ISSN 2518-7929 (Print)
ISSN 2663-5011 (Online)

BULLETIN
OF THE KARAGANDA UNIVERSITY

MATHEMATICS
Series

No. 4(112)/2023

ISSN 2518-7929 (Print)
ISSN 2663–5011(Online)
Индексі 74618
Индекс 74618

ҚАРАҒАНДЫ УНИВЕРСИТЕТІНІҢ **ХАБАРШЫСЫ**

ВЕСТНИК КАРАГАНДИНСКОГО УНИВЕРСИТЕТА

BULLETIN OF THE KARAGANDA UNIVERSITY

МАТЕМАТИКА сериясы

Серия МАТЕМАТИКА

MATHEMATICS Series

No. 4(112)/2023

1996 жылдан бастап шығады
Издается с 1996 года
Founded in 1996

Жылына 4 рет шығады
Выходит 4 раза в год
Published 4 times a year

Қарағанды / Караганда / Karaganda
2023

Postal address: 28, University Str., Karaganda, 100024, Kazakhstan
Tel.: (7212) 77-04-38 (add. 1026); fax: (7212) 35-63-98.

E-mail: vestnikku@gmail.com. Web-site: mathematics-vestnik.ksu.kz

Bulletin of the Karaganda University. Mathematics series.
ISSN 2518-7929 (Print). ISSN 2663–5011 (Online).

Proprietary: NLC «Karagandy University of the name of academician E.A. Buketov».

Registered by the Ministry of Information and Social Development of the Republic of Kazakhstan. Rediscount certificate No. KZ43VPY00027385 dated 30.09.2020.

Signed in print 29.12.2023. Format 60×84 1/8. Offset paper. Volume 25,37 p.sh. Circulation 200 copies. Price upon request. Order № 114.

Printed in the Publishing house of NLC «Karagandy University of the name of academician E.A. Buketov».
28, University Str., Karaganda, 100024, Kazakhstan. Tel.: (7212) 35-63-16. E-mail: izd_kargu@mail.ru

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DOI 10.31489/2023M4/5-20

UDC 517.925

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Well-posedness criteria for one family of boundary value problems

This paper considers a family of linear two-point boundary value problems for systems of ordinary differential equations. The questions of existence of its solutions are investigated and methods of finding approximate solutions are proposed. Sufficient conditions for the existence of a family of linear two-point boundary value problems for systems of ordinary differential equations are established. The uniqueness of the solution of the problem under consideration is proved. Algorithms for finding an approximate solution based on modified of the algorithms of the D.S. Dzhumabaev parameterization method are proposed and their convergence is proved. According to the scheme of the parameterization method, the problem is transformed into an equivalent family of multipoint boundary value problems for systems of differential equations. By introducing new unknown functions we reduce the problem under study to an equivalent problem, a Volterra integral equation of the second kind. Sufficient conditions of feasibility and convergence of the proposed algorithm are established, which also ensure the existence of a unique solution of the family of boundary value problems with parameters. Necessary and sufficient conditions for the well-posedness of the family of linear boundary value problems for the system of ordinary differential equations are obtained.

Keywords: Family of linear boundary value problems, multipoint boundary value problem, existence of solution, singular solution, well-posedness, necessary and sufficient condition.

Introduction

Problem statement and research methods

This paper is devoted to the study of a family of linear boundary value problems for differential equations

$$\frac{\partial v}{\partial t} = A(x,t)v + f(x,t), \qquad (x,t) \in [0,\omega] \times (0,T), \tag{1}$$

$$B_1(x)v(x,0) + B_2(x)v(x,T) = d(x), \qquad x \in [0,\omega], \tag{2}$$

where $(n \times n)$-matrix $A(x,t)$ e n-vector-function $f(x,t)$ are continuous on $[0,\omega] \times [0,T]$, $B_1(x)$, $B_2(x)$ and n-vector-function $d(x)$ are continuous on $[0,\omega]$, x is a parameter of the family ($x \in [0,\omega]$); $\|A(x,t)\| \le a_0$, $\|v(x,t)\| = \max\limits_{i=\overline{1,n}} \|v_i(x,t)\|$.

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In the present paper problem (1), (2) is investigated by the parameterization method [1].

The originallity of the parameterization method lies in the simple idea of introducing parameters at some points of the set on which the boundary value problem is considered, which subsequently allows us to construct an algorithm for finding a solution, obtain sufficient solvability conditions, establish solvability criteria for linear and nonlinear two-point boundary value problems, multipoint boundary value problems, boundary value problems with impulse influence, singular boundary value problems, nonlocal boundary value problems for differential equations, loaded differential equations, integro-differential Fredholm equations, differential equations with delayed argument, partial differential equations and others. These results are presented in the works of Dzhumabaev and his students (Assanova [2], Temesheva [3–7], Orumbayeva [8–10], Uteshova [11, 12], Iskakova [13, 14], Imanchiyev [15, 16], Bakirova [17], Kadirbayeva [18], Tleulessova [19], Abildayeva [20], Abdimanapova [21]).

Dzhumabaev and Assanova [22] studied a nonlocal boundary value problem for systems of linear hyperbolic equations with mixed derivative. A special substitution allowed to reduce this problem to an equivalent boundary value problem, which can be considered as a family of two-point boundary value problems for systems of ordinary differential equations, where the spatial variable servers as a parameter of the family.

This approach can also be used to study the linear nonlocal boundary value problem for a system of partial differential equations ($m = 1, 2, \dots$)

$$\frac{\partial^{m+1}u}{\partial t \partial x^m} = A(x,t)\frac{\partial^m u}{\partial x^m} + f(x,t), \quad u \in \mathbb{R}^n, \quad (x,t) \in [0,\omega] \times (0,T),$$

$$\left.\frac{\partial^k u}{\partial x^k}\right|_{x=0} = \psi_k(t), \quad t \in [0,T], \quad k = 0,1,\dots,m-1, \quad \frac{\partial^0 u}{\partial x^0} = 0,$$

$$B_1(x)\left.\frac{\partial^m u(x,t)}{\partial x^m}\right|_{t=0} + B_2(x)\left.\frac{\partial^m u(x,t)}{\partial x^m}\right|_{t=T} = d(x).$$

This fact motivated us to investigate problem (1), (2).

In this paper problem (1), (2) is investigated by the parameterization method with a modified algorithm. Sufficient conditions for the existence of a unique solution are obtained. The well-posedness criteria for problem (1), (2) are established.

Notation

- N is a natural number;
- ν is a natural number;
- $\Omega_r = [0,\omega] \times [(r-1)h, rh)$, $h = T/N$, $r = \overline{1,N}$;
- $C([0,\omega],\mathbb{R}^n)$ is the space of continuous functions $d : [0,\omega] \to \mathbb{R}^n$ with the norm $\|d\|_0 = \max\limits_{x\in[0,\omega]} \|d(x)\|$;
- $C([0,\omega] \times [0,T], \mathbb{R}^n)$ is the space of continuous functions $v : [0,\omega] \times [0,T] \to \mathbb{R}^n$ with the norm $\|v\|_1 = \max\limits_{(x,t)\in[0,\omega]\times[0,T]} \|v(x,t)\|$;
- the index r takes on the values $1, 2, \dots, N$;
- the index s takes on the values $1, 2, \dots, N+1$;
- $C([0,\omega]\times[0,T], \Omega_r, \mathbb{R}^{nN})$ is the space of systems of functions $v(x,[t]) = (v_1(x,t), v_2(x,t), \dots, v_N(x,t))$ with the norm $\|v\|_2 = \max\limits_{r=\overline{1,N}} \sup\limits_{(x,t)\in\Omega_r} \|v_r(x,t)\|$, where the function $v_r : \Omega_r \to \mathbb{R}^n$ is continuous and has a finite limit at $t \to rh - 0$ uniformly with respect to $x \in [0, \omega]$ for all r;
- $C([0, \omega], \mathbb{R}^{n(N+1)})$ is the space of functions $\lambda(x) = (\lambda_1(x), \lambda_2(x), \dots, \lambda_{N+1}(x))$ with the norm $\|\lambda\|_3 = \max\limits_{s=\overline{1,N+1}} \max\limits_{x\in[0,\omega]} \|\lambda_s(x)\|$, where $\lambda_s : [0,\omega] \to \mathbb{R}^n$ are continuous for all s;

- $C([0,T], \mathbb{R}^n)$ is the space of continuous functions $v : [0,T] \to \mathbb{R}^n$ with the norm $\|v\|_4 = \max\limits_{t\in[0,T]} \|v(t)\|$;
- I is the identity matrix of size n;
- O is the zero matrix of size $n \times n$;
- $O^{\langle 1\rangle}$ is the first column of the matrix O.

1 Solvability of a family problems (1), (2)

Definition 1. $v^*(x,t) \in C([0,\omega] \times [0,T], \mathbb{R}^n)$, continuously differentiable with respect to t and satisfying equation (1) and boundary conditions (2) for each fixed $x \in [0,\omega]$, is called a solution of the problem (1), (2).

Problem (1), (2) is investigated by the parameterization method [1]. For a fixed N, we make the partition $[0,\omega] \times [0,T) = \bigcup\limits_{r=1}^{N} \Omega_r$.

According to the scheme of the parameterization method, the problem (1), (2) is transformed into the equivalent family of multipoint boundary value problems with parameter for systems of differential equations

$$\frac{\partial \widetilde{v}_r}{\partial t} = A(x,t)(\widetilde{v}_r + \lambda_r(x)) + f(x,t), \tag{3}$$

$$\widetilde{v}_r(x,(r-1)h) = 0, \tag{4}$$

$$B_1(x)\lambda_1(x) + B_2(x)\lambda_{N+1}(x) = d(x), \tag{5}$$

$$\lambda_r(x) + \lim_{t\to rh-0} \widetilde{v}_r(x,t) - \lambda_{r+1}(x) = 0,\ r = \overline{1,N}, \tag{6}$$

where $(x,t) \in \Omega_r$, $x \in [0,\omega]$, $\lambda_r(x) = v(x,(r-1)h)$, $\lambda_{N+1}(x) = \lim\limits_{t\to T-0} v(x,t)$, $\widetilde{v}_r(x,t) = v(x,t) - \lambda_r(x)$, $r = \overline{1,N}$. A solution of problem (3)–(6) is a pair $(\lambda^*(x), \widetilde{v}^*(x,[t]))$ $\Big(\lambda^*(x) \in C([0,\omega], \mathbb{R}^{n(N+1)})$, $\widetilde{v}^*(x,[t]) \in C([0,\omega]\times[0,T], \Omega_r, \mathbb{R}^{nN})\Big)$ such that for each r is continuous and continuously differentiable with respect to t on Ω_r function $\widetilde{v}_r^*(x,t)$ at $\lambda_r(x) = \lambda_r^*(x)$ satisfies equation (3), condition (4), and $\lambda_1^*(x)$, $\lambda_{N+1}^*(x)$, $\lambda_r^*(x)$, $\lim\limits_{t\to rh-0} \widetilde{v}_r^*(x,t)$, satisfy (5), (6).

If the family of pairs $(\lambda^*(x), \widetilde{v}^*(x,[t]))$ is a solution of the family of problems (3)–(6), then the family of functions

$$v^*(x,t) = \begin{cases} \lambda_r^*(x) + \widetilde{v}_r^*(x,t) & \text{for} \quad (x,t) \in \Omega_r, \quad r = \overline{1,N}, \\ \lambda_{N+1}^*(x) & \text{for} \quad x \in [0,\omega], \quad t = T \end{cases}$$

is a solution to the family of boundary value problems (1), (2).

If the family of systems of functions $\widehat{v}(x,[t]) = (\widehat{v}_1(x,t), \widehat{v}_2(x,t), \ldots, \widehat{v}_N(x,t))$ is a solution to problem (1)-(2), then the solution to problem (3)–(6) is the pair $(\widehat{\lambda}(x), \widehat{\widetilde{v}}(x,[t]))$ with elements $\widehat{\lambda}(x) = (\widehat{\lambda}_1(x), \widehat{\lambda}_2(x), \ldots, \widehat{\lambda}_{N+1}(x))$, $\widehat{\lambda}_r(x) = \widehat{v}_r(x,(r-1)h)$, $r = \overline{1,N}$, $\widehat{\lambda}_{N+1}(x) = \lim\limits_{t\to T-0} \widehat{v}_N(x,t)$, $x \in [0,\omega]$, $\widehat{\widetilde{v}}(x,[t]) = (\widehat{\widetilde{v}}_1(x,t), \widehat{\widetilde{v}}_2(x,t), \ldots, \widehat{\widetilde{v}}_2(x,t))$, $\widehat{\widetilde{v}}_r = \widehat{v}_r(x,t) - \widehat{v}_r(x,(r-1)h)$, $(x,t) \in \Omega_r$, $r = \overline{1,N}$.

In problem (3)–(6), the initial conditions (4) appeared for elements of the family of systems of functions $\widetilde{v}(x,[t])$. For a known $\lambda_r(x)$, the Cauchy problem (3), (4) on Ω_r is equivalent to the family of Volterra integral equations of the second kind:

$$\widetilde{v}_r(x,t) = \int\limits_{(r-1)h}^{t} A(x,\tau)\widetilde{v}_r(x,\tau)d\tau + \int\limits_{(r-1)h}^{t} A(x,\tau)d\tau \cdot \lambda_r(x) + \int\limits_{(r-1)h}^{t} f(x,\tau)d\tau. \tag{7}$$

In (7), replacing $\widetilde{v}_r(x,\tau)$ by the right hand side of (7) and repeating this process ν times, we obtain the following representation of the function $\widetilde{v}_r(x,t)$:

$$\widetilde{v}_r(x,t) = D_{\nu,r}(x,t)\cdot\lambda_r(x) + F_{\nu,r}(x,t) + G_{\nu,r}(x,t,\widetilde{v}), \tag{8}$$

where

$$D_{\nu,r}(x,t) = \int\limits_{(r-1)h}^{t} A(x,\tau_1)d\tau_1 + \int\limits_{(r-1)h}^{t} A(x,\tau_1)\int\limits_{(r-1)h}^{\tau_1} A(x,\tau_2)d\tau_2 d\tau_1 + \ldots +$$
$$+\int\limits_{(r-1)h}^{t} A(x,\tau_1)\int\limits_{(r-1)h}^{\tau_1} A(x,\tau_2)\ldots\int\limits_{(r-1)h}^{\tau_{\nu-1}} A(x,\tau_\nu)d\tau_\nu\ldots d\tau_2 d\tau_1,$$

$$F_{\nu,r}(x,t) = \int\limits_{(r-1)h}^{t} f(x,\tau_1)d\tau_1 + \int\limits_{(r-1)h}^{t} A(x,\tau_1)\int\limits_{(r-1)h}^{\tau_1} f(x,\tau_2)d\tau_2 d\tau_1 + \ldots +$$
$$+\int\limits_{(r-1)h}^{t} A(x,\tau_1)\ldots\int\limits_{(r-1)h}^{\tau_{\nu-2}} A(x,\tau_{\nu-1})\int\limits_{(r-1)h}^{\tau_{\nu-1}} f(x,\tau_\nu)d\tau_\nu d\tau_{\nu-1}\ldots d\tau_1,$$

$$G_{\nu,r}(t,x,\widetilde{v}) = \int\limits_{(r-1)h}^{t} A(x,\tau_1)\ldots\int\limits_{(r-1)h}^{\tau_{\nu-1}} A(x,\tau_\nu)\widetilde{v}_r(x,\tau_\nu)d\tau_\nu\ldots d\tau_1,$$

$t\in[(r-1)h,rh)$, $r=\overline{1,N}$.

Determining from (8) the limits

$$\lim_{t\to rh-0}\widetilde{v}_r(x,t) = D_{\nu,r}(x,rh)\cdot\lambda_r(x) + F_{\nu,r}(x,rh) + G_{\nu,r}(rh,x,\widetilde{v}), \quad x\in[0,\omega], \quad r=\overline{1,N},$$

substituting them into (5), (6) and multiplying (5) by $h>0$, we obtain the family of systems of linear algebraic equations with respect to $\lambda_r(x)$, $x\in[0,\omega]$:

$$hB_1(x)\lambda_1(x) + hB_2(x)\lambda_{N+1}(x) = hd(x), \tag{9}$$

$$(I + D_{\nu,r}(x,rh))\lambda_r(x) - \lambda_{r+1}(x) = -F_{\nu,r}(x,rh) - G_{\nu,r}(rh,x,\widetilde{v}), \quad r=\overline{1,N}. \tag{10}$$

We write system (9), (10) in the form:

$$Q_\nu(h,x)\lambda(x) = -F_\nu(h,x) - G_\nu(h,x,\widetilde{v}), \quad \lambda(x)\in C([0,\omega],\mathbb{R}^{n(N+1)}),$$

where

$$Q_\nu(h,x) = \begin{pmatrix} hB_1(x) & O & O & \ldots & O & hB_2(x)\\ I+D_{\nu,1}(x,h) & -I & O & \ldots & O & O\\ O & I+D_{\nu,2}(x,2h) & -I & \ldots & O & O\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ O & O & O & \ldots & -I & O\\ O & O & O & \ldots & I+D_{\nu,N}(x,Nh) & -I \end{pmatrix},$$

$$F_\nu(h,x) = (-hd(x), F_{\nu,1}(h,x), F_{\nu,2}(2h,x), \dots, F_{\nu,N}(Nh,x)),$$
$$G_\nu(h,x,\widetilde{v}) = (O^{\langle 1\rangle}, G_{\nu,1}(h,x,\widetilde{v}), G_{\nu,2}(2h,x,\widetilde{v}), \dots, G_{\nu,N}(Nh,x,\widetilde{v})).$$

As can be seen, the process of finding a solution to problem (1), (2) is reduced to solving a family of systems of linear algebraic equations (10) for some $\widetilde{v}(x,[t])$ and solving the family of Cauchy problems (3), (4) on Ω_r when $\lambda_r(x)$, $r=\overline{1,N}$ is found.

Let us describe the algorithm for finding a solution to problem (3)–(6). Let the matrix $Q_\nu(h,x)$ be reversible for all $x \in [0,\omega]$.

Step 0. (a) The family of parameters $\lambda^{(1)}(x)$ is found from the equation $Q_\nu(h,x)\lambda(x) = -F_\nu(h,x)$.

(b) We determine the components of the system of functions $\widetilde{v}^{(0)}(x,[t])$ by solving the Cauchy problems (3), (4) on Ω_r at $\lambda_r(x) = \lambda_r^{(0)}(x)$, $r=\overline{1,N}$.

(c) On $[0,\omega]\times[0,T]$ we define the function

$$v^{(0)}(x,t) = \begin{cases} \lambda_r^{(0)}(x) + \widetilde{v}_r^{(0)}(x,t) & \text{for} \quad (x,t)\in\Omega_r, \quad r=\overline{1,N}, \\ \lambda_{N+1}^{(0)}(x) & \text{for} \quad x\in[0,\omega], \quad t=T. \end{cases}$$

Step 1. (a) The family of parameters $\lambda^{(1)}(x)$ is found from the equation $Q_\nu(h,x)\lambda(x) = -F_\nu(h,x) - G_\nu(h,x,\widetilde{v}^{(0)})$.

(b) We determine the components of the system of functions $\widetilde{v}^{(1)}(x,[t])$ by solving the Cauchy problems (3), (4) on Ω_r at $\lambda_r(x) = \lambda_r^{(1)}(x)$, $r=\overline{1,(N+1)}$.

(c) On $[0,\omega]\times[0,T]$ we define the function

$$v^{(1)}(x,t) = \begin{cases} \lambda_r^{(1)}(x) + \widetilde{v}_r^{(1)}(x,t) & \text{for} \quad (x,t)\in\Omega_r, \quad r=\overline{1,N}, \\ \lambda_{N+1}^{(1)}(x) & \text{for} \quad x\in[0,\omega], \quad t=T. \end{cases}$$

At the k-th step, we find the pair $(\lambda^{(k)}(x), \widetilde{v}^{(k)}(x,[t]))$, $k=0,1,2,\dots$. On $\bar{\Omega}$ we define the piecewise continuous function

$$v^{(k)}(x,t) = \begin{cases} \lambda_r^{(k)}(x) + \widetilde{v}_r^{(k)}(x,t) & \text{for} \quad (x,t)\in\Omega_r, \quad r=\overline{1,N}, \\ \lambda_{N+1}^{(k)}(x) & \text{for} \quad x\in[0,\omega], \quad t=T. \end{cases}$$

Condition 1. For some $h>0 : Nh=T$, ν and for any $x\in[0,\omega]$ the matrix $Q_\nu(h,x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible and the following inequalities are satisfied:

$$\|(Q_\nu(h,x))^{-1}\| \le \gamma_\nu(h,x) \le \gamma_\nu(h),$$
$$q_\nu(h) = \gamma_\nu(h)\Big\{e^{a_0h} - \sum_{j=0}^{\nu}\frac{(a_0h)^j}{j!}\Big\} < 1. \tag{11}$$

The following statement establishes sufficient conditions for the feasibility and convergence of the proposed algorithm. It should be noted that this statement ensures the existence of a unique solution of the family of boundary value problems with parameters (3)–(6).

Theorem 1. Let Condition 1 be met. Then the sequence of pairs $(\lambda^{(k)}(x), \widetilde{v}^{(k)}(x,[t]))$ converges to the unique solution $(\lambda^*(x), \widetilde{v}^*(x,[t]))$ of problem (3)–(6) and the following estimates hold true:

$$\|\lambda^* - \lambda^{(k)}\|_3 \leqslant \frac{q_\nu(h)}{1-q_\nu(h)}\|\lambda^{(k)} - \lambda^{(k-1)}\|_3, \tag{12}$$

$$\|\widetilde{v}_r^*(x,t) - \widetilde{v}_r^{(k)}(x,t)\| \le \big(e^{a_0(t-(r-1)h)} - 1\big)\|\lambda_r^*(x) - \lambda_r^{(k)}(x)\|, \tag{13}$$

where $k=1,2,\dots$, $(x,t)\in\Omega_r$, $r=\overline{1,N}$.

Proof. The continuity of the matrices $A(x,t)$ and $B_1(x)$, $B_2(x)$ on $[0,\omega]\times[0,T]$ and $[0,\omega]$, respectively, implies the continuity of the matrix $Q_\nu(h,x): \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ on $[0,\omega]$. Let us fix $\widetilde{x}$, $\widehat{x} \in [0,\omega]$. The matrix $(Q_\nu(h,x))^{-1} : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is continuous for all $x \in [0,\omega]$, since the inequality $\|(Q_\nu(h,\widetilde{x}))^{-1} - (Q_\nu(h,\widehat{x}))^{-1}\| \le \gamma_\nu^2(h))\|Q_\nu(h,\widehat{x}) - Q_\nu(h,\widetilde{x})\|$ holds.

The solution of problem (3)-(6) is found by the algorithm. Solving the equation $Q_\nu(h,x)\lambda(x) = -F_\nu(h,x)$, we find $\lambda^{(0)}(x)$. Since the matrix $(Q_\nu(h,x))^{-1}$ and the vector $F_\nu(h,x)$ are continuous for all $x \in [0,\omega]$, we have $\lambda^{(0)}(x) \in C([0,\omega],\mathbb{R}^{n(N+1)})$ and

$$\|\lambda^{(0)}\|_3 \le \gamma_\nu(h)h \max\Big\{1, \sum_{j=0}^{\nu-1} \frac{(a_0h)^j}{j!}\Big\} \max\{\|d\|_0, \|f\|_1\}.$$

For any r and $x \in [0,\omega]$, we find the function $\widetilde{v}_r^{(0)}(x,t)$ from the Cauchy problem (3), (4) with $\lambda_r(x) = \lambda_r^{(0)}(x)$:

$$\frac{\partial \widetilde{v}_r}{\partial t} = A(x,t)\widetilde{v}_r + A(x,t)\lambda_r^{(0)}(x) + f(x,t), \quad \widetilde{v}_r(x,(r-1)h) = 0, \quad r = \overline{1,N}.$$

Then for $\widetilde{v}_r^{(0)}(x,t)$ we have the estimate

$$\|\widetilde{v}_r^{(0)}(x,t)\| \le \big(e^{a_0(t-(r-1)h)} - 1\big)\|\lambda_r^{(0)}(x)\| + (t-(r-1)h)e^{a_0(t-(r-1)h)}\|f\|_1,$$

whence it follows that

$$\|\widetilde{v}^{(0)}\|_2 \le \big(e^{a_0h} - 1\big)\|\lambda^{(0)}\|_3 + he^{a_0h}\|f\|_1.$$

Then, following the algorithm, we solve the equation $Q_\nu(h,x)\lambda(x) = -F_\nu(h,x) - G_\nu(h,x,\widetilde{v}^{(0)})$ and find $\lambda^{(1)}(x)$. We have

$$\|\lambda^{(1)} - \lambda^{(0)}\|_3 = \| - (Q_\nu(h,x))^{-1} \cdot G_\nu(h,x,\widetilde{v}^{(0)})\| \le \gamma_\nu(h) \max_{r=\overline{1,N}} \|G_{\nu,r}(rh,x,\widetilde{v}^{(0)})\|\} \le$$

$$\le \gamma_\nu(h) \max_{r=\overline{1,N}} \Big\{ \int_{(r-1)h}^{rh} a_0 \ldots \int_{(r-1)h}^{\tau_{\nu-1}} a_0\|\widetilde{v}_r^{(0)}(x,\tau_\nu)\|d\tau_\nu \ldots d\tau_1\Big\} \le \gamma_\nu(h)\frac{(a_0h)^\nu}{\nu!}\|\widetilde{v}^{(0)}\|_2.$$

We define the components of the system of functions $\widetilde{v}^{(1)}(x,[t]) = (\widetilde{v}_1^{(1)}(x,t), \widetilde{v}_2^{(1)}(x,t), \ldots, \widetilde{v}_N^{(1)}(x,t))$ by solving the Cauchy problem (3), (4) with $\lambda_r(x) = \lambda_r^{(1)}(x)$:

$$\frac{\partial \widetilde{v}_r}{\partial t} = A(x,t)\widetilde{v}_r + A(x,t)\lambda_r^{(1)}(x) + f(x,t), \quad \widetilde{v}_r(x,(r-1)h) = 0, \quad r = \overline{1,N}.$$

The difference $(\widetilde{v}_r^{(1)}(x,t) - \widetilde{v}_r^{(0)}(x,t))$ is estimated as follows:

$$\|\widetilde{v}_r^{(1)}(x,t) - \widetilde{v}_r^{(0)}(x,t)\| \le \big(e^{a_0(t-(r-1)h)} - 1\big)\|\lambda_r^{(1)}(x) - \lambda_r^{(0)}(x)\|.$$

We assume that the pair $(\lambda^{(k-1)}(x), \widetilde{v}^{(k-1)}(x,[t]))$ is determined and for all $(x,t) \in \Omega_r$ the following inequalities hold:

$$\|\lambda^{(k-1)} - \lambda^{(k-2)}\|_3 \le q_\nu(h)\|\lambda^{(k-2)} - \lambda^{(k-3)}\|_3,$$

$$\|\widetilde{v}_r^{(k-1)}(x,t) - \widetilde{v}_r^{(k-2)}(x,t)\| \le \big(e^{a_0(t-(r-1)h)} - 1\big)\|\lambda_r^{(k-1)}(x) - \lambda_r^{(k-2)}(x)\|. \qquad (14)$$

At the k-th step of the algorithm, solving the equation $Q_\nu(h,x)\lambda(x) = -F_\nu(h,x) - G_\nu(h,x,\widetilde{v}^{(k-1)})$, we find $\lambda^{(k)}(x)$. Taking into account (14), we establish that

$$\|\lambda^{(k)} - \lambda^{(k-1)}\|_3 \le q_\nu(h)\|\lambda^{(k-1)} - \lambda^{(k-2)}\|_3, \quad k = 2,3,\ldots. \qquad (15)$$

We define the components of the system of functions $\widetilde{v}^{(k)}(x,[t]) = (\widetilde{v}_1^{(k)}(x,t), \widetilde{v}_2^{(k)}(x,t), \ldots, \widetilde{v}_N^{(k)}(x,t))$ by solving the Cauchy problem (3), (4) with $\lambda_r(x) = \lambda_r^{(k)}(x)$:

$$\frac{\partial \widetilde{v}_r}{\partial t} = A(x,t)\widetilde{v}_r + A(x,t)\lambda_r^{(k)}(x) + f(x,t), \quad \widetilde{v}_r(x,(r-1)h) = 0, \quad r = \overline{1,N}.$$

For all $(x,t) \in \Omega_r$, $r = \overline{1,N}$ $(k = 1,2,3,\ldots)$ we estimate the difference $(\widetilde{v}_r^{(k)}(x,t) - \widetilde{v}_r^{(k-1)}(x,t))$:

$$\|\widetilde{v}_r^{(k)}(x,t) - \widetilde{v}_r^{(k-1)}(x,t)\| \le \left(e^{a_0(t-(r-1)h)} - 1\right)\|\lambda_r^{(k)}(x) - \lambda_r^{(k-1)}(x)\|. \tag{16}$$

By the condition of Theorem, $q_\nu(h) < 1$, so it follows from (15), (16) that the pair $(\lambda^{(k)}(x), \widetilde{v}^{(k)}(x,[t]))$, $k = 0,1,2,\ldots$, converges to $(\lambda^*(x), \widetilde{v}^*(x,[t]))$, the solution of problem (3)–(6) in $C([0,\omega], \mathbb{R}^{n(N+1)}) \times C([0,\omega] \times [0,T], \Omega_r, \mathbb{R}^{nN})$.

It is not difficult to establish the validity of the inequalities:

$$\|\lambda^{(k+\ell)} - \lambda^{(k)}\|_3 \leqslant \frac{q_\nu(h)}{1 - q_\nu(h)}\|\lambda^{(k)} - \lambda^{(k-1)}\|_3, \tag{17}$$

$$\|\lambda^{(k)} - \lambda^{(0)}\|_3 \leqslant \frac{1 - q_\nu^k(h)}{1 - q_\nu(h)}\gamma_\nu(h)\frac{(a_0 h)^\nu}{\nu!}\|\widetilde{v}^{(0)}\|_2,$$

$$\|\widetilde{v}_r^{(k+\ell)}(x,t) - \widetilde{v}_r^{(k)}(x,t)\| \le \left(e^{a_0(t-(r-1)h)} - 1\right)\|\lambda_r^{(k+\ell)}(x) - \lambda_r^{(k)}(x)\|, \tag{18}$$

$$\|\widetilde{v}_r^{(k)}(x,t) - \widetilde{v}_r^{(0)}(x,t)\| \le \left(e^{a_0(t-(r-1)h)} - 1\right)\|\lambda_r^{(k)}(x) - \lambda_r^{(0)}(x)\|,$$

$(x,t) \in \Omega_r$, $r = \overline{1,N}$, $k = 1,2,\ldots$. In the inequalities (17), (18), letting $\ell \to \infty$, we establish the validity of the estimates (12), (13).

Let us show the uniqueness of the solution of problem (3)–(6). Let $v^*(x,t)$ and $\widehat{v}(x,t)$ be two solutions of problem (1), (2). Then the pairs $(\lambda^*(x), \widetilde{v}^*(x,[t]))$ and $(\widehat{\lambda}(x), \widehat{\widetilde{v}}(x,[t]))$ are solutions to the boundary value problem (3)–(6), here

$$\lambda^*(x) \in C([0,\,\omega],\, \mathbb{R}^{n(N+1)}), \quad \lambda_s^*(x) = v^*(x,(s-1)h), s = \overline{1,N+1},$$

$$\widetilde{v}_r^*(x,[t]) \in C([0,\omega] \times [0,T],\, \Omega_r, \mathbb{R}^{nN}),$$

$$\widetilde{v}_r^*(x,t) = v^*(x,t) - v^*(x,(r-1)h), \quad (x,t) \in \Omega_r, \quad r = \overline{1,N},$$

$$\widehat{\lambda}(x) \in C([0,\,\omega],\, \mathbb{R}^{n(N+1)}), \quad \widehat{\lambda}_s(x) = \widehat{v}(x,(s-1)h), \quad s = \overline{1,N+1},$$

$$\widehat{\widetilde{v}}(x,[t]) \in C([0,\omega] \times [0,T],\, \Omega_r, \mathbb{R}^{nN}),$$

$$\widehat{\widetilde{v}}_r(x,t) = \widehat{v}(x,t) - \widehat{v}(x,(r-1)h), \quad (x,t) \in \Omega_r, \quad r = \overline{1,N}.$$

Under our assumptions, the following equations hold:

$$\widetilde{v}_r^*(x,t) = \int\limits_{(r-1)h}^{t} A(x,\tau)\widetilde{v}_r^*(x,\tau)d\tau + \int\limits_{(r-1)h}^{t} A(x,\tau)d\tau \cdot \lambda_r^*(x) + \int\limits_{(r-1)h}^{t} f(x,\tau)d\tau,$$

$$\widehat{\widetilde{v}}_r(x,t) = \int\limits_{(r-1)h}^{t} A(x,\tau)\widehat{\widetilde{v}}_r(x,\tau)d\tau + \int\limits_{(r-1)h}^{t} A(x,\tau)d\tau \cdot \widehat{\lambda}_r(x) + \int\limits_{(r-1)h}^{t} f(x,\tau)d\tau,$$

$$Q_\nu^{-1}(h,x)\lambda^*(x) = -\left(F_\nu(h,x) + G_\nu(h,x,\widetilde{v}^*)\right),$$

$$Q_\nu^{-1}(h,x)\widehat{\lambda}(x) = -\big(F_\nu(h,x) + G_\nu(h,x,\widehat{\widetilde{v}})\big).$$

Then the following inequalities are true

$$\|\widetilde{v}^* - \widehat{\widetilde{v}}\|_2 \le \left(e^{a_0 h} - 1\right) \cdot \|\lambda^* - \widehat{\lambda}\|_3, \tag{19}$$

$$\|\lambda^* - \widehat{\lambda}\|_3 \le q_\nu(h)\|\lambda^* - \widehat{\lambda}\|_3.$$

Hence, by virtue of inequality (11), $\lambda^*(x) = \widehat{\lambda}(x)$. Then from (19) we obtain that $v^*(x,t) = \widehat{v}(x,t)$ for $(x,t) \in [0,\omega] \times [0,T]$. Theorem 1 is proved.

Since problem (1), (2) and problem (3)–(6) are equivalent, the following statement holds true.

Corollary 1. Let Condition 1 be met. Then the sequence $v^{(k)}(x,t)$ $(k = 0,1,2,\ldots)$ converges to the unique solution $v^*(x,t)$ of problem (1), (2) and the following estimates are true:

$$\|v^* - v^{(0)}\|_1 \le \frac{\gamma_\nu(h)e^{a_0 h}}{1 - q_\nu(h)} \cdot \frac{(a_0 h)^\nu}{\nu!} \left(\left(e^{a_0 h} - 1\right) \max_{s=\overline{1,N+1}} \max_{x\in[0,\omega]} \|v^{(0)}(x,(s-1)h)\| + h e^{a_0 h}\|f\|_1 \right).$$

2 *Well-posedness criteria for the family of problems* (1), (2)

Definition 2. The boundary value problem (1), (2) is called well-posed if for any $f(x,t) \in C([0,\omega] \times [0,T], \mathbb{R}^n)$, $d(x) \in C([0,\omega], \mathbb{R}^n)$ it has a unique solution $v(x,t)$ and

$$\|v\|_1 \le K \max\Big\{\|d\|_1, \|f\|_1\Big\},$$

where K is a constant, independent of $f(x,t)$ and $d(x)$. The number K is called the well-posedness constant of problem (1), (2).

Let us consider the equation

$$\frac{1}{h} Q_*(h,x)\lambda(x) = -F_*(h,A,f,d,x), \qquad \lambda(x) \in C([0,\omega], \mathbb{R}^{n(N+1)}),$$

where $Q_*(h,x) = \lim\limits_{\nu\to\infty} Q_\nu(h,x)$, $\quad F_*(h,A,f,d,x) = \lim\limits_{\nu\to\infty} \dfrac{1}{h} F_\nu(h,x)$.

Theorem 2. The boundary value problem (1), (2) is well-posed for all $x \in [0,\omega]$ if and only if there exists $h_0 \in (0,T]$ such that for any $h \in (0,h_0] : Nh = T$ there is a number $\nu = \nu(h)$, such that the matrix $Q_\nu(h,x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible and the following inequalities hold:

$$\|(Q_\nu(h,x))^{-1}\| \le \gamma_\nu(h), \tag{20}$$

$$q_\nu(h) = \gamma_\nu(h)\Big\{e^{a_0 h} - \sum_{j=0}^{\nu} \frac{(a_0 h)^j}{j!}\Big\} < 1. \tag{21}$$

Proof. The sufficiency of the conditions of Theorem 2 for the well-posedness of problem (1), (2) follows from Corollary 1.

Necessity. Let problem (1), (2) be well-posed with a constant K. Problem (1), (2) for every fixed $\widehat{x} \in [0,\omega]$ is a linear two-point boundary value problem for the ordinary differential equation:

$$\frac{d\widehat{v}}{dt} = \widehat{A}(t)\widehat{v} + \widehat{f}(t), \quad t \in (0,T), \quad \widehat{v} \in \mathbb{R}^n, \tag{22}$$

$$\widehat{B}_1\widehat{v}(0) + \widehat{B}_2\widehat{v}(T) = \widehat{d}. \tag{23}$$

Here $\widehat{v}(t) = v(\widehat{x}, t)$, $\widehat{A}(t) = A(\widehat{x}, t)$, $\widehat{f}(t) = f(\widehat{x}, t)$,$\widehat{B}_1 = B_1(\widehat{x})$, $\widehat{B}_2 = B_2(\widehat{x})$, $\widehat{d} = d(\widehat{x})$.

Since for $f(x,t) = \widehat{f}(t)$, $d(x) = \widehat{d}$ we have:

$$\|\widehat{v}^*\|_4 = \max_{t\in[0,T]} \|v^*(\widehat{x}, t)\| \leq \max_{(x,t)\in[0,\omega]\times[0,T]} \|v^*(x,t)\| \leq K \max\{\|d\|_0, f\|_1\} = K \max\{\|\widehat{d}\|, \|\widehat{f}\|_4\},$$

then the correct solvability of problem (1), (2) follows from the correct solvability of problem (22), (23) with constant K for every fixed $\widehat{x} \in [0,\omega]$.

For any $\varepsilon > 0$ there is $h_0 \in (0,T]$, satisfying the inequality

$$\frac{1}{a_0 h_0}(e^{a_0 h_0} - 1 - a_0 h_0) \leq \frac{\varepsilon}{(2+\varepsilon)(1+\varepsilon)}.$$

Then, by Theorem 3 [1; p. 42], we obtain the following estimate for all $h \in (0, h_0] : Nh = T$:

$$\|(Q_*(h, \widehat{x}))^{-1}\| \leqslant \frac{(1+\varepsilon)K}{h}.$$

In view of the arbitrariness of $\widehat{x} \in [0,\omega]$, we obtain

$$\|(Q_*(h, x))^{-1}\| \leqslant \frac{(1+\varepsilon)K}{h}, \quad \forall x \in [0,\omega].$$

Let us choose ν_1 such that:

$$\frac{2(1+\varepsilon)K}{h}\Big\{e^{a_0 h} - \sum_{j=0}^{\nu_1} \frac{(a_0 h)^j}{j!}\Big\} < 1.$$

For any ν, we have there is the inequality

$$\|Q_*(h,x) - Q_\nu(h,x)\| \leqslant \sum_{j=\nu+1}^{\infty} \frac{(a_0 h)^j}{j!} = \Big\{e^{a_0 h} - \sum_{j=0}^{\nu} \frac{(a_0 h)^j}{j!}\Big\}.$$

Then it follows from the theorem on small perturbations of boundedly invertible operators that for all $\nu \geq \nu_1$ the matrix $Q_\nu(h,x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible and

$$\|(Q_\nu(h,x))^{-1}\| \leqslant \frac{\|(Q_*(h,x))^{-1}\|}{1 - \|(Q_*(h,x))^{-1}\| \cdot \|Q_*(h,x) - Q_\nu(h,x)\|} < \frac{2(1+\varepsilon)K}{h}.$$

Thus, for all $\nu \geq \nu_1$, $h \in (0, h_0] : Nh = T$ and $x \in [0,\omega]$, taking $\gamma_\nu(h) = \dfrac{2(1+\varepsilon)K}{h}$, we obtain that the inequalities (20), (21). Theorem 2 is proved.

Theorem 3. The boundary value problem (1), (2) is well-posed if and only if for any ν there exists $h = h(\nu) : Nh = T$, such that the matrix $Q_\nu(h,x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible for all $x \in [0,\omega]$ and the inequalities (20), (21) are true.

Proof. Sufficiency. The well-posedness of problem (1), (2) under the conditions of Theorem follows from Corollary 1.

Necessity. Let the problem (1), (2) be well-posed with constant K. Reasoning as in the proof of Theorem 2, for a given $\varepsilon > 0$ we find $h_0 = h_0(\varepsilon)$ such that for all $h \in (0, h_0] : Nh = T$ and $x \in [0,\omega]$ the matrix $Q_*(h,x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible and

$$\|(Q_*(h,x))^{-1}\| \leqslant \frac{(1+\varepsilon)K}{h}.$$

We choose $h_1 \in (0, h_0]$ such that the relation is satisfied:

$$\frac{2(1+\varepsilon)K}{h_1}\Big\{e^{a_0 h_1} - \sum_{j=0}^{\nu}\frac{(a_0h_1)^j}{j!}\Big\} < 1. \tag{24}$$

Since $\|(Q_*(h,x))^{-1}\| \cdot \|Q_*(h,x) - Q_\nu(h,x)\| < 0.5$, then, by virtue of (24), by the small perturbation theorem of boundedly reversible operators, for all $h \in (0, h_1] : Nh = T$ and $x \in [0,\omega]$ the inequality holds $\|(Q_\nu(h,x))^{-1}\| < \dfrac{2(1+\varepsilon)K}{h}$.

Taking $\gamma_\nu(h) = \dfrac{2(1+\varepsilon)K}{h}$, by virtue of choosing $h \in (0, h_1] : Nh = T$, we obtain the fulfillment of inequalities (20) and (21). Theorem 3 is proved.

Theorem 4. Let for some ν there exist $h_0 = h_0(\nu)$ such that for all $h \in (0, h_0] : Nh = T$ and $x \in [0,\omega]$ the matrix $Q_\nu(h,x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible and

$$\|(Q_\nu(h,x))^{-1}\| \le \frac{\gamma}{h},$$

where γ is a constant, independent of h and x. Then problem (1), (2) is well-posed with constant $K = \gamma$.

Proof. For any $\varepsilon > 0$ there is $h_0 \in (0, T]$ satisfying the inequality

$$\frac{1}{a_0h_0}(e^{a_0h_0} - 1 - a_0h_0) \le \frac{\varepsilon}{(2+\varepsilon)(1+\varepsilon)}.$$

We choose $h_1 \in (0, h_0] : Nh_1 = T$ such that the following inequality is satisfied:

$$\frac{\gamma}{h_1}\Big\{e^{a_0h_1} - \sum_{j=0}^{\nu}\frac{(a_0h_1)^j}{j!}\Big\} < 1.$$

Then $q_\nu(h) \le q_\nu(h_1) < 1$ for all $h \in (0, h_1] : Nh = T$ and, by Corollary 1, the problem (1), (2) has a unique solution $v^*(x,t)$ and

$$\max_{(x,t)\in[0,\omega]\times[0,T]}\|v^*(x,t)\| \le e^{a_0h}\Big(\Big(\frac{\gamma}{1-q_\nu(h)}\cdot\frac{(a_0h)^\nu}{\nu!}\cdot\frac{e^{a_0h}-1}{h} + 1\Big)\times$$
$$\times\gamma\max\Big\{1, \sum_{j=0}^{\nu-1}\frac{(a_0h)^j}{j!}\Big\} + \frac{\gamma}{1-q_\nu(h)}\frac{(a_0h)^\nu}{\nu!}e^{a_0h}\Big)\max\{\|d\|_0, \|f\|_1\} + he^{a_0h}\|f\|_1.$$

Letting $h \to 0$ in the above inequality, we obtain that

$$\max_{(x,t)\in[0,\omega]\times[0,T]}\|v^*(x,t)\| \le \gamma\max\{\|d\|_0, \|f\|_1\}.$$

Theorem 4 is proved.

Theorem 5. Let problem (1), (2) be well-posed with constant K. Then for any ν and $\varepsilon > 0$ there exists $h_0 = h_0(\nu, \varepsilon)$ such that for all $h \in (0, h_0] : Nh = T$ and $x \in [0,\omega]$ the matrix $Q_\nu(h,x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible and

$$\|(Q_\nu(h,x))^{-1}\| \le \frac{(1+\varepsilon)K}{h}.$$

Proof. For a given $\varepsilon > 0$, find $h_0 = h_0(\varepsilon)$ such that for all $h \in (0, h_0] : Nh = T$ and $x \in [0, \omega]$ the matrix $Q_*(h, x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible and the following estimate holds true:

$$\|(Q_*(h,x))^{-1}\| \leqslant \frac{(2+\varepsilon)K}{2h}.$$

Let us choose $h_1 \in (0, h_0]$ satisfying the inequality:

$$\frac{(2+\varepsilon)K}{h_1}\Big\{e^{a_0 h_1} - \sum_{j=0}^{\nu} \frac{(a_0 h_1)^j}{j!}\Big\} < \frac{\varepsilon}{1+\varepsilon}.$$

Since $\|(Q_*(h,x))^{-1}\| \cdot \|Q_*(h,x) - Q_\nu(h,x)\| < \dfrac{1}{2} \cdot \dfrac{\varepsilon}{1+\varepsilon}$ then, the theorem on small perturbations of boundedly invertible operators, for all $h \in (0, h_1] : Nh = T$ and $x \in [0, \omega]$ the following estimate holds $\|(Q_\nu(h,x))^{-1}\| < \dfrac{(1+\varepsilon)K}{h} = \gamma_\nu(h)$ and, based on (24),

$$q_\nu(h) = \gamma_\nu(h)\Big\{e^{a_0 h} - \sum_{j=0}^{\nu} \frac{(a_0 h)^j}{j!}\Big\} < \frac{\varepsilon}{2+\varepsilon} < 1.$$

Then, according Corollary 1, there exists a unique solution $v^*(x,t)$ of problem (1), (2) and the following estimate holds:

$$\max_{(x,t)\in[0,\omega]\times[0,T]} \|v^*(x,t)\| \le e^{a_0 h}\Big(\Big(\frac{(1+\varepsilon)K}{1-q_\nu(h)} \cdot \frac{(a_0 h)^\nu}{\nu!} \cdot \frac{e^{a_0 h}-1}{h} + 1\Big)(1+\varepsilon)K \times$$
$$\times \max\Big\{1, \sum_{j=0}^{\nu-1} \frac{(a_0 h)^j}{j!}\Big\} + \frac{(1+\varepsilon)K}{1-q_\nu(h)} \frac{(a_0 h)^\nu}{\nu!} e^{a_0 h}\Big) \max\{\|d\|_0, \|f\|_1\} + h e^{a_0 h}\|f\|_1.$$

Letting $h \to 0$, we obtain the estimate $\max\limits_{(x,t)\in[0,\omega]\times[0,T]} \|v^*(x,t)\| \le (1+\varepsilon)K \max\{\|d\|_0, \|f\|_1\}$.
Theorem 5 is proved.

Conclusion

The paper proposes a modified algorithm of the parameterization method: an additional parameter is introduced and at the last point of the segment on which the boundary value problem is considered. This is the difference between the proposed modified algorithm and the classical algorithm of the parameterization method. This modification allows us to simplify the structure of the linear operator equation with respect to the introduced parameters. Sufficient conditions for the existence of a single solution of the problem (1), (2) and criteria of correct solvability of the family of linear boundary value problems for the system of ordinary differential equations are obtained. Note that the idea of the methodology used in this paper has wide prospects of development for the study of problems of solutions of linear and nonlinear boundary value problems.

Acknowledgments

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19675193)

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Шеттік есептің бір үйірінің қисынды шешімділік критерийлері туралы

Мақалада дифференциалдық теңдеулер жүйелері үшін сызықтық екі нүктелі шеттік есептер үйірі карастырылған. Оның шешімдерінің бар болу сұрақтары зерттеліп, жуық шешімді табу әдістері ұсынылған. Жәй дифференциалдық теңдеулер жүйесі үшін сызықтық екі нүктелі шеттік есептер үйірінің жеткілікті шарттары анықталған. Қарастырылған есептің шешімінің жалғыздығы дәлелденді. Д.С. Жұмабаевтың параметрлеу әдісінің алгоритмдерінің бір модификациясы негізінде зерттелетін есептің жуық шешімін табу алгоритмдері берілген және олардың жинақтылығы дәлелденген. Параметрлеу әдісінің схемасы бойынша есеп дифференциалдық теңдеулер жүйелері үшін көп нүктелі шеттік есептерінің эквивалентті үйіріне түрлендірілген. Жаңа белгісіз функцияларды енгізу арқылы біз зерттелетін есепті баламалы есепке, екінші текті Вольтерра интегралдық теңдеуіне келтіреміз. Параметрметрлі шеттік есептер үйірінің жалғыз шешімінің бар болуын қамтамасыз ететін ұсынылған алгоритмнің орындылығы мен жинақтылығының жеткілікті шарттары анықталды. Жәй дифференциалдық теңдеулер жүйесі үшін сызықтық шеттік есептер үйірінің қисынды шешімділігінің қажетті және жеткілікті шарттары алынды.

Кілт сөздер: сызықтық шеттік есептер үйірі, көпнүктелі шеттік есеп, шешімнің бар болуы, жалғыз шешім, қисынды шешімділік, қажетті және жеткілікті шарт.

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О критериях корректной разрешимости одного семейства краевых задач

В статье рассмотрено семейство линейных двухточечных краевых задач для систем дифференциальных уравнений. Исследованы вопросы существования его решений и предложены методы нахождения приближенных решений. Установлены достаточные условия существования семейства линейных двухточечных краевых задач для системы обыкновенных дифференциальных уравнений. Доказана единственность решения рассматриваемой задачи. Даны алгоритмы нахождения приближенного решения исследуемой задачи, основанные на одной модификации алгоритмов метода параметризации Д.С. Джумабаева, и доказана их сходимость. По схеме метода параметризации задача будет преобразована в эквивалентное семейство многоточечных краевых задач для систем дифференциальных уравнений. Введя новые неизвестные функции, сведем исследуемую задачу к эквивалентной задаче, интегральному уравнению Вольтерра второго рода. Установлены достаточные условия осуществимости, сходимости предложенного алгоритма, вместе с тем обеспечивающие существование единственного решения семейства краевых задач с параметрами. Получены необходимые и достаточные условия корректной разрешимости семейства линейных краевых задач для системы обыкновенных дифференциальных уравнений.

Ключевые слова: семейство линейных краевых задач, многоточечная краевая задача, существование решения, единственное решение, корректная разрешимость, необходимое и достаточное условие.

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DOI 10.31489/2023M4/21-29

UDC 517.51

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Coefficients of multiple Fourier-Haar series and variational modulus of continuity

In this paper, we introduce the concept of a variational modulus of continuity for functions of several variables, give an estimate for the sum of the coefficients of a multiple Fourier-Haar series in terms of the variational modulus of continuity, and prove theorems of absolute convergence of series composed of the coefficients of multiple Fourier-Haar series. In this paper, we study the issue of the absolute convergence for multiple series composed of the Fourier-Haar coefficients of functions of several variables of bounded p-variation. We estimate the coefficients of a multiple Fourier-Haar series in terms of the variational modulus of continuity and prove the sufficiency theorem for the condition for the absolute convergence of series composed of the Fourier-Haar coefficients of the considered function class. This paper researches the question: under what conditions, imposed on the variational modulus of continuity of the fractional order of several variables functions, there is the absolute convergence for series composed of the coefficients of multiple Fourier-Haar series.

Keywords: Fourier-Haar series, variational modulus of continuity, coefficients of multiple Fourier-Haar series.

Introduction

It is known that the definition of p-variation functions for one variable was introduced by Wiener [1], for functions of two variables this definition was given by Clarkson and Adams [2]. Similar questions for trigonometric and multiplicative systems were considered in the works [3, 4]. Let us give the necessary definitions.

Let $f(x_1, ...x_n)$ be defined on the set $[0,1]^N$ and $\rho = \rho_1 \times \rho_2 \times ... \times \rho_N$, here $\rho_j = \{0 = x_j^0 < x_j^1 < ... < x_j^s = 1\}$, $s_j \geq 1$, $j = 1, ..., N$ is an arbitrary partition of a set $[0,1]^N$. Variational sum of order p of the function $f(x_1, ...x_n)$ with respect to the partitions ρ is called the quantity $(1 \leq \ p \leq \infty)$

$$\aleph_\rho^p(f) = \left(\sum_{r_1=1}^{s_1} ... \sum_{r_N=1}^{s_N} |\Delta_1(f; x_1^{r_1-1}, ..., x_N^{r_N-1}; h_1^{r_1}, ..., h_N^{r_N})|^p \right)^{1/p},$$

here

$$\Delta_1(f; x_1, ..., x_N; h_1, ..., h_N) := \textstyle\sum_{\eta_1=0}^{1} ... \sum_{\eta_n=0}^{1} (-1)^{\eta_1+...+\eta_N} f(x_1 + \eta_1 h_1, ..., x_N + \eta_N h_N),$$

$$(x_1, ..., x_N) \in [0,1]^N, h_j > 0, h_j^{r_j} := x_j^{r_j} - x_j^{r_j-1}, r_j = 1, 2, ...s_j, j = 1, 2, ..., n.$$

Variational modulus of continity $\omega_{1-1/p}(f, \delta_1, ..., \delta_N)$ of an order $1 - \frac{1}{p}$ of the function $f(x_1, ...x_n)$ is called the value

$$\omega_{1-1/p}(f, \delta_1, ..., \delta_N) = \sup_{|\rho_j| \leq \delta_j} \aleph_\rho^p(f), \tag{1}$$

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here $|\rho_j| = \max\limits_{1\leq\rho_j\leq s_j}(x_j^{r_j} - x_{j-1}^{r_j})$.

We say that $f \in V_p[0,1]^N$, $1 \leq \ p \leq \infty$, if $V_p\left(f, [0,1]^N\right) \equiv \omega_{1-1/p}(f, 1, ..., 1) < \infty$, and if $f \in C_p[0,1]^N$, $1 \leq \ p < \infty$, if $\lim\limits_{\delta_i\to 0}\omega_{1-1/p}(f, \delta_1, ..., \delta_N) = 0.$. The properties of a variational modulus of continuity was studied by A.P. Terekhin (see [5, 6]).

Modulus of continuity $\omega(f, \delta_1, ..., \delta_N)$ for function $f(x_1, ...x_n)$ is called the value

$$\omega(f, \delta_1, ..., \delta_N) = \sup_{0<h_i\leq\delta_i}|f(x_1 + h_1, ..., x_N + h_N)... - f(x_1, ..., x_i + h_i, ..., x_N) - ... + f(x_1, ..., x_N)|.$$

The functions of the Haar system on the semi-open interval $[0, 1)$ is defined by $h_0(x) = 1$ if $x \in [0, 1)$; if $n = 2^k + j$, $k \in P = N \cup \{0\}$, $0 \leq j < 2^k$ and $\Delta_j^{(k)} = \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right)$, then

$$h_n(x) = \begin{cases} 2^{k/2}, x \in \Delta_{2j}^{(k+1)} \\ -2^{k/2}, x \in \Delta_{2j+1}^{(k+1)} \\ 0, x \in [0,1) \setminus \Delta_j^{(k)} \end{cases},$$

(see [7]).
Then the multiplicative Haar system is defined as follows:

$$h_{k_1,...,k_n}(x_1, ..., x_n) = h_{k_1}(x_1)...h_{k_n}(x_n),$$

$$(x_1, ..., x_n) \in [0, 1)^N.$$

The Fourier-Haar coefficients for functions of several variables are determined by the equality: $a_{n_1,...,n_N}(f) = \int_0^1 ... \int_0^1 f(x_1, ..., x_N)h_{n_1}(x_1)...h_{n_N}(x_N)dx_1...dx_N$, $n_1, ..., n_N \in N$.

This paper researches the question: under what conditions, imposed on the variational modulus of continuity of the fractional order of several variables functions, does the series converge?

$$\sum_{n_1=1}^{\infty} ... \sum_{n_N=1}^{\infty} |a_{n_1,...,n_N}(f)|^{\beta}, \beta > 0,$$

where $a_{n_1,...,n_N}(f)$ are the Fourier-Haar coefficients of the function f. For the case of functions of one variable, such questions were considered by S.S. Volosivets [8].

1 *Formulas and theorems*

Theorem 1. Let $f \in C_p[0,1]$, $1 < p < \infty$ and $a_n(f) = \int_0^1 f(x)h_n(x)dx$, $n \in N$. The following inequality is valid

$$\left(\sum_{i=2^k}^{2^{k+1}-1}|a_i(f)|^p\right)^{\frac{1}{p}} \leq \omega_{1-1/p}\left(f, \frac{1}{2^k}\right)2^{-\frac{k}{2}-1}.$$

You can see the proof of Theorem 1 in [8].

Further, we are considering the functions of several variables. We need the following auxiliary statements.

Lemma 1. Let $f \in V_p[0,1]^N$, $1 \leq \ p < \infty$ и $0 < \delta_1, \delta_2 < 1$. The following inequality is valid

$$\omega(f, \delta_1, ..., \delta_N)_{L_p} \leq \omega_{1-1/p}(f, \delta_1, ..., \delta_N)\delta_1^{\frac{1}{p}}...\delta_N^{\frac{1}{p}}.$$

This lemma is an analogue of the corresponding lemma from the work [5], it is proved for the case of functions of one variable, for functions of several variables, the proof is proved similarly to the one variable case.

The following theorem gives an estimate for the Fourier-Haar coefficients of two variables functions in terms of the variational modulus of continuity of the order $(1-1/p)$.

Theorem 2. Let $f \in C_p[0,1]^N$, $1<p<\infty$ and

$$a_{n_1,\dots,n_N}(f)=\int_0^1 \dots \int_0^1 f(x_1,\dots,x_N)h_{n_1}(x_1)\dots h_{n_N}(x_N)dx_1\dots dx_N, \quad n_1,\dots,n_N\in N,$$

the following inequality is valid

$$\left(\sum_{i=2^{k_1}}^{2^{k_1+1}-1}\dots\sum_{i_N=2^{k_N}}^{2^{k_N+1}-1}|a_{i_1\dots i_N}(f)|^p\right)^{\frac{1}{p}} \leq \omega_{1-1/p}\left(f,\frac{1}{2^{k_1}},\dots,\frac{1}{2^{k_N}}\right)2^{-\frac{k_1+\dots+k_N}{2}-2}. \tag{2}$$

Proof of Theorem 2. We present the proof for the two variables case [9, 10]. In many variables it is proved in a similar way. Using the definition of the Haar function $h_{n_1,n_2}(x,y)=h_{n_1}(x)\,h_{n_2}(x)$ if $n_1=2^{k_1}+m_1$,$n_2=2^{k_2}+m_2$, we have

$$a_{n_1,n_2}(f)=\int_0^1\int_0^1 f(x,y)h_{n_1}(x)h_{n_2}(y)dxdy=$$

$$=\int_{\frac{m_1}{2^{k_1}}}^{\frac{m_1+1}{2^{k_1}}}\int_{\frac{m_2}{2^{k_2}}}^{\frac{m_2+1}{2^{k_2}}} f(x,y)h_{n_1}(x)h_{n_2}(y)dxdy=$$

$$=2^{\frac{k_1+k_2}{2}}\left(\int_{\frac{m_1}{2^{k_1}}}^{\frac{2m_1+1}{2^{k_1+1}}}\int_{\frac{m_2}{2^{k_2}}}^{\frac{2m_2+1}{2^{k_2+1}}} f(x,y)dxdy-\int_{\frac{2m_1+1}{2^{k_1+1}}}^{\frac{m_1+1}{2^{k_1}}}\int_{\frac{2m_2+1}{2^{k_2}}}^{\frac{2m_2+1}{2^{k_2+1}}} f(x,y)dxdy-\right.$$

$$\left.-\int_{\frac{m_1}{2^{k_1}}}^{\frac{2m_1+1}{2^{k_1+1}}}\int_{\frac{2m_2+1}{2^{k_2+1}}}^{\frac{m_2+1}{2^{k_2}}} f(x,y)dxdy+\int_{\frac{2m_1+1}{2^{k_1+1}}}^{\frac{m_1+1}{2^{k_1}}}\int_{\frac{2m_2+1}{2^{k_2+1}}}^{\frac{m_2+1}{2^{k_2}}} f(x,y)dxdy\right),$$

then, replacing the variables taking the shift of the arguments, we get

$$a_{n_1,n_2}(f)=2^{\frac{k_1+k_2}{2}}\left(\int_{\frac{m_1}{2^{k_1}}}^{\frac{2m_1+1}{2^{k_1+1}}}\int_{\frac{m_2}{2^{k_2}}}^{\frac{2m_2+1}{2^{k_2+1}}} f(x,y)dxdy-\int_{\frac{2m_1+1}{2^{k_1+1}}}^{\frac{m_1+1}{2^{k_1}}}\int_{\frac{2m_2+1}{2^{k_2}}}^{\frac{2m_2+1}{2^{k_2+1}}} f(x+2^{-k_1-1},y)dxdy-\right.$$

$$-\int_{\frac{m_1}{2^{k_1}}}^{\frac{2m_1+1}{2^{k_1+1}}}\int_{\frac{2m_2+1}{2^{k_2+1}}}^{\frac{m_2+1}{2^{k_2}}} f(x,y+2^{-k_2-1})dxdy+$$

$$\left.+\int_{\frac{2m_1+1}{2^{k_1+1}}}^{\frac{m_1+1}{2^{k_1}}}\int_{\frac{2m_2+1}{2^{k_2+1}}}^{\frac{m_2+1}{2^{k_2}}} f(x+2^{-k_1-1},y+2^{-k_2-1})dxdy\right)=$$

$$=2^{\frac{k_1+k_2}{2}}\left(\int_{\frac{m_1}{2^{k_1}}}^{\frac{2m_1+1}{2^{k_1+1}}}\int_{\frac{m_2}{2^{k_2}}}^{\frac{2m_2+1}{2^{k_2+1}}}(f(x,y)-f(x+2^{-k_1-1},y)-\right.$$

$$\left.-f(x,y+2^{-k_2-1})+f(x+2^{-k_1-1},y+2^{-k_2-1}))dxdy\right).$$

Now, based on Holder and Lemma 1, we have (here $\frac{1}{p}+\frac{1}{q}=1$)

$$a_{n_1,n_2}(f) \le 2^{\frac{k_1+k_2}{2}} \left(\int_{\Delta_{2m_1}^{(k_1+1)}} \int_{\Delta_{2m_2}^{(k_2+1)}} |(f(x,y) - f(x+2^{-k_1-1},y) - \right.$$

$$\left. -f(x,y+2^{-k_2-1}) + f(x+2^{-k_1-1},y+2^{-k_2-1}))|^p dxdy \right)^{\frac{1}{p}} \left(\frac{1}{2^{k_1+k_2+2}} \right)^{\frac{1}{q}} \le$$

$$\le 2^{\frac{k_1+k_2}{2}} \left(\sup_{\substack{h_1 \le \frac{1}{2^{k_1+1}} \\ h_2 \le \frac{1}{2^{k_2+1}}}} \int_{\Delta_{2m_1}^{(k_1+1)}} \int_{\Delta_{2m_2}^{(k_2+1)}} |\Delta_2(f,x,y,h_1,h_2)|^p dxdy \right)^{\frac{1}{p}} \left(\frac{1}{2^{k_1+k_2+2}} \right)^{\frac{1}{q}} \le$$

$$\le 2^{\frac{k_1+k_2}{2}} V_p \left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}} \right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}} \right] \right) \times$$

$$\times \left(\frac{1}{2^{k_1+k_2+2}} \right)^{\frac{1}{q}} \left(\frac{1}{2^{k_1+k_2+2}} \right)^{\frac{1}{p}} =$$

$$= 2^{\frac{k_1+k_2}{2}} V_p \left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}} \right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}} \right] \right) \left(\frac{1}{2^{k_1+1}} \right)^{\frac{1}{q}+\frac{1}{p}} \left(\frac{1}{2^{k_2+1}} \right)^{\frac{1}{q}+\frac{1}{p}} =$$

$$= 2^{\frac{k_1+k_2}{2}} V_p \left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}} \right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}} \right] \right) \left(\frac{1}{2^{k_1+1}} \right) \left(\frac{1}{2^{k_2+1}} \right) =$$

$$= 2^{\frac{k_1+k_2}{2}-k_1-k_2-2} V_p \left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}} \right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}} \right] \right) =$$

$$= 2^{-\frac{k_1+k_2}{2}-2} V_p \left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}} \right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}} \right] \right).$$

Therefore

$$|a_{n_1,n_2}(f)|^p \le 2^{-\frac{p}{2}(k_1+k_2)-2p} \left(V_p \left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}} \right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}} \right] \right) \right)^p. \tag{3}$$

We take $\varepsilon > 0$ such that $m_1 = 0,1,...,2^{k_1}-1$ and $m_2 = 0,1,...,2^{k_2}-1$ and find partition ξ_{m_1} and η_{m_2} squares $\left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}}\right]$ and $\left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}}\right]$ (look (1)) such that

$$\left(\aleph^p_{\xi_{m_1},\eta_{m_2}}(f) \right)^p \ge \left(V_p^p \left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}} \right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}} \right] \right) \right)^p - \frac{\varepsilon}{2^{k_1-k_2}}.$$

Combining all these partitions of the square $[0,1]^2$ with a diameter no more $\frac{1}{2^{k_1}}$, $\frac{1}{2^{k_2}}$ accordingly and, summing inequalities (3), we get

$$\sum_{i=2^{k_1}}^{2^{k_1+1}-1} \sum_{i=2^{k_2}}^{2^{k_2+1}-1} |a_{ij}(f)|^p \le \omega^p_{1-1/p} \left(f, \frac{1}{2^{k_1}}, \frac{1}{2^{k_2}} - \varepsilon \right) 2^{-\frac{k_1+k_2}{2}p-2p},$$

and since ε we can be made arbitrarily small, then inequality (2) is proved. Theorem 2 is proved.
In the case of one variable functions, a similar estimate for the Fourier-Haar coefficients was obtained in [8].

The following theorem gives sufficient conditions for the convergence of double series, composed of Fourier-Haar coefficients.

Theorem 3. Let $f \in C_p[0,1]^N$, $1 < p < \infty$ and

$$a_{n_1,...,n_N}(f) = \int_0^1 ... \int_0^1 f(x_1,...,x_N)h_{n_1}(x_1)...h_{n_N}(x_N)dx_1...dx_N, \quad n_1,...,n_N \in N.$$

1) Let $\beta > 0$, $p \geq \beta$. Then, under the condition of convergence for the series

$$\sum_{n_1=1}^{\infty} ... \sum_{n_N=1}^{\infty} (n_1...n_N)^{-\frac{\beta}{2}-\frac{\beta}{p}} \omega_{1-1/p}^{\beta}\left(f, \frac{1}{n_1}, ..., \frac{1}{n_N}\right),$$

the following series converges

$$\sum_{n_1=1}^{\infty} ... \sum_{n_N=1}^{\infty} |a_{n_1...n_N}(f)|^{\beta}.$$

2) Let $\beta > 0$, $p \geq \beta$, $\gamma > \frac{1}{p} + \frac{1}{2}$, $\gamma \in R$. Then, under the condition of convergence for the series

$$\sum_{n_1=1}^{\infty} ... \sum_{n_N=1}^{\infty} (n_1...n_N)^{\gamma-\frac{1}{p}-\frac{1}{2}} \omega_{1-1/p}^{p}\left(f, \frac{1}{n_1}, ..., \frac{1}{n_N}\right),$$

the following series converges

$$\sum_{n_1=1}^{\infty} ... \sum_{n_N=1}^{\infty} (n_1...n_N)^{\gamma}|a_{n_1...n_N}(f)| < \infty.$$

Proof of Theorem 3. We present the proof for the two variables case [9, 10]. In many variables it is proved in a similar way. Consider the case 1).

Using Holder's inequality and Theorem 2, we have

$$\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1} |a_{mn}(f)|^{\beta} \leq \left(\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1} |a_{mn}(f)|^{\beta\frac{p}{\beta}}\right)^{\frac{\beta}{p}} \left(\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1} 1\right)^{1-\frac{\beta}{p}} =$$

$$= (2^k 2^l)^{1-\frac{\beta}{p}} \left(\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1} |a_{mn}(f)|^{\beta\frac{p}{\beta}}\right)^{\frac{\beta}{p}} \leq (2^{k+l})^{1-\frac{\beta}{p}} 2^{(-\frac{k+l}{2}-2)\beta} \omega_{1-1/p}^{\beta}\left(f, \frac{1}{2^k}, \frac{1}{2^l}\right) =$$

$$= 2^{(k+l)\left(1-\frac{\beta}{p}-\frac{\beta}{2}\right)} \omega_{1-1/p}^{\beta}\left(f, \frac{1}{2^k}, \frac{1}{2^l}\right).$$

Summing up both sides of the resulting inequality, we have

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1} |a_{mn}(f)|^{\beta}\right) \leq$$

$$\leq C\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2^{(k+l)\left(1-\frac{\beta}{p}-\frac{\beta}{2}\right)} \omega_{1-1/p}^{\beta}\left(f, \frac{1}{2^k}, \frac{1}{2^l}\right) \sum_{k=1}^{\infty}\sum_{l=l}^{\infty} |a_{mn}(f)|^{\beta} \leq$$

$$\leq C\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2^{(k+l)\left(1-\frac{\beta}{p}-\frac{\beta}{2}\right)} \omega_{1-1/p}^{\beta}\left(f, \frac{1}{2^k}, \frac{1}{2^l}\right) \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-\frac{\beta}{2}-\frac{\beta}{p}} \omega_{1-1/p}^{\beta}\left(f, \frac{1}{m}, \frac{1}{n}\right) =$$

$$= C\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=2^m}^{2^{m+1}-1}\sum_{l=2^n}^{2^{n+1}-1}(kl)^{-\frac{\beta}{p}-\frac{\beta}{2}}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{k},\frac{1}{l}\right) \geq$$

$$\geq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2^{-(n+m)}2^{-\frac{\beta}{2}-\frac{\beta}{p}}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{2^m},\frac{1}{2^n}\right)2^{(n+m)} =$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2^{(n+m)\left(1-\frac{\beta}{2}-\frac{\beta}{p}\right)}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{2^m},\frac{1}{2^n}\right).$$

Therefore

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-\frac{\beta}{2}-\frac{\beta}{p}}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{m},\frac{1}{n}\right) < \infty,$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{mn}(f)|^{\beta} < \infty.$$

Now we consider the case 2)

$$\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1}(mn)^{\gamma}|a_{mn}(f)|^{\beta} \leq \left(\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1}|a_{mn}(f)|^{p}\right)^{\frac{1}{p}}\left(\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1}(mn)^{\gamma q}\right)^{\frac{1}{q}} \leq$$

$$\leq \left(2^{(k+1)\gamma q}2^{(l+1)\gamma q}2^{(k+l)}\right)^{\frac{1}{q}}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{2^k},\frac{1}{2^l}\right)2^{-\frac{k+l}{2}-2} =$$

$$= 2^{\gamma-2}2^{(k+l)\left(\gamma+\frac{1}{q}-\frac{1}{2}\right)}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{2^k},\frac{1}{2^l}\right)2^{-\frac{k+l}{2}-2}.$$

Summing up both sides

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1}(mn)^{\gamma}|a_{mn}(f)|^{\beta}\right) \leq$$

$$\leq \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}2^{\gamma-2}2^{(k+l)\left(\gamma+\frac{1}{q}-\frac{1}{2}\right)}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{2^k},\frac{1}{2^l}\right)2^{-\frac{k+l}{2}-2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{\gamma}|a_{mn}(f)| \leq$$

$$\leq C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{\gamma-\frac{1}{p}-\frac{1}{2}}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{m},\frac{1}{n}\right).$$

Theorem 3 is proved.

Theorem 3 is an extension to the two-dimensional case of the corresponding theorem from the work [8].

Acknowledgments

This work is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP15473253).

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Еселі Фурье-Хаар қатарының коэффициенттері және вариациялық үзіліссіздік модулі

Мақалада көп айнымалы функциялар үшін вариациялық үзіліссіздік модулінің ұғымы енгізілген, Фурье-Хаар коэффициенттерінен құрылған еселі қатарларды вариациялық үзіліссіздік модулі арқылы бағалау және Фурье-Хаар коэффициенттерінен құрылған еселі қатарлардың абсолютті жинақталуының теоремалары дәлелденген. Авторлар Фурье-Хаар коэффициенттерінен құрылған еселі қатарлардың вариациялық үзіліссіздік модулі арқылы бағалануын және қарастырылып отырған функциялар класынан алынған Фурье-Хаар коэффициенттерінен құрылған еселі қатарлардың абсолютті жинақталуының жеткілікті шартын дәлелдеген. Көп айнымалы функциялардың $(1-1/p)$ ретті вариациялық үзіліссіздік модуліне қандай шарттар қойғанда, Фурье-Хаар коэффициенттерінен құрылған еселі қатарлардың абсолютті жинақталу деген мәселе зерттелген.

Кілт сөздер: Фурье-Хаар қатары, вариациялық үзіліссіздік модулі, еселі Фурье-Хаар коэффициенттері.

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Коэффициенты кратного ряда Фурье–Хаара и вариационный модуль непрерывности

В статье введено понятие вариационного модуля непрерывности для функций многих переменных, приведены оценка суммы коэффициентов кратного ряда Фурье–Хаара через вариационный модуль непрерывности, и доказаны теоремы об абсолютной сходимости рядов, составленных из коэффициентов кратных рядов Фурье–Хаара. Авторами исследован вопрос об абсолютной сходимости кратных рядов, составленных из коэффициентов Фурье–Хаара функций многих переменных ограниченной p-вариации. Приведена оценка коэффициентов кратного ряда Фурье–Хаара через вариационный модуль непрерывности, и доказана теорема достаточности условия абсолютной сходимости рядов, составленных из коэффициентов Фурье–Хаара рассматриваемого класса функции. Здесь изучен вопрос: «При каких условиях, накладываемых на вариационный модуль непрерывности дробного порядка функций многих переменных, имеет место абсолютная сходимость кратных рядов, составленных из коэффициентов Фурье–Хаара?»

Ключевые слова: ряды Фурье–Хаара, вариационный модуль непрерывности, коэффициенты кратного ряда Фурье–Хаара.

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Bulletin of the Karaganda University. Mathematics Series, No. 4(112), 2023, pp. 30–40

DOI 10.31489/2023M4/30-40

UDC 517.954

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On a boundary problem for the fourth order equation with the third derivative with respect to time

In this paper, we consider a boundary value problem in a rectangular domain for a fourth-order homogeneous partial differential equation containing the third derivative with respect to time. The uniqueness of the solution of the stated problem is proved by the method of energy integrals. Using the method of separation of variables, the solution of the considered problem is sought as a multiplication of two functions $X(x)$ and $Y(y)$. To determine $X(x)$,we obtain a fourth-order ordinary differential equation with four boundary conditions at the segment boundary $[0, p]$, and for a $Y(y)$ – third-order ordinary differential equation with three boundary conditions at the boundary of the segment $[0, q]$. Imposing conditions on the given functions, we prove the existence theorem for a regular solution of the problem. The solution of the problem is constructed in the form of an infinite series, and the possibility of term-by-term differentiation of the series with respect to all variables is substantiated. When substantiating the uniform convergence, it is shown that the "small denominator" is different from zero.

Keywords: Initial boundary problem, Fourier method, uniqueness, existence, eigenvalue, eigenfunction, functional series, absolute and uniform convergence.

Introduction

Problems about the vibrations of rods, beams and plates, which are of great importance in structural mechanics, lead to differential equations with a higher order than the string equation.

The study of many problems of gas dynamics, the theory of elasticity, the theory of plates and shells comes to the consideration of differential equations with higher order partial derivatives. From the point of view of physical applications, the fourth order differential equations are also of great interest (see [1–6]).

In the field of modern science and technology, initial-boundary value problems for fourth-order equations are of great importance. For example, aircraft wings, bridge slabs, floor systems, and window panes are modeled as plates with various types of end supports, which are successfully described in terms of fourth-order equations [7–9].

The monograph by T.D. Dzhuraev, A. Sopuev [10] is devoted to the classification of differential equations with partial derivatives of the fourth order, the formulation and solution of boundary value problems for such equations.

In the paper [11], a problem with boundary conditions for a non-homogeneous fourth-order equation with multiple characteristics and one lower term was considered.

In [12], a boundary value problem for a fourth-order equation of the form

$$u_{xxxx} - u_{tt} = f(x, t)$$

was investigated.

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In [13], a problem was solved with initial and boundary conditions for the beam oscillation equation of the form

$$a^2 u_{xxxx} + u_{tt} = 0,$$

in which a beam of length l is clamped with ends in a massive vise.

In [14], a boundary value problem for a degenerate higher order equation with lower terms was studied.

In [15–19], the boundary value problems for a third-order equation with multiple characteristics containing second derivatives with respect to time were discussed.

The boundary value problems for fourth-order equations with the third derivative in time have been little studied [20, 21].

1 Formulation of the problem

In the domain $D = \{(x, y) : 0 < x < p, 0 < y < q\}$ we consider the equation

$$Ł\left[u\right] \equiv \frac{\partial^4 u}{\partial x^4} - \frac{\partial^3 u}{\partial y^3} = 0, \tag{1}$$

where $p, q \in R$.

Problem A. Find a solution to equation (1) in the domain D from the class $u\left(x, y\right) \in C^{4,3}_{x,y}\left(D\right) \cap C^{3,2}_{x,y}\left(\overline{D}\right)$ such that satisfies the following boundary conditions:

$$u\left(0, y\right) = u\left(p, y\right) = u_{xx}\left(0, y\right) = u_{xx}\left(p, y\right) = 0, \quad 0 \le y \le q, \tag{2}$$

$$u_y\left(x, 0\right) = \psi_1\left(x\right), \quad u_{yy}\left(x, 0\right) = \psi_2\left(x\right), \quad u_{yy}\left(x, q\right) = \psi_3\left(x\right), \quad 0 \le x \le p, \tag{3}$$

where $\psi_i\left(x\right)$, $i = \overline{1, 3}$ are the given sufficiently smooth functions, and

$$\psi_1\left(0\right) = \psi_1\left(p\right) = \psi_1^{''}\left(0\right) = \psi_1^{''}\left(p\right) = 0, \qquad \psi_i\left(0\right) = \psi_i\left(p\right) = 0, \quad i = 2, 3. \tag{4}$$

2 The uniqueness of the solution to the problem A

Theorem 1. If the problem A has a solution, then it is unique.

Proof. Let the problem A have two solutions $u_1\left(x, y\right)$ and $u_2\left(x, y\right)$. Then the function $u\left(x, y\right) = u_1\left(x, y\right) - u_2\left(x, y\right)$ satisfies equation (1) and the uniform boundary conditions. Let us prove that $u\left(x, y\right) \equiv 0$ in $\bar{D}$.

In the domain D the following identity is valid:

$$uL\left[u\right] \equiv \frac{\partial}{\partial x}\left(u u_{xxx} - u_x u_{xx}\right) - \frac{\partial}{\partial y}\left(u u_{yy} - \frac{1}{2} u_y^2\right) + u_{xx}^2 = 0.$$

Integrating the identity over the domain D, we have

$$\int_0^q \left[u\left(p, y\right) u_{xxx}\left(p, y\right) - u\left(0, y\right) u_{xxx}\left(0, y\right)\right] dy -$$
$$- \int_0^q \left[u_x\left(p, y\right) u_{xx}\left(p, y\right) - u_x\left(0, y\right) u_{xx}\left(0, y\right)\right] dy -$$

$$- \int_0^p \left[u\left(x, q\right) u_{yy}\left(x, q\right) - u\left(x, 0\right) u_{yy}\left(x, 0\right)\right] dx +$$
$$+ \frac{1}{2} \int_0^p \left[u_y^2\left(x, q\right) - u_y^2\left(x, 0\right)\right] dx + \int_0^p \int_0^q u_{xx}^2 dx dy = 0.$$

Taking the homogeneous boundary conditions into consideration we obtain

$$\frac{1}{2}\int\limits_0^p u_y^2\left(x,q\right)dx+\int\limits_0^p\int\limits_0^q u_{xx}^2dxdy=0.$$

From the second term we obtain

$$u_{xx}=0\quad\Rightarrow\quad u\left(x,y\right)=x\cdot f_1\left(y\right)+f_2\left(y\right),\quad\left(x,y\right)\in D.$$

Assuming $x=0$ we get

$$u\left(0,y\right)=f_2\left(y\right)=0\quad\Rightarrow\quad f_2\left(y\right)=0,$$

and supposing $x=p$ we attain

$$u\left(p,y\right)=p\cdot f_1\left(y\right)=0\quad\Rightarrow\quad f_1\left(y\right)=0.$$

Hence, $u\left(x,y\right)\equiv 0$, $\left(x,y\right)\in\overline{D}$.

Theorem 1 is proved.

3 Existence of a solution to the problem A

In order to prove the existence of the solution of the problem A, we will first consider the following auxiliary problem: find a nontrivial solution of equation (1) such that satisfies conditions (2) and can be represented as

$$u\left(x,y\right)=X\left(x\right)\;Y\left(y\right).\tag{5}$$

Substituting (5) into equation (1) and separating the variables, we find the following ordinary differential equations with respect to the functions $X\left(x\right)$ and $Y\left(y\right)$:

$$X^{(4)}\left(x\right)-\lambda^4X\left(x\right)=0,\tag{6}$$

$$Y^{(3)}\left(y\right)-\lambda^4Y\left(y\right)=0,\tag{7}$$

where λ^4 is the split parameter.

Considering the boundary conditions (2), we generate the following problem for equation (6):

$$\begin{cases}X^{(4)}-\lambda^4X=0,\\ X\left(0\right)=X\left(p\right)=X''\left(0\right)=X''\left(p\right)=0.\end{cases}\tag{8}$$

A nontrivial solution to problem (8) exists if and only if

$$\lambda_n^4=\left(\frac{\pi n}{p}\right)^4,\quad n=1,2,3,\ldots.$$

These numbers are the eigenvalues of problem (8), and their corresponding eigenfunctions have the following form:

$$X_n\left(x\right)=\sqrt{\frac{2}{p}}\sin\frac{\pi n}{p}x.\tag{9}$$

A general solution (7) has the form

$$Y_n\left(y\right)=C_1e^{k_ny}+e^{-\frac{1}{2}k_ny}\left(C_2\cos\left(\frac{\sqrt{3}}{2}k_ny\right)+C_3\sin\left(\frac{\sqrt{3}}{2}k_ny\right)\right),\tag{10}$$

where $k_n = \sqrt[3]{\lambda_n^4} = \left(\frac{\pi n}{p}\right)^{4/3}$, $n \in N$ and $C_i,\ i = \overline{1,3}$ are unknown constants for now.

According to (9) and (10), it follows from equation (5) that the functions

$$u_n\left(x,y\right) = \sqrt{\frac{2}{p}}\left(C_1 e^{k_n y} + e^{-\frac{1}{2}k_n y}\left(C_2\cos\left(\frac{\sqrt{3}}{2}k_n y\right) + C_3\sin\left(\frac{\sqrt{3}}{2}k_n y\right)\right)\right)\sin\frac{\pi n}{p}x$$

are the particular solutions of equation (1), which satisfy homogeneous conditions (2).

Due to the linearity and homogeneity of (1), the sum of the particular solutions can also be the solution of equation (1). Taking this into account we will seek the solution of problem A in the form

$$u\left(x,y\right) = \sqrt{\frac{2}{p}}\sum_{n=1}^{\infty}\left(C_1 e^{k_n y} + e^{-\frac{1}{2}k_n y}\left(C_2\cos\left(\frac{\sqrt{3}}{2}k_n y\right) + C_3\sin\left(\frac{\sqrt{3}}{2}k_n y\right)\right)\right)\sin\frac{\pi n}{p}x. \tag{11}$$

Assuming temporarily that the series in (11) and its derivatives converge uniformly and requiring the function defined by the series (11) to satisfy the boundary conditions (3) we obtain

$$\begin{aligned}
u_y\left(x,0\right) &= \psi_1\left(x\right) = \sqrt{\frac{2}{p}}\sum_{n=1}^{\infty}\psi_{1n}\sin\frac{\pi n}{p}x,\\
u_{yy}\left(x,0\right) &= \psi_2\left(x\right) = \sqrt{\frac{2}{p}}\sum_{n=1}^{\infty}\psi_{2n}\sin\frac{\pi n}{p}x,\\
u_{yy}\left(x,q\right) &= \psi_3\left(x\right) = \sqrt{\frac{2}{p}}\sum_{n=1}^{\infty}\psi_{3n}\sin\frac{\pi n}{p}x,
\end{aligned}$$

where

$$\begin{cases}
k_n C_1 - \frac{1}{2}k_n C_2 + \frac{\sqrt{3}}{2}k_n C_3 = \psi_{1n},\\
k_n^2 C_1 - \frac{1}{2}k_n^2 C_2 - \frac{\sqrt{3}}{2}k_n^2 C_3 = \psi_{2n},\\
k_n^2 e^{k_n q} C_1 + k_n^2 e^{-\frac{1}{2}k_n q}\cos\left(\frac{\sqrt{3}}{2}k_n q - \frac{2\pi}{3}\right)C_2 + k_n^2 e^{-\frac{1}{2}k_n q}\sin\left(\frac{\sqrt{3}}{2}k_n q - \frac{2\pi}{3}\right)C_3 = \psi_{3n}.
\end{cases} \tag{12}$$

We can see from (12) that the numbers ψ_{in} are the Fourier coefficients of the function $\psi_i\left(x\right)$ when they are expanded into the Fourier series in terms of sines on the interval $(0,p)$, i.e.

$$\psi_{in} = \sqrt{\frac{2}{p}}\int_0^p \psi_i\left(\xi\right)\sin\frac{\pi n}{p}\xi d\xi, \quad i = \overline{1,3}.$$

Let us calculate the determinant of the system (12), i.e.

$$\Delta = \begin{vmatrix}
k_n & -\frac{1}{2}k_n & \frac{\sqrt{3}}{2}k_n\\
k_n^2 & -\frac{1}{2}k_n^2 & -\frac{\sqrt{3}}{2}k_n^2\\
k_n^2 e^{k_n q} & k_n^2 e^{-\frac{1}{2}k_n q}\cos\left(\frac{\sqrt{3}}{2}k_n q - \frac{2\pi}{3}\right) & k_n^2 e^{-\frac{1}{2}k_n q}\sin\left(\frac{\sqrt{3}}{2}k_n q - \frac{2\pi}{3}\right)
\end{vmatrix} =$$

$$= \sqrt{3}k_n^5 e^{k_n q}\bar{\Delta}, \quad \bar{\Delta} = \frac{1}{2} + e^{-\frac{3}{2}k_n q}\sin\left(\frac{\sqrt{3}}{2}k_n q - \frac{\pi}{6}\right).$$

Lemma. For an arbitrary positive q, the inequality $\Delta > 0$ holds .

Proof. Write the determinant in the form

$$\Delta = \sqrt{3}k_n^5 e^{k_n q}\bar{\Delta}(x_n), \quad \bar{\Delta}(x_n) = \frac{1}{2} + e^{-\sqrt{3}x_n}\sin\left(x_n - \frac{\pi}{6}\right),$$

where, $x_n = \dfrac{\sqrt{3}}{2}k_n q = \dfrac{\sqrt{3}}{2}\sqrt[3]{\lambda_n^4}q > 0$, $n \in N$.

Find the minimum value of $\bar{\Delta}(x_n)$. To do this, calculate the first order derivative

$$\frac{d\bar{\Delta}(x_n)}{dx_n} = 2e^{-\sqrt{3}x_n}\sin\left(\frac{\pi}{3} - x_n\right).$$

1) When $0 < x_n < \frac{\pi}{3}$, we have $\bar{\Delta}'(x_n) > 0$ .This means that $\bar{\Delta}(x_n)$ increases at finite discrete values of x_n , but it does not reach its maximum value. Then the function $\bar{\Delta}(x_n)$ takes its minimum value at $n = 1$ and we have the estimation

$$\bar{\Delta}(x_n) \geq \frac{1}{2} + e^{-\sqrt{3}x_1}\sin\left(x_1 - \frac{\pi}{6}\right) = \delta_1 > 0,$$

where $x_1 = \dfrac{\sqrt{3}}{2}\sqrt[3]{\lambda_1^4}q.$

2) If $x_n \geq \dfrac{\pi}{3}$, then the function $\bar{\Delta}(x_n)$ takes its first minimum when the argument is $\dfrac{4\pi}{3}$ and we achieve

$$\overline{\Delta}\,(x_n) \geq \frac{1}{2}\left(1 - e^{-4/\sqrt{3}\pi}\right) = \delta_2 > 0.$$

3) For sufficiently large values of x , it is obvious that the function $\bar{\Delta}(x)$ tends to $\frac{1}{2}$. From here we find

$$\bar{\Delta} \geq \delta = \min\{\delta_1; \delta_2\} > 0.$$

Considering the above considerations we conclude that $\Delta_n > 0$. The lemma is proved.

Hence, the system of equations (12) has a unique solution.

Below, we determine all unkown numbers $C_i,\ i = \overline{1,3}$:

$$C_1 = \frac{1}{\Delta}\left[\psi_{1n}k_n^4 e^{-\frac{1}{2}k_n q}\sin\left(\frac{\sqrt{3}}{2}k_n q\right) - \psi_{2n}k_n^3 e^{-\frac{1}{2}k_n q}\cos\left(\frac{\sqrt{3}}{2}k_n q + \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}\psi_{3n}k_n^3\right],$$

$$C_2 = \frac{1}{\Delta}\left[\psi_{1n}k_n^4\left(e^{-\frac{1}{2}k_n q}\sin\left(\frac{\sqrt{3}}{2}k_n q + \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2}e^{k_n q}\right) - \right.$$
$$\left. - \psi_{2n}k_n^3\left(e^{-\frac{1}{2}k_n q}\sin\left(\frac{\sqrt{3}}{2}k_n q + \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}e^{k_n q}\right) + \sqrt{3}\psi_{3n}k_n^3\right],$$

$$C_3 = \frac{1}{\Delta}\left[\psi_{1n}k_n^4 e^{k_n q}\left(\frac{1}{2} - e^{-\frac{3}{2}k_n q}\cos\left(\frac{\sqrt{3}}{2}k_n q + \frac{\pi}{3}\right)\right) + \psi_{2n}k_n^3 e^{k_n q}\left(e^{-\frac{3}{2}k_n q}\cos\left(\frac{\sqrt{3}}{2}k_n q + \frac{\pi}{3}\right) - \frac{1}{2}\right)\right].$$

In what follows, the maximum value of all found positive known numbers in estimates will be denoted by M .

Taking account condition (4), we integrate $\psi_1(x)$ by parts four times, and $\psi_i(x)$, $i = 2, 3$ by parts two times we get the following estimates:

$$|\psi_1| \leq M\frac{|\Psi_{1n}|}{n^4}, \quad |\psi_i| \leq M\frac{|\Psi_{in}|}{n^2}, \quad i = 2, 3, \tag{13}$$

where

$$\Psi_{1n} = \int\limits_0^p \psi_1^{(4)}(\xi) X_n(\xi) d\xi, \qquad \Psi_{in} = \int\limits_0^p \psi_i''(\xi) X_n(\xi) d\xi, \quad i = 2,3.$$

For $C_i,\ i = \overline{1,3}$ we can write the following estimations:

$$|C_1| e^{k_n q} \leq M \left(\frac{|\psi_{1n}|}{k_n} e^{-\frac{1}{2}k_n q} + \frac{|\psi_{2n}|}{k_n^2} e^{-\frac{1}{2}k_n q} + \frac{|\psi_{3n}|}{k_n^2} \right) \leq M \left(\frac{|\Psi_{1n}|}{n^{\frac{16}{3}}} + \frac{|\Psi_{2n}|}{n^{\frac{14}{3}}} + \frac{|\Psi_{3n}|}{n^{\frac{14}{3}}} \right),$$

$$|C_2| \leq M \left(\left(\frac{|\psi_{1n}|}{k_n} + \frac{|\psi_{2n}|}{k_n^2} \right) \left(e^{-\frac{3}{2}k_n q} + \frac{\sqrt{3}}{2} \right) + \frac{|\psi_{3n}|}{k_n^2} e^{-k_n q} \right) \leq M \left(\frac{|\Psi_{1n}|}{n^{\frac{16}{3}}} + \frac{|\Psi_{2n}|}{n^{\frac{14}{3}}} + \frac{|\Psi_{3n}|}{n^{\frac{14}{3}}} \right),$$

$$|C_3| \leq M \left(\frac{|\psi_{1n}|}{k_n} \left(\frac{1}{2} + e^{-\frac{3}{2}k_n q} \right) + \frac{|\psi_{2n}|}{k_n^2} \left(\frac{1}{2} + e^{-\frac{3}{2}k_n q} \right) \right) \leq M \left(\frac{|\Psi_{1n}|}{n^{\frac{16}{3}}} + \frac{|\Psi_{2n}|}{n^{\frac{14}{3}}} \right).$$

Theorem 2. If $\psi_1(x) \in C^4[0,p]$, $\psi_i(x) \in C^2[0,p]$, $i = 2,3$ and the corresponding conditions (4) are satisfied, then the solution of the problem A exists and it is represented by the series (11).

Proof. If the series (11) and its derivatives u_{xxxx}, u_{yyy} converge uniformly in the region $\bar{D}$, then the function $u(x,y)$ defined by this series will be the solution of the problem A.

From (11) we have

$$|u(x,y)| \leq \sum_{n=1}^{\infty} \Big(|C_1| e^{k_n q} + |C_2| + |C_3| \Big) |X_n(x)|. \tag{14}$$

Then, taking account of (13) it is obtained from (14) that

$$|u(x,y)| \leq M \left(\sum_{n=1}^{\infty} \frac{|\Psi_{1n}|}{n^{\frac{16}{3}}} + \sum_{n=1}^{\infty} \frac{|\Psi_{2n}|}{n^{\frac{14}{3}}} + \sum_{n=1}^{\infty} \frac{|\Psi_{3n}|}{n^{\frac{14}{3}}} \right) < \infty.$$

This implies that the series (11) converges absolutely and uniformly.

Now let us prove that the partial derivatives of the series (11) with respect to both variables included in the equation also converge absolutely and uniformly in the region $\bar{D}$. Calculating the derivatives with respect to y, it follows from (11) we obtain

$$\frac{\partial^3 u}{\partial y^3} = \sum_{n=1}^{\infty} k_n^3 \left[C_1 e^{k_n y} + e^{-\frac{1}{2}k_n y} \left(C_2 \cos\left(\frac{\sqrt{3}}{2} k_n y \right) + C_3 \sin\left(\frac{\sqrt{3}}{2} k_n y \right) \right) \right] X_n(x). \tag{15}$$

From (15) we determine the estimation

$$\left| \frac{\partial^3 u}{\partial y^3} \right| \leq \sum_{n=1}^{\infty} k_n^3 \Big(|C_1| e^{k_n q} + |C_2| + |C_3| \Big) |X_n(x)| \leq M \left(\sum_{n=1}^{\infty} \frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \sum_{n=1}^{\infty} \frac{|\Psi_{2n}|}{n^{\frac{2}{3}}} + \sum_{n=1}^{\infty} \frac{|\Psi_{3n}|}{n^{\frac{2}{3}}} \right). \tag{16}$$

Using the Cauchy-Bunyakovsky and Bessel inequality we attain

$$\left| \frac{\partial^3 u}{\partial y^3} \right| \leq M \left(\sum_{n=1}^{\infty} \frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \sqrt{\sum_{n=1}^{\infty} |\Psi_{2n}|^2} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}} + \sqrt{\sum_{n=1}^{\infty} |\Psi_{3n}|^2} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}} \right) \leq$$
$$\leq M \left(\sum_{n=1}^{\infty} \frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \|\Psi_{2n}\| \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}} + \|\Psi_{3n}\| \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}} \right) < \infty,$$

where

$$\sum_{n=1}^{\infty} |\Psi_{1n}|^2 \leq \left\| \psi_1^{(4)}(x) \right\|^2_{L_2[0;p]}, \quad \sum_{n=1}^{\infty} |\Psi_{in}|^2 \leq \left\| \psi_i^{(2)}(x) \right\|^2_{L_2[0,p]}, \quad i = 2, 3.$$

Therefore, the series (16) converges absolutely and uniformly. The absolute and uniform convergence of the partial derivative of the fourth order in x series (11) follows from the equality $\dfrac{\partial^4 u}{\partial x^4} = \dfrac{\partial^3 u}{\partial y^3}$.

Theorem 2 is proved.

Conclusion

The article considers an initial boundary value problem for a fourth-order equation containing the third time derivative with multiple characteristics. Uniqueness theorems are proved using the method of energy integrals. The existence of a solution is shown with the help of conditions imposed on given functions constructed as a series by the Fourier method and a regular solution of this series.

Acknowledgments

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Уақыт бойынша үшінші туындысы бар төртінші ретті теңдеу үшін шекаралық есеп жайында

Мақалада уақыт бойынша үшінші туындысы бар біртекті төртінші ретті дербес туындылы дифференциалдық теңдеу үшін тікбұрышты облыстағы шекаралық есеп қарастырылды. Қойылған есептің шешімінің жалғыздығы энергия интегралдары әдісімен дәлелденген. Айнымалыларды бөлу әдісін қолданып, есептің шешімі $X(x)$ және $Y(y)$ екі функцияның көбейтіндісі түрінде іздестіріледі. $X(x)$ анықтау үшін $[0,p]$ кесіндісінің шекарасында төрт шекаралық шарты бар төртінші ретті қарапайым дифференциалдық теңдеуін, ал $Y(y)$ анықтау үшін $[0,q]$ кесіндісінің шекарасында үш шекаралық шарты бар үшінші ретті қарапайым дифференциалдық теңдеуді аламыз. Берілген функцияларға шарттарды қою арқылы есептің тұрақты шешімінің бар екендігі туралы теорема дәлелденеді. Қойылған есептің шешімі шексіз қатар түрінде құрылып, қатардың барлық айнымалыларға қатысты қатарды мүшелеп дифференциалдау мүмкіндігі анықталған. Бірқалыпты жинақталуды табу кезінде «кіші бөлгіш» нөлге тең емес екені анықталды.

Кілт сөздер: бастапқы-шеттік есеп, Фурье әдісі, шешімнің жалғыздығы, бар болуы, меншікті мән, меншікті функция, функционалдық қатар, абсолютті және бірқалыпты жинақталу.

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О граничной задаче для уравнения четвёртого порядка, содержащего третью производную по времени

В статье рассмотрена краевая задача в прямоугольной области для однородного дифференциального уравнения в частных производных четвёртого порядка, содержащего третью производную по времени. Единственность решения поставленной задачи доказана методом интегралов энергии. Используя метод разделения переменных, решение задачи ищется в виде произведения двух функций $X(x)$ и $Y(y)$. Для определения $X(x)$ получаем обыкновенное дифференциальное уравнение четвёртого порядка с четырьмя граничными условиями на границе сегмента $[0,p]$, а для $Y(y)$ – обыкновенное дифференциальное уравнение третьего порядка с тремя граничными условиями на границе сегмента $[0,q]$. Налагая определенные условия на заданные функции, доказана теорема существования регулярного решения задачи. Решение поставленной задачи построено в виде бесконечного ряда, обоснована возможность почленного дифференцирования ряда по всем переменным. При доказательстве равномерной сходимости установлена отличность от нуля «малого знаменателя».

Ключевые слова: начально-краевая задача, метод Фурье, единственность, существование, собственное значение, собственная функция, функциональный ряд, абсолютная и равномерная сходимость.

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DOI 10.31489/2023M4/41-55

UDC 517.956.32

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Boundary value problems with displacement for one mixed hyperbolic equation of the second order

The paper studies two nonlocal problems with a displacement for the conjugation of two equations of second-order hyperbolic type, with a wave equation in one part of the domain and a degenerate hyperbolic equation of the first kind in the other part. As a non-local boundary condition in the considered problems, a linear system of FDEs is specified with variable coefficients involving the first-order derivative and derivatives of fractional (in the sense of Riemann-Liouville) orders of the desired function on one of the characteristics and on the line of type changing. Using the integral equation method, the first problem is equivalently reduced to a question of the solvability for the Volterra integral equation of the second kind with a weak singularity; and a question of the solvability for the second problem is equivalently reduced to a question of the solvability for the Fredholm integral equation of the second kind with a weak singularity. For the first problem, we prove the uniform convergence of the resolvent kernel for the resulting Volterra integral equation of the second kind and we prove that its solution belongs to the required class. As to the second problem, sufficient conditions are found for the given functions that ensure the existence of a unique solution to the Fredholm integral equation of the second kind with a weak singularity of the required class. In some particular cases, the solutions are written out explicitly.

Keywords: wave equation, degenerate hyperbolic equation of the first kind, Volterra integral equation, Fredholm integral equation, Tricomi method, method of integral equations, methods of fractional calculus theory.

Introduction. Notation. Formulation of the problem

In the Euclidean plane with independent variables x and y we consider the equation

$$0 = \begin{cases} (-y)^m u_{xx} - u_{yy} + \lambda (-y)^{\frac{m-2}{2}} u_x, & y < 0, \\ u_{xx} - u_{yy} + f, & y > 0, \end{cases} \tag{1}$$

where λ, m are given numbers, and $m > 0$, $|\lambda| \le \frac{m}{2}$; $f = f(x, y)$ is the given function; $u = u(x, y)$ is the desired function.

Equation (1) for $y < 0$ coincides with the equation form

$$(-y)^m u_{xx} - u_{yy} + \lambda (-y)^{\frac{m-2}{2}} u_x = 0, \tag{2}$$

and for $y > 0$ equation (1) is a inhomogeneous wave equation

$$u_{xx} - u_{yy} + f(x, y) = 0. \tag{3}$$

Equation (2) belongs to the class of degenerate hyperbolic equations of the first kind [1; 21]. An important property in equation (2) is that at $|\lambda| \le \frac{m}{2}$ the Cauchy problem is valid in its ordinary formulation with type degeneration along the line $y = 0$, even though it violates Protter condition [2].

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At $m = 2$ equation (2) turns into the Bitsadze-Lykov equation [3; 37], [4], [5; 234], while when $\lambda = 0$ equation (2) turns into the Gellerstedt equation [6], applicable to determine the shape of a slot in a dam [7; 234]. A special case of equation (2) is also the Tricomi equation, which plays an important role in the theory of aerodynamics and gas dynamics [8; 38], [9; 280], [10; 373] .

Equation (1) is considered in the domain $\Omega = \Omega_1 \cup \Omega_2 \cup I$, where Ω_1 is the domain bounded by characteristics $\sigma_1 = AC : \ x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0$, $\sigma_2 = CB : \ x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = r$ of equation (2) at $y < 0$, emanating from the point $C = (r/2, y_c)$, $y_c = -\left[\frac{(m+2)r}{4}\right]^{\frac{2}{m+2}}$, passing through the points $A = (0, 0)$ and $B = (r, 0)$, and the segment $I = AB$ of the line $y = 0$; Ω_2 is the domain bounded by characteristics $\sigma_3 = AD : \ x - y = 0$, $\sigma_4 = BD : \ x + y = r$ of equation (3), emanating from the points A and B, intersecting at the point $D = \left(\frac{r}{2}, \frac{r}{2}\right)$ and the line segment $I = AB$.

By a *regular* solution to Eq. (1) in the domain Ω we mean the function $u = u(x, y)$ which belongs to the class $C\left(\bar{\Omega}\right) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$, $\ u_x, u_y \in L_1(I)$, substitution of which turns Eq. (1) into an identity.

Problem 1. Find a regular solution of equation (1) in the domain Ω that satisfies the conditions

$$u\left[\theta_{r1}(x)\right] = \psi_1(x), \qquad 0 \le x \le r, \tag{4}$$

$$\alpha_1(x)\,(r-x)^{\beta_2}\, D_{rx}^{1-\beta_1}\left\{u\left[\theta_{r0}(t)\right]\right\} + \alpha_2(x) D_{rx}^{1-\beta} u(t, 0) + \alpha_3(x) u_y(x, 0) = \psi_2(x), \ \ 0 < x < r, \tag{5}$$

where $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ are defined functions on the segment $0 \le x \le r$ and $\alpha_1^2(x) + \alpha_2^2(x) + \alpha_3^2(x) \ne 0 \ \forall\, x \in [0, r]$.

Problem 2. Find a regular solution to equation (1) in the domain Ω that satisfies the nonlocal condition (5) and the boundary condition

$$u\left[\theta_{01}(x)\right] = \psi_1(x), \qquad 0 \le x \le r, \tag{6}$$

where $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ defined functions on the segment $0 \le x \le r$ while $\alpha_1^2(x) + \alpha_2^2(x) + \alpha_3^2(x) \ne 0 \ \forall\, x \in [0, r]$.

Hence $\theta_{00}(x) = \left(\frac{x}{2}, -(2-2\beta)^{\beta-1} x^{1-\beta}\right)$; $\theta_{01}(x) = \left(\frac{x}{2}, \frac{x}{2}\right)$; $\theta_{r0}(x) = \left(\frac{r+x}{2}, -(2-2\beta)^{\beta-1}(r-x)^{1-\beta}\right)$; $\theta_{r1}(x) = \left(\frac{r+x}{2}, \frac{r-x}{2}\right)$ are affixes of intersection of characteristics emanating from the point $(x, 0)$ with characteristics of AC, AD, BC, BD correspondingly; affixes of points $\beta_1 = \frac{m-2\lambda}{2(m+2)}$, $\beta_2 = \frac{m+2\lambda}{2(m+2)}$, $\beta = \beta_1 + \beta_2 = \frac{m}{m+2}$; $D_{cx}^{\alpha} g(t)$ is a fractional integro-differential operator (in the sense of Riemann-Liouville) of an order $|\alpha|$ with origin at the point c [5], [7], [11].

The Goursat problem for a hyperbolic equation degenerating inside a domain was previously studied in [12, 13]. In [12], the criterion for the continuity of the solution to the Goursat problem for an equation of form (2) is studied and in [13], the solution to the Goursat problem for a model equation that degenerates inside the domain is written explicitly. Paper [14] considers the first boundary value problem for a hyperbolic equation degenerating inside a domain. Papers [15–17] study boundary value problems for degenerate hyperbolic equations in a characteristic quadrangle with data on opposite characteristics.

Problems 1 and 2 formulated above and studied in this paper belong to the class of boundary problems with the Zhegalov-Nakhushev displacement [18–20]. Problems with a displacement for hyperbolic equations degenerate inside the domain were previously studied in [21–25]. Previously, various problems with a displacement for parabolic-hyperbolic type equations of the second and third orders were studied in the works [26, 27]. A more complete scientific literature review on boundary value problems with a displacement one can find in monographs [28–34]. As part of this work, we established sufficient conditions for the given functions $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ and $f(x, y)$, for a unique regular solution to *problems 1 and 2* in the considered domain. In some special cases, the solutions are written out explicitly.

Study of Problem 1

The study of problem 1. The following Theorem holds.

Theorem 1. Assume the given functions $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ and $f(x, y)$ are such that

$$\alpha_1(x), \alpha_2(x), \alpha_3(x), \psi_2(x) \in C[0, r] \cap C^2(0, r), \tag{7}$$

$$\psi_1(x) \in C^1[0, r] \cap C^2(0, r), \tag{8}$$

$$f(x, y) \in C\left(\bar{\Omega}_2\right), \tag{9}$$

and one of the below conditions is met: either

$$\alpha_3(x) - \gamma_2\alpha_1(x) \neq 0 \quad \forall\, x \in [0, r] \tag{10}$$

or

$$\alpha_3(x) - \gamma_2\alpha_1(x) \equiv 0,\ \alpha_2(x) + \gamma_1\alpha_1(x) \neq 0 \quad \forall\, x \in [0, r]. \tag{11}$$

Then there exists a unique regular solution to Problem 1 in the domain Ω.

Proof. Let there is a solution to problem (1), (4), (5) and let

$$u(x, 0) = \tau(x), \quad 0 \leq x \leq r, \tag{12}$$

$$u_y(x, 0) = \nu(x), \quad 0 < x < r. \tag{13}$$

At $|\lambda| \leq \frac{m}{2}$ the solution to Cauchy problem (12)-(13) for equation (2) is written out according to one of the formulas [35; 14]

$$u(x, y) = \frac{1}{B(\beta_1, \beta_2)} \int\limits_0^1 \tau\left[x + (1-\beta)(-y)^{1/(1-\beta)}(2t-1)\right] t^{\beta_2 - 1}\,(1-t)^{\beta_1 - 1}\,dt +$$

$$+ \frac{y}{B(1-\beta_1, 1-\beta_2)} \int\limits_0^1 \nu\left[x + (1-\beta)(-y)^{1/(1-\beta)}(2t-1)\right] t^{-\beta_1}\,(1-t)^{-\beta_2}\,dt, \quad |\lambda| < \frac{m}{2}, \tag{14}$$

$$u(x, y) = \tau\left[x + (1-\beta)(-y)^{1/(1-\beta)}\right] +$$

$$+ (1-\beta)\, y \int\limits_0^1 \nu\left[x + (1-\beta)(-y)^{1/(1-\beta)}(2t-1)\right] (1-t)^{-\beta}\,dt, \quad \lambda = \frac{m}{2}, \tag{15}$$

$$u(x, y) = \tau\left[x - (1-\beta)(-y)^{1/(1-\beta)}\right] +$$

$$+ (1-\beta)\, y \int\limits_0^1 \nu\left[x + (1-\beta)(-y)^{1/(1-\beta)}(1-2t)\right] (1-t)^{-\beta}\,dt, \quad \lambda = -\frac{m}{2}, \tag{16}$$

where $\tau(x) \in C[0, r] \cap C^2(0, r)$, $\nu(x) \in C^1(0, r) \cap L_1(0, r)$; $\beta_1 = \frac{m-2\lambda}{2(m+2)}$, $\beta_2 = \frac{m+2\lambda}{2(m+2)}$, $\beta = \beta_1 + \beta_2 = \frac{m}{m+2}$; $\Gamma(p) = \int\limits_0^\infty \exp(-t)\, t^{p-1} dt$, $B(p, q) = \int\limits_0^1 t^{p-1}(1-t)^{q-1}\,dt$ are Euler integrals of the first and second kind, $B(p, q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$.

Consider first the case for $|\lambda| < \frac{m}{2}$. By (14) and taking into account (8), we have

$$u\left[\theta_{r0}(x)\right] = u\left(\frac{r+x}{2},\, -\left(2-2\beta\right)^{\beta-1}\left(r-x\right)^{1-\beta}\right) =$$

$$= \frac{1}{B\left(\beta_1,\, \beta_2\right)} \int\limits_0^1 \tau\left[x + \left(r-x\right)t\right]\, t^{\beta_2-1}\, \left(1-t\right)^{\beta_1-1} dt -$$

$$-\frac{1}{B\left(1-\beta_1,\, 1-\beta_2\right)} \left(2-2\beta\right)^{\beta-1}\left(r-x\right)^{1-\beta} \int\limits_0^1 \nu\left[x + \left(r-x\right)t\right]\, t^{-\beta_1}\, \left(1-t\right)^{-\beta_2} dt.$$

Introducing a new variable $z = x + (r-x)\, t$, we can rewrite the last equality as follows

$$u\left[\theta_{r0}(x)\right] = \frac{\left(r-x\right)^{1-\beta}}{B\left(\beta_1,\, \beta_2\right)} \int\limits_x^r \frac{\tau\left(z\right)\, \left(r-z\right)^{\beta_1-1}}{\left(z-x\right)^{1-\beta_2}} dz - \frac{\left(2-2\beta\right)^{\varepsilon-1}}{B\left(1-\beta_1,\, 1-\beta_2\right)} \int\limits_x^r \frac{\nu\left(z\right)\, \left(r-z\right)^{-\beta_2}}{\left(z-x\right)^{\beta_1}} dz.$$

In terms of fractional differentiation operators (in the sense of Riemann-Liouville) defined above, we rewrite the last equality

$$u\left[\theta_{r0}(x)\right] = \frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta_1\right)} \left(r-x\right)^{1-\beta} D_{rx}^{-\beta_2} \left[\tau\left(t\right)\, \left(r-t\right)^{\beta_1-1}\right] -$$

$$-\frac{\Gamma\left(2-\beta\right)}{\Gamma\left(1-\beta_2\right)} \left(2-2\beta\right)^{\beta-1}\, D_{rx}^{\beta_1-1} \left[\nu\left(t\right)\, \left(r-t\right)^{-\beta_2}\right]. \tag{17}$$

Next, let us use the laws of weighted composition operators of fractional differentiation and integration with the same origins [5], [7; 18], [36; 20]

$$D_{cx}^{-\gamma}\, D_{ct}^{\gamma} \varphi\left(s\right) = \varphi(x), \tag{18}$$

$$D_{cx}^{\alpha} \left|t-c\right|^{\alpha+\gamma}\, D_{ct}^{\gamma} \varphi\left(s\right) = \left|x-c\right|^{\gamma}\, D_{cx}^{\alpha+\gamma} \left|t-c\right|^{\alpha} \varphi\left(t\right), \tag{19}$$

where $0 < \alpha \leq 1$, $\gamma < 0$, $\alpha + \gamma > -1$; $\varphi(x) \in L\left[a, b\right]$, and for $\alpha + \gamma > 0$ the function $\varphi(x)$ has a fractional derivative $D_{cx}^{\alpha+\gamma} \varphi\left(t\right)$.

Applying the operator $D_{rx}^{1-\beta_1}$ to both parts of equality (17) and using composition laws (18), and (19) we obtain

$$\left(r-x\right)^{\beta_2}\, D_{rx}^{1-\beta_1} u\left[\theta_{r0}\left(t\right)\right] = \gamma_1 D_{rx}^{1-\beta} \tau\left(t\right) - \gamma_2 \nu(x), \tag{20}$$

where $\gamma_1 = \frac{\Gamma(\beta)}{\Gamma(\beta_1)}$, $\gamma_2 = \frac{\Gamma(2-\beta)\,(2-2\beta)^{\beta-1}}{\Gamma(1-\beta_2)}$.

Substituting $(r-x)^{\beta_2}\, D_{rx}^{1-\beta_1} u\left[\theta_{r0}\left(t\right)\right]$ by (20) we can specify condition (5) as follows

$$\left[\alpha_2(x) + \gamma_1 \alpha_1(x)\right] D_{rx}^{1-\beta} \tau\left(t\right) + \left[\alpha_3(x) - \gamma_2 \alpha_1(x)\right] \nu(x) = \psi_2(x). \tag{21}$$

Relation (21) is fundamental between the desired functions $\tau(x)$ and $\nu(x)$, the domain Ω_1 to the line $y = 0$ for $|\lambda| < \frac{m}{2}$. At $\lambda = \frac{m}{2}$ by (15) under condition (5) we arrive again at (21) but in this case for $\beta_1 = 0$, $\beta_2 = \beta = \frac{m}{m+2}$, $\gamma_1 = 0$, $\gamma_2 = 2^{\beta-1}\left(1-\beta\right)^{\beta}$, while for $\lambda = -\frac{m}{2}$ by (16) and (5) we get (21) for $\beta_1 = \beta = \frac{m}{m+2}$, $\beta_2 = 0$, $\gamma_1 = 1$, $\gamma_2 = \Gamma\left(2-\beta\right)\left(2-2\beta\right)^{\beta-1}$.

Next, we should obtain the fundamental relation between $\tau(x)$ and $\nu(x)$ transferred to the line$y = 0$ from Ω_2.

For this purpose, we study a representation of the regular solution to problem (12), (13) in Ω_2 for equation (3), which is written out by the d'Alembert formula [37; 59]:

$$u(x,y)=\frac{\tau(x+y)+\tau(x-y)}{2}+\frac{1}{2}\int\limits_{x-y}^{x+y}\nu(t)\,dt+\frac{1}{2}\int\limits_{0}^{y}\int\limits_{x-y+t}^{x+y-t}f(s,t)\,ds\,dt, \tag{22}$$

where $\tau(x)\in C[0,r]\cap C^2(0,r)$, $\nu(x)\in C^1(0,r)\cap L_1(0,r)$, $f(x,\,y)\in C\left(\bar{\Omega}_2\right)$.

Satisfying condition (4) in (22) obtain

$$u\left[\theta_{r1}(x)\right]=u\left(\frac{r+x}{2},\,\frac{r-x}{2}\right)=\frac{\tau(r)+\tau(x)}{2}+\frac{1}{2}\int\limits_{x}^{r}\nu(t)\,dt+\frac{1}{2}\int\limits_{0}^{\frac{r-x}{2}}\int\limits_{x+t}^{r-t}f(s,\,t)\,ds\,dt=\psi_1(x),$$

whence, using the differentiation, we get the relation form

$$\tau'(x)-\nu(x)-\int\limits_{0}^{\frac{r-x}{2}}f(x+t,\,t)\,dt=2\psi_1'(x). \tag{23}$$

Relation (23) is the fundamental relation between $\tau(x)$ and $\nu(x)$, transferred from Ω_2 to the segment I of the straight line $y=0$.

Eliminating the desired function $\nu(x)$ from the above relations (21) and (23) and taking into account the matching condition $\tau(r)=\psi_1(r)$ and condition (10) of *Theorem 1*, with respect to $\tau(x)$ we arrive at the first-order ordinary differential equation with a fractional-order derivative in lower terms

$$\tau'(x)+a(x)D_{rx}^{1-\beta}\tau(t)=2\psi_1'(x)+\frac{\psi_2(x)}{\alpha_3(x)-\gamma_2\alpha_1(x)}+\int\limits_{0}^{\frac{r-x}{2}}f(x+t,\,t)\,dt,\ \ 0<x<r, \tag{24}$$

$$\tau(r)=\psi_1(r), \tag{25}$$

where $a(x)=\frac{\alpha_2(x)+\gamma_1\alpha_1(x)}{\alpha_3(x)-\gamma_2\alpha_1(x)}$.

We integrate equation (24) from x to r, in view of the initial condition (25), and get the integral equation corresponding to problem (24)-(25)

$$\tau(x)-\frac{1}{\Gamma(\beta)}\int\limits_{x}^{r}K(x,\,t)\;\tau(t)\,dt=F_1(x), \tag{26}$$

where $K(x,\,t)=\frac{a(x)}{(t-x)^{1-\beta}}+\int\limits_{x}^{t}\frac{a'(s)ds}{(t-s)^{1-\beta}}$,

$F_1(x)=2\psi_1(x)-\psi_1(r)-\int\limits_{x}^{r}\frac{\psi_2(t)}{\alpha_3(t)-\gamma_2\alpha_1(t)}dt-\int\limits_{x}^{r}\int\limits_{0}^{(r-t)/2}f(t+s,s)\,ds\,dt.$

The properties of the given functions (7), (8), (9) suggest that equation (26) is a Volterra integral equation of the second kind with the kernel $K(x,\,t)\in L_1([0,r]\times[0,r])$ having a weak singularity for $x=t$ and the right side $F_1(x)\in C[0,r]\cap C^2(0,r)$. According to the general theory of Volterra integral equations, a solution to Eq. (26), is the unique solution, and can be written out by the formula

$$\tau(x)=F_1(x)+\int\limits_{x}^{r}K(x,\,t)\;F_1(t)\,dt, \tag{27}$$

where $R\,(x,\,t) = \sum\limits_{n=0}^{\infty} \frac{K_n(x,t)}{\Gamma^{n+1}(\beta)}$ is the resolvent kernel $K\,(x,\,t)$; $K_0\,(x,\,t) = K\,(x,\,t)$, $K_{n+1}\,(x,\,t) = \int\limits_t^x K\,(x,\,s)\;K_n\,(s,\,t)\,ds$ are the iterated kernels.

Let us show that the resolvent $R\,(x,\,t)$, like the kernel $K\,(x,\,t)$ of Eq. (26), belongs to the class $R\,(x,\,t) \in L_1\,([0,r]\times[0,r])$ and has a weak singularity at $x = t$, and the solution to Eq. (27), and its right-hand side $F_1(x)$, belongs to $\tau(x) \in C[0,r] \cap C^2(0,r)$.

Indeed, considering $\alpha(x) \in C^1[0,r] \cap C^2(0,r)$ we get estimates for iterated kernels $\frac{K_n(x,t)}{\Gamma^{n+1}(\beta)}$. Let $|\alpha(x)| \le M_1$ and $|\alpha'(x)| \le M_2 \quad \forall\, x \in [0,r]$. Then for the first iterated kernel we have the estimate

$$\left|\frac{K_0\,(x,\,t)}{\Gamma\,(\beta)}\right| = \left|\frac{K\,(x,\,t)}{\Gamma\,(\beta)}\right| = \frac{1}{\Gamma\,(\beta)}\left|\frac{\alpha(x)}{(t-x)^{1-\beta}} + \int\limits_x^t \frac{\alpha'\,(s)\,dt}{(t-s)^{1-\beta}}\right| \le \frac{M_1\,(t-x)^{\beta-1}}{\Gamma\,(\beta)} + \frac{M_2\,(t-x)^{\beta}}{\Gamma\,(\beta+1)}.$$

Next

$$\left|\frac{K_1\,(x,\,t)}{\Gamma^2\,(\beta)}\right| = \left|\int\limits_x^t \frac{K\,(x,\,s)}{\Gamma\,(\beta)}\,\frac{K_0\,(s,\,t)}{\Gamma\,(\beta)}\right| \le \int\limits_x^t \left[\frac{M_1\,(s-x)^{\beta-1}}{\Gamma\,(\beta)} + \frac{M_2\,(s-x)^{\beta}}{\Gamma\,(\beta+1)}\right] \times$$

$$\times \left[\frac{M_1\,(t-s)^{\beta-1}}{\Gamma\,(\beta)} + \frac{M_2\,(t-s)^{\beta}}{\Gamma\,(\beta+1)}\right] dt = \frac{M_1^2}{\Gamma^2\,(\beta)}\int\limits_x^t (s-x)^{\beta-1}\,(t-s)^{\beta-1}\,dt +$$

$$+\frac{M_1 M_2}{\Gamma\,(\beta)\,\Gamma\,(\beta+1)}\int\limits_x^t (s-x)^{\beta-1}\,(t-s)^{\beta}\,dt + \frac{M_1 M_2}{\Gamma\,(\beta)\,\Gamma\,(\beta+1)}\int\limits_x^t (s-x)^{\beta}\,(t-s)^{\beta-1}\,dt +$$

$$+\frac{M_2^2}{\Gamma^2\,(\beta+1)}\int\limits_x^t (s-x)^{\beta}\,(t-s)^{\beta}\,dt = \frac{M_1^2\;(t-x)^{2\beta-1}}{\Gamma\,(2\beta)} + \frac{2M_1M_2\;(t-x)^{2\beta}}{\Gamma\,(2\beta+1)} + \frac{M_2^2\;(t-x)^{2\beta+1}}{\Gamma\,(2\beta+2)}.$$

Similarly,

$$\left|\frac{K_2\,(x,\,t)}{\Gamma^3\,(\beta)}\right| = \left|\int\limits_t^x \frac{K\,(x,\,s)}{\Gamma\,(\beta)}\,\frac{K_1\,(s,\,t)}{\Gamma^2\,(\beta)}\right| \le \frac{M_1^3\;(t-x)^{3\beta-1}}{\Gamma\,(3\beta)} + \frac{3M_1^2M_2\;(t-x)^{3\beta}}{\Gamma\,(3\beta+1)} +$$

$$+\frac{3M_1M_2^2\;(t-x)^{3\beta+1}}{\Gamma\,(3\beta+2)} + \frac{M_2^3\;(t-x)^{3\beta+2}}{\Gamma\,(3\beta+3)}.$$

It's clear that

$$\left|\frac{K_{n-1}\,(x,\,t)}{\Gamma^n\,(\beta)}\right| \le \sum_{k=0}^{n} \frac{C_n^k\,M_1^{n-k}M_2^k\;(t-x)^{n\beta+k-1}}{\Gamma\,(n\beta+k)}, \tag{28}$$

where $C_n^k = \frac{n!}{k!\,(n-k)!}$ is a number of combinations of n elements taken k.

Noting that $\Gamma\,(n\beta+k) \ge \Gamma\,(n\beta)\ \forall\ k = 0,\,1,\,2,\,...$ from (28) we obtain the estimate

$$\left|\frac{K_{n-1}\,(x,\,t)}{\Gamma^n\,(\beta)}\right| < \frac{(t-x)^{n\beta-1}}{\Gamma\,(n\beta)}\sum_{k=0}^{n} C_n^k\,M_1^{n-k}M_2^k\;(t-x)^k = \frac{(t-x)^{n\beta-1}}{\Gamma\,(n\beta)}\;[M_1 + M_2\,(t-x)]^n\,. \tag{29}$$

For sufficiently large n index $n\beta-1$ at $(t-x)$ in (29) is positive. And in this case, the difference $(t-x)$ can be replaced by a higher numerical value r. Thus, for the resolvent $R\,(x,\,t)$ of the kernel $K\,(x,\,t)$ we obtain the estimate:

$$|R\,(x,\,t)| = \left|\sum_{n=1}^{\infty}\frac{K_{n-1}\,(x,\,t)}{\Gamma^n\,(\beta)}\right| < \sum_{n=1}^{\infty}\frac{(M_1+M_2r)^n\;r^{n\beta-1}}{\Gamma\,(n\beta)}\,. \tag{30}$$

Using the Stirling formula for the Gamma function:

$$\Gamma(n) = \frac{1}{\sqrt{2\pi n}} n^n e^{-n+\frac{\eta}{12n}}, 0 < \eta < 1.$$

Cauchy criterion for the convergence of numerical series, it is easy to see that the right side series of inequality (30) converges. Thus, the series for the resolvent $R(x, t)$ of the kernel $K(x, t)$ in Eq. (26) converges absolutely and uniformly, and we can conclude that the resolvent of the kernel is continuous for any $0 < \beta < 1$ and any $x \neq t \in [0, r]$, having a weak singularity for $x = t$.

Further, by representation (27) and estimates (29), (30) with a continuous right-hand side, obtain the estimate

$$|\tau(x)| = \left| F_1(x) + \int_x^r K(x, t)\ F_1(t)\, dt \right| < M_3 \left[1 + \sum_{n=1}^{\infty} \frac{(M_1 + M_2 r)^n\, r^{n\beta}}{\Gamma(n\beta)} \right], \tag{31}$$

where $M_3 = \max\limits_{0 \le x \le r} |F_1(x)|$.

The convergence of the majorizing sequences (the right side of inequality (31)) implies the absolute and uniform convergence according to the Weierstrass test. Whence we conclude the continuity of the limit function $\tau(x) \in C[0, r]$.

Now $F_1(x) \in C^2(0, r)$. In this case, by double integration by parts on the right side of representation (27), we can see clearly that $\tau(x) \in C^2(0, r)$ that is, the solution to Eq. (26), as well as its right side is belong to $\tau(x) \in C[0, r] \cap C^2(0, r)$.

When $a(x) = a = const$, the solution to (26) is written out explicitly using the formula:

$$\tau(x) = F_1(x) + a \int_x^r R(x, t; a)\ F_1(t)\, dt,$$

where $R(x, t; a) = (t - x)^{\beta - 1}\ E_{\frac{1}{\beta}} \left[a\ (t - x)^{\beta};\ \beta \right]$, and $E_\rho(z; \mu) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu + n\rho^{-1})}$ is the Mittag-Leffler type function [38; 117], which coincides with the Mittag-Leffler function $E_\rho(z; 1) = E_{1/\rho}(z)$ at $\mu = 1$.

If condition (11) is satisfied, then using system (21), (23) we can immediately get:

$$\tau(x) = D_{rx}^{\beta-1} \left[\frac{\psi_2(t)}{\alpha_2(t) + \gamma_1 \alpha_1(t)} \right], \nu(x) = D_{rx}^{\beta} \left[\frac{\psi_2(t)}{\alpha_2(t) + \gamma_1 \alpha_1(t)} \right] - 2\psi_1'(x) - \int_0^{(r-x)/2} f(x + t, t)\, dt.$$

Study of Problem 2

Now we proceed to the study of problem 2. Satisfying condition (6) for (22) we obtain:

$$u[\theta_{01}(x)] = u\left(\frac{x}{2}, \frac{x}{2}\right) = \frac{\tau(x) + \tau(0)}{2} + \frac{1}{2} \int_0^x \nu(t)\, dt + \frac{1}{2} \int_0^{x/2} \int_t^{x-t} f(s, t)\, ds dt = \psi_1(x),$$

then by differentiation, we get

$$\nu(x) + \tau'(x) + \int_0^{x/2} f(x - t, t)\, dt = 2\psi_1'(x). \tag{32}$$

Relation (32) is the fundamental relation between $\tau(x)$ and $\nu(x)$, transferred from Ω_2 to the segment I of the strait line $y = 0$, in case with *Problem 2.*

Thus, when $|\lambda| < \frac{m}{2}$ with respect to the desired $\tau(x)$ and $\nu(x)$ one gets a system of equations expressed through (21) and (32). Eliminating from (21) and (32) the unknown $\nu(x)$ with respect to $\tau(x)$, in view of the matching condition $\tau(0) = \psi_1(0)$, the same way as with *problem 1*, we arrive at the following boundary value problem for a first-order ordinary differential equation with a fractional-order derivative in lower terms

$$\tau'(x) - a(x)\, D_{rx}^{1-\beta}\tau(t) = F_2(x), \qquad 0 < x < r, \tag{33}$$

$$\tau(0) = \psi_1(0), \tag{34}$$

where $F_2(x) = 2\psi_1'(x) - \frac{\psi_2(x)}{\alpha_3(x)-\gamma_2\alpha_1(x)} - \int\limits_0^{x/2} f(x-t,\, t)\, dt$.

Integrating equation (33) with respect to the variable x from x to r, considering condition (34), we obtain the integral equation corresponding to problem (33)-(34)

$$\tau(x) + \frac{1}{\Gamma(\beta)}\int\limits_0^r L(x,\, t)\,\tau(t)\, dt = \psi_1(0) + \int\limits_0^x F_2(t)\, dt, \tag{35}$$

where $L(x,\, t) = \begin{cases} K(0,\, t) - K(x,\, t), & 0 \le x < t, \\ K(0,\, t), & t < x \le r. \end{cases}$

If the given functions $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ and $f(x, y)$ have the properties (7)–(9) listed in Theorem 1, then equation (35) is a Fredholm integral equation of the second kind with the kernel $L(x,\, t) \in L_1([0, r] \times [0, r])$, with a weak singularity at $x = t$, and a right-hand side of $C[0, r] \cap C^2(0, r)$.

Let us further find sufficient conditions that ensure the unique solvability to Eq. (35). To this end, let's consider a homogeneous problem corresponding to *Problem 2*, setting $\psi_1(x) \equiv 0$, $\psi_2(x) \equiv 0$ $\forall\, x \in [0, r]$ and $f(x,\, y) \equiv 0$ $\forall\, (x, y) \in \bar{\Omega}_2$. In this case, problem (33)-(34) turns into the corresponding homogeneous problem

$$\frac{1}{a(x)}\,\tau'(x) - D_{rx}^{1-\beta}\tau(t) = 0, \qquad 0 < x < r, \tag{36}$$

$$\tau(0) = 0. \tag{37}$$

Multiplying equation (36) by the function $\tau(x)$, and integrating the resulting equality with respect to the variable x from 0 to r, with condition (37) we have

$$\int\limits_0^r a^{-1}(x)\,\tau(x)\,\tau'(x)dx - \int\limits_0^r \tau(x)\, D_{rx}^{1-\beta}\tau(t)\, dx =$$

$$= \frac{\tau^2(r)}{2a(r)} + \int\limits_0^r \frac{a'(x)}{2a^2(x)}\,\tau^2(x)dx - \int\limits_0^r \tau(x)\, D_{rx}^{1-\beta}\tau(t)\, dx = 0. \tag{38}$$

To estimate $\int\limits_0^r \tau(x)\, D_{rx}^{1-\beta}\tau(t)\, dx$, we use Lemma 1 by [39], according to which $\tau(x)\, D_{rx}^{\alpha}\varphi(t) \ge \frac{1}{2}\, D_{rx}^{\alpha}\varphi^2(t)$, $0 < \alpha \le 1$. With this inequality, we have

$$\int\limits_0^r \tau(x)\, D_{rx}^{1-\beta}\tau(t)\, dx \ge \frac{1}{2}\int\limits_0^r D_{rx}^{1-\beta}\tau^2(t)\, dx = \frac{1}{2\Gamma(\beta)}\int\limits_0^r t^{\beta-1}\,\tau^2(t)\, dt \ge 0. \tag{39}$$

If the function $a(x)$ is a nonincreasing negative, then, as follows by (39), equality (38) can take place if and only if $\tau(x) \equiv 0 \,\forall x \in [0,r]$. Then by (21) and (32) at $\psi_1(x) \equiv 0$, $\psi_2(x) \equiv 0 \,\forall x \in [0,r]$, $f\,(x,\,y) \equiv 0 \quad \forall\,(x,\,y) \in \bar{\Omega}_2$ and $[\alpha_3(x) - \gamma_2\alpha_1(x)]\,[\alpha_2(x) + \gamma_1\alpha_1(x)] \neq 0 \,\forall x \in [0,r]$ it follows that $\nu(x) \equiv 0 \,\forall x \in [0,r]$ as well. Therefore, under the above conditions Eq. (32) has a unique solution within $\tau(x) \in C[0,r] \cap C^2(0,r)$.

Thus, we have proved the following theorem.

Theorem 2. Let the given functions $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ and $f\,(x,\,y)$ be such that they have properties (7)–(9) and let

$$a(x) < 0,\, a'(x) \leq 0 \ \ \forall x \in [0,r], \tag{40}$$

$$[\alpha_3(x) - \gamma_2\alpha_1(x)]\,[\alpha_2(x) + \gamma_1\alpha_1(x)] \neq 0 \ \ \forall x \in [0,r]. \tag{41}$$

Then there exists a unique regular solution to Problem 2 in Ω.

In the case when $a(x) = a = const$ the solution to problem (33)-(34) is written out explicitly according to

$$\tau(x) = \frac{E_\beta\left[-a\,(r-x)^\beta\right]}{E_\beta\left[-a\,r^\beta\right]}\psi_1\,(0) + \frac{E_\beta\left[-a\,(r-x)^\beta\right]}{E_\beta\left[-a\,r^\beta\right]}\int\limits_0^r E_\beta\left[-at^\beta\right]\,F_2\,(t)\,dt - \int\limits_x^r E_\beta\left[-a\,(r-x)^\beta\right]\,F_2\,(t)\,dt,$$

and

$$E_\beta\left[-ar^\beta\right] \neq 0. \tag{42}$$

As follows from conditions (40)-(41) Theorem 2, inequality (42) will be satisfied, for example, for all $a < 0$.

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Екінші ретті аралас-гиперболалық теңдеу үшін ығысуы бар шеттік есептер

Мақалада облыстың бір бөлігінде толқындық теңдеуден және екінші бөлігінде бірінші типті гиперболалық теңдеуден тұратын екінші ретті гиперболалық типті екі теңдеудің түйіндесуінде ығысуы бар екі бейлокалды есеп зерттелген. Зерттелген есептердегі бейлокалды шеттік шарт ретінде сипаттамалардың бірінде және типті өзгерту сызығында қажет функцияның бірінші ретті туынды және бөлшек ретті туынды (Риман-Лиувилль мағынасында) мәндерінің айнымалы коэффициенттері бар сызықтық комбинациясы берілген. Интегралдық теңдеулер әдісін қолдана отырып, бірінші есептің шешімділігі екінші текті әлсіз сингулярлығы бар Вольтерра интегралдық теңдеуінің шешімділігіне, ал екінші есептің шешілетіндігі туралы мәселе әлсіз сингулярлығы бар екінші текті Фредгольм интегралдық теңдеуінің шешімділігіне көшеді. Бірінші есеп үшін екінші текті Вольтерра интегралдық теңдеуінің нәтижесінде алынған ядроның резольвентасына бірқалыпты жинақтылығын және оның шешімі қажетті класқа жататыны дәлелденген. Екінші есеп үшін талап етілетін кластан әлсіз ерекшелікпен екінші текті Фредгольм интегралдық теңдеуінің жалғыз шешімінің болуын қамтамасыз ететін берілген функциялар үшін жеткілікті шарттар табылды. Кейбір ерекше жағдайлар үшін есептердің шешімдері анық жазылған.

Кілт сөздер: толқындық теңдеу, бірінші текті өзгешеленген гиперболалық теңдеу, Вольтерра интегралдық теңдеуі, Фредгольм интегралдық теңдеуі, Трикоми әдісі, интегралдық теңдеулер әдісі, бөлшекті есептеу теориясының әдістері.

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Краевые задачи со смещением для одного смешанно-гиперболического уравнения второго порядка

В статье исследованы две нелокальные задачи со смещением на сопряжение двух уравнений гиперболического типа второго порядка, состоящего из волнового уравнения в одной части области и вырождающегося гиперболического уравнения первого рода — в другой. В качестве нелокального граничного условия в исследуемых задачах задана линейная комбинация с переменными коэффициентами значений производной первого порядка и производных дробного (в смысле Римана–Лиувилля) порядка от искомой функции на одной из характеристик и на линии изменения типа. С использованием метода интегральных уравнений вопрос разрешимости первой задачи эквивалентным образом редуцирован к вопросу разрешимости интегрального уравнения Вольтерра второго рода со слабой особенностью, а вопрос разрешимости второй задачи — к вопросу разрешимости интегрального уравнения Фредгольма второго рода со слабой особенностью. По первой задаче доказаны равномерная сходимость резольвенты ядра получающегося интегрального уравнения Вольтерра второго рода и принадлежность его решения требуемому классу. По второй задаче найдены достаточные условия на заданные функции, обеспечивающие существование единственного решения интегрального уравнения Фредгольма второго рода со слабой особенностью из требуемого класса. В некоторых частных случаях решения задач выписаны в явном виде.

Ключевые слова: волновое уравнение, вырождающееся гиперболическое уравнение первого рода, интегральное уравнение Вольтерра, интегральное уравнение Фредгольма, метод Трикоми, метод интегральных уравнений, методы теории дробного исчисления.

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DOI 10.31489/2023M4/56-65

UDC 517.51

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Best approximation by «angle» and the absolute Cesàro summability of double Fourier series

This article is devoted to the topic of absolute summation of series or Cesaro summation. The relevance of this article lies in the fact that a type of absolute summation with vector index which has not been previously studied is considered. In this article, a sufficient condition for the vector index absolute summation method was obtained in terms of the best approximation by «angle» of the functions from Lebesgue space. The theorem that gives a sufficient condition proves the conditions that are sufficient in different cases, which may depend on the parameters. From this proved theorem, a sufficient condition on the term mixed smoothness modulus of the function from Lebesgue space, which is easily obtained by a well-known inequality, is also presented.

Keywords: trigonometric series, Fourier series, Lebesgue space, best approximation by «angle», absolute summability of the series.

Introduction and preliminaries

Let $I_2 = \{(x_1, x_2) \in \mathbf{R}^2 : 0 \le x_j < 2\pi\}$.

We denote by $L_q(I_2)$ the space of all measurable by Lebesque, 2π-periodic on each variable functions $f(x_1, x_2)$, such that

$$\|f\|_q = \left(\int_0^{2\pi}\int_0^{2\pi} |f(x_1, x_2)|^q dx_1 dx_2\right)^{\frac{1}{q}} < +\infty, 1 \le q < +\infty.$$

Let $Y_{n_1 n_2}(f)_q$ is two-dimensional best approximation by «angle» of function $f \in L_q(I_2)$. By definition [1–3],

$$Y_{n_1 n_2}(f)_q = \inf_{T_{n_1,\infty}, T_{\infty,n_2}} \|f - T_{n_1,\infty} - T_{\infty,n_2}\|_q\,,$$

where the function $T_{n_1,\infty} \in L_q(I_2)$ is a trigonometric polynomial of degree at most n_1 in x_1, and the function $T_{\infty,n_2} \in L_q(I_2)$ is a trigonometric polynomial of degree at most n_2 in x_2.

Let $r \in \mathbf{N}$, $h_1, h_2 \in \mathbf{R}$. For a function $f \in L_q(I_2)$, the difference of order $r \in \mathbf{N}$ with respect to the variable x_1 and the difference of order $r \in \mathbf{N}$ with respect to the variable x_2 are defined as follows [1–3]:

$$\Delta^r_{h_1,x_1} f(x_1, x_2) = \sum_{\nu_1=0}^{r} (-1)^{r-\nu_1} \cdot C_r^{\nu_1} \cdot f(x_1 + h_1\nu_1, x_2),$$

and, respectively

$$\Delta^r_{h,x_2} f(x_1, x_2) = \sum_{\nu_2=0}^{r} (-1)^{r-\nu_2} \cdot C_r^{\nu_2} \cdot f(x_1, x_2 + h_2\nu_2).$$

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Denote by $\Omega_r(f;t_1,t_2)_q$ the mixed module of smoothness of an order r of function $f\in L_q(I_2)$ [1–3]:

$$\Omega_r(f;t_1,t_2)_q=\sup_{\substack{|h_j|\le\ t_j\\ j=1,2}}\|\Delta^r_{h_2,x_2}\left(\Delta^r_{h_1,x_1}(f)\right)\|_q.$$

Consider a double trigonometric series

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}(a_{n_1,n_2}\cos n_1x_1\cos n_2x_2+b_{n_1,n_2}\sin n_1x_1\cos n_2x_2+$$

$$+c_{n_1,n_2}\cos n_1x_1\sin n_2x_2+d_{n_1,n_2}\sin n_1x_1\sin n_2x_2)\equiv\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}B_{n_1,n_2}(x_1,x_2). \qquad (1)$$

Let's write it down

$$A_n^{(\beta)}=\frac{(\beta+1)(\beta+2)\dots(\beta+n)}{n!},$$

where β is a real number, n is a natural number.

The sum

$$\sigma^{(\beta_1,\beta_2)}_{n_1,n_2}(x_1,x_2)=\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}\prod_{j=1}^{2}A^{(\beta_j-1)}_{n_j-k_j}\left(A^{(\beta_j)}_{n_j}\right)^{-1}B_{k_1,k_2}(x_1,x_2)$$

called $(C;\beta_1,\beta_2)$ mean of series (1).

The series (1) called $|C;\beta_1,\beta_2|_{\lambda_1,\lambda_2}$-summable (or absolute summable with vector index), $\lambda_j\ge 1$, $j=1,2$, at the point $(x_1,x_2)\in I_2$, if the following series converges:

$$\sum_{n_2=1}^{\infty}n_2^{\lambda_2-1}\left[\sum_{n_1=1}^{\infty}n_1^{\lambda_1-1}\left|\sigma^{(\beta_1,\beta_2)}_{n_1,n_2}(x_1,x_2)-\sigma^{(\beta_1,\beta_2)}_{n_1-1,n_2}(x_1,x_2)-\right.\right.$$

$$\left.\left.-\sigma^{(\beta_1,\beta_2)}_{n_1,n_2-1}(x_1,x_2)+\sigma^{(\beta_1,\beta_2)}_{n_1-1,n_2-1}(x_1,x_2)\right|^{\lambda_1}\right]^{\frac{\lambda_2}{\lambda_1}}. \qquad (2)$$

Let

$$\tau^{(\beta_1,\beta_2)}_{n_1,n_2}(x_1,x_2)=\left(\prod_{j=1}^{2}A^{(\beta_j)}_{n_j}\right)^{-1}\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}\left(\prod_{j=1}^{2}k_jA^{(\beta_j-1)}_{n_j-k_j}\right)B_{k_1,k_2}(x_1,x_2).$$

Then the convergence of the series (2) is equivalent to the convergence of the following series

$$\sum_{n_2=1}^{\infty}n_2^{-1}\left[\sum_{n_1=1}^{\infty}n_1^{-1}\left|\tau^{(\beta_1,\beta_2)}_{n_1,n_2}(x_1,x_2)\right|^{\lambda_1}\right]^{\frac{\lambda_2}{\lambda_1}}.$$

In case $\lambda_2=\lambda_1=\lambda$ we will write $|C;\beta_1,\beta_2|_\lambda$ (absolute summability with scalar index) instead of $|C;\beta_1,\beta_2|_{\lambda_1,\lambda_2}$.

Issues related to the absolute Cesàro summability of series began to be studied intensively in the twentieth century. Among the many scientists we can mention the work of F.T. Wang [4], I.E. Zhak and M.F. Timan [5], K. Tandori [6], L. Leindler [7], M.F. Timan [8], Yu.A. Ponomarenko and M.F. Timan [9],

I. Szalay [10, 11], G. Sunouchi [12], who studied the conditions of the absolute Cesàro summability of trigonometric and orthogonal series. In recent years, various generalizations of absolute Cesàro summability have been defined, and the former classical results have been proved for these generalizations. For example, we can cite the works of H. Bor [13–15], Yu. Dansheng and Zhou Guanzhen [16], S. Sonker and A. Munjal [17, 18], E. Savaş [19, 20], B. Rhoades and E. Savaş [21]. In addition, L. Leindler [22] and H. Bor [23] gave a new application of power increasing sequence by applying absolute Cesàro summability for an infinity series. Problems of absolute Cesàro summability of multiple trigonometric Fourier series of functions from different spaces studied in works [24–29]. Almost all of this work is devoted to the topic of absolute Cesàro summability with scalar index. The absolute Cesàro summability with vector index was first defined in [24]. In the article [24] the condition $\beta_1 = \beta_2$ is considered and only sufficient conditions are obtained. Feature of our work is that under sufficient conditions $\beta_1 \neq \beta_2$.

Main results

Now we prove the main results.

We denote $\rho_{k_1k_2} = \sqrt{a^2_{k_1k_2} + b^2_{k_1k_2} + c^2_{k_1k_2} + d^2_{k_1k_2}}$.

Theorem 1. Let $1 < q \leq 2$, $1 \leq \lambda_2 \leq \lambda_1 \leq q$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then for $|C;\beta_1,\beta_2|_{\lambda_1,\lambda_2}$-summability almost everywhere on I_2 Fourier series of function $f(x_1,x_2) \in L_q(I_2)$ is sufficiently,

1) in case of $\frac{1}{q'} < \beta_1 < +\infty$, $\frac{1}{q'} = \beta_2$, for the condition to be met:

$$\sum_{n_2=2}^{\infty} (\ln n_2)^{\frac{\lambda_2}{q}} n_2^{\lambda_2\left(\frac{2}{q}-1\right)-1} \left[\sum_{n_1=1}^{\infty} n_1^{\lambda_1\left(\frac{2}{q}-1\right)-1} \cdot Y^{\lambda_1}_{n_1n_2}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}} < +\infty;$$

2) in case of $\frac{1}{q'} < \beta_1 < +\infty$, $-1 < \beta_2 < \frac{1}{q'}$, for the condition to be met:

$$\sum_{n_2=1}^{\infty} n_2^{\left(\frac{1}{q}-\beta_2\right)\lambda_2-1} \left[\sum_{n_1=1}^{\infty} n_1^{\left(\frac{2}{q}-1\right)\lambda_1-1} \cdot Y^{\lambda_1}_{n_1n_2}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}} < +\infty;$$

3) in case of $-1 < \beta_1 < \frac{1}{q'}$, $\frac{1}{q'} = \beta_2$ for the condition to be met:

$$\sum_{n_2=2}^{\infty} n_2^{\lambda_2\left(\frac{2}{q}-1\right)-1} (\ln n_2)^{\frac{\lambda_2}{q}} \left[\sum_{n_1=1}^{\infty} n_1^{\left(\frac{1}{q}-\beta_1\right)\lambda_1-1} \cdot Y^{\lambda_1}_{n_1n_2}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}} < +\infty.$$

Proof of item 1). It was proved in work [29] that in case of $\frac{1}{q'} < \beta_1 < +\infty$, $\frac{1}{q'} = \beta_2$, if the next series

$$\sum_{n_2=0}^{\infty} \left[\sum_{n_1=0}^{\infty} \left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1} \sum_{k_2=2^{n_2}}^{2^{n_2+1}-1} \rho^q_{k_1k_2} \ln k_2\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}}$$

converges, then series (1) is $|C;\beta_1,\beta_2|_{\lambda_1,\lambda_2}$-summable almost everywhere on I_2.

By simple calculations, we get

$$S = \sum_{n_2=0}^{\infty} \left[\sum_{n_1=0}^{\infty} \left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1} \sum_{k_2=2^{n_2}}^{2^{n_2+1}-1} \rho^q_{k_1k_2} \ln k_2\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}} =$$

$$= \sum_{n_2=0}^{\infty}\left[\sum_{n_1=0}^{\infty}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}(k_1k_2)^{q-2}\ln k_2(k_1k_2)^{2-q}\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C\sum_{n_2=0}^{\infty}\left[\sum_{n_1=0}^{\infty}\left(2^{n_1(2-q)}2^{n_2(2-q)}n_2\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}(k_1k_2)^{q-2}\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C\sum_{n_2=1}^{\infty}2^{n_2\lambda_2\left(\frac{2}{q}-1\right)}n_2^{\frac{\lambda_2}{q}}\left[\sum_{n_1=0}^{\infty}2^{n_1\lambda_1\left(\frac{2}{q}-1\right)}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}(k_1k_2)^{q-2}\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}}.$$

Hence, using the Hardy-Littlewood theorem [30], we obtain

$$S \leq C\sum_{n_2=1}^{\infty}2^{n_2\lambda_2\left(\frac{2}{q}-1\right)}n_2^{\frac{\lambda_2}{q}}\left[\sum_{n_1=0}^{\infty}2^{n_1\lambda_1\left(\frac{2}{q}-1\right)}\left\|\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}B_{k_1k_2}(\cdot,\cdot)\right\|_q^{\lambda_1}\right]^{\frac{\lambda_2}{\lambda_1}}. \tag{3}$$

Now, using inequality [1]

$$\left\|\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}B_{k_1k_2}(\cdot,\cdot)\right\|_q \leq C\cdot Y_{2^{n_1}-1,2^{n_2}-1}(f)_q, \tag{4}$$

due to the monotonicity of the best approximation by an «angle» from (3), we have

$$S \leq C\sum_{n_2=1}^{\infty}2^{n_2\lambda_2\left(\frac{2}{q}-1\right)}n_2^{\frac{\lambda_2}{q}}\left[\sum_{n_1=0}^{\infty}2^{n_1\lambda_1\left(\frac{2}{q}-1\right)}Y_{2^{n_1}-1,2^{n_2}-1}^{\lambda_1}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C\sum_{n_2=2}^{\infty}(\ln n_2)^{\frac{\lambda_2}{q}}\,n_2^{\lambda_2\left(\frac{2}{q}-1\right)-1}\left[\sum_{n_1=1}^{\infty}n_1^{\lambda_1\left(\frac{2}{q}-1\right)-1}Y_{n_1n_2}^{\lambda_1}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}}.$$

Proof of item 2). In case of $\frac{1}{q'} < \beta_1 < +\infty, \quad -1 < \beta_2 < \frac{1}{q'}$, by Theorem 2 in [29], the convergence of series

$$\sum_{n_2=0}^{\infty}\left[\sum_{n_1=0}^{\infty}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}}$$

implies the $|C;\beta_1,\beta_2|_{\lambda_1,\lambda_2}$-summability of series (1) almost everywhere on I_2.

Carrying out simple calculations, using the Hardy-Littlewood theorem [30] and inequality (4), we obtain

$$\sum_{n_2=0}^{\infty}\left[\sum_{n_1=0}^{\infty}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}} =$$

$$= \sum_{n_2=0}^{\infty} \left[\sum_{n_1=0}^{\infty} \left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1} \sum_{k_2=2^{n_2}}^{2^{n_2+1}-1} \rho_{k_1 k_2}^{q} (k_1 k_2)^{q-2} k_2^{q(1-\beta_2)-1} (k_1 k_2)^{2-q} \right)^{\frac{\lambda_1}{q}} \right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C \sum_{n_2=0}^{\infty} 2^{n_2 \lambda_2 \left(\frac{1}{q} - \beta_2\right)} \left[\sum_{n_1=0}^{\infty} 2^{n_1 \lambda_1 \left(\frac{2}{q} - 1\right)} \left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1} \sum_{k_2=2^{n_2}}^{2^{n_2+1}-1} \rho_{k_1 k_2}^{q} (k_1 k_2)^{q-2} \right)^{\frac{\lambda_1}{q}} \right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C \sum_{n_2=0}^{\infty} 2^{n_2 \lambda_2 \left(\frac{1}{q} - \beta_2\right)} \left[\sum_{n_1=0}^{\infty} 2^{n_1 \lambda_1 \left(\frac{2}{q} - 1\right)} \left\| \sum_{k_1=2^{n_1}}^{2^{n_1+1}-1} \sum_{k_2=2^{n_2}}^{2^{n_2+1}-1} B_{k_1 k_2}(\cdot, \cdot) \right\|_q^{\lambda_1} \right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C \sum_{n_2=0}^{\infty} 2^{n_2 \lambda_2 \left(\frac{1}{q} - \beta_2\right)} \left[\sum_{n_1=0}^{\infty} 2^{n_1 \lambda_1 \left(\frac{2}{q} - 1\right)} \cdot Y_{2^{n_1}-1, 2^{n_2}-1}^{\lambda_1}(f)_q \right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C \sum_{n_2=1}^{\infty} n_2^{\left(\frac{1}{q} - \beta_2\right)\lambda_2 - 1} \left[\sum_{n_1=1}^{\infty} n_1^{\left(\frac{2}{q} - 1\right)\lambda_1 - 1} \cdot Y_{n_1 n_2}^{\lambda_1}(f)_q \right]^{\frac{\lambda_2}{\lambda_1}}.$$

Proof of item 3). Let $-1 < \beta_1 < \frac{1}{q'}$, $\frac{1}{q'} = \beta_2$. Then by Theorem 2 in [29] the convergence of series

$$\sum_{n_2=0}^{\infty} \left[\sum_{n_1=0}^{\infty} \left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1} \sum_{k_2=2^{n_2}}^{2^{n_2+1}-1} \rho_{k_1 k_2}^{q} k_1^{q(1-\beta_1)-1} \cdot \ln k_2 \right)^{\frac{\lambda_1}{q}} \right]^{\frac{\lambda_2}{\lambda_1}}$$

implies the $|C; \beta_1, \beta_2|_{\lambda_1, \lambda_2}$-summability of series (1) almost everywhere on I_2.

In a similar way to the proof of the previous points, using the Hardy-Littlewood theorem [30] and inequality (4), we get

$$\sum_{n_2=0}^{\infty} \left[\sum_{n_1=0}^{\infty} \left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1} \sum_{k_2=2^{n_2}}^{2^{n_2+1}-1} \rho_{k_1 k_2}^{q} k_1^{q(1-\beta_1)-1} \cdot \ln k_2 \right)^{\frac{\lambda_1}{q}} \right]^{\frac{\lambda_2}{\lambda_1}} =$$

$$= \sum_{n_2=0}^{\infty} \left[\sum_{n_1=0}^{\infty} \left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1} \sum_{k_2=2^{n_2}}^{2^{n_2+1}-1} \rho_{k_1 k_2}^{q} (k_1 k_2)^{q-2} k_1^{q(1-\beta_1)-1} (k_1 k_2)^{2-q} \cdot \ln k_2 \right)^{\frac{\lambda_1}{q}} \right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C \sum_{n_2=1}^{\infty} 2^{n_2 \lambda_2 \left(\frac{2}{q} - 1\right)} n_2^{\frac{\lambda_2}{q}} \left[\sum_{n_1=0}^{\infty} 2^{n_1 \lambda_1 \left(\frac{1}{q} - \beta_1\right)} \left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1} \sum_{k_2=2^{n_2}}^{2^{n_2+1}-1} \rho_{k_1 k_2}^{q} (k_1 k_2)^{q-2} \right)^{\frac{\lambda_1}{q}} \right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C \sum_{n_2=1}^{\infty} 2^{n_2 \lambda_2 \left(\frac{2}{q} - 1\right)} n_2^{\frac{\lambda_2}{q}} \left[\sum_{n_1=0}^{\infty} 2^{n_1 \lambda_1 \left(\frac{1}{q} - \beta_1\right)} \left\| \sum_{k_1=2^{n_1}}^{2^{n_1+1}-1} \sum_{k_2=2^{n_2}}^{2^{n_2+1}-1} B_{k_1 k_2}(\cdot, \cdot) \right\|_q^{\lambda_1} \right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C \sum_{n_2=2}^{\infty} n_2^{\lambda_2\left(\frac{2}{q}-1\right)-1} (\ln n_2)^{\frac{\lambda_2}{q}} \left[\sum_{n_1=1}^{\infty} n_1^{\left(\frac{1}{q}-\beta_1\right)\lambda_1-1} \cdot Y_{n_1 n_2}^{\lambda_1}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}}.$$

Thus, the theorem is fully proved.

Using the following inequality [1]:

$$Y_{n_1 n_2}(f)_q \leq C \cdot \Omega_r(f; \frac{1}{n_1+1}, \frac{1}{n_2+1})_q$$

we can formulate another result.

Theorem 2. Let $1 < q \leq 2$, $1 \leq \lambda_2 \leq \lambda_1 \leq q$, $\frac{1}{q} + \frac{1}{q'} = 1$ and r is a natural number. Then for $|C; \beta_1, \beta_2|_{\lambda_1, \lambda_2}$-summability almost everywhere on I_2 Fourier series of function $f(x_1, x_2) \in L_q(I_2)$ is sufficiently,

1) in case of $\frac{1}{q'} < \beta_1 < +\infty$, $\frac{1}{q'} = \beta_2$, for the condition to be met:

$$\sum_{n_2=2}^{\infty} (\ln n_2)^{\frac{\lambda_2}{q}} n_2^{\lambda_2\left(\frac{2}{q}-1\right)-1} \left[\sum_{n_1=1}^{\infty} n_1^{\lambda_1\left(\frac{2}{q}-1\right)-1} \cdot \Omega_r^{\lambda_1}(f; \frac{1}{n_1}, \frac{1}{n_2})_q\right]^{\frac{\lambda_2}{\lambda_1}} < +\infty;$$

2) in case of $\frac{1}{q'} < \beta_1 < +\infty$, $-1 < \beta_2 < \frac{1}{q'}$, for the condition to be met:

$$\sum_{n_2=1}^{\infty} n_2^{\left(\frac{1}{q}-\beta_2\right)\lambda_2-1} \left[\sum_{n_1=1}^{\infty} n_1^{\left(\frac{2}{q}-1\right)\lambda_1-1} \cdot \Omega_r^{\lambda_1}(f; \frac{1}{n_1}, \frac{1}{n_2})_q\right]^{\frac{\lambda_2}{\lambda_1}} < +\infty;$$

3) in case of $-1 < \beta_1 < \frac{1}{q'}$, $\frac{1}{q'} = \beta_2$ for the condition to be met:

$$\sum_{n_2=2}^{\infty} n_2^{\lambda_2\left(\frac{2}{q}-1\right)-1} (\ln n_2)^{\frac{\lambda_2}{q}} \left[\sum_{n_1=1}^{\infty} n_1^{\left(\frac{1}{q}-\beta_1\right)\lambda_1-1} \cdot \Omega_r^{\lambda_1}(f; \frac{1}{n_1}, \frac{1}{n_2})_q\right]^{\frac{\lambda_2}{\lambda_1}} < +\infty.$$

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«Бұрышпен» ең жақсы жуықтау және екі еселі Фурье қатарының Чезаро бойынша абсолютті қосындылануы

Мақала қатарлардың абсолютті қосындылануы немесе Чезаро бойынша қосындылану тақырыбына арналған. Бұл жұмыстың өзектілігі мынада: бұрын көп зерттелмеген векторлық индекстің абсолютті қосындылану түрі қарастырылатындығында. Авторлар векторлық индекстің абсолютті қосындылану тәсілі үшін Лебег кеңістігі функциясының «бұрышпен» ең жақсы жуықтауы терминіндегі жеткілікті шартты алған. Жеткілікті шартты беретін теорема параметрлерге байланысты әртүрлі жағдайларда жеткілікті шарттарды дәлелдейді. Осы дәлелденген теоремадан белгілі теңсіздіктің көмегімен Лебег кеңістігі функциясының аралас тегістік модулі терминіндегі жеткілікті шарт алынады.

Кілт сөздер: тригонометриялық қатар, Фурье қатары, Лебег кеңістігі, «бұрышпен» ең жақсы жуықтау, қатардың абсолютті қосындылануы.

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Наилучшее приближение «углом» и абсолютная суммируемость по Чезаро двойных рядов Фурье

Статья посвящена теме абсолютного суммирования рядов, или суммирования Чезаро. Актуальность данной работы заключается в том, что рассматривается не изученный ранее вид абсолютного суммирования с векторным индексом. Авторами получено достаточное условие для метода абсолютного суммирования с векторным индексом в терминах наилучшего приближения «углом» функций из пространства Лебега. Теорема, дающая достаточное условие, доказывает достаточные условия в различных случаях, которые могут зависеть от параметров. Из этой доказанной теоремы выводится также достаточное условие в термине смешанного модуля гладкости функции из пространства Лебега, которое легко получается с помощью известного неравенства.

Ключевые слова: тригонометрический ряд, ряд Фурье, пространство Лебега, наилучшее приближение «углом», абсолютная суммируемость ряда.

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DOI 10.31489/2023M4/66-78
UDC 517.19

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Kelvin-Voigt equations with memory: existence, uniqueness and regularity of solutions

In general, the study of inverse problems is realizable only in the case when the corresponding direct problems have the unique solution with some necessary properties such as continuity and regularity. In this paper, we study initial-boundary value problems for the system of 2D-3D nonlinear Kelvin-Voigt equations with memory, which describes a motion of an incompressible homogeneous non-Newtonian fluids with viscoelastic and relaxation properties. The investigation of these direct problems is related to the study of inverse problems for this system, which requires the continuity and regularity of solutions to these direct problems and their derivatives. The system, in addition to the initial condition, is supplemented with one of the boundary conditions: stick and slip boundary conditions. In both cases of these boundary conditions, the global in time existence and uniqueness of strong solutions to these initial-boundary value problems were proved. Moreover, under suitable assumptions on the data, the regularity of solutions and their derivatives were established.

Keywords: Kelvin-Voigt system, slip and stick boundary conditions, strong solutions, global existence and uniqueness, smoothness.

Introduction

Let $\Omega \in \mathbb{R}^d,\ d = 2, 3$, be a bounded domain with a smooth boundary $\partial\Omega$, and $Q_T = \Omega \times (0, T)$ be a cylinder with a lateral $\Gamma_T = \partial\Omega \times [0, T]$. Let us consider the following initial-boundary value problem for the system of nonlinear Kelvin-Voigt (Navier-Stokes-Voigt) equations with memory

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \varkappa\Delta\mathbf{v}_t - \nu\Delta\mathbf{v} - \int_0^t K(t-\tau)\Delta\mathbf{v}(\mathbf{x},\tau)d\tau + \nabla p = f, \quad (x,t) \in Q_T, \tag{1}$$

$$\operatorname{div}\mathbf{v}(\mathbf{x},t) = 0, \quad (x,t) \in Q_T, \tag{2}$$

supplemented with the initial condition

$$\mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega \tag{3}$$

and one of the following boundary conditions: stick boundary condition

$$\mathbf{v}(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \Gamma_T \tag{4}$$

or slip boundary condition

$$\mathbf{v}_{\mathbf{n}}(\mathbf{x},t) = \mathbf{v}\cdot\mathbf{n} = 0, \quad \operatorname{rot}\mathbf{v}\times\mathbf{n} = 0, \ (\mathbf{x},t) \in \Gamma_T. \tag{5}$$

System (1)-(2) is called a Kelvin-Voigt (also called Navier-Stokes-Voigt) system with memory or an integro-differential Kelvin-Voigt system, and models a motion of viscoelastic incompressible

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non-Newtonian fluids [1–5]. Most of hydrodynamics problems were considered with stick-boundary condition (4), however, in recent years works have appeared on initial-boundary value problems with a slip-boundary condition like (5), see for instance [6–8] et al. Because this is related to the fact that these boundary conditions have an important meaning for non-Newtonian fluids [9, 10]. In the case of slip boundary condition (5), we assume that Ω is a simply connected bounded domain [11]. System (1)-(2), in some particular cases, can be considered as a nonlinear pseudoparabolic equation due to the term $\Delta \mathbf{v}_t$, therefore all established below results will be hold true also for initial-boundary value problems for such type PDEs.

The issue of study of problems (1)–(4) and (1)–(3), (5) is aroused due to the investigation of inverse problems for system (1)-(2) that is supplemented with some additional conditions on solutions of the corresponding direct problem. In generally, the study of inverse problems are realizable only there is information such as unique solvability of the corresponding direct problems and smoothness of their solutions [12–14]. The direct problems for (1)-(2) with various statements have been studied before in some works as [5, 7, 15, 16], where the existence and uniqueness of weak solutions were established. The existence, uniqueness, and the regularities of smooth solutions of the initial-boundary value problems for system (1)-(2) without the memory term have been investigated in [17] for homogeneous fluids, and in [18], in the case for non-homogeneous fluids. However, by our knowledge, there is not work for smooth solutions for problems (1)–(4) and (1)–(3), (5). By this purpose, in this paper, we investigate the existence and uniqueness of strong solutions of problems (1)–(4) and (1)–(3), (5), and their regularities. First we work on problem (1)–(4) and the study problem (1)–(3), (5) is similar to the first one, therefore, we omit some details of proofs.

1 Preliminaries

In this section, we introduce the main functional spaces and some useful inequalities related to boundary conditions (4) and (5) from [8]. We distinguish vectors from scalars by using boldface letters. The symbol C will denote a generic constant – generally a positive one, value of which will not be specified; it can change from one inequality to another. We denote by $\mathbf{L}^2(\Omega)$ the usual Lebesgue space of square integrable vector-valued functions on Ω, and by $\mathbf{W}^{m,2}(\Omega)$ the Sobolev space of functions in $\mathbf{L}^2(\Omega)$ whose weak derivatives of an order not greater than m are in $\mathbf{L}^2(\Omega)$. The norm and inner product in $\mathbf{L}^2(\Omega)$ denoted by $\|\cdot\|_{2,\Omega}$ and $(\cdot,\cdot)_{2,\Omega}$, respectively.

Let us introduce the function spaces regarding to the slip and stick boundary conditions (5) and (4), respectively (see [3, 6]):

$$\mathbf{H_n}(\Omega) \equiv \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \operatorname{div} \mathbf{v} = 0,\ \mathbf{v_n}|_{\partial\Omega} = 0\};\quad \mathbf{H}(\Omega) \equiv \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \operatorname{div} \mathbf{v} = 0, \mathbf{v}|_{\partial\Omega} = 0\};$$
$$\mathbf{H}^1_\mathbf{n}(\Omega) \equiv \{\mathbf{v} \in \mathbf{W}^1_2(\Omega) : \operatorname{div} \mathbf{v} = 0, \mathbf{v_n}|_{\partial\Omega} = 0\};\quad \mathbf{H}^1(\Omega) \equiv \{\mathbf{v} \in \mathbf{W}^1_2(\Omega) : \operatorname{div} \mathbf{v} = 0, \mathbf{v}|_{\partial\Omega} = 0\};$$
$$\mathbf{H}^2_\mathbf{n}(\Omega) \equiv \{\mathbf{v} \in \mathbf{H}^1_\mathbf{n}(\Omega) \cap \mathbf{W}^{2,2}(\Omega) : (\operatorname{rot} \mathbf{v} \times \mathbf{n})|_{\partial\Omega} = 0\};\quad \mathbf{H}^2(\Omega) \equiv \{\mathbf{v} \in \mathbf{H}^1(\Omega) \cap \mathbf{W}^{2,2}(\Omega)\}$$

and for the simplicity, we use the following common notation for both cases

$$\mathbf{V} := \begin{cases} \mathbf{H}(\Omega), \text{ in the case (4);} \\ \mathbf{H_n}(\Omega), \text{ in the case (5),} \end{cases} \quad \mathbf{V}^i := \begin{cases} \mathbf{H}^i(\Omega), \text{ in the case (4);} \\ \mathbf{H}^i_\mathbf{n}(\Omega), \text{ in the case (5), } i = 1, 2. \end{cases}$$

The scalar product and the norm in $\mathbf{V}^1_\mathbf{n}(\Omega)$ we define by $(\operatorname{rot} \mathbf{v}, \operatorname{rot} \mathbf{u})_{2,\Omega}$ and $\|\mathbf{v}\|_{\mathbf{V}^1_\mathbf{n}(\Omega)} := \|\operatorname{rot}\ \mathbf{v}\|_{2,\Omega}$, respectively. According to [3, 6, 8, 11] and the references cited in them (see for example [9, 19]), the following inequalities are hold:

Poincare's inequality

$$\|\mathbf{v}\|_{2,\Omega} \le C_1(\Omega)\, \|\nabla \mathbf{v}\|_{2,\Omega}\,,\ \ \mathbf{v} \in \mathbf{V}^1(\Omega); \tag{6}$$

$$N_1(\Omega)\, \|\mathbf{v}\|_{\mathbf{W}^{1,2}(\Omega)} \le \|\operatorname{rot} \mathbf{v}\|_{2,\Omega} \le N_2(\Omega)\, \|\mathbf{v}\|_{\mathbf{W}^{1,2}(\Omega)}\,,\quad \forall \mathbf{v} \in \mathbf{V}^1(\Omega);$$

$$N_3(\Omega)\,\|\mathbf{v}\|_{\mathbf{W}^{2,2}(\Omega)} \le \|\Delta\mathbf{v}\|_{2,\Omega} = \|\mathrm{rot}\,\mathrm{rot}\,\mathbf{v}\|_{2,\Omega} \le N_4(\Omega)\,\|\mathbf{v}\|_{\mathbf{W}^{2,2}(\Omega)}\,, \quad \forall\mathbf{v}\in\mathbf{V}^2(\Omega); \tag{7}$$

and Ladyzhenskaya inequalities [6].

Let us introduce a bilinear and continuous form $\mathbf{a}$ on $\mathbf{V}^1$, associated with the operator Δ:

$$\mathbf{a}(\mathbf{v},\mathbf{u}) = (\nabla\mathbf{v},\nabla\mathbf{u})_{2,\Omega}\,, \quad \forall\mathbf{v},\mathbf{u}\in\mathbf{V}^1(\Omega) \tag{8}$$

in case (4), and

$$\mathbf{a}(\mathbf{v},\mathbf{u}) = (\mathrm{rot}\,\mathbf{v},\mathrm{rot}\,\mathbf{u})_{2,\Omega}\,, \quad \forall\mathbf{v},\mathbf{u}\in\mathbf{V}^1(\Omega) \tag{9}$$

in case (5). It is clear that $\mathbf{a}(\mathbf{v},\mathbf{v})$ is a norm on $\mathbf{V}^1(\Omega)$, which is equivalent to $\mathbf{W}^{1,2}(\Omega)$-norm. In particular, due to (6), in $\mathbf{V}^1$ the norm $\|\,\mathrm{rot}\,\mathbf{v}\|_{2,\Omega}$ is equivalent to the norm $\|\mathbf{v}\|_{\mathbf{W}^{1,2}(\Omega)}$, and therefore equivalent to the norm $\|\nabla\mathbf{v}\|_{2,\Omega}$.

Thus, $\mathbf{a}$ defines an isomorphism A from $\mathbf{V}^1(\Omega)$ to $\mathbf{V}^{-1}(\Omega)$,

$$\langle A\mathbf{v},\mathbf{u}\rangle \equiv \mathbf{a}(\mathbf{v},\mathbf{u}), \quad \forall\mathbf{v},\mathbf{u}\in\mathbf{V}^1(\Omega),$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing of $\mathbf{V}^1$ and $\mathbf{V}^{-1}$. There hold the following continuous inclusions

$$\mathbf{V}^1(\Omega)\hookrightarrow\mathbf{L}^2(\Omega)\hookrightarrow\mathbf{V}^{-1}(\Omega),$$

where each of the first two spaces is dense in the next one.

It follows from (7) also that in $\mathbf{V}^2$ the norm $\|\Delta\mathbf{v}\|_{2,\Omega} = \|\mathrm{rotrot}\,\mathbf{v}\|_{2,\Omega}$ is equivalent to the norm $\|\mathbf{v}\|_{\mathbf{W}^{2,2}(\Omega)}$.

Regarding to sliding condition (5), we have the Green formulas (see [6] and [8, 9]):

$$\begin{aligned}
&(-\Delta\mathbf{v},\mathbf{u})_{2,\Omega} = -\left(\nabla\,\mathrm{div}\,\mathbf{v},\mathbf{u}\right)_{2,\Omega} + \left(\mathrm{rot}^2\,\mathbf{v},\mathbf{u}\right)_{2,\Omega} = -\int_{\partial\Omega}\mathrm{div}\,\mathbf{v}\cdot\mathbf{u}_n\,dS+\\
&+(\mathrm{div}\,\mathbf{v},\mathrm{div}\,\mathbf{u})_{2,\Omega} + \int_{\partial\Omega}\mathbf{u}\cdot(\mathrm{rot}\,\mathbf{v}\times\mathbf{n})\;dS + (\mathrm{rot}\,\mathbf{v},\mathrm{rot}\,\mathbf{u})_{2,\Omega} = (\mathrm{rot}\,\mathbf{v},\mathrm{rot}\,\mathbf{u})_{2,\Omega}
\end{aligned} \tag{10}$$

in case $d=3$, and

$$\begin{aligned}
&(-\Delta\mathbf{v},\mathbf{u})_{2,\Omega} = (\mathrm{div}\,\mathbf{v},\mathrm{div}\,\mathbf{u})_{2,\Omega} + \left(\overline{\mathrm{rot}}(\mathrm{rot}\,\mathbf{v}),\mathbf{u}\right)_{2,\Omega} =\\
&= \int_{\partial\Omega}(\mathrm{rot}\,\mathbf{v}\times\mathbf{n})\,\mathbf{u}\,dS + (\mathrm{rot}\,\mathbf{v},\mathrm{rot}\,\mathbf{u})_{2,\Omega} = (\mathrm{rot}\,\mathbf{v},\mathrm{rot}\,\mathbf{u})_{2,\Omega}\,,
\end{aligned} \tag{11}$$

in case $d=2$, where $\overline{\mathrm{rot}}\varphi$ is the vector $(\varphi_{x_2},-\varphi_{x_1})_{2,\Omega}$ for the scalar function φ.

The regularity properties of solutions will be proved under the following lemma, which the proof is given in [20].

Lemma 1. If $f\in L^p(0,T;X)$ and $\frac{\partial f}{\partial t}\in L^p(0,T;X)$ $(1\le p\le\infty)$, then f, after, which can be changing on a set of measure zero (from segment $(0,T)$) be a continuous mapping $[0,T]\to X$.

Definition 1. A vector function $\mathbf{v}(\mathbf{x},t)$ is a strong solution to problem (1)–(4) ((1)–(3), (5)) if:

1 $\mathbf{v}(\mathbf{x},t)\in\mathbf{C}(0,T;\mathbf{V}^1(\Omega)\cap\mathbf{V}^2(\Omega))\cap\mathbf{W}^1_2(0,T;\mathbf{V}^1(\Omega)\cap\mathbf{V}^2(\Omega))$;

2 Each equation in (1)–(4) ((1)–(3), (5)) holds in the distribution sense in the their corresponding domain.

2 Main results

Throughout the work, we assume that

$$K(t) \in L^2([0,T]) \text{ and } \|K\|_{L^2([0,T])} \equiv K_0 < \infty. \tag{12}$$

For the problems (1)–(4) and (1)–(3), (5) the following results are hold.

Theorem 1. Suppose that

$$\mathbf{f} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega)), \quad \mathbf{v}_0 \in \mathbf{V}^1(\Omega) \cap \mathbf{V}^2(\Omega),$$

and (12) are hold. Then problems (1)–(4) and (1)–(3), (5) have a unique strong solution and the following estimate is valid

$$\|\mathbf{v}\|^2_{\mathbf{L}^\infty(0,T;\mathbf{V}^1(\Omega)\cap\mathbf{V}^2(\Omega))} + \|\mathbf{v}_t\|^2_{\mathbf{L}^2(0,T;\mathbf{V}^1(\Omega)\cap\mathbf{V}^2(\Omega))} + \|\mathbf{v}\|^2_{\mathbf{L}^2(0,T;\mathbf{V}^1(\Omega))} \le C < \infty,$$

where C is a positive constant depending on data of the problem.

Proof. The proof consists of following steps: by Gelerkin's method constructing a sequence of approximated solutions; obtaining a priori estimates and passage to limit.

2.1 Galerkin's approximations

To prove the existence of a strong solution to problem (1)–(5), we use the Faedo–Galerkin method with a special basis of eigenfunctions of the spectral problem

$$-\Delta\varphi_j + \nabla q = \lambda_j\varphi_j, \ \varphi_j \in \mathbf{V}^2(\Omega)$$

is closely connected with problems (1)–(4) and (1)–(3), (5). In case (5), it is equivalent to the problem [6, 8]

$$\mathbb{A}\varphi_j \equiv -\Delta\varphi_j = \lambda_j\varphi_j, \ \varphi_j \in \mathbf{V}^2(\Omega)$$

since $\nabla q \equiv 0$ due to the fact

$$(\Delta\varphi, \nabla p) = 0, \text{ for any } \varphi \in \mathbf{V}^2(\Omega) \text{ and any } p \in W^1_2(\Omega).$$

For the problem (1)–(3), (4), $\mathbb{A}\varphi_j \equiv \tilde{\Delta}\varphi_j$ [21]. Given $m \in N$, let us consider the m-dimensional spaces $\mathbf{X}^m$ spanned by the first m eigenfunctions $\varphi_1, ..., \varphi_m$. For each $m \in N$, we search for approximate solutions in the form

$$\mathbf{v}^m(x,t) = \sum_{j=1}^{m} c_j^m(t)\varphi_j(x), \quad \varphi_j \in \mathbf{X}^m,$$

where unknown coefficients $c_j^m(t)$, $j = 1, ..., m$ are defined as solutions of the following system of ordinary differential equations derived from

$$\begin{aligned} &\frac{d}{dt}\Big((\mathbf{v}^m, \varphi_k)_{2,\Omega} - \varkappa\,(\Delta\mathbf{v}^m, \varphi_k)_{2,\Omega}\Big) + ((\mathbf{v}^m \cdot \nabla)\,\mathbf{v}^m, \varphi_k)_{2,\Omega} - \nu\,(\Delta\mathbf{v}^m, \varphi_k)_{2,\Omega} - \\ &\quad - \int_0^t K(t-\tau)\,(\Delta\mathbf{v}^m, \varphi_k)_{2,\Omega}\,d\tau = (\mathbf{f}, \varphi_k)_{2,\Omega}\,, \end{aligned} \tag{13}$$

for $k = 1,\ 2,\ \ldots,\ m$. System (13) is supplemented with the Cauchy data

$$\mathbf{v}^m(0) = \mathbf{v}_0^m, \tag{14}$$

where

$$\mathbf{v}_0^m = \sum_{j=1}^{m} (\mathbf{v}_0, \varphi_j)_{2,\Omega}\, \varphi_j$$

is a sequence in $\mathbf{L}^2(\Omega) \cap \mathbf{V}^1(\Omega)$ such that

$$\mathbf{v}_0^m \to \mathbf{v}_0(x) \text{ strong as } m \to \infty \ \text{ in } \ \mathbf{V}^1(\Omega) \cap \mathbf{V}^2(\Omega). \tag{15}$$

According to a general theory of ordinary differential equations, Cauchy problem (13)–(14) has a solution $c_j^m(t)$ in $[0, T_*]$. By a priori estimates which we shall establish below, $[0, T_*]$ can be extended to $[0, T]$.

2.2 A priori estimates

Lemma 2. Assume that

$$\mathbf{f} \in \mathbf{L}^2(0, T; \mathbf{L}^2(\Omega)),\ \mathbf{v}_0(x) \in \mathbf{V}^1(\Omega),$$

and the conditions (12) and (15) are fulfilled. Then, for all $t \in [0, T]$, the following a priori estimate is valid

$$\|\mathbf{v}^m\|^2_{\mathbf{L}^\infty(0,T;\mathbf{V}(\Omega)\cap\mathbf{V}^1(\Omega))} + \|\mathbf{v}^m\|^2_{\mathbf{L}^2(0,T;\mathbf{V}^1(\Omega))} \le M_0 < \infty, \tag{16}$$

where M_0 is a positive constant depending only on data of the problem.

Proof. Multiply k-th equation of (13) by $c_k^m(t)$ and summing up from 1 to m, then using Green's formulas (10)-(11), we obtain

$$\frac{d}{dt}\left(\|\mathbf{v}^m\|^2_{2,\Omega} + \varkappa \|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)}\right) + \nu \|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)} =$$

$$= \int_0^t K(t-\tau)\mathbf{a}\left(\mathbf{v}^m(t), \mathbf{v}^m(\tau)\right) d\tau + (\mathbf{f}, \mathbf{v}^m)_{2,\Omega} \equiv I_1, \tag{17}$$

where $\mathbf{a}$ is defined by (8) and (9), regarding to the boundary conditions. Next, we estimate the terms on the right-hand side of (17) by Hölder's and Young's inequalities

$$I_1 \le \int_0^t |K(t-\tau)|\, \|\mathbf{v}^m(\tau)\|_{\mathbf{V}^1(\Omega)} \|\mathbf{v}^m(t)\|_{\mathbf{V}^1(\Omega)}\, d\tau + \|\mathbf{v}^m\|_{2,\Omega} \|\mathbf{f}\|_{2,\Omega} \le$$

$$\le \frac{\nu}{2}\|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)} + \frac{K_0^2}{2\nu}\int_0^t \|\mathbf{v}^m(\tau)\|^2_{\mathbf{V}^1(\Omega)}\, d\tau + \frac{1}{2}\|\mathbf{v}^m\|^2_{2,\Omega} + \frac{1}{2}\|\mathbf{f}\|^2_{2,\Omega}.$$

Substituting last inequality into (17), and integrating by s from 0 to t, we obtain

$$\begin{gathered}\|\mathbf{v}^m\|^2_{2,\Omega} + \varkappa\|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)} + \nu\int_0^t \|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)}\, ds \le \|\mathbf{v}_0(x)\|^2_{2,\Omega} + \varkappa\|\mathbf{v}_0(x)\|^2_{\mathbf{V}^1(\Omega)} + \int_0^t \|\mathbf{v}^m\|^2_{2,\Omega}\, ds + \\ + \frac{K_0^2}{\nu}\int_0^t\int_0^s \|\mathbf{v}^m(s)\|^2_{\mathbf{V}^1(\Omega)}\, d\tau ds + \|\mathbf{f}\|^2_{2,Q_T} \le C_1\int_0^t\left(\|\mathbf{v}^m\|^2_{2,\Omega} + \varkappa\|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)}\right) ds + C_2,\end{gathered} \tag{18}$$

where

$$C_1 = \max\{1, \frac{K_0^2 T}{\nu\varkappa}\}, \quad C_2 = \|\mathbf{v}_0(x)\|_{2,\Omega}^2 + \varkappa \|\mathbf{v}_0(x)\|_{\mathbf{V}^1(\Omega)}^2 + \|\mathbf{f}\|_{2,Q_T}^2 .$$

Omitting the third term on the left hand side of (18) and applying Grönwall's lemma and taking supremum, we get

$$\sup_{t\in(0,T]} \left(\|\mathbf{v}^m\|_{2,\Omega}^2 + \|\mathbf{v}^m\|_{\mathbf{V}^1(\Omega)}^2 \right) \le C_3 < \infty, \tag{19}$$

where $C_3 = C_3(\nu, \varkappa, T, C_1, C_2)$. Plugging (19) with (18), we obtain the first energy estimate (16).

Lemma 3. Assume that all conditions of Lemma 2 are fulfilled. Then the following estimate is valid

$$\|\mathbf{v}^m\|_{\mathbf{L}^\infty(0,T;\mathbf{V}(\Omega)\cap\mathbf{V}^1(\Omega))}^2 + \|\mathbf{v}_t^m\|_{\mathbf{L}^2(0,T;\mathbf{V}(\Omega)\cap\mathbf{V}^1(\Omega))}^2 \le M_1 < \infty, \quad \forall t \in [0,T], \tag{20}$$

where M_1 is a positive constant depending on data of the problem.

Proof. Multiplying both sides of k-th equation of (13) by $\frac{d c_k^m}{dt}$ and summing up from $k=1$ to $k=m$, we obtain

$$\begin{gathered} \|\mathbf{v}_t^m\|_{2,\Omega}^2 + \varkappa \|\mathbf{v}_t^m\|_{\mathbf{V}^1(\Omega)}^2 + \frac{\nu}{2}\frac{d}{dt}\|\mathbf{v}^m\|_{\mathbf{V}^1(\Omega)}^2 = ((\mathbf{v}^m\cdot\nabla)\,\mathbf{v}_t^m, \mathbf{v}^m)_{2,\Omega} - \\ - \int_0^t K(t-\tau)\mathbf{a}\left(\mathbf{v}^m(\tau), \mathbf{v}_t^m(t)\right) d\tau + (\mathbf{f}, \mathbf{v}_t^m)_{2,\Omega} \equiv I_{21} + I_{22}. \end{gathered} \tag{21}$$

Using Hölder and Ladyzhenskaya together with Young inequalities, we have

$$I_{21} \le |((\mathbf{v}^m\cdot\nabla)\,\mathbf{v}_t^m, \mathbf{v}^m)| \le \|\mathbf{v}_t^m\|_{4,\Omega}\|\mathbf{v}^m\|_{4,\Omega}^2 \le \frac{\varepsilon_1}{2}\|\mathbf{v}_t^m\|_{\mathbf{V}^1(\Omega)}^2 + \frac{C^4(\Omega)}{2\varepsilon_1}\|\mathbf{v}^m\|_{\mathbf{V}^1(\Omega)}^4, \tag{22}$$

$$I_{22} \le \frac{\varepsilon_2}{2}\|\mathbf{v}_t^m(t)\|_{\mathbf{V}^1(\Omega)}^2 + \frac{K_0^2}{2\varepsilon_2}\int_0^t \|\mathbf{v}^m(\tau)\|_{\mathbf{V}^1(\Omega)}^2 d\tau + \frac{\varepsilon_3}{2}\|\mathbf{v}_t^m(t)\|_{2,\Omega}^2 + \frac{1}{2\varepsilon_3}\|\,\mathbf{f}\|_{2,\Omega}^2. \tag{23}$$

Plugging (22)-(23) with $\varepsilon_i = \frac{\varkappa}{3}, i = 1,2,3$ into (21), and integrating the result by s in $[0,t]$, $\ t \le T$, we have

$$\begin{aligned} &\nu\|\mathbf{v}^m\|_{\mathbf{V}^1(\Omega)}^2 + 2\|\mathbf{v}_t^m(t)\|_{2,Q_T}^2 + \varkappa\|\mathbf{v}_t^m(t)\|_{\mathbf{L}^2(0,T;\mathbf{V}^1(\Omega))}^2 \le \\ &\qquad \le \nu\|\mathbf{v}_0\|_{\mathbf{V}^1(\Omega)}^2 + \frac{3C^4(\Omega)}{\varkappa}M_0^2 + \frac{3K_0^2}{\varkappa}M_0 T + \frac{3}{\varkappa}\|\mathbf{f}\|_{2,Q_T}^2 := C_4 < \infty \end{aligned}$$

which follows that (20).

Lemma 4. Assume that in addition to the conditions of Lemma 2 holds

$$\mathbf{v}_0 \in \mathbf{V}(\Omega) \cap \mathbf{V}^2(\Omega).$$

Then for all $t \in [0,T]$, the estimate is valid

$$\sup_{t\in[0,T]} \|\mathbb{A}\mathbf{v}^m\|_{2,\Omega}^2 + \|\mathbb{A}\mathbf{v}_t^m\|_{2,Q_T}^2 \le M_2 < \infty, \tag{24}$$

where $\mathbb{A} \equiv \tilde{\Delta}$ for the problem with (4), and $\mathbb{A} \equiv \Delta$ for the problem with (5).

Proof. Let us multiply the k^{th} equation of (13) by $-\mu_k \frac{dc_k^m(t)}{dt}$ and sum with respect to k, from 1 to m to obtain

$$\begin{aligned}\|\mathbf{v}_t^m\|^2_{\mathbf{V}^1(\Omega)} + \varkappa \|\mathbb{A}\mathbf{v}_t^m\|^2_{2,\Omega} + \frac{\nu}{2}\frac{d}{dt}\|\mathbb{A}\mathbf{v}^m\|^2_{2,\Omega} = ((\mathbf{v}^m\cdot\nabla)\,\mathbf{v}^m, \mathbb{A}\mathbf{v}_t^m)_{2,\Omega} - \\ -\int_0^t K(t-\tau)\,(\boldsymbol{\Delta}\mathbf{v}^m(\tau), \mathbb{A}\mathbf{v}_t^m(t))_{2,\Omega}\,d\tau + (\mathbf{f}(t), \mathbb{A}\mathbf{v}_t^m)_{2,\Omega} \equiv I_{31} + I_{32}.\end{aligned} \tag{25}$$

Estimating the terms on right hand side (25) by using Hölder, Ladyzhenskaya, Sobolev and Young inequalities, we get the following inequalities

$$I_{31} \le \|\mathbb{A}\mathbf{v}_t^m\|_{2,\Omega}\,\|\mathbf{v}^m\|_{4,\Omega}\,\|\nabla\mathbf{v}^{\mathbf{m}}\|_{4,\Omega} \le \frac{\varepsilon_1}{2}\|\mathbb{A}\mathbf{v}_t^m\|^2_{2,\Omega} + \frac{C^2(\Omega)}{2\varepsilon_1}\|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)}\|\mathbb{A}\mathbf{v}^m\|^2_{2,\Omega},$$

$$I_{32} \le \frac{\varepsilon_2}{2}\|\mathbb{A}\mathbf{v}_t^m(t)\|^2_{2,\Omega} + \frac{K_0^2}{2\varepsilon_2}\int_0^t \|\mathbb{A}\mathbf{v}^m(\tau)\|^2_{2,\Omega}\,d\tau + \frac{\varepsilon_3}{2}\|\mathbb{A}\mathbf{v}_t^m(t)\|^2_{2,\Omega} + \frac{1}{2\varepsilon_3}\|\mathbf{f}\|^2_{2,\Omega}, \tag{26}$$

where $C(\Omega)$ is a constant from embedding inequalities.

Substituting (26) with $\varepsilon_i = \frac{\varkappa}{3}, i = 1, 2, 3$ into (25) and integrating the result by $\tau \in (0, t)$ and using estimates (16), (20), we have

$$\nu\|\mathbb{A}\mathbf{v}^m\|^2_{2,\Omega} + \int_0^t \|\mathbf{v}_t^m(\tau)\|^2_{\mathbf{V}^1(\Omega)}\,d\tau + \varkappa\int_0^t \|\mathbb{A}\mathbf{v}_t^m(\tau)\|^2_{2,\Omega}\,d\tau \le C_5\int_0^t \|\mathbb{A}\mathbf{v}^m(\tau)\|^2_{2,\Omega}\,d\tau + C_6, \tag{27}$$

where

$$C_5 = \frac{3K_0}{\varkappa}\left(C^2(\Omega) + K_0 T\right), \quad C_6 = \nu\|\mathbf{v}_0\|^2_{\mathbf{V}^2(\Omega)} + \frac{3}{\varkappa}\|\mathbf{f}\|^2_{2,Q_T}.$$

By applying Granwall's lemma and the standard techniques, we get from (27) estimate (24).

Along with the above estimates, one can establish the following more regular estimate assuming an additional smoothness for data.

Lemma 5. Assume that in addition to the conditions of Lemma 5 holds

$$\mathbf{f} \in \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)).$$

Then for all $t \in [0, T]$ the following estimate is valid

$$\sup_{t\in[0,T]}\|\mathbf{v}_t^n\|^2_{\mathbf{V}^1(\Omega)} + \sup_{t\in[0,T]}\|\mathbb{A}\mathbf{v}_t^n\|^2_{2,\Omega} \le M_3 < \infty. \tag{28}$$

Proof. Let us multiply the k^{th} equation of (13) by $-\mu_k \frac{dc_k^m(t)}{dt}$, and sum with respect to k, from 1 to m. Then we have

$$\begin{aligned}\|\mathbf{v}_t^m\|^2_{\mathbf{V}^1(\Omega)} + \varkappa\|\mathbb{A}\mathbf{v}_t^m\|^2_{2,\Omega} = ((\mathbf{v}^m\cdot\nabla)\,\mathbf{v}^m, \mathbb{A}\mathbf{v}_t^m)_{2,\Omega} - \\ -\int_0^t K(t-\tau)\,(\boldsymbol{\Delta}\mathbf{v}^m(\tau), \mathbb{A}\mathbf{v}_t^m(t))_{2,\Omega}\,d\tau + (\mathbf{f}(t), \mathbb{A}\mathbf{v}_t^m)_{2,\Omega} - \nu\,(\boldsymbol{\Delta}\mathbf{v}^m, \mathbb{A}\mathbf{v}_t^m)_{2,\Omega} \equiv I_{41} + I_{42},\end{aligned} \tag{29}$$

where

$$I_{41} = ((\mathbf{v}^m\cdot\nabla)\,\mathbf{v}^m, \mathbb{A}\mathbf{v}_t^m)_{2,\Omega},$$

$$I_{42} = -\int_0^t K(t-\tau)\,(\boldsymbol{\Delta}\mathbf{v}^m(\tau), \mathbb{A}\mathbf{v}_t^m(t))_{2,\Omega}\,d\tau + (\mathbf{f}(t), \mathbb{A}\mathbf{v}_t^m)_{2,\Omega} - \nu\,(\boldsymbol{\Delta}\mathbf{v}^m, \mathbb{A}\mathbf{v}_t^m)_{2,\Omega}\,.$$

Estimating I_{41} and I_{42} by using Hölder and Cauchy inequalities as above, we obtain the following inequality

$$I_{41} \le \|\mathbb{A}\mathbf{v}_t^m\|_{2,\Omega}\,\|\mathbf{v}^m\|_{4,\Omega}\,\|\nabla\mathbf{v}^m\|_{4,\Omega} \le \frac{\varepsilon_1}{2}\,\|\mathbb{A}\mathbf{v}_t^m\|_{2,\Omega}^2 + \frac{C^2(\Omega)}{2\varepsilon_1}\,\|\mathbf{v}^m\|_{\mathbf{V}^1(\Omega)}^2\,\|\mathbb{A}\mathbf{v}^m\|_{2,\Omega}^2\,, \tag{30}$$

$$\begin{aligned} I_{42} \le{}& \frac{\varepsilon_2}{2}\,\|\mathbb{A}\mathbf{v}_t^m(t)\|_{2,\Omega}^2 + \frac{K_0^2}{2\varepsilon_2}\int_0^t \|\boldsymbol{\Delta}\mathbf{v}^m(\tau)\|_{2,\Omega}^2\,d\tau + \frac{\varepsilon_3}{2}\,\|\mathbb{A}\mathbf{v}_t^m(t)\|_{2,\Omega}^2 + \frac{1}{2\varepsilon_3}\,\|\mathbf{f}\|_{2,\Omega}^2 + \\ &+ \frac{\nu}{2\varepsilon_4}\,\|\boldsymbol{\Delta}\mathbf{v}^m\|_{2,\Omega}^2 + \frac{\varepsilon_4}{2}\,\|\mathbb{A}\mathbf{v}_t^m(t)\|_{2,\Omega}^2\,. \end{aligned} \tag{31}$$

Choosing $\varepsilon_i = \frac{\varkappa}{4}, i = 1, 2, 3, 4$ in (30), (31), and substituting into (29), we get

$$\begin{aligned} \|\mathbf{v}_t^m\|_{\mathbf{V}^1(\Omega)}^2 + \frac{\varkappa}{2}\,\|\mathbb{A}\mathbf{v}_t^m\|_{2,\Omega}^2 \le{}& \frac{2C^2(\Omega)}{\varkappa}\,\|\mathbf{v}^m\|_{\mathbf{V}^1(\Omega)}^2\,\|\mathbb{A}\mathbf{v}^m\|_{2,\Omega}^2 + \\ &+ \frac{2K_0^2}{\varkappa}\int_0^t \|\boldsymbol{\Delta}\mathbf{v}^m(\tau)\|_{2,\Omega}^2\,d\tau + \frac{2}{\varkappa}\,\|\mathbf{f}\|_{2,\Omega}^2 + \frac{2\nu}{\varkappa}\,\|\boldsymbol{\Delta}\mathbf{v}^m\|_{2,\Omega}^2\,. \end{aligned} \tag{32}$$

Now, taking the supremum by $t \in [0, T]$ on both sides of (32), and using (20) and (24), we obtain

$$\sup_{t\in[0,T]}\left(\|\mathbf{v}_t^m\|_{\mathbf{V}^1(\Omega)}^2 + \frac{\varkappa}{2}\,\|\mathbb{A}\mathbf{v}_t^m\|_{2,\Omega}^2\right) \le \frac{2C^2(\Omega)}{\varkappa}M_1M_2 + \frac{2K_0^2}{\varkappa}M_2T + \frac{2}{\varkappa}\,\|\mathbf{f}\|_{L^\infty(0,T;L(\Omega))}^2 + \frac{2\nu}{\varkappa}M_2 \le K < \infty.$$

2.3 Passage to the limit as $m \to \infty$

By means of reflexivity and up to some subsequences, estimates (16), (20), (28) imply that

$$\mathbf{v}^m \rightharpoonup \mathbf{v} \quad \text{weakly-}* \text{ in } L^\infty(0, T; \mathbf{V}(\Omega) \cap \mathbf{V}^1(\Omega)), \quad \text{as } m \to \infty, \tag{33}$$
$$\mathbf{v}^m \rightharpoonup \mathbf{v} \quad \text{weakly in } L^2(0, T; \mathbf{V}(\Omega) \cap \mathbf{V}^1(\Omega)), \quad \text{as } m \to \infty, \tag{34}$$
$$\mathbf{v}_t^m \rightharpoonup \mathbf{v}_t \quad \text{weakly in } L^2(0, T; \mathbf{V}(\Omega) \cap \mathbf{V}^1(\Omega)), \quad \text{as } m \to \infty, \tag{35}$$
$$\mathbf{v}^m \rightharpoonup \mathbf{v} \quad \text{weakly-}* \text{ in } L^\infty(0, T; \mathbf{V}^2(\Omega)), \quad \text{as } m \to \infty, \tag{36}$$
$$\mathbf{v}^m \rightharpoonup \mathbf{v} \quad \text{weakly in } L^2(0, T; \mathbf{V}^2(\Omega)), \quad \text{as } m \to \infty, \tag{37}$$
$$\mathbf{v}_t^m \rightharpoonup \mathbf{v}_t \quad \text{weakly in } L^2(0, T; \mathbf{V}^2(\Omega)), \quad \text{as } m \to \infty. \tag{38}$$

On the other hand, due to the compact embedding $\mathbf{W}_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega)$ and the Aubin-Lions compactness lemma, it follows that

$$\mathbf{v}^m \longrightarrow \mathbf{v} \quad \text{strongly in } \mathbf{L}^2(Q_T) \quad \text{as } m \to \infty. \tag{39}$$

Let $\zeta(t) \in C_0^\infty([0, T])$ be an arbitrary function. Multiplying (13) by $\zeta(t)$ and integrating the result by t from 0 to T, we obtain

$$\begin{aligned} &\int_{Q_T} \mathbf{v}_t^m \cdot \varphi_k\zeta d\mathbf{x}dt + \int_{Q_T} (\mathbf{v}^m \cdot \nabla)\,\mathbf{v}^m \cdot \varphi_k\zeta d\mathbf{x}dt + \nu\int_{Q_T} \boldsymbol{\Delta}\mathbf{v}^m \cdot \varphi_k\zeta d\mathbf{x}dt + \\ &+ \varkappa\int_{Q_T} \boldsymbol{\Delta}\mathbf{v}_t^m \cdot \varphi_k\zeta d\mathbf{x}dt = \int_{Q_T}\int_0^\tau K(\tau - s)\boldsymbol{\Delta}\mathbf{v}^m \cdot \varphi_k\zeta ds d\tau + \int_{Q_T} \mathbf{f} \cdot \varphi_k\zeta d\mathbf{x}dt \end{aligned} \tag{40}$$

for $k \in \{1, \dots, m\}$. Then, fixing k, we can pass in equation (40) to the limit $m \to \infty$, by using the convergence results (33)–(39). Then, we obtain

$$\begin{gathered}\int_{Q_T} \mathbf{v}_t \cdot \varphi_k \zeta d\mathbf{x}dt + \int_{Q_T} (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \varphi_k \zeta d\mathbf{x}dt + \nu \int_{Q_T} \Delta \mathbf{v} \cdot \varphi_k \zeta d\mathbf{x}dt + \\ + \varkappa \int_{Q_T} \Delta \mathbf{v}_t \cdot \varphi_k \zeta d\mathbf{x}dt = \int_{Q_T} \int_0^{\tau} K(\tau - s) \Delta \mathbf{v} \cdot \varphi_k \zeta ds d\tau + \int_{Q_T} \mathbf{f} \cdot \varphi_k \zeta d\mathbf{x}dt.\end{gathered} \tag{41}$$

for $k \in \{1, \dots, m\}$.

By linearity, equation (41) holds for any finite linear combination of $\{\mathbf{z}_k = \varphi_k \cdot \zeta(t)\}_{k=1}^m$ with $\zeta(t) \in C_0^{\infty}([0,T])$, and, by a continuity argument, it is still true for any $\mathbf{z} \in L^2(0,T; \mathbf{V}(\Omega))$. Hence, we can see that $\mathbf{v}$ satisfies to

$$\begin{gathered}\int_{Q_T} \mathbf{v}_t \cdot \mathbf{z} d\mathbf{x}dt + \int_{Q_T} (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{z} d\mathbf{x}dt + \nu \int_{Q_T} \Delta \mathbf{v} \cdot \mathbf{z} \zeta d\mathbf{x}dt + \\ + \varkappa \int_{Q_T} \Delta \mathbf{v}_t \cdot \mathbf{z} d\mathbf{x}dt = \int_{Q_T} \int_0^{\tau} K(\tau - s) \Delta \mathbf{v} \cdot \mathbf{z} ds d\tau + \int_{Q_T} \mathbf{f} \cdot \mathbf{z} d\mathbf{x}dt,\end{gathered}$$

i.e. $\mathbf{v}$ is a strong solution to problem (1)–(4).

3 Regularity of solutions

Theorem 2. Let all conditions of Theorem 1 be fulfilled. Then

$$\mathbf{v} \in C(0,T; \mathbf{V}(\Omega) \cap \mathbf{V}^2(\Omega)), \ p \in C(0,T; G(\Omega)).$$

If, in addition,

$$\mathbf{f} \in C(0,T; L^2(\Omega))$$

holds, then for all $t \in (0,T)$

$$\mathbf{v} \in C^1(0,T; \mathbf{V}(\Omega) \cap \mathbf{V}^2(\Omega)), \ p \in C(0,T; G(\Omega)) \tag{42}$$

holds.

Proof. Embedding (42) follows from Lemma 1, under estimates (16), (20), (24). The second assertion follows from the embedding theorems under the estimates from [20, 22].

Acknowledgments

This research work has been funded by Grant number AP19676624 of the Ministry of Science and Higher Education of the Republic of Kazakhstan.

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Жады мүшесі бар Кельвин-Фойгт теңдеулері: шешімдердің бар болуы, жалғыздығы және регулярлығы

Жалпы алғанда кері есептерді зерттеу оларға сәйкес келетін тура есептердің бірмәнді шешімділігі және шешімдерінің үзіліссіздігі мен жоғары регулярлығы сияқты кейбір қажетті қасиеттерге ие болған жағдайда ғана жүзеге асырылады. Мақалада тұтқыр серпімді және релаксациялық қасиеттері ескерілген сығылмайтын біртекті Ньютондық емес сұйықтардың қозғалысын сипаттайтын жады мүшесі бар 2D-3D өлшемді сызықты емес Кельвин-Фойгт теңдеулер жүйесі үшін қойылған бастапқы-шектік есептер зерттелген. Бұл тура есептерді зерттеу осы жүйе үшін қойылған кері есептерді зерттеумен байланысты. Себебі, онда осы тура есептердің шешімдерінің және олардың туындыларының үзіліссіздігі мен регулярлығы сияқты қасиеттері қажет етіледі. Қарастырылып отырған есептерде теңдеулер жүйесі бастапқы шартпен қатар жұғу және сырғанау сияқты шекаралық шарттарының бірімен толықтырылады. Осы екі шекаралық шарттар жағдайында бастапқы-шекаралық есептердің әлді шешімдерінің уақыт бойынша глобальды бар болуы және жалғыздығы дәлелденген. Сонымен қатар, есептің берілгендері үшін қолайлы ұйғарымдар жасай отырып, шешімдермен олардың туындыларының регулярлығы көрсетілді.

Кілт сөздер: Кельвин-Фойгт жүйесі, жұғу және сырғанаудың шекаралық шарттары, әлді шешімдер, глобальды бар болуы және жалғыздығы, тегістік.

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Уравнения Кельвина–Фойгта с памятью: существование, единственность и регулярность решений

В общем случае изучение обратных задач осуществимо только в том случае, когда соответствующие прямые задачи имеют единственное решение, обладающее некоторыми необходимыми свойствами, такими как непрерывность и регулярность. В статье исследованы начально-краевые задачи для системы 2D–3D нелинейных уравнений Кельвина–Фойгта с памятью, описывающей движение несжимаемой однородной неньютоновской жидкости с вязкоупругими и релаксационными свойствами. Исследование таких прямых задач связано с изучением соответствующих обратных задач для данной системы, которое требует свойств как непрерывности и регулярности решения и их производных этих прямых задач решений. Система уравнений, помимо начального условия, дополняется одним из граничных условий: условием прилипания или скольжения. В обоих случаях доказаны глобальное во времени существование и единственность сильных решений этих начально-краевых задач. Более того, при соответствующих предположениях на данные была установлена регулярность решений и их производных.

Ключевые слова: система Кельвина–Фойгта, граничные условия скольжения и прилипания, сильные решения, глобальное существование и единственность, гладкость.

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Bulletin of the Karaganda University. Mathematics Series, No. 4(112), 2023, pp. 79–94

DOI 10.31489/2023M4/79-94

UDC 515.122

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A study on new classes of binary soft sets in topological rough approximation space

Soft binary relation is used to define new classes of soft sets, namely BR-soft simply open set and BR-soft simply* alpha open set, in topological rough approximation space over two different universes. The defined set provides information on the missing elements of a BR-soft set and can help in simplifying decision making. Approximation operators are defined and the characteristics of the proposed sets are studied with examples. The relationship between the defined sets and other soft sets is brought out. An accuracy check was done to compare the proposed method with other methods. It is identified that the proposed method is more accurate.

Keywords: Soft set, rough set, simply open, approximation space, topological space.

Introduction

Decision making becomes complicated while handling problems with inappropriate or uncertain data. To deal with complex problems with uncertainty, many researchers developed various mathematical tools and theories. Soft set theory, one of the prominent theories of uncertainty, is highly helpful in decision making due to the presence of its parameters. This theory was developed by Molodtsov [1] in 1999. Further developments in soft set theory and its application were done by many researchers [2–6]. Though soft and rough set theories are different for handling problems with uncertainty, efforts have been made to combine both for solving complex problems [7, 8]. The relationship between soft sets, soft rough sets and topologies were investigated by Li [9]. Covering soft rough sets and their topologies were also studied by many other researchers [10,11]. While dealing with soft rough sets, Feng [12] used parametrized subsets to find upper and lower approximations of a subset. These soft rough sets, soft β rough sets, soft rough approximations, soft pre-rough approximations etc., are further studied by many researchers in decision making [13–17].

Soft set theory was also extended over rough approximation space, nearness approximation spaces and ideal rough topological spaces in [18–20]. In recent years, theories of uncertainty have been extended over two universal sets. However, approximation operators between two different universal sets are less explored. Zhang and Wu [21] were the first to study approximation operators between two different universal sets by the constructive approach of a random approximation space. Following them, a few other authors started working over two different universes using fuzzy rough set, intuitionistic fuzzy rough set, neutrosophic set, etc. [22–25]. In [26], the author constructed a topological space, using the fuzzy b-q neighbourhood of one fuzzy topology and fuzzy b-closure of another fuzzy topology.

The concepts of simply-open and its irresoluteness were studied by Dontchev et al. [27]. Continuous functions, separation axioms of the e-$\mathcal{I}$ set and many other concepts like a-local function are studied in ideal topological spaces by Al-Omeri et al. [28–31]. El Sayed et al. [32] extended simply-open to soft set theory. In addition, El Safty et al. [33] defined the concept of Simply* alpha open sets in rough set theory which is a union of an alpha open set and nowhere dense set. This set is useful in the field of

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decision making as it contributes to attribute reduction. Though it is studied in the rough set, since the soft set contains a parametrization tool, it is appropriate to study simply* alpha open set over soft set. The choice of soft sets in decision making problems varies among different researchers. It is also seen from the literature that every soft set may not include all elements of the universal set. In such cases, information regarding the left-out elements of the universal sets is not emphasized. A decision-making process is highly reliable only by considering every option (element) related to the problem. For this, new classes of soft sets will have to be defined.

In this paper, BR soft simply* alpha open set and BR-soft simply open set are defined in BR-soft topological rough approximation space. The complement of soft sets is taken as in [34]. Apart from this, other classes of BR-soft near sets, namely BR-soft delta, BR-soft nowhere dense, BR-soft alpha open, BR-soft beta open, etc., are also defined over different universes to obtain their relationship between the defined sets. The topological properties of defined sets are studied. In addition, the accuracy of the proposed method is demonstrated using example problems and compared with the methods of Feng [12] and Yao [35].

In the following section (section 2) of the paper, the required preliminary definitions are given. In section 3, the sets are defined and their properties are studied. Example problems illustrating the application of the sets and their accuracy measures are given in section 4. This is followed by a conclusion and scope for future work.

1 *Preliminaries*

Definition 1.1. A soft set m_k is a mapping from a subset of a parameter set $(A \subseteq E)$ to the power set of a universal set U. The collection of soft sets m_{k_i} over U forms soft topology τ, if the following conditions are satisfied.

i $\emptyset, U \in \tau$.

ii The arbitrary union of soft sets in τ is in τ.

iii The finite intersection of soft sets in τ is in τ.

Then, (U, E, τ) is said to be a soft topological space.

Proposition 1.2. The following conditions hold in the soft topological space (U, E, τ).

i $\emptyset, U$ are soft closed sets over U.

ii The arbitrary intersection of a soft closed set is soft closed.

iii A finite union of soft closed set is soft closed.

Proposition 1.3. [34] The following conditions hold in the soft topological space (U, E, τ).

i $\emptyset^C, m_k^C$ are soft closed sets over U.

ii The arbitrary intersection of soft closed set is soft closed.

iii A finite union of soft closed set is soft closed.

Definition 1.4. $(U, \mathcal{R})$ denotes Pawlak's approximation space, $\mathcal{R}$ is an equivalence relation and $X \subseteq S$. Using R following operators were defined.

$$\underline{\mathcal{R}(X)} = \{x \in S : [x]\mathcal{R} \subseteq X\},$$
$$\overline{\mathcal{R}(X)} = \{x \in S : [x]\mathcal{R} \cap X \neq \emptyset\}.$$

If $\underline{\mathcal{R}(X)} \neq \overline{\mathcal{R}(X)}$, X is a rough set. Otherwise, X is definable.

2 *Soft set over "n" different nonempty finite sets*

Definition 2.1. $S_1, S_2, ...S_n$ be nonempty finite sets. K be the subset of a parameter set E. A pair (m, K) or m_K is called a soft binary relation over $S_1, ...S_n$, if (m, K) is a soft set (BR-soft set) over $S_1 \times S_2 \times ... \times S_n$.

Throughout this paper, we consider n=2, i.e., two non-empty finite sets say, S and T.

Definition 2.2. Let $(S, T, R_{m(s,t)})$ be a rough approximation space and τ_{BR} be a soft topology obtained from a soft binary relation over S, T. Thus, $(S, T, R_{m(s,t)}, \tau_{BR})$ is said to be BR-topological rough approximation space where the elements of τ_{BR} are BR-soft open and its complements are closed.

Example 2.3. Let $S = \{2, 3, 5\}, T = \{4, 6\}, E = \{e_1, e_2\} = K$. Let $S\times T = \{(2, 4), (2, 6), (3, 4), (3, 6), (5, 4), (5, 6)\}$. Thus, the soft binary relation over $S\times T$ is $m_k = \{(e_1, \{(3, 4), (5, 4)\}), (e_2, \{(2, 4), (3, 6)\})\}$. The soft relations induced from soft binary relation are as follows:

$$R(3,4) = \{(e_1, \{(5,4)\})\},$$
$$R(5,4) = \{(e_1, \{(3,4)\})\},$$
$$R(2,4) = \{(e_2, \{(3,6)\})\},$$
$$R(3,6) = \{(e_2, \{(2,4)\})\}.$$

Subbasis $S_B = \{\{(e_1, \{(5,4)\})\}, \{(e_1, \{(3,4)\})\}, \{(e_2, \{(3,6)\})\}, \{(e_2, \{(2,4)\})\}\}$.

The topology obtained by taking the finite intersection of an arbitrary union of elements of a subbasis is as follows:

$$\tau_{BR} = \{\emptyset, m_k, \{(e_1, \{(5,4)\})\}, \{(e_1, \{(5,4)\})\}, \{(e_1, \{(5,4)\})\}, \{(e_1, \{(5,4)\})\}, \{e_1, \{(5,4), (3,4)\}\}, \{(e_1, \{(5,4)\}), (e_2, \{(3,6)\}\}, \{(e_1, \{(5,4)\}), (e_2, \{(2,4)\}\}, \{(e_1, \{(3,4)\}), (e_2, \{(3,6)\}\}, \{(e_1, \{(3,4)\}), (e_2, \{(2,4)\}\}, \{(e_2, \{(2,4), (3,6)\})\}, \{(e_1, \{(5,4), (3,4)\}), (e_2, \{(2,4)\})\}, \{(e_1, \{(5,4), (3,4)\}), (e_2, \{(3,6)\})\}, \{(e_1, \{(5,4)\}), (e_2, \{(2,4), (3,6)\})\}, \{(e_1, \{(3,4)\}), (e_2, \{(2,4), (3,6)\})\}\}.$$

Then, $(S, T, R_{m(s,t)}, \tau_{BR})$ is BR-topological rough approximation space.

Definition 2.4. Let $(S, T, R_{m(s,t)}, \tau_{BR})$ be a BR-topological rough approximation space. For each $m_{ki} \subseteq m_k$, the BR-topological approximation operators are defined as follows:

$$\underline{\tau}_{BR}(m_{ki}) = \cup\{m_{kj} \in \tau_{BR}; m_{kj} \subseteq m_{ki}\},$$
$$\overline{\tau}_{BR}(m_{ki}) = \cap\{m_{kj} \in {\tau_{BR}}^C; m_{ki} \subseteq m_{kj}\}.$$

In other words, $\underline{\tau}_{BR}, \overline{\tau}_{BR}$ is considered as interior and closure of the BR-topological approximation space respectively.

3 *BR-Soft simply open, BR-Soft simply* alpha open sets*

Definition 3.1. In a BR-topological rough approximation space, a BR-soft subset is called BR-soft nowhere dense if $\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) = \emptyset$.

Definition 3.2. In a BR-topological rough approximation space a BR-soft subset is called a BR-soft simply* alpha open set if $m_{ki} \in \{\emptyset, m_k, (m_{kj} \cup m_{kl}) : m_{kj}$ is BR-soft α open, m_{kl} is BR-soft nowhere dense and m_k is BR-soft set$\}$.

The collection of BR-soft simply* alpha open set is denoted by $BR_S S^*\alpha O(m_{ki})$, the complement is BR-soft simply* alpha closed.

Definition 3.3. In a BR-topological rough approximation space, a BR-soft subset is called

i BR-soft simply open set if $m_k = (m_{ki}) \cup (m_{kj})$, where (m_{ki}) is BR-soft open and (m_{kj}) is BR-soft nowhere dense.

ii BR-soft delta set if $\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \subseteq \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki}))$.

iii BR-soft semi locally closed set if m_k equals the intersection of BR-soft semi open set and BR-soft semi closed set.

iv BR-soft b open set if $m_{ki} \subseteq (\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \cup \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki})))$.

v BR-soft alpha open if $m_{ki} \subseteq \underline{\tau}_{BR}(\overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki})))$.

vi BR-soft beta open if $m_{ki} \subseteq \overline{\tau}_{BR}(\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})))$.

BR-soft simply open set, BR-soft delta set, BR-soft nowhere dense, and BR-soft b open are denoted as $BR_sSO(m_{ki}), BR_s\delta O(m_{ki}), BR_sNO(m_{ki})$, and $BR_sbO(m_{ki})$ respectively.

Proposition 3.4. Every BR-soft open set is BR-soft alpha open.

Example 3.5. Considering the topology taken in Example 2.3, where $m_k = \{(e_1, \{(3,4),(5,4)\}), (e_2, \{(2,4),(3,6)\})\}$. Here, $BR_sNO(m_k) = \{\emptyset, \{(e_1, \{(5,6)\}, (e_2, \{(2,6)\})\}\}$. It is clear that the BR-soft set $m_{k_1} = \{(e_1, \{(3,4),(5,6)\}), (e_2, \{(2,4),(2,6)\})\}$ is BR-soft simply open, $m_{k_2} = \{(e_1, \{(3,4),(5,6)\}), (e_2, \{(2,4),(2,6),(3,6)\})\}$ is a BR-soft δ set.

Theorem 3.6. Every BR-soft simply open is BR-soft simply* alpha open.

Proof. Let m_k be a BR-soft simply open set. That is, m_k is a union of BR-soft open set and BR-soft nowhere dense set. Since every BR-soft open set is BR-soft alpha open, we denote m_k as a union of BR-soft alpha open set and BR-soft nowhere dense set. This proves the theorem.

The converse of Theorem 3.6 need not be true and is explained in the following example.

Example 3.7. Consider the topology taken in Example 2.3. Let the BR-soft subset be $m_k = \{(e_1, \{(3,4)\}), (e_2, \{(2,4)\})\}$ which is a BR-soft alpha set.
Thus, $\{(e_1, \{(3,4),(5,6)\}), (e_2, \{(2,4),(2,6),(3,6)\})\}$, $m_{k_1} = \{(e_1, \{(3,4),(5,6)\}), (e_2, \{(2,4),(2,6)\})\}$ are both BR-soft simply* alpha open and BR-soft simply open. The BR-soft nowhere dense set $m_{k_2} = \{(e_1, \{(5,6)\}), (e_2, \{(2,6)\})\}$ is BR-soft simply open but not BR-soft simply* alpha open.

Theorem 3.8. Every BR-soft open set is BR-soft simply open and BR-soft simply* alpha open set.

Proof. The proof is obvious for BR-soft simply open and by Theorem 3.6, the BR-soft open set is BR-soft simply* alpha open.

Remark 3.9. Though the union of the BR-soft alpha open set and the BR-soft nowhere dense set is BR-soft simply* alpha open, a BR-soft simply* alpha open set need not be BR-soft alpha open.

The following example explains Remark 3.9.

Example 3.10. The BR-soft set $m_{k_1} = \{(e_1, \{(3,4),(5,6)\}), (e_2, \{(2,4),(2,6)\})\}$ taken in example 3.7 is BR-soft simply* alpha open but not BR-soft alpha open. Since τ_{BR} is obtained by considering m_k, τ_{BR}^C is also taken with respect to m_k. Let us consider $m_{k2} = \{(e_1, \{(5,6)\}), (e_2, \{(2,6)\})\}$.
Then, we have $\overline{BR}_S(\underline{BR}_S(m_{k2})) = \emptyset$, and $\overline{BR}_S(\underline{BR}_S(\overline{BR}_S(m_{k2}))) = \emptyset$ which implies
$\overline{BR}_S(\underline{BR}_S(m_{k2})) \subset m_{k2}$, $\overline{BR}_S(\underline{BR}_S(\overline{BR}_S(m_{k2}))) \subset m_{k2}$.

Theorem 3.11. For a BR-soft subset m_{ki} in a BR-topological rough approximation space, the following conditions are equivalent:

i m_{k_i} is BR-soft simply open.

ii m_{k_i} is BR-soft semi locally closed.

iii m_{k_i} is BR-soft delta.

iv m_{k_i} is BR-soft nowhere dense.

Proof. (i) $\iff$ (ii) is obvious.

(ii) $\iff$ (iii) Let m_{ki} be BR-soft semi locally closed.
Then,

$$\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \subseteq (\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \cap \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki})))$$

and

$$\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \subseteq \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki})).$$

Thus, m_{ki} is a BR-soft delta set.

(iii) $\Longleftrightarrow$ (iv) Let m_{ki} be a BR-soft delta set. Then,

$$\begin{aligned}\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) &= \underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \cap \underline{\tau}_{BR}(\overline{\tau}_{BR}(S \times T \setminus m_{ki}))\\ &= \underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \cap (S \times T \setminus \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki})))\\ &= \underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \setminus \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki}))\\ &= \emptyset.\end{aligned}$$

Theorem 3.12. Consider the BR-topological rough approximation space, then

i The arbitrary union of a BR-soft simply open set is BR-soft simply open.

ii The finite intersection of BR-soft simply open set is BR-soft simply open.

Proof.

i Let m_{k1}, m_{k2} be two BR-soft simply open sets. Then

$$\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{k1})) \subseteq \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{k1}))$$

and

$$\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{k2})) \subseteq \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{k2})).$$

By taking union we get,

$$\begin{aligned}\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{k1})) \cup \underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{k2})) &\subseteq \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{k1})) \cup \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{k2}))\\ \Longrightarrow \underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{k1} \cup m_{k2})) &\subseteq \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{k1} \cup m_{k2})).\end{aligned}$$

Let $m_{k1} \cup m_{k2}$ be m_{k3}. Then, $\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{k3})) \subseteq \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{k3}))$.

ii m_{k_i} be the collection of BR-soft simply open sets where $i = 1, 2, ...$
Then,

$$\underline{\tau}_{BR}(\overline{\tau}_{BR}(\cap_{i=1}^{n}(m_{k_i}))) \subseteq \overline{\tau}_{BR}(\underline{\tau}_{BR}(\cap_{i=1}^{n}(m_{k_i}))).$$

Hence, $\cap_{i=1}^{n}(m_{k_i})$ is BR-soft simply open.

Remark 3.13. In the BR-topological rough approximation space,

i BR-soft simply open, BR-soft simply* alpha open and BR-soft beta open are not comparable.

ii BR-soft simply open, BR-soft simply* alpha open and BR-soft b open are not comparable.

iii BR-soft simply open, BR-soft simply* alpha open and BR-soft preopen are not comparable.

Proposition 3.14. In the BR-topological rough approximation space,

i The union of the BR-soft simply* alpha open set is BR-soft simply* alpha open.

ii The finite intersection of the BR-soft simply* alpha open set is BR-soft simply* alpha open.

Remark 3.15. Every BR-soft delta set is BR-soft nowhere dense.

Theorem 3.16. In a BR-topological rough approximation space, every BR-soft simply* alpha open set is BR-soft alpha closed.

Proof. The proof is obvious from Definition 3.2.

Theorem 3.17. In a BR-topological rough approximation space, every BR-soft simply* alpha open set is BR-soft pre closed (resp. BR-soft beta closed).

Proof. Let m_{ki} be BR-soft simply* alpha open. From Theorem 3.16,

$$\underline{\tau}_{BR}(\overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki})) \subseteq m_{ki}.$$

Since,

$$\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \subseteq \underline{\tau}_{BR}(\overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki})) \subseteq m_{ki}.$$

Thus,

$$\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \subseteq m_{ki}.$$

The proof is similar for BR-soft beta closed.

Theorem 3.18. For a BR-topological rough approximation space, the following conditions are equivalent:

i Every BR-soft simply* alpha open set is a BR-soft δ set.

ii Every BR-soft simply* alpha open set is BR-soft beta closed.

iii Every BR-soft simply* alpha open set is BR-soft pre closed.

iv Every BR-soft simply* alpha open set is BR-soft b closed.

Proof. (i) $\Longleftrightarrow$ (ii) m_{ki} be BR-soft simply* alpha open, BR-soft delta set.
Thus, we have

$$\begin{aligned} \underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) &\subseteq \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki})). \\ \underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) &\subseteq \overline{\tau}_{BR}(\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki}))) \\ &\subseteq \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki})) \\ &\subseteq m_{ki}. \end{aligned}$$

Therefore, $\overline{\tau}_{BR}(\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki}))) \subseteq m_{ki}$.

(ii) $\Longleftrightarrow$ (iii) It is obvious from the above proof that $\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \subseteq m_{ki}$.

(iii) $\Longleftrightarrow$ (iv) Since m_{ki} is preclosed and $\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \subseteq \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki}))$.

We have,

$$\begin{aligned} \underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \cap \overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki})) &= \underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) \\ &\subseteq m_{ki}. \end{aligned}$$

A diagrammatic representation of the above-mentioned concepts is given below (Fig)

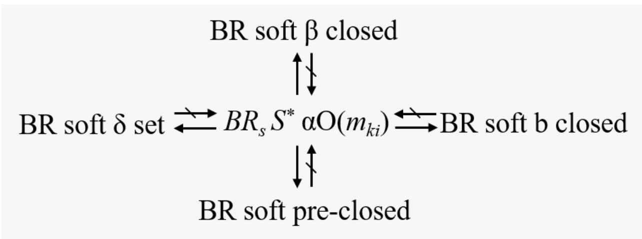

Figure. Diagrammatic representation

Proposition 3.19. Every BR-soft simply* alpha open is BR-soft semi open (resp. BR-soft beta open) based on *Proposition 1.3*.

Proof. For every BR-soft simply* alpha open m_{ki},

$$\begin{aligned} \underline{BR}_S(m_{ki}) \subseteq m_{ki} &\subseteq \overline{BR}_S(m_{ki}) \\ \Longrightarrow \overline{BR}_S(\underline{BR}_S(m_{ki})) &\subseteq \overline{BR}_S(m_{ki}) \\ \Longrightarrow m_{ki} &\subseteq \overline{BR}_S(\underline{BR}_S(m_{ki})). \end{aligned}$$

Hence, BR-soft simply* alpha open is BR-soft semi open.

The proof is similar for BR-soft beta open.

Example 3.20. Consider the topology taken in Example 2.3.
Let $m_k = \{(e_1, \{(3,4),(5,4)\}),(e_2,\{(2,4),(3,6)\})\}$ be the BR-soft simply* alpha open set. Then, by taking interior and closure with respect to Proposition 1.3, we have
$\overline{BR}_S(\underline{BR}_S(m_k)) = m_E, \overline{BR}_S(\underline{BR}_S(\overline{BR}_S(m_k))) = m_E$. That is, m_k is BR-soft semi open, BR-soft beta open.

Definition 3.21. Let $(S, T, R_{m(s,t)}, \tau_{BR})$ be a BR-topological rough approximation space. For each BR-soft simply* alpha open sets $m_{ki} \subseteq m_k$, the BR-topological approximation operators are defined as follows:

$$\underline{BR}_S(m_{ki}) = \cup\{m_{kj} \in \tau_{BR}; m_{kj} \subseteq m_{ki}\},$$
$$\overline{BR}_S(m_{ki}) = \cap\{m_{kj} \in \tau_{BR}^C; m_{ki} \subseteq m_{kj}\},$$

where m_{kj} is the BR-soft set, $\underline{BR}_S(m_{ki})$, $\overline{BR}_S(m_{ki})$ are the interior and closure of BR soft simply* alpha open sets in BR-topological rough approximation space respectively.

Theorem 3.22. The collection of all BR soft simply* alpha open sets obtained from BR-topological rough approximation space forms a BR-soft topology τ^*_{BR}.

Proof. It is obvious from *Definition 3.2* and *Proposition 3.12*.

Definition 3.23. Let τ^*_{BR} be the BR-soft topology obtained from the collection of BR soft simply* alpha open sets. For each BR-soft simply* alpha open sets $m_{ki} \subseteq m_k$ and m_{kj}, the approximation operators are defined as follows:

$$\underline{BR}_S(m_{ki}) = \cup\{m_{kj} \in \tau^*_{BR}; m_{kj} \subseteq m_{ki}\},$$
$$\overline{BR}_S(m_{ki}) = \cap\{m_{kj} \in {\tau^*_{BR}}^C; m_{ki} \subseteq m_{kj}\}.$$

Here, $\underline{BR}_S(m_{ki})$ and $\overline{BR}_S(m_{ki})$ are lower approximation (interior) and upper approximation (closure) of BR soft simply* alpha open sets in the BR-soft topology obtained from the collection of BR soft simply* alpha open sets.

The boundary region is the difference between the upper and lower approximation operators.

Concerning the quality of the approximation, accuracy is defined as the ratio of cardinality of the lower approximation (interior) and cardinality of the upper approximation (closure).

Proposition 3.24. Let τ^*_{BR} be a BR-soft topology and m_{ki}, m_{kj} be two BR-soft simply* alpha open subsets of a BR-soft simply* alpha open set m_k, then the BR-topological operators satisfy the following properties:

i $\underline{BR}_S(\emptyset) = \overline{BR}_S(\emptyset) = \emptyset$,
ii $\underline{BR}_S(m_{ki}) = \overline{BR}_S(m_{ki}) = m_{ki}$,
iii If $m_{ki} \subseteq m_{kj}$, then $\underline{BR}_S(m_{ki}) \subseteq \underline{BR}_S(m_{kj})$,
iv If $m_{ki} \subseteq m_{kj}$, then $\overline{BR}_S(m_{ki}) \subseteq \overline{BR}_S(m_{kj})$,
v $\underline{BR}_S(m_{ki} \cap m_{kj}) = \underline{BR}_S(m_{ki}) \cap \underline{BR}_S(m_{kj})$,
vi $\underline{BR}_S(m_{ki}) \cup \underline{BR}_S(m_{kj}) \subseteq \underline{BR}_S(m_{ki}) \cup (m_{kj}))$,
vii $\overline{BR}_S(m_{ki} \cup m_{kj}) = \overline{BR}_S(m_{ki}) \cup \overline{BR}_S(m_{kj})$,
viii $\overline{BR}_S(m_{ki} \cap m_{kj}) \subseteq \overline{BR}_S(m_{ki}) \cap \overline{BR}_S(m_{kj})$.

Example 3.25. Let $S = \{2,3,5\}, T = \{4,6\}$, and $E = \{e_1, e_2\} = K$. Let $S \times T = \{(2,4),(2,6),(3,4),(3,6),(5,4),(5,6)\}$. Thus, the soft binary relation over $S \times T$ is $m_k = \{(e_1, \{(3,4),(5,4)\}),(e_2,\{(2,4),(3,6)\})\}$.

The soft relations induced from soft binary relation are as follows:

$$\begin{aligned}R(3,4) &= \{(e_1, \{(5,4)\})\},\\ R(5,4) &= \{(e_1, \{(3,4)\})\},\\ R(2,4) &= \{(e_2, \{(3,6)\})\},\\ R(3,6) &= \{(e_2, \{(2,4)\})\}.\end{aligned}$$

Subbasis $S_B = \{\{(e_1, \{(5,4)\})\}, \{(e_1, \{(3,4)\})\}, \{(e_2, \{(3,6)\})\}, \{(e_2, \{(2,4)\})\}\}$.
The topology obtained by taking the finite intersection of an arbitrary union of elements of subbasis is as follows:

$$\begin{aligned}\tau_{BR} = \{&\emptyset, m_k, \{(e_1, \{(5,4)\})\}, \{(e_1, \{(5,4)\})\}, \{(e_1, \{(5,4)\})\}, \{(e_1, \{(5,4)\})\}, \{e_1, \{(5,4), (3,4)\}\},\\ &\{(e_1, \{(5,4)\}), (e_2, \{(3,6)\}\}, \{(e_1, \{(5,4)\}), (e_2, \{(2,4)\}\}, \{(e_1, \{(3,4)\}), (e_2, \{(3,6)\}\}, \{(e_1,\\ &\{(3,4)\}), (e_2, \{(2,4)\}\}, \{(e_2, \{(2,4), (3,6)\})\}, \{(e_1, \{(5,4), (3,4)\}), (e_2, \{(2,4)\})\},\\ &\{(e_1, \{(5,4), (3,4)\}), (e_2, \{(3,6)\})\}, \{(e_1, \{(5,4)\}), (e_2, \{(2,4), (3,6)\})\}, \{(e_1, \{(3,4)\}), (e_2,\\ &\{(2,4), (3,6)\})\}\}.\end{aligned}$$

Let $\{(e_1, \{(5,6)\}), (e_2, \{(2,6)\})\}$ be BR-soft nowhere dense set. Then, τ^*_{BR} is the BR-soft topology obtained by taking the union of BR-soft alpha open sets and BR-soft nowhere dense set. Consider BR-soft simply* alpha open sets $m_{k1} = \{(e_1, \{(5,6)\})\}$ and $m_{k2} = \{(e_1, \{(5,6)\}), (e_2, \{(2,6)\})\}$ where $m_{k1} \subset m_{k2}$. Therefore, we have,

$$\begin{aligned}&\overline{BR}_S(m_{k1}) = m_{k1}, \underline{BR}_S(m_{k1}) = m_{k1}.\\ &\overline{BR}_S(m_{k2}) = m_{k2}, \underline{BR}_S(m_{k2}) = m_{k2}.\\ \implies &\overline{BR}_S(m_{k1}) \subset \overline{BR}_S(m_{k2}) \ and\\ &\underline{BR}_S(m_{k1}) \subset \underline{BR}_S(m_{k2}).\end{aligned}$$

Now, let $m_{k1} = \{(e_1, \{(5,6)\})\}$ and $m_{k2} = \{(e_2, \{(2,6)\})\}$. Then,

$$\begin{aligned}\overline{BR}_S(m_{k1}) = m_{k1}, \underline{BR}_S(m_{k1}) = m_{k1}.\\ \overline{BR}_S(m_{k2}) = m_{k2}, \underline{BR}_S(m_{k2}) = m_{k2}.\end{aligned}$$

Thus,

$$m_{k1} \cap m_{k2} = \emptyset, \tag{1}$$

$$\underline{BR}_S(m_{k1} \cap m_{k2}) = \emptyset, \tag{2}$$

$$\underline{BR}_S(m_{k1}) \cap \underline{BR}_S(m_{k2}) = \emptyset. \tag{3}$$

From (1), (2) and (3) we have, $\underline{BR}_S(m_{k1} \cap m_{k2}) = \underline{BR}_S(m_{k1}) \cap \underline{BR}_S(m_{k2})$.
Similarly, $\overline{BR}_S(m_{k1} \cap m_{k2}) = \overline{BR}_S(m_{k1}) \cap \overline{BR}_S(m_{k2})$.

Theorem 3.26. Let m_{ki} and m_{kj} be two BR-soft subsets of BR-topological rough approximation space. If m_{ki} is BR-soft simply* alpha closed, then $\overline{BR}_S(m_{ki} \cap m_{kj}) \subseteq m_{ki} \cap \overline{BR}_S(m_{kj})$.

Proof. Let m_{ki} be BR-soft simply* alpha closed, such that $\overline{BR}_S(m_{ki}) = m_{ki}$. From Proposition 3.24 we have,

$$\begin{aligned}\overline{BR}_S(m_{ki} \cap m_{kj}) &\subseteq \overline{BR}_S(m_{ki}) \cap \overline{BR}_S(m_{kj})\\ &\subseteq m_{ki} \cap \overline{BR}_S(m_{kj}).\end{aligned}$$

Example 3.27. Consider the topology taken in Example 2.3. Let $m_{k1} = \{(e_1, \{(3,4), (5,4)\})\}$ be a BR-soft simply* alpha closed and $m_{k2} = \{(e_1, \{(3,4)\})\}$ be BR-soft subset. Then,

$$m_{k1} \cap m_{k2} = m_{k2}, \tag{4}$$

$$\overline{BR}_S(m_{k1} \cap m_{k2}) = m_{k2}, \tag{5}$$

$$\overline{BR}_S(m_{k2}) = \{(e_1, \{(3,4), (5,6)\}, (e_2, \{(2,6)\})\}. \tag{6}$$

From (4),(5),(6) we have, $\overline{BR}_S(m_{k1} \cap m_{k2}) = m_{k1} \cap \overline{BR}_S(m_{k2})$.

Theorem 3.28. Let m_{ki} be a BR-soft subset of BR-topological rough approximation space, then $BR_S S^* \alpha d(m_{ki}) = \emptyset$.

Proof. Let $(s,t) \in (S, T, R_{m(s,t)}, \tau_{BR})$ and BR-topological rough approximation space be discrete, then every BR-soft subset is BR-soft open and BR-soft simply* alpha open. Thus, every (s_i, t_j) is BR-soft simply* alpha open. Let $m_{kj} = (s,t)$. Then, $m_{ki} \cap m_{kj} = m_{ki} \cap (s,t) \subseteq (s,t)$.

Hence, (s,t) is not a BR-soft simply* alpha limit point of m_{ki} which implies $BR_S S^* \alpha d(m_{ki}) = \emptyset$.

Theorem 3.29. For any BR-soft simply* alpha subsets m_{ki} of $(S, T, R_{m(s,t)}, \tau_{BR})$, $\overline{BR}(m_{ki}) = m_{ki} \cup BR_S S^* \alpha d(m_{ki})$.

Proof. Let $(s,t) \in \overline{BR}(m_{ki})$. Assume $(s,t) \notin m_{ki}$ and $m_{kj} \in \tau^*_{BR}$ with $(s,t) \in m_{kj}$. Then, $(m_{ki} \cap m_{kj}) - (s,t) \neq \emptyset$ implies $(s,t) \in BR_S S^* \alpha d(m_{ki})$. Hence, $\overline{BR}(m_{ki}) \subseteq m_{ki} \cup BR_S S^* \alpha d(m_{ki})$.

Let $(s,t) \in m_{ki} \cup BR_S S^* \alpha d(m_{ki})$ implies $m_{ki} \subseteq \overline{BR}(m_{ki})$. Since, all BR-soft simply* alpha limit points of m_{ki} are soft preclosure of m_{ki}. $m_{ki} \cup BR_S S^* \alpha d(m_{ki}) \subseteq \overline{BR}(m_{ki})$.
Hence, $\overline{BR}(m_{ki}) = m_{ki} \cup BR_S S^* \alpha d(m_{ki})$.

4 Application

To observe the accuracy of the proposed method, two examples have been demonstrated in this section.

Example 4.1. Decision making on the infections of COVID-19 in humans is taken as an application of our approach. Since we use soft binary relation, this method helps to find people affected by COVID and the reasons for getting affected at the same time.

Let $S = \{S_1, S_2, S_3, S_4\}$ be four people considered and $T = \{T_1, T_2, T_3, T_4\}$ be the reasons for getting affected where,
T_1 = stay at home.
T_2 = go out and contact infected people.
T_3 = low immunity; rarely go out.
T_4 = Stay at home but any one in family go out.

Let $E = \{e_1\ (fever), e_2\ (fatigue), e_3\ (loss\ of\ smell/taste), e_4\ (Cough)\}$ be the parameter set and $A = \{e_1, e_3\}$ subset of E.

$$S \times T = \{(S_1,T_1),(S_1,T_2),(S_1,T_3),(S_1,T_4),(S_2,T_1),(S_2,T_2),(S_2,T_3),(S_2,T_4),(S_3,T_1),(S_3,T_2),(S_3,T_3),\\ (S_3,T_4),(S_4,T_1),(S_4,T_2),(S_4,T_3),(S_4,T_4)\}.$$

The following table (Table 1) represents the BR-soft set over $S \times T$ with respect to E.

Table 1

Soft matrix

$S\times T$	e_1	e_2	e_3	e_4	Yes/No
(S_1,T_1)	1	0	0	1	0
(S_1,T_2)	1	1	1	0	1
(S_1,T_3)	0	1	0	1	0
(S_1,T_4)	1	0	1	1	1
(S_2,T_1)	0	1	0	0	0
(S_2,T_2)	0	1	1	1	1
(S_2,T_3)	1	1	0	0	0
(S_2,T_4)	0	1	0	1	0
(S_3,T_1)	1	0	0	0	0
(S_3,T_2)	0	0	1	0	1
(S_3,T_3)	0	0	0	0	0
(S_3,T_4)	1	0	1	0	1
(S_4,T_1)	0	0	0	1	0
(S_4,T_2)	1	1	1	1	1
(S_4,T_3)	0	0	0	0	0
(S_4,T_4)	1	0	0	0	0

Let the soft set over the parameter set A be

$$F(e_1)=\{(S_1,T_1),(S_1,T_2),(S_1,T_4),(S_2,T_3),(S_3,T_1),(S_3,T_4),(S_4,T_2),(S_4,T_4)\},$$
$$F(e_3)=\{(S_1,T_2),(S_1,T_4),(S_2,T_2),(S_3,T_2),(S_3,T_4),(S_4,T_2)\}$$

in which the BR-soft set represents the people infected with COVID and their reason.

Let the BR-soft subset be

$$m_{ki}=\{(e_1,\{(S_1,T_1),(S_1,T_2),(S_2,T_3),(S_4,T_2)\}),(e_3,\{(S_1,T_2),(S_3,T_2),(S_4,T_2)\})\}.$$

According to Feng's method,

$$\underline{S}_{apr}(m_{ki})=\{(s,t)\in S\times T: for\ every\ k\in K,(s,t)\in m(k)\subseteq S\times T\},$$
$$\overline{S}_{apr}(m_{ki})=\{(s,t)\in S\times T:\ for\ every\ k\in K,(s,t)\in m(k)\cap S\times T\neq\emptyset\},$$

where $\underline{S}_{apr}(m,k_i),\overline{S}_{apr}(m,k_i)$ are lower and upper approximation operators. Thus we have,

$$\underline{S}_{apr}(m_{ki})=\emptyset,$$
$$\overline{S}_{apr}(m_{ki})=\{(S_1,T_1),(S_1,T_2),(S_1,T_4),(S_2,T_2),(S_2,T_3),(S_3,T_1),$$
$$(S_3,T_2),(S_3,T_4),(S_4,T_2),(S_4,T_4)\}.$$

Accuracy = cardinality of $\underline{S}_{apr}(m,k_i)/$ cardinality of $\overline{S}_{apr}(m,k_i)=0/10=0$ which implies that no patient is infected with COVID which contradicts the data given in Table 1.

To find the approximation operators of the proposed method, the subbase are obtained from Soft relations as follows:

$$\begin{aligned}
&R(S_1,T_3)=R(S_2,T_1)=R(S_2,T_4)=R(S_3,T_3)=R(S_4,T_1)=R(S_4,T_3)=\emptyset,\\
&R(S_1,T_1)=\{(e_1,\{(S_1,T_2),(S_1,T_4),(S_2,T_3),(S_3,T_1),(S_3,T_4),(S_4,T_2),(S_4,T_4)\})\},\\
&R(S_1,T_2)=\{(e_1,\{(S_1,T_1),(S_1,T_4),(S_2,T_3),(S_3,T_1),(S_3,T_4),(S_4,T_2),(S_4,T_4)\},\\
&\qquad\qquad(e_3,\{(S_1,T_4),(S_2,T_2),(S_3,T_2),(S_3,T_4),(S_4,T_2)\})\},\\
&R(S_1,T_4)=\{(e_1,\{(S_1,T_1),(S_1,T_2),(S_2,T_3),(S_3,T_1),(S_3,T_4),(S_4,T_2),(S_4,T_4)\},\\
&\qquad\qquad(e_3,\{(S_1,T_2),(S_2,T_2),(S_3,T_2),(S_3,T_4),(S_4,T_2)\})\},\\
&R(S_2,T_2)=\{(e_3,\{(S_1,T_2),(S_1,T_4),(S_3,T_2),(S_3,T_4),(S_4,T_2)\})\},\\
&R(S_2,T_3)=\{(e_1,\{(S_1,T_1),(S_1,T_2),(S_1,T_4),(S_3,T_1),(S_3,T_4),(S_4,T_2),(S_4,T_4)\})\},\\
&R(S_3,T_1)=\{(e_1,\{(S_1,T_1),(S_1,T_2),(S_1,T_4),(S_2,T_3),(S_3,T_4),(S_4,T_2),(S_4,T_4)\})\},\\
&R(S_3,T_2)=\{(e_3,\{(S_1,T_2),(S_1,T_4),(S_2,T_2),(S_3,T_4),(S_4,T_2)\})\},\\
&R(S_3,T_4)=\{(e_1,\{(S_1,T_1),(S_1,T_2),(S_1,T_4),(S_2,T_3),(S_3,T_1),(S_4,T_2),(S_4,T_4)\},\\
&\qquad\qquad(e_3,\{(S_1,T_2),(S_2,T_2),(S_3,T_2),(S_4,T_2)\})\},\\
&R(S_4,T_2)=\{(e_1,\{(S_1,T_1),(S_1,T_2),(S_1,T_4),(S_2,T_3),(S_3,T_1),(S_3,T_4),(S_4,T_4)\},\\
&\qquad\qquad(e_3,\{(S_1,T_2),(S_1,T_4),(S_2,T_2),(S_3,T_2),(S_3,T_4)\})\},\\
&R(S_4,T_4)=\{(e_1,\{(S_1,T_1),(S_1,T_2),(S_1,T_4),(S_2,T_3),(S_3,T_1),(S_3,T_4),(S_4,T_2)\})\}.
\end{aligned}$$

Thus, topological rough approximation space τ_{BR} with soft binary relation over S, T as subbase is obtained by taking an arbitrary union of finite intersection of elements of a subbasis.
According to proposed method,

$$\begin{aligned}
&\underline{\tau}_{BR}(m_{ki})=\{(e_1,\emptyset),(e_3,\{(S_1,T_2),(S_3,T_2),(S_4,T_2)\})\},\\
&\overline{\tau}_{BR}(m_{ki})=m_k.
\end{aligned}$$

Accuracy = cardinality of $\underline{\tau}_{BR}(m_{ki})$/ cardinality of $\overline{\tau}_{BR}(m_{ki})$ = 3/5=0.6.

If the people infected with COVID and their reasons be

$$\begin{aligned}
m_{kj}=\{&(e_1,\{(S_1,T_1),(S_1,T_2),(S_1,T_4),(S_2,T_3),(S_3,T_1),(S_3,T_4),(S_4,T_2),(S_4,T_4)\}),\\
&(e_3,\{(S_1,T_2),(S_1,T_4),(S_2,T_2),(S_3,T_2),(S_3,T_4),(S_4,T_2)\})\}.
\end{aligned}$$

Then, according to Feng's method,

$$\begin{aligned}
&\underline{S}_{apr}(m_{kj})=S\times T,\\
&\overline{S}_{apr}(m_{kj})=S\times T.
\end{aligned}$$

Accuracy is one.

Similarly, according to the proposed method,

$$\begin{aligned}
&\underline{\tau}_{BR}(m_{kj})=m_{kj},\\
&\overline{\tau}_{BR}(m_{kj})=m_{kj}.
\end{aligned}$$

Accuracy is one.

From the above two cases, it is obvious that, in the case of soft topological approximation space, accuracy of the proposed method is higher than Feng's method.

Example 4.2. Consider *Example 2.3* where S is the set of all prime numbers less than or equal to 6, T is the set of all composite numbers less than or equal to 6. The soft relation induced from soft binary relation are as follows:

$$\begin{aligned} R(3,4) &= \{(e_1, \{(5,4)\})\}, \\ R(5,4) &= \{(e_1, \{(3,4)\})\}, \\ R(2,4) &= \{(e_2, \{(3,6)\})\}, \\ R(3,6) &= \{(e_2, \{(2,4)\})\}. \end{aligned}$$

Subbasis $S_B = \{\{(e_1, \{(5,4)\})\}, \{(e_1, \{(3,4)\})\}, \{(e_2, \{(3,6)\})\}, \{(e_2, \{(2,4)\})\}\}$.

The topology obtained by taking the finite intersection of an arbitrary union of elements of subbasis is as follows:

$$\begin{aligned} \tau_{BR} = \{&\emptyset, m_k, \{(e_1, \{(5,4)\})\}, \{(e_1, \{(3,4)\})\}, \{(e_2, \{(3,6)\})\}, \{(e_2, \{(2,4)\})\}, \{e_1, \{(5,4), (3,4)\}\}, \\ &\{(e_1, \{(5,4)\}), (e_2, \{(3,6)\}\}, \{(e_1, \{(5,4)\}), (e_2, \{(2,4)\}\}, \{(e_1, \{(3,4)\}), (e_2, \{(3,6)\}\}, \\ &\{(e_1, \{(3,4)\}), (e_2, \{(2,4)\}\}, \{(e_2, \{(2,4), (3,6)\})\}, \{(e_1, \{(5,4), (3,4)\}), (e_2, \{(2,4)\})\}, \\ &\{(e_1, \{(5,4), (3,4)\}), (e_2, \{(3,6)\})\}, \{(e_1, \{(5,4)\}), (e_2, \{(2,4), (3,6)\})\}, \\ &\{(e_1, \{(3,4)\}), (e_2, \{(2,4), (3,6)\})\}\}, \end{aligned}$$

where

$$\begin{aligned} m_k &= \{(e_1, \{(3,4), (5,4)\}), (e_2, \{(2,4), (3,6)\}\}, \\ m_E &= \{(e_1, \{(3,4), (5,4), (5,6)\}), (e_2, \{(2,4), (2,6), (3,6)\}\}. \end{aligned}$$

Then, $(S, T, R_{m(s,t)}, \tau_{BR})$ is BR-topological rough approximation space.

$$\begin{aligned} \tau^C_{BR} = \{&m_E, \{(e_1, \{(5,6)\}), (e_2, \{(2,6)\}, \{(e_1, \{(3,4), (5,6)\}), (e_2, \{(2,4), (2,6), (3,6)\})\}, \\ &\{(e_1, \{(5,4), (5,6)\}), (e_2, \{(2,4), (2,6), (3,6)\})\}, \{(e_1, \{(3,4), (5,4), (5,6)\}), (e_2, \\ &\{(2,4), (2,6)\})\}, \{(e_1, \{(3,4), (5,4), (5,6)\}), (e_2, \{(2,6), (3,6)\})\}, \{(e_1, \{(5,6)\}), \\ &(e_2, \{(2,4), (2,6), (3,6)\})\}, \{(e_1, \{(3,4), (5,6)\}), (e_2, \{(2,4), (2,6)\})\}, \{(e_1, \{(5,4), \\ &(3,4)\}), (e_2, \{((2,6), (3,6)\})\}, \{(e_1, \{(5,4), (5,6)\}), (e_2, \{(2,4), (2,6)\})\}, \{(e_1, \{(5,4), \\ &(5,6)\}), (e_2, \{(2,6), (3,6)\})\}, \{(e_1, \{(3,4), (5,4), (5,6)\}), (e_2, \{(2,6)\})\}, \{(e_1, \{(5,6)\}), \\ &(e_2, \{(2,4), (2,6)\})\}, \{(e_1, \{(5,6)\}), (e_2, \{(2,6), (3,6)\})\}, \{(e_1, \{(3,4), (5,6)\}), (e_2, \{(2,6)\})\}, \\ &\{(e_1, \{(5,4), (5,6)\}), (e_2, \{(2,6)\})\}\}. \end{aligned}$$

The BR-soft nowhere dense sets are $\emptyset$, $\{(e_1, \{(5,6)\}), (e_2, \{(2,6)\})\}$, $\{(e_1, \{(5,6)\})\}$ and $\{(e_2, \{(2,6)\})\}$. Then, the collection of BR-soft simply* alpha open sets are as follows:
$\tau'_{BR} = \tau_{BR}$, when $\emptyset$ is the BR-soft nowhere dense set (Since all BR-soft open sets are BR-soft alpha open sets).

$\tau''_{BR} = \tau_{BR} \cup \{(e_1, \{(5,6)\}), (e_2, \{(2,6)\})\}$.

$\tau'''_{BR} = \tau_{BR} \cup \{(e_1, \{(5,6)\})\}$.

$\tau''''_{BR} = \tau_{BR} \cup \{(e_2, \{(2,6)\})\}$.

$\tau^*_{BR} = \cup\{\tau^i_{BR}\}$ where $i = ', '', ''', ''''$.

The accuracy of BR-soft subsets of $S \times T$ is obtained by the Pawlak accuracy measure as follows:

Table 2 describes the accuracy of BR-soft subsets in τ_{BR} containing BR-soft open sets based on Yao's method where accuracy = cardinality of $int(m_k)$/ cardinality of $cl(m_k)$.
Table 3 describes the accuracy of BR-soft subsets in τ^*_{BR} containing BR-soft simply* alpha open sets. Accuracy = cardinality of $\underline{BR}_S$/ cardinality of $\overline{BR}_S$, $\underline{BR}_S$ and $\overline{BR}_S$ are lower and upper approximation operators.

Table 2

Accuracy of BR-soft subsets

BR-soft subsets	Yao's method		Accuracy
	int	cl	
$A = \{(e_1, \{(5,6)\})\}$	$\emptyset$	$\{(e_1, \{(5,6)\}), (e_2, \{(2,6)\})\}$	0
$B = \{(e_2, \{(2,6)\})\}$	$\emptyset$	$\{(e_1, \{(5,6)\}), (e_2, \{(2,6)\})\}$	0
$C = \{(e_1, \{(2,4)\})\}$	C	$\{(e_1, \{(5,6)\}), (e_2, \{(2,4), (2,6)\})\}$	0.3
$D = \{(e_1, \{(3,4)\}), (e_2, \{(2,4)\})\}$	D	$\{(e_1, \{(3,4), (5,6)\}), (e_2, \{(2,4), (2,6)\})\}$	0.5
$E = \{(e_1, \{(3,4)\}), (e_2, \{(2,6)\})\}$	$\{(e_1, \{(3,4)\})\}$	$\{(e_1, \{(3,4), (5,6)\}), (e_2, \{(2,6)\})\}$	0.3
$F = \{(e_1, \{(3,4), (5,4)\}), (e_2, \{(2,4)\})\}$	F	$\{(e_1, \{(3,4), (5,4), (5,6)\}), (e_2, \{(2,6)\})\}$	0.75
$G = \{(e_1, \{(3,4), (5,4)\}), (e_2, \{(2,4), (3,6)\})\}$	G	m_E	0.7
$H = \{(e_1, \{(3,4)\}), (e_2, \{(2,4), (3,6)\})\}$	H	$\{(e_1, \{(3,4), (5,6)\}), (e_2, \{(2,4), (3,6), (2,6)\})\}$	0.6
$I = \{(e_1, \{(5,4), (5,6)\}), (e_2, \{(3,6)\})\}$	$\{(e_1, \{(5,4)\}), (e_2, \{(3,6)\})\}$	$\{(e_1, \{(5,4), (5,6)\}), (e_2, \{(3,6), (2,6)\})\}$	0.5
$J = \{(e_1, \{(3,4), (5,6)\}), (e_2, \{(2,6)\})\}$	$\{(e_1, \{(3,4)\})\}$	J	0.3
$K = m_E$	m_k	m_E	0.7

Table 3

Accuracy of BR-soft subsets

BR-soft subsets	Proposed method		Accuracy
	$\underline{BR}_S$	$\overline{BR}_S$	
$A = \{(e_1, \{(5,6)\})\}$	A	A	1
$B = \{(e_2, \{(2,6)\})\}$	B	B	1
$C = \{(e_1, \{(2,4)\})\}$	C	C	1
$D = \{(e_1, \{(3,4)\}), (e_2, \{(2,4)\})\}$	D	D	1
$E = \{(e_1, \{(3,4)\}), (e_2, \{(2,6)\})\}$	E	E	1
$F = \{(e_1, \{(3,4), (5,4)\}), (e_2, \{(2,4)\})\}$	F	F	1
$G = \{(e_1, \{(3,4), (5,4)\}), (e_2, \{(2,4), (2,6)\})\}$	G	G	1
$H = \{(e_1, \{(3,4)\}), (e_2, \{(2,4), (3,6)\})\}$	H	H	1
$I = \{(e_1, \{(5,4), (5,6)\}), (e_2, \{(3,6)\})\}$	I	I	1
$J = \{(e_1, \{(3,4), (5,6)\}), (e_2, \{(2,6)\})\}$	J	J	1
$K = m_E$	m_E	m_E	1

From the above tables (Table 2 and 3), it is obvious that the accuracy of the proposed method is higher than that of Yao's model.

Conclusion

In the current research, new classes of BR-soft open sets are introduced in BR-topological rough approximation space and their properties are studied. The accuracy measure of BR-soft subsets in BR-soft topology obtained from the collection of BR-soft simply* alpha open sets is evaluated. It is shown that the accuracy of the proposed method is high in comparison with the methods proposed by Feng and Yao. From Example 4.2, it is observed that, by using the proposed method, properties of the missing elements can also be studied. This gives a new view on solving decision making problems for a reliable solution.

Further to this work, efforts are being taken to study other topological properties like continuity, compactness, filters etc. Statistical properties of the defined set are being studied and attempts are made to develop new methods for attribute reduction. In addition, the proposed method can be extended to other areas like fuzzy, intuitionistic fuzzy, hesitant fuzzy etc., and their properties can be studied in advanced topological areas. The proposed method can also be implemented for problems with missing information.

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Аппроксимациясы шамамен алынған топологиялық кеңістігінде бинарлы жұмсақ жиындардың жаңа кластарын зерттеу

Жұмсақ бинарлы қатынас жұмсақ жиындардың жаңа кластарын анықтау үшін қолданылады, атап айтқанда, екі түрлі универсумның аппроксимациясы шамамен алынған топологиялық кеңістігінде BR-жұмсақ қарапайым ашық жиынды және BR-жұмсақ қарапайым альфа ашық жиынын анықтау үшін қолданылады. Анықталған жиын BR бағдарламалық құрал жинағының жетіспейтін элементтері туралы ақпаратты береді және шешім қабылдауды жеңілдетуге көмектеседі. Аппроксимация операторлары анықталып, ұсынылған жиындардың сипаттамалары мысалдар арқылы зерттелген. Анықталатын жиындар мен басқа жұмсақ жиындар арасында байланыс табылды. Ұсынылған әдісті басқа әдістермен салыстыру үшін дәлдік сынағы жүргізілді. Ұсынылған әдістің дәлірек екені анықталды.

Кілт сөздер: жұмсақ жиын, шамамен алынған жиын, қарапайым ашық, жуықтау кеңістігі, топологиялық кеңістік.

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Исследование новых классов бинарных мягких множеств в топологическом пространстве грубой аппроксимации

Мягкое бинарное отношение используется для определения новых классов мягких множеств, а именно BR-мягкого просто открытого множества и BR-мягкого просто* альфа открытого множества, в топологическом пространстве грубой аппроксимации двух разных универсумов. Определенный набор предоставляет информацию о недостающих элементах программного набора BR и может помочь упростить принятие решений. Определены операторы аппроксимации и на примерах изучены характеристики предложенных множеств. Выявлена связь между определимыми множествами и другими мягкими множествами. Проведена проверка точности для сравнения предложенного метода с другими методами. Установлено, что предложенный метод является более точным.

Ключевые слова: мягкое множество, грубое множество, просто открытое, аппроксимационное пространство, топологическое пространство.

DOI 10.31489/2023M4/95-104

UDC 517.588

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Implementation of summation theorems of Andrews and Gessel-Stanton

Generalized hypergeometric functions and their natural generalizations in one and several variables appear in many mathematical problems and their applications. Solving partial differential equations encountered in many applied problems of mathematics physics is expressed in terms of such generalized hypergeometric functions. In particular, the Srivastava-Daoust double hypergeometric function (S-D function) has proved its practical utility in representing solutions to a wide range of problems in pure and applied mathematics. In this paper, we introduce two general double-series identities involving bounded sequences of arbitrary complex numbers employing the finite summation theorems of Gessel-Stanton and Andrews for terminating ${}_3F_2$ hypergeometric series with arguments 3/4 and 4/3, respectively. Using these double-series identities, we establish two reduction formulas for the (S-D function) with arguments $z, 3z/4$ and $z, -4z/3$ expressed in terms of two generalized hypergeometric function of arguments proportional to z^3 and $-z^3$ respectively. All the results mentioned in the paper are verified numerically using Mathematica Program.

Keywords: Generalized hypergeometric function; Srivastava-Daoust double hypergeometric function; Reduction formulas; Mathematica Program.

1 Introduction and preliminaries

The ${}_pF_q$ $(p,\, q \in \mathbb{N}_0)$ is the generalized hypergeometric series defined by (see, e.g., [1; Section 1.5]):

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1,\,\ldots,\,\alpha_p\,;\\ \beta_1,\,\ldots,\,\beta_q\,;\end{matrix}\; z\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1,\,\ldots,\,\alpha_p;\,\beta_1,\,\ldots,\,\beta_q;\,z), \end{aligned} \tag{1}$$

being a natural generalization of the Gaussian hypergeometric series ${}_2F_1$, where $(\lambda)_\nu$ denotes the Pochhammer symbol (for $\lambda,\,\nu \in \mathbb{C}$) defined by

$$\begin{aligned} (\lambda)_\nu &:= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \qquad\qquad (\lambda,\, \nu+\lambda \in \mathbb{C}\backslash\mathbb{Z}_0^-) \\ &= \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\backslash\mathbb{Z}_0^-), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C})\,. \end{cases} \end{aligned}$$

Here Γ is the familiar Gamma function (see, e.g., [1; Section 1.1]) and it is assumed that $(0)_0 := 1$, an empty product as 1, and that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$ and the denominator parameters $\beta_1,\,\ldots,\,\beta_q$ take on complex values, provided that no zero appear in the denominator of (1), that is, that

$$(\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ j = 1,\,\ldots,q).$$

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Here and elsewhere, let $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ be respectively the sets of integers, real numbers, and complex numbers, and let

$$\mathbb{N} := \{1, 2, 3 \ldots\};\ \mathbb{N}_0 := \mathbb{N} \cup \{0\}\,;\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} = \{0, -1, -2, -3, \cdots\}\,.$$

For more details of ${}_pF_q$ including its convergence, its various special and limiting cases, and its further diverse generalizations, one may referred, for example [2, 3].

Whenever the generalized hypergeometric function ${}_pF_q$, including ${}_2F_1$, can be expressed in terms of Gamma functions through summation of its specified argument, which may include unit or $\frac{1}{2}$ argument, the outcome holds significant value from both theoretical and practical perspectives.

The generalized hypergeometric series has classical summation theorems, including those of Gauss, Gauss second, Kummer, and Bailey for the ${}_2F_1$ series, as well as Watson's, Dixon's, Whipple's, and Saalschütz's summation theorems for the ${}_3F_2$ series and others. These theorems have significant importance in both theory and application.

From 1992 to 1996, Lavoie et al. [4–6] published a series of works that generalized the aforementioned classical summation theorems for the ${}_3F_2$ series of Watson, Dixon, and Whipple. They also presented many special and limiting cases of their results, which have been further extended and generalized by Rakha-Rathie [7], Kim et al. [8], and more recently by Qureshi et al. [9]. These results have also been verified, using computer programs such as Mathematica.

Srivastava and Daoust [10; 199] introduced a generalization of the Kampé de Fériet function [11; 150] by means of the double hypergeometric series (see also [12, 13]):

$$F^{A:\ B;\ B'}_{C:\ D;\ D'}\left(\begin{array}{l}[(a_A):\vartheta,\ \varphi]:\ \ [(b_B):\psi];\ \ [(b'_{B'}):\psi'];\\ \ [(c_C):\delta,\ \varepsilon]:\ \ [(d_D):\eta];\ \ [(d'_{D'}):\eta'];\end{array} x,\ y\right)$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod\limits_{j=1}^{A}(a_j)_{m\vartheta_j+n\varphi_j}\prod\limits_{j=1}^{B}(b_j)_{m\psi_j}\prod\limits_{j=1}^{B'}(b'_j)_{n\psi'_j}}{\prod\limits_{j=1}^{C}(c_j)_{m\delta_j+n\varepsilon_j}\prod\limits_{j=1}^{D}(d_j)_{m\eta_j}\prod\limits_{j=1}^{D'}(d'_j)_{n\eta'_j}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{2}$$

where the coefficients

$$\vartheta_1, ..., \vartheta_A;\ \varphi_1, ..., \varphi_A;\ \psi_1, ..., \psi_B;\ \psi'_1, ..., \psi'_{B'};\ \delta_1, ..., \delta_C;$$
$$\varepsilon_1, ..., \varepsilon_C;\ \eta_1, ..., \eta_D;\ \eta'_1, ..., \eta'_{D'}$$

are real and positive. Let

$$\Delta_1 := 1 + \left(\sum_{j=1}^{C}\delta_j + \sum_{j=1}^{D}\eta_j\right) - \left(\sum_{j=1}^{A}\vartheta_j + \sum_{j=1}^{B}\psi_j\right)$$

and

$$\Delta_2 := 1 + \left(\sum_{j=1}^{C}\varepsilon_j + \sum_{j=1}^{D'}\eta'_j\right) - \left(\sum_{j=1}^{A}\varphi_j + \sum_{j=1}^{B'}\psi'_j\right).$$

Then

(i) The double power series in (2) converges for all complex values of x and y when $\Delta_1 > 0$ and $\Delta_2 > 0$.

(ii) The double power series in (2) is convergent for suitably constrained values of $|x|$ and $|y|$ when $\Delta_1 = 0$ and $\Delta_2 = 0$.

(iii) The double power series in (2) would diverge except when, trivially, $x = y = 0$ when $\Delta_1 < 0$ and $\Delta_2 < 0$.

Qureshi et al. [14] provided insightful remarks on previous studies, specifically [15–17]. They employed a double-series manipulation technique, utilizing Whipple's transformation (see [18; 266, Eq.(6.6)]):

$$
{}_5F_4\left[\begin{matrix} -\frac{m}{2},\ \frac{-m+1}{2},\ E,\ 1-m-B-C,\ 1-m-D\,; \\ 1-m-B,\ 1-m-C,\ \frac{1+E-D-m}{2},\ \frac{2+E-D-m}{2}\,; \end{matrix}\ 1\right]
$$
$$
= \frac{(D)_m}{(D-E)_m}\,{}_4F_3\left[\begin{matrix} -m,\ B,\ C,\ E\,; \\ 1-m-B,\ 1-m-C,\ D\,; \end{matrix}\ 1\right]
$$
$$
\left(m \in \mathbb{N}_0;\ B,\ C\, D,\ \tfrac{1+E-D-m}{2},\ \tfrac{2+E-D-m}{2} \in \mathbb{C}\setminus\mathbb{Z}_0^-\right),
$$

and (see [19; p. 537, Eq.(10.11)]; see also [17; Eq. (2.5)])

$$
{}_4F_3\left[\begin{matrix} -m,\ X,\ Y,\ Z\,; \\ U,\ W,\ X+Y+Z+1-U-W-m\,; \end{matrix}\ 1\right]
$$
$$
= \frac{(U-X)_m(Y+Z+1-U-W-m)_m}{(U)_m(X+Y+Z+1-U-W-m)_m}
$$
$$
\times\,{}_4F_3\left[\begin{matrix} -m,\ W-Y,\ W-Z,\ X\,; \\ 1-m+X-U,\ U+W-Y-Z,\ W\,; \end{matrix}\ 1\right]
$$
$$
\left(\begin{matrix} m \in \mathbb{N}_0;\ U,\ W,\ X+Y+Z+1-U-W-m, \\ 1-m+X-U,\ U+W-Y-Z,\ W \in \mathbb{C}\setminus\mathbb{Z}_0^- \end{matrix}\right).
$$

Through this approach, they introduced three double-series identities, which incorporated a bounded sequence of complex numbers. In addition, they [14] demonstrated that the application of double-series identities enables the provision of numerous reduction formulas, whether they are already known or newly discovered. Subsequently and concurrently, a number of papers have utilized series manipulation techniques along with, among several others, transformation formulas for ${}_2F_1$ in Chan et al. [20], the reduction and transformation formulas of Kampé de Fériet and Srivastava-Daoust functions [21], implications of Bailey transformations in double-series and their consequences [22], the reduction formula for ${}_2F_1$ in Karlsson [23], terminating ${}_3F_2\left(\frac{4}{3}\right)$ [24; Eq.(1.3)] (see also Gessel-Stanton summation theorem [25; Eq.(5.21)] and terminating ${}_3F_2\left(\frac{3}{4}\right)$ [24; Eq.(1.4)] (see also [26; Eq.(1.12)]) in Qureshi et al. [24]. These papers have presented multiple or double series identities, which have been employed to derive a range of reduction formulas for the Kampé de Fériet, Srivastava-Daoust function and other intriguing identities for the ${}_pF_q$ functions.

Inspired by the aforementioned papers, especially [14, 21], and utilizing the reversing order of the finite summation theorem of Gessel-Stanton [25; 305, Eq.(5.21)])

$$
{}_3F_2\left[\begin{matrix} -n, -2b-\frac{2n}{3}, -6b-n; \\ -3b-n,\ \frac{1}{2}-3b-n; \end{matrix}\ \frac{3}{4}\right] = \begin{cases} 0 & ; n = 3m+1 \text{ and } 3m+2, \\ \frac{\left(\frac{1}{3}\right)_m\left(\frac{2}{3}\right)_m(6b+1)_{3m}(2b+1)_{2m}(3)^{3m}}{(1+2b)_m(3b+1)_{3m}(3b+\frac{1}{2})_{3m}(4)^{3m}} & ; n = 3m, \end{cases} \tag{3}
$$

where $m = 0, 1, 2, 3, ...$
(also, reversing order of the terms in finite summation theorem of George Andrews [26; 4, Eq.(1.12); see also p.16, Eq.(4.8)])

$$
{}_3F_2\left[\begin{matrix} -n,\ \frac{1-3b-2n}{2}, \frac{2-3b-2n}{2}; \\ 1-b-n, 1-3b-2n; \end{matrix}\ \frac{4}{3}\right] = \begin{cases} 0 & ; n = 3m+1 \text{ and } 3m+2, \\ \frac{(3m)!(-1)^m(b)_m}{m!(b)_{3m}(3)^{3m}} & ; n = 3m, \end{cases} \tag{4}
$$

where $m = 0, 1, 2, 3, ...$

Our objective is to introduce two double-series identities. These identities incorporating bounded sequences of complex numbers are derived using series rearrangement techniques and Pochhammer symbol identities. These issues are further discussed in Section 2. In Section 3, we employ these general double-series identities to establish two reduction formulas for Srivastava-Daoust double hypergeometric function in terms of generalized hypergeometric functions with arrangements proportional to z^3 and $-z^3$. We achieve this by using Cauchy's double series identity (see, e.g., [27; 56])

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Theta(n,r)=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\Theta(n-r,r), \tag{5}$$

provided that the associated double series are absolutely convergent. We also have the following identities involving the Pochhammer symbol:

$$(-n)_r=\frac{n!(-1)^r}{(n-r)!}; \qquad 0\le r\le n, \tag{6}$$

$$\prod_{i=1}^{D}(d_i)_{3n}=3^{3nD}\prod_{i=1}^{D}\left(\frac{d_i}{3}\right)_n\prod_{i=1}^{D}\left(\frac{1+d_i}{3}\right)_n\prod_{i=1}^{D}\left(\frac{2+d_i}{3}\right)_n, \tag{7}$$

$$\prod_{j=1}^{E}(e_j)_{3n}=3^{3nE}\prod_{j=1}^{E}\left(\frac{e_j}{3}\right)_n\prod_{j=1}^{E}\left(\frac{1+e_j}{3}\right)_n\prod_{j=1}^{E}\left(\frac{2+e_j}{3}\right)_n. \tag{8}$$

Remark 1.1 Wolfram's MATHEMATICA has implemented the ${}_pF_q$ function as Hypergeometric PFQ, which is appropriate for performing both symbolic and numerical computations.

Throughout this article, we assume that any values of parameters and arguments, which would render the results in Sections 2 to 3 invalid or undefined, are tacitly excluded.

2 Two general double-series identities

This section demonstrates two double-series identities that involve bounded sequences by primarily utilizing Gessel-Stanton and George Andrews (3) and (4). The first identity takes the following form:

Theorem 1. Let $\{\Psi(\mu)\}_{\mu=1}^{\infty}$ be a bounded sequence of essentially arbitrary complex numbers or real numbers such that $\Psi(0)\neq 0$. Then, the following general double-series identity holds true:

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(n+r)\frac{(-2b)_{-\frac{2n}{3}+\frac{r}{3}}(1+6b)_{n+r}\left(\frac{1}{2}+3b\right)_n(1+3b)_n(3)^r z^{n+r}}{(-2b)_{-\frac{2n}{3}-\frac{2r}{3}}\left(\frac{1}{2}+3b\right)_{n+r}(1+3b)_{n+r}(1+6b)_n(4)^r\ r!n!}$$

$$=\sum_{n=0}^{\infty}\Psi(3n)\frac{\left(\frac{6b+1}{3}\right)_n\left(\frac{6b+2}{3}\right)_n z^{3n}}{\left(\frac{3b+1}{3}\right)_n\left(\frac{3b+2}{3}\right)_n\left(\frac{6b+1}{6}\right)_n\left(\frac{6b+5}{6}\right)_n(432)^n\ n!} \tag{9}$$

provided $\left(-2b,\frac{1}{2}+3b,1+3b,1+6b,\frac{3b+1}{3},\frac{3b+2}{3},\frac{6b+1}{6},\frac{6b+5}{6}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$, and the infinite series occurring on both sides of equation (9), are absolutely convergent.

Proof.

$$\text{Let}\qquad \Xi_1(z)=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(n+r)\frac{(-2b)_{-\frac{2n}{3}+\frac{r}{3}}(1+6b)_{n+r}\left(\frac{1}{2}+3b\right)_n(1+3b)_n(3)^r z^{n+r}}{(-2b)_{-\frac{2n}{3}-\frac{2r}{3}}\left(\frac{1}{2}+3b\right)_{n+r}(1+3b)_{n+r}(1+6b)_n(4)^r\ r!n!}. \tag{10}$$

Replacing n by $(n-r)$ in equation (10) and using Cauchy's double-series identity (5), we have

$$\Xi_1(z)=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\Psi(n)\frac{\left(-2b-\frac{2n}{3}\right)_r(-6b-n)_r(-3)^r z^n}{(-3b-n)_r\left(\frac{1}{2}-3b-n\right)_r(4)^r(n-r)!r!}. \tag{11}$$

Multiplying numerator and denominator by $n!$ and using Pochhammer symbol identity (6) to the right hand side of equation (11), we obtain

$$\Xi_1(z)=\sum_{n=0}^{\infty}\Psi(n)\frac{z^n}{n!}\sum_{r=0}^{n}\frac{(-n)_r\left(-2b-\frac{2n}{3}\right)_r(-6b-n)_r(3)^r}{(-3b-n)_r\left(\frac{1}{2}-3b-n\right)_r(4)^r\ r!}$$

$$=\sum_{n=0}^{\infty}\Psi(n)\frac{z^n}{n!}\ {}_3F_2\left[\begin{array}{c}-n,-2b-\frac{2n}{3},-6b-n;\\ \frac{1}{2}-3b-n,-3b-n;\end{array}\ \frac{3}{4}\right]. \tag{12}$$

We now apply the decomposition identity

$$\sum_{n=0}^{\infty}\Phi(n)=\sum_{n=0}^{\infty}\Phi(3n)+\sum_{n=0}^{\infty}\Phi(3n+1)+\sum_{n=0}^{\infty}\Phi(3n+2),$$

provided that each of the sums is absolutely convergent, to the right-hand side of (12). This produces

$$\Xi_1(z)=\sum_{n=0}^{\infty}\Psi(3n)\frac{z^{3n}}{(3n)!}\ {}_3F_2\left[\begin{array}{c}-(3n),-2b-2n,-6b-3n;\\ -3b-3n,\frac{1}{2}-3b-3n;\end{array}\ \frac{3}{4}\right]+\sum_{n=0}^{\infty}\Psi(3n+1)\frac{z^{3n+1}}{(3n+1)!}\times$$

$$\times{}_3F_2\left[\begin{array}{c}-(3n+1),-2b-2n-\frac{2}{3},-6b-3n-1;\\ -3b-3n-1,-3b-3n-\frac{1}{2};\end{array}\ \frac{3}{4}\right]+\sum_{n=0}^{\infty}\Psi(3n+2)\frac{z^{3n+2}}{(3n+2)!}\times$$

$$\times{}_3F_2\left[\begin{array}{c}-(3n+2),-6b-2n-\frac{4}{3},-6b-3n-2;\\ -3b-3n-2,-3b-3n-\frac{3}{2};\end{array}\ \frac{3}{4}\right]. \tag{13}$$

Finally, using the summation theorem (3) to the right hand side of equation (13), we get

$$\Xi_1(z)=\sum_{n=0}^{\infty}\Psi(3n)\frac{z^{3n}}{(3n)!}\left[\frac{\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n(6b+1)_{3n}(2b+1)_{2n}(3)^{3n}}{(2b+1)_n(3b+1)_{3n}\left(\frac{6b+1}{2}\right)_{3n}(4)^{3n}}\right].$$

After further simplification, we get the required result (9).

The second identity is given by the following theorem:

Theorem 2. Let $\{\Psi(\mu)\}_{\mu=1}^{\infty}$ be a bounded sequence of essentially arbitrary complex numbers or real numbers such that $\Psi(0)\neq 0$. Then, the following general double-series identity holds true:

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(n+r)\frac{(3b)_{2n+r}\left(\frac{1+3b}{2}\right)_{n+r}\left(\frac{3b}{2}\right)_{n+r}(b)_n(-4)^r z^{n+r}}{(3b)_{2n+2r}(b)_{n+r}\left(\frac{1+3b}{2}\right)_n\left(\frac{3b}{2}\right)_n(3)^r\ r!\ n!}$$

$$=\sum_{n=0}^{\infty}\Psi(3n)\frac{(b)_n(-z^3)^n}{\left(\frac{b}{3}\right)_n\left(\frac{b+1}{3}\right)_n\left(\frac{b+2}{3}\right)_n(729)^n\ n!} \tag{14}$$

provided $\left(3b,\frac{1+3b}{2},b,\frac{3b}{2},\frac{b}{3},\frac{b+1}{3},\frac{b+2}{3}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$, and the infinite series occurring on both sides of equation (14) are absolutely convergent.

Proof.

Let $$\Xi_2(z)=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(n+r)\frac{(3b)_{2n+r}\left(\frac{1+3b}{2}\right)_{n+r}\left(\frac{3b}{2}\right)_{n+r}(b)_n(-4)^r z^{n+r}}{(3b)_{2n+2r}(b)_{n+r}\left(\frac{1+3b}{2}\right)_n\left(\frac{3b}{2}\right)_n(3)^r\ r!\ n!}. \tag{15}$$

Replacing n by $(n-r)$ in equation (15) and using Cauchy's double-series identity (5), we have

$$\Xi_2(z)=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\Psi(n)\frac{\left(\frac{1-3b-2n}{2}\right)_r\left(\frac{2-3b-2n}{2}\right)_r(-4)^r z^n}{(1-b-n)_r(1-3b-2n)_r(3)^r\ r!\ (n-r)!}. \tag{16}$$

Multiplying numerator and denominator by $n!$ and using Pochhammer symbol identity (6) to right hand side of equation (16), we obtain

$$\Xi_2(z)=\sum_{n=0}^{\infty}\Psi(n)\frac{z^n}{n!}\sum_{r=0}^{n}\frac{(-n)_r\left(\frac{1-3b-2n}{2}\right)_r\left(\frac{2-3b-2n}{2}\right)_r(4)^r}{(1-b-n)_r(1-3b-2n)_r(3)^r\ r!}$$

$$=\sum_{n=0}^{\infty}\Psi(n)\frac{z^n}{n!}\ {}_3F_2\left[\begin{array}{c}-n,\frac{1-3b-2n}{2},\frac{2-3b-2n}{2};\\ 1-b-n,1-3b-2n;\end{array}\ \frac{4}{3}\right] \tag{17}$$

We now apply the decomposition identity

$$\sum_{n=0}^{\infty}\Phi(n)=\sum_{n=0}^{\infty}\Phi(3n)+\sum_{n=0}^{\infty}\Phi(3n+1)+\sum_{n=0}^{\infty}\Phi(3n+2),$$

provided that each of the sums is absolutely convergent, to the right-hand side of (17). This produces

$$\Xi_2(z)=\sum_{n=0}^{\infty}\Psi(3n)\frac{z^{3n}}{(3n)!}\ {}_3F_2\left[\begin{array}{c}-3n,\frac{1-3b-6n}{2},\frac{2-3b-6n}{2};\\ 1-b-3n,1-3b-6n;\end{array}\ \frac{4}{3}\right]+\sum_{n=0}^{\infty}\Psi(3n+1)\frac{z^{3n+1}}{(3n+1)!}\times$$

$$\times{}_3F_2\left[\begin{array}{c}-(3n+1),\frac{-3b-6n-1}{2},\frac{-3b-6n}{2};\\ -b-3n,-3b-6n-1;\end{array}\ \frac{4}{3}\right]+\sum_{n=0}^{\infty}\Psi(3n+2)\frac{z^{3n+2}}{(3n+2)!}\times$$

$$\times{}_3F_2\left[\begin{array}{c}-(3n+2),\frac{-3b-6n-3}{2},\frac{-3b-6n-2}{2};\\ -b-3n-1,-3b-6n-3;\end{array}\ \frac{4}{3}\right]. \tag{18}$$

Finally, using the summation theorem (4) to the right hand side of equation (18), we get

$$\Xi_2(z)=\sum_{n=0}^{\infty}\Psi(3n)\frac{z^{3n}}{(3n)!}\left[\frac{(3n)!(-1)^n(b)_n}{n!(b)_{3n}(3)^{3n}}\right]$$

$$=\sum_{n=0}^{\infty}\Psi(3n)\frac{(b)_n(-z^3)^n}{(3)^{3n}(b)_{3n}n!}.$$

After simplification, we get the result (14).

3 Certain consequences of general double-series identities (9) and (14)

In this section, we establish a result for reducibility of Srivastava-Daoust double hypergeometric function as in the following theorem.

Theorem 3. The following results hold true:

$$F^{D+2:2;0}_{E+3:1;0}\left(\begin{array}{c} [(d_D):1,1],[-2b:-\frac{2}{3},\frac{1}{3}],[1+6b:1,1]:[\frac{1}{2}+3b:1],[1+3b:1]\,;-\,; \\ [(e_E):1,1],[-2b:-\frac{2}{3},-\frac{2}{3}],[\frac{1}{2}+3b:1,1],[1+3b:1,1]:[1+6b:1]\,;-\,; \end{array} z,\frac{3z}{4}\right)$$

$$= {}_{2+3D}F_{4+3E}\left[\begin{array}{c} \Delta[3;(d_D)],\frac{6b+1}{3},\frac{6b+2}{3}; \\ \Delta[3;(e_E)],\frac{3b+1}{3},\frac{3b+2}{3},\frac{6b+1}{6},\frac{6b+5}{6}; \end{array} \frac{z^3}{16\times(27)^{(1+E-D)}}\right], \tag{19}$$

and

$$F^{D+3:1;0}_{E+2:2;0}\left(\begin{array}{c} [(d_D):1,1],[3b:2,1],[\frac{1+3b}{2}:1,1],[\frac{3b}{2}:1,1]:\quad [b:1]\quad ;-\,; \\ [(e_E):1,1],[3b:2,2],[b:1,1]:[\frac{3b}{2}:1],[\frac{1+3b}{2}:1];-\,; \end{array} z,-\frac{4z}{3}\right)$$

$$= {}_{1+3D}F_{3+3E}\left[\begin{array}{c} \Delta[(3;(d_D)],b; \\ \Delta[3;(e_E)],\frac{b}{3},\frac{b+1}{3},\frac{b+2}{3}; \end{array} \frac{-z^3}{(27)^{(2+E-D)}}\right], \tag{20}$$

where $(e_1,e_2,...,e_E,b,-2b,3b,1+6b,\frac{3b}{2},\frac{1+3b}{2},\frac{1+6b}{2},\frac{b}{3},\frac{b+1}{3},\frac{b+2}{3},\frac{3b+1}{3},\frac{3b+2}{3},\frac{6b+1}{6},\frac{6b+5}{6}\in\mathbb{C}\backslash\mathbb{Z}_0^-)$. When $D\leqslant E$ then above transformations are always convergent for $|z|<\infty$. When $D=1+E$ then above transformations are convergent for suitably constrained values of $|z|$.

Proof.

$$\text{Put}\qquad \Psi(\mu)=\frac{(d_1)_\mu(d_2)_\mu...(d_D)_\mu}{(e_1)_\mu(e_2)_\mu...(e_E)_\mu}=\frac{\prod_{i=1}^{D}(d_i)_\mu}{\prod_{i=1}^{E}(e_i)_\mu};\qquad \mu=0,1,2,3,...,$$

on the both sides of general double-series identity (9), we obtain

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{D}(d_i)_{n+r}(-2b)_{-\frac{2n}{3}+\frac{r}{3}}(1+6b)_{n+r}\left(\frac{1}{2}+3b\right)_n(1+3b)_n(3)^r z^{n+r}}{\prod_{i=1}^{E}(e_i)_{n+r}(-2b)_{-\frac{2n}{3}-\frac{2r}{3}}\left(\frac{1}{2}+3b\right)_{n+r}(1+3b)_{n+r}(1+6b)_n\;(4)^r\;r!n!}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{D}(d_i)_{3n}\left(\frac{6b+1}{3}\right)_n\left(\frac{6b+2}{3}\right)_n z^{3n}}{\prod_{i=1}^{E}(e_i)_{3n}\left(\frac{3b+1}{3}\right)_n\left(\frac{3b+2}{3}\right)_n\left(\frac{6b+1}{6}\right)_n\left(\frac{6b+5}{6}\right)_n(432)^n n!}. \tag{21}$$

Now applying the definition of double hypergeometric function (2) of Srivastava-Daoust to the left hand side of equation (21) and definition of the generalized hypergeometric function (1), together with the Pochhammer symbol identities (7) and (8) to the right hand side of equation (21), we get the desired result (19).

The proof of (20) follows exactly the same procedure and will be omitted. This completes the proof of Theorem 3.

4 Conclusions and Remarks

In our present investigation, we have obtained two general double-series identities by using the finite summation theorems of Gessel-Stanton and George Andrews for the terminating hypergeometric series ${}_3F_2$ with arguments 3/4 and 4/3 respectively. These results have been used to derive two reduction formulas for the (S-D function) with arguments $(z, 3z/4)$ and $(z, -4z/3)$ in terms of two generalized hypergeometric functions ${}_{2+3D}F_{4+3E}$ and ${}_{1+3D}F_{3+3E}$ with arguments $\frac{z^3}{16\times 27^{(1+E-D)}}$ and $\frac{-z^3}{(27)^{(2+E-D)}}$ respectively. We believe that the results established in this paper have not appeared in the literature and represent a contribution to the theory of generalized hypergeometric functions of one and two variables. The various results, which we have presented in this article, are potentially useful in mathematical analysis and applied mathematics.

Acknowledgments

The authors wish to extend their heartfelt appreciation to the anonymous reviewers for their invaluable feedback. Their constructive and encouraging comments have greatly contributed to enhancing the quality of this paper.

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Эндрюс пен Гессель-Стэнтонның қосындылау теоремаларын жүзеге асыру

Жалпыланған гипергеометриялық функциялар мен олардың бір және бірнеше айнымалылардағы заңды жалпылануы көптеген математикалық есептер мен олардың қосымшаларында кездеседі. Математикалық физиканың көптеген қолданбалы есептері бар дербес туындылы дифференциалдық теңдеулердің шешімі осындай жалпыланған гипергеометриялық функциялар арқылы өрнектеледі. Атап айтқанда, Шривастава-Дausттың қос гипергеометриялық функциясы (S-D функциясы) іргелі және қолданбалы математикадағы кең ауқымды есептердің шешімдерін ұсыну үшін өзінің практикалық пайдалылығын дәлелдеді. Мақалада сәйкесінше $3/4$ және $4/3$ аргументтері бар ${}_3F_2$ аяқталатын гипергеометриялық қатарға арналған Гессель-Стэнтон және Эндрюстің ақырлы қосындылар теоремаларын пайдалана отырып, еркін кешенді сандардың шектелген тізбегінен тұратын қос қатарлар үшін екі жалпы сәйкестендіру енгізілген. Осы қос қатар сәйкестіктерін пайдалана отырып, $z, 3z/4$ және $z, -4z/3$ аргументтері бар (S-D функциясы) екі келтіру формуласы дәлелденген, олар z^3 және $-z^3$ аргументіне пропорционал екі жалпыланған гипергеометриялық функциялар арқылы өрнектеледі. Сонымен қатар мақалада айтылған барлық нәтижелер «Mathematica» бағдарламасы арқылы сандық түрде тексерілді.

Кілт сөздер: жалпыланған гипергеометриялық функция, Шривастава-Дауcттың қос гипергеометриялық функциясы, келтіру формулалары, «Mathematica» бағдарламасы.

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Реализация теорем суммирования Эндрюса и Гесселя–Стэнтона

Обобщенные гипергеометрические функции и их естественные обобщения от одной и нескольких переменных встречаются во многих математических задачах и их приложениях. Решение уравнений в частных производных, возникающих во многих прикладных задачах математической физики, выражается через такие обобщенные гипергеометрические функции. В частности, двойная гипергеометрическая функция Шриваставы–Дауста (S–D-функция) доказала свою практическую полезность для представления решений широкого круга задач фундаментальной и прикладной математики. В настоящей статье мы вводим два общих тождества двойных рядов, включающие ограниченные последовательности произвольных комплексных чисел, используя теоремы конечного суммирования Гесселя–Стэнтона и Эндрюса для завершающих гипергеометрических рядов ${}_3F_2$ с аргументами $3/4$ и $4/3$ соответственно. Используя данные тождества двойного ряда, устанавливаем две формулы приведения для (S–D-функции) с аргументами $z, 3z/4$ и $z, -4z/3$, выраженными через две обобщенные гипергеометрические функции с аргументами, пропорциональными z^3 и $-z^3$ соответственно. Все результаты, упомянутые в статье, проверены численно с использованием программы «Mathematica».

Ключевые слова: обобщенная гипергеометрическая функция, двойная гипергеометрическая функция Шриваставы–Дауста, формулы приведения, программа «Mathematica».

Bulletin of the Karaganda University. Mathematics Series, No. 4(112), 2023, pp. 105–118

DOI 10.31489/2023M4/105-118

UDC 512

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Roughness in Fuzzy Cayley Graphs

Rough set theory is a worth noticing approach for inexact and uncertain system modelling. When rough set theory accompanies with fuzzy set theory, which both are a complementary generalization of set theory, they will be attended by potency in theoretical discussions. In this paper a definition for fuzzy Cayley subsets is put forward as well as fuzzy Cayley graphs of fuzzy subsets on groups inspired from the definition of Cayley graphs. We introduce rough approximation of a Cayley graph with respect to a fuzzy normal subgroup. We introduce the approximation rough fuzzy Cayley graphs and fuzzy rough fuzzy Cayley graphs. The last approximation is the mixture of the other approximations. Some theorems and properties are investigated and proved.

Keywords: fuzzy subset, rough set, Cayley graph, fuzzy Cayley graph, lower and upper approximations.

1 *Introduction and preliminaries*

Rough sets have been investigated in many papers. For details we refer to [1–7]. In particular, in [8], rough approximations of Cayley graphs are studied. It has intended to build up a rational connection between rough set theory [7], fuzzy set theory [9] and Cayley graphs. Cayley fuzzy graphs are studied in [10–12]. We present a new definition of fuzzy Cayley sets and so, fuzzy Cayley graphs of generators of the Cayley graph of a group. For a finite group G and a fuzzy subset μ on G, the fuzzy subset μ is called fuzzy Cayley subset, if the subset

$$S_\mu = \{a \in G \ \mid \ \mu(a) < 1\}$$

is a Cayley subset of G. It means that $1_G \notin S_\mu$ (where 1_G represents the identity element of G) and if $s \in S_\mu$, then $s^{-1} \in S_\mu$. We define the triple $(G; S_\mu; \mu)$ as a fuzzy Cayley graph. In fact, the fuzzy Cayley graph $(G; S_\mu; \mu)$ is a Cayley graph where the fuzzy Cayley subset μ constructs the Cayley subset of it.

The outline on the paper is as follows. First, we recall some notation and definitions about the simple graph. We also recall the definitions and concepts of the fuzzy subset, fuzzy subgroup, t-level relation and lower approximation operator and upper approximation operator for a fuzzy approximation space that we need for the paper in this section. In Section 2, we present the definitions of fuzzy Cayley subset and fuzzy Cayley graph for fuzzy subsets of groups and some few results for them. In Sections 3 and 4, we deal the concept of fuzzy lower and upper approximations of a Cayley graph and lower and upper approximations of a fuzzy Cayley graph with respect to a fuzzy normal subgroup. Finally, in Section 5, we combine the concept of the lower and upper approximations of a Cayley graph and lower and upper approximations of a fuzzy Cayley graph and present the fuzzy lower and upper approximations of a fuzzy Cayley graph with respect to a fuzzy normal subgroup on a finite group.

For the benefit of the reader, we collect in this section some of the basic concepts and facts that we need in this paper.

Let us introduce some basic notation and definitions about the simple graph. We consider simple graphs, which are undirected, with no loops or multiple edges.

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Now, we recall the definition's fuzzy subset, fuzzy subgroup, fuzzy normal subgroup and some proportion of them [9, 13]. Suppose that X is a universe set. A *fuzzy subset* μ on X is a function $\mu : X \to [0,1]$ mapping all elements x of X into a real number $\mu(x)$ in the closed interval $[0,1]$. Taking fuzzy subsets μ and λ on X. $\mu \subseteq \lambda$ if and only if all $x \in X$ satisfying $\mu(x) \leq \lambda(x)$. Fuzzy subset γ is called the *union* of fuzzy subsets μ and λ, if and only if $\gamma(x) = \max\{\mu(x), \lambda(x)\}$ for all $x \in X$, and γ is denoted by $\mu \cup \lambda$. Fuzzy subset φ is called the *intersection* of fuzzy subsets μ and λ, if and only if $\varphi(x) = \min\{\mu(x), \lambda(x)\}$ for all $x \in X$, and φ is denoted by $\mu \cap \lambda$.

A fuzzy subsets μ on a group G is called a *fuzzy subgroup* of G [13], if the following conditions hold:

1 $\forall a, b \in G,\ \mu(ab) \geq \min\{\mu(a), \mu(b)\}$;

2 $\forall a \in G,\ \mu(a^{-1}) \geq \mu(a)$;

3 $\mu(1_G) = 1$.

For every a in G, $\mu(a^{-1}) = \mu(a)$. This follows at once from part 2. A fuzzy subgroup μ of G, is called a *fuzzy normal subgroup* of G if for any arbitrary elements a and b of G, have to $\mu(ab) = \mu(ba)$.

We recall the t-level relation for fuzzy normal subgroups and some properties and theorems 1 and 2, that we need in the work from [4]. Let μ be a fuzzy normal subgroup of G. For each $t \in [0,1]$, the set

$$\mu_t = \{(a,b) \in G \times G \ \mid \ \mu(ab^{-1}) \geq t\}$$

is called a *t-level relation* of μ. For each t, μ_t is a congruence relation on G. We denote by $[x]_\mu$ the congruence class of μ_t containing the element x of G. Let A be a non-empty subset of G. Then the sets

$$\mu_{t-}(A) = \{x \in G \ \mid \ [x]_\mu \subseteq A\},$$
$$\mu_{t^\wedge}(A) = \{x \in G \ \mid \ [x]_\mu \cap A \neq \emptyset\}$$

are called, respectively, the *lower and upper approximations* of the set A with respect to μ_t. The pair $\mu(A) = (\mu_{t-}(A), \mu_{t^\wedge}(A))$ is called a *rough set* of A in G. A non-empty subset A of a group G is called a $\mu_{t^\wedge}$*-fuzzy rough (normal) subgroup* of G if the upper approximation of A is a (normal) subgroup of G. Similarly, a non-empty subset A of G is called a $\mu_{t-}(A)$*-fuzzy rough (normal) subgroup* of G if lower approximation is a (normal) subgroup of G. Note that, if μ and λ are fuzzy normal subgroups of a group G, then $\mu \cap \lambda$ is also a fuzzy subgroup G.

Theorem 1. Suppose that μ and λ are fuzzy normal subgroups of a group G and $t \in [0,1]$. Let A and B be any non-empty subsets of G. Then

(1) $\mu_{t-}(A) \subseteq A \subseteq \mu_{t^\wedge}(A)$,

(2) $\mu_{t-}(A \cap B) = \mu_{t-}(A) \cap \mu_{t-}(B)$,

(3) $\mu_{t^\wedge}(A \cup B) = \mu_{t^\wedge}(A) \cup \mu_{t^\wedge}(B)$,

(4) $A \subseteq B$ implies $\mu_{t-}(A) \subseteq \mu_{t-}(B)$,

(5) $A \subseteq B$ implies $\mu_{t^\wedge}(A) \subseteq \mu_{t^\wedge}(B)$,

(6) $\mu_{t-}(A \cup B) \supseteq \mu_{t-}(A) \cup \mu_{t-}(B)$,

(7) $\mu_{t^\wedge}(A \cap B) \subseteq \mu_{t^\wedge}(A) \cap \mu_{t^\wedge}(B)$,

(8) $\mu \subseteq \lambda$, implies $\lambda_{t-}(A) \subseteq \mu_{t-}(A)$,

(9) $\mu \subseteq \lambda$, implies $\mu_{t^\wedge}(A) \subseteq \lambda_{t^\wedge}(A)$,

(10) $(\mu \cap \lambda)_t = \mu_t \cap \lambda_t$,

(11) $(\mu \cap \lambda)_{t-}(A) \supseteq \mu_{t-}(A) \cap \lambda_{t-}(A)$,

(12) $(\mu \cap \lambda)_{t^\wedge}(A) \subseteq \mu_{t^\wedge}(A) \cap \lambda_{t^\wedge}(A)$.

Theorem 2. Let μ be a fuzzy normal subgroup of a group G and $t \in [0,1]$. If A is a (normal) subgroup of G, then $\mu_{t^\wedge}(A)$ is a (normal) subgroup of G. Moreover, if the lower approximation of A is non-empty, then it is a (normal) subgroup of G.

Given a continuous triangular norm T on the unit interval $I = [0, 1]$. A fuzzy binary relation R on X is called a T-similarity relation if for all $x, y, z \in X$, R satisfies the following conditions:
(1) $R(x, x) = 1$;
(2) $R(x, y) = R(y, x)$;
(3) $R(x, z)TR(x, y) \leq R(z, y)$.
The pair (X, R) is called a fuzzy approximation space (see, for example [14] and [6]). Morsi and Yakout in [6] define the lower approximation operator and upper approximation operator for a fuzzy approximation space (X, R), respectively, for $\mu \in [0, 1]^X$, as follows:

$$\underline{A}_R\mu(x) = \inf_{u\in X} \vartheta_T(R(u, x), \mu(u)) \text{ for every } x \in X,$$

$$\overline{A}_R\mu(x) = \sup_{u\in X}(R(u, x)T\mu(u)) \text{ for every } x \in X,$$

when $\vartheta_T(a, b) = \sup\{\theta \in [0, 1] \mid aT\theta \leq b\}$, for every $a, b \in [0, 1]$. Let G be a group and $C \in I_G$. If C satisfies the following conditions:
(1) $C(xy) \geq C(x)TC(y)$;
(2) $C(x^{-1}) \geq C(x)$;
(3) $C(e) = 1$,
then C is called a T-fuzzy subgroup of G. If $C(xy) = C(yx)$ for every $x, y \in G$, then C is called a T-fuzzy normal subgroup of G. It easily can be verified that the binary relation,

$$B : G \times G \to [0, 1],$$

$$B(x, y) = C(xy^{-1}), \text{ for every } x, y \in G$$

is T-similarity relation. Jiashang, Congxin and Degang in [14] define the upper approximation operator $\overline{A}_B$ and the lower approximation operator $\underline{A}_B$ with respect to B on G. In this paper, we limited the triangular norm T, the simplest triangular norm, Min. Let μ be a fuzzy subset and β be a fuzzy normal subgroup on G. We call the fuzzy subsets $\underline{A}_B\mu, \overline{A}_B\mu$ as respectively, the *lower and upper approximations of the fuzzy subset μ on G with respect to the fuzzy normal subgroup B*.

$$\underline{A}_B\mu(x) = \inf_{u\in G} \vartheta_{\min}(B(u, x), \mu(u)), \text{ for every } x \in G,$$

$$\overline{A}_B\mu(x) = \sup_{u\in G}\{\min\{B(u, x), \mu(u)\}\}, \text{ for every } x \in G.$$

The pair$(\underline{A}_B\mu, \overline{A}_B\mu)$ is called a *rough fuzzy set* of μ. The fuzzy subset μ on a group G is called a $\overline{A}_B$ *rough fuzzy (normal) subgroup*, if the upper approximation of μ is a fuzzy (normal) subgroup on G. Similarity, the fuzzy subset μ on a group G is called a $\underline{A}_B$ *rough fuzzy (normal) subgroup*, if the lower approximation of μ is a fuzzy (normal) subgroup on G.

Note that $\vartheta_{\min}(a, b) = 1$, if and only if $a \leq b$, if not it is equal to b.

The next proposition follows at once from [14; Proposition 2.4].

Theorem 3. Let G be a finite group, μ and λ be fuzzy subsets. Let B and C be fuzzy normal subgroups on G. Then
(1) $\underline{A}_B\mu \subseteq \mu \subseteq \overline{A}_B\mu$,
(2) $\overline{A}_B\overline{A}_B\mu = \underline{A}_B\overline{A}_B\mu = \overline{A}_B\mu$,
(3) $\overline{A}_B\underline{A}_B\mu = \underline{A}_B\underline{A}_B\mu = \underline{A}_B\mu$,
(4) $\underline{A}_B\mu = \mu$ if and only if $\overline{A}_B\mu = \mu$,
(5) $\overline{A}_B(\mu \cup \lambda) = \overline{A}_B\mu \cup \overline{A}_B\lambda$,
(6) $\overline{A}_B(\mu \cap \lambda) \subseteq \overline{A}_B\mu \cap \overline{A}_B\lambda$,

(7) $\underline{A}_B(\mu \cup \lambda) \supseteq \underline{A}_B\mu \cup \underline{A}_B\lambda$,
(8) $\underline{A}_B(\mu \cap \lambda) = \underline{A}_B\mu \cap \underline{A}_B\lambda$,
(9) $B \subseteq C$ then $\overline{A}_B\mu \subseteq \overline{A}_C\mu$,
(10) $B \subseteq C$ then $\underline{A}_C\mu \subseteq \underline{A}_B\mu$.

The next corollary easily can be verified based upon the parts (6) and (7) of Theorem 3.

Corollary 1. Let G be a finite group, μ and λ be fuzzy subsets. Let B be a fuzzy normal subgroup on G. If $\mu \subseteq \lambda$, then
(1) $\overline{A}_B\mu \subseteq \overline{A}_B\lambda$,
(2) $\underline{A}_B\mu \subseteq \underline{A}_B\lambda$.

The fuzzy subset $B\mathrm{Min}C$ is defined based on fuzzy subsets B and C as $B\mathrm{Min}C(x) = \min\{B(x), C(x)\}$, $\forall x \in G$. The next theorem follows from [14; Lemma 3.4, Propositions 3.5, 3.6, 4.1 and 4.2].

Theorem 4. Let G be a finite group. Suppose that B and C are fuzzy normal subgroups of G. The following properties hold.
(1) The fuzzy set $B\mathrm{Min}C$ is a fuzzy normal subgroup.
(2) $\underline{A}_B\mu\mathrm{Min}\underline{A}_C\mu \subseteq \underline{A}_{B\mathrm{Min}C}\mu$.
(3) $\overline{A}_{B\mathrm{Min}C}\mu \subseteq \overline{A}_B\mu\mathrm{Min}\overline{A}_C\mu$.
(4) If μ is a fuzzy (normal) subgroup of G, then $\overline{A}_B\mu$ is a fuzzy (normal) subgroup of G.
(5) If μ is a fuzzy (normal) subgroup of G and $B \subseteq \mu$, then $\underline{A}_B\mu$ is a fuzzy (normal) subgroup of G.

Throughout the paper, we will make frequently use of the above mentioned results.

2 *Fuzzy Cayley subsets and graphs*

In this section, we present the definitions of fuzzy Cayley subset and fuzzy Cayley graph for fuzzy subsets on groups.

Let G be a finite group and μ be a fuzzy subset on G. The fuzzy subset μ is called *fuzzy Cayley subset*, if the subset

$$S_\mu = \{a \in G \mid \mu(a) < 1\}$$

is a Cayley subset of G. It follows that $\mu(1_g) = 1$ and if $\mu(a) < 1$, then $\mu(a^{-1}) < 1$. Obviously, every fuzzy group is a fuzzy Cayley subset. Since S_μ is a Cayley set, $(G; S_\mu)$ is a Cayley graph. When μ_S is a fuzzy Cayley subset, we define the triple $(G; S_\mu; \mu)$ and called it *fuzzy Cayley graph*. In fact, the fuzzy Cayley graph $(G; S_\mu; \mu)$ is a Cayley graph where the fuzzy Cayley subset μ constructs the Cayley subset of it.

The next lemma yields that if $\mu(a) \neq \mu(b)$, then $\mu(ab) = \min\{\mu(a), \mu(b)\}$, for some $a, b \in G$, when μ is a fuzzy subgroup on G.

Lemma 1. Suppose that μ is a fuzzy subgroup on G. If $\mu(a) \neq \mu(b)$ then $\mu(ab) = \min\{\mu(a), \mu(b)\}$, for every $a, b \in G$.

Proof. Without less of generality, suppose that $\mu(b) > \mu(a)$. Since μ is a fuzzy subgroup, we get $\mu(a) = \mu(abb^{-1}) \geq \min\{\mu(ab), \mu(b^{-1})\}$. Since $\mu(b^{-1}) = \mu(b)$ and $\mu(b) > \mu(a)$, the last argument yields that $\mu(a) \geq \mu(ab)$. On the other hand, $\mu(ab) \geq \min\{\mu(a), \mu(b)\} = \mu(a)$. Therefore, $\mu(ab) = \mu(a) = \min\{\mu(a), \mu(b)\}$. Similarity, if $\mu(b) < \mu(a)$, then $\mu(ab) = \mu(b)$. Thus, we have $\mu(ab) = \min\{\mu(a), \mu(b)\}$.

Lemma 2. Suppose that μ_1 and μ_2 are fuzzy Cayley subsets on a group G. The following properties hold.
(1) If $\mu_1 \subseteq \mu_2$, then $S_{\mu_2} \subseteq S_{\mu_1}$.
(2) The fuzzy subset $\mu_1 \cup \mu_2$ is a fuzzy Cayley subset and $S_{\mu_1 \cup \mu_2} = S_{\mu_1} \cap S_{\mu_2}$.
(3) The fuzzy subset $\mu_1 \cap \mu_2$ is a fuzzy Cayley subset and $S_{\mu_1 \cap \mu_2} = S_{\mu_1} \cup S_{\mu_2}$.

Proof. (1) If $x \notin S_{\mu_1}$, then $\mu_1(x) = 1$. Since $\mu_1 \leq \mu_2$, we have $\mu_2(x) = 1$, and, thus, $x \notin S_{\mu_2}$. Therefore, $S_{\mu_2} \subseteq S_{\mu_1}$.

(2) It easily can be verified that $S_{\mu_1 \cup \mu_2} = S_{\mu_1} \cap S_{\mu_2}$. Now, suppose that $x \in S_{\mu_1 \cup \mu_2}$. Then $x \in S_{\mu_1} \cap S_{\mu_2}$. Since μ_1 and μ_2 are fuzzy Cayley subsets, we have $x^{-1} \in S_{\mu_1} \cap S_{\mu_2}$ and, thus, $x^{-1} \in S_{\mu_1 \cup \mu_2}$. Similarly, if $1 \in S_{\mu_1 \cup \mu_2}$, then $1 \in S_{\mu_1} \cap S_{\mu_2}$, a contradiction. Therefore, $\mu_1 \cup \mu_2$ is a fuzzy Cayley subset.

(3) In a similar way as last part.

Lemma 3. Let $X_1 = (G; S_1)$ and $X_2 = (G; S_2)$ be Cayley graphs. The following properties hold.
(1) $X_1 \cup X_2 = (G; S_1 \cup S_2)$.
(2) $X_1 \cap X_2 = (G; S_1 \cap S_2)$.
(3) $X_1 \subseteq X_2$ if and only if $S_1 \subseteq S_2$.

Proof. (1) Let e be an edge of $(G; S_1 \cup S_2)$. Then there exist $g \in G$ and $s \in S_1 \cup S_2$ such that e is an edge between two vertices g and gs. Since $s \in S_1 \cup S_2$, we have $s \in S_1$ or $s \in S_2$ and, thus, $e \in E(X_1)$ or $e \in E(X_2)$. Therefore, $e \in E(X_1 \cup X_2)$. Similarly, any edge of $E(X_1 \cup X_2)$ is an edge of $(G; S_1 \cup S_2)$. The result follows.

(2) In a similar way as last part.

(3) Suppose that $S_1 \subseteq S_2$. If $e \in E(X_1)$, then there exist elements $g \in G$ and $s_1 \in S_1$ such that $e = (g, gs_1)$. Since $s_1 \in S_1$ and $S_1 \subseteq S_2$, we obtain $e \in E(X_2)$. Therefore, $X_1 \subseteq X_2$. Now, suppose that $E(X_1) \subseteq E(X_2)$. Let $g \in G$. If $s_1 \in S_1$, then $(g, gs_1) \in E(X_1)$. Therefore, $(g, gs_1) \in E(X_2)$. Then $(g, gs_1) = (g', g's'_1)$ for some $g' \in G$ and $s'_1 \in S_2$. Since $g = g'$, we obtain $s_1 = s'_1$ and, thus, $s_1 \in S_2$. The result follows.

Notice that, if $V(X_1) = V(X_2)$ then $X_1 \cup X_2$ and $X_1 \cap X_2$ are obviously Cayley graphs. The Lemma 2 follows us to define subgraph, union and intersection of fuzzy Cayley graphs.

Definition 1. Suppose that $X = (G; S_\mu; \mu)$ and $Y = (G; S_\lambda; \lambda)$ are fuzzy Cayley graphs. Then
(1) $X \subseteq Y$ if and only if $\lambda \subseteq \mu$;
(2) $X \cup Y = (G; S_\mu \cup S_\lambda; \mu \cap \lambda)$;
(3) $X \cap Y = (G; S_\mu \cap S_\lambda; \mu \cup \lambda)$.

Lemma 4. Suppose that G is a finite group and μ is a fuzzy Cayley subset on G. If μ is a fuzzy subgroup and $S_\mu \neq \emptyset$ then S_μ generates G.

Proof. Suppose that $g \in G$. If $g \notin S_\mu$, then $\mu(g) = 1$. Now, if $a \in S_\mu$, then $\mu(a) < 1$. By Lemma 1, $\mu(ga^{-1}) = \mu(a)$. It follows that $ga^{-1} \in S_\mu$ and, thus, $g = ga^{-1}a \in \langle S_\mu \rangle$. Therefore, $G = \langle S_\mu \rangle$.

The following theorem is easily verified by Lemma 4.

Theorem 5. Suppose that $X = (G; S_\mu; \mu)$ is a fuzzy Cayley graph. If μ is a fuzzy subgroup, then the Cayley graph $(G; S_\mu)$ is connected.

3 Fuzzy rough Cayley graphs

Suppose that G is a finite group with identity 1_G, μ is a fuzzy normal subgroup, $0 \leq t \leq 1$, and $X = (G; S)$ is a Cayley graph. Then the following graphs (we will prove these graphs are Cayley graphs)

$$\overline{X}_{\mu_t} = (G; \mu_{t^\wedge}(S)^*) \ \ (\mu_{t^\wedge}(S)^* = \mu_{t^\wedge}(S) \setminus \{1_G\}) \ \text{ and } \ \underline{X}_{\mu_t} = (G; \mu_{t-}(S))$$

are called, respectively, *fuzzy upper and lower approximations* of the Cayley graph X with respect to the fuzzy normal subgroup μ and integer t.

Theorem 6. $\underline{X}_{\mu_t}$ and $\overline{X}_{\mu_t}$ are Cayley graphs.

Proof. By Theorem 1(1), we have $\mu_{t-}(S) \subseteq S$, and, thus, $1_G \notin \mu_{t-}(S)$. Suppose that $s \in \mu_{t-}(S)$. Then $[s]_\mu \subseteq S$. If $x \in [s^{-1}]_\mu$ then $(x, s^{-1}) \in \mu_t$ and, thus, $(x^{-1}, s) \in \mu_t$, because μ is a fuzzy normal subgroup. Thus $x^{-1} \in [s]_\mu \subseteq S$. Since S is a Cayley set, we obtain $x \in S$ and, thus, $[s^{-1}]_\mu \subseteq S$. Hence, $s^{-1} \in \mu_{t-}(S)$. Therefore, $\mu_{t-}(S)$ is a Cayley set, and $\underline{X}_{\mu_t}$ is a Cayley graph.

Now, suppose that $s \in \mu_{t^\wedge}(S)^*$. Then $[s]_\mu \cap S \neq \emptyset$ which implies that there exists $a \in [s]_\mu \cap S$. Since $a \in [s]_\mu \cap S$, we obtain $(a, s) \in \mu_t$. As μ is a fuzzy normal subgroup, $(a^{-1}, s^{-1}) \in \mu_t$. Then $a^{-1} \in [s^{-1}]_\mu$. Since S is a Cayley set, we have $[s^{-1}]_\mu \cap S \neq \emptyset$ and, thus, $s^{-1} \in \mu_{t^\wedge}(S)$. Therefore, $\mu_{t^\wedge}(S)^*$ is a Cayley set, and $\overline{X}_{\mu_t}$ is a Cayley graph.

Let G be a group congruence modulo 16 integral number $\mathbb{Z}$. Let B be a fuzzy normal subgroup of G presented in Table, and t be 0.3. Let $X = (G; S)$ be a Cayley graph such that S equals to $\{\underline{1}, \underline{2}, \underline{6}, \underline{10}, \underline{14}, \underline{15}\}$. The congruence relation $B_{0.3}$ partitions G to four classes $\{0, 4, 8, 12\}, \{1, 5, 9, 13\}$, $\{2, 6, 10, 14\}$ and $\{3, 7, 11, 15\}$. Then we have

$$\overline{X}_{B_{0.3}} = (G; \{\underline{1}, \underline{2}, \underline{3}, \underline{5}, \underline{6}, \underline{7}, \underline{9}, \underline{10}, \underline{11}, \underline{13}, \underline{14}, \underline{15}\}) \text{ and } \underline{X}_{B_{0.3}} = (G; \{\underline{2}, \underline{6}, \underline{10}, \underline{14}\}).$$

T a b l e

The fuzzy normal subgroup B

$B(\underline{1}) = 0.1$	$B(\underline{2}) = 0.2$	$B(\underline{3}) = 0.1$	$B(\underline{4}) = 0.4$
$B(\underline{5}) = 0.1$	$B(\underline{6}) = 0.2$	$B(\underline{7}) = 0.1$	$B(\underline{8}) = 0.8$
$B(\underline{9}) = 0.1$	$B(\underline{10}) = 0.2$	$B(\underline{11}) = 0.1$	$B(\underline{12}) = 0.4$
$B(\underline{13}) = 0.1$	$B(\underline{14}) = 0.2$	$B(\underline{15}) = 0.1$	$B(\underline{0}) = 1$

Theorem 7. Suppose that μ and λ are fuzzy normal subgroups of a group G and $t \in [0, 1]$. Let $X = (G; S)$, $X_1 = (G; S_1)$ and $X_2 = (G; S_2)$ be Cayley graphs. The following properties hold.

(1) $\underline{X}_{\mu_t} \subseteq X \subseteq \overline{X}_{\mu_t}$,

(2) $\overline{X_1 \cup X_2}_{\mu_t} = \overline{X_1}_{\mu_t} \cup \overline{X_2}_{\mu_t}$,

(3) $\underline{X_1 \cap X_2}_{\mu_t} = \underline{X_1}_{\mu_t} \cap \underline{X_2}_{\mu_t}$,

(4) $X_1 \subseteq X_2 \Rightarrow \underline{X_1}_{\mu_t} \subseteq \underline{X_2}_{\mu_t}$,

(5) $X_1 \subseteq X_2 \Rightarrow \overline{X_1}_{\mu_t} \subseteq \overline{X_2}_{\mu_t}$,

(6) $\underline{X_1 \cup X_2}_{\mu_t} \supseteq \underline{X_1}_{\mu_t} \cup \underline{X_2}_{\mu_t}$,

(7) $\overline{X_1 \cap X_2}_{\mu_t} \subseteq \overline{X_1}_{\mu_t} \cap \overline{X_2}_{\mu_t}$,

(8) $\mu_t \subseteq \lambda_t \Rightarrow \overline{X}_{\mu_t} \subseteq \overline{X}_{\lambda_t}$,

(9) $\mu_t \subseteq \lambda_t \Rightarrow \underline{X}_{\lambda_t} \subseteq \underline{X}_{\mu_t}$,

(10) $\overline{X}_{(\mu\cap\lambda)_t} \subseteq \overline{X}_{\mu_t} \cap \overline{X}_{\lambda_t}$,

(11) $\underline{X}_{(\mu\cap\lambda)_t} \supseteq \underline{X}_{\mu_t} \cap \underline{X}_{\lambda_t}$.

Proof. (1) By Theorem 1(1), $\mu_{t-}(S) \subseteq S \subseteq \mu_{t^\wedge}(S)$. Then $\mu_{t-}(S) \subseteq S \subseteq \mu_{t^\wedge}(S)^*$. It follows that $\underline{X}_{\mu_t} \subseteq X \subseteq \overline{X}_{\mu_t}$.

(2) Based on Lemma 3, $\overline{X_1} \cup \overline{X_2} = (G; \mu_{t^\wedge}(S_1)^* \cup \mu_{t^\wedge}(S_2)^*)$. By Theorem 1(5), we have $\mu_{t^\wedge}(S_1)^*$ and $\mu_{t^\wedge}(S_2)^* \subseteq \mu_{t^\wedge}(S_1 \cup S_2)^*$. Now, by Lemma 3(3), we have $\overline{X_1}_{\mu_t} \cup \overline{X_2}_{\mu_t} \subseteq \overline{X_1 \cup X_2}_{\mu_t}$. Conversely, by Theorem 1(3), $\mu_{t^\wedge}(S_1)^* \cup \mu_{t^\wedge}(S_2)^* = \mu_{t^\wedge}(S_1 \cup S_2)^*$. Suppose that (g, gs) is an edge of $E(\overline{X_1 \cup X_2}_{\mu_t})$. It follows that $s \in \mu_{t^\wedge}(S_1 \cup S_2)^*$. Then $s \in \mu_{t^\wedge}(S_1)^* \cup \mu_{t^\wedge}(S_2)^*$ and, thus, $s \in \mu_{t^\wedge}(S_1)^*$ or $s \in \mu_{t^\wedge}(S_2)^*$. Therefore, (g, gs) is an edge of $\overline{X_1}_{\mu_t}$ or $\overline{X_2}_{\mu_t}$. Finally, we have $\overline{X_1 \cup X_2}_{\mu_t} = \overline{X_1}_{\mu_t} \cup \overline{X_2}_{\mu_t}$.

(3) By Theorem 1(2), the proof is similar to part (2).

(4) Assume that $X_1 \subseteq X_2$. Then $S_1 \subseteq S_2$ and, thus, $\mu_{t-}(S_1) \subseteq \mu_{t-}(S_2)$. Hence, $\underline{X_1}_{\mu_t} \subseteq \underline{X_2}_{\mu_t}$.

(5) By Theorem 1(5), the proof is similar to part (4).

(6) By Theorem 1(6), we have $\mu_{t-}(S_1) \cup \mu_{t-}(S_2) \subseteq \mu_{t-}(S_1 \cup S_2)$. Then $\mu_{t-}(S_1) \subseteq \mu_{t-}(S_1 \cup S_2)$ and $\mu_{t-}(S_2) \subseteq \mu_{t-}(S_1 \cup S_2)$. Therefore, we obtain $\underline{X_1 \cup X_2}_{\mu_t} \supseteq \underline{X_1}_{\mu_t}$ and $\underline{X_1 \cup X_2}_{\mu_t} \supseteq \underline{X_2}_{\mu_t}$. And finally, $\underline{X_1 \cup X_2}_{\mu_t} \supseteq \underline{X_1}_{\mu_t} \cup \underline{X_2}_{\mu_t}$.

(7) By Theorem 1(7), the proof is similar to part (6).

(8) Assume that $\mu_t \subseteq \lambda_t$. Theorem 1(9) yields $\mu_{t^\wedge}(S) \subseteq \lambda_{t^\wedge}(S)$. Then $\mu_{t^\wedge}(S)^* \subseteq \lambda_{t^\wedge}(S)^*$ and, thus, $\overline{X}_{\mu_t} \subseteq \overline{X}_{\lambda_t}$.

(9) By Theorem 1(8), the proof is similar to part (8).

(10) By Theorem 1(12), we have

$$\begin{aligned}\overline{X}_{(\mu\cap\lambda)_t} &= (G; (\mu \cap \lambda)_{t^\wedge}(S))\\ &\subseteq (G; \mu_{t^\wedge}(S) \cap \lambda_{t^\wedge}(S))\\ &= (G; \mu_{t^\wedge}(S)) \cap (G; \lambda_{t^\wedge}(S))\\ &= \overline{X}_{\mu_t} \cap \overline{X}_{\lambda_t}.\end{aligned}$$

(11) By Theorem 1(12), the proof is similar to part (11).

Remark 1. A subset S of G is a minimal Cayley set if it generates G and if $S \setminus \{s, s^{-1}\}$ generates a proper subgroup of G for all $s \in S$.

The pair $(\underline{X}_{\mu_t}, \overline{X}_{\mu_t})$ is called a *fuzzy rough set* of the Cayley graph X. A Cayley graph $X = (G; S)$ is called a $\mu_{t^\wedge}$*-fuzzy rough generating*, if the subset $\mu_{t^\wedge}(S)^*$ is a generating set for G. Similarly, a Cayley graph $X = (G; S)$ is called an μ_{t-}*-fuzzy rough generating*, if the subset $\mu_{t-}(S)$ is a generating set for G. A Cayley graph $X = (G; S)$ is called a $\mu_{t^\wedge}$*-fuzzy rough optimal connected*, if the subset $\mu_{t^\wedge}(S)^*$ is a minimal Cayley set for G. Similarly, a Cayley graph $X = (G; S)$ is called a μ_{t-}*-fuzzy rough optimal connected*, if the subset $\mu_{t-}(S)$ is a minimal Cayley set for G.

Theorem 8. Suppose that $X = (G; S)$ is a Cayley graph.

(1) If X is a $\mu_{t^\wedge}$-fuzzy rough generating, then $\overline{X}_{\mu_t}$ is connected.
(2) If X is a μ_{t-}-fuzzy rough generating, then $\underline{X}_{\mu_t}$ is connected.
(3) If X is a $\mu_{t^\wedge}$-fuzzy rough optimal connected, then $\overline{X}_{\mu_t}$ is optimal connected.
(4) If X is a μ_{t-}-fuzzy rough optimal connected, then $\underline{X}_{\mu_t}$ is optimal connected.

Proof. It is straightforward.

4 Rough fuzzy Cayley graphs

Let G be a finite group with identity 1_G, B a fuzzy normal subgroup on G and $X = (G; S_\mu; \mu)$ be a fuzzy Cayley graph. The following fuzzy Cayley graphs (we will prove these are fuzzy Cayley graphs)

$$\overline{X}_B = (G; S_{\underline{A}_B\mu^\star}; \underline{A}_B\mu^\star) \text{ and } \underline{X}_B = (G; S_{\overline{A}_B\mu}; \overline{A}_B\mu)$$

are called, respectively, *lower and upper approximations* of the fuzzy Cayley graph X with respect to B. In the above definition, $\underline{A}_B\mu^\star(x)$ is similar to $\underline{A}_B\mu(x)$ in all elements, except for 1_G, where $\underline{A}_B\mu^\star(1_G)$ is 1.

Theorem 9. The triples $\underline{X}_B$ and $\overline{X}_B$ are fuzzy Cayley graphs.

Proof. Suppose that $\underline{A}_B\mu(x) = 1$ for some $x \in [0,1]$. Thus

$$\inf_{u\in G} \vartheta_{\min}(B(u,x), \mu(u)) = 1.$$

Therefore, for all elements $u \in G$, $\vartheta_{\min}(B(u,x), \mu(u)) = 1$ and, thus, $B(ux^{-1}) \le \mu(u)$ for every $u \in G$. On the other hand, μ is a fuzzy subgroup, and we have $\mu(u^{-1}) = \mu(u)$. Then $B(ux^{-1}) \le \mu(u^{-1})$. Since B is a fuzzy normal subgroup, we obtain $B(ux^{-1}) = B(x^{-1}u)$ and consequently, are equal to $B(u^{-1}x)$. So $B(u^{-1}x) \le \mu(u^{-1})$. Hence for all u of G, $\vartheta_{\min}(B(u^{-1}, x^{-1}), \mu(u^{-1})) = 1$ and, thus,

$$\inf_{u\in G} \vartheta_{\min}(B(u^{-1}, x^{-1}), \mu(u^{-1})) = 1.$$

Then

$$\inf_{u\in G} \vartheta_{\min}(B(u, x^{-1}), \mu(u)) = 1.$$

So $\underline{A}_B\mu(x^{-1}) = 1$. Therefore, $\underline{A}_B\mu^\star$ is a fuzzy Cayley subset and $\overline{X}_B$ is a fuzzy Cayley graph.

Theorem 3(1) leads $\mu \subseteq \overline{A}_B\mu$. Since $\mu(1_G) = 1$, we obtain $\overline{A}_B\mu(1_G) = 1$. Now suppose that $\overline{A}_B\mu(x) = 1$. Then

$$\sup_{u\in G}\{\min\{B(ux^{-1}), \mu(u)\}\} = 1.$$

Since G is finite, there exists an element u of G such that $\min\{B(ux^{-1}), \mu(u)\} = 1$. Then $B(ux^{-1}) = \mu(u) = 1$. Since μ is a fuzzy subgroup, we obtain $\mu(u^{-1}) = \mu(u)$. Now as B is a fuzzy normal subgroup, $B(ux^{-1}) = B(x^{-1}u)$ and since B is a fuzzy subgroup, we obtain $B(ux^{-1}) = B(u^{-1}x)$. Thus $\min\{B(u^{-1}x), \mu(u^{-1})\} = 1$, and $\overline{A}_B\mu(x^{-1}) = 1$. Consequently, $\overline{A}_B\mu$ is a fuzzy Cayley subset and, thus, $\underline{X}_B$ is a fuzzy Cayley graph.

Lemma 5. Suppose that G is a finite group and B is a fuzzy normal subgroup of G. If $X = (G; S_\mu; \mu)$ and $Y = (G; S_\lambda; \lambda)$ are fuzzy Cayley graphs, then:

(1) $S_{\underline{A}_B(\mu\cup\lambda)^\star} \subseteq S_{\underline{A}_B\mu^\star} \cap S_{\underline{A}_B\lambda^\star}$,
(2) $S_{\underline{A}_B(\mu\cap\lambda)^\star} = S_{\underline{A}_B\mu^\star} \cup S_{\underline{A}_B\lambda^\star}$,
(3) $S_{\overline{A}_B(\mu\cup\lambda)} = S_{\overline{A}_B\mu} \cap S_{\overline{A}_B\lambda}$,
(4) $S_{\overline{A}_B(\mu\cap\lambda)} \supseteq S_{\overline{A}_B\mu} \cup S_{\overline{A}_B\lambda}$.

Proof. (1) Suppose that $x \in S_{\underline{A}_B(\mu\cup\lambda)^\star}$. Then $\underline{A}_B(\mu\cup\lambda)^\star(x) < 1$ and $x \ne 1_G$. By Theorem 3(7), $\underline{A}_B\mu(x), \underline{A}_B\lambda(x) < 1$. Hence, $x \in S_{\underline{A}_B\mu^\star} \cap S_{\underline{A}_B\lambda^\star}$.

According to Theorem 3, items (2), (3) and (4) are straightforward.

Theorem 10. Suppose that G is a finite group and B and C are fuzzy normal subgroups of G. Let $X = (G; S_\mu; \mu)$ and $Y = (G; S_\lambda; \lambda)$ be fuzzy Cayley graphs. Then

(1) $\underline{X}_B \subseteq X \subseteq \overline{X}_B$,
(2) $\overline{X \cup Y}_B = \overline{X}_B \cup \overline{Y}_B$,
(3) $\overline{X \cap Y}_B \subseteq \overline{X}_B \cap \overline{Y}_B$,
(4) $\underline{X \cup Y}_B \supseteq \underline{X}_B \cup \underline{Y}_B$,
(5) $\underline{X \cap Y}_B = \underline{X}_B \cap \underline{Y}_B$,
(6) $\mu \subseteq \lambda \Rightarrow \underline{Y}_B \subseteq \underline{X}_B$,
(7) $\mu \subseteq \lambda \Rightarrow \overline{Y}_B \subseteq \overline{X}_B$,
(8) $B \subseteq C \Rightarrow \underline{X}_C \subseteq \underline{X}_B$,
(9) $B \subseteq C \Rightarrow \overline{X}_B \subseteq \overline{X}_C$.

Proof. (1) By Theorem 3(1), we have $\underline{A}_B\mu \subseteq \mu \subseteq \overline{A}_B\mu$. Hence $\underline{A}_B\mu^\star \subseteq \mu \subseteq \overline{A}_B\mu$. Lemma 2(1) implies that $\underline{X}_B \subseteq X \subseteq \overline{X}_B$.

(2) By Definition 1(2), $X \cup Y = (G; S_{\mu\cap\lambda}; \mu \cap \lambda)$. Then we have

$$\overline{X \cup Y}_B = (G; S_{\underline{A}_B(\mu\cap\lambda)^\star}; \underline{A}_B(\mu \cap \lambda)^\star).$$

By Theorem 3(8), $\underline{A}_B(\mu \cap \lambda) = \underline{A}_B\mu \cap \underline{A}_B\lambda$ and, thus,

$$\overline{X \cup Y}_B = (G; S_{\underline{A}_B\mu^\star\cap\underline{A}_B\lambda^\star}; \underline{A}_B\mu^\star \cap \underline{A}_B\lambda^\star).$$

Now by 1(2), $\overline{X \cup Y}_B = \overline{X}_B \cup \overline{Y}_B$. The result follows.

(3) By Theorem 3(7), the proof is similar to part (2).

(4) By Theorem 3(6), the proof is similar to part (2).

(5) By Theorem 3(5), the proof is similar to part (2).

(6) If $\mu \subseteq \lambda$, then by Corollary 1(1), $\overline{A}_B\mu \subseteq \overline{A}_B\lambda$. Now, by Definition 1(1), $\underline{Y}_B \subseteq \underline{X}_B$.

(7) By Corollary 1(2), the proof is similar to part (6).

(8) Assume that $B \subseteq C$. By Theorem 3(9), $\overline{A}_B\mu \subseteq \overline{A}_C\mu$. Therefore, we have $\underline{X}_C \subseteq \underline{X}_B$.

(9) According to Theorem 3(10), the proof is similar to part (8).

Theorem 11. Suppose that G is a finite group. If B and C are fuzzy normal subgroups and μ is a fuzzy subset on G, then the following statement hold.

(1) $(G; S_{\underline{A}_{BMinC}\mu}; \underline{A}_{BMinC}\mu) \subseteq (G; S_{\underline{A}_B\mu Min\underline{A}_C\mu}; \underline{A}_B\mu Min\underline{A}_C\mu)$,
(2) $(G; S_{\overline{A}_B\mu Min\overline{A}_C\mu}; \overline{A}_B\mu Min\overline{A}_C\mu) \subseteq (G; S_{\overline{A}_{BMinC}\mu}; \overline{A}_{BMinC}\mu)$.

Proof. According to Theorem 4, the proof of both parts are clear.

The pair $(\underline{X}_B, \overline{X}_B)$ is called a *rough set* of a fuzzy Cayley graph $X = (G; S_\mu; \mu)$. A fuzzy Cayley graph $X = (G; S_\mu; \mu)$ is called an $\overline{A}_B$ *rough generating*, if the subset $S_{\overline{A}_B\mu}$ generates G. Likewise a fuzzy Cayley graph $X = (G; S_\mu; \mu)$ is called an $\underline{A}_B$ *rough generating*, if the subset $S_{\underline{A}_B\mu}$ generates G. A fuzzy Cayley graph $X = (G; S_\mu; \mu)$ is called an $\overline{A}_B$ *rough optimal connected*, if the subset $S_{\overline{A}_B\mu}$ is a minimal Cayley set of G. Similarly a fuzzy Cayley graph $X = (G; S_\mu; \mu)$ is called an $\underline{A}_B$ *rough optimal connected*, if the subset $S_{\underline{A}_B\mu}$ is a minimal Cayley set of G.

Theorem 12. Suppose that G is a finite group, and B is a fuzzy normal subgroup of G. Let $X = (G; S_\mu; \mu)$ be a fuzzy Cayley graph. The following properties hold.

(1) If μ is a fuzzy subgroup of G, then X is a $\overline{A}_B$ rough generating.
(2) If $B \subseteq \mu$ and μ is a fuzzy subgroup of G, then X is a $\underline{A}_B$ rough generating.

Proof. According to Theorems 4 and 4, the proof is straightforward.

5 Fuzzy rough fuzzy Cayley graphs

In this section, we get the t-level relation μ_t, for each $t \in [0,1)$, as follows:

$$\mu_t = \{(a,b) \in G \times G \mid \mu(ab^{-1}) > t\}.$$

Similarly, all results related to the t-level relation μ_t are same. Let B be a fuzzy normal subgroup on G and $X = (G; S_\mu; \mu)$ be a fuzzy Cayley graph. The following fuzzy Cayley graphs (we will prove these are fuzzy Cayley graphs)

$$\underline{X}'_B = (G; B_{t_\mu -}(S_\mu); \overline{A}_B\mu^\sharp) \text{ and } \overline{X}'_B = (G; B^\wedge_{t_\mu}(S_\mu)^\star; \underline{A}_B\mu^\sharp)$$

are called, respectively, *fuzzy lower and upper approximations* of the fuzzy Cayley graph X with respect to B. The definitions of t_μ, $\underline{A}_B\mu^\sharp$ and $\overline{A}_B\mu^\sharp$ are as follows:

$$t_\mu = \max\{\mu(x) \mid x \in S_\mu\},$$

$$\text{if } x \in B^\wedge_{t_\mu}(S_\mu)^\star \text{ then } \underline{A}_B\mu^\sharp(x) = \underline{A}_B\mu^\star, \text{ otherwise } \underline{A}_B\mu^\sharp(x) = 1 \text{ and}$$

$$\text{if } x \in B_{t_\mu -}(S_\mu) \text{ then } \overline{A}_B\mu^\sharp(x) = \overline{A}_B\mu(x), \text{ otherwise } \overline{A}_B\mu^\sharp(x) = 1.$$

Theorem 13. The triples $\underline{X}'_B$ and $\overline{X}'_B$ are fuzzy Cayley graphs.

Proof. In the proof Theorem 6, it proved that the subsets $B_{t_\mu -}(S_\mu)$ and $B^\wedge_{t_\mu}(S_\mu)^\star$ are Cayley sets. To prove that the $\underline{X}'_B$ and $\overline{X}'_B$ are fuzzy Cayley graphs, it is sufficient to show that $B_{t_\mu -}(S_\mu) = S_{\overline{A}_B\mu^\sharp}$ and $B^\wedge_{t_\mu}(S_\mu)^\star = S_{\underline{A}_B\mu^\sharp}$.

Suppose that $x \in B_{t_\mu -}(S_\mu)$. Then $\overline{A}_B\mu^\sharp(x) = \overline{A}_B\mu(x)$. If $\overline{A}_B\mu(x) = 1$, then

$$\sup_{u\in G}\{\min\{B(ux^{-1}), \mu(u)\}\} = 1.$$

Since G is finite, there exists an element u in G where $\min\{B(ux^{-1}), \mu(u)\} = 1$ and, thus, $B(ux^{-1}) = \mu(u) = 1$. As $B(ux^{-1}) = 1$, we h obtain $u \in [x]_B$ and, thus, $u \in S_\mu$. Therefore, $\mu(u) < 1$, a contradiction. Then $\overline{A}_B\mu(x) \neq 1$ and, as a result, $\overline{A}_B\mu^\sharp(x) \neq 1$. Now, suppose that $x \notin B_{t_\mu -}(S_\mu)$. Based on the definition, $\overline{A}_B\mu^\sharp(x) = 1$. Therefore, $B_{t_\mu -}(S_\mu) = S_{\overline{A}_B\mu^\sharp}$.

Let x be in $B^\wedge_{t_\mu}(S_\mu)^\star$. Then $\underline{A}_B\mu^\sharp(x) = \underline{A}_B\mu^\star$. Since $x \in B^\wedge_{t_\mu}(S_\mu)^\star$, there exists an element $y \in S_\mu$ such that $y \in [x]_B$. If we have $\underline{A}_B\mu(x) = 1$, then

$$\inf_{u\in G} \vartheta_{\min}(B(ux^{-1}), \mu(u)) = 1.$$

Therefore, we have $B(ux^{-1}) \leq \mu(u)$ for every $u \in G$. Then $B(yx^{-1}) \leq \mu(y)$. Since $\mu(y) \leq t_\mu$, we obtain $B(yx^{-1}) \leq t_\mu$. As $y \in [x]_B$, $B(yx^{-1}) > t_\mu$, a contradiction. Now, suppose that $x \notin B^\wedge_{t_\mu}(S_\mu)^\star$. Based on the definition, $\underline{A}_B\mu^\sharp(x) = 1$. Hence, by above $B^\wedge_{t_\mu}(S_\mu)^\star = S_{\underline{A}_B\mu^\sharp(x)}$.

Lemma 6. Let G be a group and t_1, t_2 and t_3 be integers in the closed interval $[0,1]$. Suppose that μ is fuzzy normal subgroups of G. Let A and B be two non-empty sets. Then
(1) if $t_3 \leq t_1, t_2$, then $\mu_{t_3^\wedge}(A \cup B) \supseteq \mu_{t_1^\wedge}(A) \cup \mu_{t_2^\wedge}(B)$,
(2) if $t_3 \geq t_1, t_2$, then $\mu_{t_3^\wedge}(A \cup B) \subseteq \mu_{t_1^\wedge}(A) \cup \mu_{t_2^\wedge}(B)$,
(3) if $t_3 \geq t_1, t_2$, then $\mu_{t_3^\wedge}(A \cap B) \subseteq \mu_{t_1^\wedge}(A) \cap \mu_{t_2^\wedge}(B)$,
(4) if $t_3 \leq t_1, t_2$, then $\mu_{t_3 -}(A \cap B) \subseteq \mu_{t_1 -}(A) \cap \mu_{t_2 -}(B)$,
(5) if $t_3 \geq t_1, t_2$, then $\mu_{t_3 -}(A \cap B) \supseteq \mu_{t_1 -}(A) \cap \mu_{t_2 -}(B)$,
(6) if $t_3 \geq t_1, t_2$, then $\mu_{t_3 -}(A \cup B) \supseteq \mu_{t_1 -}(A) \cup \mu_{t_2 -}(B)$.

Proof. (1) Let $x \in \mu_{t_1^\wedge}(A) \cup \mu_{t_2^\wedge}(B)$. Then $x \in \mu_{t_1^\wedge}(A)$ or $x \in \mu_{t_2^\wedge}(B)$. Suppose that $x \in \mu_{t_1^\wedge}(A)$. Thus, $[x]_{\mu_{t_1}} \cap A \neq \emptyset$ and consequently, there exists $a \in A$ such that $\mu(xa^{-1}) > t_1$. Since $t_1 \geq t_3$, we have $\mu(xa^{-1}) > t_3$ and, thus, $[x]_{\mu_{t_3}} \cap A \neq \emptyset$. The result gives us that $x \in \mu_{t_3^\wedge}(A \cup B)$. Similarity, if $x \in \mu_{t_2^\wedge}(B)$, the same result can be gained.

(2) Let $x \in \mu_{t_3^\wedge}(A \cup B)$. Thus, $[x]_{\mu_{t_3}} \cap (A \cup B) \neq \emptyset$. Then $[x]_{\mu_{t_3}} \cap A \neq \emptyset$ or $[x]_{\mu_{t_3}} \cap B \neq \emptyset$. Suppose that $[x]_{\mu_{t_3}} \cap A \neq \emptyset$. Hence, there exists $a \in A$ such that $\mu(xa^{-1}) > t_3$. Since $t_3 \geq t_1$, we have $\mu(xa^{-1}) > t_1$ and, thus, $[x]_{\mu_{t_1}} \cap A \neq \emptyset$. The result gives us that $x \in \mu_{t_1^\wedge}(A)$. The result follows.

(3) Let $x \in \mu_{t_3^\wedge}(A \cap B)$. Then, $[x]_{\mu_{t_3}} \cap (A \cap B) \neq \emptyset$ and, thus, $[x]_{\mu_{t_3}} \cap A \neq \emptyset$ and $[x]_{\mu_{t_3}} \cap B \neq \emptyset$. Hence, there exist elements $a \in A$ and $b \in B$ such that $\mu(xa^{-1}) > t_3$ and $\mu(xb^{-1}) > t_3$. Since $t_3 \geq t_1, t_2$, we have $\mu(xa^{-1}) > t_1$ and $\mu(xb^{-1}) > t_2$ and, thus, $[x]_{\mu_{t_1}} \cap A \neq \emptyset$ and $[x]_{\mu_{t_2}} \cap B \neq \emptyset$. The result gives us that $\mu_{t_1^\wedge}(A) \cap \mu_{t_2^\wedge}(B)$. The result follows.

(4) Let $x \in \mu_{t_3-}(A \cap B)$. Then $[x]_{\mu_{t_3}} \subseteq A \cap B$. If $y \in [x]_{\mu_{t_1}}$, then $\mu(yx^{-1}) > t_1$ and, thus, $\mu(yx^{-1}) > t_3$. Then $y \in A$. It follows that $x \in \mu_{t_1-}(A)$. Similarly, we have $x \in \mu_{t_2-}(B)$.

(5) Let $x \in \mu_{t_1-}(A) \cap \mu_{t_2-}(B)$. Then $[x]_{\mu_{t_1}} \subseteq A$ and $[x]_{\mu_{t_2}} \subseteq B$. If $y \in [x]_{\mu_{t_3}}$, then $\mu(yx^{-1}) > t_3$ and, thus, $\mu(yx^{-1}) > t_1$ and $\mu(yx^{-1}) > t_2$. Then $y \in A \cap B$. It follows that $x \in \mu_{t_3-}(A \cap B)$.

(6) Let $x \in \mu_{t_1-}(A) \cup \mu_{t_2-}(B)$. Hence, $x \in \mu_{t_1-}(A)$ or $x \in \mu_{t_2-}(B)$. Suppose that $x \in \mu_{t_1-}(A)$. Hence, $[x]_{\mu_{t_1}} \subseteq A$. If $y \in [x]_{\mu_{t_3}}$, then $\mu(yx^{-1}) > t_3$ and, thus, $\mu(yx^{-1}) > t_1$. Then $y \in A$. It follows that $x \in \mu_{t_3-}(A \cup B)$. The result follows.

Theorem 14. Let G be a finite group. Taking any fuzzy normal subgroups B and C on G. If $X = (G; S_\mu; \mu)$ and $Y = (G; S_\lambda; \lambda)$ are fuzzy Cayley graphs. The following properties hold.

(1) $\underline{X}'_B \subseteq X \subseteq \overline{X}'_B$,
(2) $\overline{X \cap Y}'_B \subseteq \overline{X}'_B \cap \overline{Y}'_B$,
(3) $\overline{X \cup Y}'_B \supseteq \overline{X}'_B \cup \overline{Y}'_B$,
(4) $\underline{X \cap Y}'_B \subseteq \underline{X}'_B \cap \underline{Y}'_B$,
(5) $\mu \subseteq \lambda \Rightarrow \underline{Y}'_B \subseteq \underline{X}'_B$,
(6) $\mu \subseteq \lambda \Rightarrow \overline{Y}'_B \subseteq \overline{X}'_B$,
(7) $B \subseteq C \Rightarrow \underline{X}'_C \subseteq \underline{X}'_B$,
(8) $B \subseteq C \Rightarrow \overline{X}'_B \subseteq \overline{X}'_C$.

Proof. (1) By Theorems 3(1) and 1(1), we have, respectively, $\underline{A}_B\mu \subseteq \mu \subseteq \overline{A}_B\mu$, $B_{t_\mu-}(S_\mu) \subseteq S_\mu \subseteq B^\wedge_{t_\mu}(S_\mu)$. If $x \in B^\wedge_{t_\mu}(S_\mu)^\star$, then $\underline{A}_B\mu^\sharp(x) = \underline{A}_B\mu(x)$ and, thus, $\underline{A}_B\mu^\sharp(x) \leq \mu(x)$. If $x \notin B^\wedge_{t_\mu}(S_\mu)^\star$, then $x \notin S_\mu$ and, thus, $\mu(x) = 1$. So we have $\underline{A}_B\mu^\sharp(x) \leq \mu(x)$. Now by Definition 1, we have $X \subseteq \overline{X}'_B$.

If $x \in B_{t_\mu-}(S_\mu)$, then $\overline{A}_B\mu^\sharp(x) = \overline{A}_B\mu(x)$ and, thus, $\mu(x) \leq \overline{A}_B\mu^\sharp(x)$. If $x \notin B_{t_\mu-}(S_\mu)$, then $\overline{A}_B\mu^\sharp(x) = 1$ and again $\mu(x) \leq \overline{A}_B\mu^\sharp(x)$. Then, we have $\underline{X}'_B \subseteq X$.

(2) We have

$$\begin{aligned}\overline{X \cap Y}'_B &= \overline{(G; S_{\mu\cup\lambda}; \mu \cup \lambda)}'_B = (G; B^\wedge_{t_{\mu\cup\lambda}}(S_\mu \cap S_\lambda); \underline{A}_B(\mu \cup \lambda)^\sharp),\\ \overline{X}'_B \cap \overline{Y}'_B &= \overline{(G; S_\mu; \mu)}'_B \cap \overline{(G; S_\lambda; \lambda)}'_B\\ &= (G; B^\wedge_{t_\mu}(S_\mu); \underline{A}_B(\mu)^\sharp) \cap (G; B^\wedge_{t_\lambda}(S_\lambda); \underline{A}_B(\lambda)^\sharp)\\ &= (G; B^\wedge_{t_\mu}(S_\mu) \cap B^\wedge_{t_\lambda}(S_\lambda); \underline{A}_B(\mu)^\sharp \cup \underline{A}_B(\lambda)^\sharp).\end{aligned}$$

Since $t_{\mu\cup\lambda} \geq t_\mu, t_\lambda$, by Theorem 6(3), we have $B^\wedge_{t_{\mu\cup\lambda}}(S_\mu \cap S_\lambda) \subseteq B^\wedge_{t_\mu}(S_\mu) \cap B^\wedge_{t_\lambda}(S_\lambda)$. Also, by Theorem 3(7), we have $\underline{A}_B(\mu \cup \lambda) \supseteq \underline{A}_B\mu \cup \underline{A}_B\lambda$. If $x \in B^\wedge_{t_\mu}(S_\mu) \cap B^\wedge_{t_\lambda}(S_\lambda)$ then

$$\underline{A}_B\mu^\sharp(x) = \underline{A}_B\mu(x), \underline{A}_B\lambda^\sharp(x) = \underline{A}_B\lambda(x)$$

and, thus,

$$\underline{A}_B\mu^\sharp(x) \cup \underline{A}_B\lambda^\sharp(x) \leq \underline{A}_B(\mu \cup \lambda)(x).$$

Since $\underline{A}_B(\mu\cup\lambda)(x) \leq \underline{A}_B(\mu\cup\lambda)^\sharp(x)$, we obtain $\underline{A}_B\mu^\sharp(x)\cup\underline{A}_B\lambda^\sharp(x) \leq \underline{A}_B(\mu\cup\lambda)^\sharp(x)$. If $x \notin B^\wedge_{t_\mu}(S_\mu)\cap B^\wedge_{t_\lambda}(S_\lambda)$, then $x \notin B^\wedge_{t_{\mu\cup\lambda}}(S_\mu\cap S_\lambda)$ and, thus, $\underline{A}_B(\mu\cup\lambda)^\sharp(x) = 1$. Therefore, $\underline{A}_B(\mu\cup\lambda)^\sharp(x) \geq \underline{A}_B\mu^\sharp(x)\cup \underline{A}_B\lambda^\sharp(x)$ and Definition 1 yields $\overline{X\cap Y}'_B \subseteq \overline{X}'_B \cap \overline{Y}'_B$.

(3) We have

$$\begin{aligned}
\overline{X\cup Y}'_B &= \overline{(G;S_{\mu\cap\lambda};\mu\cap\lambda)}'_B = (G;B^\wedge_{t_{\mu\cap\lambda}}(S_\mu\cup S_\lambda);\underline{A}_B(\mu\cap\lambda)^\sharp),\\
\overline{X}'_B\cup\overline{Y}'_B &= \overline{(G;S_\mu;\mu)}'_B \cup \overline{(G;S_\lambda;\lambda)}'_B\\
&= (G;B^\wedge_{t_\mu}(S_\mu);\underline{A}_B(\mu)^\sharp)\cup(G;B^\wedge_{t_\lambda}(S_\lambda);\underline{A}_B(\lambda)^\sharp)\\
&= (G;B^\wedge_{t_\mu}(S_\mu)\cup B^\wedge_{t_\lambda}(S_\lambda);\underline{A}_B(\mu)^\sharp\cap\underline{A}_B(\lambda)^\sharp).
\end{aligned}$$

Since $t_{\mu\cap\lambda} \leq t_\mu, t_\lambda$, by Theorem 6(1), we have $B^\wedge_{t_{\mu\cap\lambda}}(S_\mu\cup S_\lambda) \supseteq B^\wedge_{t_\mu}(S_\mu)\cup B^\wedge_{t_\lambda}(S_\lambda)$. Also, by Theorem 3(8), we have $\underline{A}_B(\mu\cap\lambda) = \underline{A}_B\mu\cap\underline{A}_B\lambda$. If $x\in B^\wedge_{t_\mu}(S_\mu)\cup B^\wedge_{t_\lambda}(S_\lambda)$ then $x \in B^\wedge_{t_{\mu\cap\lambda}}(S_\mu\cup S_\lambda)$ and, thus,

$$\underline{A}_B(\mu\cap\lambda)^\sharp(x) = \underline{A}_B(\mu\cap\lambda)(x) = \underline{A}_B\mu(x)\cap\underline{A}_B\lambda(x) \leq \underline{A}_B\mu^\sharp(x)\cap\underline{A}_B\lambda^\sharp(x).$$

Now, suppose that $x\notin B^\wedge_{t_\mu}(S_\mu)\cup B^\wedge_{t_\lambda}(S_\lambda)$. Then

$$\underline{A}_B\mu^\sharp(x) = \underline{A}_B\lambda^\sharp(x) = 1$$

and, thus,

$$\underline{A}_B(\mu\cap\lambda)^\sharp(x) \leq \underline{A}_B\mu^\sharp(x)\cap\underline{A}_B\lambda^\sharp(x) = 1.$$

Therefore, $\overline{X\cup Y}'_B \supseteq \overline{X}'_B\cup\overline{Y}'_B$.

(4) We have

$$\begin{aligned}
\underline{X\cap Y}'_B &= \underline{(G;S_{\mu\cup\lambda};\mu\cup\lambda)}'_B = (G;B_{t_{\mu\cup\lambda}-}(S_\mu\cap S_\lambda);\overline{A}_B(\mu\cup\lambda)^\sharp),\\
\underline{X}'_B\cap\underline{Y}'_B &= \underline{(G;S_\mu;\mu)}'_B\cap\underline{(G;S_\lambda;\lambda)}'_B\\
&= (G;B_{t_\mu-}(S_\mu);\overline{A}_B(\mu)^\sharp)\cap(G;B_{t_\lambda-}(S_\lambda);\overline{A}_B(\lambda)^\sharp)\\
&= (G;B_{t_\mu-}(S_\mu)\cap B_{t_\lambda-}(S_\lambda);\overline{A}_B(\mu)^\sharp\cup\overline{A}_B(\lambda)^\sharp).
\end{aligned}$$

Since $t_{\mu\cup\lambda} \geq t_\mu, t_\lambda$, by Theorem 6(5), we have $B_{t_{\mu\cup\lambda}-}(S_\mu\cap S_\lambda) \supseteq B_{t_\mu-}(S_\mu)\cap B_{t_\lambda-}(S_\lambda)$. Also, by Theorem 3(5), we have $\overline{A}_B(\mu\cup\lambda) = \overline{A}_B(\mu)\cup\overline{A}_B(\lambda)$. If $x\in B_{t_\mu-}(S_\mu)\cap B_{t_\lambda-}(S_\lambda)$ then

$$\overline{A}_B\mu^\sharp(x) = \overline{A}_B\mu(x), \overline{A}_B\lambda^\sharp(x) = \overline{A}_B\lambda(x)$$

and, thus,

$$\overline{A}_B\mu^\sharp(x)\cup\overline{A}_B\lambda^\sharp(x) = \overline{A}_B(\mu\cup\lambda)(x) \leq \overline{A}_B(\mu\cup\lambda)^\sharp(x).$$

If $x\notin B_{t_\mu-}(S_\mu)\cap B_{t_\lambda-}(S_\lambda)$, then $x\notin B_{t_{\mu\cup\lambda}-}(S_\mu\cap S_\lambda)$ and, thus, $\overline{A}_B(\mu\cup\lambda)^\sharp(x) = 1$. Therefore, $\overline{A}_B\mu^\sharp(x)\cup\overline{A}_B\lambda^\sharp(x) \leq \overline{A}_B(\mu\cup\lambda)^\sharp(x)$ and, thus, $\underline{X\cap Y}'_B \subseteq \underline{X}'_B\cap\underline{Y}'_B$.

(5) If $\mu\subseteq\lambda$, then by Corollary 1(1), $\overline{A}_B\mu\subseteq\overline{A}_B\lambda$. In the other hand, by Lemma 1, we have $S_\lambda\subseteq S_\mu$ and, thus, by Theorem 1(4), $B_{t_X-}(S_\lambda)\subseteq B_{t_X-}(S_\mu)$. Then $\underline{Y}'_B\subseteq\underline{X}'_B$.

(6) The proof is similar to part (5).

(7) Since $B\subseteq C$, by Theorem 3(9) we have $\overline{A}_B\mu\subseteq\overline{A}_C\mu$. Also, Theorem 1(8) gives $C_{t_\mu-}(S)\subseteq B_{t_\mu-}(S)$. The result follows.

(8) The proof is similar to part (7).

Conclusion

This paper has intended to build up a rational connection between rough set theory, fuzzy set theory and Cayley graphs. First, formal definitions for fuzzy Cayley sets and fuzzy Cayley graphs have been suggested.

Some illustrative examples have also been presented. Fuzzy Cayley graphs and related approximations might be received attentions in some distributed and networked systems challenges.

Acknowledgments

A large part of this work was done while the authors were working at Yazd university Iran. The work of the first author was partially supported by CMUP, which is financed by national funds through FCT – Fundao para a Cincia e a Tecnologia, I.P., under the project UIDB/00144/2020. The first author also acknowledges FCT support through a contract based on the Lei do Emprego Cientficoḱ (DL 57/2016).

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Дәл емес Кейли графтарындағы шамадан ауытқу

Шамамен алынған жиындар теориясы бұл жүйелерді дәл емес және анықталмаған модельдеу үшін лайықты әдіс. Шамамен алынған жиындар теориясы жиындар теориясының одан әрі жалпылауы болып табылатын дәл емес жиындар теориясымен толықтырылғанда, олар теориялық талқылауларда жарамды болады. Мақалада Кэйли графтарының анықтамасынан туындайтын дәл емес Кэйли ішкі жиындарының анықтамасы, демек группалардағы дәл емес ішкі жиындардың дәл емес Кэйли графтары ұсынылған. Авторлар Кэйли графының дәл емес нормаль ішкі группасына қатысты шамамен жуықтауды, сонымен қатар аппроксимацияланатын шамамен алынған дәл емес Кейли графтары және дәл емес шамамен алынған дәл емес Кейли графтарын енгізген. Соңғы жуықтау басқа жуықтаулардың бірігуі болып табылады. Кейбір теоремалар мен қасиеттері зерттелген және дәлелденген.

Кілт сөздер: анық емес жиын, шамамен алынған жиын, Кейли графы, анық емес Кейли графы, төменгі және жоғарғы жуықтаулар.

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Грубость в нечетких графах Кэли

Грубая теория множеств — заслуживающий внимания подход для неточного и неопределенного моделирования систем. Когда грубая теория множеств дополняется теорией нечетких множеств, причем обе являются дополнительным обобщением теории множеств, они будут иметь силу в теоретических дискуссиях. В настоящей статье предложено определение нечетких подмножеств Кэли и, следовательно, нечетких графов Кэли нечетких подмножеств на группах, вдохновленное определением графов Кэли. Авторами введены грубая аппроксимация графа Кэли относительно нечеткой нормальной подгруппы, а также аппроксимационные грубые нечеткие графы Кэли и нечеткие грубые нечеткие графы Кэли. Последнее приближение представляет собой смесь других приближений. Исследованы и доказаны некоторые теоремы и свойства.

Ключевые слова: нечеткое подмножество, грубое множество, граф Кэли, нечеткий граф Кэли, нижняя и верхняя аппроксимации.

DOI 10.31489/2023M4/119-129

UDC 531.31, 519.21

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Representing a second-order Ito equation as an equation with a given force structure

The problem of constructing equivalent equations with a given structure of forces by the given system of stochastic equations is considered. The equivalence of equations in the sense of almost surely is investigated. The paper determines the conditions under which a given system of second-order Ito stochastic differential equations is represented in the form of stochastic Lagrange equations with non-potential forces of a certain structure. Necessary and sufficient conditions for the representability of stochastic equations in the form of stochastic equations with non-potential forces admitting the Rayleigh function are obtained. The obtained results are illustrated by an example of motion of a symmetric satellite in a circular orbit, assuming a change in pitch under the action of gravitational and aerodynamic forces.

Keywords: Stochastic differential equation, stochastic Lagrange equation, stochastic equations with non-potential forces, equivalence almost surely.

Introduction. Problem statement

In [1], Yerugin constructed a set of ordinary differential equations (ODEs) possessing a given integral curve. This work became seminal in the theory of inverse problems of dynamics. At present, this theory is quite fully developed in the class of ODEs (see for instance [2–10]). In [2, 3], Galiullin presented a classification of the main types of inverse problems of dynamics and developed general methods for their solution in the class of ODEs. Inverse problems of dynamics for Ito stochastic differential equations were studied in [11–18].

In recent decades, the increased interest in the Helmholtz problem [19] has given a new impetus to the study of inverse problems for differential systems (for a literature review, see [20]). The solution of the Helmholtz problem in a wider class of differential equations makes it possible to extend the well-developed mathematical methods of classical mechanics to this class of equations. A special place, in terms of the variety of aspects in the study of the Helmholtz problem, is occupied by the works of Santilli [21, 22], which are devoted to the problem of representing second-order ODEs in the form of the Lagrange, Hamilton, and Birkhoff equations. In [23–26], methods for solving the Helmholtz problem are extended to the class of partial differential equations (PDEs). The Helmholtz problem is considered in [27–29] in a probabilistic formulation. We also note the works [21, 22, 26], which, in addition to the authors' own research, mainly in the class of ODEs and PDEs, present a historical review of literature on the development and generalization of the Helmholtz problem.

Given the second-order stochastic equation

$$d\dot{x}_\nu = F_\nu(x, \dot{x}, t)dt + \sigma_{\nu j}(x, \dot{x}, t)d_0\xi^j, \quad \nu = \overline{1, n},\ j = \overline{1, m}, \tag{1}$$

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it is required to construct the equivalent equations of the form

$$d\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k}dt = Q_k(x,\dot{x},t)dt + \sigma'_{kj}(x,\dot{x},t)d_0\xi^j, \quad k=\overline{1,n}, \tag{2}$$

with the given structure of the forces Q_k.

We assume that the functions included in the above equations have the smoothness necessary for further reasoning and satisfy the existence and uniqueness theorem for solutions of the Cauchy problems in the class of Ito stochastic differential equations [30]. In particular, we suppose the following holds for the vector function $F(z,t)$ and the matrix $\sigma(z,t)$ (here$z=(x^T,\dot{x}^T)^T$):

(i) $F(z,t)$ and $\sigma(z,t)$ are continuous in t and satisfy the Lipschitz condition in z, i.e.

$$\left\|\sigma(z',t)-\sigma(z'',t)\right\|^2 + \left\|F(z',t)-F(z'',t)\right\|^2 \le L(1+\left|z'-z''\right|^2 \text{ for all } z', z'' \in R^{2n};$$

(ii) the linear growth condition

$$\|\sigma(z,t)\|^2 + \|F(z,t)\|^2 \le L(1+|z|^2)$$

is met for all $z \in R^{2n}$.

Let (Ω, U, P) be a probability space with a flow $\{U_t\}$. Here $\{\xi^1(t), \xi^2(t), ..., \xi^m(t)\}$ is a system of Wiener processes with the unit matrix of local variances. The equivalence of solutions of equations (1) and (2) is understood in the sense of the following definition.

Definition 1. The equations

$$d\dot{y} = Y_1(y,\dot{y},t)dt + Y_2(y,\dot{y},t)d\xi \tag{a}$$

and

$$d\dot{z} = Z_1(z,\dot{z},t)dt + Z_2(z,\dot{z},t)d\xi \tag{b}$$

are said to be equivalent almost surely (a. s.) if $y(t_0)=z(t_0),\ \dot{y}(t_0)=\dot{z}(t_0)$ a.s. imply $y(t,t_0,y_0,\dot{y}_0) = z(t,t_0,z_0,\dot{z}_0),\ \dot{y}(t,t_0,y_0,\dot{y}_0)=\dot{z}(t,t_0,z_0,\dot{z}_0)$ a. s., for all $t \ge t_0$.

The problem of construction of equation (2) by the given equation (1) was considered in [31] in the case of the absence of random perturbations $\sigma_{\nu j} \equiv \sigma'_{\nu j} \equiv 0$. The case of the presence of random perturbations and $Q_k \equiv 0$ was studied in [32] by the method of additional variables.

Hereinafter, summation is assumed for the repeated indices of the factors. The indices i, k, and v run from 1 to n, and the index j runs from 1 to m.

In other words, the problem is stated as follows: for given $F_\nu, \sigma_{\nu j}$ it is required to determine the conditions on the functions L and $\sigma'_{\nu j}$, under which equation (2) is equivalent to equation (1) with the given structure of forces Q_k.

Case A. Let Q_k be arbitrary non-potential forces.

Theorem 1. Equation (1) is represented in the form of equation (2) with arbitrary non-potential forces if and only if

$$\frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} = \delta_k^\nu, \quad \text{where} \quad \delta_k^\nu = \begin{cases} 1, \nu = k \\ 0, \nu \ne k \end{cases}, \tag{3}$$

and

$$\sigma'_{kj}(x,\dot{x},t) = \sigma_{\nu j}(x,\dot{x},t). \tag{4}$$

Proof. By the Ito's rule of stochastic differentiation, we obtain

$$d\left(\frac{\partial L}{\partial \dot{x}_k}\right) = \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x^\nu}\dot{x}_\nu + \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu}F_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma_{ij}\sigma_{\nu j}\right]dt + \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu}\sigma_{\nu j}d_0\xi^j.$$

Since $F_\nu dt + \sigma_{\nu j} d_0\xi^j = d\dot{x}_\nu$, we have

$$d\left(\frac{\partial L}{\partial \dot{x}_k}\right) = \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} d\dot{x}_\nu + \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma_{ij}\sigma_{\nu j}\right] dt. \tag{5}$$

Hence, in view of (5), equation (2) takes the form

$$d\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k}dt - \sigma'_{kj}(x,\dot{x},t)d_0\xi^j \equiv$$

$$\equiv \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} d\dot{x}_\nu + \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma_{ij}\sigma_{\nu j} - \frac{\partial L}{\partial x_k} - Q_k\right] dt - \sigma'_{kj}(x,\dot{x},t)d_0\xi^j. \tag{6}$$

Then, taking into account (6) and the original equation (1), we have

$$\frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} d\dot{x}_\nu + \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma_{ij}\sigma_{\nu j} - \frac{\partial L}{\partial x_k} - Q_k\right] dt -$$

$$-\sigma'_{kj}(x,\dot{x},t)d_0\xi^j \equiv d\dot{x}_k - F_k(x,\dot{x},t)dt - \sigma_{kj}(x,\dot{x},t)d_0\xi^j. \tag{7}$$

The above equation implies the fulfillment of condition (3) of Theorem, which in turn implies

$$\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu} \equiv 0. \tag{8}$$

Equating the coefficients of dt and $d\xi^j$ in (7), on the basis of (8) we obtain the fulfillment of condition (4) of Theorem and the following expression

$$Q_k = \frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu - \frac{\partial L}{\partial x_k} + F_k(x,\dot{x},t). \tag{9}$$

Expression (9) determines the non-potential force Q_k. If instead of (2) we consider the equation

$$d(\frac{\partial L}{\partial \dot{x}_k}) - \frac{\partial L}{\partial x_k}dt = Q_k(x,\dot{x},t)dt + \sigma_{kj}(x,\dot{x},t)d_0\xi^j, \tag{$2'$}$$

we obtain the following corollary of Theorem 1.

Corollary 1. Equation (1) is represented in the form of equation ($2'$) with arbitrary non-potential forces if and only if condition (3) is met.

In particular, for $x \in R^1, \xi \in R^1$, conditions (3) and (4) for the transition from (1) to (2) take the form

$$\frac{\partial^2 L}{\partial \dot{x}^2} = 1, \quad \sigma' = \sigma,$$

and an arbitrary non-potential force is determined as

$$Q = \frac{\partial^2 L}{\partial \dot{x}\partial t} + \frac{\partial^2 L}{\partial \dot{x}\partial x}\dot{x} - \frac{\partial L}{\partial x} + F.$$

Case B. Let Q_k admit the generalized Rayleigh function $R(x,\dot{x})$, that is,

$$Q_k(x,\dot{x}) = -\frac{\partial R}{\partial x_k}. \tag{10}$$

Then equation (2) is represented in the form

$$d(\frac{\partial L}{\partial \dot{x}_k}) - \frac{\partial L}{\partial x_k}dt = -\frac{\partial R}{\partial x_k} + \sigma'_{kj}(x,\dot{x},t)d_0\xi^j. \tag{11}$$

Theorem 2. Equation (1) is represented in the form of equation (11) with non-potential forces admitting the Rayleigh function if and only if conditions (3), (4) and

$$\frac{\partial R}{\partial x_k} = \frac{\partial L}{\partial x_k} - \frac{\partial^2 L}{\partial \dot{x}_k \partial t} - \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu} \dot{x}_\nu - F_k(x, \dot{x}, t) \tag{12}$$

are met.

This theorem is proved in the same way as Theorem 1.

Theorem 2 implies the following statement for the equation

$$d(\frac{\partial L}{\partial \dot{x}_k}) - \frac{\partial L}{\partial x_k} dt = -\frac{\partial R}{\partial x_k} + \sigma_{kj}(x, \dot{x}, t) d_0 \xi^j. \tag{11$'$}$$

Corollary 2. Equation (1) is represented in the form of equation $(11^{'})$ with non-potential forces admitting the Rayleigh function if and only if conditions (3) and (12) are met.

In particular, for $x \in R^1, \xi \in R^1$, conditions (3), (4) and (12) for the transition from (1) to (11) take the forms

$$\frac{\partial^2 L}{\partial \dot{x}^2} = 1, \quad \sigma^{'} = \sigma, \quad \frac{\partial R}{\partial x} = \frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial \dot{x} \partial t} - \frac{\partial^2 L}{\partial \dot{x} \partial x} \dot{x} - F,$$

respectively.

We now extend the definition introduced by R.M. Santilli [21] to the class of Ito stochastic differential equations.

Definition 2. We say that equation

$$A_{\nu i}(x, \dot{x}, t) d\dot{x}^i + B_\nu(x, \dot{x}, t) dt = \sigma_{\nu j}(x, \dot{x}, t) d_0 \xi^j \tag{1$'$}$$

admits the analytic representation in the form of the Lagrange stochastic equation with a given structure of forces, if there exist n^2 functions $h_k^\nu(x, \dot{x}, t)$, $det(h_k^\nu) \neq 0$, such that the following identity holds:

$$d(\frac{\partial L}{\partial \dot{x}_k}) - \frac{\partial L}{\partial x_k} dt - Q_k dt - \sigma^{'}_{kj}(x, \dot{x}, t) d_0 \xi^j \equiv h_k^\nu (A_{\nu i} d\dot{x}_i + B_\nu dt - \sigma_{\nu j} d_0 \xi^j). \tag{13}$$

Let us consider the problem of analytic representation in the sense of Definition 2. In other words, given the functions $A_{\nu,j}, B_\nu, \sigma_{\nu,j}$ and the forces Q_k in equation $(2^{'})$, it is required to determine the conditions on the functions $h_k^\nu, L, \sigma^{'}_{\nu j}$, under which the relation (13) holds.

Теорема 3. For the indirect representation of the equation with arbitrary non-potential forces Q_k, the necessary and sufficient conditions are

$$\frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_i} = h_k^\nu A_{\nu i}, \tag{14}$$

$$\sigma^{'}_{kj} = h_k^\nu \sigma_{\nu j}. \tag{15}$$

Proof. We set $F_\nu^* = -A_{\nu i}^{-1} B_i$, $\sigma_{\nu j}^* = A_{\nu i}^{-1} \sigma_{ij}$. Then, we have

$$d(\frac{\partial L}{\partial \dot{x}_k}) = \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu} \dot{x}_\nu + \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} F_\nu^* + \frac{1}{2} \frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu} \sigma_{ij}^* \sigma_{\nu j}^* \right] dt + \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} \sigma_{\nu j}^* d_0 \xi^j.$$

Since $F_\nu^* dt + \sigma_{\nu j}^* d_0 \xi^j = d\dot{x}_\nu$, we obtain

$$d\left(\frac{\partial L}{\partial \dot{x}_k}\right) = \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} d\dot{x}_\nu + \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu} \dot{x}_\nu + \frac{1}{2} \frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu} \sigma_{ij}^* \sigma_{\nu j}^* \right] dt. \tag{16}$$

Hence, in view of (16), relation (13) takes the form

$$\frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} d\dot{x}_\nu + \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma^*_{ij}\sigma^*_{\nu j} - \frac{\partial L}{\partial x_k} - Q_k\right] dt -$$

$$-\sigma'_{kj} d_0\xi^j \equiv h^\nu_k \left(A_{\nu i}(x,\dot{x},t)d\dot{x}_i + B_\nu(x,\dot{x},t)dt - \sigma_{\nu j}(x,\dot{x},t)d_0\xi^j\right).$$

Equating the coefficients of $d\dot{x}_i$ and $d\xi^j_0$, we obtain relations (14) and (15) of Theorem 3. The coefficients of dt on the right- and left-hand sides of the equation are equal due to an arbitrary non-potential force Q_k of the form

$$Q_k = \frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma^*_{ij}\sigma^*_{\nu j} - \frac{\partial L}{\partial x_k} - h^\nu_k B_\nu.$$

The following statement holds in the case when non-potential forces admit the generalized Rayleigh function (10).

Theorem 4. Equation ($2'$) has an indirect representation in the form of the Lagrange equation with non-potential forces admitting the generalized Rayleigh function (10) if and only if conditions (14), (15) and

$$\frac{\partial R}{\partial x_k} = \frac{\partial L}{\partial x_k} - \frac{\partial^2 L}{\partial \dot{x}_k \partial t} - \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu - \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma^*_{ij}\sigma^*_{\nu j} + h^\nu_k B_\nu,$$

with $\sigma^*_{\nu j} = A^{-1}_{\nu i}\sigma_{ij}$, hold.

Proof. Let us apply Ito's rule of stochastic differentiation to the expression $d(\frac{\partial L}{\partial \dot{x}_k})$ and plug it into (13). Then (13) takes the following form:

$$\frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} d\dot{x}_\nu + \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma^*_{ij}\sigma^*_{\nu j} - \frac{\partial L}{\partial x_k} + \frac{\partial R}{\partial x_k}\right] dt -$$

$$-\sigma'_{kj} d_0\xi^j \equiv h^\nu_k \left(A_{\nu i}(x,\dot{x},t)d\dot{x}_i + B_\nu(x,\dot{x},t)dt - \sigma_{\nu j}(x,\dot{x},t)d_0\xi^j\right). \tag{17}$$

Equating the coefficients on the left- and right-hand sides, we obtain the fulfillment of conditions (14), (15) and (17) of Theorem 4.

In particular, for $x \in R^1$, $\sigma \in R^1$, conditions (14), (15) and (17) take the forms

$$\frac{\partial^2 L}{\partial \dot{x}^2} = hA, \quad \sigma' = h\sigma, \quad \frac{\partial R}{\partial x} = \frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial \dot{x}\partial t} - \frac{\partial^2 L}{\partial \dot{x}\partial x}\dot{x} - \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}^3}\sigma^2 + hB.$$

If the desired Lagrangian is sought, following R.M. Santilli [21], in the form

$$L = K(x,\dot{x},t) + D_\mu(x,t)\dot{x}_\mu + C(x,t), \tag{18}$$

then we obtain the following statement in terms of functions K, D^μ and C.

Theorem 5. Equation ($2'$) has an indirect representation in the form of the Lagrange equation with non-potential forces, admitting the generalized Rayleigh function (10), and the Lagrangian of the form (18) if and only if conditions (15) and

$$\left\{\begin{array}{l} \dfrac{\partial^2 K}{\partial \dot{x}_k \partial \dot{x}_i} = h^\nu_k A_{\nu i}, \quad \dfrac{\partial R}{\partial x_k} = \left(\dfrac{\partial K}{\partial x_k} + \dfrac{\partial C}{\partial x_k}\right) - \dfrac{\partial D_k}{\partial t} - \\ -\dfrac{\partial^2 K}{\partial \dot{x}_k \partial t} - \dfrac{\partial^2 K}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu - \dfrac{1}{2}\dfrac{\partial^3 K}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma^*_{ij}\sigma^*_{\nu j} + h^\nu_k B_\nu \end{array}\right.$$

are fulfilled.

Proof follows from Theorem 4 and the relations

$$\begin{cases} \dfrac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_i} = \dfrac{\partial^2 K}{\partial \dot{x}_k \partial \dot{x}_i}, \quad \dfrac{\partial^2 L}{\partial \dot{x}_k \partial t} = \dfrac{\partial^2 K}{\partial \dot{x}_k \partial t} + \dfrac{\partial D_k}{\partial t}, \\ \dfrac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu} = \dfrac{\partial^2 K}{\partial \dot{x}_k \partial x_\nu} + \dfrac{\partial D_k}{\partial x_\nu}, \quad \dfrac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu} = \dfrac{\partial^3 K}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}, \\ \dfrac{\partial L}{\partial x_k} = \dfrac{\partial K}{\partial x_k} + \dfrac{\partial D_\mu}{\partial x_k}\dot{x}_\mu + \dfrac{\partial C}{\partial x_k}. \end{cases}$$

Example

Let us consider the planar motion of a symmetric satellite in a circular orbit, assuming a change in pitch under the action of gravitational and aerodynamic forces [33]

$$\ddot{\theta} = f(\theta, \dot{\theta}) + \sigma(\theta, \dot{\theta})\dot{\xi}_0, \tag{19}$$

where θ is the pitch angle and

$$f = Ml \sin 2\theta - M[g(\theta) + \eta\dot{\theta}], \sigma = M\delta[g(\theta) + \eta\dot{\theta}]. \tag{20}$$

Case A. Let Q be arbitrary non-potential forces.

A(i). Assume that the Lagrange function is of the form $L = \dfrac{1}{2}\dot{\theta}^2$. Then equation (2) takes the form $\ddot{\theta} = Q + \sigma'\xi_0$. Hence, by Theorem 1, condition (19) for the representation in the form (2) with the given L is written as $\sigma' = \sigma$ for $Q = f$.

A(ii). Let $L = \dfrac{1}{2}\dot{\theta}^2 + \alpha(\theta)\dot{\theta}) + \beta(\theta)$. Then equation (2) takes the form $\ddot{\theta} - \beta_\theta = Q + \sigma'\xi_0$. Taking into account the form (20) of the function f, we determine $\beta_\theta = Ml \sin 2\theta - Mg(\theta)$, or $\beta = -\dfrac{1}{2}Ml\cos 2\theta - MG(\theta)$, where $G = \int g(\theta)d\theta$. Let us now assume $\sigma' = \sigma = M\delta[g(\theta) + \eta\dot{\theta}]$. Then, by Theorem 1, we conclude that the Lagrangian $L = \dfrac{1}{2}\dot{\theta}^2 - M[\dfrac{1}{2}l\cos 2\theta + G(\theta)]$ for $Q = -Mg\dot{\theta}$ provides the representation of equation (19) in the form (2).

Case B. Let Q admit the generalized Rayleigh function R (10).

B(i). By Theorem 2, for $L = \dfrac{1}{2}\dot{\theta}^2$ the function R takes the form $R = -[N(\theta)\dot{\theta} + \dfrac{1}{2}H\dot{\theta}^2]$, where $N(\theta) = Ml\sin 2\theta - Mg(\theta)$ and $H = -M\eta$. Hence, (24) is represented in the form (11$'$).

B(ii). For $L = \dfrac{1}{2}\dot{\theta}^2 + \alpha(\theta)\dot{\theta}) + \beta(\theta)$, as in the case A(ii), we determine β and, by Theorem 2, conclude that for $R = \dfrac{1}{2}M\eta\dot{\theta}^2$ and $L = \dfrac{1}{2}\dot{\theta}^2 - M[\dfrac{1}{2}l\cos 2\theta + G(\theta)]$ equation (19) can be represented in the form (11$'$).

Acknowledgments

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19677693).

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Екінші ретті Ито теңдеуін берілген құрылымы бар күштердің теңдеуі түрінде құру

Берілген стохастикалық теңдеулер жүйесінен берілген құрылымы бар күштердің эквивалентті теңдеулерді құру есебі қарастырылған. Теңдеулердің эквиваленттілігі шамамен ықтимал мағынада зерттеледі. Екінші ретті Ито стохастикалық дифференциалдық теңдеулер жүйесі белгілі бір құрылымның потенциалды емес күштері бар стохастикалық Лагранж теңдеулері ретінде ұсынылу шарттары анықталған. Стохастикалық теңдеулердің Рэлей функциясын қабылдайтын потенциалды емес күштері бар стохастикалық теңдеулер түріндегі бейнеленуінің қажетті және жеткілікті шарттары алынды. Зерттеу нәтижелері ауырлық күші мен аэродинамикалық күштердің әсерінен тангаждық өзгерістерге ұшыраған дөңгелек орбитадағы симметриялық жерсеріктің қозғалысының мысалында көрсетілген.

Кілт сөздер: стохастикалық дифференциалдық теңдеу, стохастикалық Лагранж теңдеуі, потенциалдық емес күштері бар стохастикалық теңдеулер, шамамен ықтимал эквиваленттілік.

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Представление уравнения Ито второго порядка в виде уравнения с заданной структурой сил

Рассмотрена задача построения по заданной системе стохастических уравнений, эквивалентных уравнений с заданной структурой сил. Исследована эквивалентность уравнений в смысле почти наверное. Определены условия, при которых заданная система стохастических дифференциальных уравнений Ито второго порядка представима в виде стохастических уравнений Лагранжа с непотенциальными силами определенной структуры. Получены необходимые и достаточные условия представимости стохастических уравнений в виде стохастических уравнений с непотенциальными силами, допускающими функцию Рэлея. Результаты исследования проиллюстрированы на примере движения симметричного спутника по круговой орбите в предположении изменения тангажа под действием аэродинамических сил и тяготения.

Ключевые слова: стохастическое дифференциальное уравнение, стохастическое уравнение Лагранжа, стохастические уравнения с непотенциальными силами, эквивалентность почти наверное.

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DOI 10.31489/2023M4/130-143
UDC 510.67

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Similarities of Jonsson spectra's classes

The study of syntactic and semantic properties of a first-order language, generally speaking, for incomplete theories, is one of the urgent problems of mathematical logic. In this article we study Jonsson theories, which are satisfied by most classical examples from algebra and which, generally speaking, are not complete. A new and relevant method for studying Jonson theories is to study these theories using the concepts of syntactic and semantic similarities. The most invariant concept is the concept of syntactic similarity of theories, because it preserves all the properties of the theories under consideration. The main result of this article is the fact that any perfect Jonson theory which are complete for existential sentences, is syntactically similar to some polygon theory (S-polygon, where S is a monoid). This result extends to the corresponding classes of Jonsson theories from the Jonsson spectrum of an arbitrary model of an arbitrary signature.

Keywords: Jonsson theory, semantic model, perfectness, cosemanticness, S-act, Jonsson spectrum, syntactic and semantic similarities.

Introduction

In the work [1], was proved the fact that any complete theory is similar in some sense to a certain polygon theory (S-act). Moreover, in that work [1] two types of similarity were precisely defined: syntactic and semantic similarities. The value of this result speaks about the universality in the sense of such an algebra as a polygon (S-act). The subject of studying various model-theoretic properties of polygons (S-act) is sufficiently completely studied in [2, 3]. Considering these properties in itself imagines certain essential task. The considering of these properties in itself imagines certaining essential task.

In this article, we want to show that the fact proved in [1] is also true in the class of Jonsson theories, which, generally speaking, are not complete. On the other hand, the class of Jonsson theories includes in itself such basic classical examples from algebra, such as groups, Abelian groups, modules, fields of fixed characteristic, linear orders, Boolean algebras, various classes of lattices and polygons (S-act). Thus, it becomes clear that the class of Jonsson theories is a fairly wide class of theories and the study of their theoretical-model properties is an interesting and relevant task.

In the well-known monograph by J. Barwise «Handbook of mathematical logic» the specialist in logic H.J. Keisler in the review article «Fundamentals of model theory» conditionally divided the content of model theory into two main priorities: «western» and «eastern» model theory [4]. But at the same time, he emphasizes the unity and integrity of these priorities in the framework of the development of the general model theory.

These names are not accidental and are associated with the geographical place of residence of the founders of model theory in North America. Namely, Alfred Tarski and Abraham Robinson lived respectively on the western and eastern coasts of the United States. The tasks that determined these directions differed from each other in two fundamental ways. The first point related to the syntax is that the theories that A. Tarski's school dealt with were complete theories. The followers of A. Robinson

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were engaged in theories with a prenix length of not more than two and, as a rule, Jonsson theories. The second point is related to semantics, more exactly that are regards restrictions of morphisms between models and kinds of models.

In the «western» way actually one has dealt with complete theories, where elementary morphisms were considered. In the case of Jonsson theories logicians dealt with isomorphic embeddings and homomorphisms. Also, in connection with the semantic aspect, it should be noted that in the «eastern» version of model theory, logicians deal mainly with the class of existentially closed models of some fixed inductive theory. The difference in the development of these two directions at the moment of the state of model theory is such that the technique for studying complete theories is much more developed and multilateral. The main stages of development and differences in these directions can be found in the following works [5–25].

One of the methods for studying Jonsson theories is the method of transfer of first-order properties, which is semantic. A first-order property is called semantic if it is invariant with respect to the semantic similarity of Jonsson theories. Thus, when researching two Jonsson theories using the transfer method, the object under study will be a preimage, and the known object will be the image of some mapping that will play the role of a syntactic similarity of these two Jonsson theories. The object under study is unknown and we will be interested in those first-order properties that are formulaic and are preserved under syntactic similarity.

1 Basic concepts and results concerning Jonsson theories

We give the following necessary definitions concerning Jonsson theories and their semantic models.

Definition 1. [4] A theory T is called Jonsson if:
1) the theory T has an infinite model;
2) the theory T is inductive;
3) the theory T has the joint embedding property (JEP);
4) the theory T has the amalgamation property (AP).

Definition 2. [26] Let $\kappa \geq \omega$. Model $\mathcal{M}$ of theory T is called:
- κ-universal for T, if each model of theory T with the power strictly less κ isomorphically imbedded in $\mathcal{M}$;
- κ-homogeneous for T, if for any two models $\mathcal{A}$ and $\mathcal{A}_1$ of theory T, which are submodels of $\mathcal{M}$ with the power strictly less then κ and for isomorphism $f : \mathcal{A} \to \mathcal{A}_1$ for each extension $\mathcal{B}$ of model $\mathcal{A}$, which is a submodel of $\mathcal{M}$ and is model of T with the power strictly less then κ there exists the extension $\mathcal{B}_1$ of model $\mathcal{A}_1$, which is a submodel of $\mathcal{M}$ and an isomorphism $g : \mathcal{B} \to \mathcal{B}_1$ which extends f.

Definition 3. [26] Model $\mathcal{C}$ of Jonsson theory T is called semantic model, if it is ω^+-homogeneous-universal.

Definition 4. [26] The center of Jonsson theory T is called an elementary theory of its semantic model $\mathcal{C}$ and denoted through T^*, i.e. $T^* = \mathrm{Th}(\mathcal{C})$.

Definition 5. [27] Jonsson theory T is called a perfect theory, if each a semantic model of theory T is saturated model of T^*.

The criterion for the perfectness of the Jonsson theory was obtained by Yeshkeyev A.R. and it is as follows:

Theorem 1. [27] For any Jonsson theory T following conditions are equivalent:
1) T is perfect;
2) T^* is the model companion.

The following Definitions 6–8 were taken from [28], where generalized Jonsson theories were defined.

Definition 6. [28] Let $\Gamma \subset L$. Then:

1) notation $T \in \Gamma C_{\Delta}$ means, that $T \cap \Gamma \vdash \varphi$ for all $\varphi \in T$;

2) if $B \subseteq |\mathcal{A}|$, then $\mathrm{Th}_{\Gamma}(\mathcal{A}, B)$ denotes the set of all Γ-sentences of the language L_B, true in $\mathcal{A}$;

3) mapping $f : \mathcal{A} \to \mathcal{B}$ is said to be Γ-embedding, if for any $\overline{a} \in \mathcal{A}$ and $\varphi(\overline{x}) \in \Gamma$ from $\mathcal{A} \models \varphi(\overline{a})$ follows $\mathcal{B} \models \varphi(f(\overline{a}))$;

4) if $\mathcal{A} \subseteq \mathcal{B}$, then notation $\mathcal{A} \subseteq_{\Gamma} \mathcal{B}$ signify, that $\mathrm{Th}_{\Gamma}(\mathcal{A}, |\mathcal{A}|) \subseteq \mathrm{Th}_{\Gamma}(\mathcal{B}, |\mathcal{A}|)$;

5) sequence of models $\mathcal{A}_i$, $i < \beta$ called Γ-chain, if $\mathcal{A}_i \subseteq_{\Gamma} \mathcal{A}_j$, where $i < j < \beta$.

Definition 7. [28]

1) The theory T is persistent with respect to the union of Π_{α}-chains (or is α-inductive) if the union of any Π_{α}-chains of models of T is an again model of T.

2) The theory T has the α-joint embedding property (α-JEP), if for any $\mathcal{A}, \mathcal{B} \models T$ there is $\mathcal{M} \models T$ and Π_{α}-embeddings $f : \mathcal{A} \to \mathcal{M}$ and $g : \mathcal{B} \to \mathcal{M}$.

3) The theory T has the α-amalgamation property (α-AP) if for any $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \models T$ and Π_{α}-embeddings $f_1 : \mathcal{A} \to \mathcal{B}_1$ and $f_2 : \mathcal{A} \to \mathcal{B}_2$ there is $\mathcal{M} \models T$ and Π_{α}-embeddings $g_1 : \mathcal{B}_1 \to \mathcal{M}$ and $g_2 : \mathcal{B}_2 \to \mathcal{M}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

The following definition gives us generalized Jonsson theories or α-Jonsson theories.

Definition 8. [28] A theory T is called α-Jonsson ($0 \leq \alpha \leq \omega$) if:

1) the theory T has an infinite model;

2) the theory T is α-inductive;

3) the theory T has α-JEP;

4) the theory T has α-AP.

If compare Definitions 1 and 8, then can notice, that they differ with precision to α. At that in Definition 8 for $\alpha = 0$ we have Jonsson theories, and for $\alpha = \omega$ we have complete Jonsson theories. Further, when we work with 0-Jonsson theories, we will omit 0. Note that from Definition 1 it follows that Jonsson theories, generally speaking, are not complete.

Mustafin T.G. the following useful suggestions were proved in [28]: Proposition 1 and Proposition 2 actually give for us syntactic equivalents of α-JEP and α-AP notions.

Proposition 1. [28] The following conditions are equivalent:

1) T has α-JEP;

2) T has α-JEP for countable models;

3) if $\overline{x} \cap \overline{y} = \oslash$, $p(\overline{x})$ and $q(\overline{y})$ are arbitrary sets of $\Sigma_{\alpha+1}$-formulas, such that $T \cup p(\overline{x})$ and $T \cup q(\overline{y})$ are consistent, then $T \cup p(\overline{x}) \cup q(\overline{y})$ is consistent.

Proposition 2. [28] The following conditions are equivalent:

1) T has α-AP;

2) T has α-AP for countable models;

3) if $p(\overline{x})$ and $q(\overline{x})$ are such sets of $\Sigma_{\alpha+1}$-formulas, that $T \cup p(\overline{x})$, $T \cup q(\overline{x})$, $T \cup \{\neg\varphi(\overline{x}) : \varphi(\overline{x}) \in \Sigma_{\alpha+1}, \varphi(\overline{x}) \notin p(\overline{x}) \cap q(\overline{x})\}$ are consistent sets, then the set $T \cup p(\overline{x}) \cup q(\overline{x})$ is consistent.

4) for any $\mathcal{A} \models T$ and $\overline{a} \in \mathcal{A}$ set $\mathrm{Th}_{\Sigma_{\alpha+1}}(\mathcal{A}, \overline{a})$ it is contained in a unique maximal consistent with T the set $\Sigma_{\alpha+1}$-sentences of the language $L(\overline{a})$.

2 *The concepts of syntactic and semantic similarities of complete theories*

The notion of similarity between two complete theories was introduced in [1]. For Jonsson theories the similarity between two Jonsson theories was introduced in [27]. In both works were obtained some results which described syntactic and semantic similarity in both cases. We give a list of the necessary definitions of concepts and their necessary model-theoretical properties.

The following definition belongs to T.G. Mustafin [1].

Let $F_n(T)$, $n < \omega$ be the Boolean algebra of formulas of T with exactly n free variables $v_1, \ldots, v_n$ and $F(T) = \bigcup_n F_n(T)$.

Definition 9. [1] Complete theories T_1 and T_2 are syntactically similar if and only if there exists a bijection $f : F(T_1) \to F(T_2)$ such that

1) $f \restriction F_n(T_1)$ is an isomorphism of the Boolean algebras $F_n(T_1)$ and $F_n(T_2)$, $n < \omega$;
2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in F_{n+1}(T)$, $n < \omega$;
3) $f(v_1 = v_2) = (v_1 = v_2)$.

The following example of syntactic similarity of complete theories was given in [1].

Example 1. The following theories T_1 and T_2 of the signature $\sigma = \langle \varphi, \psi \rangle$ are syntactically similar, where φ, ψ are binary functions:

$$T_1 = \mathrm{Th}(\langle Z; +, \cdot \rangle), \quad T_2 = \mathrm{Th}(\langle Z; \cdot, + \rangle).$$

Definition 10. [1]

1) $\langle A, \Gamma, \mathcal{M} \rangle$ is called the pure triple, where A is not empty, Γ is the permutation group of A and $\mathcal{M}$ is the family of subsets of A such that from $M \in \mathcal{M}$ follows that $g(M) \in \mathcal{M}$ for every $g \in \Gamma$.

2) If $\langle A_1, \Gamma_1, \mathcal{M}_1 \rangle$ and $\langle A_2, \Gamma_2, \mathcal{M}_2 \rangle$ are pure triples and $\psi : A_1 \to A_2$ is a bijection then ψ is an isomorphism if:

(i) $\Gamma_2 = \{\psi g \psi^{-1} : g \in \Gamma_1\}$;
(ii) $\mathcal{M}_2 = \{\psi(E) : E \in \mathcal{M}_1\}$.

Definition 11. [1] The pure triple $\langle C, Aut(\mathcal{C}), Sub(C) \rangle$ is called the semantic triple of complete theory T, where C is carrier of Monster model $\mathcal{C}$ of theory T, $Aut(\mathcal{C})$ is the automorphism group of $\mathcal{C}$, $Sub(C)$ is a class of all subsets of C each of which is a carrier of the corresponding elementary submodel of $\mathcal{C}$.

Definition 12. [1] Complete theories T_1 and T_2 are semantically similar if and only if their semantic triples are isomorphic.

The following example of the semantic similarity of complete theories was given in [1].

Example 2. The following theories T_1 and T_2 are semantically similar, where

$$\begin{gathered} T_1 = Th(\langle \mathcal{M}_1; P_n, n < \omega; a_{nm}, n, m < \omega \rangle), \\ \mathcal{M}_1 = \{a_{nm} : n, m < \omega\}, \\ P_n(\mathcal{M}_1) = \{a_{nm} : m < \omega\}, \end{gathered}$$

and

$$\begin{gathered} T_2 = Th(\langle \mathcal{M}_2; Q_n, n < \omega; Q_{nm}, n, m < \omega; b_{nmk}, n, m, k < \omega \rangle), \\ \mathcal{M}_2 = \{b_{nmk} : n, m, k < \omega\}, \\ Q_n(\mathcal{M}_2) = \{b_{nmk} : m, k < \omega\}, \\ Q_{nm}(\mathcal{M}_2) = \{b_{nmk} : k < \omega\}. \end{gathered}$$

It turned out that the above types of similarity are not equivalent to each other.

Proposition 3. [1] If T_1 and T_2 are syntactically similar, then T_1 and T_2 semantically similar. The converse implication fails.

Let us recall the definition of semantic property.

Definition 13. [1] A property (or a notion) of theories (or models, or elements of models) is called semantic if and only if it is invariant relative to semantic similarity.

For example from [1] it is known that:

Proposition 4. The following properties and notions are semantic:
(1) type;
(2) forking;
(3) λ-stability;
(4) Lascar rank;
(5) Strong type;
(6) Morley sequence;
(7) Orthogonality, regularity of types;
(8) $I(\aleph_\alpha, T)$ – the spectrum function.

In English literature the term polygon over a monoid S usually uses the term S-acts [2, 3, 29, 30]. In this article we follow the terminology of Professor T.G. Mustafin, who first defined and formulated model-theoretical concepts and issues related to polygons topics [26, 31, 32].

Definition 14. [1] By a polygon over a monoid S (or we called as S-acts) we mean a structure with only unary functions $\langle A; f_\alpha : \alpha \in S\rangle$ such that:
1) $f_e(a)$ $\forall a \in A$, where e is the unit of S;
2) $f_{\alpha\beta}(a) = f_\alpha(f_\beta(a))$ $\forall \alpha, \beta \in S$, $\forall a \in A$.

The following results (Theorems 2, 3) show that any complete theory has some syntactic similar theory.

Theorem 2. [1] For every theory T_2 in a finite signature there is a theory T_1 of polygons such that some inessential extension of T_1 is an almost envelope of T_2.

Theorem 3. [1] For every theory T_2 in an infinite signature there is a theory T_1 of polygons such that some inessential extension of T_1 is an envelope of T_2.

3 The concepts of syntactic and semantic similarities of Jonsson theories. Main results

The following definition was introduced in the frame of Jonsson theories study by first author of this current article.

Let T be an arbitrary Jonsson theory, then $E(T) = \bigcup_{n<\omega} E_n(T)$, where $E_n(T)$ is a lattice of $\exists$-formulas with n free variables, T^* is a center of Jonsson theory T, i.e. $T^* = Th(\mathcal{C})$, where $\mathcal{C}$ is semantic model of Jonsson theory T in the sense of [26].

Definition 15. [27] Let T_1 and T_2 are arbitrary Jonsson theories. We say that T_1 and T_2 are Jonsson syntactically similar if exists a bijection $f : E(T_1) \longrightarrow E(T_2)$ such that:
1) restriction f to $E_n(T_1)$ is isomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;
2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;
3) $f(v_1 = v_2) = (v_1 = v_2)$.

We would like to give some examples of syntactic similarity of certain algebraic examples. For this, we recall the basic definitions associated with these examples following denotions from B. Poizat [33].

A Boolean ring is an associative ring with identity, in which $x^2 = x$ for any x is called a Boolean ring; we then have $(x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$, but $(x + y)^2 = x + y$; from which it follows that $xy + yx = 0$ for any x and y. Then $x^2 + x^2 = 0$, and hence $x + x = 0$, for every x, so $x = -x$; a Boolean ring therefore has characteristic 2, and since $xy = -yx = yx$, it is commutative.

To axiomatize this concept, we introduce the language consisting of two constant symbols 0 and 1 and two binary operations + and.

We write down some universal axioms, expressing, that A is the Boolean ring, without forgetting thus $0 \neq 1$. In a Boolean ring we define two binary operations $\wedge$ and $\vee$, and one unary operation $\neg$, in the following way: $x \wedge y = x \cdot y$; $x \vee y = x + y + xy$; $\neg x = 1 + x$.

The reader can check that the following properties are true for all x,y, z:
– (de Morgan's laws or duality laws): $\neg(\neg x) = x$, $\neg(x \wedge y) = \neg x \vee \neg y, \neg(x \vee y) = \neg x \wedge \neg y$;
– (associativity of $\wedge$): $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
– (associativity of $\vee$): $(x \vee y) \vee z = x \vee (y \vee z)$;
– (distributivity of $\wedge$ over $\vee$): $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
– (distributivity of $\vee$ over $\wedge$): $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;
– (commutativity of $\wedge$ and $\vee$): $x \wedge y = y \wedge x$, $x \vee y = y \vee x$;
– $x \wedge \neg x = 0,\ x \vee \neg x = 1$;
– $x \wedge 0 = 0,\ x \vee 0 = x,\ x \wedge 1 = x,\ x \vee 1 = 1$;
– $0 \neq 1, \neg 0 = 1, \neg 1 = 0$.

A structure in the language $(0, 1, \neg, \wedge, \vee)$ that satisfies these universal axioms is called a Boolean algebra.

Boolean algebras and Boolean rings defined in this way are examples of Jonsson theories that are syntactically similar in the sense of definition [29], as a consequence of the following fact:

Fact 1. [33] In each Boolean ring one can interpret a certain Boolean algebra.

It is easy to see that interpretation is a special case of syntactic similarity.

Proof. With the Boolean ring A we have connected some Boolean algebra $b(A)$; the converse is also true: $x \cdot y = x \wedge y$, $x + y = (x \vee y) \wedge (\neg x \vee \neg y)$, then we receive the Boolean ring $a(B)$; and besides $a(b(A)) = A$, $b(a(B)) = B$. Thus we see, that up to a language, the Boolean ring and Boolean algebras have the same structures, the Boolean ring canonically is transformed into a Boolean algebra and vice versa, transformations in both directions are carried out using quantifier free formulas.

As in the case of complete theories (Definition 12), we can define a semantic similarity between two Jonsson theories.

Definition 16. [27] The pure triple $\langle C, Aut(\mathcal{C}), Sub(C)\rangle$ is called the Jonsson semantic triple, where C is carrier of semantic model $\mathcal{C}$ of theory T, $Aut(\mathcal{C})$ is the automorphism group of $\mathcal{C}$, $Sub(C)$ is a class of all subsets of C which are carriers of the corresponding existentially closed submodels of $\mathcal{C}$.

Definition 17. [27] Two Jonsson theories T_1 and T_2 are called Jonsson semantically similar if their Jonsson semantic triples are isomorphic as pure triples.

The correctness of this definition follows from the fact that the perfect Jonsson theory has a unique semantic model up to isomorphism. Otherwise, all semantic models are only elementary equivalent to each other.

For the convenience of further exposition we introduce the following notation. The syntactic and semantic similarities of the complete theories T_1 and T_2 will be denoted $T_1 \overset{S}{\bowtie} T_2$ and $T_1 \underset{S}{\bowtie} T_2$ respectively. In the case when we consider Jonsson theories T_1 and T_2, through $T_1 \overset{S}{\rtimes} T_2$ will be denote the Jonsson syntactic similarity of theories T_1 and T_2, and through $T_1 \underset{S}{\rtimes} T_2$ Jonsson semantic similarity of theories T_1 and T_2.

Theorem 4. [27] Let T_1 and T_2 are $\exists$-complete perfect Jonsson theories, then following conditions are equivalent:

1) $T_1 \overset{S}{\rtimes} T_2$;

2) $T_1^* \overset{S}{\bowtie} T_2^*$.

The following lemma is a Jonsson analogue of Proposition 3.

Lemma 1. If two perfect $\exists$-complete Jonsson theories are Jonsson syntactically similar, then they are Jonsson semantically similar. The converse is, generally speaking, not true.

Proof. Follows from Theorem 4 and Proposition 3.

The following technical lemma is necessary to prove Proposition 5.

Lemma 2. Let T be $\exists$-complete theory and $T \subseteq T'$. Then if $p(\overline{x}) \cup T$ consistent, then $p(\overline{x}) \cup T'$ is also consistent ($p(\overline{x})$ is an arbitrary set $\exists$-formulas).

Proof. It is easy to show that T' will also be $\exists$ -complete, since $T \subseteq T'$.

Proposition 5. Let T be a perfect Jonsson theory, then for every sentence $\varphi \in T^* \backslash T$ the theory $T' = T \cup \{\varphi\}$ is a Jonsson.

Proof. Let us verify the fulfillment of all the conditions for the definition of the Jonsson theory. As T is a perfect Jonsson theory, then T^* is a Jonsson theory. Since $T \subset T' \subset T^*$, then T' is $\forall\exists$-axiomatizable and T' has an infinite model. From Lemma 2 and the syntactic definition of α-JEP (Proposition 1 for $\alpha = 0$) it is easy to see that T' has JEP.

Let us verify the fulfillment of condition 4) of Definition 1. Let $p(\overline{x}) \cup T'$, $q(\overline{x}) \cup T'$, $r(\overline{x}) \cup T'$ are consistent, where $p(\overline{x})$, $q(\overline{x})$, $r(\overline{x})$ the same as in Proposition 2 for $\alpha = 0$. Without loss of generality, we can consider that $\overline{x} = x$. Then by the previous lemma $p(\overline{x}) \cup T^*$ and $q(\overline{x}) \cup T^*$ are consistent. Let $h(x) = \{\varphi(x) : \varphi(x)|\varphi(x) \text{ is existential sentence}, \forall x\varphi(x) \in T^*\}$, $p'(x) = p(x) \cup h(x)$, $q'(x) = q(x) \cup h(x)$. It's obvious that $p'(x) \cup T^*$, $q'(x) \cup T^*$ are consistent. Let $r'(x) = \{\neg\varphi(x) : \varphi(x)|\varphi(x) \text{ is existential sentence}, \varphi(x) \in p'(x) \cap q'(x)\}$. We show that $r'(x) \cup T^*$ is consistent. Suppose the opposite, let $r'(x) \cup T^*$ be inconsistent, then exists $\varphi(x) \in r'(x)$ such that $\varphi(x) \cup T^*$ is inconsistent. Means, $\exists x\varphi(x) \cup T^*$ is inconsistent, then $\forall x\neg\varphi(x) \in T^*$ and $\neg\varphi(x) \in h(x)$. Consequently $\neg\varphi(x) \in p'(x) \cap q'(x)$. Got a contradiction. Thus $r'(x) \cup T^*$ is consistent. We have that $p'(x) \cup T^*$, $q'(x) \cup T^*$, $r'(x) \cup T^*$ are consistent. By virtue of the fact that theory T is Jonsson theory, we obtain, that $p'(x) \cup q'(x) \cup T^*$ is consistent, which means that, $p(x) \cup q(x) \cup T^*$ is also consistent. As $T' \subseteq T^*$ then and $p(x) \cup q(x) \cup T'$ is consistent. So, T' has AP. Thus T' is Jonsson theory.

The following definition was introduced by T.G. Mustafin.

Definition 18. We say that the Jonsson theory T_1 is cosemantic to the Jonsson theory T_2 ($T_1 \bowtie T_2$) if $\mathcal{C}_{T_1} = \mathcal{C}_{T_2}$, where $\mathcal{C}_{T_i}$ are semantic model of T_i, $i = 1, 2$.

This definition easily implies the following lemma.

Lemma 3. Any two cosemantic Jonsson theories are Jonsson semantically similar.

The proof follows from the definition.

Let $\mathcal{A}$ be an arbitrary model of countable language. The set $JSp(\mathcal{A}) = \{T/T$ is Jonsson theory in this language and $\mathcal{A} \in \mathrm{Mod}(T)\}$ is said to be the Jonsson spectrum of the model $\mathcal{A}$.

The relation of cosemanticness on a set of theories is an equivalence relation. Then $JSp(\mathcal{A})/_{\bowtie}$ is the factor set of the Jonsson spectrum of the model $\mathcal{A}$ with respect to $\bowtie$.

The concept of the Jonsson spectrum was introduced by the first author of this article in [7]. It is turned out that this notion useful in the following sense. Using the concept of $JSp(A)/_{\bowtie}$ in [7,8], cosemanticity criteria for Abelian groups and R-modules are obtained that refine the well-known theorems on elementary equivalence of Abelian groups [34] and R-modules [35].

We have the following result.

Theorem 5. For any Jonsson perfect $\exists$-complete theory T there is a Jonsson $\exists$-complete theory of the polygon T'_{Π} such that $T \overset{S}{\rtimes} T'_{\Pi}$.

Proof. Let T be perfect $\exists$-complete Jonsson theory. Since T^* is complete, according to Theorem 2 in the case of a finite signature and Theorem 3 in the case of an infinite signature, there is a complete theory of the polygon T_{Π} such that $T^* \overset{S}{\bowtie} T_{\Pi}$. But then, according to Proposition 3, it follows that $T^* \underset{S}{\bowtie} T_{\Pi}$. Since the concept of type is a semantic notion (Proposition 4), the concept of a formula is also semantic. It follows from Propositions 1 and 2 with $\alpha = 0$ that the properties of JEP and AP are equivalent to the consistently of some formulas, i.e. JEP and AP are semantic concepts. It is clear that $\forall\exists$ -axiomatizability is also a semantic property, since all axioms are true in the semantic model.

This means that the property "to be a Jonsson theory" is a semantic concept, and therefore T_Π is also a Jonsson theory.

Since T^* is a perfect Jonsson theory, then semantic model $\mathcal{C}_T$ of theory T is saturated. But $T^* \underset{S}{\bowtie} T_\Pi$ and, by definition, the semantic triples of these theories are isomorphic to each other, then $\mathcal{C}_T \cong \mathcal{C}_{T_\Pi}$, therefore $\mathcal{C}_{T_\Pi}$ is also saturated and therefore T_Π is a perfect Jonsson theory.

Consider $JSp(C_{T_\Pi})$. Since the theory T_Π is perfect then $|JSp(C_{T_\Pi})/_{\bowtie}| = 1$. Let $\Delta \in JSp(C_{T_\Pi})$, i.e. Δ is Jonsson theory and $\Delta^* = T_\Pi$. We show that Δ is perfect $\exists$-complete Jonsson theory. By virtue of $T^* \underset{S}{\bowtie} \Delta^*$, then from the definition of semantic similarity for complete theories it follows that Δ is the perfect Jonsson theory. If Δ is $\exists$-complete, then instead T'_Π we take Δ and then by Theorem 4 it follows that $T \overset{S}{\rtimes} \Delta = T'_\Pi$. If Δ is not $\exists$-complete, then we carry out the following replenishment procedure for this theory. As $\Delta \subset T_\Pi$, then for any existential sentence φ, of the signature language of Δ such that $\Delta \nvdash \varphi$ and $\Delta \nvdash \neg\varphi$, but $\varphi \in T_\Pi$, consider the theory $\Delta' = \Delta \cup \{\varphi\}$. Since $\Delta \subset \Delta' \subset T_\Pi$, and Δ, T_Π are Jonsson theories, it follows from Proposition 5 that Δ' is also a Jonsson theory. If Δ' is not $\exists$-complete, then we continue the procedure of adding existential sentences $\varphi \in T_\Pi$ until Δ' it becomes $\exists$-complete.

Let $\overline{\Delta} = \Delta \cup \{\varphi | \varphi \in \Sigma_1, \varphi \in T_\Pi\}$ is the result of replenishment procedure of the theory Δ, i.e. $\overline{\Delta}$ is $\exists$-complete and at the same time $\overline{\Delta}$ is a Jonsson theory. We show that $\overline{\Delta} \in JSp(C_{T_\Pi})$, hence the perfection of the theory of $\overline{\Delta}$ will follow from here. Suppose the contrary, let $\overline{\Delta} \notin JSp(C_{T_\Pi})$, then $C_{T_\Pi} \notin Mod(\overline{\Delta})$, but this is not true since $C_{T_\Pi} \models \Delta$ and for any sentence $\varphi \in \overline{\Delta} \backslash \Delta$, $\varphi \in T_\Pi$. Consequently, $C_{T_\Pi} \models \varphi$ and $C_{T_\Pi} \in \mathrm{Mod}(\overline{\Delta})$. We obtain a contradiction, i.e. $\overline{\Delta} \in JSp(C_{T_\Pi})$. But C_{T_Π} is saturated, therefore, $\overline{\Delta}$ is a perfect Jonsson theory. Then by Theorem 4 we have $T^* \overset{S}{\bowtie} \overline{\Delta}^* \Leftrightarrow T \overset{S}{\rtimes} \overline{\Delta}$, where $\overline{\Delta} = T'_\Pi$.

We extend the concepts of syntactic and semantic similarity to the spectra of models of arbitrary signature.

Definition 19. Let $\mathcal{A} \in \mathrm{Mod}\sigma_1$, $\mathcal{B} \in \mathrm{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is J-syntactically similar to class $[T]_2$ and denote $[T]_1 \overset{S}{\rtimes} [T]_2$ if for any theory $\Delta \in [T]_1$ there is theory $\Delta' \in [T]_2$ such that $\Delta \overset{S}{\rtimes} \Delta'$.

Definition 20. The pure triple $\langle C, Aut(\mathcal{C}), \overline{E}_{[T]}\rangle$ is called the J-semantic triple for class $[T] \in JSp(A)/_{\bowtie}$, where C is the semantic model of $[T]$, $AutC$ is the group of all automorphisms of C, $\overline{E}_{[T]}$ is the class of isomorphically images of all existentially closed models of $[T]$.

Definition 21. Let $\mathcal{A} \in \mathrm{Mod}\sigma_1$, $\mathcal{B} \in \mathrm{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is J-semantically similar to class $[T]_2$ and denote $[T]_1 \underset{S}{\rtimes} [T]_2$ if their semantically triples are isomorphic as pure triples.

Lemma 3. From syntactic similarity of two classes of Jonsson spectrum follows their semantic similarity. Converse statement does not true.

The proof follows from Lemma 1 and Definition 21.

Lemma 4. Let $\mathcal{A} \in \mathrm{Mod}\sigma_1$, $\mathcal{B} \in \mathrm{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$ are perfect $\exists$-complete classes, then

$$[T]_1 \overset{S}{\rtimes} [T]_2 \Leftrightarrow [T]_1^* \overset{S}{\bowtie} [T]_2^*.$$

Proof. Let $[T]_1 \overset{S}{\rtimes} [T]_2$, then for every theory $\Delta \in [T]_1$ there is $\overline{\Delta} \in [T]_2$ such that $\Delta \overset{S}{\rtimes} \overline{\Delta}$, where Δ and $\overline{\Delta}$ are perfect $\exists$-complete Jonsson theories. Then according to Theorem 4 $\Delta^* \overset{S}{\bowtie} \overline{\Delta}^*$. But $\Delta^* = \mathrm{Th}(C_{[T]_1}) = [T]_1^*$ and $\overline{\Delta}^* = \mathrm{Th}(C_{[T]_2}) = [T]_2^*$, therefore $[T]_1^* \overset{S}{\bowtie} [T]_2^*$.

Conversely, let $[T]_1^* \overset{S}{\bowtie} [T]_2^*$ then by Theorem 4 for any theory $\Delta \in [T]_1$ there is theory $\Delta' \in [T]_2$ such that $\Delta \overset{S}{\rtimes} \overline{\Delta}$, i.e. $[T]_1 \overset{S}{\rtimes} [T]_2$.

The following theorem is a generalization of Theorem 5 to the case of the class of the Jonsson spectrum of an arbitrary model of signature.

Theorem 6. Let $[T] \in JSp(\mathcal{A})/_{\bowtie}$, then for every perfect $\exists$-complete class $[T] \in JSp(\mathcal{A})/_{\bowtie}$ there is a class $[T_{\Pi}] \in JSp(\mathcal{B})/_{\bowtie}$, where T_{Π} is $\exists$-complete Jonsson theory of some model $\mathcal{B}$ of a polygon signature such that $[T] \overset{S}{\rtimes} [T_{\Pi}]$.

Proof. Let $[T] \in JSp(\mathcal{A})/_{\bowtie}$ be a perfect $\exists$-complete class, then by Theorem 5 for each theory $\Delta \in [T]$ there is a Jonsson $\exists$-complete polygon theory T_{Π}^{Δ} such that $\Delta \overset{S}{\rtimes} T_{\Pi}^{\Delta}$. Then by Theorem 4 $\Delta^* \overset{S}{\bowtie} (T_{\Pi}^{\Delta})^*$, but since $\Delta \in [T]$, then $\Delta^* = [T]^*$. T_{Π}^{Δ} is the Jonsson theory of some model of $\mathcal{B}$ signature, then $T_{\Pi}^{\Delta} \in JSp(\mathcal{B})$ and $T_{\Pi}^{\Delta} \in [T_{\Pi}] \in JSp(\mathcal{B})/_{\bowtie}$. But then $(T_{\Pi}^{\Delta})^* = [T_{\Pi}]^*$. Hence, we have $[T]^* \overset{S}{\bowtie} [T_{\Pi}]^*$. By Lemma 5, it follows that $[T] \overset{S}{\rtimes} [T_{\Pi}]$.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP09260237).

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Йонсондық спектрлердің кластарының ұқсастықтары

Бірінші ретті тілдің синтаксистік және семантикалық қасиеттерін, жалпы айтқанда, толық емес теорияларды зерттеу математикалық логиканың өзекті мәселелерінің бірі. Мақалада біз йонсондық теорияларды зерттейміз, олар алгебрадағы классикалық мысалдардың көп болуымен қанағаттандырылады және жалпы айтқанда, толық емес. Йонсондық теорияларды зерттеудің жаңа және өзекті әдісі — теорияларды синтаксистік және семантикалық ұқсастық ұғымдары арқылы зерттеу. Ең инвариантты ұғым — теориялардың синтаксистік ұқсастығы ұғымы, өйткені ол қарастырылып отырған теориялардың барлық қасиеттерін сақтайды. Осы мақаланың негізгі нәтижесі келесі факт болып табылады: кез келген толық экзистенциалды сөйлемдер үшін кемел йонсондық теориясының полигон теориясына синтаксистік тұрғыдан ұқсас екендігін көрсету (S-полигон, мұндағы S моноид). Бұл нәтиже кез келген сигнатураның тиісті моделінің йонсондық спектрінен алынған йонсондық теорияның сәйкес кластарына кеңейтіледі.

Кілт сөздер: йонсондық теория, семантикалық модель, кемел йонсондық теория, косемантика, S-полигон, йонсондық спектр, синтаксистік және семантикалық ұқсастық.

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Подобия классов йонсоновских спектров

Исследование синтаксических и семантических свойств языка первого порядка, вообще говоря, неполных теорий, является одной из актуальных задач математической логики. В настоящей статье мы изучаем йонсоновские теории, которым удовлетворяет большинство классических примеров из алгебры, и которые, вообще говоря, не полны. Новым и актуальным методом исследования йонсоновских теорий является изучение этих теорий с помощью понятий синтаксического и семантического подобий. Самым инвариантным понятием представляется понятие синтаксического подобия теорий, так как оно сохраняет все свойства рассматриваемых теорий. Основной результат данной статьи есть тот факт, что любая совершенная йонсоновская теория, полная для экзистенциальных предложений, синтаксически подобна некоторой теории полигонов (S-полигона, где S — моноид). Этот результат переносится на соответствующие классы йонсоновских теорий из йонсоновского спектра произвольной модели произвольной сигнатуры.

Ключевые слова: йонсоновская теория, семантическая модель, совершенность, косемантичность, S-полигон, йонсоновский спектр, синтаксическое и семантическое подобия.

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DOI 10.31489/2023M4/144-162

UDC 517.968.73

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Mixed inverse problem for a Benney–Luke type integro-differential equation with two redefinition functions and parameters

In this paper, we consider a linear Benney–Luke type partial integro-differential equation of higher order with degenerate kernel and two redefinition functions given at the endpoint of the segment and two parameters. To find these redefinition functions we use two intermediate data. Dirichlet boundary value conditions are used with respect to spatial variable. The Fourier series method of variables separation is applied. The countable system of functional-integral equations is obtained. Theorem on a unique solvability of countable system for functional-integral equations is proved. The method of successive approximations is used in combination with the method of contraction mapping. The triple of solutions of the inverse problem is obtained in the form of Fourier series. Absolutely and uniformly convergences of Fourier series are proved.

Keywords: Inverse problem, two redefinition functions, final conditions, intermediate functions, Fourier method, unique value solvability.

Introduction

Historically, differential equations arose in solving applied problems. Therefore, the development of differential equations at the initial stage was carried out by applied scientists. Gradually, this direction grew into an independent theory — the theory of differential equations. Therefore, it can be said many times that differential and integro-differential equations are great interest from the point of theoretical research and applications in the mathematical physics, engineering, chemistry and in other different fields [1–8]. Recent years, a number of new problems for ordinary and partial differential and integro-differential equations are studied and a large number of research papers are published. Problems with nonlocal conditions for differential and integro-differential equations were considered in [9–29]. In [30–38], integro-differential equations with a degenerate kernel were considered.

In this paper, we study the solvability of the mixed inverse problem for a Benney–Luke type partial integro-differential equation with a degenerate kernel, two parameters, and final conditions at the endpoint of the interval. This paper differs from existing papers in that it requires to find redefinition functions considering at the endpoint of the interval. This inverse problem has features in relation to the direct problem.

In the rectangular domain $\Omega = \left\{0 < t < T,\ 0 < x < l\right\}$ we consider the following partial integro-differential equation of a higher order

$$\frac{\partial^2 U}{\partial t^2} + (-1)^k \frac{\partial^{2k+2} U}{\partial t^2 \partial x^2} + \omega^2 \left[(-1)^k \frac{\partial^{2k} U}{\partial x^{2k}} + \frac{\partial^{4k} U}{\partial x^{4k}}\right] = \alpha(t)\, U(t,x) + \nu \int\limits_0^T K(t,s)\, U(s,x)\, ds, \qquad (1)$$

where k is a natural number, $0 < \alpha(t) \in C[0,T]$, T, l are given positive numbers, ω is a positive parameter, ν is a nonzero real parameter, $K(t,s) = \sum\limits_{i=1}^{m} a_i(t)\, b_i(s)$, $a_i(t),\ b_i(s) \in C\,[0,T]$. It is assumed that the systems of functions $\{a_i(t)\}$ and $\{b_i(s)\}$, $i = \overline{1, m}$ are linear independent.

It is known that when applying the method of separation of variables, the Dirichlet condition allows us to reduce partial differential equations to a countable system of ordinary differential equations. So, in solving partial integro-differential equation (1), we use the following Dirichlet boundary value conditions with respect to spatial variable x

$$U(t,0)=U(t,l)=\frac{\partial^2}{\partial x^2}U(t,0)=\frac{\partial^2}{\partial x^2}U(t,l)=$$

$$=\cdots=\frac{\partial^{4k-2}}{\partial x^{4k-2}}U(t,0)=\frac{\partial^{4k-2}}{\partial x^{4k-2}}U(t,l)=0. \tag{2}$$

We use two conditions at the endpoint of the given segment with respect to time variable t:

$$U(T,x)=\varphi_1(x),\quad U_t(T,x)=\varphi_2(x),\quad 0\le x\le l, \tag{3}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are redefinition functions and we assume that they are enough smooth on the segment $[0,l]$. For these functions the following conditions will be fulfilled

$$\varphi_i(0)=\varphi_i(l)=\varphi_i''(0)=\varphi_i''(l)=\cdots=\varphi_i^{(4k-2)}(0)=\varphi_i^{(4k-2)}(l)=0,\quad i=1,2.$$

In determining the redefinition functions, we use the following two intermediate conditions:

$$U(t_1,x)=\psi_1(x),\quad U_t(t_1,x)=\psi_2(x),\quad 0\le x\le l, \tag{4}$$

where $\psi_1(x)$ and $\psi_2(x)$ are known functions enough smooth on the segment $[0,l]$, $0<t_1<T$. For the functions $\psi_1(x)$ and $\psi_2(x)$ the following conditions will be fulfilled

$$\psi_i(0)=\psi_i(l)=\psi_i''(0)=\psi_i''(l)=\cdots=\psi_i^{(4k-2)}(0)=\psi_i^{(4k-2)}(l)=0,\quad i=1,2.$$

The choice of conditions (3) and (4) with the final and intermediate data are important in applications. Indeed, in real practice it is not always possible to determine the initial data for unknown functions. When studying the technological process of aluminum production, before the start of the production cycle, the raw material passes through firing and the state of the raw material by the beginning of the production cycle is not known. And the final expected state of the output will be unknown in reality. We find it from known intermediate conditions. Because after each technological cycle we can determine the quality of the product. So, we have an inverse problem to solve equation (1).

Problem statement. To find triple of functions

$$\left\{U(t,x)\in C\left(\overline{\Omega}\right)\cap C_{t,x}^{2,4k}(\Omega)\cap C_{t,x}^{2+2k}(\Omega),\ \varphi_i(x)\in C[0,l],\ i=1,2\right\},$$

the first of which satisfies partial integro-differential equation (1) and specified conditions (2)–(4), where $\overline{\Omega}=\left\{0\le t\le T,\ 0\le x\le l\right\}$.

Note that problem (1)–(4) is formulated such that direct problem (1)–(3) has a unique solution for all values of the parameter ω, and inverse problem (1)–(4) has a unique solution only for certain values of this parameter ω.

1 Construction of formal solution of the direct problem (1)–(3)

Note that the functions $\vartheta_n(x)=\sqrt{\frac{2}{l}}\sin\lambda_n x$, where $\lambda_n\in\frac{n\pi}{l}$, $n\in\mathbb{N}$, form a complete system of orthonormal eigenfunctions in the space $L_2[0,l]$. Linear equation (1) always has the trivial solution.

Therefore, by virtue of the Dirichlet condition (2), we seek nontrivial solutions to the linear partial integro-differential equation (1) of the higher order in the form of a Fourier series in sines

$$U(t,x)=\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}u_n(t)\,\sin\lambda_n x, \tag{5}$$

where

$$u_n(t)=\sqrt{\frac{2}{l}}\int_0^l U(t,x)\,\sin\lambda_n x\,dx,\quad \lambda_n=\frac{n\,\pi}{l}. \tag{6}$$

Substituting the Fourier series (5) into the given integro-differential equation (1), we obtain a linear second order countable system of ordinary differential equations

$$u_n''(t)+\omega^2\lambda_n^{2k}\,u_n(t)=\frac{\nu}{1+\lambda_n^{2k}}\sum_{i=1}^{m}a_i(t)\,\tau_{n\,i}+\frac{1}{1+\lambda_n^{2k}}\alpha(t)\,u_n(t), \tag{7}$$

where

$$\tau_{i\,n}=\int_0^T b_i(s)\,u_n(s)\,ds. \tag{8}$$

Solving the countable system of differential equations (7) by the variation method of arbitrary constants, we obtain the representation for its solution

$$u_n(t)=A_{1n}\,\cos\lambda_n^k\omega\,t+A_{2n}\,\sin\lambda_n^k\omega\,t+$$

$$+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\tau_{i\,n}\int_0^t\sin\lambda_n^k\,\omega\,(t-s)\,a_i(s)\,ds+$$

$$+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int_0^t\sin\lambda_n^k\,\omega\,(t-s)\,\alpha(s)\,u_n(s)\,ds, \tag{9}$$

where A_{1n} and A_{2n} are arbitrary coefficients, which will be determined by the final conditions (3). By differentiating (9) one times on t, we obtain

$$u_n'(t)=-\lambda_n^k\omega A_{1n}\,\sin\lambda_n^k\omega\,t+\lambda_n^k\omega A_{2n}\,\cos\lambda_n^k\omega\,t+$$

$$+\frac{\nu}{1+\lambda_n^{2k}}\sum_{i=1}^{m}\tau_{n\,i}\int_0^t\cos\lambda_n^k\,\omega\,(t-s)\,a_i\,(s)\,ds+$$

$$+\frac{1}{1+\lambda_n^{2k}}\int_0^t\cos\lambda_n^k\,\omega\,(t-s)\,\alpha(s)\,u_n(s)\,ds. \tag{10}$$

Now, supposing that the redefinition functions $\varphi_1(x)$ and $\varphi_2(x)$ were expanded into a Fourier series, and using Fourier coefficients (6), from conditions (3) we obtain

$$u_n(T)=\sqrt{\frac{2}{l}}\int_0^l U(T,x)\,\sin\lambda_n x\,dx=\sqrt{\frac{2}{l}}\int_0^l\varphi_1(x)\,\sin\lambda_n x\,dx=\varphi_{1\,n}, \tag{11}$$

$$u'_n(T)=\sqrt{\frac{2}{l}}\int\limits_0^l U_t(T,x)\,\sin\lambda_n x\,dx=\sqrt{\frac{2}{l}}\int\limits_0^l \varphi_2(x)\,\sin\lambda_n x\,dx=\varphi_{2n}. \tag{12}$$

To find the unknown coefficients A_{1n} and A_{2n} in presentations (9) and (10), we use final conditions (11) and (12). Then we arrive at a system of algebraic equations (SAE)

$$\begin{cases} A_{1n}\,\cos\lambda_n^k\omega\,T+A_{2n}\,\sin\lambda_n^k\omega\,T=\gamma_{1n}, \\ -A_{1n}\,\sin\lambda_n^k\omega\,T+A_{2n}\,\cos\lambda_n^k\omega\,T=\gamma_{2n}, \end{cases} \tag{13}$$

where

$$\gamma_{1n}=\varphi_{1n}-\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^m\tau_{in}\int\limits_0^T\sin\lambda_n^k\omega\,(T-s)\,a_i(s)\,ds-$$

$$-\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int\limits_0^T\sin\lambda_n^k\omega\,(T-s)\,\alpha(s)\,u_n(s)\,ds,$$

$$\gamma_{2n}=\varphi_{2n}-\frac{\nu}{1+\lambda_n^{2k}}\sum_{i=1}^m\tau_{in}\int\limits_0^T\cos\lambda_n^k\omega\,(T-s)\,a_i(s)\,ds-$$

$$-\frac{1}{1+\lambda_n^{2k}}\int\limits_0^T\cos\lambda_n^k\omega\,(T-s)\,\alpha(s)\,u_n(s)\,ds.$$

For uniquely solvability of SAE (13), the following condition

$$\delta_{0n}=\begin{vmatrix}\cos\lambda_n^k\omega\,T & \sin\lambda_n^k\omega\,T\\ -\sin\lambda_n^k\omega\,T & \cos\lambda_n^k\omega\,T\end{vmatrix}\neq 0$$

must be fulfilled. Since $\delta_{0n}=1$, this condition are fulfilled for all values of the parameter ω. Consequently, SAE (13) has a unique pair of solutions

$$A_{1n}=\delta_{1n}=\begin{vmatrix}\gamma_{1n} & \sin\lambda_n^k\omega\,T\\ \gamma_{2n} & \cos\lambda_n^k\omega\,T\end{vmatrix}=\varphi_{1n}\,\cos\lambda_n^k\omega\,T-\varphi_{2n}\,\sin\lambda_n^k\omega\,T+$$

$$+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^m\tau_{in}\int\limits_0^T\sin\lambda_n^k\omega\,s\,a_i(s)\,ds+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int\limits_0^T\sin\lambda_n^k\omega\,s\,\alpha(s)\,u_n(s)\,ds, \tag{14}$$

$$A_{2n}=\delta_{2n}=\begin{vmatrix}\cos\lambda_n^k\omega\,T & \gamma_{1n}\\ -\sin\lambda_n^k\omega\,T & \gamma_{2n}\end{vmatrix}=\varphi_{1n}\,\sin\lambda_n^k\omega\,T+\varphi_{2n}\,\cos\lambda_n^k\omega\,T+$$

$$+\frac{\nu}{1+\lambda_n^{2k}}\sum_{i=1}^m\tau_{in}\int\limits_0^T\cos\lambda_n^k\omega\,s\,a_i(s)\,ds+\frac{1}{1+\lambda_n^{2k}}\int\limits_0^T\cos\lambda_n^k\omega\,s\,\alpha(s)\,u_n(s)\,ds. \tag{15}$$

Substituting these values of (14) and (15) into presentation (9), we obtain

$$u_n(t,\nu,\omega)=\varphi_{1n}\,\chi_{1n}(t,\omega)+\varphi_{2n}\,\chi_{2n}(t,\omega)+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^m\tau_{in}\,\chi_{3in}(t,\omega)+$$

$$+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int\limits_0^T H_n(t,s,\omega)\,\alpha(s)\,u_n(s,\nu,\omega)\,ds, \tag{16}$$

where

$$\chi_{1n}(t,\omega)=\cos\lambda_n^k\omega\,(T-t)-\sin\lambda_n^k\omega\,(T-t),$$

$$\chi_{2n}(t,\omega)=\cos\lambda_n^k\omega\,(T+t)-\sin\lambda_n^k\omega\,(T-t),$$

$$\chi_{3in}(t,\omega)=\int\limits_0^T H_n(t,s,\omega)\,a_i(s)\,ds,$$

$$H_n(t,s,\omega)=\begin{cases}\sin z\,(t+s), \quad z=\lambda_n^k\omega, \quad t<s\le T,\\ \sin z\,(t-s)+\cos z\,t\,\sin z\,s+z\,\sin z\,t\,\sin z\,s, \;\; 0\le s<t.\end{cases}$$

Although functions (16) are Fourier coefficients of the solution to direct problem (1)–(3), it contains extra quantities τ_{in} that are still unknown. To find these quantities, we substitute representation (16) into designation (8) and arrive at a new SAE:

$$\tau_{in}-\frac{\nu}{\bar\lambda}\sum_{j=1}^m\tau_{jn}\,\sigma_{3ijn}(t)=\varphi_{1n}\,\sigma_{1in}+\varphi_{2n}\,\sigma_{2in}+\sigma_{4in}(u_n), \tag{17}$$

where

$$\sigma_{1in}=\int\limits_0^T b_i(s)\,\chi_{1n}(s,\omega)\,ds, \quad \sigma_{2in}=\int\limits_0^T b_i(s)\,\chi_{2n}(s,\omega)\,ds, \quad \bar\lambda=\lambda_n^k\left(1+\lambda_n^{2k}\right)\omega,$$

$$\sigma_{3ijn}=\int\limits_0^T b_i(s)\int\limits_0^T H_n(s,\theta,\omega)\,a_j(\theta)\,d\theta\,ds,$$

$$\sigma_{4in}(u_n)=\frac{1}{\bar\lambda}\int\limits_0^T b_i(s)\int\limits_0^T H_n(s,\theta,\omega)\,\alpha(\theta)\,u_n(\theta)\,d\theta\,ds.$$

To establish the unique solvability of SAE (17), we introduce the following matrix

$$\Theta_{0n}(\nu,\,\omega)=\begin{pmatrix}1-\frac{\nu}{\bar\lambda}\sigma_{311n} & \frac{\nu}{\bar\lambda}\sigma_{312n} & \dots & \frac{\nu}{\bar\lambda}\sigma_{31mn}\\ \frac{\nu}{\bar\lambda}\sigma_{321n} & 1-\frac{\nu}{\bar\lambda}\sigma_{322n} & \dots & \frac{\nu}{\bar\lambda}\sigma_{32mn}\\ \dots & \dots & \dots & \dots\\ \frac{\nu}{\bar\lambda}\sigma_{3m1n} & \frac{\nu}{\bar\lambda}\sigma_{3m2n} & \dots & 1-\frac{\nu}{\bar\lambda}\sigma_{3mmn}\end{pmatrix}$$

and consider the values of the parameter ν, for which the Fredholm determinant is not zero:

$$\Delta_{0n}(\nu,\omega)=\det\Theta_{0n}(\nu,\omega)\ne 0. \tag{18}$$

Determinant $\Delta_{0n}(\nu,\omega)$ in (18) is a polynomial with respect to $\frac{\nu}{\bar\lambda}$ of the degree not higher than m. The countable system of algebraic equations $\Delta_{0n}(\nu,\omega)=0$ has no more than m different real roots for every value of n. We denote them by $\mu_l(l=\overline{1,p},\ 1\le p\le m)$. Then $\nu_n=\nu_{ln}=\bar\lambda\,\mu_l=\lambda_n^k\left(1+\lambda_n^{2k}\right)\omega\,\mu_l$ are called the characteristic (irregular) values of the kernel for integro-differential equation (1). So, we introduce the following two designations

$$\Lambda_1=\left\{(\nu_n,\,\omega):\,\nu_n=\lambda_n^k\left(1+\lambda_n^{2k}\right)\omega\,\mu_l, \;\; \omega\in(0,\infty)\right\},$$

$$\Lambda_2=\left\{(\nu_n,\,\omega):\,|\,\Delta_{0n}(\nu,\omega)\,|>0,\ \ \nu_n\neq\lambda_n^k\left(1+\lambda_n^{2k}\right)\omega\,\mu_l,\ \ \omega\in(0,\infty)\right\}.$$

On the number set Λ_2 we consider a matrix

$$\Theta_{ijn}(\nu,\omega)=\begin{pmatrix} 1-\frac{\nu}{\lambda}\sigma_{311n} & \ldots & \frac{\nu}{\lambda}\sigma_{31(i-1)n} & \sigma_{j1n} & \frac{\nu}{\lambda}\sigma_{31(i+1)n} & \ldots & \frac{\nu}{\lambda}\sigma_{31mn} \\ \frac{\nu}{\lambda}\sigma_{321n} & \ldots & \frac{\nu}{\lambda}\sigma_{32(i-1)n} & \sigma_{j2n} & \frac{\nu}{\lambda}\sigma_{32(i+1)n} & \ldots & \frac{\nu}{\lambda}\sigma_{32mn} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\nu}{\lambda}\sigma_{3m1n} & \ldots & \frac{\nu}{\lambda}\sigma_{3m(i-1)n} & \sigma_{jmn} & \frac{\nu}{\lambda}\sigma_{3m(i+1)n} & \ldots & 1-\frac{\nu}{\lambda}\sigma_{3mmn} \end{pmatrix},$$

$j=1,2,4$. Taking into account the known properties of the matrix $\Theta_{ijn}(\nu,\omega)$, we modified the Cramer method on the set Λ_2 and obtain solutions of SAE (17) in the form

$$\tau_{in}=\varphi_{1n}\frac{\Delta_{1i}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}+\varphi_{2n}\frac{\Delta_{2in}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}+\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)},\quad i=\overline{1,\,m},\ \ (\nu,\omega)\in\Lambda_2, \tag{19}$$

where $\Delta_{ijn}(\nu,\omega)=\det\Theta_{ijn}(\nu,\omega),\ j=1,2,4$.

Substituting solutions (19) into function (16), we obtain

$$u_n(t,\nu,\omega)=\varphi_{1n}\,h_{1n}(t,\nu,\omega)+\varphi_{2n}\,h_{2n}(t,\nu,\omega)+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)}\,\chi_{3in}(t)+$$

$$+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int_0^T H_n(t,s,\omega)\,u_n(s,\nu,\omega)\,ds,\quad(\nu,\omega)\in\Lambda_2, \tag{20}$$

where

$$h_{jn}(t,\nu,\omega)=\chi_{jn}(t,\omega)+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{jn}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}\,\chi_{3in}(t,\omega),\ \ j=1,2,$$

$$\chi_{1n}(t,\omega)=\cos\lambda_n^k\omega\,(T-t)-\sin\lambda_n^k\omega\,(T-t),\quad \chi_{2n}(t,\omega)=\cos\lambda_n^k\omega\,(T+t)-\sin\lambda_n^k\omega\,(T-t),$$

$$\chi_{3in}\,(t,\omega)=\int_0^T H_n(t,s,\omega)\,a_i(s)\,ds,$$

$$H_n(t,s,\omega)=\begin{cases}\sin z\,(t+s), & z=\lambda_n^k\omega,\quad t<s\leq T,\\ \sin z\,(t-s)+\cos z\,t\,\sin z\,s+z\,\sin z\,t\,\sin z\,s, & 0\leq s<t.\end{cases}$$

Representation (20) is a countable system of functional-integral equations. Substituting representation (20) into the Fourier series (5), we obtain a formal solution of direct problem (1)–(3) on the domain Ω

$$U(t,x)=\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\sin\,\lambda_n x\times$$

$$\times\left[\varphi_{1n}\,h_{1n}(t,\nu,\omega)+\varphi_{2n}\,h_{2n}(t,\nu,\omega)+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)}\,\chi_{3in}(t)+\right.$$

$$\left.+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int_0^T H_n(t,s,\omega)\,u_n(s,\nu,\omega)\,ds\right],\quad(\nu,\omega)\in\Lambda_2. \tag{21}$$

But, there are two unknown quantities φ_{1n} and φ_{2n} in (21).

2 Formal solution of the inverse problem (1)–(4)

We will now formally define the redefinition functions $\varphi_1(x)$ and $\varphi_2(x)$. We subordinate function (20) to intermediate conditions (4). For this purpose, we differentiate (21) one times on the time-variable t:

$$U_t(t,x)=\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\sin\lambda_n x\ \Big[\varphi_{1n}\,h'_{1n}(t,\nu,\omega)+\varphi_{2n}\,h'_{2n}(t,\nu,\omega)+$$

$$+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)}\,\chi'_{3in}(t)+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int_0^T H'_n(t,s,\omega)\,u_n(s,\nu,\omega)\,ds\Big],\tag{22}$$

where

$$h'_{jn}(t,\nu,\omega)=\chi'_{jn}(t,\omega)+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{jn}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}\,\chi'_{3in}(t,\omega),\ \ j=1,2,$$

$$\chi'_{1n}(t,\omega)=\lambda_n^k\omega\,\big(\sin\lambda_n^k\omega\,(T-t)+\cos\lambda_n^k\omega\,(T-t)\big),$$

$$\chi'_{2n}(t,\omega)=-\lambda_n^k\omega\,\big(\sin\lambda_n^k\omega\,(T+t)+\cos\lambda_n^k\omega\,(T-t)\big),$$

$$\chi'_{3in}(t,\omega)=\int_0^T H'_n(t,s,\omega)\,a_i(s)\,ds,$$

$$H'_n(t,s,\omega)=\begin{cases} z\,\cos z\,(t+s), \quad z=\lambda_n^k\omega, \quad t<s\leq T,\\ z\,\cos z\,(t-s)-z\,\sin z\,t\,\sin z\,s+z^2\cos z\,t\,\sin z\,s,\ \ 0\leq s<t.\end{cases}$$

Then, applying intermediate conditions (4) to functions (21) and (22), we arrive at the solution of the following SAE:

$$\begin{cases}\varphi_{1n}\,[\chi_{1n}(t_1,\omega)+\varepsilon_{11n}]+\varphi_{2n}\,[\chi_{2n}(t_1,\omega)+\varepsilon_{12n}]=\bar\psi_{1n},\\ \varphi_{1n}\,[\chi'_{1n}(t_1,\omega)+\varepsilon_{21n}]+\varphi_{2n}\,[\chi'_{2n}(t_1,\omega)+\varepsilon_{22n}]=\bar\psi_{2n},\end{cases}\tag{23}$$

where

$$\varepsilon_{1jn}=\frac{\nu}{\bar\lambda}\sum_{i=1}^{m}\frac{\Delta_{jn}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}\,\chi_{3in}(t_1,\omega),\quad \varepsilon_{2jn}=\frac{\nu}{\bar\lambda}\sum_{i=1}^{m}\frac{\Delta_{jn}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}\,\chi'_{3in}(t_1,\omega),\ \ j=1,2,$$

$$\bar\psi_{1n}=\psi_{1n}-\frac{\nu}{\bar\lambda}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}\,(\nu,\omega)}\,\chi_{3in}(t_1,\omega)+\frac{1}{\bar\lambda}\int_0^T H_n(t_1,s,\omega)\,u_n(s,\nu,\omega)\,ds,\tag{24}$$

$$\bar\psi_{2n}=\psi_{2n}-\frac{\nu}{\bar\lambda}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}\,(\nu,\omega)}\,\chi'_{3i}(t_1,\omega)+\frac{1}{\bar\lambda}\int_0^T H'_n(t_1,s,\omega)\,u_n(s,\nu,\omega)\,ds,\tag{25}$$

$$\bar\lambda=\lambda_n^k\left(1+\lambda_n^{2k}\right)\omega.$$

The fulfillment of the following condition ensures the unique solvability of SAE (23):

$$V_{0n}(\omega)=\begin{vmatrix}\chi_{1n}(t_1,\omega)+\varepsilon_{11n} & \chi_{2n}(t_1,\omega)+\varepsilon_{12n}\\ \chi'_{1n}(t_1,\omega)+\varepsilon_{21n} & \chi'_{2n}(t_1,\omega)+\varepsilon_{22n}\end{vmatrix}=$$

$$=-z\,\sin 2zT-z\,\cos 2zT+2z\,\sin z(T-t_1)\,\cos z(T-t_1)-z\,\cos 2z(T-t_1)-$$

$$-z\,\varepsilon_{11n}[\sin z(T+t_1)+\cos z(T-t_1)]-z\varepsilon_{12n}[\sin z(T-t_1)+\cos z(T-t_1)]-$$

$$-\varepsilon_{21n}[\cos z(T+t_1) - z\,\sin z(T-t_1)] - \varepsilon_{22n}[\sin z(T-t_1) - z\,\cos z(T-t_1)]+$$
$$+\varepsilon_{11n}\varepsilon_{22n} - \varepsilon_{21n}\varepsilon_{12n} \neq 0. \tag{26}$$

Before proceeding to find the solution of SAE (23), we consider nonzero condition (26). To do this, we suppose the opposite:

$$-z\,\sin 2zT - z\,\cos 2zT + 2z\,\sin z(T-t_1)\,\cos z(T-t_1) - z\,\cos 2z(T-t_1)-$$
$$-z\,\varepsilon_{11n}[\sin z(T+t_1) + \cos z(T-t_1)] - z\varepsilon_{12n}[\sin z(T-t_1) + \cos z(T-t_1)]-$$
$$-\varepsilon_{21n}[\cos z(T+t_1) - z\,\sin z(T-t_1)] - \varepsilon_{22n}[\sin z(T-t_1) - z\,\cos z(T-t_1)]+$$
$$+\varepsilon_{11n}\varepsilon_{22n} - \varepsilon_{21n}\varepsilon_{12n} = 0, \quad z = \lambda_n^{\,k}\,\omega. \tag{27}$$

Condition (27) is a transcendental equation, and the set of its solutions with respect to ω is denoted by $\Im$. So, on the set

$$\Lambda_3 = \left\{(\nu_n, \omega):\, |\,\Delta_{0n}(\nu,\omega)\,| > 0,\;\; \nu_n \neq \lambda_n^k\left(1+\lambda_n^{2k}\right)\omega\,\mu_l,\;\; \omega \in \Im\right\}$$

SAE (23) is not uniquely solvable. But, on the other set

$$\Lambda_4 = \left\{(\nu_n,\,\omega):\, |\,\Delta_{0n}(\nu,\omega)\,| > 0,\;\; |\,V_{0n}(\omega)\,| > 0,\;\; \nu_n \neq \lambda_n^k\left(1+\lambda_n^{2k}\right)\omega\,\mu_l,\;\; \omega \in (0,\infty)\setminus\Im\right\}$$

SAE (23) is uniquely solvable. So, taking into account notations (24) and (25), we obtain

$$\varphi_{jn} = \psi_{1n}w_{j1n}(\omega) + \psi_{2n}w_{j2n}(\omega) + \frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^m \frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)}\,w_{j3in}(\omega)+$$
$$+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int_0^T W_{j\,n}(s,\omega)\,u_n(s,\nu,\omega)\,ds,\;\; j=1,2,\;\; (\nu,\omega)\in\Lambda_4, \tag{28}$$

where

$$w_{11n}(\omega) = V_{0n}^{-1}\left(\chi'_{2n}(t_1,\omega) + \varepsilon_{22n}(\omega)\right),\;\; w_{12n}(\omega) = V_{0n}^{-1}\left(-\chi_{2n}(t_1,\omega) + \varepsilon_{12n}(\omega)\right),$$
$$w_{21n}(\omega) = V_{0n}^{-1}\left(\chi'_{1n}(t_1,\omega) + \varepsilon_{21n}(\omega)\right),\;\; w_{22n}(\omega) = V_{0n}^{-1}\left(\chi_{1n}(t_1,\omega) + \varepsilon_{11n}(\omega)\right),$$
$$w_{13n}(\omega) = -\left[\chi_{3i\,n}(t_1,\omega)w_{11n}(\omega) + \chi'_{3in}(t_1,\omega)\,w_{12n}(\omega)\right],$$
$$w_{23n}(\omega) = -\left[\chi_{3in}(t_1,\omega)\,w_{21n}(\omega) + \chi'_{3i\,n}(t_1,\omega)\,w_{22n}(\omega)\right],$$
$$W_{1n}(s,\omega) = H_n(t_1,s)\,w_{11n}(\omega) + H'_n(t_1,s)\,w_{12n}(\omega),$$
$$W_{2n}(s,\omega) = H_n(t_1,s)\,w_{21n}(\omega) + H'_n(t_1,s)\,w_{22n}(\omega).$$

Since φ_{1n} and φ_{2n} are Fourier coefficients, from presentations (28) we obtain the following Fourier series

$$\varphi_j(x) = \sqrt{\frac{2}{l}}\sum_{n=1}^\infty \sin\lambda_n x\,\left[\psi_{1n}w_{j1n} + \psi_{2n}w_{j2n} + \frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^m\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)}\,w_{j3in}+\right.$$
$$\left.+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int_0^T W_{jn}(s,\omega)\,u_n(s,\nu,\omega)\,ds\right],\;\; (\nu,\omega)\in\Lambda_4. \tag{29}$$

The functions $u_n(t,\nu,\omega)$ in series (29) are Fourier coefficients of the unknown function $U(t,x,\nu,\omega)$. Therefore, we need to define the Fourier coefficients $u_n(t,\nu,\omega)$ uniquely. Substituting representation (28) into equations (20), we obtain the following countable system of functional-integral equations in the final form

$$u_n(t,\nu,\omega) = S(t,\nu,\omega;u_n) \equiv \psi_{1n}\, g_{1n}(t,\nu,\omega) + \psi_{2n}\, g_{2n}(t,\nu,\omega)+$$
$$+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)}\,g_{3in}(t,\omega)+$$
$$+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int\limits_0^T G_n(t,s,\nu,\omega)\,u_n(s,\nu,\omega)\,ds, \quad (\nu,\omega)\in\Lambda_4, \tag{30}$$

where

$$g_{1n}(t,\nu,\omega) = w_{11n}(\omega)h_{1n}(t,\nu,\omega) + w_{21n}(\omega)h_{2n}(t,\nu,\omega),$$
$$g_{2n}(t,\nu,\omega) = w_{12n}(\omega)\,h_{1n}(t,\nu,\omega) + w_{22n}(\omega)\,h_{2n}(t,\nu,\omega),$$
$$g_{3in}(t,\omega) = g_{1n}(t,\nu,\omega)\,\chi_{3in}(t_1,\omega) + g_{2n}(t,\nu,\omega)\,\chi'_{3in}(t_1,\omega) + \chi_{3in}(t,\omega),$$
$$G_n(t,s,\nu,\omega) = g_{1n}(t,\nu,\omega)\,H_n(t_1,s,\omega) + g_{2n}(t,\nu,\omega)\,H'(t_1,s,\omega) + H_n(t,s,\omega).$$

Note that this functional-integral equation (30) makes sense only for values of parameters ν, ω from the set Λ_4. In addition, in the countable system of functional-integral equations (30), the unknown function $u_n(t,\nu,\omega)$ is under the sign of the determinant and under the sign of the integral.

3 Solvable of the countable system of functional-integral equations (30)

Let us investigate the system of equations (30) in the sense of the unique solvability. To this, we consider the following well-known Banach spaces, in which we need in our further actions [26,32,33,36]. We consider the space B_2 of function sequences $\{\,u_n(t)\,\}_{n=1}^{\infty}$ on the segment $[0,T]$ with the norm

$$\|\,u(t)\,\|_{B_2} = \sqrt{\sum_{n=1}^{\infty}\left(\max_{t\in[0,T]}\,|\,u_n(t)\,|\right)^2} < \infty;$$

the space ℓ_2 of number sequences $\{\varphi_n\}_{n=1}^{\infty}$ with the norm

$$\|\,\varphi\,\|_{\ell_2} = \sqrt{\sum_{n=1}^{\infty}|\,\varphi_n\,|^2} < \infty;$$

the space $L_2[0,l]$ of square-integrable functions on an interval $[0,l]$ with norm

$$\|\,\vartheta(x)\,\|_{L_2[0,l]} = \sqrt{\int\limits_0^l |\,\vartheta(x)\,|^2\,dx} < \infty.$$

Smoothness conditions. Let on the segments $[0,l]$ there exist peace-wise continuous derivatives with respect to x up $(4k+2)$-th order for the functions $\psi_i(x)\in C^{4k+1}[0,l]$, $\;i=1,2$. Then, after integration the integrand functions $\psi_{in} = \sqrt{\frac{2}{l}}\int\limits_0^l \psi_i(x)\,\sin\lambda_n x\,dx$, $\;i=1,2$ by part $(4k+2)$ times on the variable x, we obtain the following relation

$$|\,\psi_{i,n}\,| = \left(\frac{l}{\pi}\right)^{4k+2}\frac{\left|\,\psi_{i,n}^{(4k+2)}\,\right|}{n^{4k+2}}, \; i=1,2, \tag{31}$$

where

$$\psi_{i,n}^{(4k+2)} = \int_0^l \frac{\partial^{4k+2} \psi_i(x)}{\partial x^{4k+2}} \vartheta_n(x)\, dx, \quad i = 1, 2.$$

Here we note that the Bessel inequality is true

$$\sum_{n=1}^{\infty} \left[\psi_{i,n}^{(4k+2)}\right]^2 \leq \left(\frac{2}{l}\right)^{4k+2} \int_0^l \left[\frac{\partial^{4k+2} \psi_i(x)}{\partial x^{4k+2}}\right]^2 dx, \quad i = 1, 2. \tag{32}$$

Theorem 1. Let the smoothness conditions and the following conditions be fulfilled:

$$\max_{t\in[0,T]} \left[|\, g_{1n}(t,\nu,\omega)\,|\,;\, |\, g_{2n}(t,\nu,\omega)\,|\right] = \delta_{1n} \leq \delta_1 < \infty, \tag{33}$$

$$\rho = |\,\nu\,|\, \delta_2 \left\| \sum_{i=1}^{m} \left| \frac{\bar{\Delta}_{4in}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)} \right| \delta_{0i} \right\|_{\ell_2} + \delta_3 < 1, \tag{34}$$

where δ_2, δ_3 and δ_{0i} will be defined from (38) and (39), while $\bar{\Delta}_{4in}(\nu,\omega)$ is defined from (41). Then the countable systems of functional-integral equations (30) is uniquely solvable in the space B_2. The desired solution can be founded from the following iterative process:

$$\begin{cases} u_n^0(t,\nu,\omega) = \psi_{1n}\, g_{1n}(t,\nu,\omega) + \psi_{2n}\, g_{2n}(t,\nu,\omega), \\ u_n^{r+1}(t,\nu,\omega) = S(t,\nu,\omega; u_n^r), \quad r = 0, 1, 2, ... \end{cases} \tag{35}$$

Proof. We use the method of contraction maps in combination with the method of successive approximations in the space B_2. Then, by virtue of smoothness condition (31) and estimate (33), applying the Cauchy–Schwartz inequality and Bessel inequality (32), from approximations (35) we obtain that the following estimate is valid:

$$\sum_{n=1}^{\infty} \max_{t\in[0,T]} |\, u_n^0(t)\,| \leq \sum_{n=1}^{\infty} \max_{t\in[0,T]} \left[|\,\psi_{1n}\,| \cdot |\, g_{1n}(t,\nu,\omega)\,| + |\,\psi_{2n}\,| \cdot |\, g_{2n}(t,\nu,\omega)\,|\right] \leq$$

$$\leq \delta_1 \left(\frac{l}{\pi}\right)^{4k+2} \left[\sum_{n=1}^{\infty} \frac{\left|\psi_{1,n}^{(4k+2)}\right|}{n^{4k+2}} + \sum_{n=1}^{\infty} \frac{\left|\psi_{2,n}^{(4k+2)}\right|}{n^{4k+2}}\right] \leq$$

$$\leq \delta_1 \left(\frac{\sqrt{2l}}{\pi}\right)^{4k+2} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{8k+4}}} \left[\left\| \frac{\partial^{4k+2} \psi_1(x)}{\partial x^{4k+2}} \right\|_{L_2[0,l]} + \left\| \frac{\partial^{4k+2} \psi_2(x)}{\partial x^{4k+2}} \right\|_{L_2[0,l]}\right] = \delta_0 < \infty. \tag{36}$$

Taking into account estimate (36), applying the Cauchy–Schwartz inequality, for the first difference of approximations (35) we obtain:

$$\sum_{n=1}^{\infty} \max_{t\in[0,T]} |\, u_n^1(t) - u_n^0(t)\,| \leq |\,\nu\,| \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{3k}\omega} \sum_{i=1}^{m} \left| \frac{\Delta_{4in}\left(\nu,\omega,u_n^0\right)}{\Delta_{0n}(\nu,\omega)} \right| \max_{t\in[0,T]} |g_{3in}(t,\omega)| +$$

$$+ \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{3k}\omega} \max_{t\in[0,T]} \left| \int_0^T G_n(t,s,\nu,\omega)\, u_n^0(s,\nu,\omega)\, ds \right| \leq$$

$$\leq |\,\nu\,|\,\delta_2 \sqrt{\sum_{n=1}^{\infty}\left[\sum_{i=1}^{m}\left|\,\frac{\Delta_{4in}\,(\nu,\omega,u_n^0)}{\Delta_{0n}(\nu,\omega)}\,\right|\,\delta_{0i}\right]^2} + \delta_3\delta_0 < \infty, \tag{37}$$

where

$$\delta_{0i} \geq \delta_{0in} = \max_{t\in[0,T]} |g_{3in}(t,\omega)|\,, \ \ \delta_2 = \sqrt{\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{6k}\omega^2}}, \tag{38}$$

$$\delta_3 = \sqrt{\sum_{n=1}^{\infty} \max_{t\in[0,T]} \left[\frac{1}{\lambda_n^{3k}\omega}\int_0^T |\,G_n(t,s,\nu,\omega)\,|\,ds\right]^2}\,. \tag{39}$$

Continuing this process, similarly to estimate (37) we obtain

$$\sum_{n=1}^{\infty} \max_{t\in[0,T]} \left|\,u_n^{r+1}(t) - u_n^r(t)\,\right| \leq$$

$$\leq |\,\nu\,|\,\delta_2 \sqrt{\sum_{n=1}^{\infty}\left[\sum_{i=1}^{m}\left|\,\frac{\Delta_{4in}\,(\nu,\omega,u_n^r) - \Delta_{4in}\left(\nu,\omega,u_n^{r-1}\right)}{\Delta_{0n}(\nu,\omega)}\,\right|\,\delta_{0i}\right]^2} +$$

$$+\delta_3\sqrt{\sum_{n=1}^{\infty} \max_{t\in[0,T]} \left|\,u_n^r(t,\nu,\omega) - u_n^{r-1}(t,\nu,\omega)\,\right|^2} \leq$$

$$\leq |\,\nu\,|\,\delta_2 \sqrt{\sum_{n=1}^{\infty}\left[\sum_{i=1}^{m}\left|\,\frac{\bar{\Delta}_{4in}\,(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}\,\right|\,\delta_{0i}\right]^2}\,\left\|\,u^r(t,\nu,\omega) - u^{r-1}(t,\nu,\omega)\,\right\|_{B_2} +$$

$$+\delta_3\left\|\,u^r(t,\nu,\omega) - u^{r-1}(t,\nu,\omega)\,\right\|_{B_2} \leq \rho\cdot\left\|\,u^r(t,\nu,\omega) - u^{r-1}(t,\nu,\omega)\,\right\|_{B_2}, \tag{40}$$

where

$$\rho = |\,\nu\,|\,\delta_2 \left\|\sum_{i=1}^{m}\left|\,\frac{\bar{\Delta}_{4in}\,(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}\,\right|\,\delta_{0i}\right\|_{\ell_2} + \delta_3,$$

$$\bar{\Delta}_{4in}\,(\nu,\omega) = \begin{vmatrix} 1-\frac{\nu}{\lambda}\sigma_{31\,1n} & \ldots & \frac{\nu}{\lambda}\sigma_{31\,(i-1)n} & \bar{\sigma}_{41n} & \frac{\nu}{\lambda}\sigma_{31(i+1)n} & \ldots & \frac{\nu}{\lambda}\sigma_{31mn} \\ \frac{\nu}{\lambda}\sigma_{321n} & \ldots & \frac{\nu}{\lambda}\sigma_{32(i-1)n} & \bar{\sigma}_{42n} & \frac{\nu}{\lambda}\sigma_{32(i+1)n} & \ldots & \frac{\nu}{\lambda}\sigma_{32mn} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\nu}{\lambda}\sigma_{3m1n} & \ldots & \frac{\nu}{\lambda}\sigma_{3m(i-1)n} & \bar{\sigma}_{4mn} & \frac{\nu}{\lambda}\sigma_{3m(i+1)n} & \ldots & 1-\frac{\nu}{\lambda}\sigma_{3mmn} \end{vmatrix}, \tag{41}$$

$$\bar{\sigma}_{4in} = \frac{1}{\bar{\lambda}}\int_0^T |\,b_i(s)\,|\int_0^T |\,H_n(s,\theta,\omega)\,\alpha(\theta)\,|\,d\theta\,ds.$$

According to the last condition (34) of the theorem, we have $\rho < 1$. Consequently, it follows from estimate (40) that the operator on the right-hand sides of the countable system of functional-integral equations (30) is contracting. It follows from estimates (36), (37) and (40) that there is a unique fixed point, which is a solution to the countable system of functional-integral equations (30) in the space B_2. Theorem 1 is proved.

4 Uniformly convergence of Fourier series

Theorem 2. Let the conditions of Theorem 1 are fulfilled. Then the series in (29) are convergence in the segment $[0, l]$.

Proof. Let $u_n(t, \nu, \omega) \in B_2$ be a solution of system (30). As in the case of estimates (36) and (40), we obtain

$$|\,\varphi_j(x)\,| \leq \sqrt{\frac{2}{l}} \left(\frac{\sqrt{2l}}{\pi} \right)^{4k+2} \delta_1 \delta_2 \left[\left\| \frac{\partial^{4k+2}\, \psi_1(x)}{\partial\, x^{4k+2}} \right\|_{L_2(\Omega_l)} + \left\| \frac{\partial^{4k+2}\, \psi_2(x)}{\partial\, x^{4k+2}} \right\|_{L_2(\Omega_l)} \right] +$$

$$+\,|\,\nu\,|\;\delta_2 \sqrt{\sum_{n=1}^{\infty} \left[\sum_{i=1}^{m} \left| \frac{\Delta_{4in}(\nu, \omega, u_n)}{\Delta_{0n}(\nu, \omega)} \right| \delta_{0i} \right]^2} + \delta_3 \,\|\, u(t, \nu, \omega)\,\|_{B_2} < \infty, \;\; j = 1, 2. \qquad (42)$$

Absolutely and uniformly convergence of the series (29) implies from estimate (42).

Substituting system (30) into Fourier series (5), we obtain

$$U(t, x, \nu, \omega) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \sin \lambda_n x \;[\psi_{1n}\, g_{1n}(t, \nu, \omega) + \psi_{2n}\, g_{2n}(t, \nu, \omega) +$$

$$+ \frac{\nu}{\lambda_n^k\,(1 + \lambda_n^{2k})\,\omega} \sum_{i=1}^{m} \frac{\Delta_{4in}(\nu, \omega, u_n)}{\Delta_{0n}(\nu, \omega)}\, g_{3in}(t, \omega) +$$

$$+ \frac{1}{\lambda_n^k\,(1 + \lambda_n^{2k})\,\omega} \int\limits_0^T G_n(t, s, \nu, \omega)\, u_n(s, \nu, \omega)\, ds \Bigg], \quad (\nu, \omega) \in \Lambda_4. \qquad (43)$$

Theorem 3. Let the conditions of Theorem 1 are fulfilled. Then the main unknown function $U(t, x, \nu, \omega)$ of inverse problem (1)–(4) is defined by Fourier series (43) and this series (43) converges absolutely and uniformly in the domain Ω for all $(\nu, \omega) \in \Lambda_4$. Moreover, function (43) belongs to the class $C\,(\overline{\Omega}) \cap C_{t,x}^{2,4k}(\Omega) \cap C_{t,x}^{2+2k}(\Omega)$.

The *proof* of Theorem 3 is similar to the proof of Theorem 2.

Conclusion

In the rectangular domain $\Omega = \{0 < t < T,\; 0 < x < l\}$ we consider a linear Benney–Luke type partial integro-differential equation (1) of a higher order with degenerate kernel and two redefinition functions (3) given at the endpoint of the segment [0,T]. With respect to spatial variable x Dirichlet boundary value conditions (2) is used. To find these redefinition functions intermediate data (4) are used. The Fourier series method of variables separation is applied. The countable system of functional-integral equations (30) is obtained. Theorem 1 on a unique solvability of countable system of functional-integral equations (30) is proved. The method of successive approximations is used in combination with the method of contraction mappings. The triple of solutions for the inverse problem is obtained in the form of Fourier series (29) and (43). The absolutely and uniformly convergence of Fourier series is proved (Theorem 2 and 3).

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Екі қайта анықтау функциясы мен параметрлері бар Бенни-Люк типті интегралдық-дифференциалдық теңдеу үшін аралас кері есеп

Мақалада сегменттің шеттерінде берілген екі қайта анықтау функциясы мен өзгешеленетін ядросы бар Бенни-Люк типті жоғары ретті сызықтық интегралдық-дифференциалды дербес туындылы дифференциалдық теңдеуі қарастырылған. Қайта анықтау функцияларын табу үшін аралық берілгендер пайдаланылған. Кеңістіктік айнымалыға қатысты Дирихле типінің шекаралық шарттары қолданылған. Айнымалыны бөліктеу үшін Фурье әдісі пайдаланылды. Функционалдық интегралдық теңдеулердің есептелетін жүйесі алынды. Функционалдық интегралдық теңдеулердің санаулы жүйесінің бірмәнді шешілетіндігі туралы теорема дәлелденді. Бұл жағдайда біртіндеп жуықтау әдісі сығылған бейнелеу әдісімен бірге қолданылады. Кері есептің шешімі Фурье қатары түрінде құрылады. Алынған Фурье қатарының абсолютті және бірқалыпты жинақтылығы нақтыланды.

Кілт сөздер: кері есеп, екі қайта анықтау функциясы, кейінгі шарттар, аралық функциялар, Фурье әдісі, бірмәнді шешім.

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Смешанная обратная задача для интегро-дифференциального уравнения типа Бенни–Люка с двумя функциями переопределения и параметрами

В статье рассмотрено линейное интегро-дифференциальное уравнение в частных производных типа Бенни–Люка высокого порядка с вырожденным ядром и двумя функциями переопределения, заданными в конце отрезка. Для нахождения этих функций переопределения использованы промежуточные данные. По отношению к пространственной переменной применены краевые условия типа Дирихле. Применяется метод разделения переменных Фурье. Получена счетная система функционально-интегральных уравнений. Доказана теорема об однозначной разрешимости счетной системы функционально-интегральных уравнений. При этом используется метод последовательных приближений в сочетании с методом сжатых отображений. Решение обратной задачи строится в виде ряда Фурье. Доказана абсолютная и равномерная сходимость полученных рядов Фурье.

Ключевые слова: обратная задача, две функции переопределения, финальные условия, промежуточные функции, метод Фурье, однозначная разрешимость.

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DOI 10.31489/2023M4/163-172

UDC 517.9

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Iterated discrete Hardy-type inequalities with three weights for a class of matrix operators

Iterated Hardy-type inequalities are one of the main objects of current research on the theory of Hardy inequalities. These inequalities have become well-known after study boundedness properties of the multi-dimensional Hardy operator acting from the weighted Lebesgue space to the local Morrie-type space. In addition, the results of quasilinear inequalities can be applied to study bilinear Hardy inequalities. In the paper, we discussed weighted discrete Hardy-type inequalities containing some quasilinear operators with a matrix kernel where matrix entries satisfy discrete Oinarov condition. The research of weighted Hardy-type inequalities depends on the relations between parameters p, q and θ, so we considered the cases $1 < p \leq q < \theta < \infty$ and $p \leq q < \theta < \infty$, $0 < p \leq 1$, criteria for the fulfillment of iterated discrete Hardy-type inequalities are obtained. Moreover, an alternative method of proof was shown in the work.

Keywords: Inequality, discrete Lebesgue space, Hardy-type operator, weight, quasilinear operator, matrix operator.

Introduction

The iterated integral Hardy-type inequality has the following form

$$\left(\int_0^\infty w^\theta(x)\left(\int_0^x \left|\varphi(t)\int_0^t f(s)ds\right|^q dt\right)^{\frac{\theta}{q}} dx\right)^{\frac{1}{\theta}} \leq C\left(\int_0^\infty |u(x)f(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall f \in L_{p,u}(0,\infty), \tag{1}$$

where $0 < q, p, \theta < \infty$, $u(\cdot)$, $\varphi(\cdot)$ and $w(\cdot)$ are positive functions and locally integrable on the interval $(0;\infty)$, $L_{p,u}(0,\infty)$ is a weighted Lebesgue space of functions for which the right side of the inequality (1) is finite.

At the beginning the inequality (1) has been studied with various quasilinear operators in the works [1, 2]. The equivalence of inequality (1) to the inequality, which defines the boundedness of the multidimensional Hardy operator from the Lebesgue space to the local Morrey-type space has been shown by V. Burenkov and R. Oinarov [3]. After this work researchers have become interested in an iterated integral Hardy-type inequality, then they began to use it intensively [4, 5]. In the last decade, researchers have studied weighted Hardy-type inequalities for the class of quasilinear operators including the kernel [6, 7].

Characterizations of inequality (1) was studied more deeply than discrete analogue. A discrete version of inequality (1) will be as follows

$$\left(\sum_{n=1}^\infty w_n^\theta\left(\sum_{k=1}^n \left|\varphi_k\sum_{i=1}^k f_i\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \leq C_1\left(\sum_{i=1}^\infty |u_i f_i|^p\right)^{\frac{1}{p}}, \quad \forall f \in l_{p.u}, \tag{2}$$

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where the positive constant C is independent from f, $0 < q,p,\theta < \infty$, and $\varphi = \{\varphi_i\}_{i=1}^{\infty}$ is a non-negative sequence, $u = \{u_i\}_{i=1}^{\infty}$, $w = \{w_i\}_{i=1}^{\infty}$ are positive sequences of real numbers. $l_{p,u}$ is the space of sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers such that

$$\|f\|_{p,u} = \left(\sum_{i=1}^{\infty}|u_i f_i|^p\right)^{\frac{1}{p}} < \infty, \ \ 1 \leq p < \infty.$$

Nowadays, inequality (2) is being considered in many works. In the papers [8–10], necessity and sufficient conditions for the fulfillment of iterated discrete Hardy-type inequalities were obtained for the different relations of parameters q, p and θ, namely, for the case $p \leq \theta < \infty$, in the sense that q can be any positive number. The most difficult cases $0 < \theta < min\{p, q\} < \infty$ and $0 < q < \theta < p < \infty$ for these inequalities was studied in the papers [11, 12]. Moreover, the paper [13] includes characterization of the following discrete iterated Hardy-type inequality

$$\left(\sum_{n\in Z} w_n \left(\sup_{i\geq n} \varphi_i \sum_{k\leq i} f_k\right)^{\theta}\right)^{\frac{1}{\theta}} \leq C \left(\sum_{n\in Z} f_n^p u_n\right)^{\frac{1}{p}}.$$

It is obvious to us that by using previously obtained results of iterated Hardy inequalities we can find characteristics of bilinear Hardy inequalities [14–16].

The aim of this paper is to characterize the iterated discrete Hardy-type inequality with matrix kernel defined as follows

$$\left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=1}^{n} \left|\varphi_k \sum_{i=1}^{k} a_{k,i} f_i\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \leq C_1 \left(\sum_{i=1}^{\infty}|u_i f_i|^p\right)^{\frac{1}{p}}, \quad \forall f \in l_{p.u} \tag{3}$$

and the dual discrete Hardy-type inequality has the following form

$$\left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=n}^{\infty} \left|\varphi_k \sum_{i=k}^{\infty} a_{i,k} f_i\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \leq C_2 \left(\sum_{i=1}^{\infty}|u_i f_i|^p\right)^{\frac{1}{p}}, \quad \forall f \in l_{p.u}, \tag{4}$$

where $(a_{k,i})$, $k \geq i \geq 1$, is a matrix non-negative entries of which satisfy the discrete Oinarov condition: there exists constant $d \geq 1$, entries $a_{k,i}$ are non-decreasing in k and non-increasing in i, such that the inequalities

$$\frac{1}{d}(a_{k,j} + a_{j,i}) \leq a_{k,i} \leq d(a_{k,j} + a_{j,i}) \tag{5}$$

hold for all $k \geq j \geq i \geq 1$.

The recent papers [17] and [18], where inequalities (3) and (4) are firstly studied for the matrix $(a_{k,i})$, $i \geq k \geq 1$, entries of which satisfy condition (5). The paper [17] contains results for only inequality (3) for the case $0 < q < p \leq \theta < \infty$. In work [18], authors have used the localization method and considered the case $0 < p \leq \theta < \infty$, $0 < q < \infty$. As we know, we can divide this case into the following three conditions

1) $0 < p \leq \theta < q < \infty$;
2) $0 < p \leq q < \theta < \infty$;
3) $0 < q \leq p \leq \theta < \infty$.

In the paper, we obtained necessity and sufficient conditions for the fulfillment of the inequalities (3) and (4) in the case $0 < p \leq q < \theta < \infty$ by using an alternative method which is different from the method in [18]. This method requires $q < \theta$ condition since we will use the dual principle in the space l_p. It is important to note that in this paper we present the results for the case $0 < p \leq 1$ which is interting because integral Hardy-type inequalities hold in trivial cases only [19].

1 Preliminaries

We need following known statements to obtain the main results. Let's start with reverse Hölder inequalities for weighted sequence l_p spaces and $1 < p < \infty$:

$$\left(\sum_{i=1}^{\infty} d_i^p z_i\right)^{\frac{1}{p}} = \sup_{h\geq 0} \left(\sum_{i=1}^{\infty} d_i h_i\right) \left(\sum_{i=1}^{\infty} h_i^{p'} z_i^{1-p'}\right)^{-\frac{1}{p'}},$$

$$\left(\sum_{i=1}^{\infty} d_i^p z_i\right)^{\frac{1}{p}} = \sup_{h\geq 0} \left(\sum_{i=1}^{\infty} d_i h_i z_i\right) \left(\sum_{i=1}^{\infty} h_i^{p'} z_i\right)^{-\frac{1}{p'}}.$$

We also apply theorems regarding discrete Hardy-type inequality for one class of matrix operators:

$$\left(\sum_{k=1}^{\infty} v_k^q \Big| \sum_{i=1}^{k} a_{k,i} f_i \Big|^q\right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{\infty} |u_i f_i|^p\right)^{\frac{1}{p}}, \quad \forall f \in l_{p,u}, \tag{6}$$

where the entries of the matrix $(a_{k,i})$ satisfy discrete Oinarov condition. The boundedness of Hardy-type operators with matrix kernel was considered in the manuscripts [20–22].

Theorem 1. [21] Let $p \leq q < \infty$ and $0 < p \leq 1$. Let the entries of the matrix $(a_{k,i})$ such that $a_{k,i}$ non-increasing in second index. Then inequality (6) holds if and only if $A_2 < \infty$, where

$$A_1 = \sup_{j\geq 1} \left(\sum_{k=j}^{\infty} a_{k,j}^q v_k^q\right)^{\frac{1}{q}} u_j^{-1} < \infty.$$

Moreover, $C \approx A_2$, where C is the best constant in (6).

Theorem 2. [22] Let $1 < p \leq q < \infty$ and the entries of the matrix $(a_{k,i})$ satisfy condition (5). Then the inequality (6) holds if and only if $A = \max\{A_3, A_4\} < \infty$, where

$$A_2 = \sup_{j\geq 1} \left(\sum_{i=1}^{j} u_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{k=j}^{\infty} a_{k,j}^q v_k^q\right)^{\frac{1}{q}},$$

$$A_3 = \sup_{j\geq 1} \left(\sum_{i=1}^{j} a_{j,i}^{p'} u_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{k=j}^{\infty} v_k^q\right)^{\frac{1}{q}}.$$

Moreover, $C \approx A$, where C is the best constant in (6).

2 The main results

Theorem 3. Let $0 < p \leq q < \theta < \infty$. Let the entries of the matrix $(a_{k,i})$ satisfy condition (5). Then inequality (3) holds if and only if

(i) If $0 < p \leq 1$, $B_1 < \infty$, where

$$B_1 = \sup_{j\geq 1} \left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{i=j}^{n} a_{i,j}^q \varphi_i^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} u_j^{-1}.$$

Moreover, $C_1 \approx B_1$, where C_1 is the best constant in (3).

(ii) If $p > 1$, $B = \max\{B_2, B_3\} < \infty$, where

$$B_2 = \sup_{j\geq 1} \left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{i=j}^{n} a_{i,j}^{q} \varphi_i^{q} \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \left(\sum_{k=1}^{j} u_k^{-p'} \right)^{\frac{1}{p'}},$$

$$B_3 = \sup_{j\geq 1} \left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{i=j}^{n} \varphi_i^{q} \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \left(\sum_{k=1}^{j} a_{j,k}^{p'} u_i^{-p'} \right)^{\frac{1}{p'}}.$$

Moreover, $C_1 \approx B$, where C_1 is the best constant in (3).

Proof. We assume that $0 < p \leq q < \theta < \infty$ and $0 \leq f \in l_{p,u}$. Then from inequality (3) we get that

$$C_1 = \sup_{f\geq 0} \left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=1}^{n} \left| \varphi_k \sum_{i=1}^{k} a_{k,i} f_i \right|^{q} \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \|f\|_{p,u}^{-1} < \infty. \tag{7}$$

By raising both sides of (7) to power q and denoting $S_k = \varphi_k^q \left(\sum\limits_{i=1}^{k} a_{k,i} f_i \right)^q$, we obtain that

$$C_1^q = \sup_{f\geq 0} \left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=1}^{n} S_k \right)^{\frac{\theta}{q}} \right)^{\frac{q}{\theta}} \|f\|_{p,u}^{-q}. \tag{8}$$

As $\frac{\theta}{q} > 1$, we can use of reverse Hölder inequality to (8). Then we find

$$C_1^q = \sup_{f\geq 0} \|f\|_{p,u}^{-q} \sup_{h\geq 0} \left(\sum_{n=1}^{\infty} h_n \sum_{k=1}^{n} S_k \right) \left(\sum_{n=1}^{\infty} h_n^{\frac{\theta}{\theta-q}} w_n^{-\frac{q\theta}{\theta-q}} \right)^{-\frac{\theta-q}{\theta}}$$

$$= \sup_{h\geq 0} \left(\sum_{n=1}^{\infty} \left(h_n w_n^{-q} \right)^{\frac{\theta}{\theta-q}} \right)^{-\frac{\theta-q}{\theta}} \sup_{f\geq 0} \|f\|_{p,u}^{-q} \left(\sum_{n=1}^{\infty} h_n \sum_{k=1}^{n} S_k \right).$$

By replacing S_k we obtain

$$C_1^q = \sup_{h\geq 0} \left(\sum_{n=1}^{\infty} \left(h_n w_n^{-q} \right)^{\frac{\theta}{\theta-q}} \right)^{-\frac{\theta-q}{\theta}} \left[\sup_{f\geq 0} \|f\|_{p,u}^{-1} \left(\sum_{n=1}^{\infty} h_n \sum_{k=1}^{n} \varphi_k^q \left(\sum_{i=1}^{k} a_{k,i} f_i \right)^q \right)^{\frac{1}{q}} \right]^q.$$

By changing the orders of sums we get that

$$C_1^q = \sup_{h\geq 0} \left(\sum_{n=1}^{\infty} \left(h_n w_n^{-q} \right)^{\frac{\theta}{\theta-q}} \right)^{-\frac{\theta-q}{\theta}} \left[\sup_{f\geq 0} \|f\|_{p,u}^{-1} \left(\sum_{k=1}^{\infty} \varphi_k^q \left(\sum_{i=1}^{k} a_{k,i} f_i \right)^q \sum_{n=k}^{\infty} h_n \right)^{\frac{1}{q}} \right]^q. \tag{9}$$

Let us define $H_k := \varphi_k^q \sum\limits_{n=k}^{\infty} h_n$. We will investigate separately the supremum which relates to f.

$$I := \sup_{f\geq 0} \frac{\left(\sum\limits_{k=1}^{\infty} H_k \left(\sum\limits_{i=1}^{k} a_{k,i} f_i \right)^q \right)^{\frac{1}{q}}}{\left(\sum\limits_{i=1}^{\infty} |u_i f_i|^p \right)^{\frac{1}{p}}}. \tag{10}$$

As you have noticed, we have obtained a discrete Hardy-type inequality for a class of matrix operators. Therefore, we consider two conditions regarding p. At first, if $0 < p < 1$, we use Theorem 1, then we have

$$\sup_{f\geq 0}\frac{\left(\sum\limits_{k=1}^{\infty} H_k\left(\sum\limits_{i=1}^{k} a_{k,i}f_i\right)^q\right)^{\frac{1}{q}}}{\left(\sum\limits_{i=1}^{\infty}|u_if_i|^p\right)^{\frac{1}{p}}} \approx \sup_{j\geq 1}\left(\sum_{k=j}^{\infty} a_{k,j}^q H_k\right)^{\frac{1}{q}} u_j^{-1}. \tag{11}$$

By inserting (11) into (9), we get that

$$C_1^q \approx \sup_{h\geq 0}\left(\sum_{n=1}^{\infty}\left(h_n w_n^{-q}\right)^{\frac{\theta}{\theta-q}}\right)^{-\frac{\theta-q}{\theta}} \sup_{j\geq 1} u_j^{-q}\sum_{k=j}^{\infty}\varphi_k^q a_{k,j}^q\sum_{i=k}^{\infty} h_i =$$

$$= \sup_{j\geq 1} u_j^{-q}\sup_{h\geq 0}\left(\sum_{n=1}^{\infty}\left(h_n w_n^{-q}\right)^{\frac{\theta}{\theta-q}}\right)^{-\frac{\theta-q}{\theta}}\sum_{k=j}^{\infty}\varphi_k^q a_{k,j}^q\sum_{i=k}^{\infty} h_i. \tag{12}$$

We denote $z_{k,j} = \varphi_k^q a_{k,j}^q$, $v_n = w_n^{-q}$ and $p_1 = \frac{\theta}{\theta-q}$. Then we rewrite (12) as follows

$$C_1^q \approx \sup_{j\geq 1} u_j^{-q}\sup_{h\geq 0}\frac{\sum\limits_{k=j}^{\infty} z_{k,j}\sum\limits_{i=k}^{\infty} h_i}{\left(\sum\limits_{n=1}^{\infty}(h_nv_n)^{p_1}\right)^{\frac{1}{p_1}}} = \sup_{j\geq 1} u_j^{-q}\sup_{h\geq 0}\frac{\sum\limits_{i=j}^{\infty} h_i\sum\limits_{k=j}^{i} z_{k,j}}{\left(\sum\limits_{n=1}^{\infty}(h_nv_n)^{p_1}\right)^{\frac{1}{p_1}}} = \sup_{j\geq 1} u_j^{-q}\sup_{h\geq 0}\frac{\sum\limits_{i=1}^{\infty}\mathcal{X}_i h_i\sum\limits_{k=j}^{i} z_{k,j}}{\left(\sum\limits_{n=1}^{\infty}(h_nv_n)^{p_1}\right)^{\frac{1}{p_1}}}, \tag{13}$$

where $\mathcal{X}_k = 0$ for $1 \leq k < j$ and $\mathcal{X}_k = 1$ for $k \geq j$. Since $p_1 = \frac{\theta}{\theta-q} > 1$ by applying reverse Hölder inequalities to (13) we get that

$$C_1^q \approx \sup_{j\geq 1} u_j^{-q}\left[\sum_{n=1}^{\infty}\left(\mathcal{X}_n\sum_{i=j}^{n} z_{i,j}\right)^{p_1'} v_n^{-p_1'}\right]^{\frac{1}{p_1'}} = \sup_{j\geq 1} u_j^{-q}\left[\sum_{n=j}^{\infty}\left(\sum_{i=j}^{n} z_{i,j}\right)^{p_1'} v_n^{-p_1'}\right]^{\frac{1}{p_1'}}.$$

Then we rewrite previously applied designations and obtain

$$C_1^q \approx \sup_{j\geq 1} u_j^{-q}\left[\sum_{n=j}^{\infty}\left(\sum_{i=j}^{n} a_{i,j}^q\varphi_i^q\right)^{\frac{\theta}{q}} w_n^{\theta}\right]^{\frac{q}{\theta}},$$

so that

$$C_1 \approx B_1.$$

Therefore, we find that $C_1 \approx B_1$ in the case $0 < p < 1$ and the constant C_1 depends only on the parameters p, q and θ.

Let us start estimating (10) for the case $p > 1$. Actually, by using Theorem 2 we obtain $I \approx \max\{B_2^*, B_3^*\}$, where

$$B_2^* = \sup_{j\geq 1}\left(\sum_{i=1}^{j} u_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{k=j}^{\infty} a_{k,j}^q H_k\right)^{\frac{1}{q}},$$

$$B_3^* = \sup_{j\geq 1} \left(\sum_{i=1}^{j} a_{j,i}^{p'} u_i^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{k=j}^{\infty} H_k \right)^{\frac{1}{q}}.$$

First we estimate (9) with B_2^* and B_3^*. By applying previously used designations and by changing the supremums' order of execution we get that

$$C_1^q \approx \max \left\{ \sup_{j\geq 1} \left(\sum_{i=1}^{j} u_i^{-p'} \right)^{\frac{q}{p'}} \sup_{h\geq 0} \left(\sum_{n=1}^{\infty} (h_n v_n)^{p_1} \right)^{-\frac{1}{p_1}} \sum_{k=j}^{\infty} z_{k,j} \sum_{i=k}^{\infty} h_i, \right.$$

$$\left. \sup_{j\geq 1} \left(\sum_{i=1}^{j} a_{j,i}^{p'} u_i^{-p'} \right)^{\frac{q}{p'}} \sup_{h\geq 0} \left(\sum_{n=1}^{\infty} (h_n v_n)^{p_1} \right)^{-\frac{1}{p_1}} \sum_{k=j}^{\infty} \varphi_k^q \sum_{i=k}^{\infty} h_i \right\}.$$

We can estimate the value of the best constant C_1^q in the same way as we calculated before. By changing the order of sums, applying the reverse Hölder inequality for these results and substituting the designations, we have

$$C_1^q \approx \max \left\{ \sup_{j\geq 1} \left(\sum_{i=1}^{j} u_i^{-p'} \right)^{\frac{q}{p'}} \left(\sum_{n=j}^{\infty} \left(\sum_{i=j}^{n} a_{i,j}^q \varphi_i^q \right)^{\frac{\theta}{q}} w_n^\theta \right)^{\frac{q}{\theta}}, \right.$$

$$\left. \sup_{j\geq 1} \left(\sum_{i=1}^{j} a_{j,i}^{p'} u_i^{-p'} \right)^{\frac{q}{p'}} \left(\sum_{n=j}^{\infty} \left(\sum_{i=j}^{n} \varphi_i^q \right)^{\frac{\theta}{q}} w_n^\theta \right)^{\frac{q}{\theta}} \right\}$$

then

$$C_1^q \approx \max \{B_2^q, B_3^q\}.$$

So we find that $C_1 \approx \max\{B_2, B_3\}$. We obtain $C_1 \approx B_1$ in the condition $0 < p < 1$, $p \leq q < \theta < \infty$ and $C_1 \approx \max\{B_2, B_3\}$ in the condition $1 < p \leq q < \theta < \infty$. Moreover, the equivalence constants depend only on p, q and θ. The proof is complete.

Theorem 4. Let $0 < p \leq q < \theta < \infty$. Let the entries of the matrix $(a_{k,i})$ satisfy condition (5). Then inequality (4) holds if and only if

(i) If $0 < p \leq 1$, $D_1 < \infty$, where

$$D_1 = \sup_{j\geq 1} \left(\sum_{n=1}^{j} w_n^\theta \left(\sum_{i=n}^{j} a_{j,i}^q \varphi_i^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} u_j^{-1}.$$

Moreover, $C_2 \approx D_1$, where C_2 is the best constant in (4).

(ii) If $p > 1$, $D = \max\{D_2, D_3\} < \infty$, where

$$D_2 = \sup_{j\geq 1} \left(\sum_{n=1}^{j} w_n^\theta \left(\sum_{i=n}^{j} a_{j,i}^q \varphi_i^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \left(\sum_{k=j}^{\infty} u_k^{-p'} \right)^{\frac{1}{p'}},$$

$$D_3 = \sup_{j\geq 1}\left(\sum_{n=1}^{j} w_n^{\theta}\left(\sum_{i=n}^{j}\varphi_i^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}\left(\sum_{k=j}^{\infty} a_{k,j}^{p'} u_i^{-p'}\right)^{\frac{1}{p'}}.$$

Moreover, $C_2 \approx D$, where C_2 is the best constant in (4).

Theorem 4 is devoted for inequality (4) and it can be proved similarly as Theorem 3.

Acknowledgments

We thank the referee and Prof. Ryskul Oinarov for some suggestions which improved the final version of this paper. This work was financially supported by the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP09259084).

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Матрицалық операторлар класы үшін үш салмақты итерацияланған дискретті Харди типті теңсіздіктер

Итерацияланған Харди тәріздес теңсіздіктер Харди теңсіздіктері теориясының қазіргі таңдағы зерттеулерінің негізгі объектілерінің бірі. Бұл теңсіздіктер көп өлшемді Харди операторының салмақты Лебег кеңістігінен локальды Морри тәріздес кеңістігінің шенелімділік қасиеттерін зерттегеннен кейін белгілі болды. Сонымен қатар, квазисызықты теңсіздіктердің нәтижелерін қоссызықты Харди теңсіздіктерін зерттеу кезінде қолдануға болады. Мақалада матрицалық ядросы бар кейбір квазисызықты операторлар қатысқан салмақты дискреттік Харди тәріздес теңсіздіктер қарастырылды, мұнда матрица элементтері дискретті Ойнаров шартын қанағаттандырады. Салмақты Харди тәріздес теңсіздіктерді зерттеу p, q және θ параметрлері арасындағы қатынастарға байланысты, сондықтан біз $1 < p \leq q < \theta < \infty$ және $p \leq q < \theta < \infty$, $0 < p \leq 1$ жағдайларын қарастырдық, итерацияланған дискреттік Харди тәріздес теңсіздіктердің орындалу критерийлері алынды. Сонымен қатар бұл жұмыста дәлелдеудің балама әдісі көрсетілген.

Кілт сөздер: теңсіздік, дискретті Лебег кеңістігі, Харди тәріздес оператор, салмақтар, квазисызықты оператор, матрицалық оператор.

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Итерационные дискретные неравенства типа Харди с тремя весами для класса матричных операторов

Итерированные неравенства типа Харди являются одним из основных объектов современных исследований теории неравенств Харди. Эти неравенства стали широко известны после изучения свойств ограниченности многомерного оператора Харди из весового пространства Лебега в локальное пространство типа Морри. Кроме того, результаты квазилинейных неравенств могут быть применены для изучения билинейных неравенств Харди. В статье рассмотрены весовые дискретные неравенства типа Харди, содержащие некоторые квазилинейные операторы с матричным ядром, где элементы матрицы удовлетворяют дискретному условию Ойнарова. Исследование весовых неравенств типа Харди зависит от соотношения параметров p, q и θ, поэтому мы рассмотрели случаи $1 < p \leq q < \theta < \infty$ и $p \leq q < \theta < \infty$, $0 < p \leq 1$; получили критерии выполнения итерационных дискретных неравенств типа Харди в случаях $1 < p \leq q < \theta < \infty$, $p \leq q < \theta < \infty$ и $0 < p \leq 1$. Более того, в работе показан альтернативный метод доказательства.

Ключевые слова: неравенство, дискретное пространство Лебега, оператор типа Харди, вес, квазилинейный оператор, матричный оператор.

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DOI 10.31489/2023M4/173-184
UDC 512.552.16

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Automorphisms of the universal enveloping algebra of a finite-dimensional Zinbiel algebra with zero multiplication

In recent years there has been a great interest in the study of Zinbiel (dual Leibniz) algebras. Let A be Zinbiel algebra over an arbitrary field K and let $e_1, e_2, \ldots, e_m, \ldots$ be a linear basis of A. In 2010 A. Naurazbekova, using the methods of Gröbner-Shirshov bases, constructed the basis of the universal (multiplicative) enveloping algebra $U(A)$ of A. Using this result, the automorphisms of the universal enveloping algebra of a finite-dimensional Zinbiel algebra with zero multiplication are described.

Keywords: Zinbiel (dual Leibniz) algebra, universal (multiplicative) enveloping algebra, basis, automorphism, affine automorphism.

Introduction

An algebra A over a field K is called (left) *dual Leibniz* or *Zinbiel* (Leibniz is written in reverse order) if it satisfies the identity

$$(xy)z = x(yz) + x(zy).$$

The Leibniz algebras form a Koszul operad in the sense of V. Ginzburg and M. Kapranov [1]. Under the Koszul duality, the operad of Lie algebras is dual to the operad of associative and commutative algebras. The notion of Zinbiel (dual Leibniz) algebra defined by J.-L. Loday [2] is precisely the dual operad of Leibniz algebras in their sense. Moreover, any dual Leibniz algebra A with respect to the symmetrization $a \circ b = ab + ba$ is an associative and commutative algebra [2].

Zinbiel algebras are also known as pre-commutative algebras [3] and chronological algebras [4]. A Zinbiel algebra is equivalent to the commutative dendriform algebra [5]. It plays an important role in the definition of pre-Gerstenhaber algebras [6]. The variety of Zinbiel algebras is a proper subvariety in the variety of right commutative algebras. Each Zinbiel algebra with the commutator multiplication gives a Tortkara algebra [7], which appeared in unexpected areas of mathematics [8, 9]. Recently, the notion of matching Zinbiel algebras was introduced in [10]. Zinbiel algebras also appeared in the study of rack cohomology [11], number theory [12] and in a construction of a Cartesian differential category [13]. In recent years there has been a great interest in the study of Zinbiel algebras.

J.-L. Loday (J.-L. Loday) [2] proved that the set of all non-associative words with right arranged parenthesis (right-normed words) form the basis of free Zinbiel algebra. It was shown that free Zinbiel algebras are precisely the shuffle product algebra [14]. A. Naurazbekova [15] proved that free Zinbiel algebras over a field of characteristic zero are the free associative-commutative algebras (without unity) with respect to the symmetrization multiplication and their free generators are found; also she constructed examples of subalgebras of the two-generated free Zinbiel algebra that are free Zinbiel algebras of countable rank. A. Dzhumadildaev and K. Tulenbaev [16] proved the analogue of Nagata-Higman's theorem [17] for the Zinbiel algebras (any Zinbiel nil-algebra is nilpotent). They also proved that every finite-dimensional Zinbiel algebra over an algebraically closed field is solvable and nilpotent

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over the complex number field. A. Naurabekova and U. Umirbaev [18] proved that in characteristic 0 any proper subvariety of variety of Zinbiel algebras is nilpotent and, as a consequence, the variety of Zinbiel algebras is Spechtian and has base rank 1. D.A. Towers [19] showed that every finite-dimensional Zinbiel algebra over an arbitrary field is nilpotent, extending a previous result by other authors that they are solvable. Filiform Zinbiel algebras were described and classified in [20–22]. The classification of complex Zinbiel algebras up to dimension 4 was obtained in [16] and [23]. A partial classification of the 5-dimensional case was done in [24]. M.A. Alvarez, R.F. Junior, I. Kaygorodov [25] proved that the variety of complex 5-dimensional Zinbiel algebras has dimension 24, it is defined by 16 irreducible components and it has 11 rigid algebras.

This paper is devoted to the description of automorphisms of the universal (multiplicative) enveloping algebra of a finite-dimensional Zinbiel algebra with zero multiplication.

The paper is organized as follows. In section 1, for convenience, we rewrite A. Naurazbekova's result [26] on the basis of the universal enveloping algebra of a Zinbiel algebra in new notation. In section 2, we describe automorphisms of the universal enveloping algebra of a finite-dimensional Zinbiel algebra with zero multiplication.

1 The basis of the universal enveloping algebra

Let K be an arbitrary field. An algebra A over a field K is called *dual Leibniz* or *Zinbiel* if it satisfies the identity

$$(xy)z = x(yz) + x(zy).$$

In [2] J.-L. Loday proved that any Zinbiel algebra with respect to multiplication $x \circ y = xy + yx$ is an associative commutative algebra. A linear basis of free Zinbiel algebras is also given in [2].

Let A be an arbitrary Zinbiel algebra over K. Let $L_A = \{L_x | x \in A\}$ and $R_A = \{R_x | x \in A\}$ be two isomorphic copies of the vector space A with the fixed isomorphisms $A \to L_A (x \mapsto L_x)$ and $A \to R_A (x \mapsto R_x)$, respectively. *The universal (multiplicative) enveloping algebra* $U(A)$ [27] is an associative algebra with the identity 1 generated by the two vector spaces L_A and R_A satisfying the defining relations

$$R_x R_y = R_{xy+yx},$$

$$R_x L_y = L_y R_x + L_y L_x,$$

$$L_{xy} = L_x L_y + L_x R_y$$

for all $x, y \in A$. Recall that every dual Leibniz A-bimodule M can be regarded as a left $U(A)$-module with respect to the action

$$L_a m = am, R_a m = ma, \ a \in A, m \in M.$$

Conversely, every left $U(A)$-module can be considered as a Zinbiel A-bimodule [27].

This definition of the universal enveloping algebra is suitable for algebras without identity element. If the identity element 1 is fixed in the signature, then we have to add the relations $L_1 = R_1 = \mathrm{Id} = 1$ and consider only unital modules. It is easy to see that a Zinbiel algebra is an algebra without an identity element. Below we rewrite A. Naurazbekova's [26] result on the basis of U(A) in new notation.

Theorem 1. Let A be a Zinbiel algebra over a field K and let

$$e_1, e_2, \ldots, e_m, \ldots$$

be a linear basis of A. Then the set of all associative words of the form

$$1, L_{e_i}, R_{e_j}, L_{e_i} R_{e_j}, \tag{1}$$

where $i, j \geq 1$, is a linear basis for $U(A)$.

Proof. We define a linear order $\leq$ on the set of associative words in the alphabet $R_{e_i}, L_{e_i}, i \geq 1$. Set $R_{e_i} < R_{e_j}$ and $L_{e_i} < L_{e_j}$ if $i < j$ and $R_{e_i} < L_{e_j}$ for all $i, j \geq 1$. If u and v are two words in the variables R_{e_i}, L_{e_j}, then set $u < v$ if one of the following conditions hold:

(i) $\deg(u) < \deg(v)$, where deg is the degree function with respect to the variables R_{e_i}, L_{e_j};

(ii) $\deg(u) = \deg(v)$, $\deg_L(u) < \deg_L(v)$, where $\deg_L$ is the degree function with respect to the variables L_{e_i};

(iii) $\deg(u) = \deg(v)$, $\deg_L(u) = \deg_L(v)$, and u precedes v with respect to the lexicographical order.

The defining relations of the algebra $U(A)$ are

$$R_{e_i}R_{e_j} - R_{e_i \circ e_j} = 0, \tag{2}$$

$$R_{e_i}L_{e_j} - L_{e_j e_i} = 0, \tag{3}$$

$$L_{e_i}L_{e_j} + L_{e_i}R_{e_j} - L_{e_i e_j} = 0 \tag{4}$$

for all $i, j \geq 1$.

The leading terms of these relations are $R_{e_i}R_{e_j}$, $R_{e_i}L_{e_j}$ and $L_{e_i}L_{e_j}$ for all $i, j \geq 1$. Consequently, the relations (2), (3) and (4) form three types of compositions.

Case 1. Set $w = (R_{e_i}R_{e_j})R_{e_k} = R_{e_i}(R_{e_j}R_{e_k})$. Then the relations (2) form a composition

$$f = (R_{e_i}R_{e_j} - R_{e_i \circ e_j})R_{e_k} - R_{e_i}(R_{e_j}R_{e_k} - R_{e_j \circ e_k}) = -R_{e_i \circ e_j}R_{e_k} + R_{e_i}R_{e_j \circ e_k}$$

with base w. Denote by $\equiv$ the comparison in the free associative algebra in the variables $R_{e_i}, L_{e_i}, i \geq 1$, modulo linear combinations of elements of the form ugv, where g is one of the left hand side of the relations (2), (3) and (4), u and v are associative words, and the leading monomial of ugv is less than w. We have

$$f \equiv -R_{e_i \circ e_j \circ e_k} + R_{e_i \circ e_j \circ e_k} = 0.$$

Case 2. Set $w = (R_{e_i}R_{e_j})L_{e_k} = R_{e_i}(R_{e_j}L_{e_k})$. The relations (2) and (3) form a composition

$$g = (R_{e_i}R_{e_j} - R_{e_i \circ e_j})L_{e_k} - R_{e_i}(R_{e_j}L_{e_k} - L_{e_k e_j}) = -R_{e_i \circ e_j}L_{e_k} + R_{e_i}L_{e_k e_j}$$

with base w. We have

$$g \equiv -L_{e_k(e_i \circ e_j)} + L_{(e_k e_j)e_i} = -L_{e_k(e_i \circ e_j)} + L_{e_k(e_j \circ e_i)} = 0.$$

Case 3. Set $w = (L_{e_i}L_{e_j})L_{e_k} = L_{e_i}(L_{e_j}L_{e_k})$. The relations (4) form a composition

$$h = (L_{e_i}L_{e_j} + L_{e_i}R_{e_j} - L_{e_i e_j})L_{e_k} - L_{e_i}(L_{e_j}L_{e_k} + L_{e_j}R_{e_k} - L_{e_j e_k}) =$$

$$= L_{e_i}R_{e_j}L_{e_k} - L_{e_i e_j}L_{e_k} - L_{e_i}L_{e_j}R_{e_k} + L_{e_i}L_{e_j e_k}$$

with base w. We have

$$h \equiv L_{e_i}L_{e_k e_j} + L_{e_i e_j}R_{e_k} - L_{(e_i e_j)e_k} + L_{e_i}R_{e_j}R_{e_k} - L_{e_i e_j}R_{e_k} + L_{e_i}L_{e_j e_k}$$

$$\equiv -L_{e_i}R_{e_k e_j} + L_{e_i(e_k e_j)} - L_{e_i(e_j e_k)} - L_{e_i(e_k e_j)} + L_{e_i}R_{e_j e_k} + L_{e_i}R_{e_k e_j} - L_{e_i}R_{e_j e_k} + L_{e_i(e_j e_k)} = 0.$$

Consequently, the relations (2), (3) and (4) are closed with respect to composition [28, 29]. This implies [28, 29] that the set of all words that are not divisible by the leading terms is a linear basis of the algebra $U(A)$. Therefore, the set of words of the form (1) is a linear basis for $U(A)$. Theorem 1 is proved.

2 Automorphisms

Let A be a finite-dimensional Zinbiel algebra with zero multiplication over an arbitrary field K. Let $e_1, e_2, ..., e_n$ be a linear basis of A. Then the universal enveloping algebra $U(A)$ of A is generated by the operators R_{e_1},..., R_{e_n}, L_{e_1},..., L_{e_n} and (2)–(4) imply the defining relations of $U(A)$

$$R_{e_i}R_{e_j} = R_{e_i}L_{e_j} = 0, \tag{5}$$

$$L_{e_i}L_{e_j} = -L_{e_i}R_{e_j} \tag{6}$$

for all i, j. By these relations and Theorem 1, the set of all associative words of the form

$$1, L_{e_i}, R_{e_j}, L_{e_i}R_{e_j},$$

where $i, j \in \{1, ..., n\}$, is a linear basis of $U(A)$ and $U(A)$ is a nilpotent algebra over field K with nilpotency index 3.

Theorem 2. Let A be the finite-dimensional Zinbiel algebra with zero multiplication over an arbitrary field K and let $e_1, ..., e_n$ be a linear basis of A. Then the affine automorphism group of the universal enveloping algebra $U(A)$ of A consists of endomorphisms of the form

$$\begin{aligned} \varphi(L_{e_i}) &= \sum_{j=1}^{n} \alpha_{ij} L_{e_j} + \sum_{j=1}^{n} \beta_{ij} R_{e_j}, \\ \varphi(R_{e_i}) &= \sum_{j=1}^{n} (\alpha_{ij} - \beta_{ij}) R_{e_j}, \end{aligned} \tag{7}$$

$1 \leq i \leq n$, $A = (\alpha_{ij}), D = (\delta_{ij})$, where $\delta_{ij} = \alpha_{ij} - \beta_{ij}$, are square matrices of order n over a field K, $\det A \neq 0$ and $\det D \neq 0$.

Proof. Let φ be an affine automorphism of the algebra $U(A)$ and let

$$\varphi(L_{e_i}) = \sum_{j=1}^{n} \alpha_{ij} L_{e_j} + \sum_{j=1}^{n} \beta_{ij} R_{e_j} + \mu_i,$$

$$\varphi(R_{e_k}) = \sum_{t=1}^{n} \gamma_{kt} L_{e_t} + \sum_{t=1}^{n} \delta_{kt} R_{e_t} + \nu_k,$$

$i, k \in \{1, ..., n\}$, $\alpha_{ij}, \beta_{ij}, \mu_i, \gamma_{kt}, \delta_{kt}, \nu_k \in K$ for all i, j, k, t. Since φ is an automorphism of $U(A)$, we have

$$\det \begin{pmatrix} \alpha_{11} & ... & \alpha_{1n} & \beta_{11} & ... & \beta_{1n} \\ ... & ... & ... & ... & ... & ... \\ \alpha_{n1} & ... & \alpha_{nn} & \beta_{n1} & ... & \beta_{nn} \\ \gamma_{11} & ... & \gamma_{1n} & \delta_{11} & ... & \delta_{1n} \\ ... & ... & ... & ... & ... & ... \\ \gamma_{n1} & ... & \gamma_{nn} & \delta_{n1} & ... & \delta_{nn} \end{pmatrix} \neq 0 \tag{8}$$

and (5), (6) imply

$$\varphi(R_{e_i})\varphi(R_{e_k}) = \varphi(R_{e_i})\varphi(L_{e_k}) = 0, \tag{9}$$

$$\varphi(L_{e_i})\varphi(L_{e_k}) = -\varphi(L_{e_i})\varphi(R_{e_k}) \tag{10}$$

for all i, j.

It is easy to see that if $i=j$, (9) and (10) give

$$\nu_i=\mu_i=0 \text{ for all } i.$$

Using (5) and (6), it follows from (9) and (10) that

$$\varphi(R_{e_i})\varphi(R_{e_k})=\sum_{j=1}^{n}\sum_{t=1}^{n}\gamma_{ij}\left(-\gamma_{kt}+\delta_{kt}\right)L_{e_j}R_{e_t}=0,$$

$$\varphi(R_{e_i})\varphi(L_{e_k})=\sum_{j=1}^{n}\sum_{t=1}^{n}\gamma_{ij}\left(-\alpha_{kt}+\beta_{kt}\right)L_{e_j}R_{e_t}=0,$$

$$\varphi(L_{e_i})\varphi(L_{e_k})+\varphi(L_{e_i})\varphi(R_{e_k})=\sum_{j=1}^{n}\sum_{t=1}^{n}\alpha_{ij}\left(-\alpha_{kt}+\beta_{kt}-\gamma_{kt}+\delta_{kt}\right)L_{e_j}R_{e_t}=0$$

for all $i,k\in\{1,...,n\}$. Hence

$$\gamma_{ij}\left(-\gamma_{kt}+\delta_{kt}\right)=\gamma_{ij}\left(-\alpha_{kt}+\beta_{kt}\right)=\alpha_{ij}\left(-\alpha_{kt}+\beta_{kt}-\gamma_{kt}+\delta_{kt}\right)=0 \tag{11}$$

for all $i,j,k,t\in\{1,...,n\}$.

Suppose that $\gamma_{ij}\neq 0$ for some i,j. It follows from (11) that

$$\gamma_{kt}=\delta_{kt},\alpha_{kt}=\beta_{kt} \text{ for all } k,t.$$

This contradicts (8). Consequently, $\gamma_{ij}=0$ for all i,j. Using this and (8), we obtain

$$\det A\neq 0,\det D\neq 0,$$

where $A=(\alpha_{ij}),D=(\delta_{ij})$ are square matrices of order n over a field K. It is clear that there exists i,j such that $\alpha_{ij}\neq 0$. It follows from (11) that

$$\delta_{kt}=\alpha_{kt}-\beta_{kt} \text{ for all } k,t.$$

Consequently, if ϕ is an affine automorphism of $U(A)$, then ϕ has the form (7).

It is obvious that any endomorphism of the form (7) is an automorphism of the algebra $U(A)$. Theorem 2 is proved.

Lemma 1. Let $A=(a_{ij})$ and $B=(b_{ks})$ be non-zero square matrices of orders n and m, respectively. Then

$$\det\begin{pmatrix} a_{11}B & a_{12}B & ... & a_{1n}B\\ a_{21}B & a_{22}B & ... & a_{2n}B\\ ... & ... & ... & \\ a_{n1}B & a_{n2}B & ... & a_{nn}B\end{pmatrix}=(\det A)^m\cdot(\det B)^n.$$

Proof. Prove the statement of the lemma by induction on $n+m$. Without loss of generality, assume $a_{11}\neq 0$. By the induction proposition, we get

$$\det\begin{pmatrix} a_{11}B & a_{12}B & ... & a_{1n}B\\ a_{21}B & a_{22}B & ... & a_{2n}B\\ ... & ... & ... & ...\\ a_{n1}B & a_{n2}B & ... & a_{nn}B\end{pmatrix}=\det\begin{pmatrix} a_{11}B & a_{12}B & ... & a_{1n}B\\ 0 & \left(\frac{a_{21}a_{12}}{a_{11}}-a_{22}\right)B & ... & \left(\frac{a_{21}a_{1n}}{a_{11}}-a_{2n}\right)B\\ ... & ... & ... & ...\\ 0 & \left(\frac{a_{n1}a_{12}}{a_{11}}-a_{n2}\right)B & ... & \left(\frac{a_{n1}a_{1n}}{a_{11}}-a_{nn}\right)B\end{pmatrix}=$$

$$= \det(a_{11}B)\cdot\det\begin{pmatrix} \left(\frac{a_{21}a_{12}}{a_{11}}-a_{22}\right)B & ... & \left(\frac{a_{21}a_{1n}}{a_{11}}-a_{2n}\right)B \\ ... & ... & ... \\ \left(\frac{a_{n1}a_{12}}{a_{11}}-a_{n2}\right)B & ... & \left(\frac{a_{n1}a_{1n}}{a_{11}}-a_{nn}\right)B \end{pmatrix} =$$

$$= a_{11}^m \det B\cdot\left(\det\begin{pmatrix} \left(\frac{a_{21}a_{12}}{a_{11}}-a_{22}\right) & ... & \left(\frac{a_{21}a_{1n}}{a_{11}}-a_{2n}\right) \\ ... & ... & ... \\ \left(\frac{a_{n1}a_{12}}{a_{11}}-a_{n2}\right) & ... & \left(\frac{a_{n1}a_{1n}}{a_{11}}-a_{nn}\right) \end{pmatrix}\right)^m (\det B)^{n-1} =$$

$$= a_{11}^m \det B\cdot\left(\frac{1}{a_{11}}\det\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ 0 & \left(\frac{a_{21}a_{12}}{a_{11}}-a_{22}\right) & ... & \left(\frac{a_{21}a_{1n}}{a_{11}}-a_{2n}\right) \\ ... & ... & ... & ... \\ 0 & \left(\frac{a_{n1}a_{12}}{a_{11}}-a_{n2}\right) & ... & \left(\frac{a_{n1}a_{1n}}{a_{11}}-a_{nn}\right) \end{pmatrix}\right)^m (\det B)^{n-1} =$$

$$= a_{11}^m \det B\cdot\frac{1}{a_{11}^m}(\det A)^m(\det B)^{n-1} = (\det A)^m(\det B)^n.$$

Lemma 1 is proved.

Theorem 3. Let A be the finite-dimensional Zinbiel algebra with zero multiplication over an arbitrary field K and let $e_1, e_2, ..., e_n$ be a linear basis of A. Then the automorphism group of the universal enveloping algebra $U(A)$ of A consists of endomorphism of the form

$$\begin{gathered} \varphi(L_{e_i}) = f_i + \sum_{j=1}^n \alpha_{ij}L_{e_j} + \sum_{j=1}^n \beta_{ij}R_{e_j}, \\ \varphi(R_{e_i}) = g_i + \sum_{j=1}^n \tau_{ij}R_{e_j}, \end{gathered} \tag{12}$$

$1 \le i \le n$, f_i, g_i are any homogeneous elements of degree 2 of $U(A)$, $A = (\alpha_{ij}), T = (\tau_{ij})$, where $\tau_{ij} = \alpha_{ij} - \beta_{ij}$, are square matrices of order n over a fielf K, $\det A \neq 0$ and $\det T \neq 0$.

Proof. Let φ be any automorphism of the algebra $U(A)$. By Theorem 2, the affine part of φ has the form (7). Since φ is an automorphism of $U(A)$, φ satisfies the equalities (9) and (10). Using (5) and (6), it is easy to see that φ has the form (12).

Let φ be an endomorphism of $U(A)$ of the form (12) and let

$$f_i = \sum_{k=1}^n\sum_{t=1}^n \gamma_{kt}^{(i)} L_{e_k}R_{e_t},$$

$$g_i = \sum_{k=1}^n\sum_{t=1}^n \delta_{kt}^{(i)} L_{e_k}R_{e_t},$$

$1 \le i \le n$. Prove that φ is an automorphism of $U(A)$, i.e., prove that φ has an inverse endomorphism φ'. To find the endomorphism φ' in the following form

$$\varphi'(L_{e_i}) = \sum_{k=1}^n\sum_{t=1}^n \gamma_{kt}'^{(i)} L_{e_k}R_{e_t} + \sum_{j=1}^n \alpha'_{ij}L_{e_j} + \sum_{j=1}^n \beta'_{ij}R_{e_j},$$

$$\varphi'(R_{e_i}) = \sum_{k=1}^n\sum_{t=1}^n \delta_{kt}'^{(i)} L_{e_k}R_{e_t} + \sum_{j=1}^n \tau'_{ij}R_{e_j}.$$

Since the affine part of φ' is inverse to the affine part of φ, it is easy to find all coefficients $\alpha'_{ij}, \beta'_{ij}, \tau'_{ij}$.

Let $h \in U(A)$. Denote by $\overline{h}$ the homogeneous part of degree 2 of the element h.

Using (5) and (6), we get

$$0 = \overline{\varphi' \circ \varphi(L_{e_i})} = \overline{\sum_{k=1}^{n} \sum_{t=1}^{n} \gamma_{kt}^{(i)} \varphi'(L_{e_k}) \varphi'(R_{e_t}) + \sum_{j=1}^{n} \alpha_{ij} \varphi'(L_{e_j}) + \sum_{j=1}^{n} \beta_{ij} \varphi'(R_{e_j})} =$$

$$= \sum_{k=1}^{n} \sum_{t=1}^{n} \gamma_{kt}^{(i)} \left(\sum_{p=1}^{n} \alpha'_{kp} L_{e_p} \right) \left(\sum_{s=1}^{n} \tau'_{ts} R_{e_s} \right) +$$

$$+ \sum_{j=1}^{n} \alpha_{ij} \left(\sum_{a=1}^{n} \sum_{b=1}^{n} \gamma'^{(j)}_{ab} L_{e_a} R_{e_b} \right) + \sum_{j=1}^{n} \beta_{ij} \left(\sum_{a=1}^{n} \sum_{b=1}^{n} \delta'^{(j)}_{ab} L_{e_a} R_{e_b} \right)$$

and

$$0 = \overline{\varphi' \circ \varphi(R_{e_i})} = \overline{\sum_{k=1}^{n} \sum_{t=1}^{n} \delta_{kt}^{(i)} \varphi'(L_{e_k}) \varphi'(R_{e_t}) + \sum_{j=1}^{n} \tau_{ij} \varphi'(R_{e_j})} =$$

$$= \sum_{k=1}^{n} \sum_{t=1}^{n} \delta_{kt}^{(i)} \left(\sum_{p=1}^{n} \alpha'_{kp} L_{e_p} \right) \left(\sum_{s=1}^{n} \tau'_{ts} R_{e_s} \right) + \sum_{j=1}^{n} \tau_{ij} \left(\sum_{a=1}^{n} \sum_{b=1}^{n} \delta'^{(j)}_{ab} L_{e_a} R_{e_b} \right).$$

Since all the coefficients α'_{ij}, τ'_{ij} are known, it follows from these equalities

$$\sum_{j=1}^{n} \left(\alpha_{ij} \gamma'^{(j)}_{ab} + \beta_{ij} \delta'^{(j)}_{ab} \right) L_{e_a} R_{e_b} = \mu_{ab}^{(i)} L_{e_a} R_{e_b},$$

$$\sum_{j=1}^{n} \tau_{ij} \delta'^{(j)}_{ab} L_{e_a} R_{e_b} = \nu_{ab}^{(i)} L_{e_a} R_{e_b},$$

for all $i, a, b \in \{1, ..., n\}$, where $\mu_{ab}^{(i)}, \nu_{ab}^{(i)}$ are some elements of K. For each a, b we obtain the following system of $2n$ linear equations with unknowns $\gamma'^{(1)}_{ab}, ..., \gamma'^{(n)}_{ab}, \delta'^{(1)}_{ab}, ..., \delta'^{(n)}_{ab}$:

$$\begin{cases} \sum_{j=1}^{n} \left(\alpha_{ij} \gamma'^{(j)}_{ab} + \beta_{ij} \delta'^{(j)}_{ab} \right) = \mu_{ab}^{(i)} \\ \sum_{j=1}^{n} \tau_{ij} \delta'^{(j)}_{ab} = \nu_{ab}^{(i)}, \end{cases}$$

$1 \le i \le n$. Since $\det A \ne 0, \det T \ne 0$, this system has the solution for each a, b . Consequently, there exists a left inverse of φ.

Using (5) and (6), we also get

$$0 = \overline{\varphi \circ \varphi'(L_{e_i})} = \overline{\sum_{k=1}^{n} \sum_{t=1}^{n} \gamma'^{(i)}_{kt} \varphi(L_{e_k}) \varphi(R_{e_t}) + \sum_{j=1}^{n} \alpha'_{ij} \varphi(L_{e_j}) + \sum_{j=1}^{n} \beta'_{ij} \varphi(R_{e_j})} =$$

$$= \sum_{k=1}^{n} \sum_{t=1}^{n} \gamma'^{(i)}_{kt} \left(\sum_{p=1}^{n} \alpha_{kp} L_{e_p} \right) \left(\sum_{s=1}^{n} \tau_{ts} R_{e_s} \right) +$$

$$+ \sum_{j=1}^{n} \alpha'_{ij} \left(\sum_{a=1}^{n} \sum_{b=1}^{n} \gamma_{ab}^{(j)} L_{e_a} R_{e_b} \right) + \sum_{j=1}^{n} \beta'_{ij} \left(\sum_{a=1}^{n} \sum_{b=1}^{n} \delta_{ab}^{(j)} L_{e_a} R_{e_b} \right)$$

and

$$0=\overline{\varphi\circ\varphi'(R_{e_i})}=\overline{\sum_{k=1}^{n}\sum_{t=1}^{n}\delta_{kt}'^{(i)}\varphi(L_{e_k})\varphi(R_{e_t})+\sum_{j=1}^{n}\tau_{ij}'\varphi(R_{e_j})}=$$

$$=\sum_{k=1}^{n}\sum_{t=1}^{n}\delta_{kt}'^{(i)}\left(\sum_{p=1}^{n}\alpha_{kp}L_{e_p}\right)\left(\sum_{s=1}^{n}\tau_{ts}R_{e_s}\right)+\sum_{j=1}^{n}\tau_{ij}'\left(\sum_{a=1}^{n}\sum_{b=1}^{n}\delta_{ab}^{(j)}L_{e_a}R_{e_b}\right).$$

Since all the coefficients $\alpha_{ij}', \beta_{ij}', \tau_{ij}'$ are known, it follows from these equalities

$$\sum_{k=1}^{n}\sum_{t=1}^{n}\gamma_{kt}'^{(i)}\alpha_{kp}\tau_{ts}L_{e_p}R_{e_s}=\lambda_{ps}^{(i)}L_{e_p}R_{e_s},$$

$$\sum_{k=1}^{n}\sum_{t=1}^{n}\delta_{kt}'^{(i)}\alpha_{kp}\tau_{ts}L_{e_p}R_{e_s}=\sigma_{ps}^{(i)}L_{e_p}R_{e_s},$$

for all $i,p,s\in\{1,...,n\}$, $\lambda_{ps}^{(i)},\sigma_{ps}^{(i)}$ are some elements of K. For each i we obtain the following two systems of n^2 linear equations with unknowns $\gamma_{11}'^{(i)},...,\gamma_{1n}'^{(i)},...,\gamma_{n1}'^{(i)},...,\gamma_{nn}'^{(i)}$ and $\delta_{11}'^{(i)},...,\delta_{1n}'^{(i)},...,\delta_{n1}'^{(i)},...,\delta_{nn}'^{(i)}$, respectively:

$$\left\{\gamma_{11}'^{(i)}\alpha_{1p}\tau_{1s}+...+\gamma_{1n}'^{(i)}\alpha_{1p}\tau_{ns}+...+\gamma_{n1}'^{(i)}\alpha_{np}\tau_{1s}+...+\gamma_{nn}'^{(i)}\alpha_{np}\tau_{ns}=\lambda_{ps}^{(i)}\right.$$

and

$$\left\{\delta_{11}'^{(i)}\alpha_{1p}\tau_{1s}+...+\delta_{1n}'^{(i)}\alpha_{1p}\tau_{ns}+...+\delta_{n1}'^{(i)}\alpha_{np}\tau_{1s}+...+\delta_{nn}'^{(i)}\alpha_{np}\tau_{ns}=\sigma_{ps}^{(i)}\right.,$$

$p,s\in\{1,...,n\}$. The coefficient matrices of these systems have the form

$$C=\begin{pmatrix}\alpha_{11}T & \alpha_{21}T & ... & \alpha_{n1}T\\ \alpha_{12}T & \alpha_{22}T & ... & \alpha_{n2}T\\ ... & ... & ... & \\ \alpha_{1n}T & \alpha_{2n}T & ... & \alpha_{nn}T\end{pmatrix}.$$

By Lemma 1, $\det C=(\det A)^n(\det T)^n$. Since $\det A\neq 0$, $\det T\neq 0$, we have $\det C\neq 0$. It follows that this systems has the solutions for each $1\le i\le n$. Consequently, there exists a right inverse of φ. Since in groups the left and right inverses coincide, there exists an inverse of φ. Hence φ is an automorphism of the algebra $U(A)$. Theorem 3 is proved.

Acknowledgments

This work was supported by the grant of the Ministry of Science and Higher Education of the Republic of Kazakhstan (project AP09261086).

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Нөлдік көбейтіндісі бар ақырлыөлшемді Зинбил алгебрасының универсалды ораушы алгебрасының автоморфизмдері

Соңғы жылдары Зинбил алгебраларын (дуалды Лейбниц алгебраларын) зерттеуге үлкен қызығушылық бар. Айталық A кез келген K өрісіндегі құрастырылған Зинбил алгебрасы және $e_1, e_2, \ldots, e_m, \ldots$ A алгебрасының сызықты базисі. 2010 жылы А. Науразбекова Грёбнер-Ширшов базистерінің әдістерін қолданып, A алгебрасының $U(A)$ универсалды (мультипликативті) ораушы алгебрасының базисін құрастырды. Осы нәтижені пайдаланып, нөлдік көбейтіндісі бар ақырлыөлшемді Зинбил алгебрасының универсалды ораушы алгебрасының автоморфизмдері сипатталған.

Кілт сөздер: Зинбил (дуалды Лейбниц) алгебрасы, универсалды (мультипликативті) ораушы алгебра, базис, автоморфизм, аффинді автоморфизм.

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Автоморфизмы универсальной обертывающей алгебры конечномерной алгебры Зинбиля с нулевым умножением

В последние годы наблюдается большой интерес к изучению алгебр Зинбиля (дуальных алгебр Лейбница). Пусть A алгебра Зинбиля над произвольным полем K и пусть $e_1, e_2, \ldots, e_m, \ldots$ линейный базис алгебры A. В 2010 году А. Науразбекова, применяя методы базисов Грёбнера–Ширшова, построила базис универсальной (мультипликативной) обертывающей алгебры $U(A)$ алгебры A. Используя данный результат, описаны автоморфизмы универсальной обертывающей алгебры конечномерной алгебры Зинбиля с нулевым умножением.

Ключевые слова: алгебра Зинбиля (дуальная алгебра Лейбница), универсальная (мультипликативная) обертывающая алгебра, базис, автоморфизм, аффинный автоморфизм.

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DOI 10.31489/2023M4/185-193

UDC 510.67

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Model-theoretic properties of semantic pairs and e.f.c.p. in Jonsson spectrum

The article is committed to the study of model-theoretic properties of stable hereditary Jonsson theories, wherein we consider Jonsson theories that retain jonssonnes for any permissible enrichment. The paper proves a generalization of stability that relates stability and classical stability for Jonsson spectrum. This paper introduces new concepts such as "existentially finite cover property" and "semantic pair". The basic properties of e.f.c.p. and semantic pairs in the class of stable perfect Jonsson spectrum are studied.

Keywords: Jonsson theory, semantic model, permissible enrichment, central type, hereditary theory, stable theory, perfect theory, fundamental order, saturated model, e.f.c.p., existentially closed pair, semantic pair.

Introduction

The concept of language enrichment plays a significant role in description the model-theoretic characteristics of both theories itself and models. Language enrichment options are limited by first-order language rules. In this article we are dealing with language enrichment using a one-place predicate symbol and some constant symbol. The next important point of novelty and relevance of this work is the fact that all new concepts and corresponding statements were concerned within the system of the study within the framework of the study generally speaking incomplete theories. Namely, in the class of Jonsson theories. This class is quite broad and its application covers many areas of modern mathematics. The remark about the incompleteness of the theories under consideration is relevant in the sense that the modern apparatus of model theories is developing within the system of the study of complete theories. This article presents results that clarify previously obtained theorems related to the classical concept of stability within the framework of complete theories and its generalizations.

In this work we are going to highlight the fact that we consider many classical concepts associated with the concept of stability for Jonsson theories and their types within the framework of such a new concept as the Jonsson spectrum of cosemanticness a model or class models. This concept allows us to classify Jonssons theories regarding the relation of cosemantic. Also, to find analogues of basic theorems from stability theory, such as with theorems associated with the concept f.c.p. [1], in our case, for these purposes, the idea of using the concept of the central type of Jonsson theories is used.

And here we present results related to the concept of stability of perfect Jonsson theories, and also obtain results regarding the Jonsson spectrum for semantic pairs. Semantic pairs are a generalization of beautiful pairs, which started to be explored deliberately in the work [1] of B. Poizat. In this work, B. Poizat investigated structures of a common form in which elementary substructures are distinguished. He formed the question of finding for conditions under which the theory of elementary pairs is complete. Subsequently, the works of [2–8] and others were devoted to the study of this issue. Commonly, reflection of the work of [2–8] played a significant role in the study of the issue of incomplete theory, that is, Jonsson theories. In the works of A.R. Yeshkeyev we can find a complete description of Jonsson theories regarding this issue [9–14].

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1 Local properties of the Jonsson spectrum in stable theory

The main result in the article is developed within the framework of the Jonsson theory. Since this work is not the first work in the study of the Jonsson theory, I did not want to rewrite the definition and related original concepts and theorems. A detailed description of the Jonsson theory and the initial concepts and theorems related to the theory can be found in work [15–19].

Further we will prove the main results for some fixed Jonsson spectrum. Before that, a number of results related to Jonsson spectrum were obtained. In particular, the generalization of the classical theorem on elementary equivalence of abelian groups and modules, which is one of the important concepts in algebra, is given in works [20–22].

Definition 1. [21] Let σ be an arbitrary signature, L be the set of all formulas of signature σ. Let $\mathcal{B}$ be an arbitrary model of some fixed signature σ, $\mathcal{B} \in Mod\sigma$. Let us call the Jonsson spectrum of the model $\mathcal{B}$ the set:

$$JSp(\mathcal{B}) = \{T/\mathcal{B} \in ModT, T \text{ is Jonsson theory of the signature } \sigma\}.$$

Next, we obtain the following factor set by the cosemantic relation

$$JSp(\mathcal{B})/_{\bowtie} = \{[T]|T \in JSp(\mathcal{B})\}.$$

Let $[T] \in JSp(\mathcal{B})/_{\bowtie}$. Since each theory $\Delta \in [T]$ has $\mathcal{C}_{\Delta} = \mathcal{C}_T$, then the semantic model of the $[T]$ class will be called the semantic model of the T theory: $\mathcal{C}_{[T]} = \mathcal{C}_T$. The center of the Jonsson class $[T]$ will be called the elementary theory $[T]^*$, its semantic model $\mathcal{C}_{[T]}$, i.e. $[T]^* = Th(C_{[T]})$ and $[T]^* = Th(\mathcal{C}_{\Delta})$ for any $\Delta \in [T]$. Denote by $E_{[T]} = \bigcup_{\Delta \in [T]} E_{\Delta}$ the class of all existentially closed models of the class $[T] \in JSp(\mathcal{B})/_{\bowtie}$. Note that $\bigcap_{\Delta \in [T]} \neq \emptyset$, since at least for each $\Delta \in [T]$ we have $\mathcal{C}_{[T]} \in E_{\Delta}$.

Definition 2. [21] The class $JSp(B)/_{\bowtie}$ is called perfect (further, $PJSp(A)/_{\bowtie}$) if each class $[T] \in JSp(B)/_{\bowtie}$ is perfect, $[T]$ is called perfect if $C_{[T]}$ is a saturated model.

$$PJSp(\mathcal{B}) = \{T|T \text{ is perfect Jonsson theory in language } \sigma \text{ and } \mathcal{B} \in ModT\}.$$

It is clear that $PJSp(\mathcal{B}) \subseteq JSp(B)$.

Theorem 1. [17]. Let T be a perfect Jonsson theory. Then the following conditions are equivalent:
1) T^* is a model companion of the T theory;
2) $ModT^* = E_T$;
3) $T^* = T^f$, where E_T is the class of T-existentially closed models T, $T^f = Th(F_T)$, where F_T is the class of generic T models (in the sense of Robinson's finite forcing).

Let T be a Jonsson theory, $S^J(X)$ the set of all existential n-complete types over X consistent with T for every finite n.

Definition 3. [17] We say that a Jonsson theory T is J-λ-stable if for any T-existentially closed model A, for any subset X of the set A, $|X| \leq \lambda \Rightarrow |S^J(X)| \leq \lambda$.

At one time, the author in [23] proved a theorem that connects the concepts of J-stability and classical stability for perfect Jonsson theories. And this result generalizes the concepts of stability. Now we want to define the concepts for the Jonsson spectrum.

Theorem 2. Let $[T]$ be a perfect Jonsson $\exists$-complete class, $\lambda \geq \omega$. Let $\mathcal{C}_{[T]}$ be its semantic model, $\mathcal{A} \preceq_{\exists_1} C_{[T]}$ and $\mathcal{A}$ is the existentially closed model of $[T]$, $[T] \in JSp(\mathcal{A})/_{\bowtie}$. The Jonsson class $[T]$ is J-λ-stable if and only if the center of the Jonsson class $[T]^*$ is λ-stable (in the classical sense).

Proof. We will work only with perfect Jonsson theories $PJSp(A)/_{\bowtie}$. Let $[T] \in PJSp(T)/_{\bowtie}$ and $E_n([T])$ be the distributive lattice of equivalence classes

$$\varphi^{[T]} = \{\psi \in E_n(L) \mid [T]^* \models \varphi \leftrightarrow \psi, \varphi \in E_n(L)\}.$$

We will call the Jonsson class $[T]$ stable if every theory $\Delta \in [T]$ is a stable theory by the Definition 3.

If $[T] \subset [T]^*$, then $E_n([T]) \subset E_n(Th(C_{[T]}))$, where $E_n([T])$, $E_n(Th(C_{[T]}))$ are the corresponding lattices of existential formulas. The class $[T]$ is complete for existential propositions, which means that if every theory in $[T]$ is a complete theory, therefore $E_n([T]) = E_n(Th(C_{[T]}))$. $[T] \in JSp(\mathcal{A})/_{\bowtie}$ is perfect, then the semantic model $\mathcal{C}_{[T]}$ is saturated, every Jonsson theory $\Delta \in [T]$ is perfect. Then, by Theorem 1, each $\Delta \in [T]$ has a model companion. Since $[T] \in JSp(\mathcal{A})/_{\bowtie}$ is perfect $[T]^*$ is model complete by Theorem 2.9.15 [17] , $[T]^* = Th(C_{[T]})$ if and only if $\forall n < \omega, \forall \varphi \in F_n(\Delta^*)\ \exists \theta \in E_n(\Delta^*) : \Delta^* \vdash \varphi \leftrightarrow \theta$.

Let the Jonsson class $[T]$ be J-λ-stable, this means that if in the class there is a theory from $[T]$ that is stable, then by Definition 3 for each model $\mathcal{A} \in E_\Delta$ we have that for each subset $X \subset A$, if $|X| \leq \lambda$ then $|S^J(X)| \leq \lambda$.

Note that if the class is perfect, then all E_Δ for $\Delta \in [T]$ are equal to each other.

Suppose that $[T]^*$ is not λ-stable. Then there exists $\mathcal{A} \in E_\Delta = Mod\Delta^*$, by Theorem 1, so there is $X \subset A$ such that $|X| < \lambda, \exists n < \omega \Rightarrow |S^J(X)| > \lambda$. For each formula $\varphi \in p$, where $p \in S_n(X)$, we replace φ with θ satisfying the properties $\Delta^* \vdash \varphi \leftrightarrow \theta$ and $\theta \in E_n([T]^*)$. Let p' be p after replacement. Then $p' \in S^J(X)$ and $|S^J(X)| > \lambda$. This contradicts the J-λ-stability of the class $[T]$.

2 The central type of a semantic pair

Since our main goal in this article is to consider the special properties of central types, we will work with some signature enrichments in which some fixed Jonsson theory is given, other questions regarding this can be found [18, 19, 24, 25].

In the future, the entire theory under consideration will be hereditary. We gave a detailed description of the hereditary theory in paper [19]. Now let's talk about the hereditary class. A class is hereditary if every theory in that class is hereditary.

Let us consider some extension of signature σ and consider the central type of this extension for all Jonsson theories $[T] \in PJSp(T)/_{\bowtie}$. And the central types here are taken from enrichment $\odot$ in the previous work [19].

Next, we consider the concept of "finite cover property" which arises from the work of Shelah [26]. In his works Shelag shows the following: an unstable theory has f.c.p., but this is not of great importance for us, since we will only consider stable theories.

Δ denotes a set of formulas of the form $\varphi(\bar{x}, \bar{y})$. An m-formula, or φ-m-formula is a formula of the form $\varphi(\bar{x}, \bar{y})$ or $\varphi(\bar{x}, \bar{a})$ where $l(\bar{x}) = m$ and we consider $\bar{y}$ as a sequence of parameters for which we will usually substitute some $\bar{a}$ and get $\varphi(\bar{x}, \bar{a})$.

Definition 4. [26; 62] Let $\varphi(\bar{x}, \bar{y}) \in L$, $\bar{a}^0, \ldots, \bar{a}^{n-1} \in A$.

(1) $\varphi(\bar{x}, \bar{y})$ has the finite cover property (f.c.p.) if for arbitrarily large natural numbers n there $\bar{a}^0, \ldots, \bar{a}^{n-1}$ such that $\models \neg(\exists \bar{x}) \bigwedge_{k<n} \varphi(\bar{x}, \bar{a}^k)$ but for every $l < n$, $\models (\exists \bar{x}) \bigwedge_{k<n, k \neq l} \varphi(\bar{x}, \bar{a}^k)$.

(2) T has the f.c.p. if there exists a formula $\varphi(x, \bar{y})$ which has the f.c.p.

And so, for a special case, we form matrices where rows consist of $\varphi - 1$-type. And this matrix will be the central type, and all partitions of the central type from the enrichment $\odot$ will be f.c.p.

$$p^c = \begin{pmatrix} \varphi_1^s(x, b_1) & \varphi_1^s(x, b_2) & \ldots & \varphi_1^s(x, b_n) & \ldots \\ \varphi_2^s(x, b_1) & \varphi_2^s(x, b_2) & \ldots & \varphi_2^s(x, b_n) & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \varphi_n^s(x, b_1) & \varphi_n^s(x, b_2) & \ldots & \varphi_n^s(x, b_n) & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} s = \{0, 1\}.$$

For example, $n = 3$, $k = 1$, $l = 2$, then f.c.p:

$$\begin{pmatrix} \varphi_1^0(x,a_1) & \varphi_1^1(x,a_2) & \varphi_1^1(x,a_3) \\ \varphi_2^1(x,a_1) & \varphi_2^0(x,a_2) & \varphi_2^1(x,a_3) \\ \varphi_3^1(x,a_1) & \varphi_3^1(x,a_2) & \varphi_3^0(x,a_3) \end{pmatrix}.$$

From this it can be seen that f.c.p. can be extended to the central type, while p^c must preserve the hereditary property. When central type = f.c.p., then the theory will be unstable.

In this work, B. Poizat's results on beautiful pairs are generalized on the case of $\exists$-complete J-λ-stable hereditary Jonsson theory. Instead of f.c.p. and a type, we consider existentially finite cover property (e.f.c.p.) and a central type, correspondingly, in a specific expansion of the signature. Professor A. Yeshkeyev first made a report on this at the conference Logic Colloquium-2023 [27]:

Definition 5. [27] Let T be the Jonsson L-theory and $f(\bar{x},\bar{y})$ be an $\exists$ formula of L language. If for any arbitrary large n exists $\bar{a}^0,\dots,\bar{a}^{n-1}$ in some existentially closed model of T and $\bar{a}^0,\dots,\bar{a}^{n-1}$ satisfies $\neg(\exists\bar{x})\bigwedge_{k<n} f(\bar{x},\bar{a}^k)$ and for any $l < n$ $\neg(\exists\bar{x})\bigwedge_{k<n} f(\bar{x},\bar{a}^k)$, then $f(\bar{x},\bar{y})$ is said to have e.f.c.p. (existentially finite cover property).

In [1], the connection between fundamental order and definability was defined.

The fundamental order is a tool of comparing types over models of a complete theory: it measures the degree of complexity of a type in the realization. This order is especially effective in the case of a stable theory. Since the center of the T^* Jonsson theory is a complete theory, and we can consider the fundamental order for central types. If the Jonsson theory is a perfect theory, then T^* will be a Jonsson theory. And also, due to the perfection of the theory of T , any formula is existential in T^*.

Definition 6. [20] Let $A \subseteq M$, $\exists$-formula $\varphi(\bar{x},\bar{y}) \in L(A)$ be called representable in $p \in S^J(M)$ if there exists a tuple $\bar{m} \in M$ such that $p \vdash \varphi(\bar{x},\bar{m})$.

Definition 7. [20] M, N are existentially closed submodels of the semantic model C_T of the theory of T. If $p \in S_1^J(M)$, and $q \in S_1^J(N)$, $p \geq q$ in the sense of fundamental order, if any formula represented by p is also represented by q.

Definition 8. [1] If p and q represent the same formulas, we say that they are equivalent, and they even have a class in fundamental order.

Theorem 3. [1] Let T be a stable theory, M, N are $|T|^+$ be saturated models of T, $p \in S_1(M)$, $q \in S_1(N)$. Let $A \subset M$ be $|A| \leq |T|$, such that p is defined for all formulas $f(x,\bar{y})$: $g(\bar{y},\bar{a})$ can be taken with parameters $\bar{a}$ in A; then p and q are equivalent in fundamental order T if there is an $A' \subset N$ of the same type as A such that q is the definable type of a formula of the form $g(\bar{y},\bar{a}')$, where a' corresponds to a.

A theory T is stable if and only if for any model M of T and all p from $S_1(M)$, p is definable.

In the framework of the study of Jonsson theories, which are generally incomplete, and in some expanded language with new unary predicate and constant symbols, we refine in such generalization the earlier result obtained on beautiful pairs for complete theories from [1] (Theorem 4).

Definition 9. [27] Let C_T be a semantic model of T and N, M be existentially closed submodels of C_T. A pair (N, M) is called existentially closed pair, if M is an existentially closed submodel of N.

Lemma 1. If theory T is a perfect Jonsson theory, then theory $Th_{\forall\exists}(\mathcal{C},\mathcal{M})$ is a perfect Jonsson theory.

Definition 10. [27] An existentially closed pair (C_T, M) is a semantic pair, if the following conditions hold:

1) M is $|T|^+$-$\exists$-saturated (it means that it is $|T|^+$-saturated restricted up to existential types);
2) for any tuple $\bar{a} \in C$ each its $\exists$-type in sense of T over $M \cup \{\bar{a}\}$ is satisfiable in C.

By definition we see that it generalizes the excellent pair in [1], but weaker, because in the definition the number of tuples is finite and by Definition 2.4.4 [17] the power of the semantic model is ω^+ and it does not reach 2^ω: $2^\omega > \omega^+$, $\omega^+ + \omega = \omega^+$. Using the following Theorem 4 we can show the elementary equivalence of semantic pairs.

Let class K be $\{(\mathcal{C}, \mathcal{M}) | \mathcal{M} \preceq_{\exists_1} \mathcal{C}, (\mathcal{C}, \mathcal{M}) \text{ is semantic pair}\}$.

Consider the Jonsson spectrum of class K:

$$JSp(K) = \{\nabla | \nabla \text{ is Jonsson theory}, \nabla = Th_{\forall\exists}(\mathcal{C}, \mathcal{M}), \text{where } (\mathcal{C}, \mathcal{M}) \in K\}.$$

It is easy to see that $JSp(K)/_{\bowtie}$ is the factor set of the Jonsson spectrum of class K by $\bowtie$, $[\nabla] \in JSp(K)/_{\bowtie}$.

Let $[\nabla]$ be $\exists$-complete and J-λ-stable Jonsson class, $\mathcal{C}_{[\nabla]}$ be a semantic model of the theory $[\nabla]$, $\overline{[\nabla]} = [\nabla]$ in the enrichment of $\odot$, $\overline{[\nabla]}^*$ is the center of the $\overline{[\nabla]}$, $p, q \in S(\overline{[\nabla]}^*)$, $\nabla' = Th_{\forall\exists}(\mathcal{C}, \mathcal{M})$.

Theorem 4. $(\mathcal{C}_{[\nabla]}, M_1)$ and $(\mathcal{C}_{[\nabla]}, M_2)$ are two semantic pairs, $\bar{a}$ and $\bar{b}$ tuples taken from each of them, M_1, $M_2 \in E_{[\nabla]}$. Then $(\mathcal{C}_{[\nabla]}, M_1) \equiv_{\forall\exists} (\mathcal{C}_{[\nabla]}, M_2)$, if their central types are equivalent by the fundamental order $\overline{\nabla}^*$.

Proof. Follows from Theorem 6 in [1] and from Theorem 3.

Theorem 5. Let $[\nabla]$ be a hereditary, $\exists$-complete perfect, and J-λ-stable Jonsson class. Then the following conditions are equivalent:

1) $\overline{[\nabla]}^*$ does not have e.f.c.p.;

2) Any $|T|^+$-saturated model from ∇' is a semantic pair;

3) Two tuples $\bar{a}$ and $\bar{b}$ from the models of $\overline{[\nabla]}^*$ have the same type if and only if their central types in sense of $\overline{[\nabla]}^*$ over $\mathcal{M}$ are equivalent by fundamental order $\overline{[\nabla]}^*$;

4) Two tuples $\bar{a}$ and $\bar{b}$ from models of ∇' and that are in $\mathcal{C}_{[\nabla]} \setminus M$ have the same central types in the sense of $\overline{[\nabla]}$ if and only if they have the same central types in the sense of $\overline{[\nabla]}^*$.

Proof. 1) $\Rightarrow$ 2). $\overline{[\nabla]}^*$ the center of $\overline{[\nabla]}$ theory in the permissible enrichment $\odot$. And by Theorem 2. it is λ-stable theory. In [26], Shelah showed that stable theories do not have f.c.p. If (N, M) is a $|T|^+$-$\exists$-saturated model from ∇', M is $|T|^+$-$\exists$-saturated.

Let us assume that $\overline{[\nabla]}^*$ is not λ-stable. Then there exists $\mathcal{M} \in E_\Delta = Mod\overline{[\nabla]}^*$, by Theorem 1, so there is $X \subset M$ such that $|X| < \lambda, \exists n < \omega \Rightarrow |S^J(X)| > \lambda$. For each formula $\varphi \in p$, where $p \in S_n(X)$, we replace φ with θ satisfying the properties $\Delta^* \vdash \varphi \leftrightarrow \theta$ and $\theta \in E_n([T]^*)$. Let p' be p after replacement. Then $p' \in S^J(X)$ and $|S^J(X)| > \lambda$. This contradicts the J-λ-stability of the class $\overline{[\nabla]}^*$. Hence $\overline{[\nabla]}^*$ is stable and has a saturated model.

2) $\Rightarrow$ 3). By Definition 10, since any sufficiently saturated model is a semantic pair.

3) $\Rightarrow$ 4). Because if $\bar{a}$ and $\bar{b}$ are in $\mathcal{C}_{[\Delta]} \setminus M$ let's say that their types are over $\mathcal{C}_{[\Delta]} \setminus M$ are fundamentally equivalent, that is, they implement a type over $\emptyset$.

4) $\Rightarrow$ 1). If $\overline{\Delta}^*$ does not have e.f.c.p., then by Definition 5 there would not exist an arbitrarily large number n. In the semantic pair $(\mathcal{C}_{[\Delta]}, M)$ for arbitrarily large n we find $\bar{a}_n$ in M [1]. Moreover, any $\bar{b}$ of a semantic pair, $\bar{b} \in M$ is of the same type as $\bar{a}$ over $\emptyset$ in the sense of $\overline{\Delta}$ would satisfy the opposite. Therefore, $\bar{a}$ and $\bar{b}$ will not implement the same type in the sense of Δ', which contradicts (4).

Theorem 6. Let $[\nabla]$ be a hereditary, $\exists$-complete perfect, and J-λ-stable Jonsson class. If $\overline{[\nabla]}^*$ does not have e.f.c.p. and λ-stable class, then the class If $[\nabla]'$ is J-λ-stable and does not have e.f.c.p.

Proof. The proof follows from Theorems 4 and 5.

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Йонсондық спектрлердегі йонсондық семантикалық қосар мен шекті жабудың экзистенциалды қасиетінің моделді-теоретикалық қасиеттері

Мақала кез келген рұқсаттылығы бар байытуда йонсондылықты сақтайтын стабилді әрі мұралы йонсондық теориялардың моделді-теоретикалық касиеттерін зерттеуге арналған. Жұмыста стабилділік пен классикалық стабилділікті байланыстыратын стабилділіктің жалпыламасы йонсондық спектрлер үшін дәлелденген. Ұсынылып отырған жұмыста "шекті жабудың экзистенциалды қасиеті" мен "семантикалық қосар" секілді жаңа ұғымдар енгізілген. Және осы семантикалық қосар мен шекті жабудың экзистенциалды қасиетінің стабилді кемел йонсондық спектрлер үшін негізгі қасиеттері зерттелген.

Кілт сөздер: йонсондық теория, семантикалық модель, рұқсатылығы бар байыту, централды тип, мұралы теория, стабилді теория, кемел теория, фундаменталды рет, қаныққан модель, шекті жабудың экзистенциалды қасиеті, экзистенциалды-тұйық қосар, семантикалық қосар.

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Теоретико-модельные свойства сематических пар и e.f.c.p. в йонсоновских спектрах

Статья посвящена изучению теоретико-модельных свойств стабильных наследственных йонсоновских теорий, при этом мы рассматриваем йонсоновские теории, которые сохраняют йонсоновость при любом допустимом обогащении. Авторами доказано обобщение стабильности, связывающее стабильность и классическую стабильность для йонсоновских спектров. Введены новые понятия, такие как «экзистенциальное свойство конечного покрытия» и «семантическая пара». Изучены основные свойства e.f.c.p. и семантических пар в классе стабильных совершенных йонсоновских спектров.

Ключевые слова: йонсоновская теория, семантическая модель, допустимое обогащение, центральный тип, наследственная теория, стабильная теория, совершенная теория, фундаментальный порядок, насыщенная модель, e.f.c.p., экзистенциально-замкнутая пара, семантическая пара.

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ANNIVERSARIES

70th anniversary of Doctor of Physical and Mathematical Sciences, Professor Baltabek Kanguzhin

Baltabek Esmatovich Kanguzhin was born on August 22, 1953, in the village of Shukurkol, North Kazakhstan region. He graduated in 1975 from the Faculty of Applied Mathematics of the Kazakh State University named after S.M. Kirov and has been working at the Mechanics and Mathematics Faculty of the Al-Farabi Kazakh National University since then.

From 1979 to 1982, he was a full-time postgraduate student at the Department of Mathematics of the Moscow State University named after M.V. Lomonosov. In 1983, under the supervision of Academician V.A. Sadovnichy, he defended his PhD thesis on "Inverse Problems of Spectral Analysis of Differential Operators" at the Dissertation Council at Moscow State University. In 2005, he defended his Doctoral thesis in "01.01.02 – Differential Equations and Mathematical Physics" at the Al-Farabi Kazakh National University.

Professor Kanguzhin's scientific interests are diverse: transformation formulas and spectral properties of higher-order differential operators on intervals, stochastic analysis. His primary applied research is associated with problems of internal boundary tasks using the methods of function theory. The main courses he teaches include mathematical analysis, geometric methods of mathematical physics, mathematical models of theoretical physics and their analysis, stochastic analysis. One of his main talents is the desire to understand the essence of ideas deeply.

Professor B.E. Kanguzhin established a scientific school in Kazakhstan on the spectral theory of differential operators and its applications. Under his guidance, 13 candidates in physical and mathematical sciences, 3 doctors of physical and mathematical sciences, and 10 PhDs have been defended. He is the chairman of the Dissertation Council in specialties: 6D060100, 8D05401 – Mathematics, 6D070500, 8D06104 – Mathematical and Computer Modeling at the Al-Farabi KazNU.

Over the last ten years, he has been and continues to be the scientific leader of more than 5 international and national funded projects, including those for the identification of defects in mechanical, pipeline, and electrical systems.

Professor Kanguzhin's fundamental research results in spectral theory of differential operators, theory of inverse problems of spectral analysis, main issues of approximation theory, generated from internal boundary tasks in multiply connected areas, theory of pseudodifferential operators, and main methods of complex variable function theory and its applications to solving fundamental mathematical problems have been published in more than 150 scientific works, most of which are in high-ranking international journals. He is the author of two monographs and six excellent textbooks, four of which are in Russian and two in Kazakh. He is an active member of the editorial board of several scientific publications, such as: Journal of Mathematics, Mechanics and Computer Science, International Journal of Mathematics and Physics, Ufa Mathematical Journal, etc.

In 2002, he was awarded the International Soros-Kazakhstan Prize.

Professor B.E. Kanguzhin successfully combines scientific activity with teaching. He has led the Department of Mathematical Analysis and the Department of Fundamental Mathematics of the Mechanics and Mathematics Faculty of the Al-Farabi KazNU, which was established to strengthen leading mathematical departments such as mathematical analysis, probability theory and functional analysis, as well as geometry, algebra, and mathematical logic. He twice received the state grant "Best University Teacher" of the Republic of Kazakhstan in 2005 and 2012, the chest badge "For Merits in the Development of Science in the Republic of Kazakhstan" of the Ministry of Education and Science of the Republic of Kazakhstan in 2009, and the Y. Altynsarin badge in 2013, which also speaks of his active and fruitful pedagogical activity.

B.E. Kanguzhin has a particular affinity for the academician V.I. Arnold, which is probably why he adheres to his aphorism in working with his students: "A student is not a sack to be filled but a torch to be lit". He has "lit" many young students such as D. Suragan, N. Tokmagambetov, etc., who are now making significant contributions to mathematics.

His numerous friends and colleagues know him as a talented scientist, a respected and responsible employee, a good, reliable friend, and a wonderful family man.

We congratulate Professor Baltabek Esmatovich Kanguzhin on his 70th birthday and wish him further creative achievements, the preservation of activity and inspiration in science and education.

Editorial board of the journal
«Bulletin of the Karaganda University. Mathematics series»

70th anniversary of Doctor of Technical Sciences, Professor, President of the National Engineering Academy of the Republic of Kazakhstan, President of the Turkic World Mathematical Society (TWMS), Academician of the NAS RK Bakytzhan Zhumagulov

Bakytzhan Tursynovich Zhumagulov (born August 18, 1953) is a distinguished mathematician, a prominent state, political and public figure, a major organizer of science and education in Kazakhstan, president of the National Engineering Academy of the Republic of Kazakhstan, president of the Mathematical Society of the Turkic World (TWMS) and the Kazakhstani Mathematical Society. He is a laureate of the State Prize of the Republic of Kazakhstan in the field of science, technology, and education, an Honored Worker of Kazakhstan, a Doctor of Technical Sciences, a professor, and an academician of the National Academy of Sciences of the Republic of Kazakhstan, and the International Engineering Academy.

Academician B.T. Zhumagulov is a leading scientist in the field of computational mathematics, development and application of information technologies, mathematical modeling, and mathematical methods in solving problems of hydrodynamics and practical issues of the oil and gas industry. He is the author of more than 400 scientific works and 12 monographs, published in Kazakhstan and abroad.

He began his career in scientific and teaching work at the Kazakh State University named after S.M. Kirov. He progressed from assistant to professor, head of the department, vice-rector (1979–1991). He worked as the First Vice-Minister of Education and Science of the Republic of Kazakhstan, head of the Internal Policy Department of the Administration of the President of the Republic of Kazakhstan, head of the Department of Socio-Cultural Development of the Government of the Republic of Kazakhstan (2001–2005).

In April 2008, by the decree of the President of the Republic of Kazakhstan, he was appointed rector of the Al-Farabi Kazakh National University.

By the decree of the President of the Republic of Kazakhstan No. 1065 dated September 22, 2010, he was appointed Minister of Education and Science of the Republic of Kazakhstan.

In 2017, by the decree of the Head of State, Bakytzhan Tursynovich was appointed a deputy of the Senate of the Parliament of the Republic of Kazakhstan.

B.T. Zhumagulov has supervised more than 20 doctors and candidates of sciences.

He has been awarded the "Parasat" order, the "Eren engbegi ushin" medal, the USSR medal "For Labor Distinction" and the Honorary Diploma of the Supreme Council of the Kazakh SSR. He has received commendations from the President of the country N.A. Nazarbayev and has been honored with the titles "Honored Worker of Science and Technology of the Republic of Kazakhstan" , "Honorary Worker of Education of the Republic of Kazakhstan" , "Honorary Engineer of Kazakhstan".

For his significant contribution to international cooperation, active work in developing integration processes in science, technology, and education, he has been awarded the Grand Gold Medal of the International Engineering Academy and UNESCO, a special sign of the Federation of Engineering Academies of Islamic Countries (FEIIC). For outstanding services in the development of science and engineering, he has been awarded the high international award – the "Engineering Glory" order.

B.T. Zhumagulov is an active participant in the international scientific community. He is the first vice-president of the International Engineering Academy (headquartered in Moscow, Russia). Since 1999, he has been the first vice-president of the Federation of Engineering Academies of Islamic

Countries (headquartered in Kuala Lumpur, Malaysia). In 2009, he was elected president of the Mathematical Society of the Turkic World.

Bakytzhan Tursynovich Zhumagulov celebrates his anniversary full of energy and creative force.

The editorial board of the scientific journal cordially congratulates Bakytzhan Tursynovich on his 70th birthday and wishes him good health and creative longevity.

Editorial board of the journal
«Bulletin of the Karaganda University. Mathematics series»

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Index of articles published
in the «Bulletin of the Karaganda University» in 2023 y.
«Mathematics» Series

MATHEMATICS

IN MEMORIAM OF SCIENTIST

ANNIVERSARIES