho, z_\rho) \\ &+ \frac{1}{\Gamma(\eta)} \int_0^\rho (\rho-\iota)^{\eta-1} \big[\widehat{A} z_\iota d\iota + \mathcal{F}(\iota, z_\iota) d\iota + \mathcal{H}(\iota, z_\iota) dW(\iota)\big]. \end{aligned}$$

Definition 3. [17] Let $z(\rho)$ be the solution of the integral equation offered by Lemma 3 then $z(\rho)$ fulfills

$$\begin{aligned} z(\rho) = \mathsf{S}_{\eta,\zeta}(\rho)\big[\alpha(0) - \aleph(z_\rho) - \eth(0, \alpha(0))\big] + \eth(\rho, z_\rho) + \int_0^\rho \mathsf{K}_\eta(\rho-\iota)\mathcal{F}(\iota, z_\iota) d\iota \\ + \int_0^\rho \mathsf{K}_\eta(\rho-\iota)\mathcal{H}(\iota, z_\iota) dW(\iota),\ \rho \in \mathfrak{D}, \end{aligned}$$

where $\mathsf{S}_{\eta,\zeta}(\rho) = I_0^{\zeta(1-\eta)} \mathsf{K}_\eta(\rho),\ \mathsf{K}_\eta(\rho) = \rho^{\eta-1}\mathsf{Q}_\eta(\rho)$ and $\mathsf{Q}_\eta(\rho) = \int_0^\infty \eta\epsilon\mathfrak{W}_\eta(\epsilon)T(\rho^\eta\epsilon)d\epsilon$.

Definition 4. [7] A stochastic process $z : (-\infty, d] \to \mathbb{Y}$ is said to be a mild solution of the system (1)-(2) if $I_{0^+}^{(1-\eta)(1-\zeta)} z(0) + \aleph(z_\rho) = \alpha \in L^2(\eta, B_{\mathrm{r}}),\ \rho \in (-\infty, 0]$ and the preceding integral equation that fulfills

$$\begin{aligned} z(\rho) = \mathsf{S}_{\eta,\zeta}(\rho)\big[\alpha(0) - \aleph(z_\rho) - \eth(0, \alpha(0))\big] + \eth(\rho, z_\rho) + \int_0^\rho (\rho-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho-\iota)\eth(\iota, z_\iota) d\iota \\ + \int_0^\rho (\rho-\iota)^{(\eta-1)}\mathsf{Q}_\eta(\rho-\iota)\mathcal{F}(\iota, z_\iota) d\iota + \int_0^\rho (\rho-\iota)^{(\eta-1)}\mathsf{Q}_\eta(\rho-\iota)\mathcal{H}(\iota, z_\iota) dW(\iota). \end{aligned}$$

Lemma 4. [21]
1 $\mathsf{K}_\eta(\rho)$ and $\mathsf{S}_{\eta,\zeta}(\rho)$ are strongly continuous, for $\rho > 0$.
2 $\mathsf{K}_\eta(\rho)$ and $\mathsf{S}_{\eta,\zeta}(\rho)$ are bounded linear operators on $\mathbb{Y}$, for all fixed $\rho \in S_{\frac{\pi}{2}-\omega}$, we obtain

$$\big\|\mathsf{K}_\eta(\rho)z\big\| \le \kappa_p \rho^{-1+\eta\vartheta}\|z\|, \qquad \big\|\mathsf{Q}_\eta(\rho)z\big\| \le \kappa_p \rho^{-\eta+\eta\vartheta}\|z\|,$$
$$\big\|\mathsf{S}_{\eta,\zeta}(\rho)z\big\| \le \frac{\Gamma(\vartheta)}{\Gamma(\zeta(1-\eta)+\eta\vartheta)}\kappa_p \rho^{-1+\zeta-\eta\zeta+\eta\vartheta}\|z\|.$$

Proposition 1. [19] Let $\eta \in (0,1),\ q \in (0,1]$ and for all $z \in D(\widehat{A})$, then there exists a $\kappa_q > 0$ such that

$$\|\widehat{A}^q \mathcal{Q}_\eta(\rho)z\| \le \frac{\eta\kappa_q\Gamma(2-q)}{\rho^{\eta q}\Gamma(1+\eta(1-q))}\|z\|,\ 0 < \rho < d.$$

The Hausdorff MNC will now be briefly discussed.

Definition 5. [29] For a bounded set X in a Banach space $\mathbb{Y}$, the Hausdorff MNC μ is represented as

$$\mu(X) = \inf\{\epsilon > 0 : X \text{ can be linked by a finite number of balls with radii } \epsilon\}.$$

Theorem 3. [8] If $\{v_k\}_{k=1}^{\infty}$ is a sequence of Bochner integrable functions from $\mathfrak{D} \to \mathbb{Y}$ with the measurement $\|v_k(\rho)\| \le \mu(\rho)$, for all $\rho \in \mathcal{V}$ and for all $k \ge 1$, where $\mu \in L^1(\mathfrak{D},\mathbb{R})$, then the function $\omega(\rho) = \mu\big(\{v(\rho) : k \ge 1\}\big)$ is in $L^1(\mathfrak{D},\mathbb{R})$ and fulfills

$$\mu\bigg(\bigg\{\int_0^\rho v_k(\iota)d\iota :\ k \ge 1\bigg\}\bigg) \le 2\int_0^\rho \omega(\iota)d\iota.$$

Lemma 5. [8] Let $X \subset \mathbb{Y}$ be a bounded set, then there exists a countable set $X_0 \subset X$ such that $\mu(X) \le 2\mu(X_0)$.

Definition 6. [29] If E^+ is the positive cone of an order Banach space $(E, \le)$. Let $\mho$ be the function represented on the family of all bounded subset of the Banach space $\mathbb{Y}$ with values in E^+ is known as MNC on $\mathbb{Y}$ if and only if $\mho(conv(\iota)) = \mho(\iota)$ for all bounded subset $\iota \subset \mathbb{Y}$, where $conv(\iota)$ denoted the closed convex hull of ι.

Lemma 6. [30] Let G be a closed convex subset of a Banach space $\mathbb{Y}$ and $0 \in G$. Suppose $F : G \to \mathbb{Y}$ continuous map which fulfils Mönch's requirements, i.e., if $G_1 \subset G$ is countable and, $G_1 \subset \overline{co}\big(\{0\} \cup F(G_1)\big) \implies \overline{G_1}$ is compact. Then F has a fixed point in G.

2 Existence

We require the succeeding hypotheses:

(H_1) Let $\widehat{A}$ be the almost sectorial operator of the analytic semigroup $T(\rho),\ \rho > 0$ in $\mathbb{Y}$ such that $\|T(\rho)\| \le \mathsf{K}_1$ where $\mathsf{K}_1 \ge 0$ be the constant.

(H_2) The function $\mathcal{F} : \mathfrak{D} \times B_{\mathrm{r}} \to \mathbb{Y}$ fulfills:
- (a) Caratheodory circumstances: $\mathcal{F}(\cdot, z)$ is strongly measurable for all $z \in B_{\mathrm{r}}$ and $\mathcal{F}(\rho,\cdot)$ is continuous for a.e. $\rho \in \mathfrak{D},\ \mathcal{F}(\cdot,\cdot) : [0,S] \to \mathbb{Y}$ is strongly measurable;
- (b) There exists a constant $0 < \eta_1 < \eta$ and $m_1 \in L^{\frac{1}{\eta_1}}(\mathfrak{D},\mathbb{R}^+)$ and non-decreasing continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\big\|\mathcal{F}(\rho,z)\big\| \le m_1(\rho)f(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}\|z\|),\ z \in \mathbb{Y},\ \rho \in \mathfrak{D}$ where f fulfills $\liminf_{k\to\infty} \frac{\psi(k)}{k} = 0$;

(c) There exists a constant $0 < \eta_2 < \eta$ and $e_1 \in L^{\frac{1}{\eta_2}}(\mathfrak{D}, \mathbb{R}^+)$ such that, for any bounded subset $M \subset \mathbb{Y},\ \mu(\mathcal{F}(\rho, M)) \leq e_1(\rho)\mu(M)$ for a.e. $\rho \in \mathfrak{D}$.

(H_3) The function $\mathcal{H} : \mathfrak{D} \times B_{\rm r} \to L_2^0(\mathcal{J}, \mathbb{Y})$ fulfills:

(a) Caratheodory circumstances: $\mathcal{H}(\cdot, z)$ is strongly measurable for all $z \in B_{\rm r}$ and $\mathcal{H}(\rho, \cdot)$ is continuous for a.e. $\rho \in \mathfrak{D},\ \mathcal{H}(\cdot, \cdot) : [0, S] \to L_2^0(\mathcal{J}, \mathbb{Y})$ is strongly measurable;

(b) There exists a constant $0 < \eta_3 < \eta$ and $m_2 \in L^{\frac{1}{\eta_3}}(\mathfrak{D}, \mathbb{R}^+)$ and non-decreasing continuous function $\hbar : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\big\|\mathcal{H}(\rho, z)\big\| \leq m_2(\rho)\hbar(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}\|z\|),\ z \in \mathbb{Y},\ \rho \in \mathfrak{D}$ where $\hbar$ fulfills $\liminf_{k\to\infty} \frac{\sigma(k)}{\sigma} = 0$;

(c) There exists a constant $0 < \eta_4 < \eta$ and $e_2 \in L^{\frac{1}{\eta_4}}(\mathfrak{D}, \mathbb{R}^+)$ such that, for any bounded subset $M \subset \mathbb{Y},\ \mu(\mathcal{H}(\rho, M)) \leq e_2(\rho)\mu(M)$ for a.e. $\rho \in \mathfrak{D}$.

(H_4) The function $\aleph : C(\mathfrak{D}, \mathbb{Y}) \to \mathbb{Y}$ is continuous, compact operator and there exists $L_1 > 0$ as the value such that $\|\aleph(z_1) - \aleph(z_2)\| \leq L_1\|z_1 - z_2\|$.

(H_5) The function $\eth : \mathfrak{D} \times B_{\rm r} \to \mathbb{Y}$ is continuous and there exists $q > 0,\ 0 < q < 1$ such that $\eth \in D(\widehat{A}^q)$ for all $z \in \mathbb{Y},\ \rho \in \mathfrak{D},\ \widehat{A}^q\eth(\cdot, z)$ is strongly measurable, then there exists $M_w > 0,\ M'_w > 0$ such that $\gamma_1, \gamma_2 \in \mathbb{Y}$ and $\widehat{A}^q\eth(\rho, \cdot)$ satisfies the following:

$$\big\|\widehat{A}^q\eth(\rho, \gamma_1(\rho)) - \widehat{A}^q\eth(\rho, \gamma_2(\rho))\big\| \leq M_w\rho^{1-\zeta+\eta\zeta-\eta\vartheta}\|\gamma_1(\rho) - \gamma_2(\rho)\|_{B_{\rm r}},$$
$$\big\|\widehat{A}^q\eth(\rho, z(\rho))\big\| \leq M'_w\big(1 + \rho^{1-\zeta+\eta\zeta-\eta\vartheta}\|z\|_{B_{\rm r}}\big).$$

Take $\|\widehat{A}^{-q}\| = M_0$.

Theorem 4. Suppose $(H_1) - (H_5)$ holds, then the HF neutral stochastic system (1)-(2) has a unique solution on $\mathfrak{D}$ presented, $\alpha(0) \in D(\widehat{A}^\theta)$ with $\theta > 1 + \vartheta$.

Proof. Consider the operator $\Psi : B'_{\rm r} \to B'_{\rm r}$, defined

$$\Psi(z(\rho)) = \begin{cases} \Psi_1(\rho),\ (-\infty, 0], \\ \mathsf{S}_{\eta,\zeta}(\rho)\big[\alpha(0) - \aleph(z_\rho) - \eth(0, \alpha(0))\big] + \eth(\rho, z_\rho) \\ + \int_0^\rho (\rho - \iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho - \iota)\eth(\iota, z_\iota)d\iota \\ + \int_0^\rho (\rho - \iota)^{(\eta-1)}\mathsf{Q}_\eta(\rho - \iota)\mathcal{F}(\iota, z_\iota)d\iota \\ + \int_0^\rho (\rho - \iota)^{\eta-1}\mathsf{Q}_\eta(\rho - \iota)\mathcal{H}\big(\iota, z_\iota\big)dW(\iota),\ \rho \in \mathfrak{D}. \end{cases}$$

For $\Psi_1 \in B_{\rm r}$, we specify $\widehat{\Psi}$ as

$$\widehat{\Psi}(\rho) = \begin{cases} \Psi_1(\rho), & \rho \in (-\infty, 0], \\ \mathsf{S}_{\eta,\zeta}(\rho)\alpha(0), & \rho \in \mathfrak{D}, \end{cases}$$

then $\widehat{\Psi} \in B'_{\rm r}$. Let $z(\rho) = \rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v(\rho) + \widehat{\Psi}(\rho)],\ \infty < \rho \leq d$. It is trivial to establish that $\mathfrak{u}$ fulfills by the Definition 4 if and only if v satisfies v_0 and

$$\begin{aligned} v(\rho) = &- \mathsf{S}_{\eta,\zeta}(\rho)\big[\aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big) + \eth(0, \alpha(0))\big] + \eth\big(\rho, \big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big)\big) \\ &+ \int_0^\rho (\rho - \iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho - \iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\ &+ \int_0^\rho (\rho - \iota)^{\eta-1}\mathsf{Q}_\eta(\rho - \iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\ &+ \int_0^\rho (\rho - \iota)^{\eta-1}\mathsf{Q}_\eta(\rho - \iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota). \end{aligned}$$

Let $B''_{\mathrm{r}} = \{v \in B'_{\mathrm{r}} : v_0 \in B_{\mathrm{r}}\}$. For any $v \in B'_{\mathrm{r}}$,

$$\begin{aligned}\|v\|_d =& \|v_0\|_{B_{\mathrm{r}}} + \sup\{\|v(\iota)\| : 0 \le \iota \le d\}\\ =& \sup\{\|v(\iota)\| : 0 \le \iota \le d\}.\end{aligned}$$

Hence, $(B''_{\mathrm{r}}, \|\cdot\|)$ is a Banach space.

For $P > 0$, take $\mathscr{B}_P = \{v \in B''_{\mathrm{r}} : \|v\|_d \le P\}$, then $\mathscr{B}_P \subset B''_{\mathrm{r}}$ is uniformly bounded, and for $v \in \mathscr{B}_P$, by Lemma 1,

$$\begin{aligned}\|v_\rho + \widehat{\Psi}_\rho\|_{B_{\mathrm{r}}} &\le \|v_\rho\|_{B_{\mathrm{r}}} + \|\widehat{\Psi}\|_{B_{\mathrm{r}}}\\ &\le l\Big(P + \frac{\Gamma(\vartheta)}{\Gamma(\zeta(1-\eta)+\eta\vartheta)}\kappa_P \rho^{-1+\zeta-\eta\zeta+\eta\vartheta}\Big) + \|\Psi_1\|_{B_{\mathrm{r}}}\\ &= P'.\end{aligned}$$

Consider an operator $\mho : B''_{\mathrm{r}} \to B''_{\mathrm{r}}$, specified by

$$\mho v(\rho) = \begin{cases} 0, \ \rho \in (-\infty, 0], \\ -\mathsf{S}_{\eta,\zeta}(\rho)\big[\aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big) + \eth(0, \alpha(0))\big] \\ \quad +\eth\big(\rho, \big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big)\big) \\ \quad + \int_0^\rho (\rho-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho-\iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\ \quad + \int_0^\rho (\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\ \quad + \int_0^\rho (\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota), \qquad \rho \in \mathfrak{D}. \end{cases}$$

Then to prove $\mho$ has a fixed point.

Step 1: To prove there exists a positive value P such that $\mho(\mathscr{B}_P(\mathfrak{D})) \subseteq \mathscr{B}_P(\mathfrak{D})$. Suppose the claim is incorrect i.e., for all $P > 0$, there exists $v^P \in \mathscr{B}_P(\mathfrak{D})$, but $\mho(v^P)$ not in $\mathscr{B}_P(\mathfrak{D})$, that is,

$$\begin{aligned}E\big\|v^P\big\|^2 \le P < E&\Big\|\sup_{\rho\in[0,d]} \rho^{1-\zeta+\eta\zeta-\eta\vartheta}\big(\mho v^P(\rho)\big)\Big\|^2\\ \le E&\Big\|\sup_{\rho\in[0,d]} \rho^{1-\zeta+\eta\zeta-\eta\vartheta}\Big\{ - \mathsf{S}_{\eta,\zeta}(\rho)\big[\aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big) + \eth(0, \alpha(0))\big]\\ &+ \eth\big(\rho, \big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big)\big)\\ &+ \int_0^\rho (\rho-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho-\iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\\ &+ \int_0^\rho (\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\\ &+ \int_0^\rho (\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota)\Big\}\Big\|^2\\ \le 5d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}&\Big[E\Big\|\mathsf{S}_{\eta,\zeta}(\rho)\big[\aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v^P_\rho + \widehat{\Psi}_\rho]\big) + \eth(0, \alpha(0))\big]\Big\|^2\\ &+ E\Big\|\eth\big(\rho, \rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v^P_\rho + \widehat{\Psi}_\rho]\big)\Big\|^2\\ &+ E\Big\|\int_0^\rho (\rho-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho-\iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v^P_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\ &+ E\Big\|\int_0^\rho (\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v^P_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2\end{aligned}$$

$$
\begin{aligned}
&+ E\bigg\| \int_0^\rho (\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota^P+\widehat{\Psi}_\iota]\big)dW(\iota)\bigg\|^2\Bigg] \\
&\le 5d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\Bigg[\|\mathsf{S}_{\eta,\zeta}(\rho)\|^2\big[L_1^2\|v_\rho^P+\widehat{\Psi}_\rho\|^2+\|\aleph(0)\|^2+M_w'^2\|\alpha\|^2\big] \\
&+ M_0^2M_w'^2(1+d^{2(1-\zeta+\eta\zeta-\eta\vartheta)})P' \\
&+ \int_0^\rho(\rho-\iota)^{2(\eta-1)}\|\widehat{A}^{1-q}\mathsf{Q}_\eta(\rho-\iota)\|^2E\|\widehat{A}^q\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota^P+\widehat{\Psi}_\iota]\big)\|^2d\iota \\
&+ \int_0^\rho(\rho-\iota)^{2(\eta-1)}\|\mathsf{Q}_\eta(\rho-\iota)\|^2m_1^2(d)f^2(P')d\iota \\
&+ Tr(Q)\int_0^\rho(\rho-\iota)^{2(\eta-1)}\|\mathsf{Q}_\eta(\rho-\iota)\|^2m_2^2(d)\hbar^2(P')d\iota\Bigg] \\
&\le 5d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\Bigg[\bigg(\frac{\Gamma(\vartheta)}{\Gamma(\zeta(1-\eta)+\eta\vartheta)}\bigg)^2\kappa_p^2d^{2(-1+\zeta-\eta\zeta+\eta\vartheta)} \\
&\qquad\qquad\big[L_1^2P'^2+\|\aleph(0)\|^2+M_w'^2\|\alpha\|^2\big] \\
&+ M_0^2M_w'^2\big(1+d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\big)P'^2 \\
&+ \bigg(\frac{M_w'\kappa_{1-q}\Gamma(1+q)}{q\Gamma(1+\eta q)}\bigg)^2\big(1+d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}P'\big)d^{2\eta q} \\
&+ \bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)^2\kappa_p^2m_1^2(d)f^2(P')+Tr(Q)\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)^2\kappa_p^2m_2^2(d)\hbar^2(P')\Bigg] \\
&\le 5d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}M^{**},
\end{aligned}
$$

where

$$
\begin{aligned}
M^{**} = &\Bigg[\bigg(\frac{\Gamma(\vartheta)}{\Gamma(\zeta(1-\eta)+\eta\vartheta)}\bigg)^2\kappa_p^2d^{2(-1+\zeta-\eta\zeta+\eta\vartheta)}\big[L_1^2P'^2+\|\aleph(0)\|^2+M_w'^2\|\alpha\|^2\big] \\
&+ M_0^2M_w'^2\big(1+d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\big)P'^2+\bigg(\frac{M_w'\kappa_{1-q}\Gamma(1+q)}{q\Gamma(1+\eta q)}\bigg)^2\big(1+d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}P'\big)d^{2\eta q} \\
&+ \bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)^2\kappa_p^2m_1^2(d)f^2(P')+Tr(Q)\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)^2\kappa_p^2m_2^2(d)\hbar^2(P')\Bigg].
\end{aligned}
$$

By dividing the aforementioned inequality by P and using the limit as $P\to\infty$, we arrive at the contradiction, which is $1\le 0$. Consequently, $\mho(\mathscr{B}_P(\mathfrak{D}))\subseteq\mathscr{B}_P(\mathfrak{D})$.

Step 2: The operator $\mho$ is continuous on $\mathscr{B}_P(\mathfrak{D})$. For $\mho:\mathscr{B}_P(\mathfrak{D})\to\mathscr{B}_P(\mathfrak{D})$ and for all v^k, $v\in\mathscr{B}_P(\mathfrak{D})$, $k=0,1,2,\cdots$ such that $\lim_{k\to\infty}v^k=v$, then we get $\lim_{k\to\infty}v^k(\rho)=v(\rho)$ and $\lim_{k\to\infty}\rho^{1-\zeta+\eta\zeta-\eta\vartheta}v^k(\rho)=\rho^{1-\zeta+\eta\zeta-\eta\vartheta}v(\rho)$.

By (H_2),

$$
\begin{aligned}
\mathcal{F}(\rho,z_k(\rho)) &= \mathcal{F}\big(\rho,\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v^k(\rho)+\widehat{\Psi}(\rho)]\big)\to\mathcal{F}\big(\rho,\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v(\rho)+\widehat{\Psi}(\rho)]\big) \\
&= \mathcal{F}(\rho,z(\rho)) \text{ as } k\to\infty.
\end{aligned}
$$

Take

$$
F_k(\iota)=\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota^k+\widehat{\Psi}_\iota]\big) \text{ and } F(\iota)=\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big).
$$

Then, using the assumption (H_2) and Lebesgue's dominated convergence theorem (LDCT), we can obtain

$$
\int_0^\rho(\rho-\iota)^{2(\eta-1)}\|\mathsf{Q}_\eta(\rho-\iota)\|^2E\|F_k(\iota)-F(\iota)\|^2d\iota\to 0 \text{ as } k\to\infty,\ \rho\in\mathfrak{D}. \tag{3}
$$

By (H_3),

$$\begin{aligned}\mathcal{H}(\rho, z_k(\rho)) &= \mathcal{H}\big(\rho, \rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v^k(\rho) + \widehat{\Psi}(\rho)]\big) \to \mathcal{H}\big(\rho, \rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v(\rho) + \widehat{\Psi}(\rho)]\big) \\ &= \mathcal{H}(\rho, z(\rho)) \text{ as } k \to \infty.\end{aligned}$$

Take

$$H_k(\iota) = \mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota^k + \widehat{\Psi}_\iota]\big) \text{ and } H(\iota) = \mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big).$$

Then, using the hypotheses (H_3) and LDCT, we can obtain

$$\int_0^\rho (\rho-\iota)^{2(\eta-1)} \|\mathsf{Q}_\eta(\rho-\iota)\|^2 E\|H_k(\iota) - H(\iota)\|^2 dW(\iota) \to 0 \text{ as } k \to \infty,\ \rho \in \mathfrak{D}. \tag{4}$$

Take $\mathcal{N}_k(\rho) = \aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho^k + \widehat{\Psi}_\rho]\big)$ and $\mathcal{N}(\rho) = \aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big)$, from (H_4), we get

$$E\|\mathcal{N}_k(\rho) - \mathcal{N}(\rho)\|^2 \to 0 \text{ as } k \to \infty. \tag{5}$$

Then, $\eth_k(\rho, z_k(\rho)) = \eth\big(\rho, \rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho^k + \widehat{\Psi}_\rho]\big)$ and $\eth(\rho, z(\rho)) = \eth\big(\rho, \rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big)$. From hypotheses (H_5), we obtain

$$E\|\eth(\rho, z_k(\rho)) - \eth(\rho, z(\rho))\|^2 \to 0 \text{ as } k \to \infty. \tag{6}$$

Now,

$$\begin{aligned}E\big\|\mho v^k - \mho v\big\|_d^2 \le{}& 4\bigg(\frac{\Gamma(\vartheta)}{\Gamma(\zeta(1-\eta)+\eta\vartheta)}\bigg)^2 \kappa_p^2 d^{2(-1+\zeta-\eta\zeta+\eta\vartheta)} E\big\|\mathcal{N}_k(\rho) - \mathcal{N}(\rho)\big\|^2 \\ &+ 4E\|\eth_k(\rho, z_k(\rho)) - \eth(\rho, z_\rho)\|^2 + 4\kappa_p^2\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)^2 E\big\|F_k(\iota) - F(\iota)\big\|^2 \\ &+ 4Tr(Q)\kappa_p^2\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)^2 E\big\|H_k(\iota) - H(\iota)\big\|^2.\end{aligned}$$

Using (3), (4), (5) and (6), we obtain

$$E\big\|\mho v^k - \mho v\big\|_d^2 \to 0 \text{ as } k \to \infty.$$

As a result, $\mho$ is continuous on $\mathscr{B}_P$.

Step 3: To prove $\mho$ is equicontinuous.

For $z \in \mathscr{B}_P(\mathfrak{D})$, and $0 \le \rho_1 < \rho_2 \le d$, we obtain

$$\begin{aligned}&E\bigg\|\mho z(\rho_2) - \mho z(\rho_1)\bigg\|^2 \\ &\quad= E\bigg\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\bigg(-\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2} + \widehat{\Psi}_{\rho_2}]\big) + \eth(0, \alpha(0))\big] \\ &\qquad+ \eth\big(\rho_2, \big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2} + \widehat{\Psi}_{\rho_2}]\big)\big) \\ &\qquad+ \int_0^{\rho_2} (\rho_2-\iota)^{\eta-1} \widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\ &\qquad+ \int_0^{\rho_2} (\rho_2-\iota)^{\eta-1} \mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{\rho_2} (\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\Big)\\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\Big(-\mathsf{S}_{\eta,\zeta}(\rho_1)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\eth(0,\alpha(0))\big]\\
&+ \eth\big(\rho_1,\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)\big)\\
&+ \int_0^{\rho_1} (\rho_1-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_1-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\
&+ \int_0^{\rho_1} (\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\
&+ \int_0^{\rho_1} (\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\Big)\Big\|^2\\
\leq{}& 5E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\Big(-\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2}+\widehat{\Psi}_{\rho_2}]\big)+\eth(0,\alpha(0))\big]\Big)\\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\Big(-\mathsf{S}_{\eta,\zeta}(\rho_1)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\eth(0,\alpha(0))\big]\Big)\Big\|^2\\
&+ 5E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\eth\big(\rho_2,\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2}+\widehat{\Psi}_{\rho_2}]\big)\big)\\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\eth\big(\rho_1,\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)\big)\Big\|^2\\
&+ 5E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_2} (\rho_2-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1} (\rho_1-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_1-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\
&+ 5E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_2} (\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1} (\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\
&+ 5E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_2} (\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1} (\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\Big\|^2\\
\leq{}& 10E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2}+\widehat{\Psi}_{\rho_2}]\big)+\eth(0,\alpha(0))\big]\\
&- \rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\eth(0,\alpha(0))\big]\Big\|^2\\
&+ 10E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\eth(0,\alpha(0))\big]\\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_1)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\eth(0,\alpha(0))\big]\Big\|^2\\
&+ 5E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\eth\big(\rho_2,\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2}+\widehat{\Psi}_{\rho_2}]\big)\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\eth\big(\rho_1, \big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1} + \widehat{\Psi}_{\rho_1}]\big)\big)\Big\|^2 \\
&+ 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_2-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&+ 15E\Big\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_1-\iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&+ 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&+ 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&+ 15E\Big\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&+ 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&+ 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota) \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota)\Big\|^2 \\
&+ 15E\Big\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota) \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota)\Big\|^2 \\
&+ 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota)\Big\|^2 \\
\leq{}& \sum_{i=1}^{12} I_i.
\end{aligned}
$$

$$
\begin{aligned}
I_1 ={}& 10E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2} + \widehat{\Psi}_{\rho_2}]\big) + \eth(0, \alpha(0))\big] \\
&- \rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1} + \widehat{\Psi}_{\rho_1}]\big) + \eth(0, \alpha(0))\big]\Big\|^2 \\
\leq{}& 10E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\Big(\aleph\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2} + \widehat{\Psi}_{\rho_2}]\big) - \aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1} + \widehat{\Psi}_{\rho_1}]\big)\Big)\Big\|^2.
\end{aligned}
$$

From hypotheses (H_4) and (5), we obtain I_1 tends to zero as $\rho_2 \to \rho_1$.

$$I_2 = 10E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\eth(0,\alpha(0))\big]$$
$$-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_1)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\eth(0,\alpha(0))\big]\Big\|^2$$
$$\leq 10\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_1)\Big\|^2$$
$$E\Big\|\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\eth(0,\alpha(0))\Big\|^2.$$

By the strong continuity of $\mathsf{S}_{\eta,\zeta}(\rho)$ and (H_4), we get $I_2 \to 0$ as $\rho_2 \to \rho_1$.

$$I_3 = 5E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\eth\big(\rho_2,\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2}+\widehat{\Psi}_{\rho_2}]\big)\big)$$
$$-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\eth\big(\rho_1,\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)\big)\Big\|^2$$
$$\leq 5M_0^2M_w'^2(1+P'^2)\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\Big\|^2.$$

From hypotheses (H_5), we obtain $I_3 \to 0$ as $\rho_2 \to \rho_1$.

$$I_4 = 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_2-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota$$
$$-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2$$
$$\leq 15E\Big\|\int_0^{\rho_1}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big)$$
$$\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2$$
$$\leq 15\Big\|\int_0^{\rho_1}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big)\Big\|^2$$
$$\times\Big\|\widehat{A}^{1-q}\mathsf{Q}_\eta(\rho_2-\iota)\Big\|^2E\Big\|\widehat{A}^q\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)\Big\|^2d\iota$$
$$\leq 15\Big(\frac{M_w'\kappa_{1-q}\eta\Gamma(1+q)}{q\Gamma(1+\eta q)}\Big)^2\big(1+\rho_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}P'^2\big)$$
$$\times\Big\|\int_0^{\rho_1}(\rho_2-\iota)^{\eta(q-1)}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big)d\iota\Big\|^2.$$

Implies $I_4 \to 0$ as $\rho_2 \to \rho_1$.

$$I_5 = 15E\Big\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota$$
$$-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_1-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2$$
$$\leq 15M_0^2M_w'^2(1+\rho_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}P'^2)\Big\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\big[\mathsf{Q}_\eta(\rho_2-\iota)-\mathsf{Q}_\eta(\rho_1-\iota)\big]d\iota\Big\|^2.$$

Since $\mathsf{Q}_\eta(\rho)$ is uniformly continuous in operator norm topology, we obtain $I_5 \to 0$ as $\rho_2 \to \rho_1$.

$$\begin{aligned}I_6 &= 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\ &\leq 15\bigg(\frac{M_w'\kappa_{1-q}\eta\Gamma(1+q)}{q\Gamma(1+\eta q)}\bigg)^2\big(1+\rho_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}P'^2\big)\rho_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{2\eta(1-q)}d\iota.\end{aligned}$$

Integrating and $\rho_2 \to \rho_1 \implies I_6 = 0$.

$$\begin{aligned}I_7 &= 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\ &\quad-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\ &\leq 15E\Big\|\int_0^{\rho_1}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big)\\ &\qquad\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\ &\leq 15\kappa_p^2\Big\|\int_0^{\rho_1}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big)\Big\|^2\\ &\qquad(\rho_2-\iota)^{2\eta(\vartheta-1)}m_1^2(d)f^2(P')d\iota.\end{aligned}$$

Implies $I_7 \to 0$ as $\rho_2 \to \rho_1$.

$$\begin{aligned}I_8 &= 15E\Big\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\ &\quad-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\ &\leq 15\rho_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_0^{\rho_1}(\rho_1-\iota)^{2(\eta-1)}\big\|\mathsf{Q}_\eta(\rho_2-\iota)-\mathsf{Q}_\eta(\rho_1-\iota)\big\|m_1^2(d)f^2(P')d\iota.\end{aligned}$$

Since $\mathsf{Q}_\eta(\rho)$ is uniformly continuous in operator norm topology, we obtain $I_8 \to 0$ as $\rho_2 \to \rho_1$.

$$\begin{aligned}I_9 &= 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\ &\leq 15\kappa_p^2\rho_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{2(\eta\vartheta-1)}m_1^2(d)f^2(P')d\iota.\end{aligned}$$

Integrating and $\rho_2 \to \rho_1 \implies I_9 = 0$.

$$\begin{aligned}I_{10} &= 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\\ &\quad-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\Big\|^2\\ &\leq 15E\Big\|\int_0^{\rho_1}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big)\end{aligned}$$

$$\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\bigg\|^2$$

$$\leq 15Tr(Q)\kappa_p^2\bigg\|\int_0^{\rho_1}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big)\bigg\|^2 (\rho_2-\iota)^{2\eta(\vartheta-1)}m_2^2(d)\hbar^2(P')d\iota.$$

Implies $I_{10}\to 0$ as $\rho_2\to\rho_1$.

$$I_{11}=15E\bigg\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)$$
$$-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\bigg\|^2$$
$$\leq 15Tr(Q)\rho_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_0^{\rho_1}(\rho_1-\iota)^{2(\eta-1)}\big\|\mathsf{Q}_\eta(\rho_2-\iota)-\mathsf{Q}_\eta(\rho_1-\iota)\big\|^2m_2^2(d)\hbar^2(P')d\iota.$$

Since $\mathsf{Q}_\eta(\rho)$ is uniformly continuous in operator norm topology, we get $I_{11}\to 0$ as $\rho_2\to\rho_1$.

$$I_{12}=15E\bigg\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\bigg\|^2$$
$$\leq 15Tr(Q)\kappa_p^2\rho_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{2(\eta\vartheta-1)}m_2^2(d)\hbar^2(P')d\iota.$$

Integrating and $\rho_2\to\rho_1\implies I_{12}=0$.

Hence, $\mho$ is equicontinuous on $\mathfrak{D}$.

Step 4: The Mönch statement is true.

Let $\mho=\mho_1+\mho_2+\mho_3+\mho_4+\mho_5$, where

$$\mho_1v(\rho)=-\mathsf{S}_{\eta,\zeta}(\rho)\big[\aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho+\widehat{\Psi}_\rho]\big)+\eth(0,\alpha(0))\big],$$
$$\mho_2v(\rho)=\eth\big(\rho,\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho+\widehat{\Psi}_\rho]\big)\big),$$
$$\mho_3v(\rho)=\int_0^{\rho}(\rho-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota,$$
$$\mho_4v(\rho)=\int_0^{\rho}(\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota,$$
$$\mho_5v(\rho)=\int_0^{\rho}(\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota).$$

Suppose $G_1\subseteq\mathfrak{B}_P$ is countable and $G_1\subset\overline{co}\big(\{0\}\cup F(G_1)\big)$. We demonstrate that $\mu(G_1)=0$, where μ is the Hausdorff MNC. Without loss of generality, suppose $G_1=\{v^k\}_{k=1}^{\infty}$. Since $\mho(G_1)$ is equicontinuous on $\mathfrak{D}$ as well.

Utilising Lemma (see [29]), and the hypotheses $(H_2)(c)$, $(H_3)(c)$, and (H_4), we have

$$\mu\big(\{\mho_1v^k(\rho)\}_{k=1}^{\infty}\big)\leq\mu\big\{-\mathsf{S}_{\eta,\zeta}(\rho)\big[\aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho^k+\widehat{\Psi}_\rho]\big)+\eth(0,\alpha(0))\big]\big\}_{k=1}^{\infty},$$

since $\aleph$ is compact, then $\mathsf{S}_{\eta,\zeta}(\rho)$ is relatively compact, we get $\mho_1v(\rho)$ becomes zero. Next consider,

$$\mu\big(\{\mho_2v^k(\rho)\}_{k=1}^{\infty}\big)\leq\mu\big\{\eth\big(\rho,\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho^k+\widehat{\Psi}_\rho]\big)\big)\big\}_{k=1}^{\infty},$$
$$\mu\big(\{\mho_3v^k(\rho)\}_{k=1}^{\infty}\big)\leq\mu\bigg\{\int_0^{\rho}(\rho-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota^k+\widehat{\Psi}_\iota]\big)d\iota\bigg\}_{k=1}^{\infty}.$$

From hypotheses (H_5) and properties of function $\eth$ and $\widehat{A}\mathsf{Q}$, we obtain, that terms are relatively compact. So $\mho_2 v(\rho)$ and $\mho_3 v(\rho)$ become zero.

$$\begin{aligned}\mu\big(\{\mho_4 v^k(\rho)\}_{k=1}^{\infty}\big) &\leq \mu\bigg\{\int_0^{\rho}(\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota^k+\widehat{\Psi}_\iota]\big)d\iota\bigg\}_{k=1}^{\infty}\\ &\leq 2\int_0^{\rho}(\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)e_1(\iota)\sup_{-\infty<\theta\leq 0}\mu\big(\{v_\rho^k(\theta)\}_{k=1}^{\infty}\big)d\iota\\ &\leq 2\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)\|e_1\|_{L^{\frac{1}{\eta_2}}(\mathfrak{D},\mathbb{R}^+)}\sup_{-\infty<\theta\leq 0}\mu\big(\{v_\rho^k(\theta)\}_{k=1}^{\infty}\big),\end{aligned}$$

$$\begin{aligned}\mu\big(\{\mho_5 v^k(\rho)\}_{k=1}^{\infty}\big) &\leq \mu\bigg\{\int_0^{\rho}(\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota^k+\widehat{\Psi}_\iota]\big)dW(\iota)\bigg\}_{k=1}^{\infty}\\ &\leq 2Tr(Q)\int_0^{\rho}(\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)e_2(\iota)\sup_{-\infty<\theta\leq 0}\mu\big(\{v_\rho^k(\theta)\}_{k=1}^{\infty}\big)d\iota\\ &\leq 2Tr(Q)\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)\|e_2\|_{L^{\frac{1}{\eta_4}}(\mathfrak{D},\mathbb{R}^+)}\sup_{-\infty<\theta\leq 0}\mu\big(\{v_\rho^k(\theta)\}_{k=1}^{\infty}\big).\end{aligned}$$

Thus, we have

$$\begin{aligned}\mu\big(\{\mho v^k(\rho)\}_{k=1}^{\infty}\big) &\leq \mu\big(\{\mho_1 v^k(\rho)\}_{k=1}^{\infty}\big)+\mu\big(\{\mho_2 v^k(\rho)\}_{k=1}^{\infty}\big)+\mu\big(\{\mho_3 v^k(\rho)\}_{k=1}^{\infty}\big)\\ &\quad+\mu\big(\{\mho_4 v^k(\rho)\}_{k=1}^{\infty}\big)+\mu\big(\{\mho_5 v^k(\rho)\}_{k=1}^{\infty}\big)\\ &\leq 2\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)\|e_1\|_{L^{\frac{1}{\eta_2}}(\mathfrak{D},\mathbb{R}^+)}\sup_{-\infty<\theta\leq 0}\mu\big(\{v_\rho^k(\theta)\}_{k=1}^{\infty}\big)\\ &\quad+2Tr(Q)\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)\|e_2\|_{L^{\frac{1}{\eta_4}}(\mathfrak{D},\mathbb{R}^+)}\sup_{-\infty<\theta\leq 0}\mu\big(\{v_\rho^k(\theta)\}_{k=1}^{\infty}\big)\\ &\leq 2\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)\big[\|e_1\|_{L^{\frac{1}{\eta_2}}(\mathfrak{D},\mathbb{R}^+)}+Tr(Q)\|e_2\|_{L^{\frac{1}{\eta_4}}(\mathfrak{D},\mathbb{R}^+)}\big]\mu\big(\{v^k(\rho)\}_{k=1}^{\infty}\big)\\ &\leq M^*\mu\big(\{v^k(\rho)\}_{k=1}^{\infty}\big),\end{aligned}$$

where $M^*=2\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)\big[\|e_1\|_{L^{\frac{1}{\eta_2}}(\mathfrak{D},\mathbb{R}^+)}+Tr(Q)\|e_2\|_{L^{\frac{1}{\eta_4}}(\mathfrak{D},\mathbb{R}^+)}\big]$.

Since G_1 and $\mho(G_1)$ are equicontinuous on $\mathfrak{D}$, it follows from Lemma (see [29]) that the constraint implies that $\mu(\mho G_1)\leq M^*\mu(G_1)$.

Therefore, given the requirements of the Mönch's, we get

$$\mu(G_1)\leq\mu\big(\overline{co}\{0\}\cup\mho(G_1)\big)=\mu(\mho G_1)\leq M^*\mu G_1.$$

Given $M^*<1$, we have $\mu(G_1)=0$. Therefore, G_1 is relatively compact. As a result, $\mho$ has a fixed point v in G_1 from Lemma 6.

Hence, completed the proof.

3 Example

Consider the HF neutral stochastic differential systems with infinite delay of the form

$$\begin{cases} D_{0+}^{\frac{2}{3},\zeta}\left[z(\rho,\tau)+\int_0^\pi \rho(\beta,\tau)z(\rho,\tau)d\beta\right] = z_{\tau\tau}(\rho,\tau)+\gamma\Big(\rho,\int_\infty^\rho \chi_1(\iota-\rho)z(\rho,\tau)d\iota\Big) \\ \qquad\qquad\qquad\qquad\qquad +\chi\Big(\rho,\int_\infty^\rho \chi_2(\iota-\rho)z(\rho,\tau)dW(\iota)\Big), \\ z(\rho,0)=z(\rho,\pi)=0,\ \rho\in\mathfrak{D}, \\ I_{0+}^{(1-\frac{2}{3})(1-\zeta)}z(0,\tau)+\int_0^\pi \mathcal{N}(\beta,\tau)z(\rho,\tau)d\beta = z(0,\tau),\ \tau\in[0,\pi],\ \rho\in(-\infty,0), \end{cases} \tag{7}$$

where $D_{0+}^{\frac{2}{3},\zeta}$ denoted the HFD of order $\eta = 2/3$, type ζ and $\chi,\ \chi_1,\ \rho$ and $\mathcal{N}$ are the necessary functions. Consider, $W(\rho)$ is the one-dimensional Brownian movements in $\mathbb{Y}$ represented on the filtered probability space $(\Delta, \mathscr{F}, P)$ and with $\|\cdot\|_{\mathbb{Y}}$ to write the system (7) in the abstract form of (1)-(2). Let $\mathbb{Y} = L^2[0,\pi]$, to transform this structure into an abstract structure, and $\widehat{A} : D(\widehat{A}) \subset \mathbb{Y} \to \mathbb{Y}$ is classified as $\widehat{A}x = x'$ with

$$D(\widehat{A}) = \big\{x\in\mathbb{Y} : x, x' \text{ are absolutely continuous},\ x''\in\mathbb{Y},\ x(0)=x(\pi)=0\big\}$$

and

$$\widehat{A}x = \sum_{k=1}^{\infty} k^2\langle x, \varrho_k\rangle \varrho_k,\ \ \varrho\in D(\widehat{A}),$$

where the orthogonal set of eigen vectors of $\widehat{A}$ is $\varrho_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx),\ k\in\mathbb{N}$.

Here, $\widehat{A}$ is the almost sectorial operator of the analytic semigroup $\{T(\rho),\ \rho\geq 0\}$ in $\mathbb{Y}$, $T(\rho)$ is noncompact semigroup on $\mathbb{Y}$ with $\zeta(T(\rho)B) \leq \zeta(B)$, where ζ denoted the Hausdorff measure of noncompactness and there exists a constant $\mathcal{K}_1 \geq 1$, satisfy $\sup_{\rho\in\mathfrak{D}}\|T(\rho)\| \leq \mathcal{K}_1$.

Specify, $\mathcal{F} : \mathfrak{D}\times B_{\mathrm{r}} \to \mathbb{Y},\ \mathcal{H} : \mathfrak{D}\times B_{\mathrm{r}} \to L_2^0(\mathcal{J},\mathbb{Y}),\ \eth : \mathfrak{D}\times B_{\mathrm{r}} \to \mathbb{Y}$ and $\aleph : B_{\mathrm{r}} \to \mathbb{Y}$ are the suitable functions, which fulfils the assumptions $(H_1)-(H_5)$,

$$\begin{aligned} \mathcal{F}\big(\rho,z_\rho\big)(\tau), &= \gamma\Big(\rho,\int_\infty^\rho \chi_1(\iota-\rho)z(\rho,\tau)d\iota\Big), \\ \mathcal{H}\big(\rho,z_\rho\big)(\tau), &= \chi\Big(\rho,\int_\infty^\rho \chi_2(\iota-\rho)z(\rho,\tau)dW(\iota)\Big), \\ \eth(\rho,z_\rho)(\tau) &= \int_0^\pi \rho(\beta,\tau)z(\rho,\tau)d\beta, \\ \aleph(z_\rho)(\tau) &= \int_0^\pi \mathcal{N}(\beta,\tau)z(\rho,\tau)d\beta. \end{aligned}$$

We also establish some acceptable requirements for the above-mentioned functions in order to validate all of the Theorem 4's hypotheses, and we confirm that the HF stochastic system (1)-(2) has a mild solution.

Conclusion

The existence of a mild solution to HF neutral stochastic differential systems was the main emphasis of this research. Almost sectorial operators, fractional calculus, MNC, and the fixed point approach are used to establish the key conclusions. We offered an example to further illustrate the idea. In the following years, we'll use the fixed point approach to examine the exact controllability of HF stochastic differential systems with delay.

Acknowledgments

The authors thank the referees very much for their valuable advice on this paper.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Pazy, A. (1983). Semigroups of Linear Operators and Applications to Partial Differential Equations. *Applied Mathematical Sciences, Vol. 44*. Springer, New York. https://doi.org/10.1007/978-1-4612-5561-1

2 Podlubny, I. (1999). *Fractional Differential Equations.* Academic Press, San Diego.

3 Zhou, Y. (2014). *Basic Theory of Fractional Differential Equations.* World Scientific, Singapore. https://doi.org/10.1142/9069

4 Agarwal, R.P., Lakshmikanthan, V. & Nieto, J.J. (2010). On the concept of solution for fractional differential equations with uncertainty. *Nonlinear Analysis, Theory Methods and Applications 72*(6), 2859–2862. https://doi.org/10.1016/j.na.2009.11.029

5 Ahmad, B., Alsaedi, A., Ntouyas, S.K., & Tariboon, J. (2017). *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*. Springer International Publishing AG. https://doi.org/10.1007/978-3-319-52141-1

6 Diethelm, K. (2010). The analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type. *Lecture Notes in Mathematics.* Springer-Verlag, Berlin.

7 Wang, J., & Zhou, Y. (2011). Existence and Controllability results for fractional semilinear differential inclusions. *Nonlinear Analysis: Real World Applications, 12*, 3642–3653. https://doi.org/10.1016/j.nonrwa.2011.06.021

8 Wang, J.R., Fan, Z., & Zhou, Y. (2012). Nonlocal controllability of semilinear dynamic systems with fractional derivative in Banach spaces. *Journal of Optimization Theory and Applications, 154*(1), 292–302. https://doi.org/10.1007/s10957-012-9999-3

9 Guo, Y., Shu, X.B., Li, Y., & Xu, F. (2019). The existence and Hyers-Ulam stability of solution for an impulsive Riemann-Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$. *Boundary Value Problems, 59*, 1–18. https://doi.org/10.1186/s13661-019-1172-6.

10 Li, F., Xiao, T.J., & Xu, H.K. (2012). On nonlinear neutral fractional integro-differential inclusions with infinite delay. *Journal of Applied Mathematics, 2012*, 916543, 1–19. https://doi.org/10.1155/2012/916543

11 Ma, X., Shu, X.B., & Mao, J. (2020). Existence of almost periodic solutions for fractional impulsive neutral stochastic differential equations with infinite delay. *Stochastics and Dynamics, 20*(1), 1–31. https://doi.org/10.1142/S0219493720500033

12 Boudaoui, A., & Slama, A. (2016). Approximate controllability of nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delay. *Nonlinear Dynamics and Systems Theory, 16*(1), 35–48.

13 Evans, L.C. (2013). *An Introduction to Stochastic Differential Equations.* Berkeley, CA: University of California, Berkeley.

14 Mao, X. (1997). *Stochastic Differential Equations and Applications.* Horwood, Chichester, UK.

15 Sivasankar, S., & Udhayakumar, R. (2022). A note on approximate controllability of second-order neutral stochastic delay integro-differential evolution inclusions with impulses. *Mathematical Methods in the Applied Sciences, 45*(11), 6650–6676. https://doi.org/10.1002/mma.8198

16 Hilfer, R. (2000). *Application of fractional calculus in physics.* World Scientific, Singapore.

17 Gu, H., & Trujillo, J.J. (2015). Existence of integral solution for evolution equation with Hilfer fractional derivative. *Applied Mathematics and Computation, 257*, 344–354. http://dx.doi.org/10.1016/j.amc.2014.10.083

18 Sivasankar, S., & Udhayakumar, R. (2022). Hilfer Fractional Neutral Stochastic Volterra Integro-Differential Inclusions via Almost Sectorial Operators. *Mathematics, 10*(12), 2074. https://doi.org/10.3390/math10122074.

19 Yang, M., & Wang, Q. (2017). Existence of mild solutions for a class of Hilfer fractional evolution equations with nonlocal conditions. *Fractional Calculus and Applied Analysis, 20*(3), 679–705. https://doi.org/10.1515/fca-2017-0036

20 Bedi, P., Kumar, A., Abdeljawad, T., Khan, Z.A., & Khan, A. (2020). Existence and approximate controllability of Hilfer fractional evolution equations with almost sectorial operators. *Advances in Differential Equations, 615*, 1–15.

21 Jaiswal, A., & Bahuguna, D. (2020). Hilfer fractional differential equations with almost sectorial operators. *Differential Equations and Dynamical Systems, 31*, 301–317. https://doi.org/10.1007/s12591-020-00514-y.

22 Karthikeyan, K., Debbouche, A., & Torres, D.F.M. (2021). Analysis of Hilfer fractional integro-differential equations with almost sectorial operators. *Fractal and Fractional, 5*(1), 1–14. https://doi.org/10.3390/fractalfract5010022

23 Varun Bose, C.S., & Udhayakumar, R. (2022). A note on the existence of Hilfer fractional differential inclusions with almost sectorial operators. *Mathematical Methods in the Applied Sciences, 45*(5), 2530–2541. https://doi.org/10.1002/mma.7938

24 Li, F. (2013). Mild solutions for abstract differential equations with almost sectorial operators and infinite delay. *Advances in Differential Equations, 2013*(327), 1–11.

25 Periago, F., & Straub, B. (2002). A functional calculus for almost sectorial operators and applications to abstract evolution equations. *Journal of Evolution Equations, 2*, 41–68. https://doi.org/10.1007/s00028-002-8079-9

26 Wang, R.N., De-Han Chen, & Xiao, T.J. (2012). Abstract fractional Cauchy problems with almost sectorial operators. *Journal of Differential Equations, 252*(1), 202–235. https://doi.org/10.1016/j.jde.2011.08.048

27 Zhang, L., & Zhou, Y. (2014). Fractional Cauchy problems with almost sectorial operators. *Applied Mathematics and Computation, 257*, 145–157. https://doi.org/10.1016/j.amc.2014.07.024

28 Zhou, M., Li, C., & Zhou, Y. (2022). Existence of mild solutions for Hilfer fractional differential evolution equations with almost sectorial operators. *Axioms, 11*, 144. https://doi.org/10.3390/axioms11040144.

29 Ji, S., Li, G., & Wang, M. (2011). Controllability of impulsive differential systems with nonlocal conditions. *Applied Mathematics and Computation, 217*, 6981–6989. https://doi.org/10.1016/j.amc.2011.01.107

30 Mönch, H. (1980). Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. *Nonlinear Analysis, 4*(5), 985–999.

Шексіз кешігуі бар бөлшек-нейтралды стохастикалық Хильфер дифференциалдық жүйелерінің болуы

С. Сивасанкар[1], Р. Удхаякумар[1], В. Мутхумаран[2], Г. Гокул[1], Ш. Аль-Омари[3]

[1] *Жетілдірілген ғылымдар мектебі, Веллуру технологиялық институты, Веллуру, Үндістан;*
[2] *Инженерлік және технологиялар колледжі, SRM Ғылым және технология институты, Каттанкулатур, Үндістан;*
[3] *Ғылым факультеті, Әл-Балқа қолданбалы университеті, Амман, Иордания*

Зерттеудің мақсаты әйгілі Риман-Лиувилл бөлшек туындысын жалпылайтын Гильберт кеңістігіндегі Хильфер бөлшек туындысы қатысатын дерлік секторлық операторлары бар кешігетін бөлшек-нейтралды стохастикалық дифференциалдық жүйелер үшін жұмсақ шешімдердің болуын ұсыну. Негізгі әдістер бөлшек есептеудің, жартылай группа теориясының, дерлік секторлық операторлардың, стохастикалық талдаудың және компактылы емес өлшемі арқылы Мёнхтің қозғалмайтын нүкте теоремасының негізгі қағидалары мен тұжырымдамаларына негізделген. Атап айтқанда, теңдеудің бар болуының нәтижесі әлсіз компактылықтың белгілі бір жағдайында алынды. Мақаланың соңында алынған абстрактылы нәтижелердің қолдану аясын көрсететін мысал бар.

Кілт сөздер: Хильфердің бөлшек эволюциялық жүйесі, нейтралды жүйе, компактылы емес өлшем, қозғалмайтын нүкте теоремасы.

Существование дробно-нейтральных стохастических дифференциальных систем Хильфера с бесконечным запаздыванием

С. Сивасанкар[1], Р. Удхаякумар[1], В. Мутхумаран[2], Г. Гокул[1], Ш. Аль-Омари[3]

[1] *Школа передовых наук, Технологический институт Веллуру, Веллуру, Индия;*
[2] *Инженерно-технологический колледж; Институт науки и технологий SRM, Каттанкулатур, Индия;*
[3] *Прикладной университет Аль-Балка, Амман, Иордания*

Цель данного исследования — предложить существование мягких решений для запаздывающих дробно-нейтральных стохастических дифференциальных систем с почти секториальными операторами, включающими дробную производную Хильфера в гильбертовом пространстве, которая обобщает знаменитую дробную производную Римана-Лиувилля. Основные методы построены на базовых принципах и концепциях дробного исчисления, теории полугрупп, почти секториальных операторах, стохастическом анализе и теореме Мёнха о неподвижной точке через меру некомпактности. В частности, результат существования уравнения получен при некоторых условиях слабой компактности. В конце статьи приведен пример, демонстрирующий применение полученных абстрактных результатов.

Ключевые слова: дробная эволюционная система Хильфера, нейтральная система, мера некомпактности, теорема о неподвижной точке.

Author Information[*]

Sivajiganesan Sivasankar — Master of science, Research Scholar, Department of Mathematics, Vellore Institute of Technology, Vellore- 632014, Tamilnadu, India; e-mail: *sivajisivasankar@gmail.com*; https://orcid.org/0000-0003-2422-9670

Ramalingam Udhayakumar (*corresponding author*) — Doctor of philosophy, Assistant Professor Senior Grade1, Department of Mathematics, Vellore Institute of Technology, Vellore- 632014, Tamilnadu, India; e-mail: *udhayaram.v@gmail.com*; https://orcid.org/0000-0002-7020-3466

Venkatesan Muthukumaran — Doctor of Philosophy, Assistant professor, Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur-603203, India; e-mail: *muthu.v2404@gmail.com*; https://orcid.org/0000-0002-3393-5596

Gunasekaran Gokul — Master of science, Research Scholar, Department of Mathematics, Vellore Institute of Technology, Vellore- 632014, Tamilnadu, India; e-mail: *gokulg2110@gmail.com.*

Shrideh Al-Omari — Doctor of philosophy, Professor, Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Amman 11134, Jordan; e-mail: *shridehalomari@bau.edu.jo*; https://orcid.org/0000-0001-8955-5552

[*]The author's name is presented in the order: First, Middle and Last Names.

https://doi.org/10.31489/2024M1/194-207 *Research article*

A Novel Numerical Scheme for a Class of Singularly Perturbed Differential-Difference Equations with a Fixed Large Delay

E. Srinivas, K. Phaneendra*

University College of Science, Osmania University, Hyderabad, India
(E-mail: srinivas.erla@gmail.com, kollojuphaneendra@yahoo.co.in)

A trigonometric spline based computational technique is suggested for the numerical solution of layer behavior differential-difference equations with a fixed large delay. The continuity of the first order derivative of the trigonometric spline at the interior mesh point is used to develop the system of difference equations. With the help of singular perturbation theory, a fitting parameter is inserted into the difference scheme to minimize the error in the solution. The method is examined for convergence. We have also discussed the impact of shift or delay on the boundary layer. The maximum absolute errors in comparison to other approaches in the literature are tallied, and layer behavior is displayed in graphs, to demonstrate the feasibility of the suggested numerical method.

Keywords: singularly perturbed differential-difference equation, delay, trigonometric spline, fitting parameter.

2020 Mathematics Subject Classification: 65L11, 65L12.

Introduction

Delay differential equations (DDEs) are frequently encountered in a wide range of application disciplines and are also explained in technological components like control circuits. DDEs are widely occurred in various branches of physiological control systems [1], models of red blood cell system [2], pupil light reflex behaviour [3], hybrid optically bistable devices with delayed feedback [4] and the navigational control of ships and aircraft and in more general control problems [5]. Time delays are virtually always present in systems with feedback controls. These happen because detecting information and responding to it both take time. If the argument for the delay does not appear in the highest order term, the DDE is of the retarded type. Delay differential equations of the retarded type are obtained by restricting the class in which the highest order derivative term is multiplied by a small parameter. Bender and Orszag [6], O'Malley [7], Doolan and Miller [8], Miller et al. [9], Roos et al. [10] have written books detailing several approaches to addressing singularly perturbed problems (SPPs). Driver [11], Bellman and Cooke [12], provided books that explained differential-difference equations. In [13], the researchers elucidated analysis of a class of singularly perturbed differential-difference equations [SPDDEs]. In [14], problems with solutions having a layer structure at one or both of the boundaries are addressed. The layer can alter its nature and possibly be destroyed when the shifts rise but stay small, as demonstrated by the study of the layer equations using Laplace transforms. The same researchers handle two situations in [15]. The first is concerned with the magnitude of the shifts that affect the solution, while the second is concerned with the SPDE's oscillatory solutions. Kadalbajoo and Sharma [16], provided a numerical procedure for solving SPDE with larger or smaller delay argument. To handle the delay term, a mesh is generated so that the delay term falls on nodal points. Kadalbajoo et al. [17], utilize Shishkin mesh to derive the fitted mesh approach to solve singularly perturbed general DDEs. Gabil and Erkan [18], devised a fitted difference scheme for convection-diffusion problems by employing exponential basis functions, integral identities and interpolating quadrature procedures. The authors

*Corresponding author. *E-mail: kollojuphaneendra@yahoo.co.in*
Received: 07 June 2023; *Accepted:* 07 December 2023.

in [19], suggested adaptive grid methods for the solutions to problems with boundary or interior layers. A grid with equidistributing arc-length monitor function is constructed to solve the problem. In [20], a first-order uniform convergence fitted difference approach is built in the discrete maximum norm. Ravi Kanth and Murali [21], devised a numerical scheme for solving nonlinear SPDE. The Quasilinearization technique is implemented on the nonlinear SPDE to get a sequence of linear SPDEs. A fitted spline method is implemented for the solution of these problems. In [22], it is established that the family of periodic boundary value issues for the system of ordinary differential equations with delayed argument and the periodic boundary value problem for the system of hyperbolic equations with delayed argument are related. The construction and convergence of algorithms for solving the comparable problem are demonstrated. The author of [23] investigated a boundary value problem with the Sturm-Liouville type conditions using Green's function method for a linear ordinary differential equation of fractional order with delay. In [24], authors proposed a scheme for the solution of a differential equation with delay and advanced parameters having an interior layer behaviour using a non-standard finite difference method.

1 Statement of problem

Consider the following SPDDE with a fixed delay

$$\varepsilon z''(\vartheta) + p(\vartheta)\, z'(\vartheta) + q(\vartheta)\, z(\vartheta - 1) = f(\vartheta), \ \vartheta \in [0, 2] \tag{1}$$

subject to the boundary constraints

$$z(\vartheta) = \varphi(\vartheta), \ \vartheta \in [-1, 0];$$

$$z(2) = \beta, \tag{2}$$

where, $0 < \varepsilon \ll 1$ and $p(\vartheta) \geqslant \alpha > 0$, $\theta \leqslant q(\vartheta) < 0$ and $f(\vartheta)$ are smooth functions on $[0, 2]$, $\varphi(\vartheta)$ is smooth functions on $[-1, 0]$ and β is given constant. The solution of Eq. (1) with Eq. (2) reveals a boundary layer at $\vartheta = 2$ with the small values of ε.

2 Numerical method using a trigonometric spline

The domain of the integration $[0\,,\,2]$ is partitioned into L equal sub intervals with mesh length $h = \frac{2}{L}$, so that $\vartheta_i = ih$, $i = 0, 1, 2, ..., L$ are the nodes with $0 = \vartheta_0$, $2 = \vartheta_L$. Let $z(\vartheta)$ be the exact solution and ϑ_i be an approximation to $z(\vartheta_i)$ by the trigonometric spline $S_i(\vartheta)$ passing through the points $(\vartheta_i,\, z_i)$ and $(\vartheta_{i+1},\ z_{i+1})$. Here $S_i(\vartheta)$ satisfies the conditions of interpolation at ϑ_i and ϑ_{i+1} and also the first order derivative continuity at the common nodes $(\vartheta_i,\, z_i)$ is satisfied. For each i^{th} subinterval, the trigonometric spline function $S_i(\vartheta)$ has the form

$$S_i(\vartheta) = a_i + b_i(\vartheta - \vartheta_i) + c_i \sin\tau(\vartheta - \vartheta_i) + d_i\, cos\tau\, (\vartheta - \vartheta_i), \ i = 0, 1, \ldots, L - 1. \tag{3}$$

Here a_i, b_i, c_i and d_i are constants and τ is a free parameter.

The trigonometric spline $S_i(\vartheta)$ of class $C^2[0\,,\,2]$ interpolating $z(\vartheta)$ at the points ϑ_i, $i = 0, 1, ..., L$ depends on τ and deduces to cubic spline in $[0,\, 2]$ as $\tau \to 0$. The following are defined to obtain an expression for the coefficients of Eq. (3) in terms of z_i, z_{i+1}, ψ_i and ψ_{i+1}

$$S_i(\vartheta_i) = z_i, \quad S_i(\vartheta_{i+1}) = z_{i+1}, \quad S_i''(\vartheta_i) = \psi_i, \quad S_i''(\vartheta_{i+1}) = \psi_{i+1}.$$

Using these conditions, the following expressions are obtained:

$$a_i = z_i + \frac{\psi_i}{\tau^2}\ , \ b_i = \frac{z_i - z_{i+1}}{h} + \frac{\psi_{i+1} - \psi_i}{\tau\theta},$$

$$c_i = \frac{\psi_i cos\theta \, - \psi_{i+1}}{\tau^2 sin\theta} \text{ and } d_i = -\frac{\psi_i}{\tau^2},$$

where $\theta = \tau h$, for $i = 0, 1, \ldots, L-1$. Using the continuity of the first order derivative at $(\vartheta_i,\, z_i)$, that is $S^{'}_{i+1}(\vartheta_i) = S^{'}_i(\vartheta_i)$, we get the following relation for $i = 1, 2, \ldots, L-1$.

$$\alpha\psi_{i+1} + 2\beta\psi_i + \alpha\psi_{i-1} = \frac{z_{i-1} - 2z_i + z_{i+1}}{h^2}, \tag{4}$$

where

$$\alpha = \frac{-1}{\theta^2} + \frac{1}{\theta sin\theta} \text{ and } \beta = \frac{1}{\theta^2} - \frac{cos\theta}{\theta sin\theta},$$

$$\psi_j = z^{''}(\vartheta_j)\,,\, j = \, i \pm 1,\, i.$$

At the mesh point ϑ_j, the suggested approach can be discretized by the convection-diffusion equation (1) as

$$\psi_j = \frac{1}{\varepsilon}\Big(\, f(\vartheta_j) - p(\vartheta_j)\, z^{'}(\vartheta_j) - q(\vartheta_j)\, z(\vartheta_j - 1)\Big) \text{ for } j = \, i \pm 1,\, i \tag{5}$$

using Eq. (5), Eq. (4) can be represented as

$$\frac{\alpha}{\varepsilon}\Big(\, f(\vartheta_{i+1})\, {-}\mathrm{p}(\vartheta_{i+1})\, z^{'}(\vartheta_{i+1})\, {-}\mathrm{q}(\vartheta_{i+1})\, z(\vartheta_{i+1}{-}1)\Big) +$$

$$+\frac{2\beta}{\varepsilon}\Big(\, f(\vartheta_i)\, {-}\mathrm{p}(\vartheta_i)\, z^{'}(\vartheta_i)\, {-}\mathrm{q}(\vartheta_i)\, z(\vartheta_i{-}1)\Big) +$$

$$+\frac{\alpha}{\varepsilon}\Big(\, f(\vartheta_{i-1})\, {-}\mathrm{p}(\vartheta_{-1})\, \mathrm{z}^{'}(\vartheta_{j-1})\, {-}\mathrm{q}(\vartheta_{i-1})\, z(\vartheta_{i-1}{-}1)\Big) = \left(\frac{z_{i-1} - 2z_i + z_{i+1}}{h^2}\right),$$

$$\alpha\Big(\, f(\vartheta_{i+1})\, {-}\mathrm{p}(\vartheta_{i+1})\, z^{'}(t_{i+1})\, {-}\mathrm{q}(\vartheta_{i+1})\, z(\vartheta_{i+1}{-}1)\Big) +$$

$$+2\beta\Big(\, f(\vartheta_i)\, {-}\mathrm{p}(\vartheta_i)\, z^{'}(\vartheta_i)\, {-}\mathrm{q}(\vartheta_i)\, z(\vartheta_i{-}1)\Big) +$$

$$+\alpha\Big(\, f(\vartheta_{i-1})\, {-}\mathrm{p}(\vartheta_{i-1})\, z^{'}(\vartheta_{i-1})\, {-}\mathrm{q}(\vartheta_{i-1})\, z(\vartheta_{i-1}{-}1)\Big) = \frac{\varepsilon}{h^2}(z_{i-1} - 2z_i + z_{i+1})\,. \tag{6}$$

Using the finite differences

$$z^{'}(\vartheta_{i+1}) = \left(\frac{z_{\mathrm{i-1}}{-}4z_{\mathrm{i}}{-}3z_{\mathrm{i+1}}}{2\mathrm{h}}\right),\; z^{'}(\vartheta_i) = \left(\frac{z_{\mathrm{i+1}}{-}z_{\mathrm{i-1}}}{2\mathrm{h}}\right),$$

$$z^{'}(\vartheta_{i-1}) = \left(\frac{z_{\mathrm{i+1}}{-}4z_{\mathrm{i}}{-}3z_{\mathrm{i-1}}}{2\mathrm{h}}\right).$$

Eq. (6) is reduced to

$$\alpha\left(\mathrm{f}(\vartheta_{\mathrm{i+1}})\, {-}\mathrm{p}(\vartheta_{\mathrm{i+1}})\left(\frac{z_{\mathrm{i-1}}{-}4z_{\mathrm{i}}{-}3z_{\mathrm{i+1}}}{2\mathrm{h}}\right) {-}\mathrm{q}(\vartheta_{\mathrm{i+1}})\, z(\vartheta_{i+1}{-}1)\right) +$$

$$+2\beta\left(\mathrm{f}(\vartheta_{\mathrm{i}})\, {-}\mathrm{p}(\vartheta_{\mathrm{i}})\left(\frac{z_{\mathrm{i+1}}{-}z_{\mathrm{i-1}}}{2\mathrm{h}}\right) {-}\mathrm{q}(\vartheta_{\mathrm{i}})\; z(\vartheta_i{-}1)\right) +$$

$$+\alpha\left(\mathrm{f}(\vartheta_{\mathrm{i-1}})\, {-}\mathrm{p}(\vartheta_{\mathrm{i-1}})\left(\frac{z_{\mathrm{i+1}}{-}4z_{\mathrm{i}}{-}3z_{\mathrm{i-1}}}{2\mathrm{h}}\right) {-}\mathrm{q}(\vartheta_{\mathrm{i-1}})\; z(\vartheta_{i-1}{-}1)\right) =$$

$$= \frac{\varepsilon}{h^2}(z_{\mathrm{i-1}}{-}2z_{\mathrm{i}}{+}z_{\mathrm{i+1}})\,,$$

$$\left(\frac{\varepsilon}{h^2}-\frac{\beta a_i}{h}+\left(\frac{\alpha}{2h}\left(p_{i+1}-3p_{i-1}\right)\right)\right)z_{i-1}+\left(\frac{-2\varepsilon}{h^2}-\frac{2\alpha}{h}\left(p_{i+1}-p_{i-1}\right)\right)z_i+$$
$$+\left(\frac{\varepsilon}{h^2}+\frac{\beta a_i}{h}+\frac{\alpha}{2h}\left(3p_{i+1}-p_{i-1}\right)\right)z_{i+1}=\alpha f(\vartheta_{i-1})-\alpha \mathrm{q}(\vartheta_{i-1})z(\vartheta_{i-1}-1)+$$
$$+2\beta f(\vartheta_i)-2\beta \mathrm{q}(\vartheta_i)z(\vartheta_i-1)+\alpha f\left(\vartheta_{i+1}\right)-\alpha \mathrm{q}\left(\vartheta_{i+1}\right)z\left(\vartheta_{i+1}-1\right). \qquad (7)$$

To reduce the error value in the solution over the domain $\Omega_1=(0, 1)$, we insert a fitting parameter $\sigma(\rho)$ in the above numerical scheme Eq. (7) for the equation

$$\varepsilon\sigma(\rho)\, z''(\vartheta)+p(\vartheta)\, z'(\vartheta)+q(\vartheta)\, z(\vartheta-1)=f(\vartheta).$$

The value of the fitting parameter is $\sigma(\rho)=\rho(\alpha+\beta)\coth\left(\frac{\rho p_i}{2}\right)$, where $\rho=\frac{h}{\varepsilon}$.

The scheme Eq. (7) with a fitting factor can be written as

$$E_i z_{i-1}+F_i z_i+G_i z_{i-1}-H_i=0 \quad \text{for } i=1,2,...,L-1, \qquad (8)$$

where

$$E_i=\left(\frac{\sigma\varepsilon}{h^2}-\frac{\beta a_i}{h}+\left(\frac{\alpha}{2h}\left(p_{i+1}-3p_{i-1}\right)\right)\right),$$
$$F_i=\left(\frac{-2\sigma\varepsilon}{h^2}-\frac{2\alpha}{h}\left(p_{i+1}-p_{i-1}\right)\right),$$
$$G_i=\left(\frac{\varepsilon\sigma}{h^2}+\frac{\beta a_i}{h}+\frac{\alpha}{2h}\left(3p_{i+1}-p_{i-1}\right)\right)$$

and

$$H_i=\alpha\left(f\left(\vartheta_{i-1}\right)-q\left(\vartheta_{i-1}\right)\varphi_{i-1}\right)+2\beta\left(f\left(\vartheta_i\right)-q\left(\vartheta_i\right)\varphi_i\right)+\alpha\left(f\left(\vartheta_{i+1}\right)-q\left(\vartheta_{i+1}\right)\varphi_{i+1}\right),$$

here

$$\varphi_{i+1}=z\left(\vartheta_{i+1}-1\right), \quad \varphi_i=z\left(\vartheta_i-1\right), \quad \varphi_{i-1}=z\left(\vartheta_{i-1}-1\right) \text{ in } [0,1].$$

Now, to find the solution in $\Omega_2=(1, 2)$, we consider the finite difference scheme Eq. (7) with fitting factor $\sigma(\rho)$ in the equation

$$\varepsilon\sigma(\rho)\, z''(\vartheta)+p(\vartheta)\, z'(\vartheta)+q(\vartheta)\, z(\vartheta-1)=f(\vartheta).$$

Here the value of $\sigma(\rho)$ is $\sigma(\rho)=\rho(\alpha+\beta)\coth\left(\frac{\rho p_i}{2}\right)$, where $\rho=\frac{h}{\varepsilon}$.

Then, the scheme Eq. (7) with a fitting factor can be written as

$$E_i z_{i-1}+F_i z_i+G_i z_{i+1}-H_i=0 \quad \text{for } i=L, L+1, ..., 2L-1, \qquad (9)$$

where

$$E_i=\left(\frac{\sigma\varepsilon}{h^2}-\frac{\beta a_i}{h}+\left(\frac{\alpha}{2h}\left(p_{i+1}-3p_{i-1}\right)\right)\right),$$
$$F_i=\left(\frac{-2\sigma\varepsilon}{h^2}-\frac{2\alpha}{h}\left(p_{i+1}-p_{i-1}\right)\right),$$
$$G_i=\left(\frac{\varepsilon\sigma}{h^2}+\frac{\beta a_i}{h}+\frac{\alpha}{2h}\left(3p_{i+1}-p_{i-1}\right)\right)$$

and

$$H_i=\alpha\left(f\left(\vartheta_{i-1}\right)-\mathrm{q}\left(\vartheta_{i-1}\right)z\left(\vartheta_{i-1}-L\right)\right)+\alpha\left(f\left(\vartheta_{i+1}\right)-\mathrm{q}\left(\vartheta_{i+1}\right)z\left(\vartheta_{i+1}-L\right)\right)+$$
$$+2\beta\left(f\left(\vartheta_i\right)-\mathrm{q}\left(\vartheta_i\right)z\left(\vartheta_i-L\right)\right).$$

To solve the system of equations Eq. (8) and Eq. (9), the condition $z(L)$ is required. To get the value of $z(L)$, we utilize the reduced problem of Eq. (1) by setting $\varepsilon=0$ and Runge-Kutta 4^{th} order method is used to solve the reduced differential equation.

3 Local error estimate

The local error estimate for the numerical scheme of Eq. (8) is

$$\mathcal{T}_i\left(h\right) = \left[2\alpha + 2\beta - 1\right]\varepsilon h^2 z_i'' + \left(\left(\alpha - \frac{1}{12}\right)\varepsilon z_i^{iv} - \left(\frac{-2\alpha}{3} + \frac{\beta}{3}\right)p_i z_i'''\right)h^4 + O(h^6). \tag{10}$$

Hence, with $\alpha = \frac{1}{12}$ and $\alpha + \beta = \frac{1}{2}$, the truncation error is fourth order.

4 Convergence analysis

Considering the matrix version of Eq. (8) with the boundary conditions, we have

$$\left(A + P\right)Z + Q + \mathcal{T}(h) = 0, \tag{11}$$

where

$$A = \begin{bmatrix} -2\varepsilon\sigma & \varepsilon\sigma & 0 & 0 & \dots & 0 \\ \varepsilon\sigma & -2\varepsilon\sigma & \varepsilon\sigma & .. & \dots & 0 \\ 0 & .. & .. & .. & \dots & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & .. & \varepsilon\sigma & -2\varepsilon\sigma \end{bmatrix}$$

and

$$P = \left[l_i, m_i, k_i\right] = \begin{bmatrix} m_1 & k_1 & 0 & 0 & \dots\dots.. & 0 \\ l_2 & m_2 & k_2 & 0 & \dots\dots.. & 0 \\ 0 & l_3 & m_3 & k_3 & \dots\dots.. & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & l_{L-1} & m_{L-1} \end{bmatrix},$$

where

$$l_i = \frac{h}{2}\left(-3\alpha p_{i-1} - 2\beta p_i + \alpha p_{i+1}\right) + h^2\alpha q_{i-1}\varphi_{i-1},$$

$$m_i = \frac{h}{2}\left(4\alpha p_{i-1} - \alpha p_{i+1}\right) + 2h^2\beta q_i\varphi_i,$$

$$k_i = \frac{h}{2}\left(-\alpha p_{i-1} + 2\beta p_i + 3\alpha p_{i+1}\right) + h^2\alpha q_{i+1}\varphi_{i+1}, \text{ for } 1 \leqslant i \leqslant L-1$$

and

$$Q = \left[r_1 + \left(\varepsilon\sigma + k_1\right)\varphi_0, r_2, r_3, ..., r_{L-2}, r_{L-1} + (\varepsilon\sigma + k_{L-1})\gamma\right]^T,$$

where

$$q_i = h^2\left[\alpha k_{i+1} + 2\beta k_i + \alpha k_{i-1}\right], \quad 1 \leqslant i \leqslant L-1,$$

$\mathcal{T}\left(h\right) = O\left(h^4\right)$ and $Z = \left[Z_1, Z_2, \dots, Z_{L-1}\right]^T, \mathcal{T}\left(h\right) = \left[\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_{L-1}\right]^T$, $O = [0, 0, \dots, 0]^T$ are associated vectors of Eq. (11).

Let $n = \left[n_1, n_2, ..., n_{L-1}\right]^T \cong Z$ satisfy the equation

$$\left(A + P\right)n + Q = 0. \tag{12}$$

Let $e_i = n_i - Z_i$, $i = 1, 2..., L-1$ be the discretized error $E = [e_1, e_2, ..., e_{L-1}]^T = n - Z$.
Using Eq. (12) and Eq. (11), we get the error equation as

$$(A + P)\, E = \mathcal{T}(h). \tag{13}$$

Let $|p(s)| \leq D_1$ and $|q\,(s)| \leq D_2$, where $D_1,\ D_2$ are positive constants. Let $(i,j)^{th}$ element of the matrix $(A + P)$ be $\zeta_{i,j}$ then

$$|\zeta_{i,i+1}| \leqslant (\varepsilon) + h(\alpha + \beta) D_1 + h^2(\alpha q_{i+1}\varphi_{i+1} + 2\beta q_i \varphi_i),\ 1 \leqslant i \leqslant L-2,$$

$$|\zeta_{i,i-1}| \leqslant (\varepsilon) + h(\alpha + \beta) D_1 + h^2(\alpha q_{i-1}\varphi_{i-1} + 2\beta q_i \varphi_i),\ 1 \leqslant i \leqslant L-1.$$

Hence, for small values of h, we have

$$|\zeta_{i,i+1}| < \varepsilon\sigma, \quad 1 \leqslant i \leqslant L-2$$

and

$$|\zeta_{i,i-1}| < \varepsilon\sigma, \quad 2 \leqslant i \leqslant L-1.$$

Hence $(A + P)$ is irreducible [25].

Let $\mathrm{S_i}$ be the i^{th} row elements sum, of the matrix $(A + P)$, then we have

$$S_i = -\varepsilon + h(\alpha + \beta) p_i + h^2(\alpha q_{i+1}\varphi_{i+1} + 2\beta q_i \varphi_i) \quad \text{for} \quad i = 1,$$

$$S_i = h^2\left(\alpha q_{i-1}\varphi_{i-1} + 2\beta q_i \varphi_i + \alpha q_{i+1}\varphi_{i+1}\right) \quad \text{for} \quad i = 2, 3, ..., L-2,$$

$$S_i = -\varepsilon - h(\alpha + \beta) p_i + h^2(\alpha q_{i-1}\varphi_{i-1} + 2\beta q_i \varphi_i) \quad \text{for} \quad i = L-1.$$

Let $D_{1^*} = |p(s)|$ and $D_1^* = |p(s)|$, $D_{2^*} = |q(s)|$ and $D_2^* = |q(s)|$. Since $0 < \varepsilon \ll 1$, and $\varepsilon \propto O\,(h)$ it is verified that for sufficiently small $h.\,(A + P)$ is monotone [25, 26]. Hence $(A + P)^{-1}$ exists and $(A + P)^{-1} \geqslant 0$. Thus using Eq. (13), we have

$$\|E\| \leqslant \left\|(A + P)^{-1}\right\| \|\mathcal{T}\|\,. \tag{14}$$

Let $(A + P)^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(A + P)^{-1}$ and define

$$\|\,(A + P)^{-1}\,\| = \max_{1 \leq i \leq L-1} \sum_{k=1}^{L-1} (A + P)^{-1}_{i,k} \quad \text{and} \quad \|\mathcal{T}(h)\| = \max_{1 \leq i \leq L-1} |T(h)|.$$

Since

$$(A + P)^{-1}_{i,k} \geqslant 0 \ \text{ and } \sum_{k=1}^{L-1} (A + P)^{-1}_{i,k}\,, \quad S_k = 1 \ \text{ for } 1 \leqslant i \leqslant L-1, \tag{15}$$

we have

$$(A + P)^{-1}_{i,k} \leqslant \frac{1}{\max\limits_{1 \leq i \leq L-1} S_i} < \frac{1}{h^2 D_2}, \quad i = 1, \tag{16}$$

$$(A + P)^{-1}_{i,k} \leqslant \frac{1}{S_i} < \frac{1}{h^2 D_2}, \quad i = L-1. \tag{17}$$

Further

$$\sum_{k=2}^{L-2} (A + P)^{-1}_{i,k} \leqslant \frac{1}{\max\limits_{2 \leq i \leq L-2} S_i} < \frac{1}{h^2 D_2}, \text{ for } 2 \leqslant i \leqslant L-2. \tag{18}$$

From Eq. (10), Eq. (14) and using of Eqs. (15)–(18) we get

$$\|E\| \leqslant O\left(h^2\right).$$

Second-order convergence of the proposed scheme is thus observed in the first half of the interval. Similarly, we can demonstrate that the scheme exhibits second-order convergence in the second half of the interval by using Eq. (9).

5 Numerical examples

Three examples are used to demonstrate the proposed scheme. The maximum absolute errors (MAEs) in the solution are computed using the double mesh principle [4].

Utilizing the following formula

$$R^L = \frac{log\left|\frac{E_\varepsilon^L}{E_\varepsilon^{2L}}\right|}{log2}$$

the numerical convergence for each case has been determined.

Example 1. $\varepsilon z''(\vartheta) - 3z'(\vartheta) + z(\vartheta - 1) = 0$, with $z(\vartheta) = 1;\ -1 \leqslant \vartheta \leqslant 0,\ z(2) = 2$.

Example 2. $\varepsilon z''(\vartheta) - 2z'(\vartheta) + 5z(\vartheta - 1) = 0$, with $z(\vartheta) = 1;\ -1 \leqslant \vartheta \leqslant 0,\ z(2) = 2$.

Example 3. $\varepsilon z''(\vartheta) - 5z'(\vartheta) + \frac{1}{2}z(\vartheta - 1) = \begin{cases} -1,\ 0 \leqslant \vartheta \leqslant 1 \\ 1,\ 1 \leqslant \vartheta \leqslant 2 \end{cases}$, with

$$z(\vartheta) = 1; -1 \leqslant \vartheta \leqslant 0, z(2) = 2.$$

6 Discussions and conclusion

To solve a SPDE with a fixed large delay, a trigonometric spline-based numerical technique is proposed. The strategy is designed by utilizing the continuity of the first order derivative of the spline. The convergence of the method is investigated, and it reached second order convergence. Three examples of the scheme with the right end boundary layer are provided. The maximum absolute errors (MAEs) in the solutions are tabulated in Tables 1, 2 and 3 in comparison to the method given in [27]. The rate of convergence in the solutions is also computed. The layer structure is depicted in Figures 1, 2 and 3. In the illustration, it can be seen that the width of the right end layer similarly reduces as the perturbation value does.

7 *Tables and Figures*

Table 1

MAEs in Example 1

$\varepsilon \downarrow L \rightarrow$	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
suggested method						
2^{-5}	9.5547e-06	2.3939e-06	5.9883e-07	1.4975e-07	3.7573e-08	9.7531e-09
	1.9968	1.9991	1.9996	1.9948	1.9458	
2^{-6}	1.9455e-05	4.9061e-06	1.2292e-06	3.0749e-07	7.6942e-08	1.9565e-08
	1.9458	1.9875	1.9969	1.9987	1.9755	
2^{-7}	3.8180e-05	9.8729e-06	2.4898e-06	6.2383e-07	1.5608e-07	3.9153e-08
	1.9513	1.9874	1.9968	1.9989	1.9951	
2^{-8}	6.8209e-05	1.9242e-05	4.9774e-06	1.2552e-06	3.1452e-07	7.8749e-08
	1.8257	1.9508	1.9875	1.9967	1.9978	
2^{-9}	9.7588e-05	3.4253e-05	9.6682e-06	2.5001e-06	6.3049e-07	1.5801e-07
	1.5105	1.8249	1.9513	1.9874	1.9965	
2^{-10}	1.0755e-04	4.8915e-05	1.7177e-05	4.8460e-06	1.2532e-06	3.1606e-07
	1.1367	1.5098	1.8256	1.9512	1.9873	
2^{-11}	1.0808e-04	5.3879e-05	2.4506e-05	8.6011e-06	2.4264e-06	6.2744e-07
	1.0043	1.1366	1.5105	1.8257	1.9513	
2^{-12}	1.0808e-04	5.4147e-05	2.6966e-05	1.2265e-05	4.3037e-06	1.2141e-06
	0.9971	1.0057	1.1366	1.5109	1.8257	
2^{-13}	1.0808e-04	5.4147e-05	2.7100e-05	1.3490e-05	6.1354e-06	2.1526e-06
	0.9971	0.9986	1.0064	1.1367	1.5111	
Results in [27]						
2^{-5}	3.8774(-5)	9.6108(-6)	2.3975(-6)	5.9904(-7)	1.4972(-7)	3.7294(-8)
2^{-6}	8.2126(-5)	1.9910(-5)	4.9349(-6)	1.2310(-6)	3.0757(-7)	7.6825(-8)
2^{-7}	1.8429(-4)	4.1727(-5)	1.0104(-5)	2.5044(-6)	6.2472(-7)	1.5606(-7)
2^{-8}	4.5000(-4)	9.2878(-5)	2.1029(-5)	5.0938(-6)	1.2626(-6)	3.1493(-7)
2^{-9}	1.0778(-3)	2.2589(-4)	4.6641(-5)	1.0566(-5)	2.5585(-6)	6.3417(-7)
2^{-10}	2.3688(-3)	5.4102(-4)	1.1321(-4)	2.3389(-5)	5.2961(-6)	1.2825(-6)
2^{-11}	4.9529(-3)	1.1891(-3)	2.7104(-4)	5.6715(-5)	1.1712(-5)	2.6518(-6)
2^{-12}	1.0121(-2)	2.4862(-3)	5.9570(-4)	1.3565(-4)	2.8385(-5)	5.8601(-6)
2^{-13}	2.0458(-2)	5.0805(-3)	1.2455(-3)	2.9814(-4)	6.7859(-5)	1.4200(-5)

Table 2

MAEs in Example 2

$\varepsilon \downarrow L \rightarrow$	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
suggested method						
2^{-5}	3.5024e-04	8.7651e-05	2.1918e-05	5.4801e-06	1.3706e-06	3.4554e-07
	1.9985	1.9997	1.9998	1.9994	1.9880	
2^{-6}	7 .2423e-04	1.8177e-04	4.5489e-05	1.1375e-05	2.8445e-06	7.1278e-07
	1.9943	1.9944	1.9986	1.9996	1.9996	
2^{-7}	1.4561e-03	3.6979e-04	9.2808e-05	2.3225e-05	5.8078e-06	1.4533e-06
	1.9773	1.9944	1.9986	1.9996	1.9987	
2^{-8}	2.7805e-03	7.3682e-04	1.8705e-04	4.6948e-05	1.1749e-05	2.9382e-06
	1.9160	1.9778	1.9943	1.9985	1.9995	
2^{-9}	4.6105e-03	1.3988e-03	3.7082e-04	9.4139e-05	2.3626e-05	5.9127e-06
	1.7207	1.9154	1.9779	1.9944	1.9985	
2^{-10}	5.8590e-03	2.3143e-03	7.0214e-04	1.8607e-04	4.7237e-05	1.1855e-05
	1.3401	1.7207	1.9159	1.9779	1.9944	
2^{-11}	6.0756e-03	2.9352e-03	1.1594e-03	3.5176e-04	9.3215e-05	2.36604e-05
	1.0496	1.3401	1.7207	1.9160	1.9779	
2^{-12}	6.0797e-03	3.0438e-03	1.4691e-03	5.8029e-04	1.7606e-04	4.6657e-05
	0.9981	1.0509	1.3401	1.7207	1.9159	
2^{-13}	6.05797e-03	3.0458e-03	1.5234e-03	7.3489e-04	2.9031e-04	8.8084e-05
	0.9920	0.9953	1.0517	1.3399	1.7206	
Results in [27]						
2^{-5}	1.4101(-3)	3.5121(-4)	8.7724(-5)	2.1926(-5)	5.4809(-6)	1.3713(-6)
2^{-6}	2.9715(-3)	7.3183(-4)	1.8226(-4)	4.5523(-5)	1.1378(-5)	2.8438(-6)
2^{-7}	6.3962(-3)	1.5166(-3)	3.7366(-4)	9.3055(-5)	2.3241(-5)	5.8086(-6)
2^{-8}	1.4877(-2)	3.2366(-3)	7.6743(-4)	1.8901(-4)	4.7072(-5)	1.1757(-5)
2^{-9}	3.6799(-2)	7.4974(-3)	1.6281(-3)	3.8622(-4)	9.5120(-5)	2.3688(-5)
2^{-10}	8.4871(-2)	1.8472(-2)	3.7635(-3)	8.1727(-4)	1.9379(-4)	4.7729(-5)
2^{-11}	1.8184(-1)	4.2603(-2)	9.2539(-3)	1.8854(-3)	4.0944(-4)	9.7084(-5)
2^{-12}	3.7579(-1)	9.1277(-2)	2.1343(-2)	4.6315(-3)	9.4363(-4)	2.0493(-4)
2^{-13}	7.6369(-1)	1.8863(-1)	4.5728(-2)	1.0682(-2)	2.3169(-3)	4.7207(-4)

Table 3

MAEs in Example 3

$\varepsilon \downarrow L \rightarrow$	2^{7}	2^{8}	2^{9}	2^{10}	2^{11}	2^{12}
suggested method						
2^{-5}	4.7341e-05	1.3845e-06	3.7561e-06	9.7905e-07	2.5003e-07	6.3369e-08
	1.7737	1.8821	1.9398	1.9693	1.9803	
2^{-6}	7.1268e-05	2.3726e-05	6.9366e-06	1.8816e-06	4.9045e-07	1.2535e-07
	1.5868	1.7742	1.8823	1.9398	1.9681	
2^{-7}	8.9077e-05	3.5691e-05	1.1878e-05	3.4721e-06	9.4179e-07	2.4552e-07
	1.3195	1.5873	1.7744	1.8823	1.9396	
2^{-8}	9.4225e-05	4.4593e-05	2.2312e-05	5.9429e-06	1.7371e-06	4.7118e-07
	1.0793	0.9990	1.9086	1.7745	1.8823	
2^{-9}	9.4324e-05	4.7161e-05	2.3593e-05	8.9346e-06	2.9725e-06	8.6884e-07
	1.0000	0.9992	1.4009	1.5877	1.7745	
2^{-10}	9.4287e-05	4.7190e-05	2.3593e-05	1.1160e-05	4.4684e-06	1.4866e-06
	0.9986	0.9996	1.0800	1.3205	1.5877	
2^{-11}	9.4287e-05	4.7172e-05	2.3602e-05	1.1799e-05	5.5810e-06	2.2345e-06
	0.9991	0.9996	1.0002	1.0801	1.3206	
2^{-12}	9.4287e-05	4.7172e-05	2.3593e-05	1.1803e-05	5.9004e-06	2.7908e-06
	0.9991	0.9996	0.9992	1.0003	1.0801	
2^{-13}	9.4287e-05	4.7172e-05	2.3593e-05	1.1798e-05	5.9019e-06	2.9504e-06
	0.9991	0.9996	0.9998	0.9993	1.0003	
Results in [27]						
2^{-5}	1.9126(-4)	6.2523(-6)	1.7063(-5)	4.3683(-6)	1.0987(-6)	2.7510(-7)
2^{-6}	2.2400(-4)	9.7185(-5)	3.1769(-5)	8.6781(-6)	2.2223(-6)	5.5895(-7)
2^{-7}	2.2703(-4)	1.1381(-4)	4.8981(-5)	1.6049(-5)	4.3824(-6)	1.1219(-6)
2^{-8}	2.2705(-4)	1.1535(-4)	5.7355(-5)	2.4657(-5)	8.0719(-6)	2.2029(-6)
2^{-9}	2.2705(-4)	1.1536(-4)	5.8131(-5)	2.8790(-5)	1.2377(-5)	4.0479(-6)
2^{-10}	2.2705(-4)	1.1536(-4)	5.8136(-5)	2.9180(-5)	1.4423(-5)	6.2006(-6)
2^{-11}	2.2705(-4)	1.1536(-4)	5.8136(-5)	2.9182(-5)	1.4618(-5)	7.2188(-6)
2^{-12}	2.2705(-4)	1.1536(-4)	5.8136(-5)	2.9182(-5)	1.4620(-5)	7.3164(-6)
2^{-13}	2.2705(-4)	1.1536(-4)	5.8136(-5)	2.9182(-5)	1.4620(-5)	7.3171(-6)

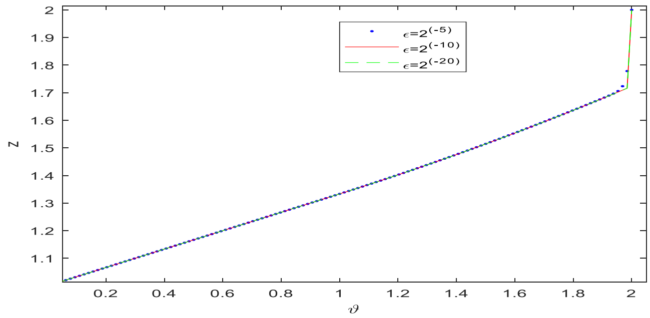

Figure 1. Layer profile in the solution Example 1 with $\varepsilon = 2^{-5}, 2^{-10}, 2^{-20}$.

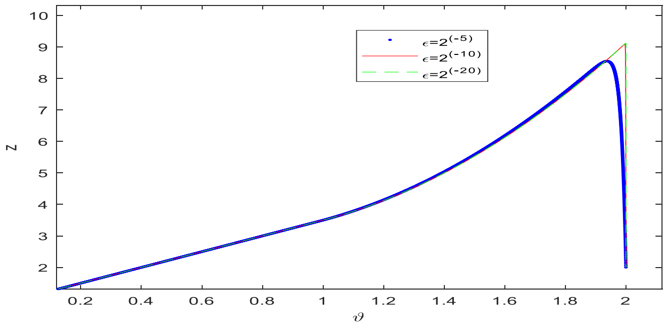

Figure 2. Layer profile in the solution Example 2 with $\varepsilon = 2^{-5}, 2^{-10}, 2^{-20}$.

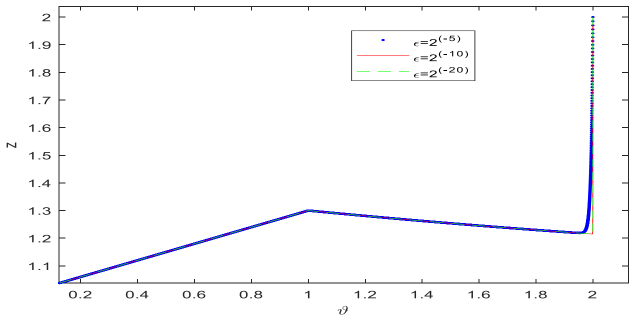

Figure 3. Layer profile in the solution Example 3 with $\varepsilon = 2^{-5}, 2^{-10}, 2^{-20}$.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Mackey, M.C., & Glass, L. (1977). Oscillations and chaos in physiological control systems. *Science 197*, 287–289.

2 Wazewska-Czyzewska, M., & Lasota, A. (1976). Mathematical models of the red cell system. *Mat Sots., 6*, 25–40.

3 Longtin, A., & Milton, J. (1998). Complex oscillations in the human pupil light reflex with mixed and delayed feedback. *Math Biosci., 90*, 183–199.

4 Derstine, M.W., Gibbs, H.M., & Kaplan, D.L. (1982). Bifurcation gap in a hybrid optical system. *Phys. Rev. A., 26*, 3720–3722. https://doi.org/10.1103/PhysRevA.26.3720

5 Mallet-Paret, J., & Nussbaum, R.D. (1996). *A differential-delay equations arising in optics and physiology*. Singapore: World scientific. https://doi.org/10.1137/0520019

6 Bender, C.M., & Orszag , S.A. (1978). *Advanced mathematical methods for scientists and engineers*. New York: Mc Graw-Hill.

7 O'Malley, R.E. (1974). *Introduction to singular perturbations*. New York: Academic press.

8 Doolan, E.P., Miller, J.J.H., & Schilders, W.H.A. (1980). *Uniform numerical methods for problems with initial and boundary layers*. Dublin: Boole Press.

9 Miller, J.J.H., O'Riordan E., & Shishkin, G.I. (1996). *Fitted numerical methods for singular perturbation problems*. Singapore: World Scientific.

10 Roos, H.G., Stynes, M., & Tobiska, L. (1996). *Numerical methods for singularly perturbed differential equations*. Berlin: Springer-Verlag.

11 Driver, R.D. (1977). *Ordinary and delay differential equations*. Heidelberg, Springer, New York, Berlin.

12 Bellman, R., & Cooke, K.L. (1963). *Differential-difference equations*. New York: Academic press.

13 Lange, C.G., & Miura, R.M. (1982). Singular perturbation analysis of boundary value problems for differential-difference equations. *SIAM J Appl Math., 42*(3), 502–530. https://doi.org/10.1137/0142036

14 Lange, C.G., & Miura, R.M. (1994). Singular perturbation analysis of boundary-value problems for differential difference equations. V. Small shifts with layer behaviour. *SIAM J. Appl. Math., 54*(1), 249–272. https://doi.org/10.1137/S0036139992228120

15 Lange, C.G., & Miura, R.M. (1994). Singular perturbation analysis of boundary-value problems for differential difference equations. VI. Small shifts with rapid oscillations. *SIAM J. Appl. Math., 54*, 273–283.

16 Kadalbajoo, M.K., Patidar, K.C., & Sharma, K.K. (2006).ε-uniformly convergent fitted methods for the numerical solution of the problems arising from singularly perturbed general DDEs. *Appl Math Comput., 182*, 119–139. https://doi.org/10.1016/j.amc.2006.03.043

17 Kadalbajoo, M.K., & Sharma, K.K. (2008). A numerical method based on finite difference for boundary value problems for singularly perturbed delay differential equations. *Appl Math Comput., 197*, 692–707. https://doi.org/10.1016/j.amc.2007.08.089

18 Amiraliyev, G.M., & Cimen, E. (2010). Numerical method for a singularly perturbed convection-diffusion problem with delay. *Appl Math Comput., 216*, 2351–2359. https://doi.org/10.1016/j.amc.2010.03.080

19 Mohapatra, J., & Natesan, S. (2010). Uniform convergence analysis of finite difference scheme for singularly perturbed delay differential equation on an adaptively generated grid. Numer. *Math. Theory Methods Appl., 3*, 1–22. https://doi.org/10.4208/nmtma.2009.m8015

20 Erdogan, F., & Amiraliyev, G.M. (2012). Fitted finite difference method for singularly perturbed delay differential equations. *Numer. Algorithms., 59*, 131–145. https://doi.org/10.1007/s11075-011-9480-7

21 Kanth, A.R., & P. Murali, M.K. (2018). A numerical technique for solving nonlinear singularly perturbed delay differential equations. *Mathematical Modelling and Analysis, 23*(1), 64–78. https://doi.org/10.3846/mma.2018.005

22 Assanova, A.T., Iskakova, N.B., & Orumbayeva, N.T. (2018). Well-posedness of a periodic boundary value problem for the system of hyperbolic equations with delayed argument. *Bulletin of the Karaganda university. Mathematics series,* 1(89), 8–14. https://doi.org/10.31489/2018m1/8-14

23 Mazhgikhova, M.G. (2020). Green function method for a fractional–order delay differential equation. *Bulletin of the Karaganda university. Mathematics series,* 1(97), 87–96. https://doi.org/10.31489/2020m1/87-96

24 Omkar, R., Lalu, M., & Phaneendra, K. (2023). Numerical solution of differential – difference equations having an interior layer using nonstandard finite differences. *Bulletin of the Karaganda university. Mathematics series,* 2(110), 104–115. https://doi.org/10.31489/2023m2/104-115

25 Varga, R.S. (1962). *Matrix iterative analysis.* Englewood Cliffs, NJ Prentice-Hall.

26 Young, D.M. (1971). *Iterative solution of large linear systems.* New York: Acadamic press.

27 Kumar, N.S., & Rao, R.N. (2020). A Second Order Stabilized Central Difference Method for Singularly Perturbed Differential Equations with a Large Negative Shift. *Differential Equations and Dynamical Systems, 31*, 787–804. https://doi.org/10.1007/s12591-020-00532-w

Бекітілген көп кешігуі бар сингулярлы ауытқыған дифференциалдық-айырымдық теңдеулер класының жаңа сандық схемасы

Э. Шринивас, К. Фанеендра

Университеттің ғылым колледжі, Османия университеті, Хайдарабад, Үндістан

Тригонометриялық сплайнға негізделген есептеу әдісі бекітілген көп кешігуі бар қабаттың әрекеті үшін дифференциалдық-айырымдық теңдеулерді сандық шешу үшін ұсынылған. Айырымдық теңдеулер жүйесін құру үшін тордың ішкі нүктесіндегі тригонометриялық сплайнның бірінші ретті туындысының үзіліссіздігі қолданылады. Сингулярлы ауытқыған теориясын қолдана отырып, шешімдегі қатені азайту үшін айырымдық схемасына сәйкестендіретін параметр енгізіледі. Әдіс жинақтылыққа тексерілген. Сонымен қатар шекаралық қабатқа ығысу немесе кешігу әсері қарастырылды. Әдебиеттерде келтірілген басқа тәсілдермен салыстырғанда максималды абсолютті қателер есептеледі және ұсынылған сандық әдістің орындылығын көрсету үшін қабаттардың өзгеруі графиктерде көрсетілді.

Кілт сөздер: сингулярлы ауытқыған дифференциалдық-айырымдық теңдеу, кешігу, тригонометриялық сплайн, сәйкестендіретін параметр.

Новая численная схема для класса сингулярно возмущенных дифференциально-разностных уравнений с фиксированным большим запаздыванием

Э. Шринивас, К. Фанеендра

Университетский колледж науки; Университет Османии, Хайдарабад, Индия

Предложен вычислительный метод на основе тригонометрического сплайна для численного решения дифференциально-разностных уравнений поведения слоя с фиксированной большой задержкой. Для построения системы разностных уравнений используется непрерывность производной первого порядка тригонометрического сплайна во внутренней точке сетки. С помощью теории сингулярных возмущений в разностную схему вводится подгоночный параметр, позволяющий минимизировать ошибку в решении. Метод проверен на сходимость. Мы также рассмотрели влияние сдвига или задержки на пограничный слой. Подсчитаны максимальные абсолютные погрешности по сравнению с другими подходами, описанными в литературе, а поведение слоев отображено на графиках, чтобы продемонстрировать осуществимость предложенного численного метода.

Ключевые слова: сингулярно возмущенное дифференциально-разностное уравнение, запаздывание, тригонометрический сплайн, подгоночный параметр.

*Author Information**

Erla Srinivas — Research Scholar, Department of Mathematics, University College of Science, Osmania University, Hyderabad, 500007, India; e-mail: *Srinivas.erla@gmail.com*; https://orcid.org/0009-0006-9380-5255

Kolloju Phaneendra (*corresponding author*) — Doctor of mathematical sciences, Associate Professor, Department of Mathematics, University College of Science, Osmania University, Hyderabad, 500007, India; e-mail: *kollojuphaneendra@yahoo.co.in*; https://orcid.org/0000-0002-2852-8494

*The author's name is presented in the order: First, Middle and Last Names.

https://doi.org/10.31489/2024M1/208-217 *Research article*

The cosemanticness of Kaiser hulls of fixed classes of models

A.R. Yeshkeyev, I.O. Tungushbayeva*, A.K. Koshekova

Institute of Applied Mathematics, Karaganda Buketov University, Karaganda, Kazakhstan
(E-mail: aibat.kz@gmail.com, intng@mail.ru, koshekova96@mail.ru)

In this article, within the framework of the study of Jonsson theories, the model-theoretic properties of cosemanticness classes belonging to the factor set of the Jonsson spectrum of an existentially closed models' subclass of some Jonsson theory in a fixed language were studied. Various results have been obtained. In particular, the properties of the cosemanticness of models and classes of models are considered; some results concerning the Jonsson equivalence in generalization for classes of existentially closed models are obtained; a criterion for the cosemanticness of J-classes in connection with their Kaiser hulls has been found.

Keywords: Jonsson theory, cosemanticness, cosemantic Jonsson theories, Jonsson spectrum, cosemanticness classes, Kaiser hull, Jonsson equivalence, J-class, cosemantic models, cosemantic classes.

2020 Mathematics Subject Classification: 03C50, 03C52.

Introduction

It is well known that in modern Model Theory, the issue of studying incomplete theories occupies a special place due to the small number of suitable methods and techniques. Anyway, this is a very difficult task, so, as a rule, model theorists use various limiting conditions to obtain results concerning incomplete theories. One of the relevant directions in this sense is studying Jonsson theories. The relevance is determined by various reasons, and the main one is that the Jonsson theories are of great applied importance in algebra due to the presence of many classical examples linking these two mathematical areas.

Traditionally, the Karaganda School of Model Theory uses the definition of the Jonsson theory given in the Russian-language edition of [1]. In recent years, the apparatus for studying Jonsson theories has been significantly expanded, which is demonstrated by the number and variety of approaches in the works [2–9].

At the same time, one of our essential areas of research in this area is not only to obtain results describing the properties of the Jonsson theories, but also to generalize these results. In 2018, Yeshkeyev A.R. introduced the concept of the Jonsson spectrum of a fixed class of models, which is a special set of Jonsson theories. When considering the Jonsson spectrum we also use the notion of the cosemanticness relation proposed by Mustafin T.G. Cosemanticness is a specific equivalence relation that generalizes and refines the elementary equivalence in terms of researching Jonsson theories. It is well known that equivalence relation is a classical instrument for studying and constructing the classification of theories in Model Theory. In this matter, in [10, 11], Yeshkeyev A.R. and Ulbrikht O.I. obtained some considerable results on abelian groups and R-modules concerning cosemanticness and other related concepts, such as consemanticness classes.

Thus, studying the properties of the cosemanticness classes of the Jonsson spectrum is of great importance not only for the development of the apparatus for the study of Jonsson theories. Firstly, this area is of interest from the point of view of research in Model Theory. In addition, it was found in [12–16] that the Jonsson theories and their cosemanticness classes in the Jonsson spectrum of fixed classes of

*Corresponding author. *E-mail: intng@mail.ru*
Received: 28 September 2023; *Accepted:* 30 November 2023.

structures have interesting structural properties that are important for Universal Algebra. In [17], the lattices of existential formulas of a fixed Jonsson theory are considered in terms of syntactic similarity. In this article, we present some basic results that demonstrate the relationship of the cosemanticness classes of the Jonsson spectrum with fixed classes of structures and the specific properties of Kaiser hulls of these classes. These results are the forerunner of the study of Jonsson theories from the point of view of lattice algebra and other related fields.

This paper consists of two sections. In Section 1, we give some basic information on Jonsson theories. In Section 2, we present our results obtained for cosemanticness classes of Jonsson spectrum, so-called J-classes of structures and their Kaiser hulls.

1 Preliminary information on Jonsson theories

In this section we describe the apparatus of the study of Jonsson theories. Let us start with some basic definitions.

Definition 1. [1] A theory T has the joint embedding property (JEP), if, for any models A and B of T, there exists a model M of T and isomorphic embeddings $f : A \to M$, $g : B \to M$.

Definition 2. [1] A theory T has the amalgamation property (AP), if for any models A, B_1, B_2 of T and isomorphic embeddings $f_1 : A \to B_1$, $f_2 : A \to B_2$ there are $M \models T$ and isomorphic embeddings $g_1 : B_1 \to M$, $g_2 : B_2 \to M$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Originally, the properties of amalgamation and joint embedding are algebraic notions. However they play a crucial role in studying various classes of structures in Model Theory, especially for incomplete theories.

There are syntactic criteria of AP and JEP. We give two classical theorems of them.

Theorem 1 (Robinson). [18] For the first order theory T of the language L (of arbitrary cardinality) the following conditions are equivalent:

1) T has JEP;

2) For all universal sentences α, β of L, if $T \vdash \alpha \vee \beta$ then $T \vdash \alpha$ or $T \vdash \beta$.

It is well known that the given theorem is equivalent to the following statement:

Theorem 2. Let T be a theory of the first-order language L. Then T has JEP iff whenever $T \cup \{\varphi\}$ and $T \cup \{\psi\}$ are consistent sets, where φ and ψ are arbitrary existential sentences of L, $T \cup \{\varphi \wedge \psi\}$ is also consistent.

Theorem 3 (Bryars). [18] The following are equivalent:

1) T has the amalgamation property;

2) for all $\alpha_1(\overline{x}), \alpha_2(\overline{x}) \in \forall_1$ with $T \vdash \alpha_1 \vee \alpha_2$ there are $\beta_1(\overline{x}), \beta_2(\overline{x}) \in \exists_1$ such that $T \vdash \beta_i \to \alpha_i$ $(i = 1, 2)$ and $T \vdash \beta_1 \vee \beta_2$.

The following theorem plays an important role in studying Jonsson Model Theory.

Theorem 4 (Hodges). [19; 363] Let T be a theory of the first-order language L, and let T have JEP. Suppose that A and B are existentially closed models of T. Then every $\forall\exists$-sentence true in A is true in B as well.

Now we recall the main definition of our study.

We are working within the framework of the following definition of Jonsson theory published in the Russian edition of [1].

Definition 3. [1; 80] A theory T is called Jonsson if the following conditions hold for T:

1. T has at least one infinite model;
2. T is an inductive theory;
3. T has the amalgam property (AP);
4. T has the joint embedding property (JEP).

There are a lot of algebraic examples of Jonsson theories. Classical examples include
1) group theory;
2) the theory of abelian groups;
3) the theory of Boolean algebras;
4) the theory of linear orders;
5) field theory of characteristic p, where p is zero or a prime number;
6) the theory of ordered fields;
7) the theory of modules.

In [20], it is proved that the theory of differentially closed fields of the fixed characteristic is a Jonsson theory as well.

It is important to note that, by Theorem 4, we can see that, for any Jonsson theory T, all existentially closed models of T are elementary equivalent by $\forall\exists$-sentences.

Further we give the notions and statements that are of great importance in research in Jonsson theories. Definitions 4, 5 and Theorem 5 were introduced by Mustafin T.G.

Definition 4. [21; 155] Let T be a Jonsson theory. A model C_T of power $2^{|T|}$ is called a semantic model of the theory T if C_T is a $|T|^+$-homogeneous $|T|^+$-universal model of the theory T.

Theorem 5. [21; 155] An inductive theory T is Jonsson iff it has a $|T|^+$-homogeneous $|T|^+$-universal model.

Definition 5. [21; 161] The elementary theory of the semantic model of the Jonsson theory T is called the center of this theory. The center is denoted by T^*, i.e. $Th(C) = T^*$.

Now we move to the central notion of this work.

Let L be a first-order language of a signature σ and let K be a class of L-structures. We consider a specific sets of theories for K that is called a Jonsson spectrum of K. The Jonsson spectrum can be described as follows.

Definition 6. [11] A set $JSp(K)$ of Jonsson theories of L, where

$$JSp(K) = \{T \mid T \text{ is a Jonsson theory and } K \subseteq Mod(T)\},$$

is said to be a Jonsson spectrum of K.

Jonsson spectra are well-described in [12, 22–24].

In terms of studying Jonsson theories, the notion of cosemanticness relation plays an important role. Let T_1 and T_2 be Jonsson theories, T_1^* and T_2^* be their centres, respectively.

Definition 7. [21; 40] T_1 and T_2 are said to be cosemantic Jonsson theories (denoted by $T_1 \bowtie T_2$), if $T_1^* = T_2^*$.

It is well known that the cosemanticness between two Jonsson theories is an equivalence relation. This means that, when introducing the relation of cosemanticness on the Jonsson spectrum $JSp(K)$, we get a partition of $JSp(K)$ into cosemanticness classes. The obtained factor set is denoted by $JSp(K)_{/\bowtie}$. This technique allows to obtain many significant generalizations when considering the cosemanticness classes instead of single theories. As it is mentioned before, applying this technique is the main idea of this paper, which will be revealed in Section 2.

2 *The properties of Kaiser hulls for J-classes*

In this section, we present the results of generalization of some well-known theorems published in different papers of the first author of this article. All of these theorems one can also find in [21].

Let T be a Jonsson theory in L, $K \subseteq E_T$. We consider the Jonsson spectrum of the given class K. Let us introduce the cosemanticness relation on $JSp(K)$. As it is well known, this relation is an equivalence relation, and therefore divides the spectrum into cosemanticness classes. Thus, we get a factor set $JSp(K)_{/\bowtie}$. Next, we will work with some fixed cosemanticness class $[T]$. It is clear that all the Jonsson theories in this class have the same semantic model, which we denote by $C_{[T]}$. In this section, we will work with this fixed class K, unless otherwise specified in the terms of the theorems or definitions.

Here we introduce the following notation. Let $T \in [T], [T] \in JSp(K)_{/\bowtie}$, A be an L-structure. Then $A \models [T]$ means that $A \models T$ for any $T \in [T]$. Similarly, this notation is generalized for the class of models as well, i.e. $K' \models [T]$ means that $A \models T$ for any $A \in K'$ and any theory $T \in [T]$.

Note that we only work within the framework of the fixed language L of signature σ.

Now we give the definitions of some necessary notions, which are actually generalizations of some well-known concepts from [21].

Definition 8. The class K' of existentially closed models of the signature σ is called a J-class, if the set of sentences $Th_{\forall\exists}(K')$ is a Jonsson theory.

Definition 9. The theory $Th_{\forall\exists}(K)$ that is a set of all $\forall\exists$-sentences of L true for each model of K, is said to be a Kaiser hull of K. We denote it by $T^0(K)$.

Note that the theory $T^0(K)$ is Jonsson, if it admits the amalgamation property, because it has infinite models, is inductive and, due to $\exists$-completeness, admits JEP. Moreover, in case of AP, $T^0(K)$ is a maximal Jonsson theory of K, and all theories T' such that $T \subseteq T' \subseteq T^0(K)$, where T is some Jonsson theory of K under consideration, are Jonsson.

Lemma 1. Let $[T] \in JSp(K)_{/\bowtie}$ consist only of $\exists$-complete theories, and let in $JSp(K)_{/\bowtie}$ there be such a class $[T']$, which consists of extensions of theories of the class $[T]$ in the same language. Then if $p(\bar{x}) \cup T$ is consistent for each theory $T \in [T]$, then $p(\bar{x}) \cup T'$ is also consistent for each theory $T' \in [T']$, where $p(\bar{x})$ is the set of $\exists$-formulas.

Proof. Let us consider an arbitrary theory $T \in [T]$. According to the condition of the Lemma, there is $T' \in [T']$ such that $T \subseteq T'$. It is obvious that if T is an $\exists$-complete theory, so is T'. Let $T \cup p(\bar{x})$ be a consistent set of formulas, for any $T \in [T]$, and $T' \cup p(\bar{x})$ be inconsistent, for any $T' \in [T']$. It means that there is $\exists\bar{y}\varphi(\bar{x}, \bar{y}) \in p(\bar{x})$, where $\varphi(\bar{x}, \bar{y})$ is a quantifier-free formula such that $T' \vdash \neg\exists\bar{x}\exists\bar{y}\varphi(\bar{x}, \bar{y})$. Consequently, $T' \vdash \forall\bar{x}\forall\bar{y}\neg\varphi(\bar{x}, \bar{y})$, and $T \vdash \forall\bar{x}\forall\bar{y}\neg\varphi(\bar{x}, \bar{y})$ as well, due to its $\forall$-completeness. The latter means that $T \cup p(\bar{x})$ is inconsistent, so we obtain a contradictory. Thus, for any theory $T \in [T]$ and any theory $T' \in [T']$, if $T \cup p(\bar{x})$ is consistent, $T' \cup p(\bar{x})$ is also consistent.

The following statement is one of the important properties of J-class.

Proposition 1. Let $[T'] \in JSp(K)_{/\bowtie}$ consist only of $\exists$-complete theories. Then any class $K' \subseteq K$ of infinite models is a J-class.

Proof. It is clear that K' is never empty, as soon as, by the conditions stated before, K consists of existentially closed models of some Jonsson theory T, which are infinite. We need to show that K' is a J-class, i.e., according to Definition 8, $Th_{\forall\exists}(K')$ is a Jonsson theory. Let us check it through Definition 3:

1) K' contains infinite models by the condition of the Proposition;

2) It is obvious that $Th_{\forall\exists}(K')$ is a set of $\forall\exists$-sentences, so this theory is inductive;

3) $Th_{\forall\exists}(K')$ is always an existentially complete theory, so it is easy to see that, by Theorem 1, it has JEP;

4) Let $Th_{\forall\exists}(K') \vdash \alpha_1 \vee \alpha_2$ for any L-formulas $\alpha_1(\overline{x}), \alpha_2(\overline{x}) \in \forall_1$. By JEP, it means that $Th_{\forall\exists}(K') \vdash \alpha_1$ or $Th_{\forall\exists}(K') \vdash \alpha_2$. Since every $T' \in [T']$ is an inductive theory, $T' \subseteq Th_{\forall\exists}(K')$, for all $T' \in [T']$. And due to the fact that each T' in this cosemanticness class is $\exists$-complete (and therefore $\forall$-complete), $T' \vdash \alpha_1$, if $Th_{\forall\exists}(K') \vdash \alpha_1$, and $T' \vdash \alpha_2$, if $Th_{\forall\exists}(K') \vdash \alpha_2$. Every theory in $[T']$ admits AP, so if $T' \vdash \alpha_1$ then $T' \vdash \alpha_1 \vee \alpha_1$ and, by Theorem 3, there are $\beta_1(\overline{x}), \beta_2(\overline{x}) \in \exists_1$ such that $T' \vdash \beta_i \to \alpha_i$ $(i = 1, 2)$ and

$T' \vdash \beta_1 \vee \beta_2$. The same is if $T' \vdash \alpha_2$. Therefore, $Th_{\forall\exists}(K') \vdash \beta_i \to \alpha_i$ $(i = 1, 2)$ and $Th_{\forall\exists}(K') \vdash \beta_1 \vee \beta_2$, which means that $Th_{\forall\exists}(K')$ admits AP.

To prove some further theorems, we need the following lemma.

Lemma 2. Let T_1 and T_2 be L-theories and let $T' = T_1 \vee T_2 = \{\varphi \vee \psi \mid \varphi \in T_1,\ \psi \in T_2\}$. Then $Mod(T') = Mod(T_1) \cup Mod(T_2)$.

Proof. Firstly, the inclusion $Mod(T_1) \cup Mod(T_2) \subseteq Mod(T')$ is true, as soon as all sentences of T' are deducible both in T_1 and T_2. Now we show the inclusion $Mod(T') \subseteq Mod(T_1) \cup Mod(T_2)$. Suppose that it is false; then there is a model $M \in Mod(T')$ such that $M \notin Mod(T_1) \cup Mod(T_2)$. It means that $M \notin Mod(T_1)$ and $M \notin Mod(T_2)$, which is equivalent to the fact that there are $\varphi \in T_1$ and $\psi \in T_2$ such that $M \nvDash \varphi$ and $M \nvDash \psi$. But according to the condition of the Lemma, for any model $M \in Mod(T')$, $M \vDash \varphi \vee \psi$ for all $\varphi \in T_1$ and $\psi \in T_2$ that is a contradiction. Hence $Mod(T') \subseteq Mod(T_1) \cup Mod(T_2)$ and $Mod(T') = Mod(T_1) \cup Mod(T_2)$.

Now we demonstrate the result that concerns to the lattices of Jonsson theories in terms of cosemanticness classes in the Jonsson spectrum.

Proposition 2. Let $K' \subseteq K$, $[T] \in JSp(K)_{/\bowtie}$, and let $C_{[T]}$ be a semantic model of $[T]$. Then $T' \in [T]$, where

$$T' = T^0(K') \vee T^0(C_{[T]}) = \{\varphi \vee \psi \mid \varphi \in T^0(K'),\ \psi \in T^0(C_{[T]})\}.$$

Proof. Firstly, we note that, according to Lemma 2, $Mod(T') = Mod(T^0(K')) \cup Mod(T^0(C_{[T]}))$. In addition, for any theory $T \in [T]$, $T \subseteq T'$, which means that T' is a Jonsson theory cosemantic to any $T \in [T]$. It remains to show that $T' \in JSp(K)$. Since $K \subseteq E_T$, $K' \subseteq K$, then $K' \equiv_{\forall\exists} K$, which means that $T^0(K) = T^0(K')$. Hence $T^0(K') \in JSp(K)$.

Now let us consider some specific relations between structures.

Definition 10. [21; 174] L-structures A and B are called Jonsson equivalent, if for any Jonsson theory T the following holds:

$$A \models T \Leftrightarrow B \models T.$$

Definition 11. [11] Structures A and B are called cosemantic, if $JSp(A) = JSp(B)$.

The following theorem also presents the result of structural approach in studying Jonsson theories and their cosemanticness classes.

Theorem 6. Let T be an arbitrary inductive L-theory such that $A \models T$ for any $A \in K$, where K is a class of infinite L-structures, and let the cosemanticness class $[T'] \in JSp(K)_{/\bowtie}$ consist only of $\exists$-complete theories. Then $[T''] \in JSp(K)_{/\bowtie}$, where

$$[T''] = \{T'' \mid T'' = T \cup T' \text{ for each } T' \in [T]\}.$$

Proof. Firstly, we should note that all theories of $[T'']$ are consistent, as soon as, for any $A \in K$, $A \models T'$ for each $T' \in [T']$, and $A \models T$, hence $A \models T''$ for any $T'' \in [T'']$. Let us consider an arbitrary $T' \in [T']$. It remains to show that $T'' = T' \cup T$ is a Jonsson theory. We do it by Definition 3.

1) All models in K are infinite, consequently T'' has infinite models;

2) Obviously, T'' is an inductive theory;

3) $T \subseteq T''$ and T is $\exists$-complete theory, hence T'' is $\exists$-complete as well. It means that T'' has JEP;

4) Here we use Theorem 3 again. Let $T'' \vdash \alpha_1 \vee \alpha_2$ for some L-formulas $\alpha_1(\overline{x}), \alpha_2(\overline{x}) \in \forall_1$. T'' has JEP, it means that $T'' \vdash \alpha_1$ or $T'' \vdash \alpha_2$. Since $T' \subseteq T''$ and T' is $\exists$-complete, $T' \vdash \alpha_1$, if $T'' \vdash \alpha_1$, and $T' \vdash \alpha_2$, if $T'' \vdash \alpha_2$. T' admits AP, so if $T' \vdash \alpha_1$ then $T' \vdash \alpha_1 \vee \alpha_1$ and, by Theorem 3, there are $\beta_1(\overline{x}), \beta_2(\overline{x}) \in \exists_1$ such that $T' \vdash \beta_i \to \alpha_i$ $(i = 1, 2)$ and $T' \vdash \beta_1 \vee \beta_2$. The same is if $T' \vdash \alpha_2$. Therefore, $T'' \vdash \beta_i \to \alpha_i$ $(i = 1, 2)$ and $T'' \vdash \beta_1 \vee \beta_2$, which means that T'' admits AP.

In [21], it was introduced the definition of cosemantic models. Now we give its analogue for classes of L-structures.

Definition 12. Let K_1 and K_2 be some classes of L-structures. Then K_1 and K_2 are said to be cosemantic ($K_1 \bowtie K_2$) if $JSp(K_1) = JSp(K_2)$.

Now we move to the main result of this paper. Theorem 7 is a criterion that connects the cosemanticness of J-classes with their Kaiser hulls.

Theorem 7. Let K_1, K_2 be J-classes. Then the following conditions are equivalent:
1) $K_1 \bowtie K_2$;
2) $T^0(K_1) = T^0(K_2)$.

Proof. Since K_1 and K_2 are J-classes, $T^0(K_1)$ and $T^0(K_2)$ are Jonsson theories. Let us prove (1) $\rightarrow$ (2). If $K_1 \bowtie K_2$, then $JSp(K_1) = JSp(K_2)$, which means that $T^0(K_1) \in JSp(K_2)$ and $T^0(K_2) \in JSp(K_1)$. But it follows that $T^0(K_1) \subseteq T^0(K_2)$ and $T^0(K_2) \subseteq T^0(K_1)$. Then $T^0(K_1) = T^0(K_2)$. The implication (2) $\rightarrow$ (1) is trivial due to inductiveness of Kaiser hulls and Jonsson theories.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Барвайс Дж. Теория моделей: справочная книга по математической логике. — Ч. 1 / Дж. Барвайс. — М.: Наука, 1982. — 392 с.

2 Yeshkeyev A.R. The J-minimal sets in the hereditary theories / A.R. Yeshkeyev, M.T. Omarova, G.E. Zhumabekova // Bulletin of the Karaganda University. Mathematics series. — 2019. — No. 2(94). — P. 92–98. https://doi.org/10.31489/2019m2/92-98

3 Poizat B. Back and Forth in Positive Logic / B. Poizat, A. Yeshkeyev // Logic in Question : book / ed. by J.-Y. Beziau, J.P. Descles, A. Moktefi, A.C. Pascu; Talks from the Annual Sorbonne Logic Workshop (2011–2019). — Birkhauser Cham, 2022. — P. 603—609. https://doi.org/10.1007/978-3-030-94452-0_31

4 Yeshkeyev A.R. Strongly minimal Jonsson sets and their properties / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2015. — No. 4(80). — P. 47–51.

5 Yeshkeyev A.R. An essential base of the central types of the convex theory / A.R. Yeshkeyev, M.T. Omarova // Bulletin of the Karaganda University. Mathematics series. — 2021. — No. 1(101). — P. 119–126. https://doi.org/10.31489/2021m1/119-126

6 Yeshkeyev A.R. On Jonsson varieties and quasivarieties / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2021. — No. 4(104). — P. 151–157. https://doi.org/10.31489/2021m4/151-157

7 Yeshkeyev A.R. Small models of hybrids for special subclasses of Jonsson theories / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University. Mathematics series. — 2019. — No. 3(95). — P. 68–73. https://doi.org/10.31489/2019m2/68-73

8 Yeshkeyev A.R. Forcing companions of Jonsson AP-theories / A.R. Yeshkeyev, I.O. Tungushbayeva, M.T. Omarova // Bulletin of the Karaganda University. Mathematics series. — 2022. — No. 3(107). — P. 163–173. https://doi.org/10.31489/2022m3/152-163

9 Yeshkeyev A.R. Existentially positive Mustafin theories of S-acts over a group / A.R. Yeshkeyev, O.I. Ulbrikht, A.R. Yarullina // Bulletin of the Karaganda University. Mathematics series. — 2022. — No. 2(106). — P. 172–185. https://doi.org/10.31489/2022m2/172-185

10 Ешкеев А.Р. JSp-косемантичность и JSB-свойство абелевых групп / А.Р. Ешкеев, О.И. Ульбрихт // Сиб. электрон. матем. изв. — 2016. — 13. — С. 861–874. https://doi.org/10.17377/semi.2016.13.068

11 Ешкеев А.Р. JSp-косемантичность R-модулей / А.Р. Ешкеев, О.И. Ульбрихт // Сиб. электрон. матем. изв. — 2019. — 16. — С. 1233–1244. https://doi.org/ 10.33048/semi.2019.16.084

12 Yeshkeyev A.R. Model-theoretical questions of the Jonsson spectrum / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2020. — No. 2(98). — P. 165–173. https://doi.org/10.31489/2020M2/165-173

13 Yeshkeyev A.R. Properties of lattices of the existential formulas of Jonsson fragments / A.R. Yeshkeyev, M.T. Kasymetova // Bulletin of the Karaganda University. Mathematics series. — 2015. — No. 3(79). — P. 25–32.

14 Yeshkeyev A.R. On lattice of existential formulas for fragment of Jonsson set / A.R. Yeshkeyev, O.I. Ulbrikht // Bulletin of the Karaganda University. Mathematics series. — 2015. — No. 3(79). — P. 33–39.

15 Yeshkeyev A.R. Model-theoretic properties of the #-companion of a Jonsson set / A.R. Yeshkeyev, M.T. Kassymetova, N.K. Shamatayeva // Eurasian Mathematical Journal. — 2018. — 9. — No. 2. — P. 68–81. https://doi.org/10.32523/2077-9879-2018-9-2-68-81

16 Yeshkeyev A.R. An algebra of the central types of the mutually model-consistent fragments / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University. Mathematics series. — 2021. — No. 1(101). — P. 111–118. https://doi.org/10.31489/2021m1/111-118

17 Yeshkeyev A.R. The properties of central types concerning enrichment by Jonsson set / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2017. — No. 1(85). — P. 36–40. https://doi.org/10.31489/2017m1/36-40

18 Bacsich P.D. Syntactic characterisations of amalgamation, convexity and related properties / P.D. Bacsich, D.R. Hughes // Journal of Symbolic Logic. — 1971. — 39. — No. 3. — P. 433–451.

19 Hodges W. Model theory / W. Hodges. — Cambridge University Press, 2008. — P. 788.

20 Yeshkeyev A.R. Connection between the amalgam and joint embedding properties / A.R. Yeshkeyev, I.O. Tungushbayeva, M.T. Kassymetova // Bulletin of the Karaganda University. Mathematics series. — 2022. — No. 1(105). — P. 127–135. https://doi.org/10.31489/2022m1/127-135

21 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во Караганд. гос. ун-та, 2016. — 370 с.

22 Yeshkeyev A.R. Existentially prime Jonsson quasivarieties and their Jonsson spectra / A.R. Yeshkeyev, I.O. Tungushbayeva, S.M. Amanbekov // Bulletin of the Karaganda University. Mathematics series. — 2022. — No. 4(108). — P. 117–124. https://doi.org/10.31489/2022m4/117-124

23 Yeshkeyev A.R. On Robinson spectrum of the semantic Jonsson quasivariety of unars / A.R. Yeshkeyev, A.R. Yarullina, M.T. Kassymetova // Bulletin of the Karaganda University. Mathematics series. — 2023. — No. 2(110). — P. 169-178. https://doi.org/10.31489/2023m2/169-178

24 Yeshkeyev A.R. The Number of Fragments of the Perfect Class of the Jonsson Spectrum / A.R. Yeshkeyev, O.I. Ulbrikht, M.T. Omarova // Lobachevskii Journal of Mathematics. — 2023. — 43. — No. 12. — P. 3658–3673.

Бекітілген модельдер класының Кайзер қабықшасының косемантикалылығы

А. Р. Ешкеев, И. О. Тунгушбаева, А. К. Кошекова

Қолданбалы математика институты, Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан

Мақалада йонсондық теорияны зерттеу барысында бекітілген тілде кейбір йонсондық теорияның экзистенциалды тұйық модельдерінің ішкі класының косемантикалылық класының модельді-теориялық қасиеттері зерттелді. Әртүрлі нәтижелер алынды. Сондай-ақ, модельдер мен модельдердің кластарының косемантикалылық қасиеттері қарастырылды; экзистенциалды тұйық модельдер класының жағдайында жалпылаудағы йонсондық эквиваленттікке қатысты кейбір нәтижелер алынды; J-класының Кайзер қабықшасына байланысты косемантикалылық критерийі табылды.

Кілт сөздер: йонсондық теория, косемантикалылық, косемантты йонсондық теориялар, йонсондық спектр, косемантикалылық кластары, Кайзер қабықшасы, йонсондық эквиваленттілік, J-класс, модельдердің косемантикалылығы, кластардың косемантикалылығы.

Косемантичность оболочек Кайзера фиксированных классов моделей

А.Р. Ешкеев, И.О. Тунгушбаева, А.К. Кошекова

Институт прикладной математики, Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан

В статье в рамках изучения йонсоновских теорий были рассмотрены теоретико-модельные свойства классов косемантичности, принадлежащих фактор-множеству йонсоновского спектра подкласса экзистенциально замкнутых моделей некоторой йонсоновской теории на фиксированном языке. Получены различные результаты. В частности, изучены свойства косемантичности моделей и классов моделей; получены некоторые результаты, касающиеся йонсоновской эквивалентности в обобщении на случаи классов экзистенциально замкнутых моделей; найден критерий косемантичности J-классов в связи с их оболочками Кайзера.

Ключевые слова: йонсоновская теория, косемантичные йонсоновские теории, йонсоновский спектр, классы косемантичности, оболочка Кайзера, йонсоновская эквивалентность, J-класс, косемантичность моделей, косемантичность классов.

References

1 Barwise, J. (1982). *Teoriia modelei: spravochnaia kniga po matematicheskoi logike. Chast 1 [Model theory: Handbook of mathematical logic. Part 1].* Moscow: Nauka [in Russian].

2 Yeshkeyev, A.R., Omarova, M.T., & Zhumabekova, G.E. (2019). The J-minimal sets in the hereditary theories. *Bulletin of the Karaganda University. Mathematics series,* 2(94), 92–98. https://doi.org/10.31489/2019m2/92-98

3 Poizat, B., & Yeshkeyev, A.R. (2022). Back and Forth in Positive Logic. *Logic in Question: book / ed. by J.-Y. Beziau, J.P. Descles, A. Moktefi, A.C. Pascu; Talks from the Annual Sorbonne Logic Workshop (2011–2019).* Birkhauser Cham, 603—609. https://doi.org/10.1007/978-3-030-94452-0_31

4 Yeshkeyev, A.R. (2015). Strongly minimal Jonsson sets and their properties. *Bulletin of the Karaganda University. Mathematics series,* 4(80), 47–51.

5 Yeshkeyev, A.R., & Omarova, M.T. (2021). An essential base of the central types of the convex theory. *Bulletin of the Karaganda University. Mathematics series,* 1(101), 119–126. https://doi.org/10.31489/2021m1/119-126

6 Yeshkeyev, A.R. (2021). On Jonsson varieties and quasivarieties. *Bulletin of the Karaganda University. Mathematics series,* 4(104), 151–157. https://doi.org/10.31489/2021m4/151-157

7 Yeshkeyev, A.R., & Mussina, N.M. (2019). Small models of hybrids for special subclasses of Jonsson theories. *Bulletin of the Karaganda University. Mathematics series,* 3(95), 68–73. https://doi.org/10.31489/2019m2/68-73

8 Yeshkeyev, A.R., Tungushbayeva, I.O., & Omarova, M.T. (2022). Forcing companions of Jonsson AP-theories. *Bulletin of the Karaganda University. Mathematics series,* 3(107), 163–173. https://doi.org/10.31489/2022m3/152-163

9 Yeshkeyev, A.R., Ulbrikht O.I., & Yarullina, A.R. (2022). Existentially positive Mustafin theories of S-acts over a group. *Bulletin of the Karaganda University. Mathematics series,* 2(106), 172–185. https://doi.org/10.31489/2022m2/172-185

10 Yeshkeyev, A.R., & Ulbrikht, O.I. (2016). JSp-kosemantichnost i JSB-svoistvo abelevyh grupp [JSp-cosemanticness and JSB-property of abelian groups]. Sibirskie elektronnye matematicheskie izvestiia *Siberian Electronic Mathematical Reports*, 13, 861–874. https://doi.org/10.17377/semi.2016.13.068 [in Russian].

11 Yeshkeyev, A.R., & Ulbrikht, O.I. (2019). JSp-kosemantichnost R-modulei [JSp-cosemanticness of R-modules]. Sibirskie Elektronnye Matematicheskie Izvestiia *Siberian Electronic Mathematical Reports*, 16, 1233–1244 [in Russian]. https://doi.org/ 10.33048/semi.2019.16.084

12 Yeshkeyev, A.R. (2020). Model-theoretical questions of the Jonsson spectrum. *Bulletin of the Karaganda University. Mathematics series,* 2(98), 165–173. https://doi.org/10.31489/2020M2/165-173

13 Yeshkeyev, A.R., & Kasymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University. Mathematics series,* 3(79), 25–32.

14 Yeshkeyev, A.R., & Ulbrikht, O.I. (2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University. Mathematics series,* 3(79), 33–39.

15 Yeshkeyev, A.R., Kassymetova, M.T., & Shamatayeva, N.K. (2018). Model-theoretic properties of the #-companion of a Jonsson set. *Eurasian Mathematical Journal, 9*(2), 68–81. https://doi.org/10.32523/2077-9879-2018-9-2-68-81

16 Yeshkeyev, A.R., & Mussina, N.M. (2021). An algebra of the central types of the mutually model-consistent fragments. *Bulletin of the Karaganda University. Mathematics series,* 1(101), 111–118. https://doi.org/10.31489/2021m1/111-118

17 Yeshkeyev, A.R. (2017). The properties of central types concerning enrichment by Jonsson set. *Bulletin of the Karaganda University. Mathematics series,* 1(85), 36–40. https://doi.org/10.31489/2017m1/36-40

18 Bacsich, P.D. (1971). Syntactic characterisations of amalgamation, convexity and related properties. *Journal of Symbolic Logic, 39*(3), 433–451.

19 Hodges, W. (2008). *Model theory.* Cambridge University Press.

20 Yeshkeyev, A.R., Tungushbayeva, I.O., & Kassymetova, M.T. (2022). Connection between the amalgam and joint embedding properties. *Bulletin of the Karaganda University. Mathematics series,* 1(105), 127–135. https://doi.org/10.31489/2022m1/127-135

21 Yeshkeyev, A.R. (2016). *Ionsonovskie teorii i ikh klassy modelei: monografiia [Jonsson theories and theirs classes of models: monograph].* Karaganda: Izdatelstvo Karagandinskogo gosudarstvennogo universiteta [in Russian].

22 Yeshkeyev, A.R., Tungushbayeva, I.O., & Amanbekov, S.M. (2022). Existentially prime Jonsson quasivarieties and their Jonsson spectra. *Bulletin of the Karaganda University. Mathematics series,* 4(108), 117–124. https://doi.org/10.31489/2022m4/117-124

23 Yeshkeyev, A.R.,Yarullina, A.R., & Kassymetova, M.T. (2023). On Robinson spectrum of the semantic Jonsson quasivariety of unars. *Bulletin of the Karaganda University. Mathematics series,* 2(110), 169–178. https://doi.org/10.31489/2023m2/169-178

24 Yeshkeyev, A.R., Ulbrikht, O.I., & Omarova, M.T. (2023). The Number of Fragments of the Perfect Class of the Jonsson Spectrum. *Lobachevskii Journal of Mathematics, 43*(12), 3658–3673.

*Author Information**

Aibat Rafhatovich Yeshkeyev — Doctor of physical and mathematical sciences, Professor, Professor of the Department of Algebra, Mathematic Logic and Geometry named after Professor T.G. Mustafin, Karaganda Buketov University, 28a Universitetskaya Street, Karaganda, 100028, Kazakhstan; e-mail: *aibat.kz@gmail.com*; https://orcid.org/0000-0003-0149-6143

Indira Orazbekovna Tungushbayeva (*corresponding author*) — Master of natural sciences, PhD student of Karaganda Buketov University, 28a Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *intng@mail.ru*; https://orcid.org/0000-0002-0432-9917

Aziza Kairatovna Koshekova — Master of pedagogical sciences, PhD student of Karaganda Buketov University, 28a Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *koshekova1998@mail.ru*.

*The author's name is presented in the order: First, Middle and Last Names.

ERRATA

https://doi.org/10.31489/2024M1/218

Erratum to: "Coefficients of multiple Fourier-Haar series and variational modulus of continuity" [*Bulletin of the Karaganda University. Mathematics series,* No. 4(112), 2023, pp. 21–29]

T.B. Akhazhanov*, N.A. Bokayev, D.T. Matin

L.N. Gumilyov Eurasian National University, Astana, Kazakhstan
(E-mail: talgat_a2008@mail.ru, bokayev2011@yandex.ru, d.matin@mail.ru)

Prof. Tuncay Aktosun requests retraction of his co-authorship for the following reason.

Prof. Tuncay Aktosun has not been involved in the publication process and has not participated in any discussions, therefore, he cannot hold responsibility for any results in the published paper.

*Corresponding author. *E-mail: talgat_a2008@mail.ru*

The Original Article was published on 29 December 2023: https://doi.org/10.31489/2023m4/21-29

ANNIVERSARIES

75th anniversary of Doctor of Physical and Mathematical Sciences, Professor M.I. Ramazanov

Murat Ibraevich Ramazanov, Doctor of Physical and Mathematical Sciences, Professor, was born on February 24, 1949, in Bulaevo, North Kazakhstan region. After graduating from the Faculty of Mechanics and Mathematics at the Kazakh State University named after S.M. Kirov (now Al-Farabi Kazakh National University) in 1971, he began his career at Karaganda State University and continues to make significant contributions to the development of the university to this day. He defended his PhD thesis in 1981 and his Doctoral thesis in 2006. M.I. Ramazanov is a highly qualified specialist in the field of loaded partial differential equations, integral equations, and their applications to applied problems. He regularly presents scientific reports at international congresses, conferences, and symposiums dedicated to discussing contemporary problems of mathematics, held both in the Republic of Kazakhstan and abroad. The scientific achievements of M.I. Ramazanov were published in high-ranking journals indexed in Web of Science and Scopus. Professor M.I. Ramazanov described the resolvent set and the spectrum for the spectrally loaded parabolic operator in terms of the (complex) spectral parameter, which is the coefficient of the loaded term, and characterized the multiplicity of eigenfunctions in the space of bounded and continuous functions depending on the value of the spectral parameter. Based on these studies, he published the monograph "Loaded Equations as Perturbations of Differential Equations".

Murat Ibraevich's contribution to the training of scientific personnel is invaluable. Under his supervision, 3 PhD theses and 5 doctoral dissertations were defended. Currently, M.I. Ramazanov is the scientific supervisor of PhD doctoral candidates in the field of "Mathematics". His name is also associated with scientific research conducted within the framework of grant funding from the Ministry of Education and Science of the Republic of Kazakhstan. For many years, he served as the scientific editor of the journal "Bulletin of the Karaganda University. Mathematics series", chairman of the Dissertation Council in the specialty 6D060100 – "Mathematics" at the Karaganda State University named after academician E.A. Buketov, and is currently a member of the editorial boards of the journals "Bulletin of KRASEC. Physical and Mathematical Sciences" (Russia), "Bulletin of Karaganda University. Mathematics series", and deputy chairman of the dissertation council for the educational program 8D05401 – Mathematics for the defense of PhD at the Karaganda Buketov University.

For his significant contribution to science, M.I. Ramazanov was awarded the State Scientific Scholarship for scientists and specialists who have made an outstanding contribution to the development of science and technology for 2008–2010, was decorated with the "For Merits in the Development of Science of the Republic of Kazakhstan" badge, twice received the grant for the title of "Best University Teacher" (2009, 2020). He is a laureate of the Prize named after D.Sc, Professor T.G. Mustafin, an honored worker of the Karaganda Buketov University; he has received several honorary diplomas from the akim of the Karaganda region "For active participation in the socio-political life of the region and personal labor contribution to the construction of a new Kazakh society", the National Chamber of Entrepreneurs of the Republic of Kazakhstan "For great merits to Kazakh science and invaluable contribution to the development of higher education, training of highly professional specialists for the Republic of Kazakhstan". M.I. Ramazanov is included in the TOP-50 of the General Rating of TPS of universities of the Republic of Kazakhstan (National Rating of Demand for Universities of the Republic of Kazakhstan, Astana).

The editorial board of the scientific journal "Bulletin of the Karaganda University. Mathematics series" and the faculty of mathematics and information technologies of the Karaganda Buketov University warmly congratulate Murat Ibraevich with his 75th anniversary and wish him strong health and creative longevity.

Editorial board of the journal
«Bulletin of the Karaganda University. Mathematics series»

EVENTS

The 15th International ISAAC Congress

Dear Colleagues,

We are pleased to announce that Nazarbayev University in Astana, Kazakhstan, will host the 15th International ISAAC Congress from July 21–25, 2025. The International Society for Analysis, its Applications, and Computation (ISAAC) Congress is a prestigious event that continues a successful series of meetings previously held across the globe.

The congress will cover a wide range of topics, including but not limited to:

1. Application of Dynamical Systems Theory in Biology
2. Complex Analysis and Partial Differential Equations
3. Complex Variables and Potential Theory
4. Constructive Methods in Boundary Value Problems and Applications
5. Function Inequalities: New Perspectives and New Applications
6. Function Spaces and their Applications to Nonlinear Evolutional Equations
7. Fractional Calculus and Fractional Differential Equations
8. Generalized Functions and Applications
9. Harmonic Analysis and Partial Differential Equations
10. Integral Transforms and Reproducing Kernels
11. Partial Differential Equations on Curved Spacetimes
12. Pseudo Differential Operators
13. Quaternionic and Clifford Analysis
14. Recent Progress in Evolution Equations
15. Wavelet Theory and its Related Topics

The conference will feature plenary and sectional talks, as well as poster presentations. The official language of the conference is English. We plan to publish the abstracts prior to the conference's commencement.

Registration Fees:

- ISAAC Members:
  - Before April 30, 2025: 150 EUR or 73,155 KZT
  - From May 1, 2025: 200 EUR or 97,540 KZT
- Non-Members:
  - Before April 30, 2025: 200 EUR or 97,540 KZT
  - From May 1, 2025: 250 EUR or 121,925 KZT
- Students and Participants from Developing Countries:
  - Before April 30, 2025: 80 EUR or 39,016 KZT
  - From May 1, 2025: 130 EUR or 63,400 KZT

Further Information:

Details on registration procedures, abstract submission guidelines, accommodation options in Astana, and information about the Programme and Organizing Committees, as well as invited speakers, will be announced in due course.

Contact Information:

Organizing Committee, School of Science and Humanities, Nazarbayev University, Qabanbay Batyr Ave 53, Astana 010000, Kazakhstan.
Email: info@isaac2025.org
Website: https://isaac2025.org/

Important Dates:

- Registration and abstract submission deadline: May 1, 2025
- Arrival day: July 20, 2025
- Departure day: July 26, 2025

We encourage you to share this information with interested colleagues. We look forward to welcoming you to Astana for an engaging and fruitful congress.

Warm regards,

Prof. Durvudkhan Suragan, Nazarbayev University, Chairman of Organizing Committee
Dr. Bolys Sabitbek, Queen Mary University of London, Member of Organizing Committee