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CONTENT

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Statistical convergence in vector lattices

The statistical convergence is defined for sequences with the asymptotic density on the natural numbers, in general. In this paper, we introduce the statistical convergence in vector lattices by using the finite additive measures on directed sets. Moreover, we give some relations between the statistical convergence and the lattice properties such as the order convergence and lattice operators.

Keywords: statistical convergence of nets, order convergence, vector lattice, directed set measure.

Introduction

The statistical convergence of sequences is handled together with the asymptotic (or, natural) density of subsets on the natural numbers $\mathbb{N}$. On the other hand, Connor introduced the notion of statistical convergence of sequences with finitely additive set functions [1, 2]. After then, some similar works have been done [3–5]. Also, several applications and generalizations of the statistical convergence of sequences have been investigated by several authors [6–13]. However, as far as we know, the concept of statistical convergence related to nets has not been done except for the paper [14], in which the asymptotic density of a directed set $(D, \leq)$ was introduced by putting a special and strong rule on the directed sets such that the set $\{\alpha \in D : \alpha \leq \beta\}$ is finite and the set $\{\alpha \in D : \alpha \geq \beta\}$ is infinite for each element β in $(D, \leq)$. We aim to introduce a general concept of statistical convergence for nets with a new notion called a directed set measure.

Recall that a binary relation "$\leq$" on a set A is called a *preorder* if it is reflexive and transitive. A non-empty set A with a preorder binary relation "$\leq$" is said to be a *directed upwards* (or, for short, *directed set*) if for each pair $x, y \in A$ there exists $z \in A$ such that $x \leq z$ and $y \leq z$. Unless otherwise stated, we consider all directed sets as infinite. For given elements a and b in a preorder set A such that $a \leq b$, the set $\{x \in A : a \leq x \leq b\}$ is called an *order interval* in A. A subset I of A is called an *order bounded set* whenever I is contained in an order interval.

A function domain of which is a directed set is said to be a *net*. A net is briefly abbreviated as $(x_\alpha)_{\alpha \in A}$ with its directed domain set A. Let $(A, \leq_A)$ and $(B, \leq_B)$ be directed sets. Then a net $(y_\beta)_{\beta \in B}$ is said to be a *subnet* of a net $(x_\alpha)_{\alpha \in A}$ in a non empty set X if there exists a function $\phi : B \to A$ such that $y_\beta = x_{\phi(\beta)}$ for all $\beta \in B$, and also, for each $\alpha \in A$ there exists $\beta_\alpha \in B$ such that $\alpha \leq \phi(\beta)$ for all $\beta \geq \beta_\alpha$ (Definition 3.3.14 [15]). It can be seen that $\{\phi(\beta) \in A : \beta_\alpha \leq \beta\} \subseteq \{\alpha' \in A : \alpha \leq \alpha'\}$ holds for subnets.

A real vector space E with an order relation "$\leq$" is called an *ordered vector space* if, for each $x, y \in E$ with $x \leq y$, $x + z \leq y + z$ and $\alpha x \leq \alpha y$ hold for all $z \in E$ and $\alpha \in \mathbb{R}_+$. An ordered vector space E is called a *Riesz space* or *vector lattice* if, for any two vectors $x, y \in E$, the infimum and the supremum

$$x \wedge y = \inf\{x, y\} \quad \text{and} \quad x \vee y = \sup\{x, y\}$$

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exist in E, respectively. A vector lattice is called *Dedekind complete* if every nonempty bounded from the above set has a supremum (or, equivalently, whenever every nonempty bounded below subset has an infimum). A subset I of a vector lattice E is said to be a *solid* if, for each $x \in E$ and $y \in I$ with $|x| \leq |y|$, it follows that $x \in I$. A solid vector subspace is called an *order ideal*. A vector lattice E has the *Archimedean* property provided that $\frac{1}{n}x \downarrow 0$ holds in E for each $x \in E_+$. In this paper, unless otherwise stated, all vector lattices are assumed to be real and Archimedean. We remind the following crucial notion of vector lattices [16–20].

Definition 1. A net $(x_\alpha)_{\alpha\in A}$ in a vector lattice E is called *order convergent* to $x \in E$ if there exists another net $(y_\alpha)_{\alpha\in A} \downarrow 0$ (i.e., $\inf y_\alpha = 0$ and $y_\alpha \downarrow$) such that $|x_\alpha - x| \leq y_\alpha$ holds for all $\alpha \in A$.

We refer the reader to some different types of order convergence and some relations among them [21]. Throughout this paper, the vertical bar of a set will stand for the cardinality of the given set and $\mathcal{P}(A)$ is the power set of A.

1 *The μ-statistical convergence*

We remind that a map from a field $\mathcal{M}$ (i.e., $M_1, M_2, \dots \in \mathcal{M}$ implies $\cup_{i=1}M_n \in \mathcal{M}$ and $A^c \in \mathcal{M}$ for all $A \in \mathcal{M}$) to $[0, \infty]$ is called *finitely additive measure* whenever $\mu(\emptyset) = 0$ and $\mu(\cup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$ for all finite disjoint sets $\{E_i\}_{i=1}^n$ in $\mathcal{M}$ [22; 25]. Now, we introduce the notion of measuring on directed sets.

Definition 2. Let A be a directed set and $\mathcal{M}$ be a subfield of $\mathcal{P}(A)$ (i.e., it satisfies the properties of field). Then

(1) an order interval $[a, b]$ of A is said to be a *finite order interval* if it is a finite subset of A;

(2) $\mathcal{M}$ is called an *interval field* on A whenever it includes all finite order intervals of A;

(3) a finitely additive measure $\mu : \mathcal{M} \to [0, 1]$ is said to be a *directed set measure* if $\mathcal{M}$ is an interval field and μ satisfies the following facts: $\mu(I) = 0$ for each finite order interval $I \in \mathcal{M}$ and $\mu(A) = 1$.

It is clear that $\mu(C) = 0$ whenever $C \subseteq B$ and $\mu(B) = 0$ holds for $B, C \in \mathcal{M}$ because μ is finitely additive.

Example 1. Consider the directed set $\mathbb{N}$ and define a measure μ from $2^{\mathbb{N}}$ to $[0, 1]$ denoted by $\mu(A)$ as the Banach limit of $\frac{1}{k}|A \cap \{1, 2, \dots, k\}|$ for all $A \in 2^{\mathbb{N}}$. Then one can see that $\mu(I) = 0$ for all finite order interval sets because of $\frac{1}{k}|I \cap \{1, 2, \dots, k\}| \to 0$. Also, it follows from the properties of the Banach limit that $\mu(\mathbb{N}) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ for disjoint sets A and B. Thus, μ is finitely additive, and so, it is a directed set measure.

Let's give an example of a directed set measure for an arbitrary uncountable set.

Example 2. Let A be an uncountable directed set. Consider a field $\mathcal{M}$ consisting of countable or co-countable (i.e., the complement of set is countable) subsets of A. Then $\mathcal{M}$ is an interval field. Thus, a map μ from $\mathcal{M}$ to $[0, 1]$ defined by $\mu(C) := 0$ if C is a countable set, otherwise $\mu(C) = 1$. Hence, μ is a directed set measure.

In this paper, unless otherwise stated, we consider all nets with a directed set measure on interval fields of the power set of the index sets. Moreover, in order to simplify the presentation, a directed set measure on an interval field $\mathcal{M}$ of directed set A will be expressed briefly as a measure on the directed set A. Motivated from [23; 302], we give the following notion.

Recall that the asymptotic density of a subset K of natural numbers $\mathbb{N}$ is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in A\}| .$$

We refer the reader for an exposition on the asymptotic density of sets in $\mathbb{N}$ to [24, 25]. We give the following observation.

Remark 1. It is clear that the asymptotic density of subsets on $\mathbb{N}$ satisfies the conditions of a directed set measure when $\mathcal{P}(\mathbb{N})$ is considered as an interval field on the directed set $\mathbb{N}$. Thus, it can be seen that the directed set measure is an extension of the asymptotic density.

Remind that a sequence (x_n) in a vector lattice E is called *statistically monotone decreasing* to $x \in E$ if there exists a subset K of $\mathbb{N}$ such that $\delta(K) = 1$ and the subsequence $(x_k)_{k\in K}$ is decreasing to x, i.e., $x_k \downarrow$ and $\inf\limits_{k\in K} x_k = x$ (see for example [8]). Now, by using the notions of measure on directed sets and the statistical monotone decreasing which was introduced in [25] for real sequences, we introduce the concept of statistical convergence of nets on vector lattices.

Definition 3. Let E be a vector lattice and $(p_\alpha)_{\alpha\in A}$ be a net in E with a measure μ on the index set A. Then $(p_\alpha)_{\alpha\in A}$ is said to be *μ-statistical decreasing* to $x \in E$ whenever there exists a subnet $q_\delta = (p_{\phi(\delta)})_{\beta\in\Delta}$ such that $\mu(\Delta) = 1$ and $(q_\delta)_{\delta\in\Delta} \downarrow x$. Then it is abbreviated as $(p_\alpha)_{\alpha\in A} \downarrow^{\mathrm{st}_\mu} x$.

We denote the class of all μ-statistical decreasing nets on a vector lattice E by $E_{st_\mu\downarrow}$, and also, the set $E_{st_\mu\downarrow}\{0\}$ denotes the class of all μ-statistical decreasing null nets on E. It is clear that $\mu(\Delta^c) = \mu(A-\Delta) = 0$ whenever $\mu(\Delta) = 1$ because of $\mu(A) = \mu(\Delta\cup\Delta^c) = \mu(\Delta)+\mu(\Delta^c)$. We consider Example 2 for the following example.

Example 3. Let E be a vector lattice and $(p_\alpha)_{\alpha\in A}$ be a net in E. Take $\mathcal{M}$ and μ from Example 2. Thus, if $(p_\alpha)_{\alpha\in A} \downarrow x$ then $(p_\alpha)_{\alpha\in A} \downarrow^{\mathrm{st}_\mu} x$ for some $x \in E$.

For the general case of Example 3, we give the following work proof of which follows directly from the basic definitions and results.

Proposition 1. If $(p_\alpha)_{\alpha\in A}$ is an order decreasing null net in a vector lattice then $(p_\alpha)_{\alpha\in A} \downarrow^{\mathrm{st}_\mu} 0$.

Now, we introduce the crucial notion of this paper.

Definition 4. A net $(x_\alpha)_{\alpha\in A}$ in a vector lattice E is said to be *μ-statistical convergent* to $x \in E$ if there exists a net $(p_\alpha)_{\alpha\in A} \downarrow^{\mathrm{st}_\mu} 0$ with a subnet $q_\delta = (p_{\phi(\delta)})_{\beta\in\Delta}$ such that $\mu(\Delta) = 1$ and $(q_\delta)_{\delta\in\Delta} \downarrow 0$ and $|x_{\phi(\delta)} - x| \leq q_\delta$ for every $\delta \in \Delta$. Then it is abbreviated as $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$.

It can be seen that $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ in a vector lattice means that there exists another sequence $(p_\alpha)_{\alpha\in A} \downarrow^{\mathrm{st}_\mu} 0$ such that $\mu\big(\{\alpha \in A : |x_\alpha - x| \nleq p_\alpha\}\big) = 0$. It follows from Remark 1 that the notion of statistical convergence of sequence coincides with the notion of μ-statistical convergence in the reel line. We denote the set E_{st_μ} as the family of all st_μ-convergent nets in E, and $E_{st_\mu}\{0\}$ is the family of all μ-statistical null nets in E.

Lemma 1. Every μ-statistical decreasing net is μ-statistical convergent.

Remark 2. Recall that a net $(x_\alpha)_{\alpha\in A}$ in a vector lattice E *relatively uniform converges* to $x \in E$ if there exists $u \in E_+$ such that, for any $n \in \mathbb{N}$, there is an index $\alpha_n \in A$ so that $|x_\alpha - x| \leq \frac{1}{n}u$ for all $\alpha \geq \alpha_n$ (Lemma 16.2 [18]). It is well known that the relatively uniform convergence implies the order convergence on Archimedean vector lattices (Lemma 2.2 [20]). Hence, it follows from Proposition 1 and Lemma 1 that every decreasing relatively uniform null net is μ-statistical convergent in vector lattices.

2 *Main Results*

Let μ be a measure on a directed set A. Following from Exercise 9. in [22; 27], it is clear that $\mu(\Delta\cap\Sigma) = 1$ for any $\Delta, \Sigma \subseteq A$ whenever $\mu(\Delta) = \mu(\Sigma) = 1$. We begin the section with the following proposition and skip its simple proof.

Proposition 2. Assume $x_\alpha \leq y_\alpha \leq z_\alpha$ satisfies in a vector lattices for each index α. Then $y_\alpha \xrightarrow{\mathrm{st}_\mu} x$ whenever $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ and $z_\alpha \xrightarrow{\mathrm{st}_\mu} x$.

It can be seen from Proposition 2 that if $0 \le x_\alpha \le z_\alpha$ satisfies for each index α and $(z_\alpha)_{\alpha\in A} \in E_{st_\mu}\{0\}$ then $(x_\alpha)_{\alpha\in A} \in E_{st_\mu}\{0\}$. We give a relation between the order and the μ-statistical convergences in the next result.

Theorem 1. Every order convergent net is μ-statistical convergent to its order limit.

Proof. Suppose that a net $(x_\alpha)_{\alpha\in A}$ is order convergent to x in a vector lattice E. Then there exists another net $(y_\alpha)_{\alpha\in A} \downarrow 0$ such that $|x_\alpha - x| \le y_\alpha$ holds for all $\alpha \in A$. It follows from Proposition 1 that $(y_\alpha)_{\alpha\in A} \downarrow^{\mathrm{st}_\mu} 0$. So, we obtain the desired result, $(x_\alpha)_{\alpha\in A} \xrightarrow{\mathrm{st}_\mu} x$.

The converse of Theorem 1 need not to be true. To see this, we consider Example 3. [26].

Example 4. Let us consider the set of all real numbers $\mathbb{R}$ with the usual order. Define a sequence (x_n) in $\mathbb{R}$ as n^2 whenever $n = k^2$ for some $k \in \mathbb{N}$ and $\frac{1}{n+1}$ otherwise. It is clear that (x_n) is not an order convergent sequence. However, if we choose another sequence (p_n) as n whenever $n = k^2$ for some $k \in \mathbb{N}$ and $\frac{1}{1}$ otherwise. Then we have $p_n \downarrow^{st_\mu} 0$. Setting $K = \{n \in \mathbb{N} : \text{n is not a square}\} \cup \{1\}$. Then we get $\mu(K) = 1$ and $|x_k| \le p_k$ for each $k \in K$. Thus we have $x_n \xrightarrow{\mathrm{st}_\mu} 0$.

Moreover, following from Theorem 23.2 [18], we observe the following result.

Corollary 1. Every order bounded monotone net in a Dedekind complete vector lattice is μ-statistical convergent.

By the definition of subnet given at the beginning of the paper, a subnet is based on some other set B, where the measure μ is not defined. However, for a subnet $y_\beta = x_{\phi(\beta)}$ of a net $(x_\alpha)_{\alpha\in A}$ with a measure μ on the index set A, we can consider the measure of a subset Δ of B as the measure of $\mu(\phi(\Delta))$ in A.

Proposition 3. The st_μ-convergence of subnets implies the st_μ-convergence of nets.

Proof. Let $(x_\alpha)_{\alpha\in A}$ be a net in a vector lattice E. Assume that a subnet $(x_{\phi(\delta)})_{\delta\in\Delta}$ of $(x_\alpha)_{\alpha\in A}$ μ-statistical converges to $x \in E$. Then there exists a net $(p_\alpha)_{\alpha\in A} \in E_{st_\mu\downarrow}\{0\}$ such that $|x_{\phi(\sigma)} - x| \le p_{\phi(\sigma)}$ for all some $\sigma \in \Sigma \subseteq \Delta$, $(p_{\phi(\sigma)})_{\sigma\in\Sigma} \downarrow 0$ and $\mu(\Sigma) = 1$. Since $\Sigma \subseteq A$ and $(x_{\phi(\sigma)})_{\sigma\in\Sigma}$ is also a subnet of $(x_\alpha)_{\alpha\in A}$, we can obtain the desired result.

Since every order bounded net has an order convergent subnet in atomic KB-spaces (Remark 6. [27]), we give the following result by considering Theorem 1 and Proposition 3.

Corollary 2. If E is an atomic KB-space then every order bounded net is μ-statistical convergent in E.

The lattice operations are μ-statistical continuous in the following sense.

Theorem 2. If $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ and $w_\alpha \xrightarrow{\mathrm{st}_\mu} w$ then $x_\alpha \vee w_\alpha \xrightarrow{\mathrm{st}_\mu} x \vee w$.

Proof. Assume that $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ and $w_\alpha \xrightarrow{\mathrm{st}_\mu} w$ hold in a vector lattice E. So, there are nets $(p_\alpha)_{\alpha\in A}, (q_\alpha)_{\alpha\in A} \in E_{st_\mu\downarrow}\{0\}$ with $\Delta, \Sigma \in \mathcal{M}$ and $\mu(\Delta) = \mu(\Sigma) = 1$ such that

$$|x_{\phi(\delta)} - x| \le p_{\phi(\delta)} \text{ and } |w_{\rho(\sigma)} - w| \le q_{\rho(\sigma)}$$

satisfy for all $\delta \in \Delta$ and $\sigma \in \Sigma$. On the other hand, it follows from Theorem 1.9(2) [17] that the inequality $|x_\alpha \vee w_\alpha - x \vee w| \le |x_\alpha - x| + |w_\alpha - w|$ holds for all $\alpha \in A$. Therefore, we have

$$|x_{\phi(\delta)} \vee w_{\phi(\sigma)} - x \vee w| \le p_{\phi(\delta)} + q_{\phi(\sigma)}$$

for each $\delta \in \Delta$ and $\sigma \in \Sigma$. Take $\Gamma := \Delta\cap\Sigma \in \mathcal{M}$. So, we have $\mu(\Gamma) = 1$, and also, $|x_{\phi(\gamma)} \vee w_{\phi(\gamma)} - x \vee w| \le p_{\phi(\gamma)} + q_{\phi(\gamma)}$ holds for all $\gamma \in \Gamma$. It follows from $(p_{\phi(\gamma)} + q_{\phi(\gamma)})_{\gamma\in\Gamma} \downarrow 0$ that $x_\alpha \vee w_\alpha \xrightarrow{\mathrm{st}_\mu} x \vee w$.

Corollary 3. If $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ and $w_\alpha \xrightarrow{\mathrm{st}_\mu} w$ in a vector lattice then
(i) $x_\alpha \wedge w_\alpha \xrightarrow{\mathrm{st}_\mu} x \wedge w$;
(ii) $|x_\alpha| \xrightarrow{\mathrm{st}_\mu} |x|$;
(iii) $x_\alpha^+ \xrightarrow{\mathrm{st}_\mu} x^+$;
(iv) $x_\alpha^- \xrightarrow{\mathrm{st}_\mu} x^-$.

We continue with several basic results that are motivated by their analogies from vector lattice theory.

Theorem 3. Let $(x_\alpha)_{\alpha \in A}$ be a net in a vector lattice E. Then the following results hold:
(i) $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ iff $(x_\alpha - x) \xrightarrow{\mathrm{st}_\mu} 0$ iff $|x_\alpha - x| \xrightarrow{\mathrm{st}_\mu} 0$;
(ii) the μ-statistical limit is linear;
(iii) the μ-statistical limit is uniquely determined;
(iv) the positive cone E_+ is closed under the statistical μ-convergence;
(v) $x_{\phi(\delta)} \xrightarrow{\mathrm{st}_\mu} x$ for any subnet $(x_{\phi(\delta)})_{\delta \in \Delta}$ of $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ with $\mu(\Delta) = 1$.

Proof. The properties (i), (ii) and (iii) are straightforward.

For (iv), take a non-negative μ-statistical convergent net $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ in E. Then it follows from Corollary 3 that $x_n = x_n^+ \xrightarrow{\mathrm{st}_\mu} x^+$. Moreover, by applying (ii), we have $x = x^+$. So, we obtain the desired result $x \in E_+$.

For (v), suppose that $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$. Then there is a net $(p_\alpha)_{\alpha \in A} \in E_{st_\mu \downarrow}\{0\}$ with $\Delta \in \mathcal{M}$ and $\mu(\Delta) = 1$ such that $|x_{\phi(\delta)} - x| \leq p_{\phi(\delta)}$ for each $\delta \in \Delta$. Thus, it is clear that $x_{\phi(\delta)} \xrightarrow{\mathrm{st}_\mu} x$. However, it should be shown that it is provided for all arbitrary elements in field under the assumption. Thus, take an arbitrary element $\Sigma \in \mathcal{M}$ with $\Sigma \neq \Delta$ and $\mu(\Sigma) = 1$. We show $(x_{\phi(\sigma)})_{\sigma \in \Sigma} \xrightarrow{\mathrm{st}_\mu} x$. Consider $\Gamma := \Delta \cap \Sigma \in \mathcal{M}$. So, we have $\mu(\Gamma) = 1$. Therefore, following from $|x_{\phi(\gamma)} - x| \leq p_{\phi(\gamma)}$ for each $\gamma \in \Gamma$, we get the desired result.

Proposition 1 shows that a decreasing order convergent net is μ-statistical convergent. For the converse of this fact, we give the following result.

Proposition 4. Every monotone μ-statistical convergent net is order convergent.

Proof. We show that $x_\alpha \downarrow$ and $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ implies $x_\alpha \downarrow x$ in any vector lattice E. To see this, choose an arbitrary index α_0. Then $x_{\alpha_0} - x_\alpha \in E_+$ for all $\alpha \geq \alpha_0$. It follows from Theorem 3 that $x_{\alpha_0} - x_\alpha \xrightarrow{\mathrm{st}_\mu} x_{\alpha_0} - x$, and also, $x_{\alpha_0} - x \in E_+$. Hence, we have $x_{\alpha_0} \geq x$. Then x is a lower bound of $(x_\alpha)_{(\alpha \in A)}$ because α_0 is arbitrary. Suppose that z is another lower bound of $(x_\alpha)_{\alpha \in A}$. So, we obtain $x_\alpha - y \xrightarrow{\mathrm{st}_\mu} x - y$. It means that $x - y \in E_+$, or equivalent to saying that $x \geq y$. Therefore, we get $x_\alpha \downarrow x$.

Remark 3. Let $x := (x_\alpha)_{\alpha \in A}$ be a net in a vector lattice. If $x\mathcal{X}_\Delta \xrightarrow{o} 0$ holds for some $\Delta \in \mathcal{M}$ with $\mu(\Delta) = 1$ and characteristic function $\mathcal{X}_\Delta$ on Δ then $x \xrightarrow{\mathrm{st}_\mu} 0$. Indeed, suppose that there exists $\Delta \in \mathcal{M}$ with $\mu(\Delta) = 1$ and $x\mathcal{X}_\Delta \xrightarrow{o} 0$ satisfies in a vector lattice E for the characteristic function $\mathcal{X}_\Delta$ of Δ. Thus, there is another net $(p_\alpha)_{\alpha \in A} \downarrow 0$ such that $|x\mathcal{X}_\Delta| \leq p_\alpha$ for all $\alpha \in A$. So, it follows from Proposition 1 that $(p_\alpha)_{\alpha \in A} \downarrow^{\mathrm{st}_\mu} 0$. Then there exists a subset $\Sigma \in \mathcal{M}$ such that $\mu(\Sigma) = 1$ and $(p_{\phi(\sigma)})_{\sigma \in \Sigma} \downarrow 0$. Take $\Gamma := \Delta \cap \Sigma$. Hence, we have $\mu(\Gamma) = 1$. Following from $|x\mathcal{X}_\Gamma| \leq p_{\phi(\gamma)}$ for each $\gamma \in \Gamma$, we obtain $x\mathcal{X}_\Delta \xrightarrow{\mathrm{st}_\mu} 0$. Therefore, by applying Theorem 3 (v) and Remark 3, we obtain $(x_\alpha)_{\alpha \in A} \xrightarrow{\mathrm{st}_\mu} 0$.

Proposition 5. The family of all st_μ-convergent nets E_{st_μ} is a vector lattice.

Proof. Let $(x_\alpha)_{\alpha\in A}\xrightarrow{\mathrm{st}_\mu} x$ and $(y_\beta)_{\beta\in B}\xrightarrow{\mathrm{st}_\mu} y$ in E. Then it follows from Theorem 3(ii) that $(x_\alpha+y_\beta)_{(\alpha,\beta)\in A\times B}\xrightarrow{\mathrm{st}_\mu} x+y$. So E_{st_μ} is a vector space. Take an element $x:=(x_\alpha)_{\alpha\in A}$ in E_{st_μ}. Then we have $x\xrightarrow{\mathrm{st}_\mu} z$ for some $z\in E$. Thus, it follows from Corollary 3 that $|x|\xrightarrow{\mathrm{st}_\mu}|z|$. It means that $|x|\in E_{st_\mu}$, i.e., E_{st_μ} is a vector lattice subspace Theorem 1.3 and Theorem 1.7 [16].

Proposition 6. The set of all order bounded nets in a vector lattice E is an order ideal in $E_{st_\mu}\{0\}$.

Proof. By the linearity of μ-statistical convergence, $E_{st_\mu}\{0\}$ is a vector space. Now, assume that $|y|\leq|x|$ hold for arbitrary $x:=(x_\alpha)_{\alpha\in A}\in E_{st_\mu}\{0\}$ and for an order bounded net $y:=(y_\alpha)_{\alpha\in A}$. Since $x\xrightarrow{\mathrm{st}_\mu}0$, we have $|x|\xrightarrow{\mathrm{st}_\mu}0$. Then it follows from Proposition 2 that $|y|\xrightarrow{\mathrm{st}_\mu}0$, and so, it follows from Theorem 3(i) that $y\xrightarrow{\mathrm{st}_\mu}0$. (Therefore, we get the desired result, $y\in E_{st_\mu}\{0\}$).

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Векторлық торлардағы статистикалық жинақталу

Статистикалық жинақталу, жалпы жағдайда, натурал сандардағы асимптотикалық тығыздығы бар тізбектер үшін анықталады. Мақалада бағытталған жиындардағы ақырлы аддитивті өлшемдерді қолдана отырып, векторлық торларға статистикалық жинақталу енгізілген. Сонымен қатар, статистикалық жинақталу мен тордың қасиеттері арасындағы кейбір қатынастар келтірілген, мысалы, реттік жинақталу және тор операторлары.

Кілт сөздер: желілердің статистикалық жинақталуы, реттік жинақталу, векторлық тор, бағытталған жиынның өлшемі.

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Статистическая сходимость в векторных решетках

Статистическая сходимость, в общем случае, определена для последовательностей с асимптотической плотностью на натуральных числах. В статье мы вводим статистическую сходимость в векторных решетках, используя конечные аддитивные меры на направленных множествах. Кроме того, приводим некоторые соотношения между статистической сходимостью и свойствами решетки, такими как сходимость порядка и операторы решетки.

Ключевые слова: статистическая сходимость сетей, порядковая сходимость, векторная решетка, мера направленного множества.

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Homogenization of Attractors to Ginzburg-Landau Equations in Media with Locally Periodic Obstacles: Critical Case

In this paper the Ginzburg-Landau equation is considered in locally periodic porous medium, with rapidly oscillating terms in the equation and boundary conditions. It is proved that the trajectory attractors of this equation converge in a weak sense to the trajectory attractors of the limit Ginzburg-Landau equation with an additional potential term. For this aim we use an approach from the papers and monographs of V.V. Chepyzhov and M.I. Vishik concerning trajectory attractors of evolution equations. Also we apply homogenization methods appeared at the end of the XX-th century. First, we apply the asymptotic methods for formal construction of asymptotics, then, we verify the leading terms of asymptotic series by means of the methods of functional analysis and integral estimates. Defining the appropriate axillary functional spaces with weak topology, we derive the limit (homogenized) equation and prove the existence of trajectory attractors for this equation. Then we formulate the main theorem and prove it with the help of axillary lemmas.

Keywords: attractors, homogenization, Ginzburg-Landau equations, nonlinear equations, weak convergence, perforated domain, strange term, porous medium.

Introduction

This work is connected with modelling of processes in perforated materials and porous media. Asymptotic analysis of solutions to problems in porous media is sufficiently complicated, especially in the case of a threshold value of sizes and a number of cavities with nontrivial Robin (Fourier) conditions on their boundaries, i.e. in the case of a singular perturbation of problems. In this situation the limit equation describing the effective behavior of the model, has a different structure if one compares it with the given one. We investigate the situation when an additional potential term appears in the limit Ginzburg-Landau equation and prove the Hausdorff convergence of attractors as the small parameter tends to zero. Thus, we construct the limit attractor and prove the convergence of the attractors of the given problem, to the attractor of the limit problem with an additional potential in the equation. Here we investigate the asymptotic behavior of attractors to an initial boundary value problem for complex Ginzburg-Landau equations in porous media. In many pure mathematical papers one can find the asymptotic analysis of problems in porous media (see, for example, [1–7]). Interesting homogenization results have been obtained for periodic, almost periodic and random structures. We want to mention here the basic frameworks [8–11], where one can find the detail bibliography.

About attractors see, for instance, [12–14] and the references in these monographs. Homogenization of attractors were studied in [14–17] (see also [18–21]).

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In this paper we present the proofs of weak convergence of the trajectory attractor $\mathfrak{A}_\epsilon$ to the Ginzburg-Landau equation in a perforated domain, as $\epsilon \to 0$, to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized equation in some natural functional space. Here, the small parameter ϵ characterizes the linear size of cavities and the distance between them in porous medium. We prove the appearance of a so called "strange term" (the potential term) in the limit equation (for example see works [1, 2]).

1 Statement of the problem

We start by the definition of a perforated domain. Suppose $\Omega \subset \mathbb{R}^d, d \geq 2$, is a smooth bounded domain. Denote

$$\Upsilon_\epsilon = \{j \in \mathbb{Z}^d \,:\, \mathrm{dist}\,(\epsilon j, \partial\Omega) \geq \epsilon\sqrt{d}\}, \quad \square \equiv \{\xi : -\frac{1}{2} < \xi_j < \frac{1}{2},\ j = 1, \ldots, d\}.$$

Considering a smooth function $F(x, \xi)$ 1-periodic in ξ, which satisfies $F(x,\xi)\Big|_{\xi\in\partial\square} \geq \mathrm{const} > 0$, $F(x, 0) = -1$, $\nabla_\xi F \neq 0$ as $\xi \in \square\backslash\{0\}$, we define $D_j^\epsilon = \{x \in \epsilon\,(\square + j) \mid F(x, \frac{x}{\epsilon}) \leq 0\}$. The perforated domain now is defined in the following way:

$$\Omega_\epsilon = \Omega \backslash \bigcup_{j\in\Upsilon_\epsilon} D_j^\epsilon.$$

Denote by ω the set $\{\xi \in \mathbb{R}^d \mid F(x, \xi) < 0\}$, and by S the set $\{\xi \in \mathbb{R}^d \mid F(x, \xi) = 0\}$. The boundary $\partial\Omega_\epsilon$ consists of $\partial\Omega$ and the boundary of the holes $S_\epsilon \subset \Omega$, $S_\epsilon = (\partial\Omega_\epsilon) \cap \Omega$.

We study the asymptotic behavior of attractors to the problem

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} = (1 + \alpha \mathrm{i})\Delta u_\epsilon + R(x, \dfrac{x}{\epsilon})\, u_\epsilon - \left(1 + \beta(x, \dfrac{x}{\epsilon})\mathrm{i}\right) |u_\epsilon|^2 u_\epsilon + g(x), & x \in \Omega_\epsilon, \\ (1 + \alpha \mathrm{i})\dfrac{\partial u_\epsilon}{\partial \nu} + \epsilon q(x, \dfrac{x}{\epsilon}) u_\epsilon = 0, & x \in S_\epsilon, t > 0, \\ u_\epsilon = 0, & x \in \partial\Omega, \\ u_\epsilon = U(x), & x \in \Omega_\epsilon, t = 0, \end{cases} \tag{1}$$

where α is a real constant, the vector ν is the outer unit vector to the boundary, $u = u_1 + \mathrm{i}u_2 \in \mathbb{C}$, $g(x) \in C^1(\Omega; \mathbb{C})$, a nonnegative 1-periodic in ξ function $q(x, \xi)$ belongs to $C^1(\Omega; \mathbb{R}^d)$. Suppose that

$$-\beta_1 \leq \beta(x, \xi) \leq \beta_2,\ -R_1 \leq R(x, \xi) \leq R_2 \ \ (\text{where } R_0, R_1, \beta_1, \beta_2 > 0), \tag{2}$$

for $x \in \Omega$, $\xi \in \mathbb{R}^d$ and the functions $R\,(x, \xi)$ and $\beta\,(x, \xi)$ can be averaged in $L_{\infty,*w}(\Omega)$. The averages are $\bar{R}(x)$ and $\bar{\beta}(x)$ respectively, i.e.,

$$\begin{aligned} \int_\Omega R\,(x, \xi)\, \varphi_1(x) dx \ &\to\ \int_\Omega \bar{R}(x) \varphi_1(x) dx, \\ \int_\Omega \beta\,(x, \xi)\, \varphi_1(x) dx \ &\to\ \int_\Omega \bar{\beta}(x) \varphi_1(x) dx \end{aligned} \tag{3}$$

for any $\varphi_1(x) \in L_1(\Omega)$, where $\xi = \dfrac{x}{\epsilon}$ as $\epsilon \to 0+$.

We define the following spaces: $\mathbf{H} := L_2(\Omega; \mathbb{C})$, $\mathbf{H}_\epsilon := L_2(\Omega_\epsilon; \mathbb{C})$, $\mathbf{V} := H_0^1(\Omega; \mathbb{C})$, $\mathbf{V}_\epsilon := H^1(\Omega_\epsilon; \mathbb{C}; \partial\Omega)$ is a set of functions from $H^1(\Omega_\epsilon; \mathbb{C})$ with a zero trace on $\partial\Omega$, and $\mathbf{L}_p := L_p(\Omega; \mathbb{C})$, $\mathbf{L}_{p,\epsilon} := L_p(\Omega_\epsilon; \mathbb{C})$. These spaces have, respectively, the next norms

$$\begin{aligned} &\|v\|^2 := \int_\Omega |v(x)|^2 dx, && \|v\|_\epsilon^2 := \int_{\Omega_\epsilon} |v(x)|^2 dx, && \|v\|_1^2 := \int_\Omega |\nabla v(x)|^2 dx, \\ &\|v\|_{1\epsilon}^2 := \int_{\Omega_\epsilon} |\nabla v(x)|^2 dx, && \|v\|_{\mathbf{L}_p}^p := \int_\Omega |v(x)|^p dx, && \|v\|_{\mathbf{L}_p\,\epsilon}^p := \int_{\Omega_\epsilon} |v(x)|^p dx. \end{aligned}$$

Let us denote that dual spaces to $\mathbf{V}$ by $\mathbf{V}' := H^{-1}(\Omega;\mathbb{C})$ and, moreover, $\mathbf{L}_q$ is the dual spaces of $\mathbf{L}_p$, where $q = \frac{p}{p-1}$, in analogous way $\mathbf{V}'_\epsilon$ and $\mathbf{L}_{q,\epsilon}$ are the dual spaces of $\mathbf{V}_\epsilon$ and $\mathbf{L}_{p,\epsilon}$.

As usually (see [14]) we investigate the behavior of weak solutions to initial boundary value problem (1), i.e., the functions

$$u_\epsilon(x,s) \in L_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\epsilon) \cap L_2^{loc}(\mathbb{R}_+;\mathbf{V}_\epsilon) \cap L_4^{loc}(\mathbb{R}_+;\mathbf{L}_{4,\epsilon})$$

which satisfy problem (1) in the sense of integral identity, i.e. for any function $\psi \in C_0^\infty(\mathbb{R}_+;\mathbf{V}_\epsilon \cap \mathbf{L}_{4,\epsilon})$ we have

$$-\int_0^\infty \int_{\Omega_\epsilon} u_\epsilon \frac{\partial \psi}{\partial t}\, dxdt + (1+\alpha \mathrm{i}) \int_0^\infty \int_{\Omega_\epsilon} \nabla u_\epsilon \nabla \psi \, dxdt - \int_0^\infty \int_{\Omega_\epsilon} \Bigg(\Bigg(R\Big(x,\frac{x}{\epsilon}\Big) u_\epsilon - \\ - \left(1+\beta\Big(x,\frac{x}{\epsilon}\Big)\mathrm{i}\right)|u_\epsilon|^2 u_\epsilon \Bigg)\Bigg) \psi \, dxdt + \epsilon \int_0^{+\infty} \int_{S_\epsilon} q\left(x,\frac{x}{\epsilon}\right) u_\epsilon \psi \, d\sigma dt = \int_0^\infty \int_{\Omega_\epsilon} g(x)\psi\, dxdt. \quad (4)$$

Since $u_\epsilon(x,t) \in L_4(0,M;\mathbf{L}_{4,\epsilon})$, one can get $R\left(x,\frac{x}{\epsilon}\right) u_\epsilon(x,t) - \left(1+\beta\left(x,\frac{x}{\epsilon}\right)\mathrm{i}\right)|u_\epsilon(x,t)|^2 u_\epsilon(x,t) \in L_{4/3}(0,M;\mathbf{L}_{4/3,\epsilon})$. In addition, since $u_\epsilon(x,t) \in L_2(0,M;\mathbf{V}_\epsilon)$, we have $(1+\alpha\mathrm{i})\Delta u_\epsilon(x,t) + g(x) \in L_2(0,M;\mathbf{V}'_\epsilon)$. Consequently, for any weak solution $u_\epsilon(x,s)$ to problem (1) we obtain

$$\frac{\partial u_\epsilon(x,t)}{\partial t} \in L_{4/3}(0,M;\mathbf{L}_{4/3,\epsilon}) + L_2(0,M;\mathbf{V}'_\epsilon).$$

Keeping in mind the Sobolev embedding theorem, we conclude $L_{4/3}(0,M;\mathbf{L}_{4/3,\epsilon}) + L_2(0,M;\mathbf{V}'_\epsilon) \subset L_{4/3}\left(0,M;\mathbf{H}_\epsilon^{-r}\right)$. Here the space $\mathbf{H}_\epsilon^{-r} := H^{-r}(\Omega_\epsilon;\mathbb{C})$ and $r = \max\{1, d/4\}$. Therefore, for an arbitrary weak solution $u_\epsilon(x,t)$ of (1) we get $\frac{\partial u_\epsilon(x,t)}{\partial t} \in L_{4/3}\left(0,M;\mathbf{H}_\epsilon^{-r}\right)$.

Remark 1.1. Using the standard approach from [13], one can prove the existence of weak solution $u(x,s)$ to the problem (1) for every $U \in \mathbf{H}_\epsilon$ and fixed ϵ, satisfying $u(x,0) = U(x)$.

It is possible to prove the following basic Lemma similarly to Proposition 3 from [20].

Lemma 1.1. Suppose that $u_\epsilon(x,t) \in L_2^{loc}(\mathbb{R}_+;\mathbf{V}_\epsilon) \cap L_4^{loc}(\mathbb{R}_+;\mathbf{L}_{4,\epsilon})$ is a weak solution to (1). Then

(i) $u \in C(\mathbb{R}_+;\mathbf{H}_\epsilon)$;

(ii) the function $\|u_\epsilon(\cdot,t)\|_\epsilon^2$ is absolutely continuous on $\mathbb{R}_+$ and, moreover,

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon(\cdot,t)\|_\epsilon^2 + \|\nabla u_\epsilon(\cdot,t)\|_\epsilon^2 + \|u_\epsilon(\cdot,t)\|_{\mathbf{L}_{4,\epsilon}}^4 - \int_{\Omega_\epsilon} R\left(x,\frac{x}{\epsilon}\right)|u_\epsilon(x,t)|^2 dx + \\ + \epsilon \int_{S_\epsilon} q\left(x,\frac{x}{\epsilon}\right)|u_\epsilon(x,t)|^2 d\sigma = \int_{\Omega_\epsilon} Re\left(g(x)\bar{u}_\epsilon(x,t)\right) dx,$$

for almost every $t \in \mathbb{R}_+$.

We fix ϵ. Bellow, where it is natural, we omit the index ϵ in the notation of functional spaces. Now we use the approach described in Section 2 to construct the trajectory attractor of (1), which has the form (7) if we set $E_1 = \mathbf{L}_p \cap \mathbf{V},\ E_0 = \mathbf{H}^{-r},\ E = \mathbf{H}$ and $A(u) = (1+\alpha\mathrm{i})\Delta u + R(\cdot)\,u - (1+\beta(\cdot)\mathrm{i})\,|u|^2 u + g(\cdot)$.

To define the trajectory space $\mathcal{K}_\epsilon^+$ for (1), we use the general approaches of Section 2 and for every $[t_1,t_2] \in \mathbb{R}$ we have the Banach spaces

$$\mathcal{F}_{t_1,t_2} := L_4(t_1,t_2;\mathbf{L}_4) \cap L_2(t_1,t_2;\mathbf{V}) \cap L_\infty(t_1,t_2;\mathbf{H}) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in L_{4/3}\left(t_1,t_2;\mathbf{H}^{-r}\right) \right\}$$

with the following norm

$$\|v\|_{\mathcal{F}_{t_1,t_2}} := \|v\|_{L_4(t_1,t_2;\mathbf{L}_4)} + \|v\|_{L_2(t_1,t_2;\mathbf{V})} + \|v\|_{L_\infty(0,M;\mathbf{H})} + \left\|\frac{\partial v}{\partial t}\right\|_{L_{4/3}(t_1,t_2;\mathbf{H}^{-r})}.$$

Setting $\mathcal{D}_{t_1,t_2} = L_q\left(t_1, t_2; \mathbf{H}^{-r}\right)$ we obtain $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$ and for $u(s) \in \mathcal{F}_{t_1,t_2}$ we have $A(u(s)) \in \mathcal{D}_{t_1,t_2}$. One considers now weak solutions to (1) as solutions of an equation in the general scheme of Section 2.

Consider the spaces

$$\mathcal{F}_{+}^{loc} = L_4^{loc}(\mathbb{R}_+; \mathbf{L}_4) \cap L_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap L_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in L_{4/3}^{loc}(\mathbb{R}_+; \mathbf{H}^{-r}) \right\},$$

$$\mathcal{F}_{\epsilon,+}^{loc} = L_4^{loc}(\mathbb{R}_+; \mathbf{L}_{4,\epsilon}) \cap L_2^{loc}(\mathbb{R}_+; \mathbf{V}_\epsilon) \cap L_\infty^{loc}(\mathbb{R}_+; \mathbf{H}_\epsilon) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in L_{4/3}^{loc}(\mathbb{R}_+; \mathbf{H}_\epsilon{}^{-r}) \right\}.$$

We introduce the following notation. Let $\mathcal{K}_\epsilon^+$ be the set of all weak solutions to (1). For any $U \in \mathbf{H}$ there exists at least one trajectory $u(\cdot) \in \mathcal{K}_\epsilon^+$ such that $u(0) = U(x)$. Consequently, the space $\mathcal{K}_\epsilon^+$ to (1) is not empty and is sufficiently large.

It is easy to see that $\mathcal{K}_\epsilon^+ \subset \mathcal{F}_+^{loc}$ and the space $\mathcal{K}_\epsilon^+$ is translation invariant, i.e., if $u(s) \in \mathcal{K}_\epsilon^+$, then $u(h+s) \in \mathcal{K}_\epsilon^+$ for all $h \geq 0$. Hence, $S(h)\mathcal{K}_\epsilon^+ \subseteq \mathcal{K}_\epsilon^+$ for all $h \geq 0$.

We define metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ in the spaces $\mathcal{F}_{t_1,t_2}$ by means of the norms from $L_2(t_1, t_2; \mathbf{H})$. We get

$$\rho_{0,M}(u, v) = \left(\int_0^M \|u(s) - v(s)\|_{\mathbf{H}}^2 ds \right)^{1/2} \qquad \forall u(\cdot), v(\cdot) \in \mathcal{F}_{0,M}.$$

The topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$ (respectively $\Theta_{\epsilon,+}^{loc}$ in $\mathcal{F}_{\epsilon,+}^{loc}$) is generated by these metrics. Let us recall that $\{v_k\} \subset \mathcal{F}_+^{loc}$ converges to $v \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\|v_k(\cdot) - v(\cdot)\|_{L_2(0,M;\mathbf{H})} \to 0$ $(k \to \infty)$ for each $M > 0$. Bearing in mind (8), we conclude that the topology Θ_+^{loc} is metrizable. We consider this topology in the trajectory space $\mathcal{K}_\epsilon^+$ of (1). Also it can be seen that the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}_\epsilon^+$, is continuous in this topology.

Using the scheme of Section 2, one can define bounded sets in the space $\mathcal{K}_\epsilon^+$ by means of the Banach space $\mathcal{F}_+^b$. We naturally get

$$\mathcal{F}_+^b = L_4^b(\mathbb{R}_+; \mathbf{L}_4) \cap L_2^b(\mathbb{R}_+; \mathbf{V}) \cap L_\infty(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in L_{4/3}^b(\mathbb{R}_+; \mathbf{H}^{-r}) \right\}$$

and the set $\mathcal{F}_+^b$ is a subspace of $\mathcal{F}_+^{loc}$.

Consider the translation semigroup $\{S(t)\}$ on $\mathcal{K}_\epsilon^+$, $S(t) : \mathcal{K}_\epsilon^+ \to \mathcal{K}_\epsilon^+$, $t \geq 0$.

Suppose that $\mathcal{K}_\epsilon$ is the kernel to (1), that consists of all weak complete solutions $u(s), \in \mathbb{R}$, to our system of equations, bounded in

$$\mathcal{F}^b = L_4^b(\mathbb{R}; \mathbf{L}_4) \cap L_2^b(\mathbb{R}; \mathbf{V}) \cap L_\infty(\mathbb{R}; \mathbf{H}) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in L_{4/3}^b(\mathbb{R}; \mathbf{H}^{-r}) \right\}.$$

Proposition 1.1. Problem (1) has the trajectory attractors $\mathfrak{A}_\epsilon$ in the topological space Θ_+^{loc}. The set $\mathfrak{A}_\epsilon$ is uniformly (w.r.t. $\epsilon \in (0,1)$) bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover, $\mathfrak{A}_\epsilon = \Pi_+\mathcal{K}_\epsilon$, the kernel $\mathcal{K}_\epsilon$ is non-empty and uniformly (w.r.t. $\epsilon \in (0,1)$) bounded in $\mathcal{F}^b$. Recall that the spaces $\mathcal{F}_+^b$ and Θ_+^{loc} depend on ϵ.

To prove this proposition we use the approach of the proof from [14]. To prove the existence of an absorbing set (bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}) one can use Lemma 1.1 similar to [14].

It is easy to verify, that $\mathfrak{A}_\epsilon \subset \mathcal{B}_0(R)$ for all $\epsilon \in (0,1)$. Here $\mathcal{B}_0(R)$ is a ball in $\mathcal{F}_+^b$ with a sufficiently large radius R. By means of Lemma 2.1 we have

$$\mathcal{B}_0(R) \Subset L_2^{loc}(\mathbb{R}_+; \mathbf{H}^{1-\delta}), \tag{5}$$

$$\mathcal{B}_0(R) \Subset C^{loc}(\mathbb{R}_+; \mathbf{H}^{-\delta}), \quad 0 < \delta \leq 1. \tag{6}$$

Formula (5) immediately follows, if we take $E_0 = \mathbf{H}^{-r},\ E = \mathbf{H}^{1-\delta},\ E_1 = \mathbf{H}^1 = \mathbf{V}$, and $p_1 = 2$, $p_0 = 4/3$, keeping in mind the compact embedding $\mathbf{V} \Subset \mathbf{H}^{1-\delta}$. Formula (6) follows from the compact embedding $\mathbf{H} \Subset \mathbf{H}^{-\delta}$, if we take $E_0 = \mathbf{H}^{-\mathbf{r}}(D),\ E = \mathbf{H}^{-\delta},\ E_1 = \mathbf{H}^1 = \mathbf{V}$, and $p_0 = 4/3$.

Bearing in mind (5) and (6), the attraction to the constructed trajectory attractor can be strengthen.

Corollary 1.1. For any bounded in $\mathcal{F}_+^b$ set $\mathcal{B} \subset \mathcal{K}_\epsilon^+$ we get

$$\operatorname{dist}_{L_2(0,M;\mathbf{H}^{1-\delta})} (\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\epsilon) \to 0 \ (t \to \infty),$$
$$\operatorname{dist}_{C([0,M];\mathbf{H}^{-\delta})} (\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\epsilon) \to 0 \ (t \to \infty),$$

where M is a positive constant.

2 Trajectory attractors of evolution equations

This section is devoted to the construction of trajectory attractors to autonomous evolution equations.

Consider an autonomous evolution equation of the form

$$\frac{\partial u}{\partial t} = A(u), \quad t \geq 0. \tag{7}$$

Here $A(\cdot) : E_1 \to E_0$ is a nonlinear operator, E_1, E_0 are Banach spaces and $E_1 \subseteq E_0$. As an example one can consider $A(u) = (1+\alpha\mathrm{i})\Delta u + R(\cdot)\, u - (1+\beta(\cdot)\mathrm{i})\, |u|^2 u + g(\cdot)$.

We study weak solutions $u(s)$ to (7) as functions of parameter $s \in \mathbb{R}_+$ as a whole. To be precise we say that $s \equiv t$ denotes the time. The set of solutions of (7) is said to be *a trajectory space* $\mathcal{K}^+$ of equation (7). Now, we describe the trajectory space $\mathcal{K}^+$ in detail.

Consider solutions $u(s)$ of (7) defined on $[t_1, t_2] \subset \mathbb{R}$. We consider solutions to problem (7) in a Banach space $\mathcal{F}_{t_1,t_2}$. The space $\mathcal{F}_{t_1,t_2}$ is a set $f(s), s \in [t_1, t_2]$ satisfying $f(s) \in E$ for almost all $s \in [t_1, t_2]$, where E is a Banach space, satisfying $E_1 \subseteq E \subseteq E_0$.

For instance, $\mathcal{F}_{t_1,t_2}$ can be considered as the intersection spaces $C([t_1, t_2]; E)$, or $L_p(t_1, t_2; E)$, for $p \in [1, \infty]$. Suppose that $\Pi_{t_1,t_2}\mathcal{F}_{\tau_1,\tau_2} \subseteq \mathcal{F}_{t_1,t_2}$ and $\|\Pi_{t_1,t_2} f\|_{\mathcal{F}_{t_1,t_2}} \leq C(t_1, t_2, \tau_1, \tau_2)\|f\|_{\mathcal{F}_{\tau_1,\tau_2}}\ \forall f \in \mathcal{F}_{\tau_1,\tau_2}$. Here $[t_1, t_2] \subseteq [\tau_1, \tau_2]$ and Π_{t_1,t_2} denotes the restriction operator onto $[t_1, t_2]$, constant $C(t_1, t_2, \tau_1, \tau_2)$ does not depend on f.

Suppose that $S(h)$ for $h \in \mathbb{R}$ denotes the translation operator $S(h)f(s) = f(h+s)$. It is easy to see, that if the argument s of $f(\cdot)$ belongs to the segment $[t_1, t_2]$, then the argument s of $S(h)f(\cdot)$ belongs to $[t_1 - h, t_2 - h]$ for $h \in \mathbb{R}$. Suppose that the mapping $S(h)$ is an isomorphism from F_{t_1,t_2} to F_{t_1-h,t_2-h} and $\|S(h)f\|_{\mathcal{F}_{t_1-h,t_2-h}} = \|f\|_{\mathcal{F}t_1,t_2}\ \forall f \in \mathcal{F}_{t_1,t_2}$. It is easy to see that this assumption is natural.

Suppose that if $f(s) \in \mathcal{F}_{t_1,t_2}$, then $A(f(s)) \in \mathcal{D}_{t_1,t_2}$, where $\mathcal{D}_{t_1,t_2}$ is a Banach space, which is larger, $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$. The derivative $\frac{\partial f(t)}{\partial t}$ is a distribution with values in E_0, $\frac{\partial f}{\partial t} \in D'((t_1, t_2); E_0)$ and we suppose that $\mathcal{D}_{t_1,t_2} \subseteq D'((t_1, t_2); E_0)$ for all $(t_1, t_2) \subset \mathbb{R}$. A function $u(s) \in \mathcal{F}_{t_1,t_2}$ is a *solution* of (7), if $\frac{\partial u}{\partial t}(s) = A(u(s))$ in the sense of $D'((t_1, t_2); E_0)$.

Let us define the space $\mathcal{F}_+^{loc} = \{f(s),\ s \in \mathbb{R}_+ \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2}, \quad \forall\ [t_1, t_2] \subset \mathbb{R}_+\}$. For instance, if $\mathcal{F}_{t_1,t_2} = C([t_1, t_2]; E)$, then $\mathcal{F}_+^{loc} = C(\mathbb{R}_+; E)$ and if $\mathcal{F}_{t_1,t_2} = L_p(t_1, t_2; E)$, then $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+; E)$.

A function $u(s) \in \mathcal{F}_+^{loc}$ is a solution of (7), if $\Pi_{t_1,t_2} u(s) \in \mathcal{F}_{t_1,t_2}$ and $u(s)$ is a solution of (7) for every $[t_1, t_2] \subset \mathbb{R}_+$.

Let $\mathcal{K}^+$ be a set of solutions to (7) from $\mathcal{F}_+^{loc}$. Note, that $\mathcal{K}^+$ in general is not the set of *all* solutions from $\mathcal{F}_+^{loc}$. The set $\mathcal{K}^+$ consists on elements, which are *trajectories* and the set $\mathcal{K}^+$ is the *trajectory space* of the equation (7).

Suppose that the trajectory space $\mathcal{K}^+$ is *translation invariant*, i.e., if $u(s) \in \mathcal{K}^+$, then $u(h+s) \in \mathcal{K}^+$ for every $h \geq 0$.

Consider the translation operators $S(h)$ in $\mathcal{F}_+^{loc} : S(h)f(s) = f(s+h),\ h \geq 0$. It is easy to see that the map $\{S(h), h \geq 0\}$ forms a semigroup in $\mathcal{F}_+^{loc} :\ S(h_1)S(h_2) = S(h_1 + h_2)$ for $h_1, h_2 \geq 0$ and in addition $S(0)$ is the identity operator. Next step is to change the variable h into the time variable t. The *translation semigroup* $\{S(t), t \geq 0\}$ maps the trajectory space $\mathcal{K}^+$ to itself: $S(t)\mathcal{K}^+ \subseteq \mathcal{K}^+$ for all $t \geq 0$.

We investigate attracting properties of the translation semigroup $\{S(t)\}$ acting on the trajectory space $\mathcal{K}^+ \subset \mathcal{F}_+^{loc}$. Next step is to define a topology in the space $\mathcal{F}_+^{loc}$.

One can see, that metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ is defined on $\mathcal{F}_{t_1,t_2}$ for every $[t_1,t_2] \subset \mathbb{R}$. Suppose that

$$\rho_{t_1,t_2}(\Pi_{t_1,t_2}f, \Pi_{t_1,t_2}g) \leq D(t_1,t_2,\tau_1,\tau_2)\rho_{\tau_1,\tau_2}(f,g) \quad \forall f,g \in \mathcal{F}_{\tau_1,\tau_2},\ [t_1,t_2] \subseteq [\tau_1,\tau_2],$$

$$\rho_{t_1-h,t_2-h}(S(h)f, S(h)g) = \rho_{t_1,t_2}(f,g) \quad \forall f,g \in \mathcal{F}_{t_1,t_2},\ [t_1,t_2] \subset \mathbb{R},\ h \in \mathbb{R}.$$

Now, we denote by Θ_{t_1,t_2} metric spaces on $\mathcal{F}_{t_1,t_2}$. For instance, ρ_{t_1,t_2} is metric associated with the norm $\|\cdot\|_{\mathcal{F}_{t_1,t_2}}$ of $\mathcal{F}_{t_1,t_2}$. At the other hand, in application ρ_{t_1,t_2} generates the topology Θ_{t_1,t_2} that is weaker than the strong one of the $\mathcal{F}_{t_1,t_2}$.

The *projective limit* of the spaces Θ_{t_1,t_2} defines the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$, that is, by definition, a sequence $\{f_k(s)\} \subset \mathcal{F}_+^{loc}$ tends to $f(s) \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\rho_{t_1,t_2}(\Pi_{t_1,t_2}f_k, \Pi_{t_1,t_2}f) \to 0$ as $k \to \infty$ for all $[t_1,t_2] \subset \mathbb{R}_+$. It is possible to show that the topology Θ_+^{loc} is metrizable. For this aim we use, for example, the Frechet metric

$$\rho_+(f_1,f_2) := \sum_{m\in\mathbb{N}} 2^{-m}\frac{\rho_{0,m}(f_1,f_2)}{1+\rho_{0,m}(f_1,f_2)}. \tag{8}$$

The translation semigroup $\{S(t)\}$ is continuous in Θ_+^{loc}. This statement follows from the definition of Θ_+^{loc}.

We also define the following Banach space

$$\mathcal{F}_+^b := \{f(s) \in \mathcal{F}_+^{loc} \mid \|f\|_{\mathcal{F}_+^b} < +\infty\},$$

where the norm

$$\|f\|_{\mathcal{F}_+^b} := \sup_{h\geq 0}\|\Pi_{0,1}f(h+s)\|_{\mathcal{F}_{0,1}}.$$

We remember that $\mathcal{F}_+^b \subseteq \Theta_+^{loc}$. We need from our Banach space $\mathcal{F}_+^b$ only one fact It should define bounded subsets in the trajectory space $\mathcal{K}^+$. For constructing a trajectory attractor in $\mathcal{K}^+$, instead of considering the corresponding uniform convergence topology of the Banach space $\mathcal{F}_+^b$, we use much weaker topology, i.e. the local convergence topology Θ_+^{loc}.

Assume that $\mathcal{K}^+ \subseteq \mathcal{F}_+^b$, that is, every trajectory $u(s) \in \mathcal{K}^+$ of equation (7) has a finite norm. We define an attracting set and a trajectory attractor of the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$.

Definition 2.1. A set $\mathcal{P} \subseteq \Theta_+^{loc}$ is called an *attracting set* of the semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$ in the topology Θ_+^{loc} if for any bounded in $\mathcal{F}_+^b$ set $\mathcal{B} \subseteq \mathcal{K}^+$ the set $\mathcal{P}$ attracts $S(t)\mathcal{B}$ as $t \to +\infty$ in the topology Θ_+^{loc}, i.e., for any ϵ-neighbourhood $O_\epsilon(\mathcal{P})$ in Θ_+^{loc} there exists $t_1 \geq 0$ such that $S(t)\mathcal{B} \subseteq O_\epsilon(\mathcal{P})$ for all $t \geq t_1$.

It is easy to see that the attracting property of $\mathcal{P}$ can be formulated equivalently: we have

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}S(t)\mathcal{B}, \Pi_{0,M}\mathcal{P}) \longrightarrow 0 \quad (t \to +\infty),$$

where $\mathrm{dist}_{\mathcal{M}}(X,Y) := \sup_{x\in X}\mathrm{dist}_{\mathcal{M}}(x,Y) = \sup_{x\in X}\inf_{y\in Y}\rho_{\mathcal{M}}(x,y)$ is the Hausdorff semidistance from a set X to a set Y in a metric space $\mathcal{M}$. We remember that the Hausdorff semidistance is not symmetric, for any $\mathcal{B} \subseteq \mathcal{K}^+$ bounded in $\mathcal{F}_+^b$ and for each $M > 0$.

Definition 2.2 ([14]). A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called the *trajectory attractor* of the translation semigroup $\{S(t)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, if

(i) $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc},
(ii) the set $\mathfrak{A}$ is strictly invariant with respect to the semigroup: $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$,
(iii) $\mathfrak{A}$ is an attracting set for $\{S(t)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, that is, for each $M > 0$ we have

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}S(t)\mathcal{B}, \Pi_{0,M}\mathfrak{A}) \to 0 \quad (t \to +\infty).$$

Let us formulate the main assertion on the trajectory attractor for equation (7).

Theorem 2.1 ([13, 14]). Assume that the trajectory space $\mathcal{K}^+$ corresponding to equation (7) is contained in $\mathcal{F}_+^b$. Suppose that the translation semigroup $\{S(t)\}$ has an attracting set $\mathcal{P} \subseteq \mathcal{K}^+$ which is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Then the translation semigroup $\{S(t), t \geq 0\}$ acting on $\mathcal{K}^+$ has the trajectory attractor $\mathfrak{A} \subseteq \mathcal{P}$. The set $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}.

Let us describe in detail, i.e., in terms of complete trajectories of the equation, the structure of the trajectory attractor $\mathfrak{A}$ to equation (7). We study the equation (7) on the time axis

$$\frac{\partial u}{\partial t} = A(u), \ t \in \mathbb{R}. \tag{9}$$

Note that the trajectory space $\mathcal{K}^+$ of equation (9) on $\mathbb{R}_+$ have been defined. We need this notion on the entire $\mathbb{R}$. If a function $f(s)$, $s \in \mathbb{R}$, is defined on the entire time axis, then the translations $S(h)f(s) = f(s+h)$ are also defined for negative h. A function $u(s), s \in \mathbb{R}$ is a *complete trajectory* of equation (9) if $\Pi_+ u(s+h) \in \mathcal{K}^+$ for all $h \in \mathbb{R}$. Here $\Pi_+ = \Pi_{0,\infty}$ denotes the restriction operator to $\mathbb{R}_+$.

We have $\mathcal{F}_+^{loc}, \mathcal{F}_+^b$, and Θ_+^{loc}. Let us define spaces $\mathcal{F}^{loc}, \mathcal{F}^b$, and Θ^{loc} in the same way:

$$\mathcal{F}^{loc} := \{f(s), s \in \mathbb{R} \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2} \ \forall \ [t_1, t_2] \subseteq \mathbb{R}\}; \qquad \mathcal{F}^b := \{f(s) \in \mathcal{F}^{loc} \mid \|f\|_{\mathcal{F}^b} < +\infty\},$$

where

$$\|f\|_{\mathcal{F}^b} := \sup_{h \in \mathbb{R}} \|\Pi_{0,1} f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{10}$$

The topological space Θ^{loc} coincides (as a set) with $\mathcal{F}^{loc}$ and, by definition, $f_k(s) \to f(s)$ $(k \to \infty)$ in Θ^{loc} if $\Pi_{t_1,t_2} f_k(s) \to \Pi_{t_1,t_2} f(s)$ $(k \to \infty)$ in Θ_{t_1,t_2} for each $[t_1, t_2] \subseteq \mathbb{R}$. It is easy to see that Θ^{loc} is a metric space as well as Θ_+^{loc}.

Definition 2.3. The *kernel* $\mathcal{K}$ in the space $\mathcal{F}^b$ of equation (9) is the union of all complete trajectories $u(s), s \in \mathbb{R}$, of equation (9) that are bounded in the space $\mathcal{F}^b$ with respect to the norm (10), i.e.

$$\|\Pi_{0,1} u(h+s)\|_{\mathcal{F}_{0,1}} \leq C_u \quad \forall h \in \mathbb{R}.$$

Theorem 2.2. Assume that the hypotheses of Theorem holds. Then $\mathfrak{A} = \Pi_+ \mathcal{K}$, the set $\mathcal{K}$ is compact in Θ^{loc} and bounded in $\mathcal{F}^b$.

To prove this assertion one can use the approach from [14].

In various applications, to prove that a ball in $\mathcal{F}_+^b$ is compact in Θ_+^{loc} the following lemma is useful. Let E_0 and E_1 be Banach spaces such that $E_1 \subset E_0$. We consider the Banach spaces

$$W_{p_1,p_0}(0, M; E_1, E_0) = \left\{\psi(s), s \in 0, M \mid \psi(\cdot) \in L_{p_1}(0, M; E_1), \ \psi'(\cdot) \in L_{p_0}(0, M; E_0)\right\},$$

$$W_{\infty,p_0}(0, M; E_1, E_0) = \left\{\psi(s), s \in 0, M \mid \psi(\cdot) \in L_{\infty}(0, M; E_1), \ \psi'(\cdot) \in L_{p_0}(0, M; E_0)\right\},$$

(where $p_1 \geq 1$ and $p_0 > 1$) with norms

$$\|\psi\|_{W_{p_1,p_0}} := \left(\int_0^M \|\psi(s)\|_{E_1}^{p_1} ds\right)^{1/p_1} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} ds\right)^{1/p_0},$$

$$\|\psi\|_{W_{\infty,p_0}} := \operatorname{ess\,sup}\left\{\|\psi(s)\|_{E_1} \mid s\in[0,M]\right\} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} ds\right)^{1/p_0}.$$

Lemma 2.1 (Aubin-Lions-Simon, [22]). Assume that $E_1 \Subset E \subset E_0$. Then the following embeddings are compact:

$$W_{p_1,p_0}(0,T;E_1,E_0) \Subset L_{p_1}(0,T;E), \qquad W_{\infty,p_0}(0,T;E_1,E_0) \Subset C([0,T];E).$$

In this paper we investigate evolution equations and their trajectory attractors depending on a small parameter $\epsilon > 0$.

Definition 2.4. We say that the trajectory attractors $\mathfrak{A}_\epsilon$ converge to the trajectory attractor $\overline{\mathfrak{A}}$ as $\epsilon \to 0$ in the topological space Θ_+^{loc} if for any neighborhood $\mathcal{O}(\mathfrak{A})$ in Θ_+^{loc} there is an $\epsilon_1 \geq 0$ such that $\mathfrak{A}_\epsilon \subseteq \mathcal{O}(\overline{\mathfrak{A}})$ for any $\epsilon < \epsilon_1$, that is, for each $M > 0$ we have

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M}\mathfrak{A}_\epsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0 \ (\epsilon \to 0).$$

3 Formal homogenization procedure

Let M_i be a solution to a problem

$$\begin{cases} \Delta_\xi M_i\,(x,\xi) = 0 & \text{in} \quad \omega, \\ \dfrac{\partial M_i(x,\xi)}{\partial \nu} = -\tilde{\nu}_i & \text{on} \quad S(x). \end{cases} \tag{11}$$

Denote by $\langle\cdot\rangle$ the integral over the set $\square \cap \omega$, and $Q(x) = \int_S q(x,\xi)\, d\sigma$.

The limit problem has the form

$$\begin{cases} \dfrac{\partial u_0}{\partial t} - (1+\alpha \mathrm{i}) \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle \frac{\partial u_0}{\partial x_j}\right) - & \\ \qquad -R(x)u_0 + (1+\beta(x)\mathrm{i})\,|u_0|^2\, u_0 + Q(x)u_0 = |\square\cap\omega|\, g(x), & x\in\Omega, \\ u_0 = 0, & x\in\partial\Omega, t>0, \\ u_0 = U(x), & x\in\Omega, t=0. \end{cases} \tag{12}$$

It is easy to see that system (12) also has trajectory attractor $\mathfrak{A}$ in the trajectory space $\mathcal{K}_+$ corresponding to problem (12) and $\mathfrak{A} = \Pi_+\mathcal{K}$, where $\mathcal{K}$ is the kernel of system (12) in $\mathcal{F}_+^b$.

The integral identity for problem (12) takes the form

$$-\int_{\mathbb{R}_+}\int_\Omega u_0 \frac{\partial v}{\partial t} dtdx + (1+\alpha\mathrm{i})\int_{\mathbb{R}_+}\int_\Omega \sum_{i,j=1}^{d} \left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle \frac{\partial u_0}{\partial x_i}\frac{\partial v}{\partial x_j} dtdx +$$
$$-\int_{\mathbb{R}_+}\int_\Omega \Big(R(x)u_0 - (1+\beta(x)\mathrm{i})\,|u_0|^2\, u_0 - Q(x)u_0\Big) v dtdx = \int_{\mathbb{R}_+}\int_\Omega |\square\cap\omega|\, g(x)\, v\, dtdx$$

for any function $v \in C_0^\infty(\mathbb{R}_+; \mathbf{V}\cap\mathbf{L}_4)$.

Remark 3.1. Note that $M_i(x,\xi)$ are not defined in the whole Ω. We can extend $M_i(x,\xi)$ into the interior of the cavities retaining the regularity of these functions by means of the technique of the symmetric extension, keeping the same notation for the extended functions.

4 Auxiliaries

We study the asymptotics of solution $u_\epsilon(x)$ as $\epsilon \to 0$ to the next boundary-value problem

$$\begin{cases} -(1+\alpha\mathrm{i})\Delta u_\epsilon = g(x) & \text{in} \quad \Omega_\epsilon, \\ (1+\alpha\mathrm{i})\dfrac{\partial u_\epsilon}{\partial \nu_\epsilon} + \epsilon q\left(x, \dfrac{x}{\epsilon}\right) u_\epsilon = 0 & \text{on} \quad S_\epsilon, \\ u_\epsilon = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{13}$$

Here n_ϵ is the internal normal to the boundary of cavities and $q(x,\xi)$ is a sufficiently smooth 1-periodic in ξ function.

Definition 4.1. The function $u_\epsilon \in H^1(\Omega_\epsilon, \partial\Omega)$ is a solution of problem (13), if the following integral identity

$$(1+\alpha\mathrm{i}) \int_{\Omega^\epsilon} \nabla u_\epsilon(x)\, \nabla v(x)\, dx + \epsilon \int_{S_\epsilon} q\left(x, \frac{x}{\epsilon}\right) u_\epsilon(x) v(x)\, ds = \int_{\Omega^\epsilon} g(x)\, v(x)\, dx$$

holds true for any function $v \in H^1(\Omega_\epsilon, \partial\Omega)$.

Here $H^1(\Omega^\epsilon, \partial\Omega)$ is the closure of the set of functions belonging to $C^\infty(\overline{\Omega}^\epsilon)$ and vanishing in a neighborhood of $\partial\Omega$, by the $H^1(\Omega^\epsilon)$ norm.

Here we derive the leading terms of the asymptotic expansion and, then, construct the homogranized problem. For this aim we consider the solution $u_\epsilon(x)$ to (13) as an asymptotic series

$$u_\epsilon(x) = u_0(x) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right) + \epsilon^2 u_2\left(x, \frac{x}{\epsilon}\right) + \epsilon^3 u_3\left(x, \frac{x}{\epsilon}\right) + \ldots \tag{14}$$

Substituting expression (14) in equation (13) and bearing in mind the relation

$$\frac{\partial}{\partial x}\zeta\left(x, \frac{x}{\epsilon}\right) = \left(\frac{\partial}{\partial x}\zeta(x,\xi) + \frac{1}{\epsilon}\frac{\partial}{\partial \xi}\zeta(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}},$$

we get the formula

$$\begin{aligned} -\frac{g(x)}{1+\alpha\mathrm{i}} = \Delta_x u_\epsilon(x) \cong \Delta_x u_0(x) + \epsilon \left(\Delta_x u_1(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + 2\left(\nabla_x, \nabla_\xi u_1(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \\ + \frac{1}{\epsilon}\left(\Delta_\xi u_1(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \epsilon^2\left(\Delta_x u_2(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + 2\epsilon\left(\nabla_x, \nabla_\xi u_2(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \\ + \left(\Delta_\xi u_2(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \epsilon^3\left(\Delta_x u_3(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \\ + 2\epsilon^2\left(\nabla_x, \nabla_\xi u_3(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \epsilon\left(\Delta_\xi u_3(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \ldots \end{aligned} \tag{15}$$

Similarily, substituting (14) into boundary conditions in (13), we get the relation

$$\begin{aligned} 0 = \frac{\partial u_\epsilon}{\partial \nu_\epsilon} + \epsilon\frac{q\left(x, \frac{x}{\epsilon}\right)}{1+\alpha\mathrm{i}} u_\epsilon \cong \left(\nabla_x u_0, \nu_\epsilon\right) + \epsilon\frac{q\left(x, \frac{x}{\epsilon}\right)}{1+\alpha\mathrm{i}} u_0 + \epsilon\left(\nabla_x u_1, \nu_\epsilon\right) + \\ + \left(\nabla_\xi u_1\big|_{\xi=\frac{x}{\epsilon}}, \nu_\epsilon\right) + \epsilon^2\frac{q\left(x, \frac{x}{\epsilon}\right)}{1+\alpha\mathrm{i}} u_1 + \epsilon^2\left(\nabla_x u_2, \nu_\epsilon\right) + \epsilon\left(\nabla_\xi u_2\big|_{\xi=\frac{x}{\epsilon}}, \nu_\epsilon\right) + \\ + \epsilon^3\frac{q\left(x, \frac{x}{\epsilon}\right)}{1+\alpha\mathrm{i}} u_2 + \epsilon^3\left(\nabla_x u_3, \nu_\epsilon\right) + \epsilon^2\left(\nabla_\xi u_3\big|_{\xi=\frac{x}{\epsilon}}, \nu_\epsilon\right) + \epsilon^4\frac{q\left(x, \frac{x}{\epsilon}\right)}{1+\alpha\mathrm{i}} u_3 + \ldots, \end{aligned} \tag{16}$$

which means that it satisfies the boundary condition on S_ϵ.

The normal vector ν_ϵ depends on x and $\frac{x}{\epsilon}$ in Ω_ϵ. Now, we consider x and $\xi = \frac{x}{\epsilon}$ as independent variables, and then we represent ν_ϵ in Ω_ϵ in the form

$$\nu_\epsilon(x, \frac{x}{\epsilon}) = \widetilde{\nu}(x, \xi)\Big|_{\xi=\frac{x}{\epsilon}} + \epsilon\nu'_\epsilon(x, \xi)\Big|_{\xi=\frac{x}{\epsilon}},$$

where $\widetilde{\nu}$ is a normal vector to $S(x) = \{\xi \,|\, F(x, \xi) = 0\}$,

$$\nu'_\epsilon = \nu' + O(\epsilon).$$

Collecting all the terms of order ϵ^{-1} in (15) and of order ϵ^0 in (16), we deduce the auxiliary problem

$$\begin{cases} \Delta_\xi u_1\,(x, \xi) = 0 & \text{in} \quad \omega, \\ \dfrac{\partial u_1(x, \xi)}{\partial \nu} = -\left(\nabla_x(u_0(x)), \tilde{n}\right) & \text{on} \quad S, \end{cases} \tag{17}$$

which we solve in the space of 1-periodic in ξ functions and here x is a parameter, $\omega := \{\xi \in \mathbb{T}^d \,|\, F(x, \xi) > 0\}$. This is the cell problem appearing in case of Neumann conditions on the boundary of cavities. It is easy to see that the compatibility condition $\displaystyle\int_{S(x)} (\nabla_x u_0(x), \tilde{\nu}(\xi))\;d\sigma = 0$ of (17) is satisfied, and the solution of this problem is the first corrector in (14).

At the next step we collect all the terms of order ϵ^0 in (15) and of order ϵ^1 in (16). This gives us

$$\begin{cases} \Delta_\xi u_2\,(x, \xi) &= -\dfrac{g(x)}{1+\alpha\mathrm{i}} - \Delta_x u_0(x) - 2\left(\nabla_\xi, \nabla_x u_1(x, \xi)\right) \quad \text{in} \quad \omega, \\ \dfrac{\partial u_2(x, \xi)}{\partial \nu} &= -\left(\nabla_x u_1(x, \xi), \tilde{\nu}\right) - \left(\nabla_\xi u_1(x, \xi), \nu'\right) - \\ & \quad -\left(\nabla_x u_0(x), \nu'\right) - \dfrac{q(x, \xi)}{1+\alpha\mathrm{i}} u_0(x) \quad \text{on} \quad S(x). \end{cases} \tag{18}$$

The 1-periodic in ξ solution of the latter problem is the second term of the internal asymptotic expansion of $u_\epsilon(x)$.

It is easy to see that for our analysis it is convenient to represent the solution $u_1(x, \xi)$ of problem (17) in the following form:

$$u_1(x, \xi) = \left(\operatorname{grad}_x u_0(x), M(x, \xi)\right),$$

where 1-periodic vector–function $M(x, \xi) = (M_1(x, \xi), \ldots, M_d(x, \xi))$ is a solution to (11).

Now, (18) can be rewritten as follows

$$\begin{cases} \Delta_\xi u_2\,(x, \xi) &= -\dfrac{g(x)}{1+\alpha\mathrm{i}} - \Delta_x u_0(x) - 2\displaystyle\sum_{i,j=1}^{d} \dfrac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}\;\dfrac{\partial M_i(x, \xi)}{\partial \xi_j} - \\ & \quad -2\displaystyle\sum_{i,j=1}^{d} \dfrac{\partial u_0(x)}{\partial x_i}\;\dfrac{\partial^2 M_i(x, \xi)}{\partial \xi_j\,\partial x_j} \qquad \text{in} \quad \omega, \\ \dfrac{\partial u_2(x, \xi)}{\partial \nu} &= -\displaystyle\sum_{i,j=1}^{d} \dfrac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j} M_i(x, \xi)\nu_j - \displaystyle\sum_{i,j=1}^{d} \dfrac{\partial u_0(x)}{\partial x_i}\;\dfrac{\partial M_i(x, \xi)}{\partial x_j}\;\nu_j - \\ & \quad -\dfrac{q(x, \xi)}{1+\alpha\mathrm{i}} u_0(x) - \displaystyle\sum_{i,j=1}^{d} \dfrac{\partial u_0(x)}{\partial x_i}\;\left(\dfrac{\partial M_i(x, \xi)}{\partial \xi_j} + \delta_{ij}\right)\nu'_j \qquad \text{on} \quad S(x). \end{cases}$$

Writing down the solvability condition in the last problem, we derive the equation:

$$\int_{\square\cap\omega}\left(\frac{g(x)}{1+\alpha \mathrm{i}}+\Delta_x u_0(x)+2\sum_{i,j=1}^{d}\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}\ \frac{\partial M_i(x,\xi)}{\partial \xi_j}+2\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\ \frac{\partial^2 M_i(x,\xi)}{\partial \xi_j\,\partial x_j}\right)d\xi =$$
$$=\int_Q\left(\sum_{i,j=1}^{d}\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}M_i(x,\xi)\nu_j+\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\ \frac{\partial M_i(x,\xi)}{x_j}\ \nu_j+\right.$$
$$\left.+\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\ \frac{\partial M_i(x,\xi)}{\partial \xi_j}\ \nu'_j+\sum_{i=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\ \nu'_i+\frac{q(x,\xi)}{1+\alpha \mathrm{i}}u_0(x)\right)d\sigma. \quad (19)$$

From (19) by the Stokes formula we derive the equation

$$|\square\cap\omega|\Delta_x u_0(x)+\sum_{i,j=1}^{d}\left\langle\frac{\partial^2 M_i(x,\xi)}{\partial x_j\,\partial \xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}+\sum_{i,j=1}^{d}\left\langle\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}+$$
$$+|\square\cap\omega|\frac{g(x)}{1+\alpha \mathrm{i}}=\frac{Q(x)}{1+\alpha \mathrm{i}}u_0(x)+\sum_{i=1}^{d}U_i(x)\frac{\partial u_0(x)}{\partial x_i}, \quad (20)$$

which is the limit equation in Ω. We denoted by $<\ \cdot\ >$ the integral over $\square\cap\omega$, and $Q(x)=\int_{S(x)}q(x,\xi)\,d\sigma$. Moreover, $U_i(x)=\int_{S(x)}\left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}\nu'_j+\nu'_i\right)d\sigma$.

It is not necessary to calculate $U_i(x)$, since by the selfadjointness of the operators of the given problems and the convergence of the corresponding belinear forms, we get that the G–limit operator is necessary selfadjoint. Therefore, the limit equation (20) takes the form:

$$(1+\alpha \mathrm{i})\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\right)+|\square\cap\omega|g(x)=Q(x)u_0(x) \quad (21)$$

and, consequently,

$$U_i(x)=\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left\langle\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle-\sum_{j=1}^{d}\left\langle\frac{\partial^2 M_i(x,\xi)}{\partial x_j\,\partial \xi_j}\right\rangle.$$

It is easy to see that $\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle$ is a smooth positively defined matrix (see [9]).

The next statement is about the limit behavior of the solution to (13).

Theorem 4.1. Suppose that $g(x)\in C^1(\mathbb{R}^d)$ and that $q(x,\xi)$ is smooth enough nonnegative function. Then, for any sufficiently small ϵ problem (13) has the unique solution and the following convergence

$$\|u_0-u_\epsilon\|_{H^1(\Omega_\epsilon)}\longrightarrow 0$$

takes place, where u_0 is a solution of equation (21) with zero Dirichlet conditions on $\partial\Omega$.

Remark 4.1. In fact, in the formulation of Theorem 4.1 the condition $q(x,\xi)\geq 0$ can be replaced by the weaker condition $Q(x)\geq 0$.

4.1 Preliminary Lemmas

Here we give some technical propositions, which we use in the further analysis. Some of these propositions have been proved in [3, 23]. We omit their proofs.

Lemma 4.1. If the conditions of Theorem are satisfied, then the Friederichs type inequality

$$\int_{\Omega_\epsilon} |\nabla v|^2 dx + \epsilon \int_{S_\epsilon} q\left(x, \frac{x}{\epsilon}\right) v^2 ds \geq C_1 \|v\|^2_{H^1(\Omega^\epsilon, \partial\Omega)}$$

is valid for any $v \in H^1(\Omega_\epsilon, \partial\Omega)$, where C_1 is independent of ϵ.

Now we formulate a modified version of Lemma 5 from [23].

Lemma 4.2. If we suppose

$$\frac{1}{|\Box \cap \omega|} \int_{\Box\cap\omega} Q(x)\ d\xi - \int_{S(x)} q(x,\xi)\ d\sigma \equiv 0,$$

then the following inequality

$$\left| \frac{1}{|\Box \cap \omega|} \int_{\Omega^\epsilon} Q(x)\, v(x)\ dx - \epsilon \int_{S_\epsilon} q\left(x, \frac{x}{\epsilon}\right)\ v(x)\ d\sigma \right| \leq C_2 \epsilon \|v\|_{H^1(\Omega^\epsilon)}$$

holds for any $v(x) \in H^1(\Omega_\epsilon, \partial\Omega)$; the constant C_2 is independent of ϵ.

Proof. The proof of this assertion can be found in [24].

Lemma 4.3. If y_ϵ is a solution to

$$\begin{cases} -(1+\alpha \mathrm{i})\Delta y_\epsilon = h^\epsilon(x) & \text{in} \quad \Omega_\epsilon, \\ (1+\alpha \mathrm{i})\dfrac{\partial y_\epsilon}{\partial \nu_\epsilon} + \epsilon q\left(x, \dfrac{x}{\epsilon}\right) y_\epsilon = 0 & \text{on} \quad S_\epsilon, \\ y_\epsilon = 0 & \text{on} \quad \Omega, \end{cases}$$

where $h^\epsilon(x) = g(x)$ for $x \in \Omega_\epsilon$ and 0 otherwise, then

$$\|y_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C_3 \epsilon.$$

The proposition, which is a modification of Lemma 5 from [23], formulated below.

Lemma 4.4. Suppose $w^\epsilon(x) \in L_\infty(\Omega)$, and let Π^ϵ belong to $\{x \in \Omega \,|\, \mathrm{dist}\,(x, \partial\Omega) \leq C_0 \epsilon\}$. Then the following inequality

$$\left| \int_{\Pi^\epsilon} w^\epsilon(x) \Big|_{\xi = \frac{x}{\epsilon}} \nabla_x u_0(x)\, v(x)\ dx \right| \leq C_4 \epsilon^{\frac{3}{2}} \|w\|_{L_\infty(\Omega)} \|v\|_{H^1(\Omega_\epsilon)}$$

holds for any $v(x) \in H^1(\Omega_\epsilon, \partial\Omega)$; the constant C_4 is independent of ϵ.

Proof of the Theorem 4.1. The proof of this assertion can be found in [23].

5 The main assertion

Here formulate the main proposition concerning the Ginzburg-Landau equation.

Theorem 5.1. The following limit holds in the topological space Θ_+^{loc}

$$\mathfrak{A}_\epsilon \to \overline{\mathfrak{A}} \ \text{ as } \epsilon \to 0+\,. \tag{22}$$

Moreover,

$$\mathcal{K}_\epsilon \to \overline{\mathcal{K}} \text{ as } \epsilon \to 0+ \ \text{ in } \ \Theta^{loc}. \tag{23}$$

Remark 5.1. The functions belonging the sets $\mathfrak{A}_\epsilon$ and $\mathcal{K}_\epsilon$ are defined in the perforated domains Ω_ϵ. But, all these functions can be extended insides the cavities remaining their norms in the spaces $\mathbf{H}, \mathbf{V}$, and $\mathbf{L}_p$ (without perforation) with the constants independent of the small parameter (the prolongation of functions defined in perforated domains, see, for instance, in [10; Ch.VIII]). Hence, in Theorem 5.1, we have all the distances in the spaces without perforation.

Proof. It is easy to see that (23) implies (22). Hence, it is sufficient to prove (23), i.e., for every neighborhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc} there exists $\epsilon_1 = \epsilon_1(\mathcal{O}) > 0$, such that

$$\mathcal{K}_\epsilon \subset \mathcal{O}(\overline{\mathcal{K}}) \quad \text{for} \quad \epsilon < \epsilon_1. \tag{24}$$

Assume that (24) is not true. Then there exists a neighborhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\epsilon_k \to 0+ \ (k \to \infty)$, and a sequence $u_{\epsilon_k}(\cdot) = u_{\epsilon_k}(s) \in \mathcal{K}_{\epsilon_k}$, such that

$$u_{\epsilon_k} \notin \mathcal{O}'(\overline{\mathcal{K}}) \quad \text{for all} \quad k \in \mathbb{N}.$$

The function $u_{\epsilon_k}(s), s \in \mathbb{R}$ is a solution to

$$\begin{cases} \dfrac{\partial u_{\epsilon_k}}{\partial t} = (1+\alpha \mathrm{i})\Delta u_{\epsilon_k} + R\left(x, \dfrac{x}{\epsilon_k}\right) u_{\epsilon_k} - \left(1 + \beta\left(x, \dfrac{x}{\epsilon_k}\right)\mathrm{i}\right)|u_{\epsilon_k}|^2 u_{\epsilon_k} + g\left(x\right), & x \in \Omega_{\epsilon_k}, \\ (1+\alpha \mathrm{i})\dfrac{\partial u_{\epsilon_k}}{\partial \nu} + \epsilon_k q\Big(x, \dfrac{x}{\epsilon_k}\Big) u_{\epsilon_k} = 0, & x \in S_{\epsilon_k}, t > 0, \\ u_{\epsilon_k} = 0, & x \in \partial\Omega, \\ u_{\epsilon_k} = U(x), & x \in \Omega_{\epsilon_k}, t = 0. \end{cases}$$

on the axis $t \in \mathbb{R}$. To get the uniform in ϵ estimate of the solution we use the following Lemmas (see [25; Ch. III, §5] and [26] respectively).

By means of integral identity (4) and Lemma 1.1 we derive the estimate, the sequence $\{u_{\epsilon_k}(x, s)\}$ is bounded in $\mathcal{F}^b$, i.e.,

$$\|u_{\epsilon_k}\|_{\mathcal{F}^b} = \sup_{t\in\mathbb{R}} \|u_{\epsilon_k}(t)\| + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\epsilon_k}(s)\|_1^2 ds\right)^{1/2} +$$
$$+ \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\epsilon_k}(s)\|_{\mathbf{L}_4}^4 ds\right)^{1/4} + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\|\frac{\partial u_{\epsilon_k}}{\partial t}(s)\right\|_{\mathbf{H}^{-r}}^{4/3} ds\right)^{3/4} \le C \ \text{ for all } k \in \mathbb{N}. \tag{25}$$

The constant C is independent of ϵ.

Consequently, there exists a subsequence $\{u_{\epsilon'_k}(x, s)\} \subset \{u_{\epsilon_k}(x, s)\}$, such that $u_{\epsilon_k}(x, s) \to u(s)$ as $k \to \infty$ in Θ^{loc}. Here $u(x, s) \in \mathcal{F}^b$ and $u(s)$ are the solution to (25) with the same constant C. Because of (25) we get $u_{\epsilon_k}(x, s) \rightharpoonup u(x, s)$ $(k \to \infty)$ weakly in $L_2^{loc}(\mathbb{R}; \mathbf{V})$, weakly in $L_4^{loc}(\mathbb{R}; \mathbf{L}_4)$, $*$-weakly in $L_\infty^{loc}(\mathbb{R}_+; \mathbf{H})$ and $\frac{\partial u_{\epsilon_k}(x,s)}{\partial t} \rightharpoonup \frac{\partial u(x,s)}{\partial t}$ $(k \to \infty)$ weakly in $L_{4/3,w}^{loc}(\mathbb{R}; \mathbf{H}^{-r})$. We claim that $u(x, s) \in \overline{\mathcal{K}}$. We have $\|u\|_{\mathcal{F}^b} \le C$. Hence, we have to establish that $u(x, s)$ is a weak solution to (12).

According to the auxiliary problem in the case $\theta = 1$ we have

$$(1+\alpha \mathrm{i}) \int_{-M}^{M} \int_{\Omega_{\epsilon_k}} \nabla u_{\epsilon_k} \nabla \psi \, dx dt + \epsilon_k \int_{-M}^{M} \int_{S_{\epsilon_k}} q\Big(x, \frac{x}{\epsilon_k}\Big) u_{\epsilon_k} \psi d\sigma dt + \int_{-M}^{M} \int_{\Omega_{\epsilon_k}} g(x) \psi dx dt \longrightarrow$$

$$(1+\alpha \mathrm{i}) \int_{-M}^{M} \int_{\Omega} \sum_{i,j=1}^{d} \left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right\rangle \frac{\partial u_0(x,t)}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx dt -$$
$$+ \int_{-M}^{M} \int_{\Omega} Q(x) u_0(x,t) \psi dx dt + \int_{-M}^{M} \int_{\Omega} |\Box \cap \omega| \, g(x) \psi dx dt$$

as $k \to \infty$.

The differentiation is continuous in the space of generalized functions, also $\dfrac{\partial u_\epsilon}{\partial t} \longrightarrow \dfrac{\partial u_0}{\partial t}$ as $\epsilon \to 0+$.

Now, we prove that

$$R\left(x, \frac{x}{\epsilon_k}\right) u_{\epsilon_k}(x,s) \rightharpoonup \bar{R}(x)u(x,s) \tag{26}$$

and

$$\left(1+\beta\left(x,\frac{x}{\epsilon_k}\right)\mathrm{i}\right)|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s) \rightharpoonup \left(1+\bar{\beta}(x)\mathrm{i}\right)|u(x,s)|^2 u(x,s) \tag{27}$$

as $k \to \infty$ weakly in $L^{loc}_{4/3,w}\left(\mathbb{R};\mathbf{L}_{4/3}\right)$.

Fixing an arbitrary number $M>0$, we consider the sequence $\{u_{\epsilon_k}(x,s)\}$ bounded in $L_4\left(-M,M;\mathbf{L}_4\right)$ (see (25)). Hence, the sequence $\{|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s)\}$ is bounded in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$. Because $\{u_{\epsilon_k}(x,s)\}$ is bounded in $L_2(-M,M;\mathbf{V})$ and $\left\{\dfrac{\partial u_{\epsilon_k}(x,s)}{\partial t}\right\}$ is bounded in $L_{4/3}\left(-M,M;\mathbf{H}^{-r}\right)$ we suppose that $u_{\epsilon_k}(x,s)\to u(x,s)$ as $k\to\infty$ strongly in $L_2\left(-M,M;\mathbf{H}\right)$ and hence

$$u_{\epsilon_k}(x,s)\to u(x,s) \text{ a.e. in } (x,s)\in\Omega\times(-M,M).$$

It follows that

$$|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s)\to |u(x,s)|^2 u(x,s) \text{ a.e. in } (x,s)\in\Omega\times(-M,M). \tag{28}$$

We have

$$\begin{aligned}
&\left(1+\beta\left(x,\frac{x}{\epsilon_k}\right)\mathrm{i}\right)|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s) - \left(1+\bar{\beta}(x)\mathrm{i}\right)|u(x,s)|^2 u(x,s) = \\
&\quad = \left(1+\beta\left(x,\frac{x}{\epsilon_k}\right)\mathrm{i}\right)\left(|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s) - |u(x,s)|^2 u(x,s)\right) + \\
&\quad + \left(\left(1+\beta\left(x,\frac{x}{\epsilon_k}\right)\mathrm{i}\right) - \left(1+\bar{\beta}(x)\mathrm{i}\right)\right)|u(x,s)|^2 u(x,s).
\end{aligned} \tag{29}$$

We show that both terms in the right-hand side of (29) tends to zero as $k\to\infty$ weakly in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$.

The sequence $\left(1+\beta\left(x,\frac{x}{\epsilon_k}\right)\mathrm{i}\right)\left(|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s) - |u(x,s)|^2 u(x,s)\right)$ converges to zero as $k\to\infty$ almost everywhere in $(x,s)\in\Omega\times(-M,M)$ (see (28)) and is bounded in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$ (see (2)). Consequently using Lemma 1.3 from [27] we get $\left(1+\beta\left(x,\frac{x}{\epsilon_k}\right)\mathrm{i}\right)\left(|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s) - |u(x,s)|^2 u(x,s)\right)$ $\rightharpoonup 0$ weakly in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$ as $k\to\infty$.

The sequence $\left(\left(1+\beta\left(x,\frac{x}{\epsilon_k}\right)\mathrm{i}\right) - \left(1+\bar{\beta}(x)\mathrm{i}\right)\right)|u(x,s)|^2 u(x,s)$ goes weakly in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$ to zero as $k\to\infty$, since by the assumption $\beta\left(x,\dfrac{x}{\epsilon}\right)\rightharpoonup\bar{\beta}(x)$ *-weakly in $L_{\infty,w}\left(-M,M;\mathbf{L}_\infty\right)$ as $k\to\infty$ (see (3)) and $|u(x,s)|^2 u(x,s)\in L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$.

We have proved (27). The convergence of (26) is proved similarly.

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Локальды периодты кеуектері бар орталарда Гинсбург-Ландау теңдеулерінің аттракторларын орташалау: критикалық жағдай

Жұмыста теңдеуде және шекаралық шарттарында тез тербелмелі мүшелері бар Гинсбург-Ландау теңдеуін тесік облыста қарастырылған. Бұл теңдеудің траекториялық аттракторлары әлсіз мағынада «оғаш мүшесі» (әлеуеті) бар орташаланған Гинсбург-Ландау теңдеуінің траекториялық аттракторларына жуықтайтыны дәлелденеді. Ол үшін В.В. Чепыжовтың және М.И. Вишиктің эволюциялық теңдеулердің траекториялық аттракторлары туралы мақалалары мен монографияларының әдістемесі қолданылған. Сондай–ақ, XX ғасырдың соңында пайда болған орташалау әдістері пайдаланылған. Алдымен асимптотикалық әдістерді асимптотиканы формальды құру үшін қолданылған, содан кейін асимптотикалық қатарлардың негізгі мүшелерін функционалды талдау және интегралды бағалау әдістерін қолдана отырып таңдалған. Сәйкесінше, көмекші әлсіз топологиялы функционалды кеңістікті анықтай отырып, шекті (орташаланған) теңдеуі алынған және осы теңдеудің траекториялық аттракторы бар екені дәлелденген. Содан кейін негізгі теорема тұжырымдалған, оны көмекші леммалардың көмегімен дәлелденген.

Кілт сөздер: аттракторлар, орташалау, Гинсбург-Ландау теңдеулері, сызықтық емес теңдеулер, әлсіз жинақтылық, тесік облыс, «оғаш мүше», кеуекті орта.

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Усреднение аттракторов уравнений Гинзбурга-Ландау в средах с локально периодическими препятствиями: критический случай

Мы рассматриваем уравнение Гинзбурга-Ландау с быстро осциллирующими членами в уравнении и граничных условиях в перфорированной области. Доказываем, что траекторные аттракторы этого уравнения в слабом смысле сходятся к траекторным аттракторам усредненного уравнения Гинзбурга-Ландау со «странным членом» (потенциалом). Для этого используем подход из статей и монографий В.В. Чепыжова и М.И. Вишика о траекторных аттракторах эволюционных уравнений. Также мы применяем методы усреднения, появившиеся в конце XX века. Сначала используем асимптотические методы для формального построения асимптотик, далее выверяем главные члены асимптотических рядов с помощью методов функционального анализа и интегральных оценок. Определяя соответствующие вспомогательные функциональные пространства со слабой топологией, выводим предельное (усредненное) уравнение и доказываем существование траекторного аттрактора для этого уравнения. Затем формулируем основную теорему и доказываем ее с помощью вспомогательных лемм.

Ключевые слова: аттракторы, усреднение, уравнения Гинзбурга-Ландау, нелинейные уравнения, слабая сходимость, перфорированная область, «странный член», пористая среда.

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On the time-optimal control problem for a heat equation

In previous works, we have considered some control problems for parabolic type equations, namely, control problems for parabolic type equations were studied as boundary value problems of the first type, and the weight function was expanded into a Fourier series by sines. In this paper, we consider boundary control problem for a heat equation on the interval. In the part of the bound of the given domain it is given value of a solution and it is required to find a control to get the average value of the solution. By the mathematical-physics methods it is proved that like this control exists and the estimate of a minimal time for achieving the given average temperature over some domain is found.

Keywords: heat equation, minimal time, admissible control, integral equation, initial-boundary value problem.

Introduction

Consider the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad (x,t) \in \Omega = \{(x,t) : 0 < x < l, \quad t > 0\}, \tag{1}$$

with boundary value conditions

$$u_x(0,t) \ = \ -\mu(t), \quad u_x(l,t) \ = \ 0, \quad t > 0, \tag{2}$$

and an initial condition

$$u(x,0) \ = \ 0, \quad 0 \le x \le l. \tag{3}$$

Definition 1. A function $\mu(t)$ is an *admissible control* if this function is piecewise smooth on $t \ge 0$ and satisfies the conditions

$$\mu(0) = 0, \quad |\mu(t)| \le M, \quad \text{where} \quad M = \text{const} > 0.$$

Consider the function $\rho(x) \in W_2^2[0,l]$ satisfying the conditions

$$\rho'(x) \le 0, \quad \rho''(x) \ge 0, \quad \frac{1}{l}\int\limits_0^l \rho(x)\,dx = 1. \tag{4}$$

Let

$$\rho(x) = \sum_{k=1}^{\infty} \rho_k \cos\frac{k\pi x}{l}, \quad x \in (0,l),$$

where

$$\rho_k = \frac{2}{l}\int\limits_0^l \rho(x) \cos\frac{k\pi x}{l}\,dx, \quad k = 1,2,... \tag{5}$$

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Problem H. *Let $\theta > 0$ be a given constant. Problem H consists in looking for the minimal value of $T > 0$ so that for $t > 0$ the solution $u(x,t)$ of problem* (1)–(3) *with a control function $\mu(t)$ exists and for some $T_1 > T$ satisfies the equation*

$$\int_0^l \rho(x)\, u(x,t)\, dx \;=\; \theta, \quad T \le t \le T_1. \tag{6}$$

We recall that the time-optimal control for parabolic type equations was first investigated in [1] and [2]. Recent results concerned with this problem were established in [3–12]. Some boundary control problems for hyperbolic type equations are studied in [13]. The same result as in this article was seen in detail in [5] case. Detailed information on the problems of optimal control for distributed parameter systems is given in [14] and in the monographs [15, 16] and [17]. Close to this work, boundary control problems for the pseudo-parabolic equation were studied in works [18, 19].

Overall numerical optimization and optimal control have been studied in a great number of publications such as [20]. The practical approaches to the optimal control of the heat equation are described in publications such as [21].

Theorem 1. Let

$$0 < \theta < \frac{\rho_1\, l^2\, M}{\pi^2}.$$

Set

$$T_0 = -\frac{l^2}{\pi^2} \ln\left(1 - \frac{\theta\, \pi^2}{\rho_1\, l^2\, M}\right).$$

Then a solution T_{min} of the Problem H exists and the estimate $T_{min} \le T_0$ is valid.

1 *Main integral equation*

Let $T > 0$ and B be a Banach space. Set by $C([0,T] \to B)$ the Banach space of all continuous mappings $u : [0,T] \to B$ with the norm

$$\|u\| = \max_{0 \le t \le T} \|u(t)\|.$$

Now by symbol $\widetilde{W}_2^1(\Omega)$ we denote the subspace of the Sobolev space $W_2^1(\Omega)$ formed by functions trace of which is equal to $\partial\Omega$ zero. Note that since $\widetilde{W}_2^1(\Omega)$ is closed and the sum of a series of functions from $\widetilde{W}_2^1(\Omega)$ converging in metric $W_2^1(\Omega)$ also in $\widetilde{W}_2^1(\Omega)$ (see, [10]).

Definition 2. By the solution of the problem (1) - (3) we mean function $u(x,t)$, expressed the form

$$u(x,t) = \mu(t)\, \frac{(l-x)^2}{2l} - v(x,t),$$

where the function $v(x,t)$ is a generalized solution from $C([0,T] \to \widetilde{W}_2^1(\Omega))$ of the problem

$$v_t(x,t) - v_{xx}(x,t) \;=\; \mu'(t)\, \frac{(l-x)^2}{2l} - \frac{1}{l}\, \mu(t),$$

with initial and boundary conditions

$$v_x(0,t) = v_x(l,t) = 0, \quad v(x,0) = 0, \quad 0 \le x \le l.$$

Consequently, we get (see, [22, 23])

$$v(x,t) \;=\; \frac{l}{6}\,\mu(t) - \frac{1}{l}\int\limits_0^t \mu(s)\,ds + \frac{2\,l}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\frac{k\pi x}{l}}{k^2}\int\limits_0^t e^{-(k\pi/l)^2(t-s)}\,\mu'(s)\,ds.$$

Note that the class $C([0,T]\to \widetilde{W}_2^1(\Omega))$ is a subset of the class $W_2^1(\Omega)$ considered in the monograph [24] in order to define a problem with homogeneous boundary conditions. So, the generalized solution given above is also a generalized solution in the sense of monograph [24].

Proposition 1. Let $\mu\in W_2^1(\mathbb{R}_+)$ and $\mu(0)=0$. Then the function

$$u(x,t) = \frac{1}{l}\int\limits_0^t \left(1 + 2\sum_{k=1}^{\infty} e^{-(k\pi/l)^2(t-s)}\,\cos\frac{k\pi x}{l}\right)\mu(s)\,ds \tag{7}$$

is a solution of problem (1)–(3).

Proof. We write the function $u(x,t)$ again in the form

$$u(x,t) \;=\; \mu(t)\,\frac{(l-x)^2}{2l} -$$

$$-\frac{l}{6}\,\mu(t) + \frac{1}{l}\int\limits_0^t \mu(s)\,ds - \frac{2\,l}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\frac{k\pi x}{l}}{k^2}\int\limits_0^t e^{-(k\pi/l)^2(t-s)}\,\mu'(s)\,ds.$$

Now we show that function $v(x,t)$ belongs to the class $C([0,T]\to \widetilde{W}_2^1(\Omega))$. For this, it is enough to prove that the gradient of this function, taken in $x\in\Omega$, continuously depends on $t\in[0,T]$ in the norm of the space $L_2(\Omega)$. According to Parseval's equality, the norm of this gradient is

$$\|v_x(\cdot,t)\|^2_{L_2(\Omega)} \;=\; \frac{2\,l}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\int\limits_0^t e^{-(k\pi/l)^2(t-s)}\,\mu'(s)\,ds\right)^2 \le$$

$$\le C\,\|\mu'\|^2\sum_{k=1}^{\infty}\frac{1}{k^4} \le C_1\,\|\mu'\|^2.$$

Proposition 1 is proved.

From (7) and condition (6), we can write

$$\theta(t) = \int\limits_0^l \rho(x)\,u(x,t)\,dx =$$

$$= \int\limits_0^t \left(\frac{1}{l}\int\limits_0^l \rho(x)\,dx + \frac{2}{l}\sum_{k=1}^{\infty} e^{-(k\pi/l)^2(t-s)}\int\limits_0^l \rho(x)\cos\frac{k\pi x}{l}dx\right)\mu(s)\,ds.$$

Then according to (4) and (5), we have

$$\theta(t) = \int\limits_0^t \left(1 + \sum_{k=1}^{\infty}\rho_k\,e^{-(k\pi/l)^2(t-s)}\right)\mu(s)\,ds.$$

Set

$$B(t) = 1 + \sum_{k=1}^{\infty} \rho_k \, e^{-(k\pi/l)^2 t}, \quad t > 0. \tag{8}$$

Then we get the main integral equation

$$\int_0^t B(t-s)\mu(s)ds = \theta(t), \quad t > 0.$$

Lemma 1. [6] Let $g(y) \geq 0$ and $g'(y) \leq 0$. Then the inequality holds

$$\int_0^{n\pi} g(y) \sin y dy \geq 0, \quad y \in [0, \infty), \quad n = 1, 2, ...$$

Proposition 2. For the coefficients $\{\rho_k\}_{k \in N}$ defined by (5) the estimate

$$0 \ \leq \ \rho_k \ \leq \ \frac{C}{k^2}, \quad k = 1, 2, ...$$

is valid.

Proof. From (5), we write

$$\rho_k \ = \ \frac{2}{l} \int_0^l \rho(x) \, \cos \frac{k\pi x}{l} \, dx \ = \ \frac{2}{k\pi} \, \rho(x) \, \sin \frac{k\pi x}{l} \Big|_{x=0}^{x=l} \ -$$

$$- \ \frac{2}{k\pi} \int_0^l \rho'(x) \sin \frac{k\pi x}{l} \, dx = -\frac{2}{k\pi} \int_0^l \rho'(x) \sin \frac{k\pi x}{l} \, dx. \tag{9}$$

By conditions (4) and Lemma 1 we obtain $\rho_k \geq 0$. Then, from (9) we can write

$$\rho_k \ = \ -\frac{2}{k\pi} \int_0^l \rho'(x) \sin \frac{k\pi x}{l} \, dx = \frac{2\,l}{k^2\pi^2} \, \rho'(x) \, \cos \frac{k\pi x}{l} \Big|_{x=0}^{x=l} -$$

$$-\frac{2\,l}{k^2\pi^2} \int_0^l \rho''(x) \cos \frac{k\pi x}{l} \, dx = \frac{2\,l}{k^2\,\pi^2} \, [\rho'(l) \, (-1)^k - \rho'(0)] + \frac{o(1)}{k^2},$$

where $\rho'(l) \, (-1)^k - \rho'(0) \geq 0$.

Then we obtain

$$0 \ \leq \ \rho_k \ \leq \ \frac{C}{k^2}.$$

Proposition 2 is proved.

Proposition 3. A function $B(t)$ defined by (8) is continuous on the half-line $t \geq 0$.

Proof. Indeed, from (8) and Proposition 2 we obtain

$$1 \ \leq \ B(t) \ \leq \ 1 + \text{const} \sum_{k=1}^{\infty} \frac{1}{k^2} \, e^{-(k\pi/l)^2 t}.$$

Proposition 3 is proved.

2 Estimate for the Minimal Time

Consider the Volterra integral equation

$$\int_0^t B(t-s)\,\mu(s)\,ds \;=\; \theta, \quad t \geq T,$$

where

$$B(t) \;=\; 1 + \sum_{k=1}^{\infty} \rho_k\, e^{-(k\pi/l)^2\,t}. \tag{10}$$

Proposition 4. For the function defined by Eq. (10) the following estimate

$$B(t) \geq \rho_1 e^{-(\pi/l)^2\,t}$$

is valid.

Proof. Proof of the proposition comes from functional series defined by (10) is non-negative.

Proposition 4 is proved.

We introduce a function as follows

$$Q(t) = \int_0^t B(t-s)ds = \int_0^t B(s)\,ds.$$

It is clear that physical meaning of this function $Q(t)$ equals the average temperature of Ω in case where the heater is acting unit load (see, [3, 10]). We know that $Q(0) = 0$ and $Q'(t) = B(t) > 0$. Set

$$Q^* = \lim_{t\to\infty} Q(t) = \int_0^\infty B(s)ds.$$

Proposition 5. Let $0 < \theta < MQ^*$. In that case there is $T > 0$ and a real measurable function $\mu(t)$ and the equality

$$\int_0^T B(T-s)\mu(s)ds = \theta \tag{11}$$

is valid.

Proof. Obviously, if we set $\mu(t) = M$ then we obtain

$$\int_0^t B(t-s)\mu(s)ds = M\int_0^t B(t-s)ds = MQ(t),$$

and since from (11) there exists $T > 0$ so that $MQ(T) = \theta$.

Proposition 5 is proved.

Remark 1. We know that the value T found in Proposition 5 gives a solution to the problem. Clearly, T is a root of the following equation

$$Q(T) = \frac{\theta}{M}. \tag{12}$$

Proposition 6. Let
$$0 < \theta < \frac{\rho_1\, l^2\, M}{\pi^2}. \tag{13}$$

Then there exists $T > 0$ and
$$T < -\frac{l^2}{\pi^2} \ln\left(1 - \frac{\theta\, \pi^2}{\rho_1\, l^2\, M}\right),$$
and the Eq. (12) is fulfilled.

Proof. Now we use Proposition 4. As result, we can write
$$Q(t) = \int_0^t B(s)\, ds \geq \rho_1 \int_0^t e^{-(\pi/l)^2\, s}\, ds = \rho_1\, l^2\, \frac{1 - e^{-(\pi/l)^2\, t}}{\pi^2}. \tag{14}$$

Consider the equation for the defining of T_0:
$$\rho_1\, l^2\, \frac{1 - e^{-(\pi/l)^2\, T_0}}{\pi^2} = \frac{\theta}{M}. \tag{15}$$

Then we have
$$T_0 = -\frac{l^2}{\pi^2} \ln\left(1 - \frac{\theta\, \pi^2}{\rho_1\, l^2\, M}\right).$$

From (14) and (15), we can write
$$0 < \frac{\theta}{M} \leq Q(T_0).$$

Obviously, there exists T, $0 < T < T_0$, which is a solution of Eq. (12).

Proposition 6 is proved.

Proposition 7. Let $T > 0$ satisfies Eq. (12) and condition (13). Then there exist $T_1 > T$ and the measurable function $\mu(t)$ so that $|\mu(t)| \leq M$ and the equality
$$\int_0^l \rho(x) u(x,t)\, dx = \theta, \quad T \leq t \leq T_1$$
is valid.

Proof. According to the following
$$\int_0^t B(t-s)\mu(s) ds = \theta,$$
it is enough to prove that there exists a solution of the equation
$$\int_0^t B(t-s)\mu(s) ds = f(t), \quad 0 \leq t \leq T_1, \tag{16}$$
where
$$f(t) = \begin{cases} MQ(t), & \text{if } 0 \leq t \leq T, \\ \theta, & \text{if } T < t \leq T_1. \end{cases} \tag{17}$$

Solution (17) is piecewise smooth and, according to Eq. (12), is continuous.

Set

$$\mu(t) = \begin{cases} M, & \text{if } 0 \le t \le T, \\ \mu_1(t), & \text{if } T < t \le T_1, \end{cases} \tag{18}$$

where $\mu_1(t)$ is a solution of the following integral equation

$$\int_0^T B(t-s)M ds + \int_T^t B(t-s)\mu_1(s)ds = \theta, \quad T \le t \le T_1. \tag{19}$$

Then differentiating this equation we obtain

$$B(0)\mu_1(t) + \int_T^t B'(t-s)\mu_1(s)\, ds = M\,[B(t-T) - B(t)]. \tag{20}$$

According to Proposition 2,

$$B(0) = 1 + \sum_{k=1}^{\infty} \rho_k < \infty.$$

We know that the function $B(t)$ is convergence function on given interval. Therefore, equation (20) has a unique solution $\mu_1(t)$ for $t \ge T$, which is continuous function on $t \ge T$. Besides,

$$\mu_1(T) = M\left(1 - \frac{B(T)}{B(0)}\right) < M,$$

and there exists $T_1 > T$ so that

$$|\mu_1(t)| \le M, \quad T \le t \le T_1.$$

We know that this function is the unique solution of equation (19). Hence, function (18) is piecewise continuous and satisfies equation (16). Consequently, this function $\mu(t)$, which has a jump at the point $t = T$, is the required solution.

Proposition 7 is proved.

Proof of Theorem 1 follows from Propositions 6 and 7.

Conclusions

Note that in case where the temperature θ is small enough, the value of T_0 can be replaced by the following one:

$$T_0 \;=\; \frac{\theta}{\rho_1 M}.$$

Hence, in this case the estimate of optimal time given by Theorem 1 is proportional to required temperature θ and inversely proportional to size of the rod l and to the maximum output of heat source M.

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Жылутеңдеуі үшін оңтайлы уақыт мәселесі туралы

Алдыңғы жұмыстарда параболалық типті теңдеулер үшін кейбір басқару есептері қарастырылған. Яғни параболалық типті теңдеулердің басқару есептері бірінші типті шекаралық есептер ретінде зерттеліп, салмақ функциясы синустар бойынша Фурье қатарына кеңейтілді. Мақалада интервалдағы жылу теңдеуі үшін шекті бақылау мәселесі зерттелген. Өріс шекарасының бұл бөлігінде бақылаудың мәні берілген және температураның орташа мәнін алу үшін басқару элементін табу қажет. Математикалық-физикалық әдістерді қолдана отырып, мұндай бақылаудың бар екендігі дәлелденді және белгілі бір аумақта берілген орташа температураға жету үшін ең аз уақыттың бағасы табылды.

Кілт сөздер: жылу теңдеуі, ең аз уақыт, рұқсат етілген бақылау, интегралдық теңдеу, бастапқы-шекаралық есеп.

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О задаче быстродействия для уравнения теплопроводности

В предыдущих работах мы рассмотрели некоторые задачи управления для уравнений параболического типа, а именно: задачи управления для уравнений параболического типа изучались как краевые задачи первого типа, а весовая функция разлагалась в ряд Фурье по синусам. В настоящей работе рассмотрена задача граничного управления для уравнения теплопроводности на отрезке. В части границы данной области задано значение решения и требуется найти управление, чтобы получить среднее значение решения. Методами математической физики доказано, что подобное управление существует, и находится оценка минимального времени достижения заданной средней температуры по некоторой области.

Ключевые слова: уравнение теплопроводности, минимальное время, допустимое управление, интегральные уравнения, начально-краевая задача.

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On a boundary-value problem in a bounded domain for a time-fractional diffusion equation with the Prabhakar fractional derivative

We aim to study a unique solvability of a boundary-value problem for a time-fractional diffusion equation involving the Prabhakar fractional derivative in a Caputo sense in a bounded domain. We use the method of separation of variables and in time-variable, we obtain the Cauchy problem for a fractional differential equation with the Prabhakar derivative. Solution of this Cauchy problem we represent via Mittag-Leffler type function of two variables. Using the new integral representation of this two-variable Mittag-Leffler type function, we obtained the required estimate, which allows us to prove uniform convergence of the infinite series form of the solution for the considered problem.

Keywords: Time-fractional diffusion equation, regularized Prabhakar fractional derivative, Mittag-Leffler type functions.

Introduction and formulation of a problem

Application of Fractional Calculus in mathematical modeling of real-life processes became crucial and appropriate mathematical tools have been developed [1–5].

A number of stochastic models for explaining anomalous diffusion have been introduced in literature (see, for instance, [6–9]).

There are other applications of time-fractional diffusion, for example, in the image denoising model [10].

Let us consider the following time-fractional diffusion equation

$$^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}u(t,x) - u_{xx}(t,x) = f(t,x) \tag{1}$$

in a domain $\Omega = \{(t,x):\ 0 < x < 1,\ 0 < t < T\}$. Here $f(t,x)$ is a given function and

$$^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}y(t) = {}^{P}I_{0t}^{\alpha,m-\beta,-\gamma,\delta}\frac{d^m}{dt^m}y(t)$$

represents regularized Prabhakar fractional derivative [11] and

$$^{P}I_{0t}^{\alpha,\beta,\gamma,\delta}y(t) = \int\limits_0^t (t-\xi)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left[\delta(t-\xi)^{\alpha}\right]y(\xi)d\xi,\ t>0$$

represents Prabhakar fractional integral [12]. We note that above-given definitions are valid for $\alpha,\beta,\gamma,\delta \in \mathbf{C}$ such that $\Re(\alpha) > 0$ and $m-1 < \Re(\beta) < m$, $m \in \mathbf{N}$.

We formulate a boundary-value problem for Eq.(1) in the particular case $(0 < \beta < 1)$ as follows:

Problem: To find a solution of Eq.(1) in Ω, satisfying the following conditions:

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- regularity conditions: $u(t,x) \in C(\overline{\Omega}),\ u(\cdot,x) \in C^1_{-1}(0,T),\ u(t,\cdot) \in C^2(0,1)$;
- initial condition: $u(0,x) = \psi(x),\ 0 \le x \le 1$;
- boundary conditions: $u(t,0) = u(t,1) = 0,\ 0 \le t \le T$.

Here the function $\psi(x)$ is a given function such that $\psi(0) = \psi(1) = 0$ and a class of functions C^m_μ is defined as follows:

Definition 1. [13] We say that $f \in C_\mu[a,b]$, if there is a real number $p > \mu$ $(\mu > -1)$, such that $f(x) = (x-a)^p f_1(x)$ with $f_1 \in C[a,b]$. Similarly, we say that $f \in C^m_\mu[a,b]$, if and only if $f^{(m)} \in C_\mu[a,b]$.

We would like to note related works, where the main objects are PDEs involving the above-mentioned Prabhakar fractional derivative or some generalizations.

The following Cauchy problem for the time-fractional diffusion-wave equation

$$\begin{cases} D^{\sigma}_{\nu,\gamma+\nu,-\lambda,0+} g(x,t) = C g_{xx}(x,t),\ x \in \mathbf{R},\ t > 0, \\ g(x,0^+) = \delta(x), \\ g_t(x,t)|_{t\to 0^+} = 0,\ \sigma \in \mathbf{R},\ \gamma > 0, \nu > 0,\ 0 < \gamma+\nu \le 2 \end{cases}$$

was the subject of investigation in [11]. The authors used the Laplace-Fourier transform to find a solution to this problem in an explicit form. The solution was represented via Prabhakar and Wright's functions.

The explicit solution of the Cauchy problem in $t > 0$, $x \in \mathbf{R}$ has been found for the following time-fractional heat equations [14]:

$$D^{\gamma,\mu,\nu}_{\rho,\omega,0+} u(x,t) = K u_{xx}(x,t)$$

and

$${}^C D^{\gamma,\mu,\nu}_{\rho,\omega,0+} u(x,t) = K u_{xx}(x,t),$$

where

$$D^{\gamma,\mu,\nu}_{\rho,\omega,0+} f(t) = E^{-\gamma,\nu}_{\rho,\nu(1-\mu),\omega,0+} \frac{d}{dt} E^{-\gamma,(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+} f(t),$$

$${}^C D^{\gamma,\mu,\nu}_{\rho,\omega,0+} f(t) = E^{-\gamma}_{\rho,1-\mu,\omega,0+} \frac{d}{dt} f(t),$$

$\mu \in (0,1)$, $\nu \in [0,1]$, $\gamma, \omega \in \mathbf{R}$, $\rho > 0$,

$$E^{\gamma}_{\rho,\mu,\omega,0+} f(t) = \int_0^t (t-y)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(t-y)^\rho] f(y)dy$$

is the Prabhakar fractional integral [12].

The following PDE involving the Prabhakar derivative

$$D^{\gamma}_{\alpha,\beta,\omega,0+} u(x,t) = a(x)u_{xx}(x,t) + b(x)u_x(x,t) + c(x)u(x,t) + d(x,t)$$

was investigated together with the appropriate initial conditions [15]. Using the Sumudu transform, the authors have found an approximate solution to the proposed problem.

Authors in [16] studied the following time-fractional heat conduction equation with a heat absorption term in spherical coordinates in the case of central symmetry [17]:

$${}^C D^{\gamma,\mu}_{\rho,\omega,0+} T(r,t) = a\left(T_{rr}(r,t) + \frac{2}{r} T_r(r,t)\right) - bT(r,t),\ t > 0,\ 0 \le r < R.$$

Imposing initial $T(r,0) = 0$ and boundary $T(R,t) = pt^\beta$ $(\beta > 0)$ conditions and using the Laplace transform, they found exact solutions for this problem.

The distinctive side of the present problem is that we consider the boundary-value problem in a bounded domain and use a method of separation of variables. We will get the solution to the problem in an infinite series form represented by the new Mittag-Leffler type function of two variables. In the next section, we provide the main result (a unique solvability of the problem) and corresponding proof with details.

Main result

We search solution of the problem $u(t,x)$ and the given function $f(t,x)$ as follows

$$u(t,x)=\sum_{n=0}^{\infty}U_n(t)\sin n\pi x, \tag{2}$$

$$f(t,x)=\sum_{n=0}^{\infty}f_n(t)\sin n\pi x, \tag{3}$$

where $U_n(t)$ are unknowns to be found and $f_n(t)$ are the Fourier coefficients of the function $f(t,x)$, given as

$$f_n(t)=2\int_0^1 f(t,x)\sin n\pi x.$$

Substituting (2) and (3) into (1) and considering initial condition, we obtain the following Cauchy problem:

$$\begin{cases} {}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}U_n(t)+(n\pi)^2U_n(t)=f_n(t),\\ U_n(0)=\psi_n, \end{cases}$$

where ψ_n are the Fourier coefficients of the given function $\psi(x)$, which are defined as follows

$$\psi_n=2\int_0^1 \psi(x)\sin n\pi x.$$

Let us first present some statements, required for the further stages. The first statement is devoted to finding an explicit solution to the Cauchy problem for a fractional differential equation with the regularized Prabhakar derivative.

Lemma 1. Let $\alpha,\beta\in\mathbf{R}^+$, $\gamma,\delta,a_0,a_1,...,a_{m-1}\in\mathbf{R}$, $m=[\beta]+1$, $m-1\le\beta<m$. If $f(t)\in C_\mu^m$, then for any real number λ the following Cauchy problem

$$\begin{cases} {}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}y(t)-\lambda y(t)=f(t),\\ y^k(0)=a_k,\;\; k=0,1,...,m-1 \end{cases} \tag{4}$$

has a solution represented by

$$\begin{gathered} y(t)=\sum_{k=0}^{m-1}\tfrac{a_kx^k}{k!}+\sum_{k=0}^{m-1}a_kx^{\beta+k}\Gamma(\gamma)E_2\left(\begin{matrix}\gamma,\gamma,1;1,0\\ \beta+k+1,\beta,\alpha;\gamma,\gamma;1,1\end{matrix}\;\begin{matrix}|\lambda t^\beta\\ |\delta t^\alpha\end{matrix}\right)+\\ +\Gamma(\gamma)\int_0^t(t-z)^{\beta-1}E_2\left(\begin{matrix}\gamma,\gamma,1;1,0\\ \beta,\beta,\alpha;\gamma,\gamma;1,1\end{matrix}\;\begin{matrix}|\lambda(t-z)^\beta\\ |\delta(t-z)^\alpha\end{matrix}\right)f(z)dz. \end{gathered} \tag{5}$$

Here $E_2(\cdot)$ is the Mittag-Leffler type function in two variables represented as

$$
\begin{gathered}
E_2\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2 & |x\\ \delta_1,\alpha_3,\beta_2;\delta_2,\alpha_4;\delta_3,\beta_3 & |y\end{matrix}\right)=\\
=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\gamma_1)_{\alpha_1 i+\beta_1 j}(\gamma_2)_{\alpha_2 i}}{\Gamma(\delta_1+\alpha_3 i+\beta_2 j)}\frac{x^i}{\Gamma(\delta_2+\alpha_4 i)}\frac{y^j}{\Gamma(\delta_3+\beta_3 j)}.
\end{gathered}
\tag{6}
$$

This function for the first time was mentioned in the work [18], but not studied at all.

Proof. In [19], the solution of the Cauchy problem (4) is represented in the following infinite series form:

$$
\begin{gathered}
y(t)=\sum_{k=0}^{m-1}\frac{a_k x^k}{k!}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{m-1}a_k\frac{((1+i)\gamma)_j}{j!}\frac{\lambda^{(i+1)}\delta^j t^{\alpha j+(i+1)\beta+k}}{\Gamma(\alpha j+(i+1)\beta+k+1)}+\\
+\int\limits_0^t\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{((1+i)\gamma)_j}{j!}\frac{\lambda^i\delta^j(t-z)^{\alpha j+(i+1)\beta-1}}{\Gamma(\alpha j+(i+1)\beta)}f_n(z)dz.
\end{gathered}
\tag{7}
$$

The double series in this formulae can be represented by the function defined in (6). Considering the well-known definition of the Pochhammer symbol, namely,

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$$

one can easily deduce (5) from (7) using (6) in the following particular case:

$$\gamma_1=\gamma,\ \alpha_1=\gamma,\ \beta_1=1,\ \gamma_2=1,\ \alpha_2=0,\ \delta_1=\beta+k+1,\ \alpha_3=\beta,$$

$$\beta_2=\alpha,\ \delta_2=\gamma,\ \alpha_4=\gamma,\ \delta_3=1,\ \beta_3=1.$$

The next statements are related to the estimation of the function (6), which is crucial for the proof of the uniform convergence of infinite series. First, we present an integral representation of the function (6) via known functions.

Lemma 2. Let $\Re(\delta_1)>\Re(\gamma_1)>0$. If $\alpha_3=\alpha_1$ and $\beta_2=\beta_1$, then the following integral representation holds true:

$$
\begin{gathered}
E_2\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2 & |x\\ \delta_1,\alpha_3,\beta_2;\delta_2,\alpha_4;\delta_3,\beta_3 & |y\end{matrix}\right)=\\
=\frac{1}{\Gamma(\gamma_1)\Gamma(\delta_1-\gamma_1)}\int\limits_0^1\xi^{\gamma_1-1}(1-\xi)^{\delta_1-\gamma_1-1}E^{\gamma_2,\alpha_2}_{\alpha_4,\delta_2}\left(x\xi^{\alpha_1}\right)E_{\beta_3,\delta_3}\left(y\xi^{\beta_1}\right)d\xi.
\end{gathered}
\tag{8}
$$

Here $E_{\beta_3,\delta_3}(z)$ is two-parameter Mittag-Leffler function and

$$E^{\gamma_2,\alpha_2}_{\alpha_4,\delta_2}(z)=\sum_{m=0}^{\infty}\frac{(\gamma_2)_{\alpha_2 m}z^m}{\Gamma(\alpha_4 m+\delta_2)}.$$

Proof. On the right-hand side of (8) we use the series form of functions $E^{k,p}_{m,n}(z)$ and $E_{m,n}(z)$ and will integrate term-by-term:

$$\frac{1}{\Gamma(\gamma_1)\Gamma(\delta_1-\gamma_1)}\int\limits_0^1\xi^{\gamma_1-1}(1-\xi)^{\delta_1-\gamma_1-1}\sum_{i=0}^{\infty}\frac{(\gamma_2)_{\alpha_2 i}\left(x\xi^{\alpha_1}\right)^i}{\Gamma(\alpha_4 i+\delta_2)}\sum_{j=0}^{\infty}\frac{\left(y\xi^{\beta_1}\right)^j}{\Gamma(\beta_3 j+\delta_3)}d\xi=$$

$$= \frac{1}{\Gamma(\gamma_1)\Gamma(\delta_1-\gamma_1)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\gamma_2)_{\alpha_2 i} x^i}{\Gamma(\alpha_4 i+\delta_2)} \frac{y^j}{\Gamma(\beta_3 j+\delta_3)} \int_0^1 \xi^{\alpha_1 i+\beta_1 j+\gamma_1-1}(1-\xi)^{\delta_1-\gamma_1-1} d\xi.$$

Using the definition of Beta-function and after some simplifications, we deduce the left-hand side of (8).

In particular, if $\gamma=\beta$ and $\alpha=1$, we have

$$E_2\left(\begin{matrix} \gamma,\gamma,1;1,0 & |\lambda x^{\beta} \\ \beta+k+1,\beta,\alpha;\gamma,\gamma;1,1 & |\delta x^{\alpha}\end{matrix}\right)=$$

$$= \frac{1}{\Gamma(\gamma)\Gamma(\beta+k+1-\gamma)} \int_0^1 \xi^{\gamma-1}(1-\xi)^{\beta+k-\gamma} E^{1,0}_{\gamma,\gamma}\left(\lambda x^{\beta}\xi^{\gamma}\right) E_{1,1}\left(\delta x^{\alpha}\xi\right) d\xi.$$

It is known that

$$E^{1,0}_{\gamma,\gamma}(z)=E_{\gamma,\gamma}(z),\ \ E_{1,1}(z)=e^z.$$

Hence, considering the fact that if $\lambda<0$, $\delta\le 0$, then

$$|E_{\gamma,\gamma}(\lambda x^{\beta}\xi^{\gamma})| \le \frac{C_1}{1+|\lambda x^{\beta}\xi^{\gamma}|},\ \ |e^{\delta x^{\alpha}\xi}|\le C_2,\ \ (C_1,\ C_2\in \mathbf{R}^+),$$

one can get the following:

$$\left|E_2\left(\begin{matrix} \gamma,\gamma,1;1,0 & |\lambda x^{\beta} \\ \beta+k+1,\beta,\alpha;\gamma,\gamma;1,1 & |\delta x^{\alpha}\end{matrix}\right)\right| \le$$

$$\le \frac{1}{\Gamma(\gamma)\Gamma(\beta+k+1-\gamma)} \int_0^1 \xi^{\gamma-1}(1-\xi)^{\beta+k-\gamma} \frac{C_1C_2}{1+|\lambda x^{\beta}\xi^{\gamma}|} d\xi \le$$

$$\le \frac{C_1C_2}{\Gamma(\gamma)\Gamma(\beta+k+1-\gamma)} \int_0^1 \xi^{\gamma-1}(1-\xi)^{\beta+k-\gamma} d\xi = \frac{C_1C_2}{(\beta+k-\gamma)\Gamma(\beta+k)} = C,$$

where C is any positive real number.

Based on Lemma 1, we explicitly find $U_n(t)$ as follows

$$U_n(t)=\psi_n\left[1+t^{\beta}\Gamma(\gamma)E_2\left(\begin{matrix} \gamma,\gamma,1;1,0 & |-(n\pi)^2t^{\beta} \\ \beta+1,\beta,\alpha;\gamma,\gamma;1,1 & |\delta t^{\alpha}\end{matrix}\right)\right]+$$
$$+\Gamma(\gamma)\int_0^t (t-z)^{\beta-1} E_2\left(\begin{matrix} \gamma,\gamma,1;1,0 & |-(n\pi)^2(t-z)^{\beta} \\ \beta,\beta,\alpha;\gamma,\gamma;1,1 & |\delta(t-z)^{\alpha}\end{matrix}\right) f_n(z)dz.$$

Since, in our case $\lambda=-(n\pi)^2$ and assuming that $\delta\le 0$, we can easily get when $\gamma=\beta$, $\alpha=1$ the following estimates:

$$|u(t,x)| \le \sum_{n=0}^{\infty}\left[C_1|\psi_n|+C_2|f_n(t)|\right] \le \bar{C}_1\left\|\psi(x)\right\|_2^2+\bar{C}_2\left\|f(t,x)\right\|_2^2.$$

This will be enough for the uniform convergence of the series (2), but for the infinite series corresponding to the function $u_{xx}(t,x)$ we need to impose more conditions to the given functions. Namely,

$$|u_{xx}(t,x)| \le \bar{C}_3\left\|\psi''(x)\right\|_2^2+\bar{C}_4\left\|\frac{\partial^2 f(t,x)}{\partial x^2}\right\|_2^2.$$

The following statement is valid:

Theorem 1. If $\psi(x)\in C^1[0,1], \psi''(x)\in L_2(0,1)$ and $f(\cdot,x)\in C^1_{-1}[0,T]$, $f_x(t,\cdot)\in C[0,1],\ f_{xx}(t,\cdot)\in L_2(0,1)$, then there exists a unique solution of the problem represented as (2).

Conclusion

In the bounded domain, we have considered a boundary problem for a sub-diffusion equation involving regularized Prabhakar fractional order derivative. Presenting the solution of the corresponding Cauchy problem via a two-variable Mittag-Leffler type function and using its new integral representation, we have proved a unique solvability of the formulated boundary problem. We note that the same approach can be done for the fractional wave equation. Moreover, various inverse problems can be studied by applying obtained results.

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Прабхакар бөлшек туындысы бар уақыт-бөлшек диффузия теңдеуі үшін шектелген облыстағы шекаралық есеп бойынша

Зерттеудің мақсаты шектелген облыста Капуто мағынасындағы Прабхакар бөлшек туындысын қамтитын уақыттық-бөлшек диффузиялық теңдеу үшін шекаралық есептің бірегей шешімін зерттеу. Айнымалыларды бөлу әдісі қолданылған және уақыт айнымалысында Прабхакар туындысы бар бөлшек дифференциалдық теңдеу үшін Коши есебі алынған. Осы Коши есебінің шешімі екі айнымалы Миттаг-Леффлер типті функциясы арқылы берілген. Осы екі айнымалы Миттаг-Леффлер типті функцияның жаңа интегралды көрінісін пайдалана отырып, қарастырылып отырған есептің шешімінің шексіз қатар түрінің біркелкі жинақтылығын дәлелдеуге мүмкіндік беретін қажетті баға алынған.

Кілт сөздер: уақыт-бөлшек диффузия теңдеуі, регуляризацияланған Прабхакар бөлшек туындысы, Миттаг-Леффлер типті функциялар.

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Об одной краевой задаче в ограниченной области для уравнения диффузии дробного времени с дробной производной Прабхакара

Нашей целью является изучение однозначной разрешимости краевой задачи для уравнения диффузии с дробным временем, включающего дробную производную Прабхакара по Капуто в ограниченной области. Воспользуемся методом разделения переменных и в переменной по времени получим задачу Коши для уравнения дробного дифференциала с производной Прабхакара. Решение этой задачи Коши представим через функцию типа Миттаг-Леффлера от двух переменных. Используя новое интегральное представление этой функции типа Миттаг-Леффлера с двумя переменными, мы получили требуемую оценку, которая позволяет доказать равномерную сходимость решения в виде бесконечного ряда для рассматриваемой задачи.

Ключевые слова: уравнение диффузии с дробным временем, регуляризованная дробная производная Прабхакара, функции типа Миттаг-Леффлера.

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Geometry of strongly minimal hybrids of fragments of theoretical sets

In this article, strongly minimal geometries of fragment hybrids are considered. In this article, a new concept was introduced as a family of Jonsson definable subsets of the semantic model of the Jonsson theory T, denoted by $JDef(C_T)$. The classes of the Robinson spectrum and the geometry of hybrids of central types of a fixed $RSp(A)$ are considered. Using the construction of a central type for theories from the Robinson spectrum, we formulate and prove results for hybrids of Jonsson theories. A criterion for the uncountable categoricity of a hereditary hybrid of Jonsson theories is proved in the language of central types. The results obtained can be useful for continuing research on various Jonsson theories, in particular, for hybrids of Jonsson theories.

Keywords: Jonsson theory, semantic model, fragment, hybrid of Jonsson theories, Jonsson set, theoretical set, central type, pregeometry, Robinson theory, strongly minimal type.

Introduction

The current state of development of the conceptual and technical apparatus of model theory can be described without exaggeration as a set of syntactic and semantic concepts related to the consideration of most of the complete theories of first-order languages, on the other hand, due to the meager arsenal of the capabilities of the technical apparatus, the subject of study of incomplete theories. A special class of, generally speaking, incomplete theories is singled out in the study of Jonsson theories.

By virtue of the definition of the Jonsson theory, such a theory is, generally speaking, not complete. In the class of its models, there can be infinite and finite models, and isomorphic embeddings will also be used. Thus, we see that the transformation of certain results from complete theories to Jonsson's is complicated due to the different technical arsenal of the above theories. The reason for this problem is the replacement of elementary embeddings by isomorphic embeddings and the incompleteness of Jonsson theories. Thus, the universally homogeneous models that define the semantic model of Jonsson's theory are, generally speaking, not always saturated.

This fact clearly describes an example of group theory. The class of all groups has a Jonsson theory, a semantic model that is not saturated. In this regard, this class does not have a model companion, which makes it very difficult to apply the well-established technique of model companions to this class when studying the property of the center of this class.

Thus, the study of Jonsson theories is an important task.

In the works of the following authors, such as B. Jonsson [1], M. Morley and R. Vaught [2], A. Robinson [3], G. Cherlin [4], T.G. Mustafin [5], A.R. Yeshkeyev [6–8] gave a complete description of Jonsson theories and their companions. We would like to acknowledge the following authors with their publications, who played a great role in the study of this issue for Jonsson theories [9–12].

The notion of central type, which arises during signature enrichments, is one of the new concepts in Jonsson theories [13]. Thus, within the framework of the model theory of Jonsson theories, new relationships arise between classical concepts from the theory of models for complete theories.

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Also noteworthy is the emergence of a new method for constructing a Jonsson theory from two Jonsson theories. This is obtained using the concept of a hybrid of Jonsson theories, which was first defined in [14]. Various examples of algebraic objects and their constructions can be associated with this concept. In subsequent papers [15,16], results were obtained related to hybrids of Jonsson theories, which play an important role in model theory and in universal algebra.

The paper [17] considered Jonsson theories and their many syntactic and semantic properties of the first order in language enrichments that preserve the properties of Jonsson. Such Jonsson theories are called hereditary [12].

One of the classical methods of model theory is the method of interpreting a well-studied theory into a less-studied theory. Following the ideology of this method, a new method for studying Jonsson theories was defined, namely: using the concepts of syntactic and semantic similarity of Jonsson theories, new results were obtained in the framework of the classification of Jonsson theories.

1 Local properties of the geometry of strongly minimal sets

This article discusses the basic concepts of local properties of the geometry of strongly minimal sets on theoretical subsets of some existentially closed model. By studying the combinatorial properties of the pregeometry given on Jonsson sets, we have obtained results on relatively strongly minimal Jonsson sets. Minimal structures, pregeometries and geometries of minimal structures were defined. And also, for Jonsson theories, the concepts of dimension, independence and basis in Jonsson strongly minimal structures were considered.

First, let's define a hybrid of the first type and the second type.

Definition 1 ([14], p. 102). 1) Let T_1 and T_2 be some Jonsson theories of the countable language L of the same signature σ; C_1 and C_2 are their semantic models, respectively. In the case of common signature of Jonsson theories T_1, T_2, let us call a hybrid of Jonsson theories T_1 and T_2 of the first type the following theory $Th_{\forall\exists}(C_1 \diamond C_2)$ if that theory is Jonsson in the language of signature σ and denote it by $H(T_1, T_2)$, where the operation $\diamond \in \{\times, +, \oplus\}$ and $C_1 \diamond C_2 \in Mod\,\sigma$. Here $\times$ means cartesian product, $+$ means sum and $\oplus$ means direct sum. Herewith, the algebraic construction $(C_1 \diamond C_2)$ is called a semantic hybrid of the theories T_1, T_2.

2) If T_1 and T_2 are Jonsson theories of different signatures σ_1 and σ_2, then $H(T_1, T_2) = Th_{\forall\exists}(C_1 \diamond C_2)$ will be called a hybrid of the second type, if that theory is Jonsson in the language of signature $\sigma = \sigma_1 \cup \sigma_2$ where $C_1 \diamond C_2 \in Mod\,\sigma$.

Obviously that 1) is the particular case of 2).

Since the hybrid of two Jonsson theories is a Jonsson theory, in the case when this theory is perfect, we will say for brevity – a perfect hybrid of two Jonsson theories. As the center of the hybrid $H(T_1, T_2)$, we will mean the center of the Jonsson theory $Th_{\forall\exists}(C_1 \diamond C_2)$ and denote it by $H^*(T_1, T_2)$.

Let us define the Morley rank for existentially definable subsets of the semantic model.

We want to assign to each Jonsson subset of X of the semantic model an ordinal (or perhaps -1 or ∞) - its Morley rank, denoted by r_M.

Let T be a fragment of some Jonsson set, and it is a perfect Jonsson theory, C be a semantic model, Z be a definable set of C.

Definition 2. [6] $r_M(Z) \geq 0$, if and only if, Z is not empty; $r_M(Z) \geq \lambda$, if and only if, $r_M(Z) \geq \alpha$ for all $\alpha < \lambda$ (λ is limit ordinal); $r_M(Z) \geq \alpha + 1$, if and only if, in Z there is an infinite family Z_i of pairwise disjoint $\exists$-definable subsets such that $r_M(Z_i) \geq \alpha$ for all i.

Then the Morley rank of the set Z is $r_M(Z) = sup\{\alpha \mid r_M(Z)\} \geq \alpha$,

with the convention that $r_M(Z) = -1$ and $r_M(Z) = \infty$, if $r_M(Z) \geq \alpha$ for all α (in last case, we say that Z has no rank).

Definition 3. [6] The Morley degree $r_D(Z)$ of a Jonsson set Z, having Morley rank α, is the maximum length d of its decomposition $Z = Z_1 \cup ... \cup Z_n$ into disjoint existentially definable subsets of rank α.

If the rank is 0, then the degree of an existentially defined subset is the number of its elements. The Morley degree is also undefined if an existentially definable subset has no rank. In our case, we study Jonsson minimal sets. Note that a strongly minimal set is a set of rank 1 and degree 1.

Consider the closure operator, which is defined by an algebraic closure in the model-theoretic sense. A strongly minimal set that is equipped with the above closure operator is a pregeometry. A model of a strongly minimal theory is defined up to isomorphism by its dimension as a pregeometry. Completely categorical theories are controlled by a strongly minimal set; this remark is used in the proof of Morley's theorem. Boris Zilber considered the origin of pregeometry on vector spaces and algebraically closed fields.

Consider an example of an algebraic closure in Jonsson's strongly minimal theories, which is an existentially complete perfect Jonsson's theory in a countable language L.

If K is an algebraically closed field and $Z \subseteq K$, then $acl(Z)$ is an algebraically closed subfield generated by Z.

Consider the properties of the Jonsson algebraic closure that are true for any subset S of the semantic model of the Jonsson theory T.

Let M be some existentially closed submodel of the semantic model for a fixed theory in the language L, and $S \subseteq M$ be a Jonsson strongly minimal set.

Let $S \subseteq M^n$ be an infinite ∇-definable set, where $\nabla \subseteq L$ is the set of existential formulas of a given language.

Definition 4. [6] We say that S is Jonsson minimal in M if for any ∇-definable $Y \subseteq S$ either Y is finite or $S \backslash Y$ is finite.

Definition 5. [6] We say that S and φ are Jonsson strongly minimal if φ is Jonsson minimal in any existentially closed extension N from M.

Definition 6. [6] We say that a theory T is Jonsson strongly minimal if the formula $v = v$ is Jonsson strongly minimal (that is, if $M \in ModE_T$, then M is Jonsson strongly minimal)

Consider acl_S is an algebraic closure restricted to S.

For $Z \subseteq S$ let $acl_S(Z) = \{b \in S :\ b$ be a Jonsson algebraic over $Z\}$.

In our case, the properties of the [18] algebraic closure are true for the Jonsson algebraic closure of any subset S of the semantic model of the theory.

Lemma 1. [6]

1 $acl(acl(Z)) = acl(Z) \supseteq Z$.

2 If $Z \subseteq B$, then $acl(Z) \subseteq acl(B)$.

3 If $z \in acl(Z)$, then $z \in acl(Z_0)$ for some finite $Z_0 \subseteq Z$.

Lemma 2 (Exchange). [6] Suppose that $S \subset M$ is Jonsson strongly minimal, $Z \subseteq S$ and $z, b \in S$. If $z \in acl(Z \cup \{b\}) \backslash acl(Z)$, then $b \in acl(Z \cup \{z\})$.

The concept of linear independence in vector spaces is one of the important concepts of algebra, and the concept of independence generalizes linear independence in vector spaces and in algebraically closed fields. In turn, algebraic independence is defined in the Jonsson strongly minimal set we are considering.

Let $M \in Mod\, E_T$, and S be a Jonsson strongly minimal set in M.

Definition 7. [6] We will call $Z \subseteq S$ is Jonsson independently if $a \notin acl(Z \backslash \{z\})$ for all $z \in Z$. If $C \subset S$, we say that Z is Jonsson independent over S if $z \notin acl(C \cup (Z \backslash \{z\}))$ for all $z \in Z$.

Definition 8. [6] We will call Z is a Jonsson basis for $Y \subseteq S$ if $Z \subseteq Y$ is Jonsson independent and $acl(Z) = acl(Y)$.

Note that any maximal Jonsson independent subset of Y is a Jonsson basis for Y.

Definition 9. [6] If $Y \subseteq S$, then the Jonsson dimension of the set Y is the cardinality of the Jonsson basis for Y.

Let $J \dim Y$ denote the Jonsson dimension of Y.

If S is uncountable, then $J - \dim(S) =| S |$, since the language is countable and $acl(A)$ is countable for any countable $Z \subseteq D$.

A J-pregeometry (X, cl) is a subset X of the semantic model of some fixed Jonsson theory with operator $cl : P(X) \to P(X)$ on the set of subsets X and if the following conditions are satisfied:

1) if $A \subseteq X$, then $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$;
2) if $A \subseteq B \subseteq X$, then $cl(A) \subseteq cl(B)$;
3) (exchange) $A \subseteq X$, $a, b \in X$ and $a \in cl(A \cup \{b\})$, then $a \in cl(A)$, $b \in cl(A \cup \{a\})$;
4) (finite character) If $A \subseteq X$ and $a \in cl(A)$, then there is a finite $A_0 \subseteq A$, such that $a \in cl(A_0)$.

We say that $A \subseteq X$ is *closed*, if $cl(A) = A$.

Since D is a Jonsson strongly minimal set, the Jonsson pregeometry is defined as follows $cl(A) = acl(A) \cap D$ for $A \subseteq D$ (by Theorem 12 from [6] and Lemma 1).

Definition 10. [6] If (X, cl) is a Jonsson pregeometry, we will call A is Jonsson independent if $a \notin cl(A \setminus \{a\})$ for all $a \in A$, and B is a J-basis for Y if $B \subseteq Y$ is J-independent and $Y \subseteq acl(B)$.

If $A \subseteq X$, we also consider the localization $cl_A(B) = cl(A \cup B)$.

If (X, cl) is a J-predgeometry, then we will call $Y \subseteq X$ is *Jonsson independent over* A, if Y is Jonsson independent in (X, cl_A).

$\dim(Y/A)$ is the dimension of Y in the localization (X, cl_A), $\dim(Y/A)$ is called the dimension of Y over A.

Definition 11. [6] We will call a J-pregeometry (X, cl) is a J-geometry if $cl(\varnothing) = \varnothing$ and $cl(\{x\}) = \{x\}$ for any $x \in X$.

For further study, we denote some important properties of pregeometry.

Definition 12. [6] Let (X, cl) be a J-predgeometry. We will call (X, cl) is trivial if $cl(A) = \bigcup_{a \in A} cl(\{a\})$ for any $A \subseteq X$. We will call (X, cl) is modular if, for any finite-dimensional closed sets $A, B \subseteq X$, holds $Jdim(A \cup B) = Jdim(A) + Jdim(B) - Jdim(A \cap B)$.

(X, cl) is locally modular if (X, cl_a) is modular for some $a \in X$.

Theorem 1. [6] For a J-predgeometry (X, cl) the following are equivalent:

1 (X, cl) is modular;
2 if $A \subseteq X$ is closed and non-empty, $b \in X$, $x \in cl(A, b)$, then $\exists a \in A$, such that $x \in cl(a, b)$;
3 if $A, B \subseteq X$ are closed and non-empty, $x \in cl(A, B)$, then $\exists a \in A$ and $\exists b \in B$, such that $x \in cl(a, b)$.

Proof. Similarly to the proof of Lemma 8.1.13 from [19].

2 Model-theoretical properties of the Robinson spectrum

This section is devoted to the study of the model-theoretic properties of the Robinson spectrum of an arbitrary model of an arbitrary signature. The study of ω-categorical universals by specialists in model theory and universal algebra is well known ([20], § 5 of the appendix). In this section, we will deal with Robinson theories. The Robinson theory is a special case of the Jonsson theory, namely the Jonsson universal. To study the above theory, an algorithm for working with central types of a fixed spectrum was used. The elements of this spectrum are Jonsson universals. The result will be a central type enriched with additional constants and a unary predicate. Thus, we have obtained a criterion for the uncountable categoricity of the Robinson spectrum class in the language of central types.

In [21], Hrushovski E. defined Robinson theories of a theory as universal theories admitting a quantifier separation. In the study of Robinson theories, quantifier-free types occupy the main place. In our case, we are using central types.

Considering the structure of Jonsson sets, one can easily see that they have a very simple structure in the sense of the Morley rank, i.e. elements from the set-theoretic difference (hole) of the closure and the set have rank 0, i.e., they are all algebraic.

Another advantageous point for us in considering the concept of a Jonsson set is that we can obtain some existentially closed models by closing the Jonsson set.

This fact is necessary for studying the Morley rank for an arbitrary fragment of the set under consideration. Saturation for complete theories is a condition for the correctness of the definition of the Morley rank. Imperfect Jonsson theories require saturation with existential types in the semantic model. In the case of Jonsson sets, when studying elements from the set-theoretic difference, one can consider $\forall\exists$-consequences that are true in the closures of the Jonsson set. Based on this, we can conclude that the considered set of sentences will be Jonsson theory. In this section, strongly minimal Jonsson sets have been considered and described. The basic concepts associated with the notion of strong minimality for complete theories have been carried over to Jonsson theories. In particular, the notion of strong minimality is considered for fixed formula subsets of the semantic model of the Jonsson theory. In this case, the semantic model must be saturated in its power, i.e. the theory under consideration must be perfect. As is known, Jonsson's theory has a semantic model C of sufficiently large power. The semantic models of the perfect Jonsson theory are uniquely determined by their power. In our case, we will consider Jonsson subsets.

Definition 13. [6] A Jonsson theory T is called Robinson theory if it is universally axiomatizable.

Let T be a Robinson theory, A be an arbitrary model of signature σ. The Robinson spectrum of the model A is the set:

$$RSp(A) = \{T|\ T \text{ is Robinson theory in the language of signature } \sigma \text{ and } A \in Mod(T)\}.$$

Consider $RSp(A)/_{\bowtie}$ the factor set of the Robinson spectrum of the model A with respect to $\bowtie$.

If T is an arbitrary Robinson theory in the language of signature σ, then $E_{[T]} = \bigcup\limits_{\Delta\in[T]} E_{\Delta}$ is the class of all existentially closed models of class $[T] \in RSp(A)/_{\bowtie}$.

Let A be an arbitrary model of signature σ. Let $|RSp(A)/_{\bowtie}| = |K|$, K be some index set. We say that the class $[T] \in RSp(A)/_{\bowtie}$ is a $\aleph$-categorical if any theory $\Delta \in [T]$ is a $\aleph$-categorical and, respectively, the class $RSp(A)/_{\bowtie}$ will be called a $\aleph$-categorical if for each $j \in K$ the class $[T]_j$ is a $\aleph$-categorical.

Definition 14. [9] The set X is said to be Jonsson in the theory T if it satisfies the following properties:

1) X is the Σ-definable subset of C;

2) $dcl(X)$ is a support of some existentially closed submodel C.

Definition 15. [9] Let T be some Jonsson theory, C is the semantic model of the theory T, $X \subseteq C$. A set X is called theoretical set, if

1) X is Jonsson set, and let $\varphi(x)$ be the formula that defines the set X;

2) $\varphi(x) = \exists y\phi(x, y)$ and let θ be the universal closure of the formula $\varphi(x)$, i.e. θ is the sentence $\forall x\exists y\phi(x, y)$ defines some Jonsson theory.

Definition 16. [23] We say that all $\forall\exists$-consequences of an arbitrary theory create a Jonsson fragment of this theory, if the deductive closure of these $\forall\exists$-consequences is a Jonsson theory.

Definition 17. [23] We say that all $\forall$-consequences of an arbitrary theory create a Robinson fragment of this theory, if the deductive closure of these $\forall$-consequences is a Robinson theory.

We say that a model $M \in E_T$ is *Jonsson minimal* if for any definable $X \subseteq M$ either X is finite or $M \backslash X$ is finite. We say that a theory T Jonsson strongly minimal, if every model $M \in E_T$ is minimal. A non-algebraic type containing a Jonsson strongly minimal formula is called Jonsson strongly minimal.

Theorem 2 ([22], p. 298). Let T be universal theory, complete for existential sentences, having a countably algebraically universal model. Then T has an algebraically prime model, which is (Σ, Δ)-atomic.

Definition 18. A relational structure $C_T =< C, (X_i)_{i\in I} >$ consists of a (non empty) set C, and a family $(X_i)_{i\in I}$ of subsets of $\bigcup_{n\geq 1} C_T^n$, that is, for each i, X_i is a subset of $C_T^{n_i}$ for some $n_i \geq 1$. We add the extra condition that the diagonal of C_T^2 is one of the X_i's.

Each X_i is called an basic subset of C_T.

Definition 19. Let $C_T =< C, (X_i)_{i\in I} >$ be a semantic model of the Jonsson theory in pure predicate language. We define the family of Jonsson definable subsets of the semantic model of the Jonsson theory T, denoted by $JDef(C_T)$. $Def(C_T)$ is the smallest family of subsets of $\bigcup_{n\geq 1} C_T^n$ with the following properties:

- For every $i \in I$, $B_i \in JDef(C_T)$
- $JDef(C_T)$ is closed under finite boolean combinations, i.e. if $M, N \subseteq C_T^n$, M, N are the Jonsson sets. $M, N \in JDef(C_T)$, then $M \cup N \in JDef(C_T), M \cap N \in JDef(C_T)$ and $C_T^n \setminus M \in JDef(C_T)$.
- $JDef(C_T)$ is closed under cartesian product, i.e. if $M, N \in JDef(C_T)$, $M \times N \in JDef(C_T)$.
- $JDef(C_T)$ is closed under projection, i.e. if $M \subset C_T^{n+m}$, $N \in JDef(C_T)$, if $\pi_n(M)$ is the projection of M on C_T^n, $\pi_n(N) \in JDef(C_T)$.
- $JDef(C_T)$ is closed under specialization, i.e. if $M \in Def(C_T)$, $M \subseteq C_T^{n+k}$ and if $\bar{m} \in C_T^n$ then
$$M(\bar{m}) = \{\bar{b} \in C_T^k; (\bar{m}, \bar{b}) \in M\} \in JDef(C_T).$$
- $JDef(C_T)$ is closed under permutation of coordinates, i.e. if $M \in JDef(C_T), M \subseteq C_T^n$, if σ is any permutation of $\{1, ..., n\}$,
$$\sigma(M) = \{(a_{\sigma(1)}, ..., a_{\sigma(n)};\ (a_1, ..., a_n) \in M\} \in JDef(C_T).$$

$cl : (P(C_T)) \to P(C_T)$. $P(C_T) = \{A \subseteq C_T \mid A \in JDef(C_T)\}$. When T perfect Jonsson theory, then T^* is the model complete, $\varphi(x) \in T$ follow that $\exists\psi(x),\ \ \psi(x) \in \Sigma_1$ such that $T^* \vdash \forall x(\varphi(x) \leftrightarrow \psi(x))$.

Definition 20. [17] An enrichment $\tilde{T}$ is called admissible if the ∇-type (this means that the ∇subset of the language L_σ and any formula from this type belongs to ∇) in this enrichment is definable within the framework of $\tilde{T}_\Gamma$-stability, where Γ is the enrichment of the signature σ.

Definition 21. [17] A Robinson theory T is called hereditary if, in any of its admissible enrichments, any extension is a Robinson theory. The class $[T] \in RSp(A)/_{\bowtie}$ will be called hereditary if each theory $\Delta \in [T]$ is hereditary.

Definition 22. [17] A model A is called the Δ-good algebraically prime model of the theory T if A is a countable model of the theory T and for each model B of the theory T, each $n \in \omega$ and all $a_0, \ldots, a_{n-1} \in A$, $b_0, \ldots, b_{n-1} \in B$ if $(A, a_0, \ldots, a_{n-1}) \equiv_\Delta (B, b_0, \ldots, b_{n-1})$, then for each $a_n \in A$ there is some $b_n \in B$, such that $(A, a_0, \ldots, a_n) \equiv_\Delta (B, b_0, \ldots, b_n)$.

Definition 23. [23] Let T_1 and T_2 are Jonsson theory. We will say, that T_1 and T_2 are J-syntactically similar, if there is bijection $f : E(T_1) \to E(T_2)$ such that:

1) restriction f to $E_n(T_1)$ is isomorphism lattice $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;
2) $f(\exists v_{n+1}\varphi) = \exists\varphi n + 1 f(\varphi), \varphi \in E_{n+1}(T), n < \omega$;
3) $f(v_1 = v_2) = (v_1 = v_2)$

Consider the general scheme for obtaining the central type for an arbitrary Robinson theory.

Let C_T be a semantic model of the theory T, $A \subseteq C_T$. Let $\sigma_\Gamma = \sigma \cup \Gamma$, where $\Gamma = \{P\} \cup \{c\}$. Let $\bar{T} = Th_\forall(C_T, a)_{a\in P(C_T)} \cup Th_\forall(E_T) \cup \{P(c)\} \cup \{"P \subseteq"\}$, where $P(C_T)$ is an existentially closed submodel of C_T, $\{"P \subseteq"\}$ is an infinite set of sentences, demonstrating that P is an existentially closed submodel of signature σ_Γ. This means that P is a solution to the equation $P(C_T) = M \subseteq E_T$ of signature σ_Γ. Due to the heredity of T, the theory $\bar{T}$ is also a Jonsson theory. Consider all completions of the theory $\bar{T}$ of signature σ_Γ. Since the theory $\bar{T}$ is Jonsson's, it has its own center, denoted by $\bar{T}^*$. The above mentioned center is one of the completions of the $\bar{T}$ theory. When the signature σ_Γ is restricted to $\sigma \cup P$, the constant c does not belong to this signature. Therefore, we can replace this constant with the variable x. After that, this theory will be a complete 1-type for the variable x.

Let X_1, X_2 be the strongly minimal theoretical sets. $Fr(X_1) = T_1, Fr(X_2) = T_2$ are the Robinson fragments. $H(T_1, T_2) = Th_\forall(C_{T_1} \times C_{T_2})$, $cl(X_1) = M_1, cl(X_2) = M_2; M_1, M_2 \in E_T$. $Fr(X_1) = \Delta_1, Fr(X_2) = \Delta_2$. Δ_1, Δ_2 are Jonsson syntactical similar. By virtue of Jonsson syntactical similarity of this fact $Th_\forall(M_1) = T_1, Th_\forall(M_2) = T_2$ also Jonsson syntactical similar. T_1, T_2 are the Jonsson strongly minimal theories. Then since $\bar{T}_1$ is a Jonsson theory, it has its own center, let us denote it by $\bar{T}_1^*$, this center is one of the above completions of the theory $\bar{T}_1$. Accordingly $\bar{T}_2$ is a Jonsson theory, it has its own center, let us denote it by $\bar{T}_2^*$, this center is one of the above completions of the theory $\bar{T}_2$. In the theorem we consider the hybrid $H(\bar{T}_1, \bar{T}_2)$ of the Jonsson theries T_1, T_2.

R_1 is every existential formula $\varphi(\bar{x})$ consistent with T is implied by some Δ formula $\theta(\bar{x})$ consistent with T.

Theorem 3. Let $[T]$ be class from $RSp(A)/_{\bowtie}$, complete for existential sentences, admitting R_1. Let $T_1, T_2 \in [T]$. Then the following conditions are equivalent:

1 $H(T_1, T_2)$ has an algebraically prime model;
2 $H(T_1, T_2)$ has $(\exists, \Delta)$-atomic model;
3 $H(T_1, T_2)$ has $(\Delta, \exists)$-atomic model;
4 $H(T_1, T_2)$ has a Δ-good algebraically prime model;
5 $H(T_1, T_2)$ has a single algebraically prime model.

Proof. Let $T_1, T_2 \in [T]$ satisfies the conditions of Theorem 3, then by virtue of the theorem 4.1 ([22], p.309) the $H(T_1, T_2)$ also satisfies the conditions this theorem.

Theorem 4. Let $[T]$ be hereditary class from $RSp(A)/_{\bowtie}$, $T_1, T_2 \in [T]$, then the following conditions are equivalent:

1 any countable model from $E_{H(\bar{T}_1,\bar{T}_2)}$ has an algebraically prime model extension in $E_{H(\bar{T}_1,\bar{T}_2)}$;
2 $P^c_{H(\bar{T}_1,\bar{T}_2)}$ is the strongly minimal type, where $P^c_{H(\bar{T}_1,\bar{T}_2)}$ is the central type of $H(\bar{T}_1, \bar{T}_2)$.

Proof. (1)⇒(2). For convenience of the proof, we denote $H(\bar{T}_1, \bar{T}_2) = \mathbb{T}$. Consider a semantic model $C_\mathbb{T}$ of the class $[T]$. The $C_\mathbb{T}$ model is ω-universal by virtue of the definitions of κ-universality and κ-homogeneity. In our case, the power is uncountable. Therefore, consider a countable elementary submodel D of the $C_\mathbb{T}$ model. The elementary submodel D is existentially closed since $C_\mathbb{T}$ is existentially closed by virtue of (Lemma [23], p. 162). Therefore, the elementary submodel D is countably algebraically universal. We apply the 2 theorem, according to which every theory $\overline{\Delta} \in \mathbb{T}$ has an algebraically simple model A_0. We define $A_{\delta+1}$ by induction, which is an algebraically simple extension of the A_δ model and $A_\lambda = \bigcup\{A_\delta \mid \delta < \lambda\}$. Then let $\overline{A} = \bigcup\{A_\delta | \delta < \omega_1\}$. Suppose $B \models \Delta$ and $cardB = \omega_1$. Let us show that

$B \approx A$, for this we decompose B into a chain $\{B_\delta | \delta < \omega_1\}$ of countable models. Such a decomposition is possible due to the fact that the $\overline{\Delta}$ theory is Jonsson. We define the function $g : \omega_1 \to \omega_1$ and the chain $\{f_\delta : A_{g\delta} \to B_\delta \mid 0 < \delta < \omega_1\}$ of isomorphisms by the formula induction on δ:

1) $g0 = 0$ and $f_0 : A_0 \to B_0$;

2) $g\lambda = \bigcup\{g\delta | \delta < \lambda\}$ and $f_\lambda = \bigcup\{f_\delta | \delta < \lambda\}$;

3) $f_{\delta+1}$ is equal to the union of the chain $\{f_\delta^\gamma | \gamma \leq \rho\}$, which is determined by induction on γ;

4) $f_{\delta+1}^0 = f_\delta$, $f_{\delta+1}^\lambda = \bigcup\{|f_{\delta+1}^\gamma | \gamma < \lambda\}$;

5) suppose that $f_1^\gamma : A_{g\delta+\gamma} \to B_{\delta+1}$. If $f_{\delta+1}^\gamma$ is a mapping onto, then $\rho = \gamma$. Otherwise, by virtue of the algebraic primeness of $A_{g\delta+\gamma+1}$, we can extend $f_{\delta+1}^\gamma$ to $f_{\delta+1}^{\gamma+1} : A_{g\delta+\gamma+1} \to B_{\delta+1}$;

6) $g(\delta+1) = g\delta + \rho$.

By virtue of $f = \bigcup\{| f_\delta \mid \delta < \omega_1\}$ $\overline{A}$ is mapped isomorphically to B. Now let's apply the theorem 3. B is an arbitrary model of the Δ theory. $\overline{A}$ is the only algebraic prime and existentially closed model. By virtue of the condition and construction, it follows that E_Δ for each $\overline{\Delta} \in \mathbb{T}$ has a unique model in uncountable cardinality. This condition means that the semantic model $C_\mathbb{T}$ is saturated, i.e. the class $\mathbb{T}$ will be perfect. Thus $Mod\mathbb{T}^* = E_\mathbb{T}$. Therefore, the theory $\mathbb{T}^*$ is ω_1-categorical. $\mathbb{T}^*$ has a strongly minimal formula according to the Lachlan-Baldwin theorem. Since we are dealing with a central type, we get a non-principal type that contains the Jonsson strongly minimal formula. This implies that the type is Jonsson strongly minimal.

(2) $\Rightarrow$ (1). Due to the fact that $P_\mathbb{T}^c$ is a strongly minimal type, when passing to the signature $\sigma_\Gamma = \sigma \cup \Gamma$, the type becomes $\mathbb{T}^*$ theory. As mentioned above, the theory is the center of the class $\mathbb{T}$, hence it is complete. Let us show that $\mathbb{T}^*$ is ω_1-categorical. By inductance, for any models $A, B \in Mod\mathbb{T}^*$ there are models $A', B' \in E_\mathbb{T}$ and isomorphic embeddings $f : A \to A'$, $g : B \to B'$. Suppose $|A'| = |B'| = \omega_1$. If $A \ncong B$, then $A' \ncong B'$. Therefore, there exists $\varphi(x) \in B(At)$ such that $A' \models \varphi(x)$ and $B' \models \neg\varphi(x)$. Since in our case $\mathbb{T}$ is an inherited class, then $\mathbb{T} \in RSp(A)/_{\bowtie}$. Due to the universal axiomatizability of this class and the fact that $A' \in Mod(\mathbb{T}^*)$ as an existentially closed model is isomorphically embedded into the semantic model C of the class $\mathbb{T}$. Since $\mathbb{T}^* = Th(C)$ is complete, $\mathbb{T}^* \vdash \exists x\varphi(x)$ follows. Since A' and B' are Jonsson minimal, either $\varphi(A')$ is finite or $A' \backslash \varphi(A')$ is finite. Let $\varphi(A')$ be finite, then there exists a $\forall\exists$-proposition ψ which shows that $\varphi(A')$ is finite and $\mathbb{T}^* \vdash \forall\exists(\varphi \& \psi)$ hence $B' \models \psi(x)$ but $B' \models \psi(x) \& \neg\varphi(x)$, but at the same time, since $A', B' \in E_\mathbb{T}$, $A' \equiv_{\forall\exists} B'$, then we got a contradiction with strongly minimality.

If the definable complement of the formula is finite in the model A' under consideration, then the proof is carried out in a similar way. Thus $\mathbb{T}$ is ω_1-categorical.

By virtue of Morley's uncountable categoricity theorem, $\mathbb{T}^*$ is ω_1-categorical, and hence this theory is perfect. Then, by virtue of the Jonsson theory completeness criterion $\mathbb{T}^*$ is a model complete theory and $Mod\mathbb{T}^* = E_\Delta$ for every $\overline{\Delta} \in \mathbb{T}$, i.e. $Mod\mathbb{T}^* = E_{(T)}$. If $\mathbb{T}^*$ is model complete, then any isomorphic embedding is elementary. Since $\mathbb{T}^*$ is a complete theory, by virtue of Morley's theorem we obtain what is required.

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Қатты минималды гибридтерінің фрагменттерінің теоретикалық жиындарының геометриясы

Мақала фрагмент гибридтерінің қатты минималды геометрияларын зерттеуге арналған. Авторлар $JDef(C_T)$ деп белгіленген «T йонсондық теорияның семантикалық моделінің йонсондық анықталған ішкі жиындарының үйірі» деген жаңа тұжырымдама енгізген. Робинсон спектрінің кластары және бекітілген $RSp(A)$ централдық типтерінің гибридтерінің геометриясы қарастырылған. Робинсон спектріндегі теориялар үшін централдық типті құруды пайдалана отырып, йонсондық теориялардың гибридтері үшін нәтижелерді тұжырымдалған және дәлелденген. Йонсондық теориялардың мұралық гибридінің саналымсыз категориялық критерийі централдық типтер тілінде дәлелденген. Алынған нәтижелер йонсондық әртүрлі теориялар бойынша, атап айтқанда, йонсондық теориялардың гибридтері бойынша зерттеулерді жалғастыру үшін пайдалы болуы мүмкін.

Кілт сөздер: йонсондық теория, семантикалық модель, фрагмент, йонсондық теориялардың гибриді, йонсондық жиын, теоретикалық жиын, централдық тип, алғашқы геометрия, робинсондық теория, қатты минималды тип.

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Геометрия сильно минимальных гибридов фрагментов теоретических множеств

Статья посвящена изучению сильно минимальных геометрий гибридов фрагментов. Авторами было введено новое понятие «семейство йонсоновских определимых подмножеств семантической модели йонсоновской теории T», обозначаемое через $JDef(C_T)$. Рассмотрены классы робинсоновского спектра и геометрия гибридов центральных типов фиксированного $RSp(A)$. С помощью построения центрального типа для теорий из робинсоновского спектра формулируются и доказываются результаты для гибридов йонсоновских теорий, в частности, критерий несчетной категоричности наследственного гибрида йонсоновских теорий на языке центральных типов. Полученные результаты могут быть полезны для продолжения исследований различных йонсоновских теорий, в частности, для гибридов йонсоновских теорий.

Ключевые слова: йонсоновская теория, семантическая модель, фрагмент, гибрид йонсоновских теорий, йонсоновское множество, теоретическое множество, центральный тип, предгеометрия, робинсоновская теория, сильно минимальный тип.

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Development of the fuzzy sets theory: weak operations and extension principles

The paper considers the problems that arise when using the theory of fuzzy sets to solve applied problems. Unlike stochastic methods, which are based on statistical data, fuzzy set theory methods make sense to apply when statistical data are not available. In these cases, algorithms should be based on membership functions formed by experts who are specialists in this field of knowledge. Ideally, complete information about membership functions is required, but this is an impractical procedure. More often than not, even the most experienced expert can determine only their carriers or separate sets of the α-cuts for unknown fuzzy parameters of the system. Building complete membership functions of unknown fuzzy parameters on this basis is risky and unreliable. Therefore, the paper proposes an extension of the fuzzy sets theory axiomatics in order to introduce non-traditional (less demanding on the completeness of data on membership functions) extension principles and operations on fuzzy sets. The so-called α-weak operations on fuzzy sets are proposed, which are based on the use of separate sets of the α-cuts. It is also shown that all classical theorems of Cantor sets theory apply in the extended axiomatic theory. New extension principles of generalization have been introduced, which allow solving problems in conditions of significant uncertainty of information.

Keywords: Cantor set, fuzzy set, function of belonging, set of α-cut, core of fuzzy set, α-weak operation.

Introduction

It is well known that the concept of a fuzzy set, proposed by L. Zadeh in 1965 [1], immediately arouse great interest among mathematicians and scientists of other fields and stimulated the appearance of a large number of works in this direction. Just two years later, Gauguin extended this concept to L-fuzzy sets, and further introduced the interval fuzzy line, regular fuzzy numbers and fuzzy metric spaces, fuzzy topological spaces, fuzzy relations and mappings, concepts and theorems of fuzzy algebra [2–11]. All these works with slight variations are based on the well-known maximin extension principle (MMPG) Zadeh [1], which fully satisfied the researchers. The mathematical apparatus of fuzzy set theory (FST) began to be widely used both in physics [12,13] and in applied disciplines [14–18]. At the same time, there are quite a few applied problems for which the use of the maximin extension principle prevents their solution. The fact is that the application of MMPG requires complete information about the membership functions of fuzzy defined parameters of the task, and this, unfortunately, is often the almost impossible procedure. In these cases, even the most experienced expert can determine only their cores or α-cuts for the unknown fuzzy parameters of the system. Building complete membership functions of unknown fuzzy parameters on this basis is risky and unreliable.

Thus, it seems appropriate to expand the axiomatics of the fuzzy sets theory in order to introduce non-traditional (less demanding on the completeness of data on membership functions) extension principles and operations on fuzzy sets. In works [19, 20], an unconventional class of so-called α-weak operations on fuzzy sets was proposed for the first time, further, introducing new concepts, we will follow these works.

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Statement of the problem

All problems with uncertain parameters, which should be solved using fuzzy set theory methods, can be divided into two classes:

1. Problems with non-numerical input parameters.

In these problems, each of the non-numerical parameters corresponds to a certain logical variable (term), to which the expert assigns a membership function (performs fuzzification), then certain procedures are carried out with the assigned membership functions, and the defuzzification procedure is applied to the new membership functions obtained as a result. As a rule, the quality of these calculations significantly depends on the knowledge of experts in the subject of research and on the adequacy of fuzzification and defuzzification procedures.

2. Problems with non-numerical input parameters.

As a rule, it is advisable to solve such problems using the methods of probability theory, but for this the researcher must have a sufficient amount of reliable statistical data. If these data are not available, or their number is very small, then it makes sense to apply the methods of fuzzy set theory. In this case, the uncertain parameters are given by vague numbers, the membership functions of which are formed by experts who are specialists in this field of knowledge.

The main problem of these methods is that even the most experienced expert can determine only their cores or α-cuts for unknown fuzzy parameters of the system. Building complete membership functions of unknown fuzzy parameters on this basis is risky and unreliable.

Therefore, the task of expanding the axiomatics of the fuzzy sets theory in order to introduce non-traditional (less demanding on the completeness of data on membership functions) extension principles and operations on fuzzy sets is actual. For this, the authors propose to introduce the so-called α-weak operations on fuzzy sets, which are based on the use of α-cuts.

Research results

Let's consider the basics of weak operations axiomatics. The α-cut set of the fuzzy set $\tilde{A}$ defined on the universum X is the usual Cantor set of elements $x \in X$, for which the condition $\mu_{\tilde{A}}(x) \geq \alpha$ is fulfilled, where $\alpha \geq (0, 1]$. The limiting case of the α-cut set is the so-called core (or, otherwise, the 0-cut) of the fuzzy set $\tilde{A}$, which is also a Cantor set of elements $x \in X$ for which the condition $\mu_{\tilde{A}}(x) > 0$ is fulfilled.

It is known that every operation on classical Cantor sets can be matched with many similar operations on fuzzy sets. There is only one mandatory condition that each of these operations must meet - they must reduce to the corresponding classical operation in the case of degeneracy of fuzzy sets to classical Cantor sets.

Obviously, that weak operations on fuzzy sets must have the same properties as the analogical ones on classical Cantor sets, that is the same theorems must be fair for them as for classical sets. Let's consider it on the example of the relation of loose inclusion. L. Zadeh defined this relation as: fuzzy set $\tilde{A}$, which is defined on the universum X, if and only if includes fuzzy set $\tilde{B}$, defined on this universum, when for all elements $x \in X$ the membership function $\mu_{\tilde{A}}(x)$ is more or equal to the membership function $\mu_{\tilde{B}}$

$$\tilde{A} \supseteq \tilde{B} \Leftrightarrow \forall x \geq X \left(\mu_{\tilde{A}}(x) \geq \mu_{\tilde{B}}(x)\right). \tag{1}$$

From the fuzzy theory point of view, the membership function of the classical Cantor set A in X looks like $\mu_A : X \to \{0, 1\}$, and for the set A we can write

$$A = \{(x, \mu_A(x)) \,|\, \forall x \in X (x \in A \Leftrightarrow \mu_A(x) = 1)\}.$$

The definition of relation of inclusion for classical sets A and B, expressed through their membership function is formulated as: classical set A, defined on the universum X, if and only if includes classical set B, defined at the same universum, when for all elements $x \in X$, if $\mu_B(x) = 1$, then and $\mu_A(x) = 1$, that is

$$\tilde{A} \supseteq \tilde{B} \Leftrightarrow \forall x \in X(\mu_B(x) = 1 \Rightarrow \mu_A(x) = 1). \tag{2}$$

The definition, which lessens the demands to the membership functions $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ in comparison with (1), doesn't demand the condition $\mu_{\tilde{A}}(x) \geq \mu_{\tilde{B}}(x)$ to be carried out, and is based on the sets of α-cuts of fuzzy set (which are the commom Cantor sets) and is suggested being called loose α-weak inclusion (is marked $\overset{\alpha}{\supseteq}$) and analogically can be formulated as (2): fuzzy set $\tilde{A}$, that defined on the universum X, α-weakly includes fuzzy set $\tilde{B}$, defined on the same universum, if and only if when for all elements $x \in X$, if $\mu_{\tilde{B}}(x) \geq \alpha$, then and $\mu_{\tilde{A}}(x) \geq \alpha$, or

$$\tilde{A} \overset{\alpha}{\supseteq} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{B}}(x) \geq \alpha \Rightarrow \mu_{\tilde{A}}(x) \geq \alpha\right).$$

In boundary case, the relation which is based on the cores of fuzzy sets $\tilde{A}, \tilde{B}$ is offered to call just loose weak inclusion or loose 0-weak inclusion (is marked $\overset{0}{\supseteq}$). Its definition can be formulated as: fuzzy set $\tilde{A}$, defined on the universum X, if and only if 0-weakly includes fuzzy set $\tilde{B}$, defined on the same universum, when for all elements $x \in X$, if $\mu_{\tilde{B}} > 0$, then and $\mu_{\tilde{A}} > 0$, or

$$\tilde{A} \overset{0}{\supseteq} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{B}}(x) > 0 \Rightarrow \mu_{\tilde{A}}(x) > 0\right).$$

Let's introduce the definition of the α-weak supplement operation. The traditional supplement of the fuzzy set $\tilde{A}$ in X is the accepted fuzzy set $\bar{\tilde{A}}$ is X, for which the following condition is carried out

$$\forall x \in X \left(\mu_{\bar{\tilde{A}}}(x) = 1 - \mu_{\tilde{A}}(x)\right).$$

For classical Cantor sets, the supplement of set A is considered to be the set $\overline{A}$, that is

$$\forall x \in X \left(\mu_A(x) = 1 \Leftrightarrow \mu_{\overline{A}}(x) = 0\right). \tag{3}$$

Analogically to (3) the definition of operation of α-weak supplement is offered to formulate as: fuzzy set $\overset{\alpha}{\bar{\tilde{A}}}$ in X is α-weak supplement of fuzzy set $\tilde{A}$ in X if and only if, when for all elements $x \in X$, if $\mu_{\tilde{A}}(x) \geq \alpha$, then $\mu_{\overset{\alpha}{\bar{\tilde{A}}}}(x) < \alpha$, and vice versa, that is

$$\forall x \in X \left(\mu_{\tilde{A}}(x) \geq \alpha \Leftrightarrow \mu_{\overset{\alpha}{\bar{\tilde{A}}}}(x) < \alpha\right). \tag{4}$$

It follows from (4) that

$$\forall x \in X \left(\mu_{\tilde{A}}(x) < \alpha \Leftrightarrow \mu_{\overset{\alpha}{\bar{\tilde{A}}}}(x) \geq \alpha\right).$$

Analogically to the definition (4) for the operation of weak supplement (or 0-weak supplement) we can write: fuzzy set $\overset{0}{\bar{\tilde{A}}}$ in X is a weak supplement of fuzzy set $\tilde{A}$ in X if and only if, when for all the elements $x \in X$, if $\mu_{\tilde{A}}(x) > 0$, then $\mu_{\bar{\tilde{A}}^{0}}(x) = 0$, and vice versa, that is

$$\forall x \in X \left(\mu_{\tilde{A}}(x) > 0 \Leftrightarrow \mu_{\bar{\tilde{A}}^{0}}(x) = 0 \right). \tag{5}$$

It follows from (5) that

$$\forall x \in X \left(\mu_{\tilde{A}}(x) = 0 \Leftrightarrow \mu_{\bar{\tilde{A}}^{0}}(x) > 0 \right).$$

The definition for the relation of α-weak equation between fuzzy sets $\tilde{A}, \tilde{B}$ in X is formulated as: fuzzy set $\tilde{A}$, defined on the universum X, α-weakly equal to fuzzy set $\tilde{B}$, defined on this universum, if and only if, when for all the elements $x \in X$, if $\mu_{\tilde{A}} > 0$, then and $\mu_{\bar{\tilde{A}}^{\alpha}}(x) = 0$, and vice versa, that is

$$\tilde{A} \overset{\alpha}{=} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \right).$$

For a weak equation (0-weak equation) we can write

$$\tilde{A} \overset{0}{=} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{B}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x) > 0 \right).$$

Let's consider the definition for other main relations between fuzzy sets and operations on them. It is suggested that α-weak combination of fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X is the fuzzy set $\tilde{C} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cup} \tilde{B}$ in X, if and only if, when for all elements $x \in X$, if $\mu_{\tilde{C}}(x) \geq \alpha$ then $\mu_{\tilde{A}}(x) \geq \alpha$ or $\mu_{\tilde{B}}(x) \geq \alpha$, and vice versa, that is

$$\tilde{C} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cup} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{C}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\tilde{B}}(x) \geq \alpha \right).$$

Analogically, weak (0-weak) association of fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X is the fuzzy set $\tilde{C} \overset{0}{=} \tilde{A} \overset{0}{\cup} \tilde{B}$ in X if and only if, when for all elements $x \in X$, if $\mu_{\tilde{C}}(x) > 0$ then $\mu_{\tilde{A}}(x) > 0$ or $\mu_{\tilde{B}}(x) > 0$, and vice versa, that is

$$\tilde{C} \overset{0}{=} \tilde{A} \overset{0}{\cup} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{C}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x) > 0 \vee \mu_{\tilde{B}}(x) > 0 \right).$$

At last, α-weak crossing of fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X is the fuzzy set $\tilde{C} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} \tilde{B}$ in X if and only if, when for all elements $x \in X$, if $\mu_{\tilde{C}}(x) \geq \alpha$, then $\mu_{\tilde{A}}(x) \geq \alpha$ and $\mu_{\tilde{B}}(x) \geq \alpha$, and vice versa, that is

$$\tilde{C} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{C}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\tilde{B}}(x) \geq \alpha \right).$$

Analogically, a weak (0-weak) crossing of fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X is the fuzzy set $\tilde{C} \overset{0}{=} \tilde{A} \overset{0}{\cap} \tilde{B}$ in X if and only if $\mu_{\tilde{C}}(x) > 0$, when for all elements $x \in X$, if then $\mu_{\tilde{A}}(x) > 0$ and $\mu_{\tilde{B}}(x) > 0$, and vice versa, that is

$$\tilde{C} \overset{0}{=} \tilde{A} \overset{0}{\cap} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{C}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x) > 0 \wedge \mu_{\tilde{B}}(x) > 0 \right).$$

The definition of the more complex operation of the Descartes multiplication of fuzzy sets is suggested as follows: α-weak Descartes multiplication of the fuzzy sets $\tilde{A}_i$ in X is the fuzzy set $\tilde{A} \overset{\alpha}{=} \tilde{A}_1 \overset{\alpha}{\times} \tilde{A}_2 \overset{\alpha}{\times} \ldots \cdots \overset{\alpha}{\times} \tilde{A}_n$ in $X = X_1 \times X_2 \times \ldots \ldots X_n$ if and only if, when for all elements $x = (x_1, x_2, \ldots, x_n) \in X$, if $\mu_{\tilde{A}}(x) \geq \alpha$, then simultaneously $\mu_{\tilde{A}_1}(x) \geq \alpha$, $\mu_{\tilde{A}_2}(x) \geq \alpha$, $\ldots, \mu_{\tilde{A}_n}(x) \geq \alpha$ and vice versa, that is

$$\tilde{A} \overset{\alpha}{=} \tilde{A}_1 \overset{\alpha}{\times} \tilde{A}_2 \overset{\alpha}{\times} \ldots \cdots \overset{\alpha}{\times} \tilde{A}_n \Leftrightarrow$$

$$\Leftrightarrow x = (x_1, x_2, \ldots, x_n) \in X \left(\mu_{\tilde{A}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}_1}(x) \geq \alpha \wedge \mu_{\tilde{A}_2}(x) \geq \alpha \wedge \cdots \wedge \mu_{\tilde{A}_n}(x) \geq \alpha\right).$$

Accordingly, weak (0-weak) Descartes multiplication of fuzzy sets $\tilde{A}_i$ in X is the fuzzy set $\tilde{A} \overset{0}{=} \tilde{A}_1 \overset{0}{\times} \tilde{A}_2 \overset{0}{\times} \ldots \cdots \overset{0}{\times} \tilde{A}_n$ in $X = X_1 \times X_2 \times \ldots\ldots X_n$ if and only if, when for all elements $x = (x_1, x_2, \ldots, x_n) \in X$, if $\mu_{\tilde{A}}(x) > 0$, then simultaneously $\mu_{\tilde{A}_1}(x) > 0$, $\mu_{\tilde{A}_2}(x) > 0$, $\ldots, \mu_{\tilde{A}_n}(x) > 0$, and vice versa, that is

$$\tilde{A} \overset{0}{=} \tilde{A}_1 \overset{0}{\times} \tilde{A}_2 \overset{0}{\times} \ldots \cdots \overset{0}{\times} \tilde{A}_n \Leftrightarrow$$

$$\Leftrightarrow x = (x_1, x_2, \ldots, x_n) \in X \left(\mu_{\tilde{A}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}_1}(x) > 0 \wedge \mu_{\tilde{A}_2}(x) > 0 \wedge \cdots \wedge \mu_{\tilde{A}_n}(x) > 0\right).$$

If we analyze all the above definitions of α-weak operations, we can come to the conclusion that the results of α-weak operations are ambiguous. Unlike traditional operations on fuzzy sets, the result of any α-weak operation is not a specific fuzzy set, but a set of fuzzy sets, each of which satisfies given conditions. This ambiguity makes it possible to operate with fuzzy sets, the membership functions of which are not completely specified or are specified imprecisely. Such functions are most often obtained with the help of expert procedures.

It is obvious that α-weak operations on fuzzy sets should have the same properties as similar operations on classical Cantor sets, that is, the same theorems as for classical sets should be valid for them. Let's formulate and prove analogical theorems for α-weak operations.

Theorems of idempotency.

Theorem 1. Operation of α-weak association is idempotent, that is

$$\tilde{A} \overset{\alpha}{\cup} \tilde{A} \overset{\alpha}{=} \tilde{A}.$$

Proof. Let's consider the fuzzy set $\tilde{C} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cup} \tilde{A}$ in X. According to the definition of the operation of α-weak association for an arbitrary element $x \in X$, we can write $\mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\tilde{A}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha$. Since the logical operation is idempotent, that is $\vee$, then for an arbitrary element $x \in X$, it will be fair $\mu_{\tilde{C}}(x) \geq \alpha \vee \mu_{\tilde{A}}(x) \geq \alpha$, what had to be proved.

It follows from the theorem 1, that the operation of weak association of fuzzy sets is also idempotent, that is

$$\tilde{A} \overset{0}{\cup} \tilde{A} \overset{0}{=} \tilde{A}.$$

By means of analogical considerations we can prove that the operations of α-weak and weak crossing are idempotent as well, that is

$$\tilde{A} \overset{\alpha}{\cap} \tilde{A} \overset{\alpha}{=} \tilde{A}.$$

$$\tilde{A} \overset{0}{\cap} \tilde{A} \overset{0}{=} \tilde{A}.$$

Theorems of distributivenes.

Theorem 2. Operations of α-weak crossing of fuzzy sets is distributive, that is

$$\tilde{A} \overset{\alpha}{\cap} \left(\tilde{B} \overset{\alpha}{\cup} \tilde{C}\right) \overset{\alpha}{=} \left(\tilde{A} \overset{\alpha}{\cap} \tilde{B}\right) \overset{\alpha}{\cup} \left(\tilde{A} \overset{\alpha}{\cap} \tilde{C}\right).$$

Proof. Let's consider $\tilde{C1} \overset{\alpha}{=} \tilde{B} \overset{\alpha}{\cup} \tilde{C}$, $\tilde{D1} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} \tilde{C1}$, $\tilde{C2} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} \tilde{B}$, $\tilde{C3} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} \tilde{C}$, $\tilde{D2} \overset{\alpha}{=} \tilde{C2} \overset{\alpha}{\cup} \tilde{C3}$. According to the definitions of the α-weak association and crossing operations for an arbitrary element $x \in X$ we can write

$$\mu_{\tilde{C1}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{B}}(x) \geq \alpha \vee \mu_{\tilde{C}}(x) \geq \alpha, \tag{6}$$

$$\mu_{\tilde{D1}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\tilde{C1}}(x) \geq \alpha, \tag{7}$$

$$\mu_{\tilde{C2}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\tilde{B}}(x) \geq \alpha, \tag{8}$$

$$\mu_{\tilde{C3}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\tilde{C}}(x) \geq \alpha, \tag{9}$$

$$\mu_{\tilde{D2}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{C2}}(x) \geq \alpha \vee \mu_{\tilde{C3}}(x) \geq \alpha. \tag{10}$$

Having done the substitution of the equivalent expressions for the logical variables $\mu_{\tilde{C1}}(x) \geq \alpha$, $\mu_{\tilde{C2}}(x) \geq \alpha$ and $\mu_{\tilde{C3}}(x) \geq \alpha$ from logical equations (6, 8, 9) into logical equations (7, 10) we obtain

$$\mu_{\tilde{D1}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \left(\mu_{\tilde{B}}(x) \geq \alpha \vee \mu_{\tilde{C}}(x) \geq \alpha\right),$$

$$\mu_{\tilde{D2}}(x) \geq \alpha \Leftrightarrow \left(\mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\tilde{B}}(x) \geq \alpha\right) \vee \left(\mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\tilde{C}}(x) \geq \alpha\right).$$

Since logical operation $\wedge$ is distributive, that for an arbitrary element $x \in X$ we can claim, that $\mu_{\tilde{D1}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{D2}}(x) \geq \alpha$, what had to be proved.

The operation of weak crossing of fuzzy sets is also distributive, that is

$$\tilde{A} \overset{0}{\cap} \left(\tilde{B} \overset{0}{\cup} \tilde{C}\right) \overset{0}{=} \left(\tilde{A} \overset{0}{\cap} \tilde{B}\right) \overset{0}{\cup} \left(\tilde{A} \overset{0}{\cap} \tilde{C}\right).$$

By means of analogical considerations we can prove that operations of α-weak and weak association are also distributive, that is

$$\tilde{A} \overset{\alpha}{\cup} \left(\tilde{B} \overset{\alpha}{\cap} \tilde{C}\right) \overset{\alpha}{=} \left(\tilde{A} \overset{\alpha}{\cup} \tilde{B}\right) \overset{\alpha}{\cap} \left(\tilde{A} \overset{\alpha}{\cup} \tilde{C}\right),$$

$$\tilde{A} \overset{0}{\cup} \left(\tilde{B} \overset{0}{\cap} \tilde{C}\right) \overset{0}{=} \left(\tilde{A} \overset{0}{\cup} \tilde{B}\right) \overset{0}{\cap} \left(\tilde{A} \overset{0}{\cup} \tilde{C}\right).$$

Theorems of involution.

Theorem 3: For any fuzzy set $\tilde{A}$ in X, the α-weak complement of its α-weak complement is α-weakly equal to the fuzzy set $\tilde{A}$, that is

$$\overset{\overset{\alpha}{\alpha}}{\tilde{A}} \overset{\alpha}{=} \tilde{A}.$$

Proof. Let's consider fuzzy sets $\tilde{B} \overset{\alpha}{=} \overset{\alpha}{\tilde{A}}$ and $\tilde{C} \overset{\alpha}{=} \overset{\alpha}{\tilde{B}}$ in X. According to the definition of α-weak complement, for the arbitrary element $x \in X$ we can write $\mu_{\tilde{B}}(x) < \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha$ and $\mu_{\tilde{C}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{B}}(x) < \alpha$. So, for an arbitrary element $x \in X$ the equivalency $\mu_{\tilde{C}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha$ will be fair, what had to be proved.

It follows from Theorem 3, that for any of fuzzy sets $\tilde{A}$ in X, the weak complement of its weak complement is weakly equal to the fuzzy set $\tilde{A}$, that is

$$\overset{\overset{0}{0}}{\tilde{A}} \overset{0}{=} \tilde{A}.$$

Theorems de Morgan.

Theorem 4. α-weak complement of the α-weak association of the fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X are α-weakly equals to α-weak crossing of α-weak complement of these fuzzy sets, that is

$$\overset{\overset{\alpha}{\alpha}}{\left(\tilde{A} \cup \tilde{B}\right)} \overset{\alpha}{=} \overset{\alpha}{\tilde{A}} \overset{\alpha}{\cap} \overset{\alpha}{\tilde{B}}.$$

Proof. Let's consider fuzzy sets $\tilde{C}1 \stackrel{\alpha}{=} \tilde{A} \overset{\alpha}{\cup} \tilde{B}$, and $\tilde{C}2 \stackrel{\alpha}{=} \overset{\underline{\alpha}}{\tilde{A}} \overset{\alpha}{\cap} \overset{\underline{\alpha}}{\tilde{B}}$ and $\tilde{C}3 \stackrel{\alpha}{=} \overset{\underline{\alpha}}{\tilde{C}}1$ in X. According to the definitions of the corresponding operations, for the arbitrary element $x \in X$ we can write

$$\mu_{\tilde{C}1}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\tilde{B}}(x) \geq \alpha, \tag{11}$$

$$\mu_{\tilde{C}2}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) < \alpha \wedge \mu_{\tilde{B}}(x) < \alpha, \tag{12}$$

$$\mu_{\tilde{C}3}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{C}1}(x) < \alpha. \tag{13}$$

Taking into consideration that $\mu_{\tilde{C}1}(x) \geq \alpha \Leftrightarrow \neg\mu_{\tilde{C}1}(x) \geq \alpha$, let's do the substitution of the equivalent expression for the logical variable $\mu_{\tilde{C}1}(x) \geq \alpha$ from logical equation (11) into logical equation (13), and as a result we'll obtain

$$\mu_{\tilde{C}3}(x) \geq \alpha \Leftrightarrow \neg\left(\mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\tilde{B}}(x) \geq \alpha\right). \tag{14}$$

Since $\mu_{\tilde{A}}(x) < \alpha \Leftrightarrow \neg\left(\mu_{\tilde{A}}(x) \geq \alpha\right)$ and $\mu_{\tilde{B}}(x) < \alpha \Leftrightarrow \neg\left(\mu_{\tilde{B}}(x) \geq \alpha\right)$, the expression (12) we can write as

$$\mu_{\tilde{C}2}(x) \geq \alpha \Leftrightarrow \neg\left(\mu_{\tilde{A}}(x) \geq \alpha\right) \wedge \neg\left(\mu_{\tilde{B}}(x) \geq \alpha\right). \tag{15}$$

As it follows from the similar logical de Morgan's law

$$\neg\left(\mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\tilde{B}}(x) \geq \alpha\right) \Leftrightarrow \neg\left(\mu_{\tilde{A}}(x) \geq \alpha\right) \wedge \neg\left(\mu_{\tilde{B}}(x) \geq \alpha\right),$$

and the expressions (14) and (15) we can write $\mu_{\tilde{C}3}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{C}2}(x) \geq \alpha$, what had to be proved.

It follows from the theorem 4 that the weak complement of the weak association of fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X weakly equals to the weak crossing of the weak complement of these fuzzy sets, that is

$$\overset{\frac{0}{0}}{\left(\tilde{A} \cup \tilde{B}\right)} \stackrel{0}{=} \overset{\underline{0}}{\tilde{A}} \overset{0}{\cap} \overset{\underline{0}}{\tilde{B}}.$$

By means of similar considerations we can prove the fairness of the second de Morgan' theorem for α-weak and weak operations, namely

$$\overset{\frac{\alpha}{\alpha}}{\left(\tilde{A} \cap \tilde{B}\right)} \stackrel{\alpha}{=} \overset{\underline{\alpha}}{\tilde{A}} \overset{\alpha}{\cup} \overset{\underline{\alpha}}{\tilde{B}},$$

$$\overset{\frac{0}{0}}{\left(\tilde{A} \cap \tilde{B}\right)} \stackrel{0}{=} \overset{\underline{0}}{\tilde{A}} \overset{0}{\cup} \overset{\underline{0}}{\tilde{B}}.$$

Besides above mentioned theorems, in classical theory of sets there are also theorems characterizing the operations between fuzzy sets and universum or empty set. Let's check the reality of the similar theorem for α-weak operations' class.

Theorem 5. α-weak association of the fuzzy set $\tilde{A}$ in X and the empty set $\oslash$ α-weakly equals to the fuzzy set $\tilde{A}$ in X, that is

$$\tilde{A} \overset{\alpha}{\cup} \oslash \stackrel{\alpha}{=} \tilde{A}.$$

Proof. Let's consider fuzzy set $\tilde{B} \stackrel{\alpha}{=} \tilde{A} \overset{\alpha}{\cup} \oslash$ in X. According to the definition of α-weak association operation, for the arbitrary element $x \in X$ we can write $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\oslash}(x) \geq \alpha$. Since the definition of an empty set $\oslash$ $\mu_{\oslash}(x) = 0$, then $\mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\oslash}(x) = 0 \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha$. So, $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{B}}(x) \geq \alpha$, what had to be proved.

Similarly, the weak association of fuzzy set $\tilde{A}$ in X and the empty set $\oslash$ are weakly equals to the fuzzy set $\tilde{A}$ in X, that is

$$\tilde{A} \overset{0}{\cup} \oslash \stackrel{0}{=} \tilde{A}.$$

Theorem 6. α-weak crossing of the fuzzy set $\tilde{A}$ in X and the empty set $\oslash$ is α-weakly equal to the empty set $\oslash$, that is

$$\tilde{A} \overset{\alpha}{\cap} \oslash \overset{\alpha}{=} \oslash.$$

Proof. Let's consider the fuzzy set $\tilde{B} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} \oslash$ in X. According to the definition of the α-weak crossing operation, for the arbitrary element $x \in X$ we can write $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\oslash}(x) \geq \alpha$. As to the definition of the empty $\oslash$, that $\mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\oslash}(x) = 0 \Leftrightarrow \mu_{\oslash}(x) = 0$. So, $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\oslash}(x) = 0$, what had to be proved.

Similarly, a weak crossing of the fuzzy set $\tilde{A}$ in X and the empty set $\oslash$ weakly equals the empty set $\oslash$, that is

$$\tilde{A} \overset{0}{\cap} \oslash \overset{0}{=} \oslash.$$

Theorem 7. α-weak association of the fuzzy set $\tilde{A}$ in X with the universum X α-weakly equals to the universum X, that is

$$\tilde{A} \overset{\alpha}{\cup} X \overset{\alpha}{=} X.$$

Proof. Let's consider the fuzzy set $\tilde{B} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cup} X$ in X. Acccording to the definition of the α-weak association operation, for an arbitrary element $x \in X$ we can write $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \vee \mu_X(x) = 1$. As to the definition of the universum for all of the $x \in X$ $\mu_X(x) = 1$, that $\mu_{\tilde{A}}(x) \geq \alpha \vee \mu_X(x) = 1 \Leftrightarrow \mu_X(x) = 1$. So, $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_X(x) = 1$, what had to be proved.

Similarly, weak association of the fuzzy set $\tilde{A}$ in X with the universum X weakly equal to the universum X, that is

$$\tilde{A} \overset{0}{\cup} X \overset{0}{=} X.$$

Theorem 8. α-weak crossing of the fuzzy set $\tilde{A}$ in X with the universum X α-weakly equals the fuzzy set $\tilde{A}$ in X, that is

$$\tilde{A} \overset{\alpha}{\cap} X \overset{\alpha}{=} \tilde{A}.$$

Proof. Let's consider the fuzzy set $\tilde{B} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} X$ is X. According to the definition of the α-weak crossing operation for an arbitrary element $x \in X$, we can write $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_X(x) \geq \alpha$. As to the definition of universum, for all $x \in X$ $\mu_X = 1$, that $\mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_X(x) = 1 \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha$. So, $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha$, what had to be proved. Similarly, the weak crossing of the fuzzy set $\tilde{A}$ in X with universum X weakly equals the fuzzy set $\tilde{A}$ in X, that is

$$\tilde{A} \overset{0}{\cap} X \overset{0}{=} \tilde{A}.$$

Let's consider the theorems characterizing α-weak operations between fuzzy sets and their α-weak complement. There are theorems for the Cantor sets

$$A \cup \bar{A} = X,$$

$$A \cap \bar{A} = \oslash.$$

In the traditional theory of fuzzy sets similar theorems are absent.

As for weak operations between fuzzy sets, the following theorem exists.

Theorem 9. Weak crossing of the fuzzy set $\tilde{A}$ in X with its weak complement $\overset{0}{\bar{\tilde{A}}}$ in X weakly equals the empty set $\oslash$, that is

$$\tilde{A} \overset{0}{\cap} \overset{0}{\bar{\tilde{A}}} \overset{0}{=} \oslash.$$

Proof. Let's consider fuzzy sets $\tilde{B} \stackrel{0}{=} \overset{0}{\bar{\tilde{A}}}$ and $\tilde{C} \stackrel{0}{=} \tilde{A} \overset{0}{\cap} \tilde{B}$ in X. According to the definition of α-weak crossing operation, for the arbitrary element $x \in X$ we can write

$$\mu_{\tilde{B}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x) = 0, \tag{16}$$

$$\mu_{\tilde{C}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x) > 0 \wedge \mu_{\tilde{B}}(x) > 0. \tag{17}$$

Having done the substitution of the equivalent expression for a logical variable $\mu_{\tilde{B}}(x) > 0$ from the logical equation (16) into the logical equation (17) we get $\mu_{\tilde{C}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x) > 0 \wedge \mu_{\tilde{A}}(x) = 0$.

Since $\mu_{\tilde{A}}(x) > 0 \wedge \mu_{\tilde{A}}(x) = 0 \Leftrightarrow False$, then $\mu_{\tilde{C}}(x) = 0$, what had to be proved.

Let's consider α-weak operations on binary fuzzy relations (BFR). Binary fuzzy relation $\left(\tilde{A}, X\right)$ – is a fuzzy set defined on the Descartes square $X \times X$ and for which the following is true:

$$\forall x, y \in X \left(\mu_{\tilde{A}}(x, y) \in [0, 1]\right).$$

Since BFR is a common fuzzy set and the only difference is that its elements are the ordered pairs of the Descartes square of the universum X, then for BFR all introduced beforehand α-weak operations occur (association, crossing, complement, difference etc). At the same time, for BFR one can introduce additionally operations which are absent for ordinary fuzzy sets. Therefore there is an inverted relation, its definition is in the traditional theory is written as:

$$\left(\tilde{A}^{-1}, X\right) \; is\, the\, inverted\, relation\, to \left(\tilde{A}, X\right) \; if\; and\; only\; if,\; when$$

$$\forall x, y \in X \left(\mu_{\tilde{A}^{-1}}(y, x) = \mu_{\tilde{A}}(x, y)\right).$$

Following the principles of building the class of weak operations, for the α-weak inverted relation we can write:

$$\left(\overset{\alpha}{\tilde{A}^{-1}}, X\right) is\, \alpha - weak\, inverted\, relation\, to\, \left(\tilde{A}, X\right) \; if\; and\; only\; if,\; when$$

$$\forall x, y \in X \left(\mu_{\tilde{A}^{-1}}(y, x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x, y) > 0\right).$$

Accordingly,

$$\left(\overset{0}{\tilde{A}^{-1}}, X\right) is\, \alpha - weak\, inverted\, relation\, to\, \left(\tilde{A}, X\right) \; if\; and\; only\; if,\; when$$

$$\forall x, y \in X \left(\mu_{\tilde{A}^{-1}}(y, x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x, y) > 0\right).$$

Let's formulate the definition for a weak composition of fuzzy relations. Traditional maximin composition of fuzzy relations is formulated as: fuzzy relation $\left(\tilde{A}_1 \circ \tilde{A}_2, X\right)$ is a maximin composition of fuzzy relations $\left(\tilde{A}_1, X\right)$ and $\left(\tilde{A}_2, X\right)$ as to the definition if and only if the, when

$$\forall x, y \in X \left(\mu_{\tilde{A}_1 \circ \tilde{A}_2}(x, y) = \underset{z \in X}{MaxMin}(\mu_{\tilde{A}_1}(x, z), \mu_{\tilde{A}_2}(z, y))\right).$$

The definition for the α-weak composition can be written as: fuzzy relation $\left(\tilde{A}_1 \overset{\alpha}{\circ} \tilde{A}_2, X\right)$ is the α-weak composition of fuzzy relations $\left(\tilde{A}_1, X\right)$ and $\left(\tilde{A}_2, X\right)$ according to its definition if and only if, when

$$\forall x, y \in X \left(\mu_{\tilde{A}_1 \circ \tilde{A}_2}(x, y) \geq \alpha \Leftrightarrow \exists z \in X (\mu_{\tilde{A}_1}(x, z) \geq \alpha \wedge \mu_{\tilde{A}_2}(z, y)) \geq \alpha\right), \alpha \in (0, 1]. \tag{18}$$

It follows from (18) that fuzzy relation $\left(\tilde{A}_1 \overset{0}{\circ} \tilde{A}_2, X\right)$ is the α-weak composition of fuzzy relations $\left(\tilde{A}_1, X\right)$ and $\left(\tilde{A}_2, X\right)$ if and only if, when

$$\forall x, y \in X \left(\mu_{\tilde{A}_1 \circ \tilde{A}_2}(x, y) > 0 \Leftrightarrow \exists z \in X(\mu_{\tilde{A}_1}(x, z) > 0 \wedge \mu_{\tilde{A}_2}(z, y)) > 0\right), \alpha \in (0, 1].$$

Let's proceed to the fuzzy sets reflections and the extension principles. As it is known the extension principles is the way of defining the image of fuzzy set under crisp or fuzzy reflection. There can be many such methods, but all of them must satisfy two conditions:

1. The image of any fuzzy set, regardless of the nature of the reflection, is also a fuzzy set.
2. Any extension principle should not contradict the definition of a clear representation of classical Cantor sets.

The definition of the maximin of extension principle, the most widespread in the traditional theory of fuzzy sets, for the crisp reflection of fuzzy sets can be formulated as follows: fuzzy set $f(\tilde{A})$ in Y is the image of the fuzzy set $\tilde{A}$ in X under crisp reflection $f : X \to Y$ according to the definition if and only if, when

$$\forall y \in Y \left(\mu_{f(\tilde{A})}(y) = \underset{x \in f^{-1}(y)}{Max}\ \mu_{\tilde{A}}(x)\right), \tag{19}$$

where $f^{-1}(y)$ is the proimage of the element $y \in Y$ under crisp reflection $f : X \to Y$.

Maximin of extension principle for fuzzy reflection of the fuzzy sets one can be written as: fuzzy set $\tilde{f}(\tilde{A})$ in Y is the image of the fuzzy set $\tilde{A}$ in X under fuzzy reflection $f : X \to Y$ according to the definition if and only if, when

$$\forall y \in Y \left(\mu_{\tilde{f}(\tilde{A})}(y) = \underset{x \in X}{Max} Min(\mu_{\tilde{A}}(x), \mu_{\tilde{f}}(x, y))\right), \tag{20}$$

where $\mu_{\tilde{f}} : X \times Y \to (0, 1]$ - membership function of fuzzy reflection $f : X \to Y$.

Let's formulate the extension principles for crisp and fuzzy reflections of fuzzy sets that are more general than (19, 20) and less demanding on the completeness of data on membership functions.

The definition of α-weak extension principle for crisp reflections of fuzzy sets is formulated as: fuzzy set $f(\tilde{A})$ in Y is the α-weak image of fuzzy set $\tilde{A}$ in X under crisp reflection $f : X \to Y$ according to the definition if and only if, when

$$\forall y \in Y \left(\mu_{f(\tilde{A})}(y) \geq \alpha \Leftrightarrow \exists x \in f^{-1}(y)(\mu_{\tilde{A}}(x) \geq \alpha)\right),$$

where $f^{-1}(y)$ is the proimage of the element $y \in Y$ under crisp reflection $f : X \to Y$.

Accordingly, for the principle of weak extension for crisp reflections of fuzzy sets we can write: fuzzy set $f(\tilde{A})$ in Y is a weak image of fuzzy set $\tilde{A}$ in X under crisp reflection $f : X \to Y$ according to the definition if and only if, when

$$\forall y \in Y \left(\mu_{f(\tilde{A})}(y) > 0 \Leftrightarrow \exists x \in f^{-1}(y)(\mu_{\tilde{A}}(x) > 0)\right).$$

The definition of α-weak extension principle for fuzzy reflections of fuzzy sets can be written as: fuzzy set $\tilde{f}(\tilde{A})$ in Y is the α-weak image of fuzzy set $\tilde{A}$ in X under fuzzy reflection $\tilde{f} : X \to Y$ according to the definition if and only if, when

$$\forall y \in Y \left(\mu_{\tilde{f}(\tilde{A})}(y) \geq \alpha \Leftrightarrow \exists x \in X(\mu_{\tilde{A}}(x) \geq \alpha) \wedge \mu_{\tilde{f}}(x, y) \geq \alpha\right),$$

where $\mu_{\tilde{f}} : X \times Y \to (0, 1]$ - membership function of fuzzy reflection $\tilde{f} : X \to Y$.

Accordingly, for the principle of weak extension for the fuzzy reflections of fuzzy sets we can write: fuzzy set $\tilde{f}(\tilde{A})$ in Y is a weak image of fuzzy set $\tilde{A}$ in X under fuzzy reflection $\tilde{f}: X \to Y$ according to the definition if and only if, when

$$\forall y \in Y \left(\mu_{\tilde{f}(\tilde{A})}(y) > 0 \Leftrightarrow \exists x \in X(\mu_{\tilde{A}}(x) > 0) \wedge \mu_{\tilde{f}}(x, y) > 0 \right).$$

Conclusions

1. There is a large number of applied problems for which the use of the maximin extension principle hinders their solution, since its application requires complete information about the membership functions of vaguely defined parameters of the problem, and this is often a practically impossible procedure. In these cases, even the highest-level expert can determine only cores or α-cuts for the unknown fuzzy parameters of the system. Building complete membership functions of unknown fuzzy parameters on this basis is risky and unreliable.

2. The axiomatics of the theory of fuzzy sets have been extended in order to introduce non-traditional (less demanding on the completeness of data on membership functions) extension principles and operations on fuzzy sets. The so-called α-weak operations on fuzzy sets are proposed, which are based on the use of α-cuts.

3. The axiomatics of weak operations is constructed so that each of these operations reduces to the corresponding classical operation in the case of degeneracy of fuzzy sets to classical Cantor sets.

4. For weak operations on fuzzy sets, the same theorems as for classical sets are valid, namely, theorems of idempotency, distributivity, involution, de Morgan and others.

5. Weak operations are introduced not only for fuzzy sets, but also for binary fuzzy relations, which made it possible to construct the principles of weak extension. All this makes it possible to use the mathematical apparatus of fuzzy sets to solve problems in conditions of significant uncertainty of input information.

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Анық емес жиындар теориясының дамуы: әлсіз операциялар және жалпылау принциптері

Жұмыста қолданбалы есептерді шешу үшін анық емес жиындар теориясын пайдалану кезінде туындайтын мәселелер қарастырылған. Статистикалық деректерге негізделген стохастикалық әдістерден айырмашылығы, статистикалық деректер болмаған кезде анық емес жиындар теориясы әдістерін қолданған жөн. Бұл жағдайларда алгоритмдер осы білім саласындағы мамандар болып табылатын сарапшылар жасаған тиістілік функциясына негізделуі керек. Ең дұрысы, тиістілік функциялары туралы толық ақпарат қажет, бірақ бұл практикалық процедура емес. Көбінесе, тіпті ең тәжірибелі маман тек олардың тасымалдаушыларын немесе белгісіз бұлыңғыр жүйе параметрлері үшін α-деңгейінің бөлек жиынтықтарын анықтай алады. Осы негізде белгісіз анық емес параметрлердің толық тиістілік функцияларын құру тәуекелді және сенімсіз. Сондықтан мақалада анық емес жиындар теориясының аксиоматикасын кеңейту ұсынылады (тиістілік функциялар туралы деректердің толықтығын талап етпейтін) жалпылаудың және анық емес жиындардағы операциялардың принциптерін енгізу. Бөлек α-деңгейлі жиындарды қолдануға негізделген анық емес жиындардағы α-әлсіз деп аталатын амалдар ұсынылған. Сондай-ақ, кеңейтілген аксиоматикалық теорияда Кантордың жиындар теориясының барлық классикалық теоремаларын қолдануға болатыны көрсетілген.

Ақпараттың маңызды белгісіздігі жағдайында мәселелерді шешуге мүмкіндік беретін жаңа жалпылау принциптері енгізілді.

Кілт сөздер: Кантор жиыны, анық емес жиын, тиістілік функция, α-деңгейлі жиын, анық емес жиынды қолдау, α-әлсіз операция.

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Развитие теории нечетких множеств: слабые операции и принципы обобщения

В работе рассмотрены проблемы, возникающие при использовании теории нечетких множеств для решения прикладных задач. В отличие от стохастических методов, основанных на статистических данных, методы теории нечетких множеств целесообразно применять, когда статистические данные недоступны. В этих случаях алгоритмы должны основываться на функциях принадлежности, формируемых экспертами, являющимися специалистами в данной области знаний. В идеале требуется полная информация о функциях принадлежности, но это непрактичная процедура. Чаще всего даже самый опытный специалист может определить только их носители или отдельные наборы α-уровня для неизвестных нечетких параметров системы. Построение на этой основе полных функций принадлежности неизвестных нечетких параметров рискованно и ненадежно. Поэтому в статье предложены расширение аксиоматики теории нечетких множеств с целью введения нетрадиционных (менее требовательных к полноте данных о функциях принадлежности) принципов обобщения и операций над нечеткими множествами, а также так называемые α-слабые операции над нечеткими множествами, основанные на использовании отдельных множеств α-уровня. Кроме того, показано, что все классические теоремы теории множеств Кантора применимы в расширенной аксиоматической теории. Введены новые принципы обобщения, позволяющие решать задачи в условиях значительной неопределенности информации.

Ключевые слова: множество Кантора, нечеткое множество, функция принадлежности, множество α-уровня, носитель нечеткого множества, α-слабая операция.

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Some non-standard quasivarieties of lattices

The questions of the standardness of quasivarieties have been investigated by many authors. The problem "Which finite lattices generate a standard topological prevariety?" was suggested by D.M. Clark, B.A. Davey, M.G. Jackson and J.G. Pitkethly in 2008. We continue to study the standardness problem for one specific finite modular lattice which does not satisfy all Tumanov's conditions. We investigate the topological quasivariety generated by this lattice and we prove that the researched quasivariety is not standard, as well as is not finitely axiomatizable. We also show that there is an infinite number of lattices similar to the lattice mentioned above.

Keywords: lattice, quasivariety, basis of quasi-identities, profinite algebra, topological quasivariety, profinite quasivariety.

Introduction

The problems concerning finite axiomatizability and standardness of (quasi)varieties of algebras are among the most researched and relevant topics in universal algebra.

According to R. McKenzie [1], each finite lattice has a finite identity basis. The analogous statement for quasi-identities is incorrect. V.P. Belkin in [2] proved that there is a finite lattice which has no finite quasi-identity basis. In this regard, the problem "Which finite lattices have finite quasi-identity bases" was proposed by V.A. Gorbunov and D.M. Smirnov [3]. A sufficient two-part condition under which a locally finite quasivariety of lattices does not have a finite (independent) quasi-identity basis was found by V.I. Tumanov [4].

In [5] the concept of a standard (topological) quasivariety was introduced, and the basic properties were investigated and many examples of standard and non-standard quasivarieties were provided. The standardness of algebras was further studied by D.M. Clark, B.A. Davey, R.S. Freese and M.G. Jackson in [6], who established a general condition guaranteeing the standardness of a set of finite algebras. In [7] sufficient conditions were found under which a quasivariety contains a continuum of non-standard subquasivarieties. In [6] it was proved that any finite lattice generates a standard variety. However, in [8] it was established that Belkin's lattice generates non-standard quasivariety. These naturally arose the problem "Which finite lattices generate standard topological quasivarieties?" that was suggested by D.M. Clark, B.A. Davey, M.G. Jackson and J.G. Pitkethly in [8].

In [9, 10] one specific lattice was studied and it was proved that this lattice has no finite basis of quasi-identities [9] and generates non-standard quasivariety [10], respectively. The special feature of this lattice is that it does not satisfy one of the two-part Tumanov's condition (see Theorem 2).

In this paper we continue to study the standardness problem for one specific finite modular lattice. This lattice does not satisfy all Tumanov's conditions [4] and the quasivariety generated by this lattice is not standard, as well as is not finitely based (Theorem 3). At the end we show that there is an infinite number of lattices similar to this lattice (Theorem 4).

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1 Basic concepts and preliminaries

We recall some basic definitions and results for quasivarieties that we will refer to. For more information on the basic notions of general algebra and topology introduced below and used throughout this paper, we refer to [11–13].

We assume that all classes of algebras the same fixed finite signature σ and abstract, unless we specify otherwise. Also an algebra $\langle A; \sigma \rangle$ and its carrier (its basic set) A will be identified and denoted by the same way, namely A.

A class of algebras which is closed with respect to subalgebras, direct products (including the direct product of an empty family), and ultraproducts is a *quasivariety.* In other words, a class of algebras axiomatized by a set of quasi-identities is a quasivariety. A *quasi-identity* is a universal Horn sentence with the non-empty positive part

$$(\forall \bar{x})[p_1(\bar{x}) \approx q_1(\bar{x}) \wedge \cdots \wedge p_n(\bar{x}) \approx q_n(\bar{x}) \to p(\bar{x}) \approx q(\bar{x})],$$

where $p, q,\ p_1, q_1, \ldots, p_n, q_n$ are terms. A quasivariety closed with respect to homomorphisms is a *variety.* In other words, a variety is a class of similar algebras axiomatized by a set of identities, according to Birkhoff theorem [14]. An *identity* is a sentence of the form $(\forall \bar{x})[s(\bar{x}) \approx t(\bar{x})]$ for some terms $s(\bar{x})$ and $t(\bar{x})$. A quasivariety $\mathbf{K}$ has a finite basis of quasi-identities (finitely axiomatizable) if there is a finite set Σ of quasi-identities such as $\mathbf{K} = \mathrm{Mod}(\Sigma)$. Otherwise $\mathbf{K}$ has no finite basis of quasi-identities.

By $\mathbf{Q}(\mathbf{K})$ ($\mathbf{V}(\mathbf{K})$) we denote the smallest quasivariety (variety) containing a class $\mathbf{K}$. $\mathbf{Q}(\mathbf{K})$ is called finitely generated if $\mathbf{K}$ is a finite family of finite algebras. In case when $\mathbf{K} = \{A\}$ we write $\mathbf{Q}(A)$ instead of $\mathbf{Q}(\{A\})$. By Maltsev-Vaught theorem [15], $\mathbf{Q}(\mathbf{K}) = \mathbf{SPP}_u(\mathbf{K})$, where $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{P}_u$ are operators of taking subalgebras, direct products and ultraproducts, respectively.

Let $\mathbf{K}$ be a quasivariety. A congruence α on algebra A is called a $\mathbf{K}$-*congruence* provided $A/\alpha \in \mathbf{K}$. The set $\mathrm{Con}_{\mathbf{K}}A$ of all $\mathbf{K}$-congruences of A forms an algebraic lattice with respect to inclusion $\subseteq$. An algebra $A \in \mathbf{K}$ is *subdirectly* $\mathbf{K}$-*irreducible* if an intersection of any number of nontrivial $\mathbf{K}$-congruences is nontrivial. Since for any class $\mathbf{R}$ we have $\mathbf{Q}(\mathbf{R}) = \mathbf{SPP}_u(\mathbf{R}) = \mathbf{P}_s\mathbf{SP}_u(\mathbf{R})$, where $\mathbf{P}_s$ is operator of taking subdirect products, then for finitely generated quasivariety $\mathbf{Q}(A)$, every subdirectly $\mathbf{Q}(A)$-irreducible algebra is isomorphic to some subalgebra of A.

A finite algebra A with discrete topology generates a topological quasivariety consisting of all topologically closed subalgebras of non-zero direct powers of A endowed with the product topology. An algebra A is *profinite* with respect to quasivariety $\mathbf{R}$ if A is an inverse limit of finite algebras from $\mathbf{R}$. A topological quasivariety $\mathbf{Q}_\tau(A)$ is *standard* if every Boolean topological algebra (compact, Hausdorf and totally disconnected) with the algebraic reduct in $\mathbf{Q}(A)$ is profinite with respect to $\mathbf{Q}(A)$. In this case, we say that algebra A generates a *standard topological quasivariety.* For more information on the topological quasivarieties we refer to [6] and [8].

We say that X is *pointwise non-separable* with respect to quasivariety $\mathbf{R}$ if the following condition holds: There exist $a, b \in X$, $a \neq b$, such that, for each $n \in N$, each finite structure $M \in \mathbf{R}$ and each homomorphism $\varphi : X_n \to M$, we have $\varphi(a) = \varphi(b)$.

The following theorem provides non-standardness of quasivariety.

Theorem 1.(Second inverse limit technique [8])

Let $X = \varprojlim\{X_n \mid n \in N\}$ be a surjective inverse limit of finite structures, and let $\mathbf{K}$ be a quasivariety. Assume that $X \in \mathbf{K}$ is pointwise non-separable with respect to $\mathbf{K}$ and each substructure of X_n that is generated by at most n elements belongs to $\mathbf{K}$ for all $n \in N$. Then $\mathbf{K}$ is non-standard, as well as is not finitely axiomatizable.

To formulate our main result (Theorem 3) we need some special preliminaries.

Let $(a] = \{x \in L \mid x \leq a\}$ ($[a) = \{x \in L \mid x \geq a\}$) be a principal ideal (coideal) of a lattice L. A pair $(a, b) \in L \times L$ is called *splitting* (*semi-splitting*) if $L = (a] \cup [b)$ and $(a] \cap [b) = \varnothing$ ($L = (a] \cup [b)$ and $(a] \cap [b) \neq \varnothing$).

For any semi-splitting pair (a, b) of a lattice M we define a lattice

$$M_{a-b} = \langle\{(x, 0), (y, 1) \in M \times 2 \mid x \in (a], y \in [b)\}; \vee, \wedge\rangle \leq_s M \times \mathbf{2},$$

where $\mathbf{2} = \langle\{0, 1\}; \vee, \wedge\rangle$ is a two element lattice.

Theorem 2. (Tumanov's theorem [4])

Let a locally finite quasivarieties of lattices $\mathbf{M}$ and $\mathbf{N} \subset \mathbf{M}$ satisfy the following two conditions:

a) in any finitely subdirectly $\mathbf{M}$-irreducible lattice $M \in \mathbf{M} \backslash \mathbf{N}$ there is a semi-splitting pair (a, b) such that $M_{a-b} \in \mathbf{N}$;

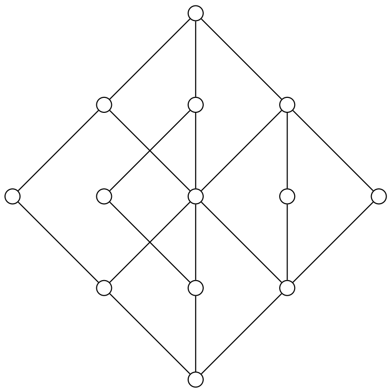
b) there is a finite simple lattice $\mathcal{P} \in \mathbf{N}$ that is not a proper homomorphic image of any subdirectly $\mathbf{N}$-irreducible lattice.

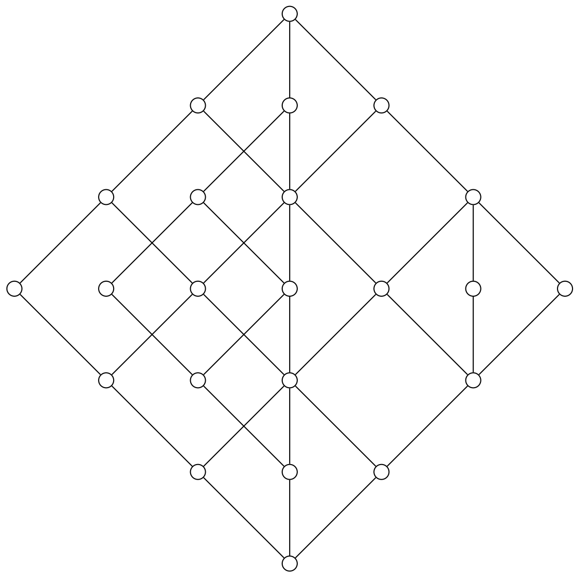
Then the quasivariety $\mathbf{N}$ has no coverings in the lattice of subquasivarieties of $\mathbf{M}$. In particular, $\mathbf{N}$ has no finite (independent) basis of quasi-identities provided $\mathbf{M}$ is finitely axiomatizable.

A quasivariety is called *proper* if it is not variety. A subalgebra B of an algebra A is called *proper* if B is not one-element (trivial) and $B \ncong A$. For an algebra A and elements $a, b \in A$, by $\theta(a, b)$ we denote the least congruence on A containing pair (a, b).

2 Main result

Let A' and A are the modular lattices displayed in Figure 1. And let $\mathbf{Q}(A)$ and $\mathbf{V}(A)$ are quasivariety and variety generated by A, respectively. Since every subdirectly $\mathbf{Q}(A)$-irreducible lattice is a sublattice of A, and A' is simple and a homomorphic image of A, and A' is not a sublattice, then $A' \in \mathbf{V}(A) \backslash \mathbf{Q}(A)$, that is $\mathbf{Q}(A)$ is a proper quasivariety. One can check that A' has no semi-splitting pair. Thus, the condition a) of Tumanov's theorem does not hold on the quasivariety $\mathbf{Q}(A)$. It is easy to see that M_3 is unique non-distributive simple lattice in $\mathbf{Q}(A)_{SI}$ and it is a homomorphic image of A. Hence, the condition b) of Tumanov's theorem is not hold on quasivarieties $\mathbf{Q}(A)$ and $\mathbf{V}(A)$.





A' A

Figure 1: Lattices A' and A

The main result of the paper is

Theorem 3. The topological quasivariety generated by the lattice A is not standard, as well as is not finitely axiomatizable.

Proof of Theorem 3.

To prove the theorem we use Theorem 1. According to this theorem we will construct $L = \varprojlim\{L_n \mid n \in N\}$ a surjective inverse limit of the finite lattices such that every n-generated sublattice of L_n belongs to $\mathbf{Q}(A)$ and L is pointwise non-separable with respect to $\mathbf{Q}(A)$.

Let S be a non-empty subset of a lattice L. Denote by $\langle S \rangle$ the sublattice of L generated by S.

We define a modular lattice L_n by induction:

$n = 0$. $L_0 \cong M_{3-3}$ and $L_0 = \langle \{a_0, b_0, c_0, a^0, b^0, c^0\} \rangle$ (Fig. 2).

$n = 1$. L_1 is a modular lattice generated by $L_0 \cup \{a_1, b_1, c_1, a^1, b^1, c^1\}$ such that $\langle \{a_1, b_1, c_1, a^1, b^1, c^1\} \rangle \cong M_{3-3}$, and $c_0 = a^1$, $a^0 \wedge b^0 = c_0 \vee b^1 = c_0 \vee c_1$ (Fig. 3).

$n > 1$. L_n is a modular lattice generated by the set $L_{n-1} \cup \{a_n, b_n, c_n, a^n, b^n, c^n\}$ such that $\langle \{a_n, b_n, c_n, a^n, b^n, c^n\} \rangle \cong M_{3-3}$, and $c_{n-1} = a^n$, $a^0 \wedge b^0 = c_0 \vee b^n = c_0 \vee c_n$ (Fig. 4).

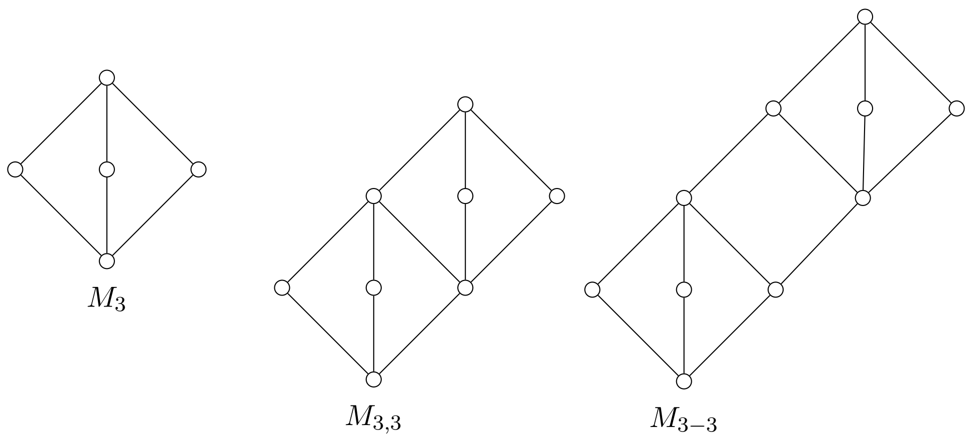


Figure 2: Lattices M_3, $M_{3,3}$ and M_{3-3}

Let L_n^- be a sublattice of L_n generated by the set $\{a_i, b_i, c_i, a^i, b^i, c^i \mid 0 < i \leq n\}$. One can see that $L_n^- \cong L_n/\theta(a_0, b_0)$ and $L_n^- \leq_s M_{3-3}^n$. Hence, $L_n^- \in \mathbf{Q}(A)$.

Claim 1. Every proper sublattice of L_n belongs to $\mathbf{Q}(A)$.

Proof of Claim 1.

It is enough to prove the claim for arbitrary maximal proper sublattices of L_n. Since L_n is generated by the set of double irreducible elements $S = \{a_0, b_0, b^0, c^0, c_n\} \cup \{b_i, b^i \mid 0 < i \leq n\}$ then every maximal proper sublattices L of L_n generated by $S - \{x\}$ for some $x \in S$, that is $L = \langle S - \{x\} \rangle$.

Suppose that $x \in \{a_0, b_0, b^0, c^0\}$. Then the lattice $\langle \{a_0, b_0, b^0, c^0\} \backslash \{x\} \rangle / \theta(c_0, a^0 \wedge b^0)$ be a homomorphic image of L with the kernel $\alpha = \theta(a_1, c_n)$ and belongs to $\mathbf{Q}(A)$.

One can see that for $\beta = \theta(a_0, b_0)$ if $x \in \{b^0, c^0\}$ and $\beta = \theta(b^0, c^0)$ if $x \in \{a_0, b_0\}$, L/β is isomorphic to a sublattice of $L_n^- \times \mathbf{2}$ and belongs to $\mathbf{Q}(A)$. Thus, α and β are $\mathbf{Q}(A)$-congruences. One can check that $\alpha \cap \beta = 0$. Hence $L \leq_s L/\alpha \times L/\beta$. Therefore, $L \in \mathbf{Q}(A)$.

Suppose that $x \in \{b_i, b^i \mid 0 < i \leq n\} \cup \{c_n\}$. Without loss of generality, assume that $x = b_n$. Let $\alpha = \theta(c_0, c_{n-1})$. Then L/α is isomorphic to the sublattice S of L_1 generated by the set $\{a_0, b_0, b^0, c^0, a_1, b_1, b^1\}$. Since the lattice $P = \langle \{a_0, b_0, b^0, c^0, b^1, c^1\} \rangle$ is a sublattice of A and $S \leq_s P \times \mathbf{2}^2$ we get $S \in \mathbf{Q}(A)$. On the other hand, $L/\theta(a_0, b_0)$ is a sublattice of L_n^-. Since $L_n^- \in \mathbf{Q}(A)$ then $L/\theta(a_0, b_0) \in \mathbf{Q}(A)$. One can see that $\alpha \cap \theta(a_0, b_0) = 0$. Hence, L is a subdirect product of two lattices from $\mathbf{Q}(A)$. Therefore, $L \in \mathbf{Q}(A)$.

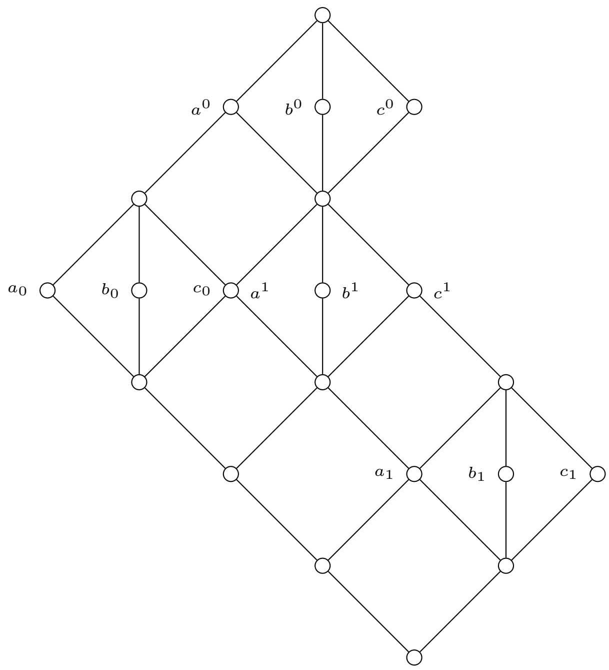


Figure 3: Lattice L_1

Let $\varphi_{n,n-1}$ be a homomorphism from L_n to L_{n-1} such that $\ker \varphi_{n,n-1} = \theta(a^n, b_n)$, and $\varphi_{n,n}$ an identity map for all $n > 1$ and $m < n$. And let $\varphi_{n,m} = \varphi_{m+1,m} \circ \cdots \circ \varphi_{n,n-1}$. It can be seen that $\{L_n; \varphi_{n,m}, N\}$ forms inverse family, where N is the linear ordered set of positive integers.

We denote $L = \varprojlim\{L_n \mid n \in N\}$ and show that $L \in \mathbf{Q}(A)$.

Claim 2. The lattice L belongs to $\mathbf{Q}(A)$.

Proof of Claim 2.

Let α be a quasi-identity of the following form

$$\&_{i \le r} p_i(x_0, \ldots, x_{n-1}) \approx q_i(x_0, \ldots, x_{n-1}) \to p(x_0, \ldots, x_{n-1}) \approx q(x_0, \ldots, x_{n-1}).$$

Assume that α is valid on $\mathbf{Q}(A)$ and

$$L \models p_i(a_0, \ldots, a_{n-1}) = q_i(a_0, \ldots, a_{n-1}) \quad \text{for all} \quad i < r,$$

for some $a_0, \ldots, a_{n-1} \in L$. From the definition of inverse limit we have that $L \le_s \prod_{i \in I} L_i$. Therefore

$$L_s \models p_i(a_0(s), \ldots, a_{n-1}(s)) = q_i(a_0(s), \ldots, a_{n-1}(s)) \quad \text{for all} \quad i < r.$$

Each at most n generated subalgebra of L_s belongs to $\mathbf{Q}(A)$ for all $s > n$, by Claim 1. Hence α is true in L_s for all $s > n$. And this in turn entails

$$L_s \models p(a_0(s), \ldots, a_{n-1}(s)) = q(a_0(s), \ldots, a_{n-1}(s)).$$

Since $a_i(m) = \varphi_{s,m}(a_i(s))$ for all $0 \le i < n$ and $m < s$, we get

$$L_m \models p(a_0(m), \ldots, a_{n-1}(m)) = q(a_0(m), \ldots, a_{n-1}(m)) \quad \text{for all} \quad m < s.$$

So

$$L \models p(a_0, \ldots, a_{n-1}) = q(a_0, \ldots, a_{n-1}).$$

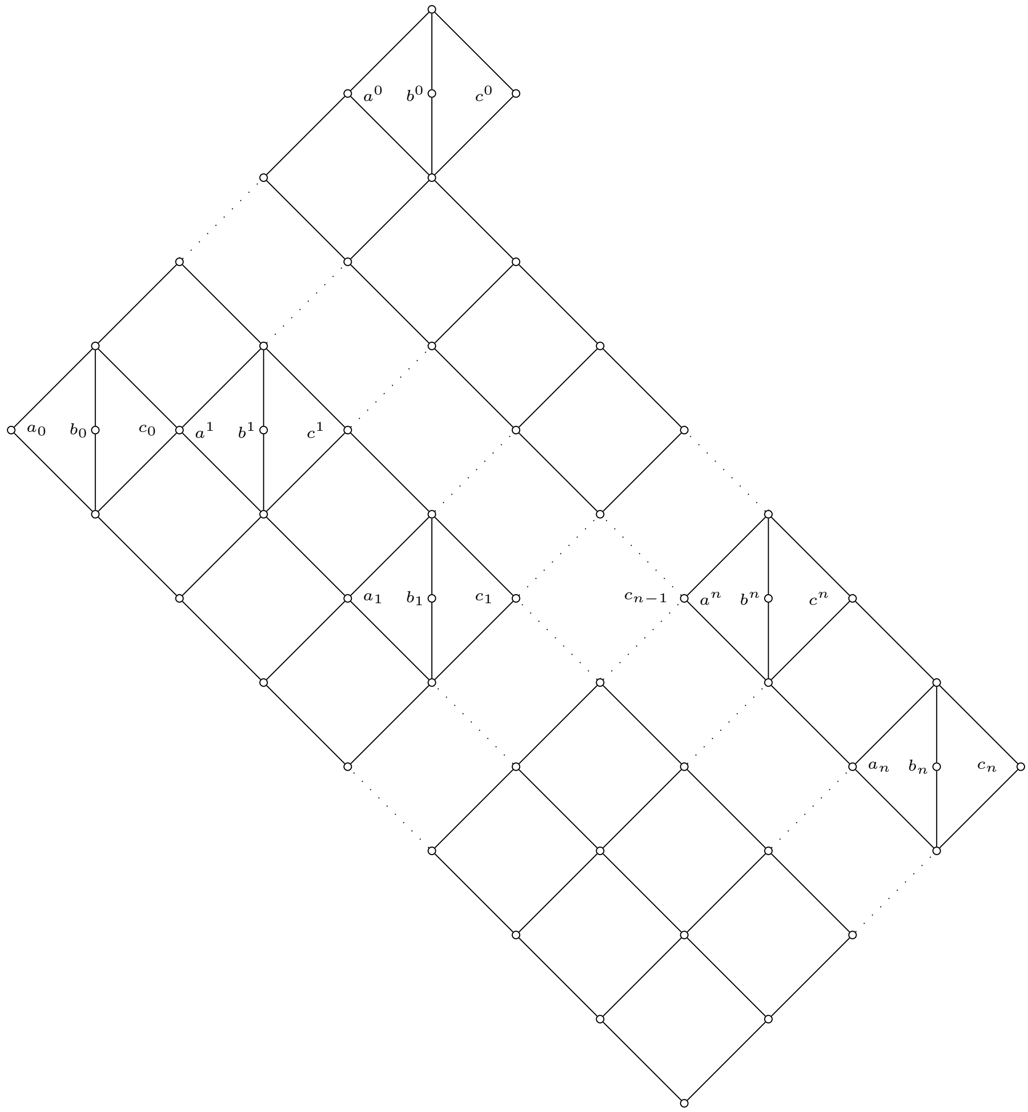


Figure 4: Lattice L_n, $n \geq 2$

Hence $L \models \alpha$, for every α that is valid on $\mathbf{Q}(A)$. This proves that $L \in \mathbf{Q}(A)$.

Claim 3. The lattice L is point-wise separable with respect to $\mathbf{Q}(A)$.

Proof of Claim 3.

We obtain $\varphi_{n,m}(a_0) = a_0$ and $\varphi_{n,m}(b_0) = b_0$, by definition of $\varphi_{n,n-1}$. And $a = (a_0, \ldots, a_0, \ldots)$, $b = (b_0, \ldots, b_0, \ldots) \in L$, by definition of inverse limit. Let $\alpha : L \to M$ be a homomorphism, $M \in \mathbf{Q}(A)$ and M finite. There is $n > 2$ and homomorphism $\psi_M : L_n \to M$ such that $\alpha = \varphi_n \circ \psi_M$ for some surjective homomorphism $\varphi_n : L \to L_n$ (by universal property of inverse limit). It is not difficult to see that any non-trivial homomorphic image of L_n is isomorphic to L_m, $m < n$, or contains $M_{3,3}$ as a sublattice. Since $L_m, M_{3,3} \notin \mathbf{Q}(A)$ and $\psi_M(L_n) \leq M \in \mathbf{Q}(A)$, then we obtain that $\psi_M(L_n)$ is trivial. That is $\psi_M(x) = const$ for all $x \in L_n$. So we get $\alpha(a) = \alpha(b)$.

Thus, the Claims 1–3 state that the conditions of Theorem 1 holds on $\mathbf{Q}(A)$. Therefore, the quasivariety $\mathbf{Q}(A)$ generated by A is not standard, as well as not finitely axiomatizable.

Remark. In the paper [16] it has been proved that the quasivariety generated by the lattice A is not finitely based. We would like to point out that we presented the proof of the Claim 1 for the sake of completeness of the proof of the main result. We also note that Claims 2 and 3 were proved by arguments of [17].

We note that there is an infinite number of lattices similar to the lattice A. This is the context of the following.

Theorem 4. Let L be a finite lattice such that $M_{3,3} \not\leq L$, $A \leq L$ and $L_n \not\leq L$ for all $n > 1$. Then the topological quasivariety generated by the lattice L is not standard, as well as is not finitely axiomatizable.

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Кейбір стандартты емес торлардың квазикөпбейнелері

Квазикөпбейнелердің стандарттылық мәселелерін көптеген авторлар зерттеді. Д.М. Кларк, Б.А. Дэйви, М.Г. Джексон және Дж.Г. Питкетли «Қандай соңғы торлар стандартты топологиялық предкөпбейнені тудырады?» деген мәселені 2008 жылы ұсынды. Тумановтың барлық жағдайларын қанағаттандырмайтын бір нақты модульдік тордың стандарттылық мәселесін зерттеу жалғастырылған. Осы тордан пайда болған топологиялық квазикөпбейне зерттелген және зерттелетін квазикөпбейне стандартты емес, сонымен қатар әрине аксиоматизацияланбайтыны дәлелденген. Сондай-ақ жоғарыда аталған торға ұқсас торлардың шексіз саны бар екені көрсетілген.

Кілт сөздер: тор, квазикөпбейне, квазисәйкестіктердің базисі, профиниттік алгебра, топологиялық квазикөпбейне, профиниттік квазикөпбейне.

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Некоторые нестандартные квазимногообразия решеток

Вопросы стандартности квазимногообразий исследовались многими авторами. Проблема «Какие конечные решетки порождают стандартное топологическое предмногообразие?» была предложена Д.М. Кларком, Б.А. Дэйви, М.Г. Джексоном и Дж.Г. Питкетли в 2008 году. Мы продолжаем изучать проблему стандартности для одной конкретной конечной модулярной решетки, которая не удовлетворяет всем условиям Туманова. Исследуем топологическое квазимногообразие, порожденное этой решеткой, и доказываем, что исследуемое квазимногообразие не является стандартным и конечно аксиоматизируемым. Кроме того, показываем, что существует бесконечное число решеток, подобных упомянутой выше.

Ключевые слова: решетка, квазимногообразие, базис квазитождеств, профинитная алгебра, топологическое квазимногообразие, профинитное квазимногообразие.

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Ranks and approximations for families of cubic theories

In this paper, we study the rank characteristics for families of cubic theories, as well as new properties of cubic theories as pseudofiniteness and smooth approximability. It is proved that in the family of cubic theories, any theory is a theory of finite structure or is approximated by theories of finite structures. The property of pseudofiniteness or smoothly approximability allows one to investigate finite objects instead of complex infinite ones, or vice versa, to produce more complex ones from simple structures.

Keywords: approximation of a theory, cube, cubic structure, cubic theory, pseudofinite theory, smoothly approximated structure.

1 Introduction

Modern mathematical models, which are large relational structures (random graphs) and at the same time time-dependent dynamic models, such as the growth of the Internet, social networks and computer security, cannot be described and explored by infinite models in standard graph theory. However, if a set of models is algorithmically well defined, then these sets exhibit general patterns that are inherent in «almost all» models in the community. These general laws for well-defined systems can be investigated using statistical and model-theoretic methods. From a model-theoretic point of view, one can approach approximations [1], definability [2], and interpretability [3].

The ranks and degrees for families of complete theories [4], similar to the Morley rank and degree for a fixed theory, and the Cantor-Bendixson rank and degree, were introduced by S. Sudoplatov. The problem arises of describing ranks and degrees for natural theory families. Ranks and degrees for families of incomplete theories are examined in [5, 6], for families of permutation theories - in [7], and for families of all theories of arbitrary languages - in [8].

The [1] examines approximations of theories both in the general context and in relation to specific natural theory families. The problem of describing the approximation forms of the natural theory families arises.

This work is devoted to the description of the ranks and degrees of families of cubic theories, as well as approximation by theories of finite cubic structures. Pseudofinite structures are mathematical structures that resemble finite structures but are not actually finite. They are important in various areas of mathematics, including model theory and algebraic geometry. Further study of pseudofinite structures will continue to reveal new insights and applications in mathematics and beyond.

1.1 Preliminaries from cubic theories

Cubic structures are defined in [9], theoretical properties of the model are discussed and included in the monograph [10], applications in discrete mathematics are presented [11]. The following necessary terminology for cubic structures was taken from [9, 11] without specifying it.

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Definition 1. An *n-dimensional cube* or an *n-cube* (where $n \in \omega$) is a graph isomorphic to the graph $\mathcal{Q}_n$ with universe $\{0,\ 1\}^n$ and such that any two vertices $(\delta_1, \ldots,\ \delta_n)$ and $(\delta'_1, \ldots,\ \delta'_n)$ are adjacent if and only if these vertices differ by exactly one coordinate.

Let λ be an infinite cardinal number. A λ-*dimensional cube* or a λ-*cube* is a graph isomorphic to a graph $\Gamma = \langle X; R\rangle$ that satisfies the following conditions:

(1) the universe $X \subseteq \{0;\ 1\}^{\lambda}$ is generated from an arbitrary function $f \in\ X$ by the operator $\langle f\rangle$ attaching, to the set $\{f\}$, all results of substitutions for any finite tuples $(f(i_1), \ldots,\ f(i_m))$ by tuples $(1 - f(i_1), \ldots, 1 - f(i_m))$;

(2) the relation R consists of edges connecting functions differing exactly in one coordinate.

The described graph $\mathcal{Q} \rightleftarrows \mathcal{Q}_f$ with the universe $\langle f\rangle$ is a canonical representative for the class of λ-cubes.

Note that the canonical representative of the class of n-cubes (as well as the canonical representatives of the class of λ-cubes) are generated by any its function: $\{0,1\}^n = \langle f\rangle$, where $f \in \{0,1\}^n$. Therefore the universes of canonical representatives $\mathcal{Q}_f$ of n-cubes like λ-cubes, will be denoted by $\langle f\rangle$.

Any graph $\Gamma =\ \langle X;\ R\rangle$, where any connected component is a cube, is called a *cubic structure*. A theory T of the graph language $\{R^{(2)}\}$ is *cubic* if $T\ =\ Th(\mathcal{M})$ for some cubic structure $\mathcal{M}$. In this case, the structure $\mathcal{M}$ is called a *cubic model* of T.

The *invariant* of a theory T is the function

$$Inv_T : \omega \cup \{\infty\} \to \omega \cup \{\infty\},$$

satisfying the following conditions:

(1) for any natural n; $Inv_T(n)$ is the number of connected components in any model of T, being n-cubes, if that number is finite, and $Inv_T(n)\ = \infty$ if that number is infinite;

(2) $Inv_T(\infty) =\ 0$ if models of T do not contain infinite-dimensional cubes (i. e., the dimensions of cubes are totally bounded), otherwise we set $Inv_T(\infty) = 1$.

The *diameter* $d(T)$ of a cubic theory T is the maximal distance between elements in models of T, if these distances are bounded, and we set $d(T) \rightleftharpoons \infty$ otherwise. The *support* (accordingly the ∞-*support*) $Supp(T)(Supp_\infty(T))$ of a theory T is the set $\{n \in \omega | Inv_T(n) \neq\ 0\}(\{n \in \omega | Inv_T(n)\ = \infty\})$.

If the diameter $d(T)$ is finite then there exists an upper estimate for dimensions of cubes, being in models of T. It means that $Supp(T)$ is finite, i. e., $Inv_T(\infty) =\ 0$. In this case the ∞-support is non-empty.

If $d(T) = \infty$ then $Inv_T(\infty) = 1$. In this case the support $Supp(T)$ can be either finite or infinite.

1.2 Preliminaries from model theory and approximations of theories

Historically, pseudofinite fields were first introduced by J. Ax and S. Kochen [12] in the form of non-principal ultraproducts of finite fields. Later, J. Ax in [13] connected the notion of pseudofiniteness and the construction of ultraproducts. The class of *pseudofinite fields* was defined in the work of J. Ax [13] and regardless of him in the work of Yu. Ershov [14] with an axiom system indicating this class.

In 1965 J. Ax [15] investigated fields F having the property that *every absolutely irreducible variety over F has an F-rational point.* It was shown that the non-principal ultraproduct of finite fields has such property. Yu. Ershov called such fields *regularly closed.* The notion of *pseudofiniteness* is credited to work in the 1968s by J. Ax [13]. He introduced the notion of *pseudofiniteness* to show the decidability of the theory of all finite fields, i.e. there is an algorithm to decide whether a given statement is true for all finite fields. It was proved that pseudofinite fields are exactly those infinite fields that have every elementary property common to all finite fields, that is, pseudofinite fields are infinite models of the theory of finite fields.

In the early 1990s, E. Hrushovski resumed research in the field of pseudofinite structures in meeting on Finite and Infinite Combinatorics in Sets and Logic [16], as well as in the joint works of E. Hrushovski and G. Cherlin and the following definition first occurs in [17], subsequently in [18]:

Definition 2. Let Σ be a language and $\mathcal{M}$ be a Σ-structure. A Σ-structure $\mathcal{M}$ is *pseudofinite* if for each Σ-sentences φ, $\mathcal{M} \models \varphi$ implies that there is a finite $\mathcal{M}_0$ such that $\mathcal{M}_0 \models \varphi$. The theory $T = Th(\mathcal{M})$ of a pseudofinite structure $\mathcal{M}$ is called pseudofinite.

In the work [1] S. Sudoplatov defined approximations relative given family $\mathcal{T}$ of complete theories.

Definition 3. [1] Let $\mathcal{T}$ be a family of theories and T be a theory such that $T \notin \mathcal{T}$. The theory T is said to be *$\mathcal{T}$-approximated,* or *approximated by the family $\mathcal{T}$*, or a *pseudo-$\mathcal{T}$-theory,* if for any formula $\varphi \in T$ there exists $T' \in \mathcal{T}$ for which $\varphi \in T'$.

If a theory T is $\mathcal{T}$-approximated, then $\mathcal{T}$ is said to be an *approximating family* for T, and theories $T' \in \mathcal{T}$ are said to be *approximations* for T. We put $\mathcal{T}_\varphi = \{T \in \mathcal{T} \mid \varphi \in T\}$. Any set $\mathcal{T}_\varphi$ is called the *φ-neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$. A family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.

Recall that the *E-closure* for a family $\mathcal{T}$ of complete theories is characterized by the following proposition.

Proposition 1. [19] *Let $\mathcal{T}$ be a family of complete theories of the language Σ. Then $Cl_E(\mathcal{T}) = \mathcal{T}$ for a finite $\mathcal{T}$, and for an infinite $\mathcal{T}$, a theory T belongs to $Cl_E(\mathcal{T})$ if and only if T is a complete theory of the language Σ and $T \in \mathcal{T}$, or $T \neq \mathcal{T}$ and for any formula φ the set $\mathcal{T}_\varphi$ is infinite.*

We denote by $\overline{\mathcal{T}}$ the class of all complete theories of relational languages, by $\overline{\mathcal{T}}_{fin}$ the subclass of $\overline{\mathcal{T}}$ consisting of all theories with finite models, and by $\overline{\mathcal{T}}_{inf}$ the class $\overline{\mathcal{T}} \backslash \overline{\mathcal{T}}_{fin}$.

Proposition 2. [1] *For any theory T the following conditions are equivalent:*
(1) T is pseudofinite;
(2) T is $\overline{\mathcal{T}}_{fin}$-approximated;
(3) $T \in Cl_E(\overline{\mathcal{T}}_{fin}) \backslash \overline{\mathcal{T}}_{fin}$.

1.3 Preliminaries from ranks for families of theories

In [4], *rank* $\mathrm{RS}(\cdot)$ is defined inductively for families of complete theories.
(1) The empty family $\mathcal{T}$ is assigned the rank $\mathrm{RS}(\mathcal{T}) = -1$.
(2) For finite nonempty families $\mathcal{T}$ set $\mathrm{RS}(\mathcal{T}) = 0$.
(3) For infinite families $\mathcal{T}$ we set $\mathrm{RS}(\mathcal{T}) \geq 1$.
(4) For the family $\mathcal{T}$ and the ordinal number we set $\alpha = \beta + 1$ $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent $\Sigma(\mathcal{T})$ sets of φ_n, $n \in \omega$ such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.
(5) If α is a limit ordinal, then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for each $\beta < \alpha$.
(6) Let $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \ngeq \alpha + 1$.
(7) If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we set $\mathrm{RS}(\mathcal{T}) = \infty$.
A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

If $\mathcal{T}$ is e-totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

Proposition 3. [4] *$\mathcal{T}$ is e-minimal $\Leftrightarrow$ $RS(\mathcal{T}) = 1$ and $ds(\mathcal{T}) = 1$*

Definition 4. [4] A family $\mathcal{T}$, with infinitely many accumulation points, is called *a-minimal* if for any sentence $\varphi \in \Sigma(T)$, $\mathcal{T}_\varphi$ or $\mathcal{T}_{\neg\varphi}$ has finitely many accumulation points.

Let α be an ordinal. A family $\mathcal{T}$ of rank α is called *α-minimal* if for any sentence $\varphi \in \Sigma(T)$, $\mathrm{RS}(\mathcal{T}_\varphi) < \alpha$ or $\mathrm{RS}(\mathcal{T}_{\neg\varphi}) < \alpha$.

Proposition 4. [4] (1) *A family $\mathcal{T}$ is* 0*-minimal $\Leftrightarrow$ $\mathcal{T}$ is a singleton.*

(2) *A family* $\mathcal{T}$ *is* 1*-minimal* $\Leftrightarrow$ $\mathcal{T}$ *is e-minimal.*
(3) *A family* $\mathcal{T}$ *is* 2*-minimal* $\Leftrightarrow$ $\mathcal{T}$ *is a-minimal.*
(4) *For any ordinal* α *a family* $\mathcal{T}$ *is* α*-minimal* $\Leftrightarrow$ $\mathrm{RS}(\mathcal{T}) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}) = 1$.

2 Ranks for families of cubic theories

Consider a language Σ composed of $R^{(2)}$. Let $\mathcal{T}_{cub}$ be the family of all cubic theories of Σ. Let T be a cubic theory and $\mathcal{Q} \models T$. For a cubic theory T we consider the above invariants and the following possibilities:

2.1 Family of cubic theories with a bounded number of $Inv_T(n)$

If for each theory T from the subfamily $\mathcal{T} \subset \mathcal{T}_{cub}$ both diameters $d(T)$ and $Inv_T(n)$ are finite, and also $Inv_T(\infty) = 0$ or $Supp(T)$ is finite, the subfamily $\mathcal{T}$ is finite, so $RS(\mathcal{T}) = 0$, and the degree of $ds(\mathcal{T})$ is equal to the number of invariants. Let's illustrate how the grades of families differ.

Example 1. Now we consider a one-element family $\mathcal{T} = \{T_1\}$. If we consider n_0-cubes with invariant $Inv_{T_1}(n_0) = m$, then $RS(\mathcal{T}) = 0$, $ds(\mathcal{T}) = 1$. And if we work with n_0-cubes and n_1-cubes with $Inv_{T_1}(n_0) = m$ and $Inv_{T_1}(n_1) = l$ for $m \neq l$, then $ds(\mathcal{T}) = 2$. For a finite number k, if we are dealing with n_k-cubes with the set of invariants $\{Inv_{T_1}(n_0), \dots, Inv_{T_1}(n_k)\}$, $n_i \neq n_j$, we still have $RS(\mathcal{T}) = 0$ and degree $ds(\mathcal{T}) = k + 1$.

Example 2. Let us deal with the finite family $\mathcal{T} \subset \mathcal{T}_{cub}$ consisting of theories $T_1, \dots, T_n$. If the number of m_i-cubes in each theory T_i is equal to k, in other words, each theory has the same number of m_i-cubes, that is, $Inv_{T_i}(m_i) = k$ with $Inv_{T_i}(m_i) \neq Inv_{T_j}(m_j)$, $i \neq j$, then $RS(\mathcal{T}) = 0$, $ds(\mathcal{T}) = n$, since $\mathcal{T}$ is represented as a disjoint union of finite subfamilies $\mathcal{T}_{\varphi_i} = \{T_i \in \mathcal{T} | \varphi_i \in T_i$ is a sentence describing m_i-cubes $\}$.

In the examples above, one can notice that the degree of the family depends on the number of invariants. If for the theories considered in Example 2 we add the conditions that each theory has the same number of invariants, let, for example, s, then $ds(\mathcal{T}) = n \cdot s$. And if for different $s_1, \dots, s_n$, in each theory T_i there are s_i invariants, then $ds(\mathcal{T}) = \sum_{i=1}^{n} s_i$.

For a family $\mathcal{T} \subset \mathcal{T}_{cub}$ such that $Inv_T(\infty) = 0$ and $Supp(T)$ is finite for every theory $T \in \mathcal{T}$, the degree varies in a similar way.

Let us now consider infinite subfamilies $\mathcal{T} \subset \mathcal{T}_{cub}$ of all cubic theories with a bounded number of $Inv_T(n) = \infty$ and $Inv_T(\infty) = 0$ for every $T \in \mathcal{T}$. In this case, $Supp(T)$ is infinite and the rank of the family increases, and for the degree of the family, we consider the number of accumulation points.

For natural numbers $n, m \in \omega$, with $n \neq m$, we denote by $\mathcal{T}_n$ the family of cubic theories from $\mathcal{T}_{cub}$ with one arbitrary value $Inv_T(n)$, where $T \in \mathcal{T}_n$ and $Inv_T(m) = 0$.

Proposition 5. Each subfamily $\mathcal{T}_n$ *of* $\mathcal{T}_{cub}$ *is e-minimal.*

Proof. By Proposition 3, it suffices to prove that $RS(\mathcal{T}_n) = 1$ and $ds(\mathcal{T}_n) = 1$. The family $\mathcal{T}_n$ consists of theories $T_1, \dots, T_s$ with $Inv_{T_i}(n) = k_i$, $k_i > 0$ $1 \leq i \leq s$ and the only theory T_∞ with $Inv_{T_\infty}(n) = \infty$. The theory T_∞ is the only accumulation point for $\mathcal{T}_n$, and the number of accumulation points is equal to the degree of the family. We get $RS(\mathcal{T}_n) = 1$ and $ds(\mathcal{T}_n) = 1$, which implies an e-minimality of $\mathcal{T}_n$.

Example 3. We are dealing with cubes of different sizes n_0 and n_1. Then we get a countable number of options $(Inv_T(n_0), Inv_T(n_1))$. Thus there is a countable set of theories with n_0-cubes and n_1-cubes forming the family $\mathcal{T}'$. Here every family with an infinite $Inv_T(n_0)$ or $Inv_T(n_1)$ has $RS = 1$, and the only accumulation point with $Inv_T(n_0) = Inv_T(n_1) = \infty$, has infinitely many n_0 cubes, infinitely many n_1 cubes and $RS(\mathcal{T}') = 2$. Thus for the given family $\mathcal{T}'$ $RS(\mathcal{T}') = 2$ and $ds(\mathcal{T}') = 1$. Hence the family is a-minimal.

Example 4. If there exists a countable number of n_i-cubes, $i \in \omega$, with countable $(Inv_T(n_0), Inv_T(n_1), \ldots, Inv_T(n_i))$ one can construct an α-minimal family $\mathcal{T}$ consisting of a countable number of e-minimal subfamilies $\mathcal{T}_i$, $i \in \omega$. According to the definition of α-minimality, the family $\mathcal{T}$ has $RS(\mathcal{T}) = \alpha$, $ds(\mathcal{T}) = 1$ and is represented as $\mathcal{T}_{\bigwedge_{i\in\omega}\varphi_i}$.

So by increasing the number of $Inv_T(n)$ invariants and the dimension of the cubes, one can unlimited increase rank to any natural number. If the set $Inv_T(n)$ is countable, then the family $\mathcal{T} \subset \mathcal{T}_{cub}$ of cubic theories is e-totally transcendental and can contain e-minimal, a-minimal, α-minimal subfamilies.

Realizations of e-minimal, a-minimal, α-minimal subfamilies of the family $\mathcal{T}_{cub}$ of all cubic theories allow one to construct a subfamily $\mathcal{T}$ with a given countable rank and degree. According to the definition of α-minimality, a family of $\mathcal{T}$ cubic theories with $RS(\mathcal{T}) = \alpha$ and $ds(\mathcal{T}) = n$ can be represented as a disjoint union of subfamilies $\mathcal{T}_{Inv_T(k_0)}, \ldots, \mathcal{T}_{Inv_T(k_{n-1})}$, somewhat differently $Inv_T(k_0), \ldots, Inv_T(k_{n-1})$, so every $\mathcal{T}_{Inv_T(k_i)}$ is α-minimal.

2.2 Family of cubic theories with an unbounded number of $Inv_T(n)$

The next result shows that the family $\mathcal{T}_{cub}$ of all cubic theories is not e-totally transcendental.

Theorem 1. $RS(\mathcal{T}_{cub}) = \infty$.

Proof. Repeating the arguments of [1; Proposition 4.4] and [8; Proposition 2.5] we can construct a 2-tree of sentences φ, φ_0, φ_1, φ_{01}, ... indicating an infinite rank.

3 Approximations of cubic theories

The following theorem shows that any cubic theory is approximated by theories of finite cubic structures.

Theorem 2. Any cubic theory T with an infinite model is pseudofinite.

Proof. Let $\mathcal{Q}$ be an infinite model of a cubic theory T. Since for finite k and n, $Inv_T(n) = k$ and $Inv_T(\infty) = 0$, the cubic model $\mathcal{Q}$ is finite and consists of a finite number of finite connected components (n-cubes), we will consider only the following cases:

Case 1. If $Inv_T(n) = \infty$ and $Inv_T(\infty) = 0$ (that is, ∞-support is a singleton), then $\mathcal{Q}$ consists of an infinite number of connected components of finite diameters. The $\mathcal{Q}$ model is approximated by the disjoint union $\bigsqcup_{i\in\omega} \mathcal{Q}_i$ of models $\mathcal{Q}_i$, $i \in \omega$ which the connected components are n-cubes. Each such n-cubes are pairwise isomorphic that implies the pseudofiniteness of T.

Case 2. If for finite k and $n \in \omega$, $Inv_T(n) = k$ and $Inv_T(\infty) = 1$, then the theory T has models $\mathcal{Q} = \mathcal{Q}_0 \bigsqcup \mathcal{Q}_1$, where $\mathcal{Q}_0$ is a finite cubic model consisting of $m \leq k$ connected components (n-cubes) of finite diameters, $\mathcal{Q}_1$ is an infinite cubic model consisting of $k - m$ connected components of infinite diameters. Since the components of the model $\mathcal{Q}_0$ do not affect the pseudofiniteness, $\mathcal{Q}_1$ is approximated by increasing the dimension, as well as the diameters of the connected components. Let $\mathcal{Q}'_n$ be a finite model with $k - m$ connected components which are n-cubes. Using $\mathcal{Q}'_i = \mathcal{Q}'_2 \cup \mathcal{Q}'_{i-1}, i > 2$ in the limit, we obtain the desired model $\mathcal{Q}_1$. The set of theories $\{Th(\mathcal{Q}'_i) | i \in \omega\}$ approximate the theory $Th(\mathcal{Q}_1)$ and theories $\{Th(\mathcal{Q}_0 \bigsqcup \mathcal{Q}'_i) | i \in \omega\}$ approximate the T theory.

We can also grow connected components to get a pseudofinite model $\mathcal{Q}'$ with $Inv_T(n) = \infty$ and $Inv_T(\infty) = 1$, having components of both finite and infinite diameters.

Case 3. Let $Inv_T(n) = \infty$ and $Inv_T(\infty) = 1$. Let the cubic model $\mathcal{Q}$ have only an infinite number of connected components of infinite diameters. For the cubic model $\mathcal{Q}$, it is true that $\mathcal{Q} = \bigsqcup_{i\in\omega} \mathcal{Q}'_i$, where $\mathcal{Q}'_i = \mathcal{Q}'_2 \cup \mathcal{Q}'_{i-1}, i > 2$. That is, first we take the finite model and increase the diameters of the connected components, we get a model with a finite number of connected components, each of which is infinite-dimensional cubes, then, increasing the number of the connected components, we get the desired model $\mathcal{Q}$.

4 Futher direction

Recently, various methods similar to the "transfer principle" have been rapidly developing, where one property of the structure or pieces of this structure is satisfied in all infinite structures or in another algebraic structure. Such methods include smoothly approximable structures, holographic structures, almost sure theory, and pseudofinite structures approximable by finite structures. Pseudofinite structures in an explicit form after J. Ax were not studied for a long time. Until the 1990s, only a few results on this topic were obtained, and the very first result is the result of B.I. Zilber [20] asserting that ω-categorical theory is not finitely axiomatizable. At the time, the property of being pseudofinite was not considered particularly important or interesting, but the proof is based on pseudofiniteness.

One of the first results in the theory of classification of pseudofinite structures is the famous theorem of G. Cherlin, L. Harrington and A. Lachlan [21], which generalizes Zilber's theorem to the class of ω-stable ω-categorical structures, stating that totally categorical theories (and in more generally, ω-categorical ω-stable theories) are pseudofinite. They also proved that such structures are smoothly approximated by finite structures.

Definition 5. [22] Let L be a countable language and let $\mathcal{M}$ be a countable and ω-categorical L-structure. L-structure $\mathcal{M}$ (or $Th(\mathcal{M})$) is said to be *smoothly approximable* if there is an ascending chain of finite substructures $A_0 \subseteq A_1 \subseteq \ldots \subseteq \mathcal{M}$ such that $\bigcup_{i\in\omega} A_i = \mathcal{M}$ and for every i, and for every $\bar{a}, \bar{b} \in A_i$ if $tp_{\mathcal{M}}(\bar{a}) = tp_{\mathcal{M}}(\bar{b})$, then there is an automorphism σ of M such that $\sigma(\bar{a}) = \bar{b}$ and $\sigma(A_i) = A_i$, or equivalently, if it is the union of an ω-chain of finite homogeneous substructures; or equivalently, if any sentence in $Th(\mathcal{M})$ is true of some finite homogeneous substructure of $\mathcal{M}$.

A. Lachlan introduced the concept of *smoothly approximable* structures to change the direction of analysis from finite to infinite, that is, to classify large finite structures that appear to be *smooth approximations* to an infinite limit.

Smoothly approximated structures were first examined in generality in [22], subsequently in [23]. The model theory of smoothly approximable structures has been developed very much further by G. Cherlin and E. Hrushovski [18]. The class of smoothly approximable structures is a class of ω-categorical supersimple structures of finite rank which properly contains the class of ω-categorical ω-stable structures (so in particular the totally categorical structures).

Recall [24,25] that a countable model $\mathcal{Q}$ of a theory T is called a *limit model* if $\mathcal{Q}$ is represented as the union of a countable elementary chain of models of the theory T that are prime over tuples, and the model $\mathcal{Q}$ itself is not prime over any tuple. A theory T is called *l-categorical* if T has a unique (up to isomorphism) limit model.

Homogeneity and l-categoricity, as well as the Morley rank for a fixed cubic theory, are studied in [9,10].

Proposition 6. Any model $\mathcal{Q}$ of the l-categorical cubic theory T is smoothly approximable by finite cubic structures.

Proof. The limit model $\mathcal{Q}$ of l-categorical cubic theories T is represented as an ascending chain of finite prime substructures $\mathcal{Q}'_0 \subseteq \mathcal{Q}'_1 \subseteq \ldots \subseteq \mathcal{Q}$ such that $\mathcal{Q} = \bigcup_{i\in\omega} \mathcal{Q}'_i$ and there is an automorphism σ of $\mathcal{Q}$ such that $\sigma(\mathcal{Q}'_i) = \mathcal{Q}'_i$.

Conclusions

In the paper the ranks and degrees for families of cubic theories are described. Several examples of families of finite rank cubic theories are given. It is proved that any cubic theory with an infinite model is pseudofinite.

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Кубтық теориялардың үйiрлерi үшiн рангiлер мен аппроксимациялар

Жұмыста кубтық теориялар үйiрлерiнiң рангтық сипаттамалары, сонымен қатар псевдоақырлы және тегiс аппроксимациялау сияқты кубтық теориялардың жаңа қасиеттерi зерттелген. Кубтық теориялар үйiрiндегi кез келген теория ақырлы құрылым теориясы болып табылатыны немесе ақырлы құрылымдардың теорияларымен аппроксимацияланатыны дәлелдендi. Псевдоақырлылық немесе тегiс аппроксимациялану қасиетi күрделi шексiз құрылымдардың орнына ақырлы объектiлердi зерттеуге немесе керiсiнше қарапайым құрылымдардан күрделi құрылымдарды тудыруға мүмкiндiк бередi.

Кiлт сөздер: теориялар аппроксимациялары, куб, кубтық құрылым, кубтық теория, псевдоақырлы теория, тегiс аппроксимацияланатын құрылым.

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Ранги и аппроксимация для семейств кубических теорий

В работе изучены ранговые характеристики семейств кубических теорий, а также новые свойства кубических теорий, такие как псевдоконечность и гладкая аппроксимируемость. Доказано, что в семействе кубических теорий любая теория является теорией конечной структуры или аппроксимируется теориями конечных структур. Свойство псевдоконечности или гладкой аппроксимируемости позволяет исследовать конечные объекты вместо сложных бесконечных или, наоборот, из простых структур производить более сложные.

Ключевые слова: аппроксимация теории, куб, кубическая структура, кубическая теория, псевдоконечная теория, гладко аппроксимируемая структура.

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Solution of a two-dimensional parabolic model problem in a degenerate angular domain

In this paper, the boundary value problem of heat conduction in a domain was considered, boundary of which changes with time, as well as there is no the problem solution domain at the initial time, that is, it degenerates into a point. To solve the problem, the method of heat potentials was used, which makes it possible to reduce it to a singular Volterra type integral equations of the second kind. The peculiarity of the obtained integral equation is that it fundamentally differs from the classical Volterra integral equations, since the Picard method is not applicable to it and the corresponding homogeneous integral equation has a nonzero solution.

Keywords: heat equation, boundary value problem, degenerate domain, Volterra singular integral equation, regularization.

Introduction

Recently, in connection with the intensive development of modern contact technology and due to the high speed of electrical devices, more reliable measurement of the temperature field of the contact system has become relevant. And, no less important, it is necessary to study the dynamics of its change over time. At the same time, the temperature field of high-current contacts must be studied taking into account the change in the size of the contact area, which changes both due to the action of electrodynamic forces and due to the melting of the contact material at high temperatures.

When the electrodes are opened on the contact surface, the melting temperature is reached and a liquid metal bridge appears between them. As a result of further opening, this bridge is divided into two parts and the contact material is transferred from one electrode to another, that is, bridge erosion occurs, which can eventually disrupt their normal operation.

To solve this kind of heat conduction problems, it is necessary to use generalized heat potentials and further reduce the original boundary value problem to singular Volterra type integral equations. From a mathematical point of view, the peculiarity of the problems under consideration is that, firstly, the domain in which solutions are sought has a moving boundary, and secondly, at the initial moment of time, the contacts are in a closed state and the problem solution domain degenerates into a point [1–14].

The problem considered in this paper is called a model one, since the case is studied when the boundary of the domain in which the solution of the problem is sought moves according to the linear law $x = t$. In the future, it is planned to study this problem in the case when the boundary of the domain will change according to an arbitrary law $x = \gamma(t)$, $\gamma(0) = 0$.

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1 Statement of the boundary value problem

We consider the following two-dimensional boundary value problem in spatial variables in a cone $Q = \left\{(x,y,t)|\ \sqrt{x^2+y^2} < t,\ t>0,\right\}$ with a lateral surface $\Gamma = \left\{(x,y,t)|\ \sqrt{x^2+y^2} = t,\ t>0\right\}$ for the equation

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - a^2\beta\left(\frac{1}{x}\cdot\frac{\partial u}{\partial x} + \frac{1}{y}\cdot\frac{\partial u}{\partial x}\right) \tag{1}$$

with a boundary condition

$$u(x,y,t)|_{\Gamma} = g(x,y,t), \tag{2}$$

where $0<\beta<1$, $g(x,y,t)$ is a given function. It is necessary to find a function $u(x,y,t)$ satisfying the equation (1) in Q and the boundary condition (2).

Such boundary value problems in domains that change with time and degenerate into a point arise, for example: when describing the heat transfer process in a moving medium velocity of which is a function of the coordinates; in mathematical modeling of thermophysical processes in the electric arc of high-current disconnecting devices, while taking into account the effect of contracting the axial section of the arc into a contact spot in the cathode field. They are also relevant in the creation of new technologies in metallurgy, the production of crystals, laser technologies, etc.

Passing in (1),(2) to cylindrical coordinates, in the domain $Q = \{(r,t)|\ 0<r<t,\ t>0\}$, we obtain the following boundary value problem for the axisymmetric case:

$$\frac{\partial u}{\partial t} = a^2\cdot\frac{1-2\beta}{r}\cdot\frac{\partial u}{\partial r} + a^2\cdot\frac{\partial^2 u}{\partial r^2},\quad 0<\beta<1, \tag{3}$$

$$u(r,t)|_{r=0} = g_1(t),\quad t>0, \tag{4}$$

$$u(r,t)|_{r=t} = g_2(t),\quad t>0. \tag{5}$$

2 Representation of a solution of the boundary problem (3)–(5) using heat potentials

The fundamental solution for the equation (3) is the function

$$G(r,\xi,t-\tau) = \frac{1}{2a^2}\cdot\frac{r^\beta\cdot\xi^{1-\beta}}{t-\tau}\cdot\exp\left[-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right]\cdot I_\beta\left(\frac{r\xi}{2a^2(t-\tau)}\right),$$

where ξ is a parameter, $I_\beta(z)$ is the modified Bessel function of order β. We will seek the solution of the problem (3) – (5) as the sum of double layer heat potentials

$$u(r,t) = \int_0^t \left.\frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\right|_{\xi=\tau}\mu(\tau)d\tau + \int_0^t \left.\frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\right|_{\xi=0}\nu(\tau)d\tau, \tag{6}$$

where $\mu(t)$ and $\nu(t)$ are potential densities to be determined.

Let's transform the function (6), for this we calculate the derivative:

$$\frac{\partial G(r,\xi,t-\tau)}{\partial \xi} = \frac{1}{4a^4}\cdot\frac{r^\beta\cdot\xi^{1-\beta}}{(t-\tau)^2}\cdot\exp\left[-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right]\cdot\left\{rI_{\beta-1}\left(\frac{r\xi}{2a^2(t-\tau)}\right) - \xi I_\beta\left(\frac{r\xi}{2a^2(t-\tau)}\right)\right\} +$$
$$+\frac{1}{2a^2}\cdot\frac{r^\beta(1-2\beta)}{(t-\tau)\xi^\beta}\cdot\exp\left[-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right]\cdot I_\beta\left(\frac{r\xi}{2a^2(t-\tau)}\right),$$

where we have used the relation [15; 975]:

$$I'_\beta(z) = I_{\beta-1}(z) - \frac{\beta}{z} I_\beta(z).$$

Next we find

$$\left.\frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\right|_{\xi=0} = \frac{1}{(2a^2)^{\beta+1}} \cdot \frac{r^{2\beta}}{2^\beta (t-\tau)^{\beta+1}} \cdot \frac{1}{\beta\Gamma(\beta)} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] \tag{7}$$

and

$$\left.\frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\right|_{\xi=\tau} =$$

$$= \frac{1}{4a^4} \cdot \frac{r^\beta \cdot \tau^{1-\beta}}{(t-\tau)^2} \cdot \exp\left[-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right] \cdot \left\{ r I_{\beta-1}\left(\frac{r\tau}{2a^2(t-\tau)}\right) - \tau I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right)\right\} +$$
$$+\frac{1}{2a^2} \cdot \frac{r^\beta(1-2\beta)}{(t-\tau)\tau^\beta} \cdot \exp\left[-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right).$$

We transform the last equality as follows:

$$\left.\frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\right|_{\xi=\tau} =$$

$$= \frac{r^\beta \tau^{1-\beta}(r-\tau)}{4a^4(t-\tau)^2} \cdot \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \cdot \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right) +$$
$$+\frac{r^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2} \cdot \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right) +$$
$$+\frac{r^\beta(1-2\beta)}{2a^2(t-\tau)\tau^\beta} \cdot \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \cdot \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right), \tag{8}$$

where we introduced the notation

$$I_{\beta-1,\beta}(z) = I_{\beta-1}(z) - I_\beta(z).$$

We substitute the obtained relations (7), (8) into the equality (6), and then we obtain the integral representation of the solution for the equation (9):

$$u(r,t) = \int_0^t \left\{ \frac{r^\beta \tau^{1-\beta}(r-\tau)}{4a^4(t-\tau)^2} \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right) +\right.$$
$$+\frac{r^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2} \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right) +$$
$$\left. + \frac{r^\beta(1-2\beta)}{2a^2(t-\tau)\tau^\beta} \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right)\right\} \mu(\tau)d\tau +$$
$$+\int_0^t \frac{1}{(2a^2)^{\beta+1}} \cdot \frac{r^{2\beta}}{2^\beta(t-\tau)^{\beta+1}} \cdot \frac{1}{\beta\Gamma(\beta)} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] \cdot \nu(\tau)d\tau, \tag{9}$$

where

$$t^{-\beta} e^{\frac{t}{4a^2}} \mu(t) \in L_\infty(0,\infty).$$

3 Reduction of the boundary value problem (3) – (5) to a singular Volterra type integral equation

We require that the function $u(r,t)$ defined by the equality (9) satisfy the boundary conditions (4),(5), which will allow us to define the functions $\mu(t)$ and $\nu(t)$.

$$\lim_{r\to 0} u(r,t) = \lim_{r\to 0}\left[\int_0^t \left\{ \frac{r^{\beta}\tau^{1-\beta}(r-\tau)}{4a^4(t-\tau)^2} \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_{\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right) + \right.\right.$$

$$+\frac{r^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2} \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right) +$$

$$\left. + \frac{r^{\beta}(1-2\beta)}{2a^2(t-\tau)\tau^{\beta}} \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] \cdot I_{\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right)\right\}\mu(\tau)d\tau +$$

$$\left. + \int_0^t \frac{1}{(2a^2)^{\beta+1}} \cdot \frac{r^{2\beta}}{2^{\beta}(t-\tau)^{\beta+1}} \cdot \frac{1}{\beta\Gamma(\beta)} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] \cdot \nu(\tau)d\tau \right] =$$

$$= \frac{1}{(2a^2)^{\beta+1}} \cdot \frac{1}{2^{\beta}} \cdot \frac{1}{\beta\Gamma(\beta)} \cdot \lim_{r\to 0}\int_0^t \frac{r^{2\beta}}{(t-\tau)^{\beta+1}} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] \cdot \nu(\tau)d\tau =$$

$$= \left\| \frac{r^2}{4a^2(t-\tau)} = z \right\| = \frac{1}{(2a^2t)^{\beta+1}} \cdot \frac{1}{2^{\beta}} \cdot \frac{1}{\beta\Gamma(\beta)} \times$$

$$\times \lim_{r\to 0}\int_{\frac{r^2}{4a^2t}}^{\infty} \frac{r^{2\beta}\cdot(4a^2)^{\beta+1}\cdot z^{\beta+1}}{r^{2\beta+2}} \cdot \frac{r^2}{4a^2z^2} \cdot e^{-z} \cdot \nu\left(t - \frac{r^2}{4a^2z}\right) dz =$$

$$= \frac{1}{(2a^2)^{\beta+1}} \cdot \frac{1}{2^{\beta}} \cdot \frac{1}{\beta\Gamma(\beta)} \cdot \frac{(4a^2)^{\beta+1}}{4a^2} \cdot \lim_{r\to 0}\int_{\frac{r^2}{4a^2t}}^{\infty} z^{\beta-1} \cdot e^{-z} \cdot \nu\left(t - \frac{r^2}{4a^2z}\right) dz =$$

$$= \frac{1}{2a^2} \cdot \frac{1}{\beta\Gamma t(\beta)} \cdot \nu(t) \cdot \int_0^{\infty} z^{\beta-1} \cdot e^{-z} dz = \frac{1}{2a^2} \cdot \frac{1}{\beta\Gamma(\beta)} \cdot \Gamma(\beta) \cdot \psi(t) = \frac{1}{2a^2\beta} \cdot \nu(t) = g_1(t).$$

from here one of the sought-for densities $\nu(t)$ is directly determined

$$\nu(t) = 2a^2\beta g_1(t).$$

Therefore,

$$u(r,t) = \sum_{i=1}^{3} u_1(r,t) + \widetilde{g_1}(r,t), \tag{10}$$

where

$$u_1(r,t) = \int_0^t \frac{r^{\beta}\tau^{1-\beta}(r-\tau)}{4a^4(t-\tau)^2} \cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}} \cdot e^{-\frac{r\tau}{2a^2(t-\tau)}} \cdot I_{\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$u_2(r,t) = \int_0^t \frac{r^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2} \cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}} \cdot e^{-\frac{r\tau}{2a^2(t-\tau)}} \cdot I_{\beta-1,\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$u_3(r,t) = \int_0^t \frac{r^{\beta}(1-2\beta)}{2a^2(t-\tau)\tau^{\beta}} \cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}} \cdot e^{-\frac{r\tau}{2a^2(t-\tau)}} \cdot I_{\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$\widetilde{g_1}(r,t) = \frac{1}{(2a^2)^{\beta}} \cdot \frac{1}{2^{\beta}} \cdot \frac{1}{\Gamma(\beta)} \int_0^t \frac{r^{2\beta}}{(t-\tau)^{\beta+1}} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] \cdot g_1(\tau)d\tau.$$

Remark 1. If $g_1(t)$ is bounded, then $\widetilde{g_1}(r,t)$ is also bounded.

Indeed,

$$\widetilde{g_1}(r,t) \leq \frac{1}{(2a^2)^\beta} \cdot \frac{1}{2^\beta} \cdot \frac{1}{\Gamma(\beta)} \cdot |g_1(t)| \int_0^t \frac{r^{2\beta}}{(t-\tau)^{\beta+1}} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] d\tau =$$

$$= \left\| \frac{r^2}{4a^2(t-\tau)} = z \right\| = \frac{1}{(2a^2)^\beta} \cdot \frac{1}{2^\beta} \cdot \frac{1}{\Gamma(\beta)} \cdot 4^\beta \cdot a^{2\beta} \cdot |g_1(t)| \int_{\frac{r^2}{4a^2 t}}^{\infty} z^{\beta-1} \cdot e^{-z} dz =$$

$$= |g_1(t)| \cdot \frac{\Gamma\left(\beta, \frac{r^2}{4a^2 t}\right)}{\Gamma(\beta)} < |g_1(t)|, \qquad \forall (r,t) \in G.$$

Now let us satisfy the boundary condition (5).

$$u(r,t)|_{r=t} = \lim_{r\to t-0} u(r,t) = g_2(t) = \widetilde{g_1}(t,t) +$$

$$+ \int_0^t \left\{ \frac{t^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2} \exp\left[-\frac{t-\tau}{4a^2}\right] \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{t\tau}{2a^2(t-\tau)}\right) + \right.$$

$$\left. + \frac{t^\beta(1-2\beta)}{2a^2(t-\tau)\tau^\beta} \exp\left[-\frac{t-\tau}{4a^2}\right] \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right) \right\} \mu(\tau) d\tau - \frac{\mu(t)}{2a^2}.$$

As a result, we obtain the following integral equation for the unknown density $\mu(t)$:

$$\mu(t) - \int_0^t \left\{ \frac{t^\beta(1-2\beta)}{(t-\tau)\tau^\beta} \exp\left[-\frac{t-\tau}{4a^2}\right] \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right) + \right.$$

$$+ \frac{t^{\beta+1}\tau^{1-\beta}}{2a^2(t-\tau)^2} \exp\left[-\frac{t-\tau}{4a^2}\right] \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{t\tau}{2a^2(t-\tau)}\right) +$$

$$\left. + \frac{t^\beta \tau^{1-\beta}}{2a^2(t-\tau)} \exp\left[-\frac{t-\tau}{4a^2}\right] \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right) \right\} \mu(\tau) d\tau = F(t). \tag{11}$$

where

$$F(t) = -2a^2 g_2(t) + 2a^2 \widetilde{g_1}(t,t).$$

We introduce the following notation

$$t^{1-\beta} \exp\left[\frac{t}{4a^2}\right] \mu(t) = \mu_1(t), \quad t^{1-\beta} \exp\left[\frac{t}{4a^2}\right] F(t) = F_1(t).$$

Then the last integral equation is transformed into the following equation:

$$\mu_1(t) - \int_0^t N(t,\tau)\mu_1(\tau) d\tau = F_1(t), \tag{12}$$

kernel of which has the form:

$$N(t,\tau) = \sum_{i=1}^{2} N_i(t,\tau),$$

and, moreover,

$$N_1(t,\tau) = \frac{t(1-2\beta)}{\tau(t-\tau)} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right) +$$

$$+ \frac{t^2}{2a^2(t-\tau)^2} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{t\tau}{2a^2(t-\tau)}\right),$$

$$N_2(t,\tau) = \frac{t}{2a^2(t-\tau)} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right).$$

A feature of this integral equation follows from

Remark 2. The kernel of the integral equation (11) satisfies the equality

$$\lim_{t\to 0}\int_0^t N(t,\tau)d\tau = \frac{1-\beta}{\beta},$$

moreover, $\forall t > 0$, $\forall \beta \in (0;1)$:

$$\int_0^t N_1(t,\tau)d\tau = \frac{1-\beta}{\beta}, \qquad \lim_{t\to 0}\int_0^t N_2(t,\tau)d\tau = 0.$$

Indeed,

$$\int_0^t N_1(t,\tau)d\tau = \int_0^t \left\{ \frac{t(1-2\beta)}{\tau(t-\tau)} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right) + \right.$$

$$\left. + \frac{t^2}{2a^2(t-\tau)^2} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{t\tau}{2a^2(t-\tau)}\right) d\tau \right\} = \left\| \frac{t\tau}{2a^2(t-\tau)} = z \right\| =$$

$$= \int_0^\infty (1-2\beta)\cdot\frac{1}{z^2}\cdot z\cdot e^{-z}\cdot I_\beta(z)dz + \int_0^\infty e^{-z}\cdot\left\{I_{\beta-1}(z) - I_\beta(z)\right\}dz =$$

$$= (1-2\beta)\int_0^\infty \frac{1}{z}\cdot e^{-z}\cdot I_\beta(z)dz + 1 = \|(2.15.4.3)\ [16;272]\| =$$

$$= \frac{(1-2\beta)}{\sqrt{\pi}}\cdot\Gamma\left[\begin{array}{c} \beta, \ \ \frac{1}{2} \\ 1+\beta \end{array}\right] + 1 = \frac{1-\beta}{\beta};$$

$$\int_0^t N_2(t,\tau)d\tau = \int_0^t \frac{t}{2a^2(t-\tau)} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right) d\tau =$$

$$= \left\| \frac{t\tau}{2a^2(t-\tau)} = z \right\| = \int_0^\infty \frac{t}{t+2a^2z}\cdot e^{-z}\cdot I_\beta(z)dz \le \frac{t}{2a^2}\int_0^\infty \frac{1}{z}\cdot e^{-z}\cdot I_\beta(z)dz =$$

$$= \frac{t}{2a^2}\cdot\frac{1}{\sqrt{\pi}}\cdot\Gamma\left[\begin{array}{c} \beta, \ \ \frac{1}{2} \\ 1+\beta \end{array}\right] = \frac{t}{2a^2\beta}\underset{t\to 0}{\longrightarrow} 0.$$

4 Solution of the characteristic integral equation

In order to find a solution of the integral equation (10), we first study the following characteristic integral equation:

$$\mu_1(t) - \int_0^t N_1(t,\tau)\mu_1(\tau)d\tau = \Phi(t). \tag{13}$$

Remark 3. Remark (12) implies that for $\frac{1}{2} < \beta < 1$, $\left(0 < \frac{1-\beta}{\beta} < 1\right)$ the integral equation (13) in the class of essentially bounded functions has a unique solution that can be found by the method of successive approximations.

By Remark 2 for $0 < \beta \le \frac{1}{2}$ $\left(\frac{1-\beta}{\beta} \ge 1\right)$ equation (13) is indeed characteristic for the equation (11). Instead of the variables t, τ we introduce new variables x, y:

$$t = \frac{1}{y}, \quad \tau = \frac{1}{x}; \qquad \mu_1(t) = \mu_1\left(\frac{1}{y}\right) = \mu_2(y), \qquad \Phi(t) = \Phi\left(\frac{1}{y}\right) = \Phi_1(y), \tag{14}$$

Then the equation (13) reduces to the following integral equation with respect to the unknown function $\mu_2(y)$:

$$\mu_2(y) - \int_y^{\infty} M_-(y-x)\mu_2(x)dx = \Phi_1(y), \tag{15}$$

where

$$M_-(y-x) = \frac{1-2\beta}{x-y} \cdot \exp\left(-\frac{1}{2a^2(x-y)}\right) \cdot I_\beta\left(\frac{1}{2a^2(x-y)}\right) +$$
$$+\frac{1}{2a^2(x-y)^2} \cdot \exp\left(-\frac{1}{2a^2(x-y)}\right) \cdot I_{\beta-1,\beta}\left(\frac{1}{2a^2(x-y)}\right).$$

Remark 4. If we find a solution to the equation (13), then we will obtain a solution to the equation (11) by applying the equivalent regularization method to the solution of the characteristic equation [17, 18].

5 Solution of the homogeneous characteristic equation

The equation (15) differs fundamentally from the Volterra equations of the second kind, for which the solution exists and is unique. The solution of the corresponding homogeneous equation

$$\mu_2(y) - \int_y^{\infty} M_-(y-x)\mu_2(x)dx = 0, \tag{16}$$

in the general case may also be non-trivial. The eigenfunctions of the integral equation (16) are determined by the roots of the following transcendental equation [18; 569] with respect to the parameter p:

$$\widehat{M_-}(-p) = \int_0^{\infty} M_-(z) \cdot e^{pz} dz = 1, \quad \mathrm{Re}p < 0, \tag{17}$$

since, by applying the Laplace transform to the equation (16), we obtain

$$\widehat{\mu_2}(p) \cdot \left[1 - \widehat{M_-}(-p)\right] = 0, \quad \mathrm{Re}p < 0. \tag{18}$$

In order to find the image of the function $\widehat{M_-}(-p)$ we use:
1) the formula (29.169) [19; 350];
2) the property: let $f(t) \risingdotseq \hat{f}(p)$, then $\frac{1}{t}f(t) \risingdotseq \int_p^{\infty} \hat{f}(p)dp$ [20; 506]. Thus, we have

$$\widehat{M_-}(-p) = 2(1-2\beta)K_\beta\left(\frac{\sqrt{-p}}{a}\right)I_\beta\left(\frac{\sqrt{-p}}{a}\right) +$$
$$+\frac{1}{a^2}\int_{-\infty}^{p}\left[K_{\beta-1}\left(\frac{\sqrt{-q}}{a}\right)I_{\beta-1}\left(\frac{\sqrt{-q}}{a}\right) - K_\beta\left(\frac{\sqrt{-q}}{a}\right)I_\beta\left(\frac{\sqrt{-q}}{a}\right)\right]dq.$$

To calculate the last integral, we use the formula (1.12.4.3) [16; 44]:

$$\frac{1}{a^2}\int_{-\infty}^{p}\left[K_{\beta-1}\left(\frac{\sqrt{-q}}{a}\right)I_{\beta-1}\left(\frac{\sqrt{-q}}{a}\right) - K_\beta\left(\frac{\sqrt{-q}}{a}\right)I_\beta\left(\frac{\sqrt{-q}}{a}\right)\right]dq =$$
$$= \left\|\frac{\sqrt{-q}}{a} = z\right\| = 2\int_{\frac{\sqrt{-p}}{a}}^{\infty} z\left[K_{\beta-1}(z)I_{\beta-1}(z) - K_\beta(z)I_\beta(z)\right]dz =$$
$$= z^2\left[\left(1 + \frac{(\beta-1)^2}{z^2}\right)I_{\beta-1}(z)K_{\beta-1}(z) - I'_{\beta-1}(z)K'_{\beta-1}(z)\right]\Bigg|_{\frac{\sqrt{-p}}{a}}^{\infty} -$$

$$- z^2 \left[\left(1 + \frac{\beta^2}{z^2} \right) I_\beta(z) K_\beta(z) - I'_\beta(z) K'_\beta(z) \right] \Big|_{\frac{\sqrt{-p}}{a}}^{\infty} =$$

$$= z^2 \left[\left(1 + \frac{(\beta-1)^2}{z^2} \right) I_{\beta-1}(z) K_{\beta-1}(z) - \right.$$

$$\left. - \left\{ I_\beta(z) + \frac{\beta-1}{z} I_{\beta-1}(z) \right\} \left\{ -K_\beta(z) + \frac{\beta-1}{z} K_{\beta-1}(z) \right\} \right] \Big|_{\frac{\sqrt{-p}}{a}}^{\infty} -$$

$$- z^2 \left[\left(1 + \frac{\beta^2}{z^2} \right) I_\beta(z) K_\beta(z) - \left\{ I_{\beta-1}(z) - \frac{\beta}{z} I_\beta(z) \right\} \left\{ -K_{\beta-1}(z) - \frac{\beta}{z} K_\beta(z) \right\} \right] \Big|_{\frac{\sqrt{-p}}{a}}^{\infty} =$$

$$= \left[\left(z^2 + (\beta-1)^2 \right) I_{\beta-1}(z) K_{\beta-1}(z) + z^2 I_\beta(z) K_\beta(z) - \right.$$

$$\left. - z\ (\beta-1) I_\beta(z) K_{\beta-1}(z) + z(\beta-1) I_\beta(z) K_\beta(z) - (\beta-1)^2 I_{\beta-1}(z) K_{\beta-1}(z) \right] \Big|_{\frac{\sqrt{-p}}{a}}^{\infty} -$$

$$- \left[(z^2+\beta^2) I_\beta(z) K_\beta(z) + z^2 I_{\beta-1}(z) K_{\beta-1}(z) + z\beta I_{\beta-1}(z) K_\beta(z) - \right.$$

$$\left. - z\beta I_\beta(z) K_{\beta-1}(z) - \beta^2 I_\beta(z) K_\beta(z) \right] \Big|_{\frac{\sqrt{-p}}{a}}^{\infty} =$$

$$= \left[z I_\beta(z) K_{\beta-1}(z) - z I_{\beta-1}(z) K_\beta(z) \right] \Big|_{\frac{\sqrt{-p}}{a}}^{\infty} =$$

$$= \left[2z I_\beta(z) K_{\beta-1}(z) - \left(z I_\beta(z) K_{\beta-1}(z) + z I_{\beta-1}(z) K_\beta(z) \right) \right] \Big|_{\frac{\sqrt{-p}}{a}}^{\infty} =$$

$$= 2z I_\beta(z) K_{\beta-1}(z) \Big|_{\frac{\sqrt{-p}}{a}}^{\infty} = 1 - 2\frac{\sqrt{-p}}{a} I_\beta \left(\frac{\sqrt{-p}}{a} \right) K_{\beta-1} \left(\frac{\sqrt{-p}}{a} \right),$$

where we used the following relations:

$$K'_\beta(z) = -K_{\beta-1}(z) - \frac{\beta}{z} K_\beta(z),$$

$$K'_{\beta-1}(z) = -K_\beta(z) + \frac{\beta-1}{z} K_{\beta-1}(z),$$

$$I'_\beta(z) = I_{\beta-1}(z) - \frac{\beta}{z} I_\beta(z),$$

$$I'_{\beta-1}(z) = I_\beta(z) + \frac{\beta-1}{z} I_{\beta-1}(z).$$

Then the equation (17) will take the form:

$$2 I_\beta \left(\frac{\sqrt{-p}}{a} \right) \left[(1-2\beta) K_\beta \left(\frac{\sqrt{-p}}{a} \right) - \frac{\sqrt{-p}}{a} K_{\beta-1} \left(\frac{\sqrt{-p}}{a} \right) \right] = 0, \qquad \mathrm{Re} p < 0,$$

where $K_\beta \left(\frac{\sqrt{-p}}{a} \right)$ is the Macdonald function.

Let's assume that $I_\beta \left(\frac{\sqrt{-p}}{a} \right) = 0$. According to the definition of the Bessel function for the imaginary argument $I_\beta \left(\frac{\sqrt{-p}}{a} \right) = e^{-\frac{\pi}{2}\beta i} J_\beta \left(\frac{i\sqrt{-p}}{a} \right)$, where $J_\beta \left(\frac{i\sqrt{-p}}{a} \right)$ is the Bessel function – cylinder function of the first kind. The function $J_\beta(z)$ for any real β has an infinite set of real roots; for $\beta > -1$ all its roots are real and equal $iz_k = \alpha_k$, $z_k = -i\alpha_k$, $\alpha_k \in \mathbb{R}$, $k \in \mathbb{Z}\backslash\{0\}$ [21], i.e. in our case $\frac{i\sqrt{-p_k}}{a} = \alpha_k$, where $\alpha_k \in \mathbb{R}$. Hence $p_k = a^2 \alpha_k^2$, which contradicts the condition $\mathrm{Re} p < 0$.

Thus, it is necessary to find the roots of the equation for $0 < \beta \leq \frac{1}{2}$

$$(1-2\beta)K_\beta\left(\frac{\sqrt{-p}}{a}\right)-\frac{\sqrt{-p}}{a}K_{\beta-1}\left(\frac{\sqrt{-p}}{a}\right)=0,\quad \mathrm{Re}p<0. \tag{19}$$

It should be noted that for $\frac{1}{2}<\beta<1$ this equation has no roots. This means that in the equation (18)

$$1-\widehat{M_-}(-p)\neq 0,$$

whence it follows that $\widehat{\mu_2}(p)=0$. That is, the homogeneous integral equation (16) has only a zero solution in this case. For $0<\beta\leq\frac{1}{2}$ the equation (19) has a unique real root $p_0\leq 0$, and the root $p_0=0$ corresponds to the case $\beta=\frac{1}{2}$. And for $0<\beta<\frac{1}{2}$ the root is $p_0<0$. This means that the equation (16) for $0<\beta<\frac{1}{2}$ has a non-zero solution $\mu_2(y)=Ce^{p_0 y}$, $p_0<0$. Then, returning to the original variables (14), we obtain that the homogeneous integral equation corresponding to the equation (13), for $0<\beta<\frac{1}{2}$ has an eigenfunction

$$\mu^{(0)}(t)=C\cdot\frac{1}{t^{1-\beta}}\cdot e^{\frac{p_0}{t}-\frac{t}{4a^2}},\quad p_0<0,\quad C=\mathrm{const}.$$

Accordingly, for $\beta=\frac{1}{2}$, the eigenfunction has the form:

$$\mu^{(0)}(t)=C\cdot\frac{1}{\sqrt{t}}\cdot e^{-\frac{t}{4a^2}},\qquad C=\mathrm{const}.$$

6 *Solution of an inhomogeneous characteristic integral equation. Construction of the resolvent.*

The equation (15) cannot be solved by directly applying the Laplace transform, since the convolution theorem is not applicable here. Let's apply the method of model solutions [18; 561]. Then the solution of the equation (15) has the form

$$\varphi_1(y)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\widehat{\Phi_1}(p)}{1-\widehat{M_-}(-p)}dp=\Phi_1(y)+\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\widehat{R_-}(-p)\widehat{\Phi_1}(p)e^{pz}dp,$$

where

$$\widehat{\Phi_1}(p)=\int_0^\infty\Phi_1(y)e^{-py}dy,\quad \widehat{R_-}(-p)=\frac{\widehat{M_-}(-p)}{1-\widehat{M_-}(-p)},\quad \mathrm{Re}p<0;$$

$$\widehat{M_-}(-p)=1+2I_\beta\left(\frac{\sqrt{-p}}{a}\right)\left[(1-2\beta)K_\beta\left(\frac{\sqrt{-p}}{a}\right)-\frac{\sqrt{-p}}{a}K_{\beta-1}\left(\frac{\sqrt{-p}}{a}\right)\right],\ \mathrm{Re}p<0.$$

If $\widehat{R_-}(-p)\risingdotseq R_-(y)$, then the solution of the equation (15) has the form

$$\varphi_1(y)=\Phi_1(y)+\frac{1}{2\pi i}\int_y^\infty R_-(y-x)\Phi_1(x)dx. \tag{20}$$

To find the resolvent $R_-(y)$, we write its image in the following form:

$$\widehat{R_-}\left(\frac{\sqrt{-p}}{a}\right)=\frac{1-2I_\beta\left(\frac{\sqrt{-p}}{a}\right)\left[\frac{\sqrt{-p}}{a}K_{\beta-1}\left(\frac{\sqrt{-p}}{a}\right)-(1-2\beta)K_\beta\left(\frac{\sqrt{-p}}{a}\right)\right]}{2I_\beta\left(\frac{\sqrt{-p}}{a}\right)\left[\frac{\sqrt{-p}}{a}K_{\beta-1}\left(\frac{\sqrt{-p}}{a}\right)-(1-2\beta)K_\beta\left(\frac{\sqrt{-p}}{a}\right)\right]},\quad \mathrm{Re}p<0,$$

and use the following properties [19; 191]:

1. If $\varphi(t)\risingdotseq\widehat{\varphi}(p)$, then

$$\varphi(\alpha t)\risingdotseq\frac{1}{\alpha}\widehat{\varphi}\left(\frac{p}{\alpha}\right),\ \alpha>0.$$

2. If $\widehat{\varphi}(p) \risingdotseq \varphi(t)$, then

$$\widehat{\varphi}\left(\sqrt{p}\right) = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{t^{\frac{3}{2}}} \int_0^\infty \tau \cdot e^{-\frac{\tau^2}{4t}} \varphi(\tau) d\tau.$$

For convenience, we introduce the notation $\frac{\sqrt{-p}}{a} = z$ and find the original expression

$$\widehat{R^*}(z) = \frac{1 - 2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right]}{2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right]}.$$

According to [20; 519]:

$$\widehat{R^*}(z) = \frac{A(z)}{B(z)} \risingdotseq \sum_{-\infty}^{+\infty} \frac{A(z_k)}{B'(z_k)} \cdot e^{-z_k y},$$

where z_k are zeros of the function

$$B(z) = 2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right].$$

1) Let $y_\beta(z) = zK_{\beta-1}(z) - (1-2\beta)K_\beta(z) = 0$. This equation, as noted earlier, has one root z_0 for $0 < \beta < \frac{1}{2}$.

2) Let $I_\beta(z) = e^{-\frac{\pi}{2}\beta i} J_\beta(iz) = 0$. Therefore, $iz_k = \alpha_k$ or $z_k = -i\alpha_k$, where $\alpha_k \in \mathbb{R}$.

Then

$$\widehat{R^*}(z) = \frac{A(z)}{B(z)} \risingdotseq \sum_{-\infty}^{+\infty} \frac{A(z_k)}{B'(z_k)} \cdot e^{-z_k y} = \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{A(z_k)}{B'(z_k)} \cdot e^{-z_k y} + \frac{A(z_0)}{B'(z_0)} \cdot e^{-z_0 y} = R^*_-(y),$$

where

$$B(z) = 2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right]$$

$$B'(z) = 2I_{\beta-1}(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right] + 2(1-2\beta)I_\beta(z)K_{\beta-1}(z) +$$
$$+\left(\frac{4\beta(1-2\beta)}{z} - 2z\right) I_\beta(z)K_\beta(z).$$

Thus, we obtain that for $0 < \beta < \frac{1}{2}$:

$$R^*_-(y) = \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{e^{-z_k y}}{2I_{\beta-1}(z_k)\left[z_k K_{\beta-1}(z_k) - (1-2\beta)K_\beta(z_k)\right]} +$$
$$+\frac{e^{-z_0 y}}{2I_\beta(z_0)K_{\beta-1}(z_0)\left[1 - \frac{1}{1-2\beta} z_0^2\right]}. \tag{21}$$

We introduce the following notations:

$$A_{\beta,k} = \frac{1}{2I_{\beta-1}(z_k)\left[z_k K_{\beta-1}(z_k) - (1-2\beta)K_\beta(z_k)\right]}, \quad A_{\beta,0} = \frac{1}{2I_\beta(z_0)K_{\beta-1}(z_0)\left[1 - \frac{1}{1-2\beta} z_0^2\right]}.$$

From equality (21) we have

$$\widehat{R_-}\left(\frac{\sqrt{-p}}{a}\right) \risingdotseq R_-(y) = \frac{a^2}{2\sqrt{\pi} y^{\frac{3}{2}}} \cdot \sum_{k \in \mathbb{Z}\setminus\{0\}} A_{\beta,k} \cdot \int_0^\infty x e^{-\frac{x^2}{4y} - ia^2\alpha_k x} dx +$$
$$+\frac{a^2}{2\sqrt{\pi} y^{\frac{3}{2}}} \cdot A_{\beta,0} \cdot \int_0^\infty x e^{-\frac{x^2}{4y} - z_0 a^2 x} dx.$$

Lemma 1. The resolvent $R_-(y)$ satisfies the estimate

$$R_-(y) \le \frac{A}{\sqrt{y}}.$$

Proof.

$$R_-(y) \le \left| \frac{a^2}{2\sqrt{\pi} y^{\frac{3}{2}}} \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k} \int_0^\infty x e^{-\frac{x^2}{4y} - i\alpha_k a^2 x} dx + A_{\beta,0} \int_0^\infty x e^{-\frac{x^2}{4y} - z_0 a^2 x} dx \right) \right| \le$$

$$\le \frac{a^2}{2\sqrt{y}} \cdot \left\{ \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k} \right| + |A_{\beta,0}| \right\}.$$

Since $|A_{\beta,0}| = C_\beta = \text{const}$, we estimate the sum $\left|\sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k}\right|$:

$$\left| \sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k} \right| = \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{2I_{\beta-1}(z_k)\left[z_k K_{\beta-1}(z_k) - (1-2\beta)K_\beta(z_k)\right]} \right| =$$

$$= \left\| \begin{array}{c} K_\beta(z) = \frac{\pi i}{2} e^{\frac{\pi}{2}\beta i} H_\beta^{(1)}(iz); \qquad I_\beta(z) = e^{-\frac{\pi}{2}\beta i} J_\beta(iz); \\ z_k = -i\alpha_k; \qquad z_{-k} = i\alpha_k; \\ J_\beta(-z) = e^{\beta\pi i} J_\beta(z); \qquad H_\beta^{(1)}(-z) = -e^{-\beta\pi i} H_\beta^{(2)}(z) \end{array} \right\| =$$

$$= \frac{1}{\pi} \left| \sum_{k=1}^\infty \left(\frac{1}{J_{\beta-1}(\alpha_k)\left[\alpha_k H_{\beta-1}^{(1)}(\alpha_k) + (1-2\beta) H_\beta^{(1)}(\alpha_k)\right]} - \right.\right.$$

$$\left.\left. - \frac{1}{e^{(\beta-1)\pi i} J_{\beta-1}(\alpha_k)\left[-\alpha_k e^{-(\beta-1)\pi i} H_{\beta-1}^{(2)}(\alpha_k) + (1-2\beta) e^{-\beta\pi i} H_\beta^{(2)}(\alpha_k)\right]} \right) \right| =$$

$$= \left\| \begin{array}{c} H_\beta^{(1)}(z) = J_\beta(z) + iN_\beta(z); \qquad H_\beta^{(2)}(z) = J_\beta(z) - iN_\beta(z) \\ J_\beta(\alpha_k) = 0 \end{array} \right\| =$$

$$= \frac{2}{\pi} \left| \sum_{k=1}^\infty \frac{\alpha_k}{(\alpha_k J_{\beta-1}(\alpha_k))^2 + (\alpha_k N_{\beta-1}(\alpha_k))^2 + 2\alpha_k(1-2\beta) N_{\beta-1}(\alpha_k) N_\beta(\alpha_k) + ((1-2\beta)N_\beta(\alpha_k))^2} \right| \le$$

$$\le \frac{2}{\pi} \left| \sum_{k=1}^\infty \frac{\alpha_k}{(\alpha_k J_{\beta-1}(\alpha_k))^2 + (\alpha_k N_{\beta-1}(\alpha_k))^2} \right| = \frac{2}{\pi} \left| \sum_{k=1}^\infty \frac{1}{\alpha_k H_{\beta-1}^{(1)}(\alpha_k) H_{\beta-1}^{(2)}(\alpha_k)} \right| \le$$

$$\le \frac{2}{\pi} \left| \int_{\alpha_1}^\infty \frac{d(\alpha_n)}{\alpha_k \cdot H_{\beta-1}^{(1)}(\alpha_k) \cdot H_{\beta-1}^{(2)}(\alpha_k)} \right| = \|(1.10.3.3)\ [16; 42]\| =$$

$$= \frac{2}{\pi} \left| -\frac{\pi}{4i} \cdot \ln \frac{H_{\beta-1}^{(2)}(\alpha_k)}{H_{\beta-1\nu}^{(1)}(\alpha_k)} \Bigg|_{\alpha_1}^\infty \right| = \frac{1}{2} \cdot \left| \left\{ \ln \left| \frac{H_{\beta-1}^{(2)}(\alpha_k)}{H_{\beta-1}^{(1)}(\alpha_k)} \right| + i \cdot \arg \frac{H_{\beta-1}^{(2)}(\alpha_k)}{H_{\beta-1}^{(1)}(\alpha_k)} \right\} \Bigg|_{\alpha_1}^\infty \right| =$$

$$= \frac{1}{2} \cdot \left| \arg H_{\beta-1}^{(2)}(\alpha_k) - \arg H_{\beta-1\nu}^{(1)}(\alpha_k) \right| \le \frac{\pi}{2}.$$

Thus, we get

$$R_-(y) \leq \frac{a^2}{2\sqrt{y}} \cdot \left\{ \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k} \right| + |A_{\beta,0}| \right\} \leq \frac{a^2(\pi + 2C_\beta)}{4\sqrt{y}} = \frac{C_\beta{}^{(1)}}{\sqrt{y}}, \; C_\beta{}^{(1)} = \text{const}.$$

$$R_-(y) \leq \frac{a^2}{2\sqrt{y}} \cdot \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k} \right| \leq \frac{a^2\pi}{4\sqrt{y}}.$$

Lemma is proved.

7 Solution of the characteristic equation

We found a solution of the equation

$$\mu_2(y) - \int_y^\infty M_-(y-x)\mu_2(x)dx = \Phi_1(y),$$

which for $0 < \beta < \frac{1}{2}$ has the form

$$\mu_2(y) = \Phi_1(y) + \int_y^\infty R_-(x-y)\Phi_1(x)dx + Ce^{p_0 y}.$$

Returning to the original variables, we write the solution of the characteristic equation (20) as follows:

$$\mu_1(t) = \Phi(t) + \int_0^t \frac{R_-(t,\tau)}{\tau^2}\Phi(\tau)d\tau + Ce^{\frac{p_0}{t}}.$$

For the convergence of the last integral it is necessary that

$$\Phi_2(t) = \frac{1}{t} \cdot \Phi(t) \in L_\infty(0,\infty).$$

Then we write the solution of the characteristic equation (20) as

$$\mu_1(t) = t \cdot \Phi_2(t) + \int_0^t \tilde{R}(t,\tau) \cdot \Phi_2(\tau)d\tau + Ce^{\frac{p_0}{t}},$$

where

$$\tilde{R}(t,\tau) \leq C \cdot \frac{\sqrt{t}}{\sqrt{\tau} \cdot \sqrt{t-\tau}}.$$

The last inequality follows from Lemma 1.

8 Solution of the initial integral equation. The Carleman-Vekua regularization

Theorem 1. Initial integral equation (11) for any function $t^{-\beta}e^{\frac{t}{4a^2}} \cdot F(t) \in L_\infty(0,\infty)$ $\left(0 < \beta < \frac{1}{2}\right)$ has the unique solution in the class of functions

$$t^{-\beta} \exp\left[\frac{t}{4a^2}\right] \mu(t) \in L_\infty(0,\infty), \qquad \left(0 < \beta < \frac{1}{2}\right)$$

which can be found by the method of successive approximations.

Proof. We rewrite the initial integral equation (11) as

$$\mu_1(t) - \int_0^t N_1(t,\tau)\mu_1(\tau)d\tau = F_1(t) + \int_0^t N_2(t,\tau)\mu_1(\tau)d\tau. \tag{22}$$

Assuming the right-hand side of the equation (22) to be temporarily known, we write it in the following form:

$$[1-\mathcal{M}]\mu_2(t) \equiv$$

$$\equiv \mu_2(t) - \int_0^t M(t,\tau)\mu_2(\tau)d\tau = \frac{1}{t}F(t) + \frac{1}{t}\int_0^t \tilde{R}(t,\tau)\cdot\frac{F(\tau)}{\tau}d\tau + \frac{C}{t}e^{\frac{p_0}{t}}, \tag{23}$$

where

$$\mu_2(t) = \frac{1}{t}\mu_1(t), M(t,\tau) = \frac{\tau}{t}N_2(t,\tau) + \frac{\tau}{t}\cdot\int_\tau^t \tilde{R}(t,\xi)\cdot\frac{N_2(\xi,\tau)}{\xi}d\xi.$$

The following estimate for the kernel $M(t,\tau)$

$$M(t,\tau) \leq \frac{\widetilde{D_1}}{\sqrt{t-\tau}} + \widetilde{D_2}, \quad \widetilde{D_1},\ \widetilde{D_2} = \text{const}$$

holds. Thus, we show that equation (23) for each $C \neq 0$ has a unique solution

$$\mu_2(t) = \mu_{2,\text{part}}(t) + C\mu_{2,\text{hom}}(t),$$

where

$$\mu_{2,\text{hom}}(t) = [1-\mathcal{M}]^{-1}\mu^{(0)}(t), \mu_2(t) = t^{-\beta}e^{\frac{t}{4a^2}}\mu(t).$$

At the same time, if $F(t)=0$, then integral equation (23) has a solution $\mu_2^{(0)}(t) = C\cdot[1-\mathcal{M}]^{-1}\mu^{(0)}(t)$. The theorem is proved.

9 *Solution of the boundary value problem (3)–(5)*

Theorem 2. If the conditions $g_1(t) \in L_\infty(0,\infty)$, $t^{-\beta}g_2(t) \in L_\infty(0,\infty)$ $\left(0<\beta<\frac{1}{2}\right)$ are satisfied, then the boundary value problem (3) – (5) has a solution $u(r,t) \in L_\infty(G)$.

Proof. From the integral representation (10) of the boundary value problem (3)–(5) we have

$$u(r,t) = \sum_{i=1}^{3} u_i(r,t) + \widetilde{g_1}(r,t),$$

where

$$u_1(r,t) = \int_0^t \frac{r^\beta \tau^{1-\beta}(r-\tau)}{4a^4(t-\tau)^2}\cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}}\cdot e^{-\frac{r\tau}{2a^2(t-\tau)}}\cdot I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$u_2(r,t) = \int_0^t \frac{r^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2}\cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}}\cdot e^{-\frac{r\tau}{2a^2(t-\tau)}}\cdot I_{\beta-1,\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$u_3(r,t) = \int_0^t \frac{r^\beta(1-2\beta)}{2a^2(t-\tau)\tau^\beta}\cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}}\cdot e^{-\frac{r\tau}{2a^2(t-\tau)}}\cdot I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$\widetilde{g_1}(r,t) = \frac{1}{(2a^2)^\beta}\cdot\frac{1}{2^\beta}\cdot\frac{1}{\Gamma(\beta)}\int_0^t \frac{r^{2\beta}}{(t-\tau)^{\beta+1}}\cdot\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right]\cdot g_1(\tau)d\tau.$$

Let $t^{-\alpha}e^{\frac{t}{4a^2}}\cdot\mu(t)\in L_\infty(0,\infty)$. Let us find out for what values of α the solution of the problem $u(r,t)$ will satisfy the condition $u(r,t)\in L_\infty(G)$. First, we estimate the first term.

$$u_1(r,t)=\int_0^t \frac{r^\beta\tau^{1-\beta}(r-\tau)}{4a^4(t-\tau)^2}\exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right]I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau\le$$

$$=\left\|\begin{array}{c}\exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right]\le\exp\left[-\frac{t-\tau}{4a^2}\right],\\ \frac{r\tau}{2a^2(t-\tau)}=\xi\end{array}\right\|=$$

$$=C_1e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta t^{1-\beta+\alpha}}{(2a^2)^2}\int_0^\infty\frac{\xi^{1-\beta+\alpha}}{\left(\frac{r}{2a^2}+\xi\right)^{2-\beta+\alpha}}\cdot e^{-\xi}I_\beta(\xi)d\xi\le$$

$$\le C_1e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta t^{1-\beta+\alpha}}{(2a^2)^2}\int_0^\infty\frac{1}{\xi}\cdot e^{-\xi}I_\beta(\xi)d\xi=e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta t^{1-\beta+\alpha}}{(2a^2)^2}\cdot\Gamma\left[\begin{array}{cc}\beta, & \frac12\\ 1+\beta & \end{array}\right]=$$

$$=C_1e^{-\frac{t}{4a^2}}\cdot\frac{\sqrt{\pi}t^{1+\alpha}}{(2a^2)^2\beta}\cdot\left(\frac{r}{t}\right)^\beta\le\frac{C_1\sqrt{\pi}}{(2a^2)^2\beta}\cdot e^{-\frac{t}{4a^2}}\cdot t^{1+\alpha}\le\widetilde{C_1}=\text{const}\quad\forall(r,t)\in Q.$$

Now we estimate the second term.

$$u_2(r,t)=\int_0^t\frac{r^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2}\cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}}\cdot e^{-\frac{r\tau}{2a^2(t-\tau)}}\cdot I_{\beta-1,\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau=$$

$$=\left\|\begin{array}{c}\exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right]\le\exp\left[-\frac{t-\tau}{4a^2}\right],\\ \frac{r\tau}{2a^2(t-\tau)}=\xi\end{array}\right\|\le$$

$$\le C_2e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta t^{\alpha-\beta}}{2a^2}\int_0^\infty\frac{\xi^{1-\beta+\alpha}}{\left(\frac{r}{2a^2}+\xi\right)^{1-\beta+\alpha}}\cdot e^{-\xi}I_{\beta-1,\beta}(\xi)d\xi\le$$

$$\le C_2e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta t^{\alpha-\beta}}{2a^2}\int_0^\infty e^{-\xi}I_{\beta-1,\beta}(\xi)d\xi\le C_2e^{-\frac{t}{4a^2}}\cdot\frac{t^\alpha}{2a^2}\le\widetilde{C_2}=\text{const},\qquad\forall(r,t)\in Q.$$

And, finally, we estimate the third term.

$$u_3(r,t)=\int_0^t\frac{r^\beta(1-2\beta)}{2a^2(t-\tau)\tau^\beta}\cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}}\cdot e^{-\frac{r\tau}{2a^2(t-\tau)}}\cdot I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau=$$

$$=\left\|\begin{array}{c}\exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right]\le\exp\left[-\frac{t-\tau}{4a^2}\right],\\ \frac{r\tau}{2a^2(t-\tau)}=\xi\end{array}\right\|\le$$

$$\le C_3e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta(1-2\beta)t^{\alpha-\beta}}{2a^2}\int_0^\infty\frac{\xi^{\alpha-\beta}}{\left(\frac{r}{2a^2}+\xi\right)^{1+\alpha-\beta}}\cdot e^{-\xi}I_\beta(\xi)\,d\xi\le$$

$$\le C_3e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta(1-2\beta)t^{\alpha-\beta}}{2a^2}\int_0^\infty\frac{1}{\xi}\cdot e^{-\xi}I_\beta(\xi)d\xi=C_3e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta(1-2\beta)t^{\alpha-\beta}}{2a^2}\cdot\Gamma\left[\begin{array}{cc}\beta, & \frac12\\ 1+\beta & \end{array}\right]=$$

$$=C_3e^{-\frac{t}{4a^2}}\cdot\frac{(1-2\beta)\sqrt{\pi}t^\alpha}{2a^2\beta}\cdot\left(\frac{r}{t}\right)^\beta\le C_3\frac{(1-2\beta)\sqrt{\pi}}{2a^2\beta}\cdot e^{-\frac{t}{4a^2}}\cdot t^\alpha\le\widetilde{C_3}=\text{const},\qquad\forall(r,t)\in Q.$$

Hence it is clear that for $\alpha\ge0$ the solution of the problem $u(r,t)\in L_\infty(G)$.

The estimate for the fourth term follows from Remark 1. This implies the validity of the main result, Theorem 2.

The results of this work will be used in solving a similar problem in a funnel-shaped degenerate domain, that is, when the boundary of the domain changes according to the law $r=\gamma(t)$, $\gamma(0)=0$.

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Бұрыштық жойылатын облыста модельдік екі өлшемді параболалық есепті шешу

Жұмыста шекарасы уақыттың өзгеруімен қозғалатын жылу өткізгіштіктің шекаралық есебі зерттелген, сонымен қатар есепті шешу облысы уақыттың бастапқы сәтінде болмайды, яғни нүктеге айналады. Берілген есепті шешу үшін жылу потенциалдары әдісі қолданылған, бұл оны екінші ретті Вольтерра типті сингулярлық интегралдық теңдеуге түрлендіруге мүмкіндік береді. Алынған интегралдық теңдеудің ерекшелігі — ол классикалық Вольтерра интегралдық теңдеулерінен түбегейлі ерекшеленеді, өйткені оған Пикар әдісі қолданылмайды және сәйкес біртекті интегралдық теңдеудің нөлдік емес шешімі бар.

Кілт сөздер: жылу өткізгіштік теңдеуі, шекаралық есеп, жойылатын облыс, Вольтерраның сингулярлық интегралдық теңдеуі, регуляризация.

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Решение модельной двумерной параболической задачи в угловой вырождающейся области

В работе исследована краевая задача теплопроводности в области, граница которой преобразуется с изменением времени, а также область решения задачи отсутствует в начальный момент времени, то есть вырождается в точку. Для решения поставленной задачи использован метод тепловых потенциалов, что позволяет редуцировать ее к сингулярному интегральному уравнению типа Вольтерра второго рода. Особенность полученного интегрального уравнения заключается в том, что оно принципиально отличается от классических интегральных уравнений Вольтерра, так как к нему неприменим метод Пикара и соответствующее однородное интегральное уравнение имеет ненулевое решение.

Ключевые слова: уравнение теплопроводности, краевая задача, вырождающаяся область, сингулярное интегральное уравнение Вольтерра, регуляризация.

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Bounded on the semi-axis multiperiodic solution of a linear finite-hereditarity integro-differential equation of parabolic type

The question of the existence of a solution of linear integro-differential systems of parabolic type limited on the semiaxis in a spatial variable and multiperiodic in time variables was considered. Sufficient conditions of multiperiodic oscillations in time variables in a linear homogeneous equation with a boundary condition and in a linear inhomogeneous equation were established. A linear homogeneous and inhomogeneous finite-hereditarity integro-differential equation of convective-diffusion type were investigated.

Keywords: integro-differential, finite-hereditarity, convection, diffusion, parabolic type, differentiation operator, Fourier series.

Problem statement

It is known [1,2] that many hereditary phenomena in biology and mechanics are described by various types of integro-differential equations. If the state of a phenomenon at the moment τ is determined by the set of states at the moments of the interval $(\tau-\varepsilon,\tau]$, then such a phenomenon is called hereditary with a finite hereditary period $\varepsilon>0$.

In the case of $\varepsilon=+\infty$, the state of the phenomenon at the moment τ depends on its states at moments in the interval $(-\infty,\tau]$. The hereditary of the phenomenon can also be related to the interval $(\tau_0,\tau]$, where τ_0 is some constant.

When the heredity of the phenomenon is bounded by the period $\varepsilon>0$, then a linear phenomenon with bounded hereditarity can be described by an integro-differential equation of the form

$$\frac{du(\tau)}{d\tau}=A(\tau)u(\tau)+\int\limits_{\tau-\varepsilon}^{\tau}K(\tau,s)u(s)ds+f(\tau). \tag{1}$$

In the case of a quasilinear phenomenon of the heredity of the period $\varepsilon>0$ we obtain the equation

$$\frac{du(\tau)}{d\tau}=A(\tau)u(\tau)+\int\limits_{\tau-\varepsilon}^{\tau}K(\tau,s)u(s)ds+f\left(\tau,u(\tau),\int\limits_{\tau-\varepsilon}^{\tau}K(\tau,s)u(s)ds\right).$$

In the linear (1) and quasi-linear equations the functions $A(\tau)$, $K(\tau,s)$ and $f(\tau,u,\overline{v})$ are known. Such equations, along with biological phenomena, describe the processes of elastic deformations, electromagnetism, and other sections of the general dynamics related to the hereditary propagation of thermal, magnetic, light, sound and other waves along the x axis. Propagations of this kind type can also be of a diffusion nature. Propagations of this kind may have a diffusive character also. Then the equation describing this phenomenon takes a form [3,4]:

$$\frac{\partial u(x,\tau)}{\partial\tau}-a^2\frac{\partial^2u(x,\tau)}{\partial x^2}=a(x,\tau)u(x,\tau)+$$

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$$+\int_{\tau-\varepsilon}^{\tau} b(x,\tau,s)u(x,s)ds + f\left(x,\tau,u(x,\tau)\right). \tag{2}$$

In the case of multi-frequency waves and fluctuations, to study the processes, following [5–19], it will be necessary to introduce a variable $t = (t_1, ..., t_m)$, varying on the vector field $\frac{dt}{d\tau} = c$ and one has to consider the equation

$$D_c u(x,t,\tau) - a^2 \frac{\partial^2 u(x,t,\tau)}{\partial x^2} = a(x,t,\tau)u(x,t,\tau) +$$

$$+\int_{\tau-\varepsilon}^{\tau} b(x,t,\tau,t-c\tau+cs,s)u(x,t-c\tau+cs,s)ds + f\left(x,t,\tau,u(x,t,\tau)\right) \tag{3}$$

with differentiation operator

$$D_c = \frac{\partial}{\partial\tau} + \sum_{j=1}^{m} c_j \frac{\partial}{\partial t_j}$$

in the direction of the vector $c = (c_1, ..., c_m)$ with constant coordinates $c_j > 0$, $j = \overline{1,m}$, and all the input data of this equation are assumed to be periodic in time variables $(t,\tau) = (t_1, ..., t_m, t_0)$, $t_0 = \tau$ period-vector $(\omega,\theta) = (\omega_1, ..., \omega_m, \omega_0)$, with incommensurable components $\omega_0 = \theta$, ω_j, $j = \overline{1,m}$.

Obviously, [5–16] along the $t = c(\tau - \tau_0)$ characteristics vector field operator D_c of the equation (3) turns into the equation (2), and its (ω,θ)-periodic on $(t,\tau) \in R^m \times R$ solutions turn into almost periodic τ solutions of the latter at $x \in R_+$.

Thus, the investigation of multiperiodic by (t,τ) solutions of equation (3) of period (ω,θ) at $x \in R_+$ is of great importance in applied problems of the theory of fluctuations and oscillations.

Note that problem studies in such a formulation are not found in the scientific literature. The research is carried out in the inductive order from the particular to general. In this connection, the problem was studied for various linear cases of equations (3).

It is clear [17,18] that the problem under consideration and its methods of investigation are closely related to some applied aspects of equations of mathematical physics of parabolic type and analytical problems of the theory of multi-frequency oscillations.

The researchers' interest in the problems for integro-differential equations, started at the end of the XIX century, has not weakened to this day [19,20]. From various points of view, where the hereditary terms of the equations are described by integrals of Volterra or Fredholm types, and the dynamics of phenomena are characterized by ordinary or partial derivatives of unknowns, developing their theory from equations to inclusions.

1 Multiperiodic zeros of the differentiation operator in the multiperiodic boundary condition

Applying the differentiation operator $\nabla_c = D_c - a^2 \frac{\partial^2}{\partial x^2}$ of the variables $x \in R_+ = (0, +\infty)$, $\tau = t_0 \in R$, $t = (t_1, ..., t_m) \in R^m$ to the function $v(x,t,\tau)$ we introduce the equation

$$\nabla_c v(x,t,\tau) = 0. \tag{4}$$

Here D_c the differentiation operator for time variables (t,τ) of the form $D_c = \frac{\partial}{\partial\tau} + \sum\limits_{j=0}^{m} c_j \frac{\partial}{\partial t_j}$, $c_0 = 1$; $a = conts > 0$; ∇_c is the differentiation operator by (x,t,τ). The equation with one-dimensional time t_j of the form $c_j \frac{\partial v_j}{\partial t_j} - a^2 \frac{\partial^2 v_j}{\partial x^2} = 0$ has solution v_j, depending on $\gamma_j \sqrt{c_j} x + \gamma_j^2 a^2 t_j$ running waves with parameter γ_j, then the solution of equation (4) with multidimensional time (t,τ) can be represented by the relations

$$v(x,t,\tau)=\alpha+\beta e^{\sum\limits_{j=0}^{m}\left(\gamma_j\sqrt{c_j}x+\gamma_j^2a^2t_j\right)} \tag{5}$$

with arbitrary differentiable functions α, β and γ_j, $j=\overline{0,m}$ vector variable $t-c\tau=(t_1-c_1\tau,...,t_m-c_m\tau)$, $c=(c_1,...,c_m)$.

Consequently, relation (5) represents zeros of the operator ∇_c at $x\in R_+$, $(t,\tau)\in R^m\times R$.

In what follows we will deal with bounded zeros of the operator ∇_c. Then by setting x to zero from (5) we obtain the limit function

$$v(x,t,\tau)|_{x=0}=\alpha+\beta e^{a^2\sum\limits_{j=0}^{m}\gamma_j^2t_j}\equiv v^0(t,\tau) \tag{6}$$

and for $x\to+\infty$, in the case of $Re\gamma_j<0$, we have

$$v(x,t,\tau)|_{x=+\infty}=\alpha\equiv v^+(t,\tau). \tag{7}$$

To ensure that the solution (5) for $t_j>0$, by virtue of (6) and (7), the functions α, β, γ_j and along with the condition $Re\gamma_j<0$, the conditions $Im\gamma_j>Re\gamma_j$, $j=\overline{0,m}$ must be bounded.

The main problem is related to the establishment of sufficient conditions for the existence of (ω,θ)-periodic on (t,τ) real-analytic at $t_j\in\Pi_\rho=\left\{t_j:\frac{2\pi}{\omega_j}|Imt_j|<\rho\right\}$, $j=\overline{0,m}$, $\omega_0=\theta$, $\omega=(\omega_1,...,\omega_m)$, $\rho=const>0$, solutions of the equations in question. Therefore in this case we assume that the boundary condition (6) is defined by the function

$$v^0(t+\omega,\tau+\theta)=v^0(t,\tau)\in A_{t,\tau}^{\omega,\theta}\left(\Pi_\rho^m\times\Pi_\rho\right). \tag{8}$$

Here $A_{t,\tau}^{\omega,\theta}\left(\Pi_\rho^m\times\Pi_\rho\right)$ is a class of (ω,θ)-periodic rea-analytic at $(t,\tau)\in\Pi_\rho^m\times\Pi_\rho$ and continuous on closures $\overline{\Pi}_\rho^m\times\overline{\Pi}_\rho$ functions, with $\omega_0=\theta,\omega_1,...,\omega_m$ are rationally incommensurable positive constants, ρ being the bandwidth Π_ρ of the interval $0<\rho<1$.

From the condition (8) we have a Fourier series representation of the function $v^0(t,\tau)$:

$$v^0(t,\tau)=\sum_{k\in Z^{m+1}}v_k^0e^{2\pi i\sum\limits_{j=0}^{m}k_j\nu_jt_j}, \tag{9}$$

where $k=(k_0,k_1,...,k_m)$, $\nu=(\nu_0,\nu_1,...,\nu_m)$, $\nu_j=\omega_j^{-1}$, $j=\overline{0,m}$; v_k^0 - are Fourier coefficients having the properties $\overline{v}_k^0=v_{-k}^0$ and satisfying the estimate

$$|v_k^0|\le||v^0||e^{-\rho|k|} \tag{10}$$

with the norm $||v^0||=\sup\limits_{\Pi_\rho^m\times\Pi_\rho}|v^0(t,\tau)|$ and $|k|=\sum\limits_{j=0}^{m}|k_j|$.

Due to rational incommensurability of frequencies $\nu_j=\omega_j^{-1}$, $j=\overline{0,m}$ parameters α, β, γ_j become constant, for the function depending on the difference $t_j-\tau$ to be ω_j and $\theta=\omega_0$ - periodic as by t_j and so by τ it is necessary and sufficient. Assuming (8) with respect to (6) we find the solution (4), (6) in the form of series

$$v(x,t,\tau)=\sum_{k\in Z^{m+1}}v_ke^{\sum\limits_{j=0}^{m}\left(\gamma_j\sqrt{c_j}x+\gamma_{jk}^2a^2t_j\right)} \tag{11}$$

with constant coefficients v_k and indicators γ_{jk}, $j = \overline{0,m}$, $k \in Z^{m+1}$.

Obviously, (11) is a generalization of the function (5) to an infinite series, which represents the solution of the equation (4) in general form.

Substituting (11) and (9) into the boundary conditions (6) formally we obtain $v_k = v_k^0$, $\gamma_{jk}^2 a^2 = 2\pi i k_j \nu_j$, $j = \overline{0,m}$, $k_j = Z_+^0$, Z_+^0 is the set of non-negative integers.

Hence $\gamma_{jk} = \pm\frac{\sqrt{2\pi\nu_j k_j}}{a} \cdot \sqrt{i} = \pm\left(\frac{\sqrt{\pi\nu_j k_j}}{a} + i\frac{\sqrt{\pi\nu_j k_j}}{a}\right)$ at $k_j \in Z_+^0$ is the set of positive integers.

Since we are interested in the solution bounded by x in R_+, we have

$$\gamma_{jk} = -\left(\frac{\sqrt{\pi\nu_j k_j}}{a} + i\frac{\sqrt{\pi\nu_j k_j}}{a}\right),\ j = \overline{0,m},\ k_j = Z_+^0. \tag{12}$$

In the case of negative $k_j = -|k_j| < 0$ we have the equation $\gamma_{jk}^2 a^2 = -2\pi i |k_j| \nu_j$, $j = \overline{0,m}$, $k_j = Z_-$ to determine the indicators γ_{jk}. Hence we find $\gamma_{jk} = \pm\sqrt{-1}\sqrt{\frac{2\pi|k_j|\nu_j}{a^2}} \cdot \sqrt{i} = \pm i\frac{\sqrt{2\pi\nu_j|k_j|}}{a} \cdot \frac{1+i}{\sqrt{2}} = \pm\left(-\frac{\sqrt{\pi\nu_j|k_j|}}{a} + i\frac{\sqrt{\pi\nu_j|k_j|}}{a}\right)$.

Hence, to ensure that the solution is bounded by $x \in R_+$ we take the roots with a plus sign:

$$\gamma_{jk} = -\frac{\sqrt{\pi\nu_j|k_j|}}{a} + i\frac{\sqrt{\pi\nu_j|k_j|}}{a},\ j = \overline{0,m},\ k_j = Z_-. \tag{13}$$

Thus, the roots (12) and (13) are mutually conjugate. Hence, combining these formulas we have

$$\gamma_{jk} = -\frac{\sqrt{\pi\nu_j|k_j|}}{a} - sign k_j\frac{\sqrt{\pi\nu_j|k_j|}}{a},\ j = \overline{0,m},\ k_j = Z, \tag{14}$$

where this formula includes the case $k_j = 0$, at which $sign\ 0 = 0$.

Substituting (14) into (11) we obtain the solution

$$v(x,t,\tau) = v_0^0 +$$

$$+\sum_{0\neq k\in Z^{m+1}} v_k^0 exp\left[-\sum_{j=0}^m \frac{\sqrt{\pi\nu_j c_j|k_j|}}{a}x + i\left(sign k_j\sum_{j=0}^m \frac{\sqrt{\pi\nu_j c_j|k_j|}}{a}x + 2\pi k_j\nu_j t_j\right)\right]. \tag{15}$$

Obviously, the series (15) converges absolutely and uniformly at $x \in \overline{R}_+$ and $(t,\tau) \in R^m \times R$, differentiable by x (a finite number of times), analyticity at (t,τ) is preserved. In support of this claim, we use the evaluation (10) and $\sum\limits_{j=0}^m |k_j|^{1/2} \leq \sqrt{m+1}\left(\sum_{j=1}^m |k_j|\right)^{1/2}$, which follows from the Bunyakovskii-Schwartz inequality.

The solution (15) is multiperiodic at (t,τ), bounded at $(x,t,\tau) \in \overline{R}_+ \times \overline{\Pi}_\rho^m \times \overline{\Pi}_\rho$ and unique in the class of bounded functions.

Theorem 1. The Problem (4), (6) under the condition (8) has at $(x,t,\tau) \in \overline{R}_+ \times \overline{\Pi}_\rho^m \times \overline{\Pi}_\rho$ the only real-analytic (ω,θ)-periodic on (t,τ) solution $v(x,t,\tau)$ of the form (15) satisfying the

$$|v(x,t,\tau)| \leq c^0||v^0||/\delta^{m+1},\ x \in \overline{R}_+,\ (t,\tau) \in \overline{\Pi}_{\rho-\delta}^m \times \overline{\Pi}_{\rho-\delta} \tag{16}$$

with an arbitrary constant δ from the interval $0 < \delta < \rho < 1$, where $c^0 = c^0(m)$is a constant, independent of δ and v^0.

The proof of all the positions of the theorem is given above. To complete it, it is necessary to verify the validity of the estimate (16).

Indeed, from (15) we have the series

$$v(x,t,\tau)=\sum_{k\in Z^{m+1}} v_k(x)e^{2\pi i\sum\limits_{j=0}^{m} k_j\nu_j t_j} \tag{17}$$

with coefficients

$$v_k(x)=v_k^0 exp\left[-\sum_{j=0}^{m}\frac{\sqrt{\pi\nu_j c_j|k_j|}}{a}\left(1+isignk_j\right)x\right], \tag{18}$$

which satisfy the inequalities

$$|v_k(x)|\le|v_k^0|,\ \ k\in Z^{m+1}. \tag{19}$$

The case of absence of t is considered in [17; 201–202]. Then, by virtue of (10), from (19) it follows that

$$|v_k(x)|\le||v^0||e^{-\rho|k|}. \tag{20}$$

Consequently, according to the properties of the Fourier coefficients of analytic functions [18; 108], the function (17) with coefficients (18) satisfying the evaluation (20) is analytic and obeys the constraint (16).

2 Multiperiodic solution of a linear diffusion equation with a multi-frequency oscillating source

Consider the equation

$$\nabla_c u(x,t,\tau)\equiv D_c u(x,t,\tau)-a^2\frac{\partial^2 u(x,t,\tau)}{\partial x^2}=f(x,t,\tau). \tag{21}$$

Here $a=const>0$, the function $f(x,t,\tau)$ is represented as a series

$$f(x,t,\tau)=\sum_{k\in Z^{m+1}} f_k e^{-\gamma_k x+2\pi i\sum\limits_{j=0}^{m} k_j\nu_j t_j} \tag{22}$$

with constants of $\gamma_k>0$, f_k, $k=(k_0,k_1,...,k_m)\in Z^{m+1}$; $\nu_j=\omega_j^{-1}$, $j=\overline{0,m}$ with

$$|f_k|\le||f||e^{-\rho|k|}, \tag{23}$$

where $||f||=\sup\limits_{\overline{R}_+\times\overline{\Pi}_\rho^m\times\overline{\Pi}_\rho}|f(x,t,\tau)|$.

The multiperiodic solution of the equation (21) will be sought in the form

$$u(x,t,\tau)=\sum_{k\in Z^{m+1}} W_k(t,\tau)e^{-\gamma_k x}. \tag{24}$$

Substituting (22) and (24) in (21) we obtain

$$\sum_{k\in Z^{m+1}}\left[D_c W_k(t,\tau)-a^2\gamma_k^2 W_k(t,\tau)\right]e^{-\gamma_k x}=\sum_{k\in Z^{m+1}} f_k e^{2\pi i\sum\limits_{j=0}^{m} k_j\nu_j t_j}e^{-\gamma_k x}.$$

Hence we have equations $D_c W_k(t,\tau) - a^2\gamma_k^2 W_k(t,\tau) = f_k e^{2\pi i \sum\limits_{j=0}^{m} k_j\nu_j t_j}$, $k \in Z^{m+1}$ which have (ω,θ)-periodic by (t,τ) solutions

$$W_k(t,\tau) = \int\limits_{+\infty}^{\tau} f_k e^{2\pi i \sum\limits_{j=0}^{m} k_j\nu_j(t_j - c_j\tau + c_j s) + a^2\gamma_k^2(\tau - s)} ds =$$

$$= \frac{f_k}{-a^2\gamma_k^2 + 2\pi i \sum\limits_{j=0}^{m} k_j\nu_j c_j} e^{2\pi i \sum\limits_{j=0}^{m} k_j\nu_j t_j} = \frac{1}{a_k + ib_k} f_k e^{2\pi i \sum\limits_{j=0}^{m} k_j\nu_j t_j}, \tag{25}$$

since conditions $\Delta_k = a_k + ib_k \neq 0$, where $a_k = -a^2\gamma_k^2$, $b_k = 2\pi \sum\limits_{j=0}^{m} k_j\nu_j c_j$, $k \in Z^{m+1}$ are satisfied. By substituting (25) into (24) we obtain solution

$$u^*(x,t,\tau) = \sum_{k\in Z^{m+1}} \frac{1}{a_k + ib_k} f_k e^{2\pi i \sum\limits_{j=0}^{m} k_j\nu_j t_j - \gamma_k x}. \tag{26}$$

To ensure the convergence of the series (26), we assume that the strong incommensurability condition is fulfilled $\widetilde{\nu}_j = \nu_j c_j$, $j = \overline{0,m}$ of the form

$$|b_k| = 2\pi \left| \sum_{j=0}^{m} k_j \widetilde{\nu}_j \right| \geq \lambda^{-1}|k|^{-1}, \quad |k| = \sum_{j=0}^{m} |k_j| > 0 \tag{27}$$

with constants $\lambda > 0$ and $l \geq m+1$, or the sequence a_k satisfies the condition of boundedness condition of the form

$$|a_k| \geq r, \quad k \in Z^{m+1} \tag{28}$$

with constant $r > 0$.

If one of the conditions (27) and (28), together with estimation (23) is satisfied, the series (26) will converge absolutely and uniformly.

Thus we distinguish two kinds of running waves $\psi_k(x,t,\tau) = 2\pi i \sum\limits_{j=0}^{m} k_j\nu_j t_j - \gamma_k x$, $k \in Z^{m+1}$, for which a) $\Delta_k = a_k + ib_k = 0$ and b) $\Delta_k = a_k + ib_k \neq 0$, $k \in Z^{m+1}$. In the case a) $u_k = e^{\psi_k(x,t,\tau)}$ will turn out to be zeros of the operator ∇_c, and in the case b) $\nabla_c u_k \neq 0$.

Note that a similar result can be obtained when the real function $f(x,t,\tau)$ is defined for complex values $\gamma_k = \alpha_k + i\beta_k$, $\beta_k \neq 0$.

So equation (21) under the conditions (22), (23) and under one of the conditions (27) and (28) admit only (ω,θ)-periodic on (t,τ) solution (26) with values $\Delta_k = a_k + ib_k \neq 0$, $k \in Z^{m+1}$.

In general, equation (21) has an infinite set of (ω,θ)-periodic solutions $u(x,t,\tau)$ by (t,τ), consisting of the sum of the solutions $v(x,t,\tau)$ of the homogeneous equations (4) with $\Delta_k = 0$, $k \in Z^{m+1}$ and the solution $u^*(x,t,\tau)$ of the nonhomogeneous equation (21) with $\Delta_k \neq 0$, $k \in Z^{m+1}$:

$$u(x,t,\tau) = v(x,t,\tau) + u^*(x,t,\tau), \tag{29}$$

where $v(x,t,\tau)$ is defined by the problem (4), (6), and $u^*(x,t,\tau)$ by the relation (26) and satisfies the boundary condition

$$u(0,t,\tau) = v(0,t,\tau) + u^*(0,t,\tau). \tag{30}$$

The solution (29) of the boundary value problem (21), (30) is singular.

Theorem 2. Under the conditions (22), (23) and (27) or (28) the equations (21) has (ω, θ)-periodic solutions represented in the form (29) with terms (15) and (26).

If for some $k^0 = (k_0^0, k_1^0, ..., k_m^0)$ we have $\Delta_{k_0} = a_{k_0} + ib_{k_0} = 0$, then we exclude the corresponding k^0-subject from relation (26) and introduce a function

$$u^0(x,t,\tau) = \frac{\alpha_0\tau + \alpha_1 t_1 + ... + \alpha_m t_m}{\alpha_0 + \alpha_1 c_1 + ... + \alpha_m c_m} f_{k^0} e^{2\pi i \sum\limits_{j=0}^{m} k_j^0 \nu_j t_j - \gamma_{k^0} x} \tag{31}$$

with an arbitrary constant vector $\alpha = (\alpha_0, \alpha_1, ..., \alpha_m)$ satisfying equation

$$\nabla_c u^0(x,t,\tau) = f_{k^0} e^{2\pi i \sum\limits_{j=0}^{m} k_j^0 \nu_j t_j - \gamma_{k^0} x}. \tag{32}$$

Then, based on (31) and (32), the solution (26) can be represented in the form

$$\widetilde{u}^*(x,t,\tau) = u^0(x,t,\tau) + \sum_{k^0 \neq k \in Z^{m+1}} \frac{f_k}{a_k + ib_k} e^{2\pi i \sum\limits_{j=0}^{m} k_j \nu_j t_j - \gamma_k x}. \tag{33}$$

Theorem 3. If $a_{k^0} + ib_{k^0} = 0$ and $a_k + ib_k \neq 0$ at $k \neq k^0$, then under the conditions of Theorem 2, equation (21) has a solution $u(x,t,\tau) = v(x,t,\tau) + \widetilde{u}^*(x,t,\tau)$, where $v(x,t,\tau)$ is defined by the formula (6), and $\widetilde{u}^*(x,t,\tau)$ by relation (33).

3 Multiperiodic solutions of a linear homogeneous integro-differential parabolic equations with finite hereditarity

Consider (ω, θ)-periodic by (t, τ) equation

$$\nabla_c u(x,t,\tau) \equiv D_c u(x,t,\tau) - a^2 \frac{\partial^2 u(x,t,\tau)}{\partial x^2} =$$

$$= a(x,t,\tau)u(x,t,\tau) + \int\limits_{\tau-\varepsilon}^{\tau} b(x,t,\tau,t - c\tau + cs, s)u(x, t - c\tau + cs, s)ds.$$

This equation describes a multi-frequency phenomenon propagating along the semi-axis R_+ 1) diffusion with constant $a^2 \neq 0$, 2) linearly hereditary with finite period $\varepsilon > 0$ and kernel $b = b(x,t,\tau,\sigma,s)$, 3) at each point $x \in R_+$ it is linearly related to the external environment by the coefficient $a = a(x,t,\tau)$ and 4) flows with speed $D_c u(x,t,\tau)$ defined by differentiation along the direction of vector field of operator $D_c = \frac{\partial}{\partial\tau} + \sum_{j=1}^m c_j \frac{\partial}{\partial t_j}$.

An important special case of the process is when its heredity and coupling to the external world do not depend on $x \in R_+$. In this regard, we introduce into consideration the equation

$$\nabla_c u(x,t,\tau) = a(t,\tau)u(x,t,\tau) + \int\limits_{\tau-\varepsilon}^{\tau} b(t,\tau,\sigma,s)u(x,\sigma,s)ds, \tag{34}$$

where the matrices $a(t,\tau)$ and $b(t,\tau,\sigma,s)$ are real-analytic functions.

Consider the null operator ∇_c, depending on $m+1$ running waves $\sum\limits_{j=0}^{m}(\gamma_j x + 2\pi i t_j \nu_j k_j)$ of the form

$$v_k(x,t,\tau) = be^{2\pi i \sum_{j=0}^{m} t_j\nu_j k_j + \gamma_j x} \tag{35}$$

with constant coefficient $b \neq 0$ and parameter $\gamma_j = \gamma_j(k_j, \nu_j, c_j, a)$.

It's obvious that $v_k(x,t,\tau)$ has the property

$$\nabla_c v_k(x,t,\tau) = 0,$$

$$v_k(x, t - c\tau + cs, s) = b^{-1} v_k(0, -c\tau + cs) v_k(x,t,\tau). \tag{36}$$

Next, by replacing

$$u(x,t,\tau) = U(t,\tau) v_k(x,t,\tau) \tag{37}$$

equation (34) on the basis of (35), (36) is reduced to

$$D_c U(t,\tau) = a(t,\tau)U(t,\tau) + \int\limits_{\tau-\varepsilon}^{\tau} b(t,\tau,\sigma,s) b^{-1} v_k(0,\sigma,s) U(\sigma,s) ds.$$

Under the conditions

$$a(t,\tau) \in A_{t,\tau}^{\omega,\theta}\left(\Pi_\rho^m \times \Pi_\rho\right), \quad b(t,\tau,\sigma,s) \in A_{t,\tau,\sigma,s}^{\omega,\theta,\omega,\theta}\left(\Pi_\rho^m \times \Pi_\rho \times \Pi_\rho^m \times \Pi_\rho\right) \tag{38}$$

it is possible to show the existence of a single solution $U_k(t,\tau,\sigma,s) \equiv U_k(t,\tau,t-c\tau+cs,s)$, satisfying the condition $U_k(t,s,t,s) = E$ at $\tau = s$ and $U_k(t,\tau,\sigma,s) \in A_{t,\tau,\sigma,s}^{\omega,\theta,\omega,\theta}\left(\Pi_\rho^m \times \Pi_\rho \times \Pi_\rho^m \times \Pi_\rho\right)$.

Suppose that $U_k(t,\tau,\sigma,s)$ satisfies the estimate

$$|U_k(t,\tau,t-c\tau+cs,s)| \leq \Lambda e^{-\lambda(\tau-s)} \tag{39}$$

with constants $\Lambda \geq 1$ and $\lambda > 0$ for any $k \in Z^{m+1}$.

Then a solution of the form (37), which is bounded at $x \in \overline{R}_+$, $t \in R^m$ and $\tau \geq s$ and satisfies the evaluation

$$|u(x,t,\tau)| \leq \Lambda e^{-\lambda(\tau-s)} |v_k(x,t,\tau)| \leq u^0 e^{-[\lambda(\tau-s)+\mu x]} \tag{40}$$

with some constant u^0, $\lambda > 0$ and $\mu > 0$. Here $\mu > 0$ is defined on the estimation of the zero (35) operator ∇_c.

Inequality (40) shows that under the condition (39) the homogeneous equation (34) has only a zero bounded (ω,θ)-periodic solution on (t,τ).

Theorem 4. Under the conditions (38) and (39), equation (34) has only zero (ω,θ)-periodic in (t,τ) solution.

4 *Multiperiodic solution of a complete linear inhomogeneous integro-differential equation of parabolic type*

Let's introduce the equation

$$\nabla_c u(x,t,\tau) = a(t,\tau)u(x,t,\tau) + \int\limits_{\tau-\varepsilon}^{\tau} b(t,\tau,\sigma,s) u(x,\sigma,s) ds +$$

$$+f(t,\tau)exp\left(\sum_{j=0}^{m}[\gamma_j x+2\pi i k_j\nu_j t_j]\right). \tag{41}$$

Here $v_j(x,t_j)=\gamma_j x+2\pi i k_j\nu_j t_j$ are the travelling waves defined by the equation

$$\nabla_c exp[v_j(x,t_j)]=0, \quad j=\overline{0,m} \tag{42}$$

with unknown parameters γ_j and constants, $\nu_j=\omega_j^{-1}$, $k_j\in Z$ with the condition that $x\to+\infty$ follows

$$expv_j(x,t_j)\to 0. \tag{43}$$

The functions $a(t,\tau)$, $b(t,\tau,\sigma,s)$ and $f(t,\tau)$ are (ω,θ)-periodic by (t,τ) and (σ,s), belong to the class $A^{\omega,\theta,\omega,\theta}_{t,\tau,\sigma,s}\left(\Pi^m_\rho\times\Pi_\rho\times\Pi^m_\rho\times\Pi_\rho\right)$.

From the conditions (42) and (43) we have $a^2\gamma_j^2=\pm 2\pi i k_j\nu_j c_j$, $j=\overline{0,m}$ and $c_0=1$ at $t_0=\tau$. Hence we have $\gamma_j=\pm\frac{\sqrt{2\pi k_j\nu_j c_j}}{a}\frac{1\pm i}{\sqrt{2}}$, $k_j>0$; $\gamma_j=\pm\frac{\sqrt{2\pi|k_j|\nu_j c_j}}{a}\frac{1\mp i}{\sqrt{2}}$, $k_j<0$. To satisfy the condition (43) we choose γ_j as

$$\gamma_j=-\frac{\sqrt{\pi|k_j|\nu_j c_j}}{a}(1-isignk_j). \tag{44}$$

Thus, by virtue of the latter relationship, the function

$$v(x,t,\tau)=exp\left[\sum_{j=0}^{m}v_j(x,t_j)\right] \tag{45}$$

has the property

$$\nabla_c v(x,t,\tau)=0,\ x\in R_+,\ (t,\tau)\in R^m\times R. \tag{46}$$

It can be shown that

$$v(x,t-c\tau+cs,s)=v(x,t,\tau)exp\left[-2\pi i\sum_{j=0}^{m}k_j\nu_j(\tau-s)\right]. \tag{47}$$

Next, enter the replacement

$$u(x,t,\tau)=U(t,\tau)v(x,t,\tau) \tag{48}$$

into the equation (41) and due to (47) we obtain

$$D_cU(t,\tau)v(x,t,\tau)+U(t,\tau)\nabla_c v(x,t,\tau)=a(t,\tau)U(t,\tau)v(x,t,\tau)+$$

$$+\int_{\tau-\varepsilon}^{\tau}b(t,\tau,\sigma,s)exp\left[-2\pi i\sum_{j=0}^{m}k_j\nu_j c_j(\tau-s)\right]U(\sigma,s)v(x,t,\tau)ds+f(t,\tau)v(x,t,\tau).$$

Then, given (46), reducing by $v(x,t,\tau)\neq 0$ we have the equation

$$D_cU(t,\tau)=a(t,\tau)U(t,\tau)+\int_{\tau-\varepsilon}^{\tau}b(t,\tau,\sigma,s)exp\left[-2\pi i\sum_{j=0}^{m}k_j\nu_j c_j(\tau-s)\right]U(\sigma,s)ds+f(t,\tau). \tag{49}$$

The solution $U(t,\tau,\sigma,s)$ of the homogeneous equation corresponding to equation (49) with initial condition $U(t,s,t,s)=E$ satisfies the evaluation (40).

Then it is easy to show that the inhomogeneous equation (49) admits a single (ω, θ)-periodic by (t, τ) solution

$$U^*(t,\tau) = \int_{-\infty}^{\tau} U(t,\tau,t-c\tau+cs_1,s_1) f(t-c\tau+cs_1,s_1) ds_1. \tag{50}$$

Then by substituting (50) in (48), we obtain a single bounded on $x \in \overline{R}_+$, (ω,θ)-periodic on (t,τ) solution

$$u^*(x,t,\tau) = U^*(t,\tau) v(x,t,\tau) \tag{51}$$

of equations (41).

Theorem 5. Let the functions a, b and f belong to the class $A^{\omega,\theta,\omega,\theta}_{t,\tau,\sigma,s}\left(\Pi^m_\rho \times \Pi_\rho \times \Pi^m_\rho \times \Pi_\rho\right)$. Then under conditions (43), (44) and (40) equation (41) has a unique bounded in $x \in \overline{R}_+$ (ω,θ)-periodic on (t,τ) solution of the form (51) with factors (50) and (45).

By the superposition method, the theorem can be generalised when the free term $f(x,t,\tau)$ equation (41) can be represented as

$$f(x,t,\tau) = \sum_{k\in Z^{m+1}} f_k(t,\tau) exp\left(\sum_{j=0}^{m}\left[\gamma_{k_j} x + 2\pi i k_j \nu_j t_j\right]\right),$$

where γ_{k_j} is a constant from (44).

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Параболалық типті ақырлы-эредитарлы сызықты интегралды-дифференциалдық теңдеудің жартылай осьте шектелген көппериодты шешімі

Параболалық типті сызықты интегралды-дифференциалдық теңдеулер жүйесінің кеңістік айнымалысы бойынша жартылай осьте шектелген және уақыт айнымалылары бойынша көппериодты шешімінің бар болуы жөнінде сұрақ қарастырылған. Шекаралық шартты сызықты біртекті теңдеуде және сызықты біртексіз теңдеуде уақыт айнымалысы бойынша көппериодты тербелістердің жеткілікті шарттары анықталған. Конвективті-диффузиялы типті ақырлы-эредитарлы интегралды-дифференциалдық сызықты біртекті және біртексіз теңдеу зерттелген.

Кілт сөздер: интегралды-дифференциалдық, ақырлы-эредитарлы, конвективті, диффузиялы, параболалық типті, дифференциалдық оператор, Фурье қатары.

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Ограниченное на полуоси многопериодическое решение линейного конечно-эредитарного интегро-дифференциального уравнения параболического типа

Рассмотрен вопрос о существовании ограниченного на полуоси по пространственной переменной и многопериодического по временным переменным решения линейной интегро-дифференциальной системы параболического типа. Установлены достаточные условия многопериодических колебаний по временным переменным в линейном однородном уравнении с граничным условием и в линейном неоднородном уравнении. Исследованы линейное однородное и неоднородное конечно-эредитарное интегро-дифференциальное уравнения конвективно-диффузионного типа.

Ключевые слова: интегро-дифференциальное, конечно-эредитарное, конвективный, диффузионный, параболический тип, дифференциальный оператор, ряд Фурье.

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Criteria for the boundedness of a certain class of matrix operators from l_{pv} into l_{qu}

One of the main aims in the theory of matrices is to find necessary and sufficient conditions for the elements of any matrix so that the corresponding matrix operator maps continuously from one normed space into another one. Thus, it is very important to find the norm of the matrix operator, at least, to find upper and lower estimates of it. This problem in Lebesgue spaces of sequences in the general case is still open. This paper deals with the problem of boundedness of matrix operators from l_{pv} into l_{qu} for $1 < q < p < \infty$, and we obtain necessary and sufficient conditions of this problem when matrix operators belong to the classes $O_2^{\pm}$ satisfying weaker conditions than Oinarov's condition.

Keywords: matrix operator, conjugate operator, weight sequence, boundedness, weight inequalities, weight Lebesgue space, Oinarov's condition, Hardy operator, Hardy inequality, matrix.

Introduction

Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Let $u = \{u_i\}, v = \{v_i\}$ be sequences of positive numbers, which will be called the weight sequences. Let l_{pv} the space of all sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers such that $\|f\|_{pv} = \left(\sum\limits_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{1}{p}}$, $1 \leq p < \infty$.

We consider the problem of boundedness for the following matrix operators

$$\left(A^+ f\right)_i = \sum_{j=1}^{i} a_{ij} f_j, \; i \geq 1, \tag{1}$$

$$\left(A^- f\right)_j = \sum_{i=j}^{\infty} a_{ij} f_i, \; j \geq 1 \tag{2}$$

from l_{pv} into l_{qu}, where $a_{ij} > 0, i \geq j \geq 1$, i.e. the vaidity of the inequality

$$\|A^{\pm} f\|_{qu} \leq C \|f\|_{pv}, \forall f \in l_{pv}. \tag{3}$$

The matrix operators (1), (2) were studied in many papers in different sequence spaces. The almost complete collection of these results is presented in the work by M. Stieglitz and H. Tietz [1]. There the mappings of matrix operators are considered in 11 sequence spaces except its mapping from l_{pv} into l_{qu}. The remaining case is still an open problem.

When $a_{ij} = 1, i \geq j \geq 1$ operators (1), (2) coincide with the discrete Hardy operators, which have been studied by many researchers, and main results were obtained in [2–7].

In the general case, the question is open on conditions on the entries of a matrix (a_{ij}) that giving boundedness of operators (1) and (2). For several classes of matrices, criteria for boundedness of the

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operators (1), (2) are known. One of the first studied classes was the class of operators matrices of which satisfy the following discrete Oinarov's condition: there exists $d \geq 1$ such that

$$\frac{1}{d}\left(a_{ik}+a_{kj}\right) \leq a_{ij} \leq d\left(a_{ik}+a_{kj}\right)$$

for all $i \geq k \geq j \geq 1$ (see [8], [9]).

In 2012 in paper [10] the wide classes O_n^+, O_n^-, $n \geq 0$ of matrices were presented, which defined by conditions on a matrix (a_{ij}) that are weaker than Oinarov's condition, and the necessary and sufficient conditions for boundedness of these operators for $1 < p \leq q < \infty$ were obtained, where their matrices belonged to these classes. However, the problem of boundedness of operators (1) and (2) with matrix from the classes O_n^+, O_n^-, $n > 1$ for the case $1 < q < p < \infty$ is still open. But the first results for this case - the criteria of boundedness for matrix operators from $O_1^\pm$ are found in [11], [12].

In the present paper, we find criteria of boundedness for operators (1), (2) from l_{pv} into l_{qu}, where their matrices belong to the class $O_2^\pm$ when $1 < q < p < \infty$.

Convention: The symbol $M << K$ means that $M \leq cK$, where c > 0 is a constant depending only on unessential parameters. If $M << K << M$, then we write $M \approx K$.

We assume $g_i = 0$ when $i < 1$ and $\Delta^- g_i = g_i - g_{i-1}, \Delta^+ g_i = g_i - g_{i+1}$.

1 *Preliminaries*

Let's give the definition of classes $O_1^\pm$, $O_2^\pm$.

Definition 1. Let (a_{ij}) be a matrix which is non-negative and non-decreasing in the first index for all $i \geq j \geq 1$. A matrix (a_{ij}) belongs to the class O_1^+, if there exist a non-negative matrix $(a_{ij}^{1,0})$, a number $r_1 \geq 1$ such that the estimates

$$\frac{1}{r_1}\left(a_{ik}^{1,0}+a_{kj}\right) \leq a_{ij} \leq r_1\left(a_{ik}^{1,0}+a_{kj}\right)$$

hold for all $i \geq k \geq j \geq 1$.

Definition 2. Let (a_{ij}) be a matrix which is non-negative and non-increasing in the second index for all $i \geq j \geq 1$. A matrix (a_{ij}) belongs to the class O_1^-, if there exist a non-negative matrix $(a_{ij}^{0,1})$, a number $\bar{r}_1 \geq 1$ such that the estimates

$$\frac{1}{\bar{r}_1}\left(a_{ik}+a_{kj}^{0,1}\right) \leq a_{ij} \leq \bar{r}_1\left(a_{ik}+a_{kj}^{0,1}\right)$$

hold for all $i \geq k \geq j \geq 1$.

Definition 3. Let (a_{ij}) be a matrix which is non-negative and non-decreasing in the first index for all $i \geq j \geq 1$. A matrix (a_{ij}) belongs to the class O_2^+, if there exist a non-negative matrices $(a_{ij}^{2,0})$, $(a_{ij}^{2,1})$, $(a_{ij}^{(1)})$, a number $r_2 \geq 1$ such that $(a_{ij}^{(1)}) \in O_1^+$,

$$\frac{1}{r_2}\left(a_{ik}^{2,0}+a_{ik}^{2,1}a_{kj}^{(1)}+a_{kj}\right) \leq a_{ij} \leq r_2\left(a_{ik}^{2,0}+a_{ik}^{2,1}a_{kj}^{(1)}+a_{kj}\right)$$

for all $i \geq k \geq j \geq 1$.

Definition 4. Let (a_{ij}) be a matrix which is non-negative and non-increasing in the second index for all $i \geq j \geq 1$. A matrix (a_{ij}) belongs to the class O_2^-, if there exist non-negative matrices $(a_{ij}^{0,2})$, $(a_{ij}^{1,2})$, $(a_{ij}^{(1)})$, a number $\bar{r}_1 \geq 1$ such that $(a_{ij}^{(1)}) \in O_1^-$,

$$\frac{1}{\bar{r}_2}\left(a_{ik}+a_{ik}^{(1)}a_{kj}^{1,2}+a_{kj}^{0,2}\right) \leq a_{ij} \leq \bar{r}_2\left(a_{ik}+a_{ik}^{(1)}a_{kj}^{1,2}+a_{kj}^{0,2}\right)$$

for all $i \geq k \geq j \geq 1$.

Let us consider some examples of matrices that belong to the classes $O_1^{\pm}$ and $O_2^{\pm}$.

Example 1. Let $\alpha > 0$. Let $\{a_i\}_{i=1}^{\infty}$ be a non-decreasing positive sequence and $\{b_i\}_{i=1}^{\infty}$ be an arbitrary positive sequence, such that $a_i \geq b_j$, $i \geq j \geq 1$. Then $a_{ij} = a_{ij}^{(1)} := \left(\ln \frac{a_i}{b_j}\right)^{\alpha} \in O_1^+$, when $i \geq j \geq 1$.

Indeed, for all $i \geq k \geq j \geq 1$

$$a_{ij}^{(1)} = \left(\ln \frac{a_i}{a_k} \cdot \frac{a_k}{b_j}\right)^{\alpha} \approx \left(\ln \frac{a_i}{a_k}\right)^{\alpha} + \left(\ln \frac{a_k}{b_j}\right)^{\alpha} = a_{ik}^{1,0} + a_{kj}^{(1)},$$

where $a_{ik}^{1,0} = \left(\ln \frac{a_i}{a_k}\right)^{\alpha}$.

Example 2. Let $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ satisfy the conditions from Example 1. Moreover, we assume that $\{\omega_i\}_{i=1}^{\infty}$ is a non-negative sequence. Then $a_{ij} = a_{ij}^{(2)} := \sum\limits_{s=j}^{i} \omega_s \left(\ln \frac{a_s}{b_j}\right)^{\alpha} \in O_2^+$, $i \geq j \geq 1$.

Indeed, for all $i \geq k \geq j \geq 1$ we have

$$a_{ij}^{(2)} = \sum_{s=j}^{i} \omega_s \left(\ln \frac{a_s}{b_j}\right)^{\alpha} \approx \sum_{s=j}^{k} \omega_s \left(\ln \frac{a_s}{b_j}\right)^{\alpha} + \sum_{s=k}^{i} \omega_s \left(\ln \frac{a_s}{b_j}\right)^{\alpha} \approx$$

$$\approx a_{kj}^{(2)} + \sum_{s=k}^{i} \omega_s \left(\ln \frac{a_s}{a_k}\right)^{\alpha} + \left(\ln \frac{a_k}{b_j}\right)^{\alpha} \sum_{s=k}^{i} \omega_s =$$

$$= a_{ik}^{2,0} + a_{ik}^{2,1} a_{kj}^{(1)} + a_{kj}^{(2)},$$

where $a_{kj}^{(1)} = \left(\ln \frac{a_k}{b_j}\right)^{\alpha} \in O_1^+$, $a_{ik}^{2,0} = \sum\limits_{s=k}^{i} \omega_s \left(\ln \frac{a_s}{a_k}\right)^{\alpha}$, $a_{ik}^{2,1} = \sum\limits_{s=k}^{i} \omega_s$, $i \geq k \geq j \geq 1$.

In the same way, one can show that $a_{ij}^{(1)} = \left(\ln \frac{a_i}{b_j}\right)^{\alpha} \in O_1^-$ and $a_{ij}^{(2)} := \sum\limits_{s=j}^{i} \omega_s \left(\ln \frac{a_s}{b_j}\right)^{\alpha} \in O_2^-$, $i \geq j \geq 1$, if $\{a_i\}_{i=1}^{\infty}$ is an arbitrary positive sequence and $\{b_i\}_{i=1}^{\infty}$ is a non-decreasing positive sequence, such that $a_i \geq b_j$, $i \geq j \geq 1$.

Remark 1. As it is shown in [10] the matrices $(a_{ij}^{2,0})$, $(a_{ij}^{2,1})$, $(a_{ij}^{(1)})$, $(a_{ij}^{0,2})$, $(a_{ij}^{1,2})$ can be considered non-decreasing in i and non-increasing in j.

Lemma A. [9] *Let $\gamma > 0$, $1 \leq n < N \leq \infty$ and let $\{h_k\}$ be a non-negative sequence. Then*

$$\left(\sum_{k=n}^{N} h_k\right)^{\gamma} \approx \sum_{k=n}^{N} \left(\sum_{i=n}^{k} h_i\right)^{\gamma-1} h_k, \tag{4}$$

$$\left(\sum_{k=n}^{N} h_k\right)^{\gamma} \approx \sum_{k=n}^{N} \left(\sum_{i=k}^{N} h_i\right)^{\gamma-1} h_k. \tag{5}$$

Let us state the necessary assertions from [5], [11] in a convenient form.

Theorem A. Let $1 < q < p < \infty$. The inequality

$$\left(\sum_{k=1}^{\infty} \left|\sum_{j=1}^{k} f_j\right|^q u_k^q\right)^{\frac{1}{q}} \leq C \left(\sum_{k=1}^{\infty} |f_k v_k|^p\right)^{\frac{1}{p}}, \forall f \in l_{pv} \tag{6}$$

holds if and only if

$$F=\left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}u_j^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_k^{-p'}\right)^{\frac{p-q}{pq}}<\infty.$$

Moreover, $F\approx C$, where C is the best constant in (6).

Theorem B. Let $1<q<p<\infty$ and the matrix (a_{ij}) belongs to the class O_1^+. Then inequality (3) for operator (1) holds if and only if $B=\max\{B_0,B_1\}<\infty$, where

$$B_0=\left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}(a_{jk}^{1,0})^q u_j^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_k^{-p'}\right)^{\frac{p-q}{pq}},$$

$$B_1=\left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}u_j^q\right)^{\frac{q}{p-q}}\left(\sum_{i=1}^{k}a_{ki}^{p'}v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}}u_k^q\right)^{\frac{p-q}{pq}}.$$

Moreover, $B\approx C$, where C is the best constant in (3).

2 Main results

Our main results read.

Theorem 1. Let $1<q<p<\infty$ and $(a_{ij})\in O_2^+$. Then operator (1) is bounded from l_{pv} into l_{qu} if and only if $M^+=\max\{M_{2,0}^+,M_{2,1}^+,M_{2,2}^+\}$, where

$$M_{2,0}^+=\left(\sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}(a_{si}^{2,0})^q u_s^q\right)^{\frac{p}{p-q}}\left(\sum_{j=1}^{i}v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_i^{-p'}\right)^{\frac{p-q}{pq}},$$

$$M_{2,1}^+=\left(\sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}\left(a_{si}^{2,1}\right)^q u_s^q\right)^{\frac{p}{p-q}}\left(\sum_{j=1}^{i}\left(a_{ij}^{(1)}\right)^{p'}v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}\Delta^-\left(\sum_{j=1}^{i}\left(a_{ij}^{(1)}\right)^{p'}v_j^{-p'}\right)\right)^{\frac{p-q}{pq}},$$

$$M_{2,2}^+=\left(\sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}u_s^q\right)^{\frac{p}{p-q}}\left(\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}\Delta^-\left(\sum_{j=1}^{i}(a_{ij})^{p'}v_j^{-p'}\right)\right)^{\frac{p-q}{pq}}.$$

Moreover, $\|A^+\|_{pv\to qu}\approx M^+$, where $\|A^+\|_{pv\to qu}$ is the norm of operator A^+ from l_{pv} into l_{qu}.

Our corresponding result for operator (2) reads as follows.

Theorem 2. Let $1<q<p<\infty$ and $(a_{ij})\in O_2^-$. Then operator (2) is bounded from l_{pv} into l_{qu} if and only if $\mathcal{M}^-=\max\{\mathcal{M}_{0,2}^-,\mathcal{M}_{1,2}^-,\mathcal{M}_{2,2}^-\}$, where

$$\mathcal{M}_{0,2}^-=\left(\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i}(a_{is}^{0,2})^q u_s^q\right)^{\frac{p}{p-q}}\left(\sum_{j=1}^{\infty}v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_i^{-p'}\right)^{\frac{p-q}{pq}},$$

$$\mathcal{M}_{1,2}^{-}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i}\left(a_{is}^{1,2}\right)^{q}u_s^q\right)^{\frac{p}{p-q}}\left(\sum_{j=i}^{\infty}\left(a_{ji}^{(1)}\right)^{p'}v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}\Delta^{+}\left(\sum_{j=i}^{\infty}\left(a_{ji}^{(1)}\right)^{p'}v_j^{-p'}\right)\right)^{\frac{p-q}{pq}},$$

$$\mathcal{M}_{2,2}^{-}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i}u_s^q\right)^{\frac{p}{p-q}}\left(\sum_{j=i}^{\infty}a_{ji}^{p'}v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}\Delta^{+}\left(\sum_{j=i}^{\infty}(a_{ji})^{p'}v_j^{-p'}\right)\right)^{\frac{p-q}{pq}}.$$

Moreover, $\|A^{-}\|_{pv\to qu}\approx\mathcal{M}^{-}$, where $\|A^{-}\|_{pv\to qu}$ is the norm of operator A^{-} from l_{pv} into l_{qu}.

Using the conjugacy of operators (1) and (2) from Theorem 1 and Theorem 2 we obtain the following results.

Theorem 3. Let $1<q<p<\infty$ and $(a_{ij})\in O_2^{+}$. Then operator (2) is bounded from l_{pv} into l_{qu} if and only if $M^{-}=\max\{M_{2,0}^{-},M_{2,1}^{-},M_{2,2}^{-}\}$, where

$$M_{2,0}^{-}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}(a_{si}^{2,0})^{p'}v_s^{-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{j=1}^{i}u_j^q\right)^{\frac{q}{p-q}}u_i^q\right)^{\frac{p-q}{pq}},$$

$$M_{2,1}^{-}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}(a_{si}^{2,1})^{p'}v_s^{-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{j=1}^{i}(a_{ij}^{(1)})^{q}u_j^q\right)^{\frac{q}{p-q}}\Delta^{-}\left(\sum_{j=1}^{i}(a_{ij}^{(1)})^{q}u_j^q\right)\right)^{\frac{p-q}{pq}},$$

$$M_{2,2}^{-}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}v_s^{-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{j=1}^{i}a_{ij}^{q}u_j^q\right)^{\frac{q}{p-q}}\Delta^{-}\left(\sum_{j=1}^{i}a_{ij}^{q}u_j^q\right)\right)^{\frac{p-q}{pq}}.$$

Moreover, $\|A^{-}\|_{pv\to qu}\approx M^{-}$, where $\|A^{-}\|_{pv\to qu}$ is the norm of operator A^{-} from l_{pv} into l_{qu}.

Theorem 4. Let $1<q<p<\infty$ and $(a_{ij})\in O_2^{-}$. Then operator (1) is bounded from l_{pv} into l_{qu} if and only if $\mathcal{M}^{+}=\max\{\mathcal{M}_{0,2}^{+},\mathcal{M}_{1,2}^{+},\mathcal{M}_{2,2}^{+}\}$, where

$$\mathcal{M}_{0,2}^{+}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i}(a_{is}^{2,0})^{p'}v_s^{-p'}\right)^{\frac{p}{p-q}}\left(\sum_{j=i}^{\infty}u_j^q\right)^{\frac{p(q-1)}{p-q}}u_i^q\right)^{\frac{p-q}{pq}},$$

$$\mathcal{M}_{1,2}^{+}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i}(a_{is}^{1,2})^{p'}v_s^{-p'}\right)^{\frac{p}{p-q}}\left(\sum_{j=i}^{\infty}(a_{ji}^{(1)})^{q}u_j^q\right)^{\frac{p(q-1)}{p-q}}\Delta^{+}\left(\sum_{j=i}^{\infty}(a_{ji}^{(1)})^{q}u_j^q\right)\right)^{\frac{p-q}{pq}},$$

$$\mathcal{M}_{2,2}^{+}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i}v_s^{-p'}\right)^{\frac{p}{p-q}}\left(\sum_{j=i}^{\infty}a_{ji}^{q}u_j^q\right)^{\frac{p(q-1)}{p-q}}\Delta^{+}\left(\sum_{j=i}^{\infty}a_{ji}^{q}u_j^q\right)\right)^{\frac{p-q}{pq}}.$$

Moreover, $\|A^{+}\|_{pv\to qu}\approx\mathcal{M}^{+}$, where $\|A^{+}\|_{pv\to qu}$ is the norm of operator A^{+} from l_{pv} into l_{qu}.

Since the proof of Theorem 2 is completely analogous to the proof of Theorem 1, we introduce the proof of Theorem 1.

Proof. Necessary. Let operator (1) be bounded from l_{pv} into l_{qu}, $\|A^+\|_{pv\to qu}<\infty$, i.e. the following inequality holds:

$$\left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i}a_{ij}f_j\right)^q u_i^q\right)^{\frac{1}{q}}\le\|A^+\|_{pv\to qu}\left(\sum_{i=1}^{\infty}f_i^p v_i^p\right)^{\frac{1}{p}},\tag{7}$$

for all non-negative sequences $f\in l_{pv}$, in particular, for non-negative finite sequences $f\in l_{pv}$. By applying (4), a relation $a_{ik}>>a_{ij}^{2,0}, i\ge j\ge k\ge 1$ from Definition 3 and using the Abel transform, we obtain

$$\sum_{i=1}^{\infty}(a_{ij}f_j)^q u_i^q\approx\sum_{i=1}^{\infty}\sum_{j=1}^{i}a_{ij}f_j\left(\sum_{s=1}^{j}a_{is}f_s\right)^{q-1}u_i^q>>$$

$$>>\sum_{i=1}^{\infty}\sum_{j=1}^{i}(a_{ij}^{2,0})^q f_j\left(\sum_{s=1}^{j}f_s\right)^{q-1}u_i^q=\sum_{j=1}^{\infty}f_j\left(\sum_{s=1}^{j}f_s\right)^{q-1}\sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q=$$

$$=\sum_{j=1}^{\infty}\Delta^-\left(\sum_{n=1}^{j}f_n\left(\sum_{s=1}^{n}f_s\right)^{q-1}\right)\sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q=$$

$$=\sum_{j=1}^{\infty}\left(\sum_{n=1}^{j}f_n\left(\sum_{s=1}^{n}f_s\right)^{q-1}\right)\Delta^+\left(\sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q\right)+\lim_{N\to\infty}\left(\sum_{n=1}^{N}f_n\left(\sum_{s=1}^{n}f_s\right)^{q-1}\right)\sum_{i=N+1}^{\infty}(a_{iN+1}^{2,0})^q u_i^q\approx$$

$$\approx\sum_{j=1}^{\infty}\left(\sum_{s=1}^{j}f_s\right)^{q}\Delta^+\left(\sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q\right)+\lim_{N\to\infty}\left(\sum_{n=1}^{N}f_n\left(\sum_{s=1}^{n}f_s\right)^{q-1}\right)\sum_{i=N+1}^{\infty}(a_{iN+1}^{2,0})^q u_i^q.$$

Due to the finiteness of f and $a_{ij}^{2,0}$ is non-increasing in j, we have

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}f_n\left(\sum_{s=1}^{n}f_s\right)^{q-1}\right)\sum_{i=N+1}^{\infty}(a_{iN+1}^{2,0})^q u_i^q=0.$$

Then

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i}a_{ij}f_j\right)^q u_i^q>>\sum_{j=1}^{\infty}\left(\sum_{s=1}^{j}f_s\right)^q\Delta^+\left(\sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q\right).$$

Hence and from (7) it follows that

$$\left(\sum_{j=1}^{\infty}\left(\sum_{s=1}^{j}f_s\right)^q\Delta^+\left(\sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q\right)\right)^{\frac{1}{q}}<<\|A^+\|_{pv\to qu}\left(\sum_{i=1}^{\infty}(f_i v_i)^p\right)^{\frac{1}{p}}.$$

Then according to Theorem A, we get

$$\infty>\|A^+\|_{pv\to qu}>>\left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}\Delta^+\left(\sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q\right)\right)^{\frac{p}{p-q}}\left(\sum_{s=1}^{k}v_s^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_k^{-p'}\right)^{\frac{p-q}{pq}}=$$

$$=\left(\sum_{k=1}^{\infty}\left(\sum_{i=k}^{\infty}(a_{ik}^{2,0})^q u_i^q\right)^{\frac{p}{p-q}}\left(\sum_{s=1}^{k}v_s^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_k^{-p'}\right)^{\frac{p-q}{pq}}=M_{2,0}^+.\tag{8}$$

Inequality (7) holds if and only if the following dual inequality

$$\left(\sum_{j=1}^{\infty}\left(\sum_{i=j}^{\infty}a_{ij}g_i\right)^{p'}v_j^{-p'}\right)^{\frac{1}{p'}} << \|A^+\|_{pv\to qu}\left(\sum_{i=1}^{\infty}(g_iu_i^{-1})^{q'}\right)^{\frac{1}{q'}} \tag{9}$$

holds for all non-negative sequences $g \in l_{q',u^{-1}}$, in particular, for non-negative finite sequences $g \in l_{q',u^{-1}}$. Using (5), a relation $a_{ij} << a_{kj}, k \geq i$ from Definition 3 and applying the Abel transform, we obtain

$$\sum_{j=1}^{\infty}\left(\sum_{i=j}^{\infty}a_{ij}g_i\right)^{p'}v_j^{-p'} \approx \sum_{j=1}^{\infty}\sum_{i=j}^{\infty}a_{ij}g_i\left(\sum_{s=i}^{\infty}a_{sj}g_s\right)^{p'-1}v_j^{-p'} >>$$

$$>> \sum_{j=1}^{\infty}\sum_{i=j}^{\infty}a_{ij}^{p'}g_i\left(\sum_{s=i}^{\infty}g_s\right)^{p'-1}v_j^{-p'} = \sum_{i=1}^{\infty}g_i\left(\sum_{s=i}^{\infty}g_s\right)^{p'-1}\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'} =$$

$$= \sum_{i=1}^{\infty}\Delta^+\left(\sum_{n=i}^{\infty}g_n\left(\sum_{s=n}^{\infty}g_s\right)^{p'-1}\right)\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'} =$$

$$= \sum_{i=1}^{\infty}\left(\sum_{n=i}^{\infty}g_n\left(\sum_{s=n}^{\infty}g_s\right)^{p'-1}\right)\Delta^-\left(\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'}\right) + \lim_{N\to\infty}\left(\sum_{n=N+1}^{\infty}g_n\left(\sum_{s=n}^{\infty}g_s\right)^{p'-1}\right)\sum_{j=1}^{N}a_{Nj}^{p'}v_j^{-p'} \approx$$

$$\approx \sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}g_s\right)^{p'}\Delta^-\left(\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'}\right) + \lim_{N\to\infty}\left(\sum_{n=N+1}^{\infty}g_n\left(\sum_{s=n}^{\infty}g_s\right)^{p'-1}\right)\sum_{j=1}^{N}a_{Nj}^{p'}v_j^{-p'}.$$

Due to the finiteness of g we have, that

$$\lim_{N\to\infty}\left(\sum_{n=N+1}^{\infty}g_n\left(\sum_{s=n}^{\infty}g_s\right)^{p'-1}\right)\sum_{j=1}^{N}a_{Nj}^{p'}v_j^{-p'} = 0.$$

Since $\Delta^-(\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'}) \geq 0$, we assume $\omega_i = \left(\Delta^-(\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'})\right)^{\frac{1}{p'}}$. Then

$$\sum_{j=1}^{\infty}\left(\sum_{i=j}^{\infty}a_{ij}g_i\right)^{p'}v_j^{-p'} >> \sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}g_s\right)^{p'}\omega_i^{p'}.$$

Hence and from (9) it follows

$$\left(\sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}g_s\right)^{p'}\omega_i^{p'}\right)^{\frac{1}{p'}} << \|A^+\|_{pv\to qu}\left(\sum_{i=1}^{\infty}(g_iu_i^{-1})^{q'}\right)^{\frac{1}{q'}}. \tag{10}$$

We pass to dual inequality (10), i.e.

$$\left(\sum_{j=1}^{\infty}\left(\sum_{s=1}^{j}f_s\right)^{q}u_j^{q}\right)^{\frac{1}{q}} << \|A^+\|_{pv\to qu}\left(\sum_{i=1}^{\infty}\left(f_i\omega_i^{-1}\right)^{p}\right)^{\frac{1}{p}}, \quad 0 \leq f \in l_{pv}.$$

Then by applying Theorem A, we obtain

$$\infty > \|A^+\|_{pv\to qu} >> \left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}u_j^q\right)^{\frac{p}{p-q}}\left(\sum_{s=1}^{k}\omega_s^{p'}\right)^{\frac{p(q-1)}{p-q}}\omega_k^{p'}\right)^{\frac{p-q}{pq}} =$$

$$= \left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}u_j^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}\Delta^-\left(\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'}\right)\right)^{\frac{p(q-1)}{p-q}}\Delta^-\left(\sum_{j=1}^{k}a_{kj}^{p'}v_j^{-p'}\right)\right)^{\frac{p-q}{pq}} =$$

$$= \left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}u_j^q\right)^{\frac{p}{p-q}}\left(\sum_{j=1}^{k}a_{kj}^{p'}v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}\Delta^-\left(\sum_{j=1}^{k}a_{kj}^{p'}v_j^{-p'}\right)\right)^{\frac{p-q}{pq}} = M_{2,2}^+. \tag{11}$$

From Definition 3 it follows, that $a_{ij} >> a_{ik}^{2,1}a_{kj}^{(1)}, i \geq k \geq j \geq 1$. Then for $i \geq k \geq j \geq 1$

$$a_{ij} >> a_{ik}^{2,1}a_{kj}^{(1)} = a_{ik}^{2,1}a_{kj}^{(1)}\theta_k, \tag{12}$$

where

$$\theta_k = \begin{cases} 1, & j \leq k \leq i, \\ 0, & k > i, \quad k < j. \end{cases}$$

Let $\varphi = \{\varphi_i\}_{i=1}^{\infty}$ be a sequence of non-negative numbers such that $\sum_{i=1}^{\infty}\varphi_i = 1$. Multiplying both parts of (12) to φ and summing up by $k \in N$, we have

$$a_{ij} >> \sum_{k=j}^{i}a_{ik}^{2,1}a_{kj}^{(1)}\varphi_k. \tag{13}$$

Then using (13) and changing the order of summation twice, we have

$$\sum_{i=1}^{\infty}u_i^q\left(\sum_{j=1}^{i}a_{ij}f_j\right)^q = \sum_{i=1}^{\infty}u_i^q\sum_{j=1}^{i}a_{ij}f_j\left(\sum_{s=1}^{i}a_{is}f_s\right)^{q-1} \geq$$

$$\geq \sum_{i=1}^{\infty}u_i^q\sum_{j=1}^{i}\left(\sum_{k=j}^{i}a_{ik}^{2,1}a_{kj}^{(1)}\varphi_k\right)f_j\left(\sum_{s=1}^{i}\left(\sum_{\tau=s}^{i}a_{i\tau}^{2,1}a_{\tau s}^{(1)}\varphi_\tau\right)f_s\right)^{q-1} =$$

$$= \sum_{i=1}^{\infty}u_i^q\sum_{k=1}^{i}\varphi_k a_{ik}^{2,1}\sum_{j=1}^{k}a_{kj}^{(1)}f_j\left(\sum_{\tau=1}^{i}\varphi_\tau a_{i\tau}^{2,1}\sum_{s=1}^{\tau}a_{\tau s}^{(1)}f_s\right)^{q-1} \geq$$

$$\geq \sum_{k=1}^{\infty}\varphi_k\left(\sum_{j=1}^{k}a_{kj}^{(1)}f_j\right)\sum_{i=k}^{\infty}u_i^q a_{ik}^{2,1}\left(\sum_{\tau=k}^{i}\varphi_\tau a_{i\tau}^{2,1}\sum_{s=1}^{\tau}a_{\tau s}^{(1)}f_s\right)^{q-1} >>$$

$$>> \sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}a_{kj}^{(1)}f_j\right)^q\sum_{i=k}^{\infty}u_i^q a_{ik}^{2,1}\left(\sum_{\tau=k}^{i}\varphi_\tau a_{i\tau}^{2,1}\right)^{q-1}\varphi_k =$$

$$= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} a_{kj}^{(1)} f_j \right)^q h_k, \tag{14}$$

where $h_k = \sum\limits_{i=k}^{\infty} u_i^q a_{ik}^{2,1} \left(\sum\limits_{\tau=k}^{i} \varphi_\tau a_{i\tau}^{2,1} \right)^{q-1} \varphi_k$. From (7) and (14) it follows

$$\left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} a_{kj}^{(1)} f_j \right)^q h_k \right)^{\frac{1}{q}} << \|A^+\|_{pv \to qu} \left(\sum_{i=1}^{\infty} (f_i v_i)^p \right)^{\frac{1}{p}}, \; 0 \le f \in l_{pv}.$$

By applying Theorem B and taking into account (4), we get

$$\|A^+\|_{pv \to qu} >> B_1 := \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} h_j \right)^{\frac{q}{p-q}} \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} h_k \right)^{\frac{p-q}{pq}}.$$

Using that $B_1 < \infty$ and $\sum\limits_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'}$ is increasing in k, we have

$$0 = \lim_{N\to\infty} \sum_{k=N}^{\infty} h_k \left(\sum_{j=k}^{\infty} h_j \right)^{\frac{q}{p-q}} \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \ge$$

$$\ge \lim_{N\to\infty} \sum_{k=N}^{\infty} h_k \left(\sum_{j=k}^{\infty} h_j \right)^{\frac{q}{p-q}} \left(\sum_{i=1}^{N} (a_{Ni}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}}.$$

Further, using this relation to the Abel transform in B_1, (5) and the following elementary estimate

$$b^\gamma - a^\gamma \approx b^{\gamma-1}(b-a), \tag{15}$$

where $b > a > 0, \gamma > 0$, we obtain

$$\|A^+\|_{pv \to qu} >> B_1 \approx \left(\sum_{k=1}^{\infty} \Delta^+ \left(\sum_{j=k}^{\infty} h_j \left(\sum_{s=j}^{\infty} h_s \right)^{\frac{q}{p-q}} \right) \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{pq}} =$$

$$= \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} h_j \left(\sum_{s=j}^{\infty} h_s \right)^{\frac{q}{p-q}} \right) \Delta^- \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{pq}} \approx$$

$$\approx \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} h_j \right)^{\frac{p}{p-q}} \Delta^- \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{pq}} \ge$$

$$\ge \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} h_j \right)^{\frac{p}{p-q}} \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} \Delta^- \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right) \right)^{\frac{p-q}{pq}},$$

where

$$\sum_{j=k}^{\infty} h_j = \sum_{j=k}^{\infty}\sum_{i=j}^{\infty} u_i^q a_{ij}^{(2,1)} \left(\sum_{\tau=j}^{i} \varphi_\tau a_{i\tau}^{(2,1)}\right)^{q-1} \varphi_j = \sum_{i=k}^{\infty} u_i^q \sum_{j=k}^{i} a_{ij}^{(2,1)} \varphi_j \left(\sum_{\tau=j}^{i} \varphi_\tau a_{i\tau}^{(2,1)}\right)^{q-1} \approx$$

$$\approx \sum_{i=k}^{\infty} u_i^q \left(\sum_{j=k}^{i} \varphi_j a_{ij}^{(2,1)}\right)^q .$$

Therefore, due to $\forall \varphi : \sum\limits_{k=1}^{\infty} \varphi_k = 1$, we have
$\|A^+\|_{pv\to qu} >>$

$$>> \sup_{\varphi} \left(\sum_{k=1}^{\infty}\left(\sum_{i=k}^{\infty} u_i^q \left(\sum_{j=k}^{i} \varphi_j a_{ij}^{2,1}\right)^q\right)^{\frac{p}{p-q}} \left(\sum_{s=1}^{k} (a_{ks}^{(1)})^{p'} v_s^{-p'}\right)^{\frac{p(q-1)}{p-q}} \Delta^- \left(\sum_{s=1}^{k} (a_{ks}^{(1)})^{p'} v_s^{-p'}\right)\right)^{\frac{p-q}{pq}} .$$

Assume, that $\varphi_j = \delta_j(m), m \geq 1$, where

$$\delta_j(m) = \begin{cases} 1, & j = m, \\ 0, & j \neq m. \end{cases}$$

Then taking into account that $a_{ij}^{2,1}$ is non-increasing in j
$\|A^+\|_{pv\to qu} >>$

$$>> \sup_{m\geq 1} \left(\sum_{k=1}^{\infty}\left(\sum_{i=k}^{\infty} u_i^q \left(\sum_{j=k}^{i} a_{ij}^{2,1} \delta_j(m)\right)^q\right)^{\frac{p}{p-q}} \left(\sum_{s=1}^{k} (a_{ks}^{(1)})^{p'} v_s^{-p'}\right)^{\frac{p(q-1)}{p-q}} \Delta^- \left(\sum_{s=1}^{k} (a_{ks}^{(1)})^{p'} v_s^{-p'}\right)\right)^{\frac{p-q}{pq}} =$$

$$= \left(\sum_{k=1}^{\infty}\left(\sum_{i=k}^{\infty} u_i^q (a_{ik}^{2,1})^q\right)^{\frac{p}{p-q}} \left(\sum_{s=1}^{k} (a_{ks}^{(1)})^{p'} v_s^{-p'}\right)^{\frac{p(q-1)}{p-q}} \Delta^- \left(\sum_{s=1}^{k} (a_{ks}^{(1)})^{p'} v_s^{-p'}\right)\right)^{\frac{p-q)}{pq}} = M_{2,1}^+. \qquad (16)$$

Thus, from (8), (11) and (16) it follows

$$M^+ = \max\{M_{2,0}^+, M_{2,1}^+, M_{2,2}^+\} << \|A^+\|_{pv\to qu} < \infty. \qquad (17)$$

Sufficiency. Let $M^+ < \infty$ and $0 \leq f \in l_{pv}$. $\mathbb{Z}$ is the set of integer numbers. Let's assume $\sum\limits_{i=k}^{n} = 0$ when $k > n$ and $a_{ij} = 0$ when $i < j$.

For all $i \geq 1$ we define the following set of integer numbers:

$$T_i = \{k \in \mathbb{Z} : (r_2 + 1)^k \leq (A^+ f)_i\},$$

where r_2 is the constant from Definition 3 and we assume that $k_i = \max T_i$. Then

$$(r_2 + 1)^{k_i} \leq (A^+ f)_i < (r_2 + 1)^{k_i+1}, \forall i \in \mathbb{N}. \qquad (18)$$

Let $m_1 = 1$ and $M_1 = \{i \in \mathbb{N} : k_i = k_1 = k_{m_1}\}$. Suppose that m_2 is such that $\sup M_1 + 1 = m_2$. Obviously $m_2 > m_1$ and if the set M_1 is upper bounded, then $m_2 < \infty$ and $m_2 - 1 = \max M_1 = \sup M_1$.

Suppose that we have found numbers $1 = m_1 < m_2 < ... < m_s < \infty$, $s \geq 1$, then we define m_{s+1} by $m_{s+1} = \sup M_s + 1$, where $M_s = \{i \in \mathbb{N} : k_i = k_{m_s}\}$.

Let $N_0 = \{s \in \mathbb{N} : m_s < \infty\}$. Further, we assume that $k_{m_s} = n_s, s \in N_0$. From the definition of m_s and from (18) it follows that, for $s \in N_0$

$$(r_2+1)^{n_s} \leq (A^+f)_i < (r_2+1)^{n_s+1}, m_s \leq i \leq m_{s+1}-1 \tag{19}$$

and $\mathbb{N} = \bigcup_{s\in N_0}[m_s, m_{s+1}-1]$, where $[m_s, m_{s+1}) \cap [m_l, m_{l+1}) \neq 0$.

By using (19), Definition 3 and $n_{s-2}+1 \leq n_s-1$, which follows from the inequality $n_{s-2} < n_{s-1} < n_s$, we can estimate the value $(r_2+1)^{n_s-1}$:

$$(r_2+1)^{n_s-1} = (r_2+1)^{n_s} - r_2(r_2+1)^{n_s-1} \leq (r_2+1)^{n_s} - r_2(r_2+1)^{n_{s-2}+1} \leq$$

$$\leq (A^+f)_{m_s} - r_2(A^+f)_{m_{s-1}-1} = \sum_{i=1}^{m_s} a_{m_s i} f_i - r_2 \sum_{i=1}^{m_{s-1}-1} a_{m_{s-1}i} f_i =$$

$$= \sum_{i=m_{s-1}}^{m_s} a_{m_s i} f_i + \sum_{i=1}^{m_{s-1}-1} [a_{m_s i} - r_2 a_{m_{s-1}-1i}] f_i <<$$

$$<< \sum_{i=m_{s-1}}^{m_s} a_{m_s i} f_i + \sum_{i=1}^{m_{s-1}-1} [r_2 a^{2,0}_{m_s m_{s-1}-1} + r_2 a^{2,1}_{m_s m_{s-1}-1} a^{(1)}_{m_{s-1}-1i}] f_i <<$$

$$<< \sum_{i=m_{s-1}}^{m_s} a_{m_s i} f_i + r_2 a^{2,0}_{m_s m_{s-1}-1} \sum_{i=1}^{m_{s-1}-1} f_i + r_2 a^{2,1}_{m_s m_{s-1}-1} \sum_{i=1}^{m_{s-1}-1} a^{(1)}_{m_{s-1}-1i} f_i. \tag{20}$$

Then taking into account (20), we get

$$\|A^+f\|^q_{qu} = \sum_{s\in N_0} \sum_{i=m_s}^{m_{s+1}-1} u_i^q (A^+f)_i^q < \sum_{s\in N_0} (r_2+1)^{(n_s+1)q} \sum_{i=m_s}^{m_{s+1}-1} u_i^q \leq$$

$$\leq (r_2+1)^{2q} \sum_{s\in N_0} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s i} f_i + r_2 a^{2,0}_{m_s m_{s-1}-1} \sum_{i=1}^{m_{s-1}-1} f_i + r_2 a^{2,1}_{m_s m_{s-1}-1} \sum_{i=1}^{m_{s-1}-1} a^{(1)}_{m_{s-1}-1i} f_i \right)^q \sum_{i=m_s}^{m_{s+1}-1} u_i^q <<$$

$$<< \sum_{s\in N_0} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s i} f_i \right)^q \sum_{i=m_s}^{m_{s+1}-1} u_i^q + \sum_{s\in N_0} \left(a^{2,0}_{m_s m_{s-1}-1} \right)^q \left(\sum_{i=1}^{m_{s-1}-1} f_i \right)^q \sum_{i=m_s}^{m_{s+1}-1} u_i^q +$$

$$+ \sum_{s\in N_0} \left(a^{2,1}_{m_s m_{s-1}-1} \right)^q \left(\sum_{i=1}^{m_{s-1}-1} a_{m_{s-1}-1i} f_i \right)^q \sum_{i=m_s}^{m_{s+1}-1} u_i^q =$$

$$= S_{2,2} + S_{2,0} + S_{2,1}. \tag{21}$$

By applying Hölder's inequality twice and (5), we estimate $S_{2,2}$.

$$S_{2,2} = \sum_{s\in N_0} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s i} f_i v_i v_i^{-1} \right)^q \sum_{i=m_s}^{m_{s+1}-1} u_i^q \leq \sum_{s\in N_0} \left(\sum_{i=m_{s-1}}^{m_s} (f_i v_i)^p \right)^{\frac{q}{p}} \left(\sum_{i=m_{s-1}}^{m_s} a^{p'}_{m_s i} v_i^{-p'} \right)^{\frac{q}{p'}} \sum_{i=m_s}^{m_{s+1}-1} u_i^q \leq$$

$$\leq \left(\sum_{s\in N_0}\sum_{i=m_{s-1}}^{m_s}(f_iv_i)^p\right)^{\frac{q}{p}}\left(\sum_{s\in N_0}\left(\sum_{i=m_{s-1}}^{m_s}a_{m_si}^{p'}v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{i=m_s}^{m_{s+1}-1}u_i^q\right)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\leq$$

$$\leq 2^{\frac{q}{p}}\|f\|_{pv}^q\left(\sum_{s\in N_0}\left(\sum_{i=m_{s-1}}^{m_s}a_{m_si}^{p'}v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}}\sum_{i=m_s}^{m_{s+1}-1}u_i^q\left(\sum_{j=i}^{m_{s+1}-1}u_j^q\right)^{\frac{q}{p-q}}\right)^{\frac{p-q}{p}}\leq$$

$$\leq \|f\|_{pv}^q\left(\sum_{i=1}^{\infty}u_i^q\left(\sum_{j=i}^{\infty}u_j^q\right)^{\frac{q}{p-q}}\left(\sum_{n=1}^{i}a_{in}^{p'}v_n^{-p'}\right)^{\frac{q(p-1)}{p-q}}\right)^{\frac{p-q}{p}}=\|f\|_{l_{pv}}^q\tilde{M}_{2,2}^q,$$

where

$$\tilde{M}_{2,2}^{\frac{pq}{p-q}}=\sum_{i=1}^{\infty}u_i^q\left(\sum_{j=i}^{\infty}u_j^q\right)^{\frac{q}{p-q}}\left(\sum_{n=1}^{i}a_{in}^{p'}v_n^{-p'}\right)^{\frac{q(p-1)}{p-q}}.$$

Using the Abel transform, (5) and (15), we have

$$\tilde{M}_{2,2}^{\frac{pq}{p-q}}=\sum_{i=1}^{\infty}\Delta^+\left(\sum_{k=i}^{\infty}u_k^q\left(\sum_{j=k}^{\infty}u_j^q\right)^{\frac{q}{p-q}}\right)\left(\sum_{n=1}^{i}a_{in}^{p'}v_n^{-p'}\right)^{\frac{q(p-1)}{p-q}}=$$

$$=\sum_{i=1}^{\infty}\left(\sum_{k=i}^{\infty}u_k^q\left(\sum_{j=k}^{\infty}u_j^q\right)^{\frac{q}{p-q}}\right)\Delta^-\left(\sum_{n=1}^{i}a_{in}^{p'}v_n^{-p'}\right)^{\frac{q(p-1)}{p-q}}\approx$$

$$\approx\sum_{i=1}^{\infty}\left(\sum_{k=i}^{\infty}u_k^q\right)^{\frac{p}{p-q}}\left(\sum_{n=1}^{i}a_{in}^{p'}v_n^{-p'}\right)^{\frac{p(q-1)}{p-q}}\Delta^-\left(\sum_{n=1}^{i}a_{in}^{p'}v_n^{-p'}\right)=M_{2,2}^+<\infty.$$

Therefore

$$S_{2,2}<<(M_{2,2}^+)^q\|f\|_{pv}^q. \tag{22}$$

To estimate $S_{2,0}$, we assume

$$\eta_i(m_{s-1}-1)=\begin{cases}\sum\limits_{s\in N_0}\left(a_{m_sm_{s-1}-1}^{2,0}\right)^q\sum\limits_{i=m_s}^{m_{s+1}-1}u_i^q, & i=m_{s-1}-1,\\ 0, & i\neq m_{s-1}-1.\end{cases}$$

and we use Theorem A.

$$S_{2,0}=\sum_{s\in N_0}\left(a_{m_sm_{s-1}-1}^{2,0}\right)^q\left(\sum_{i=1}^{m_{s-1}-1}f_i\right)^q\sum_{i=m_s}^{m_{s+1}-1}u_i^q=\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}f_i\right)^q\eta_n<<$$

$$<<\left(\sum_{n=1}^{\infty}\left(\sum_{i=n}^{\infty}\eta_j\right)^{\frac{p}{p-q}}\left(\sum_{j=1}^{n}v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_n^{-p'}\right)^{\frac{p-q}{p}}\|f\|_{pv}^q. \tag{23}$$

Taking into account Remark 1, we estimate $\sum\limits_{i=n}^{\infty}\eta_i$:

$$\sum_{i=n}^{\infty}\eta_i=\sum_{s:m_{s-1}-1\geq n}\left(a^{2,0}_{m_s m_{s-1}-1}\right)^q\sum_{i=m_s}^{m_{s+1}-1}u_i^q=\sum_{s:m_{s-1}-1\geq n}\sum_{i=m_s}^{m_{s+1}-1}\left(a^{2,0}_{m_s m_{s-1}-1}\right)^q u_i^q<<\sum_{i=n}^{\infty}\left(a^{2,0}_{in}\right)^q u_i^q.$$

Hence and from (23), we have

$$S_{2,0}<<\left(\sum_{n=1}^{\infty}\left(\sum_{i=n}^{\infty}\left(a^{2,0}_{in}\right)^q u_i^q\right)^{\frac{p}{p-q}}\left(\sum_{j=1}^{n}v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_n^{-p'}\right)^{\frac{p-q}{pq}}\|f\|^q_{pv}=(M^+_{2,0})^q\|f\|^q_{pv}. \qquad (24)$$

Now, by using Theorem B we estimate $S_{2,1}$.

$$S_{2,1}=\sum_{s\in N_0}\left(a^{2,1}_{m_s m_{s-1}-1}\right)^q\left(\sum_{i=1}^{m_{s-1}-1}a^{(1)}_{m_{s-1}-1i}f_i\right)^q\sum_{i=m_s}^{m_{s+1}-1}u_i^q<<$$

$$<<\sum_{s\in N_0}\left(\sum_{i=1}^{m_{s-1}-1}a^{(1)}_{m_{s-1}-1\,i}f_i\right)^q\sum_{i=m_s}^{m_{s+1}-1}\left(a^{2,1}_{i\,m_{s-1}-1}\right)^q u_i^q=$$

$$=\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}a^{(1)}_{ki}f_i\right)^q\theta_k\leq\left(\max\{\tilde{B}_0,\tilde{B}_1\}\right)^q\|f\|^q_{l_{pv}}, \qquad (25)$$

where $\theta_k=\sum\limits_{s\in N_0}\sum\limits_{n=m_s}^{m_{s+1}-1}\left(a^{2,1}_{n\,m_{s-1}-1}\right)^q u_n^q$ when $k=m_{s-1}-1$ and $\theta_k=0$ when $k\neq m_{s-1}-1$,

$$\tilde{B}_0=\left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}\left(a^{1,0}_{jk}\right)^q\theta_j\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_k^{-p'}\right)^{\frac{p-q}{pq}},$$

$$\tilde{B}_1=\left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}\theta_j\right)^{\frac{q}{p-q}}\left(\sum_{i=1}^{k}\left(a^{(1)}_{ki}\right)^{p'}v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}}\theta_k\right)^{\frac{p-q}{pq}}.$$

Let's evaluate the expression $\sum\limits_{j=k}^{\infty}\left(a^{1,0}_{jk}\right)^q\theta_j$ in $\tilde{B}_0$.

$$\sum_{j=k}^{\infty}\left(a^{1,0}_{jk}\right)^q\theta_j=\sum_{s:m_{s-1}-1\geq k}\left(a^{1,0}_{m_{s-1}-1\,k}\right)^q\sum_{n=m_s}^{m_{s+1}-1}\left(a^{2,1}_{n\,m_{s-1}-1}\right)^q u_n^q=$$

$$=\sum_{s:m_{s-1}-1\geq k}\sum_{n=m_s}^{m_{s+1}-1}\left(a^{2,1}_{n\,m_{s-1}-1}\right)^q\left(a^{1,0}_{m_{s-1}-1\,k}\right)^q u_n^q.$$

In [10] it is shown that $a^{2,1}_{n\,m_{s-1}-1}a^{1,0}_{m_{s-1}-1\,k}<<a^{2,0}_{n\,k}$ when $n\geq m_{s-1}-1\geq k\geq 1$. Then

$$\sum_{j=k}^{\infty}\left(a^{1,0}_{jk}\right)^q\theta_j<<\sum_{n=k}^{\infty}\left(a^{2,0}_{nk}\right)^q u_n^q.$$

Thus
$$\tilde{B}_0 << \left(\sum_{k=1}^{\infty} \left(\sum_{n=k}^{\infty} \left(a_{nk}^{2,0} \right)^q u_n^q \right)^{\frac{p}{p-q}} \left(\sum_{i=1}^{k} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} v_k^{-p'} \right)^{\frac{p-q}{pq}} = M_{2,0}^{+} < \infty. \tag{26}$$

By using the Abel transform, (5) and (15) we estimate the value $\tilde{B}_1$.
$$\tilde{B}_1^{\frac{pq}{p-q}} = \sum_{k=1}^{\infty} \Delta^{+} \left(\sum_{i=k}^{\infty} \theta_i \left(\sum_{j=i}^{\infty} \theta_j \right)^{\frac{q}{p-q}} \right) \left(\sum_{i=1}^{k} \left(a_{ki}^{(1)} \right)^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} =$$

$$= \sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} \theta_i \left(\sum_{j=i}^{\infty} \theta_j \right)^{\frac{q}{p-q}} \right) \Delta^{-} \left(\sum_{i=1}^{k} \left(a_{ki}^{(1)} \right)^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \approx$$

$$\approx \sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} \theta_i \right)^{\frac{p}{p-q}} \left(\sum_{i=1}^{k} \left(a_{ki}^{(1)} \right)^{p'} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} \Delta^{-} \left(\sum_{i=1}^{k} \left(a_{ki}^{(1)} \right)^{p'} v_i^{-p'} \right). \tag{27}$$

Since
$$\sum_{i=k}^{\infty} \theta_i = \sum_{s: m_{s-1}-1 \geq k} \sum_{n=m_s}^{m_{s+1}-1} \left(a_{n,m_{s-1}-1}^{2,1} \right)^q u_n^q \leq \sum_{n=k}^{\infty} \left(a_{n\,k}^{2,1} \right)^q u_n^q,$$

hence and from (27), it follows
$$\tilde{B}_1 << \left(\sum_{k=1}^{\infty} \left(\sum_{n=k}^{\infty} \left(a_{nk}^{2,1} \right)^q u_n^q \right)^{\frac{p}{p-q}} \left(\sum_{i=1}^{k} \left(a_{ki}^{(1)} \right)^{p'} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} \Delta^{-} \left(\sum_{i=1}^{k} \left(a_{ki}^{(1)} \right)^{p'} v_i^{-p'} \right) \right)^{\frac{p-q}{pq}} = M_{2,1}^{+}. \tag{28}$$

Thus, from (25), (26) and (28), we obtain
$$S_{2,1} << \left(\max\{M_{2,0}^{+}, M_{2,1}^{+}\} \right)^q \|f\|_{l_{pv}}^q.$$

Hence and from (21), (22), (24) we have
$$\|A^{+} f\|_{qu} << \max\{M_{2,0}^{+}, M_{2,1}^{+}, M_{2,2}^{+}\} \|f\|_{l_{pv}} = M^{+} \|f\|_{l_{pv}},$$

i.e. the operator A^{+} is bounded from l_{pv} into l_{qu} and takes place for the norm $\|A^{+}\|_{pv\to qu} << M^{+}$, which with (17) gives us $\|A^{+}\|_{pv\to qu} \approx M^{+}$.

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Матрицалық операторлар бір класының l_{pv}-дан l_{qu}-ға шенелгендік критерийі

Матрицалар теориясының негізгі міндеттерінің бірі матрицалық оператордың бір нормалы кеңістіктен басқа нормалы кеңістікке үзіліссіз өту үшін матрицалар элементтеріне қажетті және жеткілікті шарттарын анықтау. Сонымен қатар матрицалық оператордың нормасын немесе оның дәл жоғарғы және төменгі бағалауын табу маңызды. Бұл есеп жалпы жағдайда Лебег тізбектер кеңістігінде ашық есеп. Берілген мақалада матрицалық операторының l_{pv}-дан l_{qu}-ға $1 < q < p < \infty$ болғанда шенелгендігі қарастырылған және бұл есептің қажетті және жеткілікті шарттары алынды, мұндағы матрица $O_2^{\pm}$ дискретті Ойнаров класына тиісті.

Кілт сөздер: матрицалық оператор, түйіндес оператор, салмақты тізбек, шенелгендік, салмақты теңсіздіктер, Лебег салмақты кеңістігі, Ойнаров шарты, Харди операторы, Харди теңсіздігі, матрица.

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Критерий ограниченности некоторого класса матричных операторов из l_{pv} в l_{qu}

Одной из основных задач теории матриц является нахождение необходимых и достаточных условий для элементов матрицы, при которых матричный оператор непрерывно действует из одного нормированного пространства в другое. При этом очень важно найти значение нормы матричного оператора, в крайнем случае, зафиксировать точные верхние и нижние оценки. Эта задача в лебеговых пространствах последовательностей в общем случае остается открытой. В статье рассмотрена проблема ограниченности матричных операторов из l_{pv} в l_{qu} при $1 < q < p < \infty$ и получены необходимые и достаточные условия этой задачи, когда матричные операторы принадлежат классам $O_2^{\pm}$, удовлетворяющим более слабым условиям, чем условие Ойнарова.

Ключевые слова: матричный оператор, сопряженный оператор, весовая последовательность, ограниченность, весовые неравенства, весовое пространство Лебега, условие Ойнарова, оператор Харди, неравенство Харди, матрица.

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On the Fourier transform of functions from a Lorentz space $L_{\overline{2},\overline{r}}$ with a mixed metric

The classical inequalities of Bochkarev play a very important role in harmonic analysis. The meaning of these inequalities lies in the connection between the metric characteristics of functions and the summability of their Fourier coefficients. One of the most important directions of harmonic analysis is the theory of Fourier series. His interest in this direction is explained by his applications in various departments of modern mathematics and applied sciences, as well as the availability of many unsolved problems. One of these problems is the study of the interrelationships of the integral properties of functions and the properties of the sum of its coefficients. The solution of these problems was dedicated to the efforts of many mathematicians. And further research in this area are important and interesting problems and can give new, unexpected effects. In the article we receive a two-dimensional analog of the Bochkarev type theorem for the Fourier transform.

Keywords: Lorentz Space, Hausdorff-Young-Riesz theorem, Bochkarev's theorem, Cauchy-Bunyakovsky inequality, Helder's inequality.

Introduction

This article is devoted to the Hardy-Littlewood inequalities for an anisotropic Lorentz space. This inequalities characterize the connection between the Fourier coefficients and integral properties of the function. The study of relationship between the integrality of a function and the summability of its Fourier coefficients has been the subject of many papers. There are well-known classical results in this direction, such as Parseval, Bessel, Riesz, Hardy-Littlewood, Palley, Stein [1, 2], also modern works [3–11] and others. However, the Hausdorff-Young-Riesz theorem does not extend to the spaces $L_{2,r}$, if $r \neq 2$.

In 1997 Bochkarev S.V. [12] established that, in contrast to the spaces $L_{p,r}$, $1 < p < 2, 1 \leq r \leq \infty$, in the Lorentz space $L_{2,r},\ 2 < r \leq \infty$ the direct analogue of the Hausdorff - Young - Riesz theorem is not satisfied. And he derived upper bounds for the Fourier coefficients of functions from $L_{2,r}$ replacing the Hausdorff-Young-Riesz theorem and proved that for some class of multiplicative systems these estimates are unstrengthened.

Theorem (S.V. Bochkarev). Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal system of complex-valued functions on $[0, 1]$,

$$\|\phi_n\| \leq M,\ n = 1, 2, ...$$

and let $f \in L_{2,r}, \quad 2 < r \leq \infty$. Then the inequality

$$\sup_{n \in N} \frac{1}{|n|^{\frac{1}{2}} \log(n+1)^{\frac{1}{2}-\frac{1}{r}}} \sum_{m=1}^{n} a_m^* \leq C\|f\|_{L_{2,r'}}$$

holds, where a_n are the Fourier coefficients of the system $\{\phi_n\}_{n=1}^{\infty}$.

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In 2015 an analogue of Bochkarev's theorem was received for the Fourier transform of a function from the space $L_{2,r}(\mathbb{R})$.

Theorem A [13]. *Let* $\Re_N = \{A = \bigcup_{i=1}^{N} A_i$, *where* A_i *are segments in* $\mathbb{R}\}$, *then for any functions* $f \in L_{2,r}(\mathbb{R})$, $2 < r < \infty$ *the following inequality holds:*

$$\sup_{N\geq 8}\sup_{A\subset\Re_N}\frac{1}{|A|^{\frac{1}{2}}\log_2(1+N)^{\frac{1}{2}-\frac{1}{r}}}\left|\int_A \hat{f}(\xi)d\xi\right| \leq 23\|f\|_{L_{2,r}}.$$

The aim of this article is to obtain a two-dimensional analog of the Bochkarev type theorem for the Fourier transform. To do this, we need to introduce the following definitions:

Definition 1 [14]. Let $\bar{p} = (p_1, p_2)$, $\bar{r} = (r_1, r_2)$ and satisfy the following conditions: $0 < \bar{p} \leq \infty$, $0 < \bar{r} \leq \infty$. The Lorentz Space $L_{\bar{p},\bar{r}}[0,1]^2$ with a mixed metric is defined as the set of all measurable functions defined on $[0,1]^2$, for which the norms are finite:

$$\|f\|_{L_{\bar{p},\bar{r}}} = \|\|f\|_{L_{p_1,r_1}}\|_{L_{p_2,r_2}} = \left(\int_0^1\left(t_2^{\frac{1}{p_2}}\left(\int_0^1\left(t_1^{\frac{1}{p_1}}f^{*_1}(t_1,\cdot)\right)^{r_1}\frac{dt_1}{t_1}\right)_{t_2}^{*_2}\right)^{\frac{r_2}{r_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{r_2}}$$

in the case $0 < \bar{r} < \infty$, and

$$\|f\|_{L_{\bar{p},\infty}} = \sup_{t_1,t_2} t_1^{\frac{1}{p_1}} t_2^{\frac{1}{p_2}} f^{*_1*_2}(t_1,t_2)$$

in the case $\bar{r} = \infty$.

Definition 2 [15]. Let $f \in L_1(\mathbb{R}^2)$. Its two-dimensional Fourier transform is defined by the following formula:

$$\hat{f}(\xi_1,\xi_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2)e^{-2\pi i(x_1\xi_1+x_2\xi_2)}dx_1dx_2.$$

Main results

To prove the main theorem, it is necessary to prove an auxiliary lemma:

Lemma. Let $\frac{4}{3} < q_1, q_2 < 2$ and $f \in L_{\bar{q},\bar{2}}(\mathbb{R}^2)$. Then for any measurable sets A_1 and A_2 of finite measure in $\Re_N$ the inequality

$$\sup_{A_1\subset\Re_N}\sup_{A_2\subset\Re_N}\frac{1}{|A_1|^{\frac{1}{q_1}}|A_2|^{\frac{1}{q_2}}}\int_{A_1}\int_{A_2}\left|\hat{f}(\xi_1,\xi_2)\right|d\xi_1d\xi_2 \leq$$

$$\leq C\left(\frac{q_1}{2(q_1-1)}\right)^{\left(\frac{1}{q_1}-\frac{1}{2}\right)}\left(\frac{q_2}{2(q_2-1)}\right)^{\left(\frac{1}{q_2}-\frac{1}{2}\right)}\|f\|_{L_{\bar{q},\bar{2}}}$$

holds.

Proof. We consider the following inequality

$$\left|\int_{A_1}\int_{A_2}\hat{f}(\xi_1,\xi_2)d\xi_1d\xi_2\right| =$$

$$= \left| \int\limits_{A_1}\int\limits_{A_2} \left(\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x_1,x_2) e^{-2\pi x_1\xi_1 x_2\xi_2} dx_1 dx_2 \right) d\xi_1 d\xi_2 \right| \leq$$

$$\leq |A_1||A_2| \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} |f(x_1,x_2)| dx_1 dx_2 = |A_1||A_2| \|f\|_{L_{\bar{1}}}, \quad (1)$$

and from the Cauchy-Bunyakovsky inequality and from the Plancherel theorem we have:

$$\left| \int\limits_{A_1}\int\limits_{A_2} \hat{f}(\xi_1,\xi_2) d\xi_1 d\xi_2 \right| \leq |A_1|^{\frac{1}{2}} |A_2|^{\frac{1}{2}} \left(\int\limits_{A_1}\int\limits_{A_2} \left(\hat{f}(\xi_1,\xi_2) \right)^2 d\xi_1 d\xi_2 \right)^{\frac{1}{2}} =$$

$$= |A_1|^{\frac{1}{2}} |A_2|^{\frac{1}{2}} \left(\int\limits_{A_1}\int\limits_{A_2} \left(f(x_1,x_2) \right)^2 dx_1 dx_2 \right)^{\frac{1}{2}} = |A_1|^{\frac{1}{2}} |A_2|^{\frac{1}{2}} \|f\|_{L_{\bar{2}}}. \quad (2)$$

Consider representation (2) $f = f_{00} + f_{01} + f_{10} + f_{11}$ constructed like the following.

Let $0 < \tau_1, \tau_2 < \infty$, $\chi_{\Omega_{x_2}}(x_1)$ be a set characteristic function Ω_{x_2}.

$$\Omega_{x_2} = \{(x_1,x_2) : |f(x_1,x_2)| > f^{*_1}(\tau_1,x_2)\} \cup e_{x_2},$$

where e_{x_2} is a measurable subset $\{(x_1,x_2) : |f(x_1,x_2)| = f^{*_1}(\tau_1,x_2)\}$ such that:

$$\mu_1(\Omega_{x_2}) = \tau_1.$$

This set is always available, since for a fixed x_2

$$\mu_1\{(x_1,x_2) : |f(x_1,x_2)| > f^{*_1}(\tau_1,x_2)\} \leq \tau_1,$$

$$\mu_1\{(x_1,x_2) : |f(x_1,x_2)| \geq f^{*}(\tau_1,x_2)\}.$$

Denote by g_0 and g_1 the functions

$$g_0(x_1,x_2) = f(x_1,x_2)\chi_{\Omega_{x_2}}(x_1),$$

$$g_1(x_1,x_2) = f(x_1,x_2) - g_0(x_1,x_2).$$

In turn, each function g_0, g_1 can be represented as

$$g_0 = f_{00} + f_{01}, \; g_1 = f_{10} + f_{11}.$$

Let

$$W_0 = \{x_2 \in (0,\infty) : \|g_0(\cdot,x_2)\|_{L_1} > (\|g_0(\cdot,x_2)\|_{L_1})^{*_2}(\tau_2)\} \cup e_0,$$

where

$$e_0 \subset \{x_2 \in (0,\infty) : \|g_0(\cdot,x_2)\|_{L_1} = (\|g_0(\cdot,x_2)\|_{L_1})^{*_2}(\tau_2)\}, \; \mu_2(W_0) = \tau_2,$$

and

$$W_1 = \{x_2 \in (0,\infty) : \|g_1(\cdot,x_2)\|_{L_2} > (\|g_1(\cdot,x_2)\|_{L_2})^{*_2}(\tau_2)\} \cup e_1,$$

where

$$e_1 \subset \{x_2 \in (0,\infty) : \|g_1(\cdot,x_2)\|_{L_2} = (\|g_1(\cdot,x_2)\|_{L_2})\}, \; \mu_2(W_1) = \tau_2.$$

Then

$$f_{00}(x_1,x_2)=g_0(x_1,x_2)\chi_{W_0}(x_2),\ \ f_{01}(x_1,x_2)=g_0(x_1,x_2)-f_{00}(x_1,x_2),$$

$$f_{10}(x_1,x_2)=g_1(x_1,x_2)\chi_{W_1}(x_1),\ \ f_{11}(x_1,x_2)=g_1(x_1,x_2)-f_{10}(x_1,x_2).$$

Thus representation is constructed

$$f=f_{00}+f_{01}+f_{10}+f_{11}.$$

Then for an arbitrary $\tau=(\tau_1,\tau_2)\in(0,\infty)^2$, we get

$$\left|\int\limits_{A_1}\int\limits_{A_2}\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2\right|\leq$$

$$\leq\left|\int\limits_{A_1}\int\limits_{A_2}\hat{f}_{00}(\xi_1,\xi_2)d\xi_1 d\xi_2\right|+\left|\int\limits_{A_1}\int\limits_{A_2}\hat{f}_{01}(\xi_1,\xi_2)d\xi_1 d\xi_2\right|+$$

$$+\left|\int\limits_{A_1}\int\limits_{A_2}\hat{f}_{10}(\xi_1,\xi_2)d\xi_1 d\xi_2\right|+\left|\int\limits_{A_1}\int\limits_{A_2}\hat{f}_{11}(\xi_1,\xi_2)d\xi_1 d\xi_2\right|=I_1+I_2+I_3+I_4. \tag{3}$$

For I_1 we use inequality (1) and get the following estimate

$$I_1\leq|A_1||A_2|M_1M_2\int\limits_0^\infty\int\limits_0^\infty|f_{00}(x_1,x_2)|dx_1dx_2.$$

Now let us estimate I_2

$$I_2\leq|A_2|^{\frac{1}{2}}\left(\int\limits_{A_2}\left(\int\limits_{A_1}\hat{f}_{01}(\xi_1,\xi_2)d\xi_1\right)^2 d\xi_2\right)^{\frac{1}{2}}\leq$$

$$\leq|A_2|^{\frac{1}{2}}\left(\int\limits_{A_2}\left|\int\limits_{A_1}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}f_{01}(x_1,x_2)e^{-2\pi x_1\xi_1x_2\xi_2}dx_1dx_2d\xi_1\right|^2 d\xi_2\right)^{\frac{1}{2}}\leq$$

$$\leq|A_2|^{\frac{1}{2}}\left(\int\limits_{A_2}\left|\int\limits_{-\infty}^{+\infty}\left(\int\limits_{A_1}\int\limits_{-\infty}^{+\infty}f_{01}(x_1,x_2)e^{-2\pi x_1\xi_1}dx_1d\xi_1\right)e^{-2\pi ix_2\xi_2}dx_2\right|^2 d\xi_2\right)^{\frac{1}{2}}.$$

Applying Plancherel theorem, we get

$$I_2\leq|A_2|^{\frac{1}{2}}\left(\int\limits_{-\infty}^{+\infty}\left(\widehat{\int\limits_{A_1}\int\limits_{-\infty}^{+\infty}f_{01}(x_1,x_2)e^{-2\pi ix_1\xi_1}dx_1d\xi_1}\right)^2_{k_2}dx_2\right)^{\frac{1}{2}}=$$

$$=|A_2|^{\frac{1}{2}}\left(\int\limits_{-\infty}^{+\infty}\left|\int\limits_{A_1}\int\limits_{-\infty}^{+\infty}f_{01}(x_1,x_2)e^{-2\pi ix_1\xi_1}dx_1d\xi_1\right|^2dx_2\right)^{\frac{1}{2}}\leq$$

$$\leq C|A_1||A_2|^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |f_{01}(x_1, x_2)| dx_1 \right)^2 dx_2 \right)^{1/2}.$$

Let's estimate I_3

$$I_3 = \left| \int_{A_1} \int_{A_2} \widehat{f_{10}(\xi_1, \xi_2)} d\xi_1 d\xi_2 \right| =$$

$$= \left| \int_{A_1} \int_{A_2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{10}(x_1, x_2) e^{-2\pi i x_1 \xi_1 x_2 \xi_2} dx_1 dx_2 d\xi_1 d\xi_2 \right| \leq$$

$$\leq \int_{A_1} \int_{A_2} \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f_{10}(x_1, x_2) e^{-2\pi i x_1 \xi_1} dx_1 \right| d\xi_1 e^{-2\pi i x_2 \xi_2} d\xi_2 dx_2 \leq$$

$$\leq C|A_2| \int_{A_1} \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f_{10}(x_1, x_2) e^{-2\pi i x_1 \xi_1} dx_1 d\xi_1 \right| dx_2.$$

Using Cauchy-Bunyakovsky inequality, we obtain

$$I_3 \leq C|A_2||A_1|^{\frac{1}{2}} \left(\int_{A_1} \left(\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f_{10}(x_1, x_2) e^{-2\pi i x_1 \xi_1} dx_1 d\xi_1 \right| dx_2 \right)^2 \right)^{\frac{1}{2}} \leq$$

$$\leq C|A_2||A_1|^{\frac{1}{2}} \int_{-\infty}^{+\infty} \left(\int_{A_1} \left| \int_{-\infty}^{+\infty} f_{10}(x_1, x_2) e^{-2\pi i x_1 \xi_1} dx_1 d\xi_1 \right|^2 \right)^{\frac{1}{2}} dx_2.$$

Using Plancherel theorem, we get the following

$$I_3 \leq C|A_2||A_1|^{\frac{1}{2}} \int_{-\infty}^{+\infty} \left(\left| \int_{-\infty}^{+\infty} f_{10}(x_1, x_2) dx_1 \right|^2 \right)^{\frac{1}{2}} dx_2.$$

Applying for I_4 Cauchy-Bunyakovsky inequality and Plancherel equality, we get the following estimate

$$I_4 = \left| \int_{A_1} \int_{A_2} \widehat{f}_{11}(\xi_1, \xi_2) d\xi_1 d\xi_2 \right| \leq |A_1|^{1/2} |A_2|^{1/2} \left(\int_{A_1} \int_{A_2} \left| \hat{f}_{11}(\xi_1, \xi_2) d\xi_1 d\xi_2 \right|^2 \right)^{\frac{1}{2}} =$$

$$= |A_1|^{1/2} |A_2|^{1/2} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f_{11}(x_1, x_2)|^2 dx_1 dx_2 \right)^{\frac{1}{2}}.$$

Substituting the obtained estimates into relation (3), we have

$$\left|\int_{A_1}\int_{A_2}\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2\right| \le |A_1||A_2|\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f_{00}(x_1,x_2)|dx_1dx_2+$$

$$+|A_1||A_2|^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}|f_{01}(x_1,x_2)|dx_1\right)^2 dx_2\right)^{\frac{1}{2}}+$$

$$+|A_2||A_1|^{\frac{1}{2}}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}|f_{10}(x_1,x_2)|^2\,dx_1\right)^{\frac{1}{2}}dx_2+$$

$$+|A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f_{11}(x_1,x_2)|^2dx_1dx_2\right)^{\frac{1}{2}}.$$

Further

$$\left|\int_{A_1}\int_{A_2}\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2\right| \le |A_1||A_2|\int_0^\infty\left(\int_0^\infty f^{*_1}(t_1,\cdot)dt_1\right)^{*_2}_{t_2}dt_2+$$

$$+|A_1||A_2|^{\frac{1}{2}}\left(\int_0^\infty\left(\left(\int_{\tau_1}^\infty f^{*_1}(t_1,\cdot)dt_1\right)^{*_2}_{t_2}\right)^2 dt_2\right)^{\frac{1}{2}}+$$

$$+|A_1|^{\frac{1}{2}}|A_2|\int_0^\infty\left(\left(\int_0^\infty (f^{*_1}(t_1,\cdot))^2dt_1\right)^{*_2}_{t_2}\right)^{\frac{1}{2}}dt_2+$$

$$+|A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}}\left(\int_{\tau_2}^\infty\left(\int_{\tau_1}^\infty (f^{*_1}(t_1,\cdot))^2dt_1\right)^{*_2}_{t_2}dt_2\right)^{\frac{1}{2}}=$$

$$=J_1+J_2+J_3+J_4.$$

Now we estimate each term. To estimate J_1 we use Helder's inequality

$$J_1=|A_1||A_2|\int_0^\infty t_2^{\frac{1}{q_2}}t_2^{\frac{1}{q_2'}}\left(\int_0^\infty t_1^{\frac{1}{q_1}}t_1^{\frac{1}{q_1'}}f^{*_1}(t_1,\cdot)\frac{dt_1}{t_1}\right)^{*_2}_{t_2}\frac{dt_2}{t_2}\le$$

$$\le|A_1||A_2|\left(\int_0^\infty t_2^{\frac{2}{q_2}}\left(\int_0^\infty\left(t_1^{\frac{1}{q_1}}f^{*_1}(t_1,\cdot)\right)^2\frac{dt_1}{t_1}\right)^{*_2}_{t_2}\frac{dt_2}{t_2}\right)^{\frac{1}{2}}\times$$

$$\times\left(\int_0^\infty t_2^{\frac{2}{q_2'}}\left(\int_0^\infty t_1^{\frac{2}{q_1'}}\frac{dt_1}{t_1}\right)\frac{dt_2}{t_2}\right)^{\frac{1}{2}}=$$

$$= |A_1||A_2|\|f\|_{L_{\bar{q},\bar{2}}} \left(\int_0^\infty t_1^{\frac{2}{q_1'}} \frac{dt_1}{t_1} \right)^{\frac{1}{2}} \left(\int_0^\infty t_2^{\frac{2}{q_2'}} \frac{dt_2}{t_2} \right)^{\frac{1}{2}} =$$

$$= |A_1||A_2|\|f\|_{L_{\bar{q},\bar{2}}} \left(\frac{q_1'}{2} \right)^{\frac{1}{2}} \left(\frac{q_2'}{2} \right)^{\frac{1}{2}} \tau_1^{\frac{1}{q_1'}} \tau_2^{\frac{1}{q_2'}} =$$

$$= |A_1||A_2|\|f\|_{L_{\bar{q},\bar{2}}} \left(\frac{q_1}{2(q_1-1)} \right)^{\frac{1}{2}} \left(\frac{q_2}{2(q_2-1)} \right)^{\frac{1}{2}} \tau_1^{1-\frac{1}{q_1}} \tau_2^{1-\frac{1}{q_2}}.$$

To estimate the term J_2 we also use Helder's inequality

$$J_2 = |A_1||A_2|^{\frac{1}{2}} \left(\int_{\tau_2}^\infty \left(\left(\int_0^{\tau_1} f^{*_1}(t_1,\cdot) dt_1 \right)^{*_2}_{t_2} \right)^2 dt_2 \right)^{\frac{1}{2}} =$$

$$= |A_1||A_2|^{\frac{1}{2}} \left(\int_{\tau_2}^\infty \left(t_2^{\frac{1}{q_2}} t_2^{-\frac{1}{q_2}} \left(\int_0^{\tau_1} t_1^{\frac{1}{q_1}} t_1^{\frac{1}{q_1'}} f^{*_1}(t_1,\cdot) \frac{dt_1}{t_1} \right)^{*_2}_{t_2} \right)^2 \frac{t_2 dt_2}{t_2} \right)^{\frac{1}{2}} \leq$$

$$\leq |A_1||A_2|^{\frac{1}{2}} \left(\int_{\tau_2}^\infty t_2^{\frac{1}{q_2}} \left(\int_0^{\tau_1} \left(t_1^{\frac{1}{q_1}} f^{*_1}(t_1,\cdot) \right)^2 \frac{dt_1}{t_1} \right)^{*_2}_{t_2} \frac{dt_2}{t_2} \right)^{\frac{1}{2}} \times$$

$$\times \left(\int_{\tau_2}^\infty t_2^{1-\frac{2}{q_2}} \left(\int_0^{\tau_1} t_1^{\frac{2}{q_1}} \frac{dt_1}{t_1} \right) \frac{dt_2}{t_2} \right)^{1/2} =$$

$$= |A_1||A_2|^{\frac{1}{2}} \|f\|_{L_{\bar{q},\bar{2}}} \left(\int_{\tau_2}^\infty t_2^{1-\frac{2}{q_2}} \frac{dt_2}{t_2} \right)^{\frac{1}{2}} \left(\int_0^{\tau_1} \left(t_1^{\frac{2}{q_1'}} \right) \frac{dt_1}{t_1} \right)^{\frac{1}{2}}.$$

Since $q_2 < 2$, the second integral is estimated in terms of $\tau_2^{\frac{1}{2}-\frac{1}{q_2}}$. So

$$J_2 \leq |A_1||A_2|^{\frac{1}{2}} \|f\|_{L_{\bar{q},\bar{2}}} \left(\frac{q_1'}{2} \right)^{\frac{1}{2}} \tau_1^{\frac{1}{q_1'}} \tau_2^{\frac{1}{2}-\frac{1}{q_2}} =$$

$$= |A_1||A_2|^{\frac{1}{2}} \|f\|_{L_{\bar{q},\bar{2}}} \left(\frac{q_1}{2(q_1-1)} \right)^{\frac{1}{2}} \tau_1^{1-\frac{1}{q_1}} \tau_2^{\frac{1}{2}-\frac{1}{q_2}}.$$

Now we estimate J_3

$$J_3 = |A_1|^{\frac{1}{2}}|A_2| \int_0^{\tau_2} \left(\left(\int_{\tau_1}^\infty (f^{*_1}(t_1,\cdot))^2 dt_1 \right)^{*_2}_{t_2} \right)^{\frac{1}{2}} dt_2 =$$

$$= |A_1|^{\frac{1}{2}}|A_2| \int_0^{\tau_2} t_2^{-\frac{1}{q_2}} \left(\left(\int_{\tau_1}^\infty \left(t_1^{\frac{1}{q_1}} t_2^{\frac{1}{q_2}} f^{*_1}(t_1,\cdot) t_1^{-\frac{1}{q_1}} \right)^2 \frac{t_1 dt_1}{t_1} \right)^{*_2}_{t_2} \right)^{\frac{1}{2}} \frac{t_2 dt_2}{t_2^{\frac{1}{2}+\frac{1}{2}}}.$$

We use Helder's inequality

$$J_3 \le |A_1|^{\frac{1}{2}}|A_2| \sup_{\tau_1<t_1<\infty} t_1^{\frac{1}{2}-\frac{1}{q_1}} \left(\int\limits_0^{\tau_2} t_2^{-\frac{1}{q_2}} \left(\left(\left(\int\limits_{\tau_1}^{\infty} \left(t_1^{\frac{1}{q_1}} t_2^{\frac{1}{q_2}} f^{*_1}(t_1,\cdot)\right)^2 \frac{dt_1}{t_1}\right)^{*_2}_{t_2}\right) \frac{dt_2}{t_2}\right)^{\frac{1}{2}} \times$$

$$\times \left(\int\limits_0^{\tau_2} \left(t_2^{1-\frac{1}{q_2}}\right)^2 \frac{dt_2}{t_2}\right)^{\frac{1}{2}} =$$

$$= |A_1|^{\frac{1}{2}}|A_2| \|f\|_{L_{\bar{q},\bar{2}}} \sup_{\tau_1<t_1<\infty} t_1^{\frac{1}{2}-\frac{1}{q_1}} \left(\int\limits_0^{\tau_2} t_2^{2\left(1-\frac{1}{q_2}\right)} \frac{dt_2}{t_2}\right)^{\frac{1}{2}} =$$

$$= |A_1|^{\frac{1}{2}}|A_2| \|f\|_{L_{\bar{q},\bar{2}}} \tau_1^{\frac{1}{2}-\frac{1}{q_1}} \tau_2^{1-\frac{1}{q_2}} \left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{2}}.$$

It remains to estimate the last integral

$$J_4 \le |A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}} \left(\int\limits_0^{\tau_1} \left(\int\limits_{\tau_2}^{\infty} (f^{*_1}(t_1,\cdot))^2 t_1^{\frac{2}{q_1}-1} t_1^{1-\frac{2}{q_1}} t_2^{\frac{2}{q_2}-1} t_2^{1-\frac{2}{q_2}} dt_1\right)^{*_2}_{t_2} dt_2\right)^{\frac{1}{2}}.$$

Since $q_2 < 2$, we get

$$J_4 \le |A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}} \left(\int\limits_0^{\tau_1} \left(\int\limits_{\tau_2}^{\infty} \left(t_1^{\frac{1}{q_1}} t_2^{\frac{1}{q_2}} f^{*_1}(t_1,\cdot)\right)^2 \frac{dt_1}{t_1}\right)^{*_2}_{t_2} \frac{dt_2}{t_2}\right)^{\frac{1}{2}} \tau_1^{\frac{1}{2}-\frac{1}{q_1}} \tau_2^{\frac{1}{2}-\frac{1}{q_2}} \le$$

$$\le |A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}} \|f\|_{L_{\bar{q},\bar{2}}} \tau_1^{\frac{1}{2}-\frac{1}{q_1}} \tau_2^{\frac{1}{2}-\frac{1}{q_2}}.$$

Thus, we have

$$\left|\int\limits_{A_1}\int\limits_{A_2} \hat{f}(\xi_1,\xi_2)\right| \le \|f\|_{L_{\bar{q},\bar{2}}} \left(|A_1||A_2| \left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{2}} \left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{2}} \times\right.$$

$$\times\, \tau_1^{1-\frac{1}{q_1}} \tau_2^{1-\frac{1}{q_2}} +$$

$$+|A_1||A_2|^{\frac{1}{2}} \left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{2}} \tau_1^{1-\frac{1}{q_1}} \tau_2^{\frac{1}{2}-\frac{1}{q_2}} +$$

$$\left.+|A_1|^{\frac{1}{2}}|A_2| \left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{2}} \tau_1^{\frac{1}{2}-\frac{1}{q_1}} \tau_2^{1-\frac{1}{q_2}} + |A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}} \tau_1^{\frac{1}{2}-\frac{1}{q_1}} \tau_2^{\frac{1}{2}-\frac{1}{q_2}}\right).$$

Choosing $\tau_1 = \frac{\left(\frac{q_1}{2(q_1-1)}\right)^{-1}}{|A_1|}$ and $\tau_2 = \frac{\left(\frac{q_2}{2(q_2-1)}\right)^{-1}}{|A_2|}$, we get

$$\left|\int\limits_{A_1}\int\limits_{A_2} \hat{f}(\xi_1,\xi_2)\right| \le \|f\|_{L_{\bar{q}\bar{2}}} \left(|A_1||A_2| \left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{2}} \left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{2}} \times\right.$$

$$\times\left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{q_1}-1}|A_1|^{\frac{1}{q_1}-1}\left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{q_2}-1}|A_2|^{\frac{1}{q_2}-1}+$$

$$+|A_1||A_2|^{\frac{1}{2}}\left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{2}}\left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{q_1}-1}|A_1|^{\frac{1}{q_1}-1}\times$$

$$\times\left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{q_2}-\frac{1}{2}}|A_2|^{\frac{1}{q_2}-\frac{1}{2}}+$$

$$+|A_1|^{\frac{1}{2}}|A_2|\left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{2}}\left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{q_1}-\frac{1}{2}}|A_1|^{\frac{1}{q_1}-\frac{1}{2}}\times$$

$$\times\left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{q_2}-1}|A_2|^{\frac{1}{q_2}-1}+|A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}}\left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{q_1}-\frac{1}{2}}|A_1|^{\frac{1}{q_1}-\frac{1}{2}}\times$$

$$\times\left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{q_2}-\frac{1}{2}}|A_2|^{\frac{1}{q_2}-\frac{1}{2}}\Bigg).$$

We get the following inequality:

$$\left|\int\limits_{A_1}\int\limits_{A_2}\hat{f}(\xi_1,\xi_2)\right|\leq$$

$$\leq M|A_1|^{\frac{1}{q_1}}|A_2|^{\frac{1}{q_2}}\left(\frac{q_1}{2(q_1-1)}\right)^{\left(\frac{1}{q_1}-\frac{1}{2}\right)}\left(\frac{q_2}{2(q_2-1)}\right)^{\left(\frac{1}{q_2}-\frac{1}{2}\right)}\|f\|_{L_{\bar{q},\bar{2}}},$$

or

$$\frac{1}{|A_1|^{\frac{1}{q_1}}|A_2|^{\frac{1}{q_2}}}\int\limits_{A_1}\int\limits_{A_2}\left|\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2\right|\leq$$

$$\leq\left(\frac{q_1}{2(q_1-1)}\right)^{\left(\frac{1}{q_1}-\frac{1}{2}\right)}\left(\frac{q_2}{2(q_2-1)}\right)^{\left(\frac{1}{q_2}-\frac{1}{2}\right)}\|f\|_{L_{\bar{q},\bar{2}}}.$$

Taking the least upper bound over all $A_1\subset\Re_N$ and $A_2\subset\Re_N$, we obtain the assertion of the lemma, that is,

$$\sup_{A_1\subset\Re_N}\sup_{A_2\subset\Re_N}\frac{1}{|A_1|^{\frac{1}{q_1}}|A_2|^{\frac{1}{q_2}}}\int\limits_{A_1}\int\limits_{A_2}\left|\hat{f}(\xi_1,\xi_2)\right|d\xi_1 d\xi_2\leq$$

$$\leq C\left(\frac{q_1}{2(q_1-1)}\right)^{\left(\frac{1}{q_1}-\frac{1}{2}\right)}\left(\frac{q_2}{2(q_2-1)}\right)^{\left(\frac{1}{q_2}-\frac{1}{2}\right)}\|f\|_{L_{\bar{q},\bar{2}}},$$

where $|A|$ is the number of elements in A.

Theorem. Let $\Phi_{m_1,m_2}(x_1,x_2)=\varphi_{m_1}(x_1)\cdot\psi_{m_2}(x_2)$, $m_1,m_1\in\mathbb{N}$ be an orthonormal bounded system of functions. Then, for any $f\in L_{\bar{2},\bar{r}}(\mathbb{R}^2)$, where $2<r_1,r_2<\infty$ the inequality holds:

$$\sup_{\substack{|A_1|\geq 8\\ A_1\subset\mathbb{N}}}\sup_{\substack{|A_2|\geq 8\\ A_2\subset\mathbb{N}}}\frac{1}{|A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}}(\log_2(|A_1|+1))^{\frac{1}{2}-\frac{1}{r_1}}(\log_2(|A_2|+1))^{\frac{1}{2}-\frac{1}{r_2}}}\times$$

$$\times\int\limits_{A_1}\int\limits_{A_2}\left|\hat{f}(\xi_1,\xi_2)\right|d\xi_1 d\xi_2\leq\|f\|_{L_{\bar{2},\bar{r}}}.$$

Proof. Let $|A_1|, |A_2| \geq 8$. Then for any (q_1, q_2) such that $1 < q_1, q_2 < 2$ and $f \in L_{\bar{q},\bar{2}}$ the following estimate is true

$$\|f\|_{L_{\bar{q},\bar{2}}} \leq \|f\|_{L_{\bar{2},\bar{r}}} \|1\|_{L_{\bar{p},\bar{r}'}}, \tag{4}$$

where $\frac{1}{\bar{q}} = \frac{1}{2} + \frac{1}{\bar{p}}$, $\quad \frac{1}{\bar{r}'} = \frac{1}{2} - \frac{1}{\bar{r}}$. Now we consider

$$\|1\|_{L_{\bar{p},\bar{r}'}} = \left(\int_0^1 \left(\int_0^1 t_1^{\frac{r_1'}{p_1}} t_2^{\frac{r_1'}{p_2}} \frac{dt_1}{t_1} \right)^{\frac{r_2'}{r_1'}} \frac{dt_2}{t_2} \right)^{\frac{1}{r_2'}} =$$

$$= \left(\int_0^1 t_1^{r_1'\left(\frac{1}{q_1} - \frac{1}{2}\right)} \frac{dt_1}{t_1} \right)^{\frac{1}{r_1'}} \left(\int_0^1 t_2^{r_2'\left(\frac{1}{q_2} - \frac{1}{2}\right)} \frac{dt_2}{t_2} \right)^{\frac{1}{r_2'}} =$$

$$= \left(\frac{1}{r_1'\left(\frac{1}{q_1} - \frac{1}{2}\right)} \right)^{\frac{1}{r_1'}} \left(\frac{1}{r_2'\left(\frac{1}{q_2} - \frac{1}{2}\right)} \right)^{\frac{1}{r_2'}} \leq \left(\frac{2q_1}{2 - q_1} \right)^{\frac{1}{r_1'}} \left(\frac{2q_2}{2 - q_2} \right)^{\frac{1}{r_2'}}.$$

According to the previous inequality, we obtain

$$\|f\|_{L_{\bar{q},\bar{2}}} \leq \left(\frac{2q_1}{2 - q_1} \right)^{\frac{1}{r_1'}} \left(\frac{2q_2}{2 - q_2} \right)^{\frac{1}{r_2'}} \|f\|_{L_{\bar{2},\bar{r}}}.$$

Applying Lemma 1, we get

$$\frac{1}{|A_1|^{\frac{1}{q_1}} |A_2|^{\frac{1}{q_2}}} \int_{A_1} \int_{A_2} \left| \hat{f}(\xi_1, \xi_2) \right| d\xi_1 d\xi_2 \leq$$

$$\leq C \left(\frac{q_1}{2(q_1 - 1)} \right)^{\left(\frac{1}{q_1} - \frac{1}{2}\right)} \left(\frac{q_2}{2(q_2 - 1)} \right)^{\left(\frac{1}{q_2} - \frac{1}{2}\right)} \|f\|_{L_{\bar{q},\bar{2}}}.$$

Taking into account (4), we get the following inequality

$$\frac{1}{|A_1|^{\frac{1}{q_1}} |A_2|^{\frac{1}{q_2}}} \sum_{k_1 \in A_1} \sum_{k_2 \in A_2} \left| \hat{f}(\xi_1, \xi_2) \right| d\xi_1 d\xi_2 \leq C \left(\frac{q_1}{2(q_1 - 1)} \right)^{\left(\frac{1}{q_1} - \frac{1}{2}\right)} \left(\frac{q_2}{2(q_2 - 1)} \right)^{\left(\frac{1}{q_2} - \frac{1}{2}\right)} \times$$

$$\times \left(\frac{2q_1}{2 - q_1} \right)^{\frac{1}{r_1'}} \left(\frac{2q_2}{2 - q_2} \right)^{\frac{1}{r_2'}} \|f\|_{L_{\bar{2},\bar{r}}}.$$

Taking into account the arbitrariness of parameters q_1 and q_2, we set

$$q_1 = \frac{2 \log_2 |A_1|}{\log_2 |A_1| + 2} < 2,$$

$$q_2 = \frac{2 \log_2 |A_2|}{\log_2 |A_2| + 2} < 2,$$

$\frac{1}{\bar{q}} - \frac{1}{2} = \frac{1}{\log_2 N}$, $\quad |A_1|^{\frac{1}{q_1}} = |A_1|^{\frac{1}{\log_2 |A_1|} + \frac{1}{2}} = 2|A_1|^{\frac{1}{2}}$, $\quad |A_2|^{\frac{1}{q_2}} = 2|A_2|^{\frac{1}{2}}$.

$$\frac{1}{|A_1|^{\frac{1}{2}} |A_2|^{\frac{1}{2}}} \int_{A_1} \int_{A_2} \left| \hat{f}(\xi_1, \xi_2) \right| d\xi_1 d\xi_2 \leq$$

$$\leq C \left(\frac{1}{2\left(1-\frac{1}{q_1}\right)} \right)^{\left(\frac{1}{q_1}-\frac{1}{2}\right)} \left(\frac{1}{2\left(1-\frac{1}{q_2}\right)} \right)^{\left(\frac{1}{q_2}-\frac{1}{2}\right)} \times$$

$$\times \left(\frac{1}{\frac{1}{q_1}-\frac{1}{2}} \right)^{\frac{1}{r_1'}} \left(\frac{1}{\frac{1}{q_2}-\frac{1}{2}} \right)^{\frac{1}{r_2'}} \|f\|_{L_{\bar{2},\bar{r}}} \leq$$

$$\leq 4M \left(\frac{1}{2\left(\frac{1}{2}-\frac{1}{\log_2|A_1|}\right)} \right)^{\frac{1}{\log_2|A_1|}} \left(\frac{1}{2\left(\frac{1}{2}-\frac{1}{\log_2|A_2|}\right)} \right)^{\frac{1}{\log_2|A_2|}} \times$$

$$\times \left(\log_2|A_1|\right)^{\frac{1}{r_1'}} \left(\log_2|A_2|\right)^{\frac{1}{r_2'}} \|f\|_{L_{\bar{2},\bar{r}}}.$$

Considering $|A_1|, |A_2| \geq 8$, we get the following estimate

$$\frac{1}{|A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}} \left(\log_2|A_1|\right)^{\frac{1}{2}-\frac{1}{r_1}} \left(\log_2|A_2|\right)^{\frac{1}{2}-\frac{1}{r_2}}} \int\limits_{A_1}\int\limits_{A_2} \left|\hat{f}(\xi_1,\xi_2)\right| d\xi_1 d\xi_2 \leq C\|f\|_{L_{\bar{2},\bar{r}}}.$$

Taking the least upper bound over all A_1 and A_2 from $\mathbb{N}$, we get

$$\sup_{\substack{|A_1|\geq 8\\ A_1\subset\mathbb{N},}} \sup_{\substack{|A_2|\geq 8\\ A_2\subset\mathbb{N},}} \frac{1}{|A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}} \left(\log_2(|A_1|+1)\right)^{\frac{1}{2}-\frac{1}{r_1}} \left(\log_2(|A_2|+1)\right)^{\frac{1}{2}-\frac{1}{r_2}}} \times$$

$$\times \int\limits_{A_1}\int\limits_{A_2} \left|\hat{f}(\xi_1,\xi_2)\right| d\xi_1 d\xi_2 \leq C\|f\|_{L_{\bar{2},\bar{r}}}.$$

Conclusions

The results obtained in this study specifically the Bochkarev-type inequality in a space of a mixed metric, allow us to effectively address problems concerning Fourier series multipliers [16–18].

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Аралас метрикалы $L_{\overline{2},\overline{r}}$ Лоренц кеңістігіндегі Фурье функцияларының түрлендірулері жайлы

Гармоникалық талдауда классикалық Бочкарев теңсіздіктері өте маңызды рөл атқарады. Бұл теңсіздіктердің мәні функциялардың метрикалық сипаттамалары мен олардың Фурье коэффициенттерінің қосындысы арасындағы байланыста жатыр. Гармоникалық талдаудың маңызды бағыттарының бірі Фурье қатарларының теориясы. Оның бұл салаға деген қызығушылығы қазіргі математика мен қолданбалы ғылымдардың әртүрлі салаларында қолданылуына, сондай-ақ көптеген шешілмеген мәселелердің болуына байланысты. Осы мақсаттардың бірі функцияның интегралдық қасиеттері мен оның коэффициенттерінің қосындысының қасиеттері арасындағы байланысты зерттеу. Көптеген математиктердің еңбектері осы есептерді шешуге арналды. Бұл саладағы әрі қарайғы зерттеулер маңызды және қызықты зерттеу болып табылады және жаңа, күтпеген нәтижелерге әкелуі мүмкін. Мақалада Фурье түрленуі үшін Бочкарев типті теореманың екі өлшемді аналогы алынған.

Кілт сөздер: Лоренц кеңістігі, Хаусдорф–Янг–Рис теоремасы, Бочкарев теоремасы, Коши-Буняковский теңсіздігі, Хельдер теңсіздігі.

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О преобразовании Фурье функций в пространстве Лоренца $L_{\overline{2},\overline{r}}$ со смешанной метрикой

Классические неравенства Бочкарева играют очень важную роль в гармоническом анализе. Смысл этих неравенств заключается в связи между метрическими характеристиками функций и суммируемостью их коэффициентов Фурье. Одним из важнейших направлений гармонического анализа является теория рядов Фурье. Его интерес к этому направлению объясняется его приложениями в различных разделах современной математики и прикладных наук, а также наличием многих нерешенных проблем. Одной из таких задач является изучение взаимосвязей интегральных свойств функции и свойств суммы ее коэффициентов. Решению этих задач были посвящены усилия многих математиков. И дальнейшие исследования в этой области являются важными и интересными задачами и могут привнести новые, неожиданные эффекты. В данной статье мы получаем двумерный аналог теоремы типа Бочкарева для преобразования Фурье.

Ключевые слова: пространство Лоренца, теорема Хаусдорфа-Юнга-Рисса, теорема Бочкарева, неравенство Коши-Буняковского, неравенство Гельдера.

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A fragment of a theoretical set and its strongly minimal central type

The paper defines a new class of algebras, the theory of which is a special case of Jonsson theories. This class applies to both varieties and Jonsson theories. The main results of this article are the following two results. In this article, an answer is obtained to the question of the equivalence of existential closure and algebraic closure of the model of the cosemantic class of a fixed spectrum of a Robinson hereditary variety. A criterion for strong minimality is obtained in the framework of the study of central types of central classes and fragments of a fixed spectrum.

Keywords: Jonsson theory, existentially closed model, algebraically closed model, cosemanticness, Robinson spectrum, Robinson hereditary variety, central type, Jonsson fragment, theoretical set, strongly minimal type.

Introduction

This article belongs to a fairly well-known topic in the field of model theory. Namely, this topic is related to the classification of theories regarding such an important concept as categoricity. As it is well known, only 4 combinations are possible with respect to the concept of categoricity: total categoricity; ω-categoricity and not ω_1-categoricity; ω_1-categoricity and not ω-categoricity; nowhere categoricity. The notion of strong minimality is closely related to the notion of ω_1-categoricity, that is, in all four combinations of the above concept of categoricity, the concept of strong minimality is either present or absent. Thus, the study of the strong minimality property is important in classifying complete theories.

The topic studied in this article is related to the study of Jonsson theories and their classes of models [1–6]. In papers earlier than this paper, the main methods used to study Jonsson theories [1,5,7,8] were considered. One of the methods for studying complete theories is to enrich the signature with symbols that allow one to obtain new information about the models of the old signature and their theories in the language of these symbols. In the works [9–11], related to the enrichment of Jonsson theories, the notion of a central type was introduced on the basis of the notion of heredity of Jonsson theory. The concept of heredity is closely related to the concept of the stability of the center of the Jonsson theory and the Jonsson stability of the Jonsson theory itself. As is well known, the best description among Jonsson theories lends itself to the study of perfect Jonsson theories due to the existence of a model companion of such theories. The concept of stability is closely related to the concept of categoricity, which plays an important role in the theory of classification of complete theories and, accordingly, incomplete theories. Due to the fact that the concept of heredity of Jonsson theory still does not have a complete description, this topic is relevant and modern in the framework of the study of the enrichment of Jonsson theories.

Jonsson theories, in their essence, are, generally speaking, incomplete theories. That is, the technical apparatus of the study of Jonsson theories, in comparison with complete theories, is less adapted to the transfer and adaptation of the concepts and achievements of complete theories.

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A valuable concept for operating on the properties of elements and subsets of a semantic model is the Jonsson set, that is, a definable set with the help of some existential formula, the definable closure of which defines some existentially closed submodel of the considered semantic model. An interesting and important special case of the Jonsson set is the notion of a theoretical set. In fact, within the framework of the study of the Jonsson set, we get a “new and special” Jonsson theory, the axioms of which are directly related to the given Jonsson set.

Until now, an unresolved problem is a problem of characterization of the concept heredity of Jonsson theory. The relevance of this problem is confirmed by the following important counterexample: the elementary theory of an algebraically closed field ceases to be Jonsson after enrichment with an unary predicate. In this regard, the study of the model-theoretic properties of central types in predicate enrichment is an important model-theoretic task for describing hereditary Jonsson theories.

The concepts of central type and Jonsson spectrum were first introduced by Yeshkeyev A.R., respectively, in [12, 13]. With the help of these concepts, complete descriptions of Jonsson Abelian groups [13] and Jonsson modules [14] with respect to the concept of cosemanticness were obtained, thereby starting a new study in the framework of model-theoretic algebra. Later, the study of the model-theoretic properties of these concepts was continued in the works [9, 11, 14–18].

Note that another method of studying Jonsson's theories is to study these theories using the concepts of syntactic and semantic similarities. In the papers [19–21], using these concepts, some results were obtained in the framework of the study of Jonsson theories and their centers, as well as definable subsets of the semantic model.

1 Basic necessary concepts and preliminaries

Let us give the main definitions of the concepts of model-theoretic concepts that you need to know in order to understand and be able to work in the framework of studying Jonsson theories and their classes of models. The following definitions and their model-theoretic properties are generators for that part of model theory that studies the basic properties of definable subsets of the semantic model of various fixed Jonsson theories.

Definition 1. [22; 80] A theory T is called a Jonsson theory if

1) theory T has an infinite model;

2) theory T is inductive, i.e. T is equivalent to the set of $\forall\exists$-sentences;

3) theory T has the joint embedding property (JEP), i.e. any two models A, B of the theory T are isomorphically embedded in some model C of the theory T;

4) theory T has the amalgamation property (AP), i.e. if for any A, B, $C \models T$ such that $f_1 : A \to B$, $f_2 : A \to C$ are isomorphic embeddings, there are $D \models T$ and isomorphic embeddings $g_1 : B \to D$, $g_2 : C \to D$, such that $g_1 f_1 = g_2 f_2$.

Examples of Jonsson theories are the theories of well-known classical algebras such as groups, Abelian groups, Boolean algebras, linear orders, fields of fixed characteristic, and polygons.

Note that Jonsson theories, generally speaking, are not complete.

Definition 2. [23] Let $\kappa \geq \tau$. A model M of the theory T is called κ-universal for T, if each model of the theory T of cardinality strictly less than κ is isomorphically embeddable into M.

Definition 3. [23] Let $\kappa \geq \tau$. A model M of the theory T is called κ-homogeneous for T, if for any two models A and A_1 of the theory T, which are submodels of M, cardinality is strictly less than κ, and isomorphism $f : A \to A_1$, for each extension B of the model A, that is a submodel of M and the model of the theory T of cardinality is strictly less than κ there exists an extension B_1 of the model A_1, which is a submodel of M, and the isomorphism $g : B \to B_1$, continuing f.

A homogeneous-universal model for T is called a κ-homogeneous-universal model for T of cardinality κ, where $\kappa \geq \tau$.

The following concept is crucial when working with Jonsson theories.

Definition 4. [23] Let T be a Jonsson theory. A model C of the theory T is called a semantic model if it is τ^+–homogeneous and τ^+–universal simultaneously.

A semantic model plays an important role as a semantic invariant. Such a model always exists for any Jonsson theory.

The next important fact shows that any Jonsson theory is determined by its semantic model.

Fact 1. [23] Every Jonsson theory T has a κ^+-homogeneous-universal model of cardinality 2^κ. Conversely, if T is inductive, has an infinite model, and has a τ^+-homogeneous-universal model, then T is a Jonsson theory.

Theorem 1. [23] Let T be a Jonsson theory. Two κ-homogeneous-universal for T models A and B are elementarily equivalent.

Definition 5. [1; 161] Let C_T be a semantic model of the Jonsson theory T. Then the elementary theory $Th(C_T)$ of the model C_T is called the center of T and is denoted by T^*.

The following result makes it possible to describe a special subclass of Jonsson theories that have a model companion.

Fact 2. [23] Let T be the Jonsson theory. If T^* is model complete and $\kappa > \tau$, then κ-homogeneous universal models of T are κ-saturated; if T^* is not model complete, no semantic model of T is τ^+-saturated.

It follows from the Fact 2 and the mutual model compatibility of the Jonsson theory T and its center T^* that T^* is a model companion of the theory T.

Further in our article, the language will be countable, which means that $\tau = \omega$ and $\kappa > \omega$ or $\kappa \geqslant \omega$. From the Fact 2 for the notion of the perfectness of the Jonsson theory κ must be greater than τ.

Definition 6. [24] Let $\kappa > \tau$. Jonsson theory T is perfect if its semantic model is τ^+-saturated.

Thus, from the Fact 2 and the Definition 6 we can conclude that a perfect Jonsson theory is a Jonsson theory that has a model companion and it is equal to its center.

Recall that a model M of theory T is existentially closed in T if every existential sentence φ of L_M which holds in some model of T extending M holds in M.

The notion of an existentially closed model is a generalization of the notion of an algebraically closed field.

Lemma 1. [24] The semantic model C_T of the Jonsson theory T is T-existentially closed.

Proposition 1. [22; 97] If T is inductive theory, then every model of theory T can be extended to an existentially closed model.

Let us denote by E_T the class of all existentially closed models of the theory T.

Theorem 2. [24] If the Jonsson theory T is perfect, then $E_T = \mathrm{Mod}(T^*)$, where $T^* = \mathrm{Th}(C_T)$.

Definition 7. [25] Let $A \in \Sigma$, where Σ is a universal class in a countable language $L(\Sigma)$. Then A is algebraically closed if A has no proper algebraic extensions. An extension B of A is an algebraic closure of A if B is an algebraically closed algebraic extension of A.

The ability to compare complete theories is an important tool in classifying these theories. Mustafin T.G. a method of syntactic and semantic similarity was proposed for the classification of complete theories and their monster models [26]. Let us give the main definitions related to these concepts.

Let T be complete theory then $F(T) = \bigcup_{n<\omega} F_n(T)$, where $F_n(T)$ is Boolean algebra of formulas with n free variables.

Definition 8. [26] Let T_1 and T_2 be complete theories. We will say that T_1 and T_2 are syntactically similar ($T_1 \overset{S}{\bowtie} T_2$) if exists bijection $f : F(T_1) \to F(T_2)$ such that:

1) restriction f to $F_n(T_1)$ is isomorphism of Boolean algebras $F_n(T_1)$ and $F_n(T_2)$, $n < \omega$;

2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in F_{n+1}(T)$, $n < \omega$;

3) $f(v_1 = v_2) = (v_1 = v_2)$.

Definition 9. [26] 1) $\langle A, \Gamma, \mathcal{M}\rangle$ is called the pure triple, where A is not empty, Γ is the permutation group of A and $\mathcal{M}$ is the family of subsets of A such that from $M \in \mathcal{M}$ follows that $g(M) \in \mathcal{M}$ for every $g \in \Gamma$.

2) If $\langle A_1, \Gamma_1, \mathcal{M}_1\rangle$ and $\langle A_2, \Gamma_2, \mathcal{M}_2\rangle$ are pure triples and $\psi : A_1 \to A_2$ is a bijection then ψ is an isomorphism if:

(i) $\Gamma_2 = \{\psi g \psi^{-1} : g \in \Gamma_1\}$;

(ii) $\mathcal{M}_2 = \{\psi(E) : E \in \mathcal{M}_1\}$.

Definition 10. [26] The pure triple $\langle C, Aut(\mathcal{C}), Sub(C)\rangle$ is called the semantic triple of complete theory T, where C is the universe of Monster model $\mathcal{C}$ of theory T, $Aut(\mathcal{C})$ is the automorphism group of $\mathcal{C}$, $Sub(C)$ is a class of all subsets of C each of which is a carrier of the corresponding elementary submodel of $\mathcal{C}$.

Definition 11. [26] Complete theories T_1 and T_2 are semantically similar if and only if their semantic triples are isomorphic.

Proposition 2. [26] If T_1 and T_2 are syntactically similar, then T_1 and T_2 semantically similar. The converse implication fails.

In what follows, we will denote the syntactic and semantic similarities of the complete theories T_1 and T_2 as $T_1 \overset{S}{\bowtie} T_2$ and $T_1 \underset{S}{\bowtie} T_2$, respectively.

Let us recall the definition of semantic property.

Definition 12. [26] A property (or a notion) of theories (or models, or elements of models) is called semantic if and only if it is invariant relative to semantic similarity.

For example from [26] it is known that:

The ability to compare complete theories with the help of syntactic and semantic similarity was useful in describing the most important properties of the theory of stability in the study of complete theories. The following result confirms the importance of syntactic and, accordingly, semantic similarity of complete theories.

Proposition 3. [26] The following properties and notions are semantic:

(1) type;

(2) forking;

(3) λ-stability;

(4) Lascar rank;

(5) Strong type;

(6) Morley sequence;

(7) Orthogonality, regularity of types;

(8) $I(\aleph_\alpha, T)$ – the spectrum function.

The following definition was introduced Yeshkeyev A.R. in the frame of Jonsson theories study [24].

Let T be an arbitrary Jonsson theory, then $E(T) = \bigcup_{n<\omega} E_n(T)$, where $E_n(T)$ is a lattice of $\exists$-formulas with n free variables, T^* is a center of Jonsson theory T, i.e. $T^* = Th(\mathcal{C})$, where $\mathcal{C}$ is semantic model of Jonsson theory T in the sense of [23].

Definition 13. [24] Let T_1 and T_2 are arbitrary Jonsson theories. We say that T_1 and T_2 are Jonsson syntactically similar ($T_1 \overset{S}{\rtimes} T_2$) if it exists a bijection $f : E(T_1) \longrightarrow E(T_2)$ such that:

1) the restriction of f to $E_n(T_1)$ is an isomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;

2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;

3) $f(v_1 = v_2) = (v_1 = v_2)$.

In particular, a criterion was obtained that connects fixed Jonsson theories and their centers, which are complete theories. Thus, a connection is found between the concepts of syntactic and semantic similarity of complete theories and the corresponding similarities of fixed Jonsson theories.

Theorem 3. [24] Let T_1 and T_2 are $\exists$-complete perfect Jonsson theories, then following conditions are equivalent:

1) $T_1 \overset{S}{\rtimes} T_2$;

2) $T_1^* \overset{S}{\bowtie} T_2^*$.

One of the important and useful concepts of model theory is the formulaic definability of fixed subsets of the models under consideration. In particular, when studying complete theories, there are axiomatic approaches to such subsets [27]. In this article, when passing to fixed subsets of the semantic model of a fixed Jonsson theory, the concept of special definable formulaic subsets of the semantic model is used. These concepts were defined by Yeshkeyev A.R. [28], where he defined the concept of the Jonsson set and its particular case, the theoretical set. This approach is a generalization of the well-known concept of a basis in linear algebra.

Definition 14. [28] Let T be some Jonsson theory in a fixed language and C_T is its semantic model. A subset $X \subseteq C_T$ is called a Jonsson set in the theory T, if it satisfies the following properties:

1) the set X is a $\exists$-definable subset of C_T (this means that there is a $\exists$-formula, the solution of which in the C_T is the set X);

2) $cl(X) = M$, $M \in E_T$, where cl is some closure operator defining a pregeometry [29; 289] over C (for example $cl = acl$ or $cl = dcl$).

Further in our article it is assumed that $acl = dcl$.

Consider a countable language L, a complete for existential sentences perfect Jonsson theory T in the language L and its semantic model C_T. Let X be a Jonsson set in T and M be an existentially closed submodel of the semantic model C_T, where $\mathrm{dcl}(X) = M$. Then let $\mathrm{Th}_{\forall\exists}(M) = Fr(X)$, where $Fr(X)$ is the *Jonsson fragment* of the Jonsson set X.

Definition 14. A set X is called a theoretical set, if X is Jonsson set, $\varphi(C) = X$ and the universal closure of the formula $\varphi(x)$ defines some finitely axiomatizable Jonsson theory.

The concept of strong minimality, both for sets and for theories, has played an important role in the description of uncountably categorical complete theories [32]. Recall the definition of a strongly minimal type.

Let M be a structure of language L. A subset X of M is called minimal if it is definable (with parameters in M), infinite, and if for any definable (with parameters in M) subset Y of M either $X \cap Y$ or $X \setminus Y$ is finite. A formula $\varphi(x)$ (in $L(M)$) is strongly minimal if it defines a minimal set in all elementary extensions of M. A non-algebraic type is strongly minimal if it contains a strongly minimal formula.

2 Main results

Definition 15. A Jonsson theory T is called Robinson theory if it is universally axiomatizable.

Definition 16. [10] An enrichment $\tilde{T}$ is called admissible if the ∇-type (this means that $\nabla \subset L_\sigma$ and any formula from this type belongs to ∇) in this enrichment is definable within the framework of $\tilde{T}_\Gamma$-stability, where Γ is the enrichment of the signature σ.

Definition 17. [10] A Robinson theory T is called hereditary if in any of its admissible enrichments any extension is a Robinson theory.

Let T be a Robinson theory, A be an arbitrary model of signature σ. The Robinson spectrum of the model A is the set:

$$RSp(A) = \{T|\ T \text{ is Robinson theory in the language of signature } \sigma \text{ and } A \in Mod(T)\}.$$

Definition 18 (T.G. Mustafin [1]). We say that the Jonsson theory T_1 is cosemantic to the Jonsson theory T_2 ($T_1 \bowtie T_2$), if $C_{T_1} = C_{T_2}$, where C_{T_i} is the semantic model of the theory T_i, $i = 1, 2$.

It is easy to see that the cosemantic relation on a set of Jonsson theories is an equivalence relation. Since the Robinsonian theory is a special case of the Jonsson theory, then we can consider the $RSp(A)/_{\bowtie}$ factor set of the Robinson spectrum of the model A with respect to $\bowtie$.

And one can define the Robinson spectrum $RSp(K)$ of the class K structures for arbitrary signature by analog with the Robinson spectrum $RSp(A)$:

$$RSp(K) = \{T \mid T \text{ is a Robinson theory in the language } K \subseteq \mathrm{Mod}(T)\}.$$

We can note that if $A \in K$ then $RSp(A) \supset RSp(K)$.

Let $[T] \in RSp(K)/_{\bowtie}$, then $E_{[T]} = \bigcup\limits_{\Delta \in [T]} E_\Delta$ is the class of all existentially closed models of class $[T]$.

We will call a class $[T] \in RSp(K)/_{\bowtie}$ perfect if every theory $\Delta \in [T]$ is perfect.

We will call the class $[T] \in RSp(K)/_{\bowtie}$ hereditary if each theory $\Delta \in [T]$ is hereditary.

In what follows, we will work with a special class of K structures called a variety.

Recall that identities are formulas of the form $(\forall x_1, \dots x_n)\varphi(x_1, \dots x_n)$, where $\varphi(x_1, \dots x_n)$ is an atomic formula of signature σ.

Definition 19. [30] A class K of systems of signature σ is called a variety if there exists a collection F of identities of signature σ such that K consists of those and only those systems of signature σ in which all formulas from F. The collection F is called the defining collection of the variety.

Note that every variety is an axiomatizable class of algebras.

Examples of varieties are the classes of all semigroups, all groups, Abelian groups, Boolean rings, nilpotent groups of steps $\leq s$.

Let us formulate the following the well-known classical result:

Theorem 4 (Birkhoff [30], p. 337). For a non-empty class K of algebraic systems to be a variety, it is necessary and sufficient that the following conditions be satisfied:

1) the Cartesian product of an arbitrary sequence of K-systems is a K-system;

2) any subsystem of an arbitrary K-system is a K-system;

3) any homomorphic image of an arbitrary K-system is a K-system;

i.e. it is necessary and sufficient that the class K be hereditary, multiplicatively, and homomorphically closed.

Definition 20. A class of structures K_R of signature σ will be called a Robinson class if $Th_\forall(K_R)$ is a Robinson theory.

Definition 21. We will call a variety K Robinson hereditary if every Robinson class $K_R \subseteq K$ is a subvariety of the class K.

In [25] the question was formulated about the coincidence of the concepts of algebraic closure and existential closure in classes of models of a fixed variety. This question in this context is relevant to universal algebra. The concepts of algebraic closure and existential closure in the theory of models have an independent meaning, since the theory, generally speaking, may not be connected with the concept of variety. In this paper, the following result gives a positive answer to the above Forrest question in the framework of studying the cosemanticness classes of a fixed Robinson spectrum of a Robinson hereditary variety.

Theorem 5. Let K be a Robinson hereditary variety, $[T] \in RSp(K)/_{\bowtie}$ is perfect class, then for any algebraically closed model $A \in Mod[T]$ it follows that $A \in E_{[T]}$.

Proof. Suppose the opposite. Let there exist a model $A \in Mod([T])$ such that A is algebraically closed, but $A \notin E_{[T]}$. Then there exists a sentence $\theta = \exists \overline{x} \neg \varphi(\overline{x})$ and a model $B \in Mod([T])$ such that $B \supseteq A$ and $B \models \theta$, but $A \nvDash \theta$. Then $A \models \neg\theta$, that is, $A \models \forall \overline{x} \varphi(\overline{x})$. Since any theory $\Delta \in [T]$, Δ is Robinson theory, then according Proposition 1, there exists $B' \in E_{[T]}$ such that $A \to B'$, $B' \to C$, where C is a semantic model of the class $[T]$. Since class $[T]$ is perfect, then $C \models \neg\theta$. On the other hand, if $B \in E_{[T]}$, then $B \equiv_{\forall\exists} B'$ and $B' \models \theta$. If $B \notin E_{[T]}$, then there exists $B'' \in E_{[T]}$, such that $B \to B''$ and $B'' \to C$. In both cases we have $C \models \theta$. We got a contradiction. So our assumption was wrong, therefore, $A \in E_{[T]}$.

The idea of a central type allows one to study classes of models of the center of hereditary Jonsson theory in an enriched language. In this context, in the considered enrichment, we use a one-place predicate and some constant symbols, and one constant symbol is fixed in terms of the location of the interpretation of this constant relative to an existentially closed submodel of a fixed semantic model, which is an interpretation of a one-place predicate symbol. Taking into account the fact that in the pregeometry that specifies the closure of the set of types under consideration, the definable closure and algebraic closure of which are equal to each other, it allows avoiding collisions of non-preservation of the notion of Jonsson property in this enrichment.

Consider the general scheme for obtaining the central type for a hereditary cosemanticness class of Robinson theories [6].

Let A be an arbitrary model of signature σ, $[T] \in RSp(A)/_{\bowtie}$ be a hereditary class, $C_{[T]}$ be semantic model of class $[T]$. For each theory $\Delta \in [T]$, consider its enrichment $\bar{\Delta}$ in language of signature $\sigma_\Gamma = \sigma \cup \Gamma$, where $\Gamma = \{P\} \cup \{c\}$, obtained as follows:

$$\bar{\Delta} = Th_{\forall}(C_{[T]}, a)_{a \in P(C_{[T]})} \cup Th_{\forall}(E_\Delta) \cup \{P(c)\} \cup \{"P \subseteq"\},$$

where $\{"P \subseteq"\}$ is an infinite set of sentences expressing the fact that the interpretation of the symbol P is an existentially closed submodel in the language of the signature σ_Γ. That is, the interpretation of the symbol P is a solution to the equation $P(C_{[T]}) = M \subseteq E_\Delta$ in the language σ_Γ. Due to the heredity of the theory Δ, the theory $\bar{\Delta}$ is a Robinson theory. Collecting all such theories $\bar{\Delta}$, we obtain the class $[\bar{T}]$ of Robinson theories. The center $[\bar{T}]^* = Th(C_{[\bar{T}]})$ of class $[\bar{T}]$ is one of the completions for each theory $\bar{\Delta} \in [\bar{T}]$. Restricting the signature σ_Γ to $\sigma \cup \{P\}$, due to the laws of first-order logic, since the constant c does not already belong to this signature, we can replace this constant with the variable x. Then the theory $[\bar{T}]^*$ will be a complete 1-type for the variable x. We will call this type the central type of the class $[\bar{T}]$ in the above enrichment and denote it $P^c_{[\bar{T}]}$.

In work [6] was obtained criterion of uncountable categoricity for the hereditary Jonsson theory in the language of central types.

Theorem 6. [6] Let $[T]$ be hereditary class from $RSp(A)/_{\bowtie}$, then the following conditions are equivalent:

1) any countable model from $E_{[\overline{T}]}$ has an algebraically prime model extension in $E_{[\overline{T}]}$;

2) $P^c_{[\overline{T}]}$ is the strongly minimal type, where $P^c_{[\overline{T}]}$ is the central type of $[\overline{T}]$.

To prove the main result, we need a well-known fact:

Theorem 7 (Morley [31]). A theory T is ω_1-categorical if and only if any of its countable models has a simple proper elementary extension.

Obviously, we can use Morley's uncountable categoricity theorem in connection with the existence of an algebraically simple model extension for the central type in the framework of the following theorem. This means the following: the central type obtained by enriching the corresponding hereditary Robinson theory is exactly the center of the enriched Jonsson theory. If we replace the variable x with a constant that defines the central type, then we get a complete theory, which is model complete due to the perfection of the enriched Jonsson theory of this center. Thus, due to the model completeness, an algebraically simple model extension will also be a simple model extension, which allows us to consider this center as an ω_1-categorical theory, in which there is a strongly minimal formula, by virtue of the above Morley theorem.

Theorem 8. Let K be a Robinson hereditary variety, $[T] \in RSp(K)/_{\bowtie}$ be hereditary class, $X \subseteq C_{[T]}$ be a theoretical set defined by some strongly minimal $\exists$-formula $\varphi(x)$, Δ is some $\exists$-complete finitely axiomatizable Jonsson theory defined by $\forall x \varphi(x)$, then the following conditions are equivalent :

1) $\Delta \overset{S}{\rtimes} Fr(X) \overset{S}{\rtimes} T$;
2) the central type $P^c_{[\overline{T}]}$ of class $[\overline{T}]$ is strongly minimal.

Proof. If $\Delta \overset{S}{\rtimes} Fr(X) \overset{S}{\rtimes} T$, then by Theorem 3 $\Delta^* \overset{S}{\bowtie} Fr^*(X) \overset{S}{\bowtie} T^*$. But then, according to Proposition 3, these theories preserve the Morley rank, and, accordingly, the ω_1-categoricity, which is expressed in terms of the Morley rank. Thus, we have obtained that the theories Δ^*, $Fr^*(X)$ and $[T]^*$ are ω_1-categorical, i.e. all semantic models of these theories are saturated, hence the theories Δ, $Fr(X)$ and T are perfect. This means that the class $[T]$ is also perfect.

Note that in Theorem 6 item 1) is equivalent to the fact that the class $[T]$ is ω_1-categorical (this follows from Morley's theorem), and therefore perfect. Then, from Theorem 6 it follows that the central type $P^c_{[\overline{T}]}$ of the class $[\overline{T}]$ is strongly minimal.

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Теоретикалық жиынның фрагменті және оның қатты минималды централдық типі

Жұмыста алгебралардың жаңа класы анықталған, оның теориялары йонсондық теориялардың дербес жағдайы болып табылады. Бұл класс көптүрліліктерге де, йонсондық теорияларға да қолданылады. Осы мақаланың негізгі нәтижелері келесі екі нәтиже болып табылады. Авторлар робинсон мұралық көптүрлілігінің бекітілген спектрінің косеманттылық класының моделінің экзистенциалды тұйықталуы мен алгебралық тұйықталуының эквиваленттілігі туралы сұраққа жауап алынған. Централдық кластардың централдық типтерін және бекітілген спектрдің фрагменттерін зерттеу аясында қатты минималдылық критерийі алынды.

Кілт сөздер: йонсондық теория, экзистенциалды тұйық модель, алгебралық тұйық модель, косеманттылық, робинсон спектрі, робинсон мұралық көптүрлілігі, централдық тип, йонсондық фрагмент, теоретикалық жиын, қатты минималдық тип.

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Фрагмент теоретического множества и его сильно минимальный центральный тип

В работе определён новый класс алгебр, теория которых является частным случаем йонсоновских теорий. Данный класс относится и к многообразиям, и к йонсоновским теориям. Основными результатами настоящей статьи являются следующие два: авторами получены ответ на вопрос об эквивалентности экзистенциальной замкнутости и алгебраической замкнутости модели класса косемантичности фиксированного спектра робинсоновски наследственного многообразия, а также критерий сильной минимальности в рамках изучения центральных типов центральных классов и фрагментов фиксированного спектра.

Ключевые слова: йонсоновская теория, экзистенциально замкнутая модель, алгебраически замкнутая модель, косемантичность, робинсоновский спектр, робинсоновски наследственное многообразие, центральный тип, йонсоновский фрагмент, теоретическое множество, сильно минимальный тип.

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On categoricity questions for universal unars and undirected graphs under semantic Jonsson quasivariety

The article is devoted to the study of semantic Jonsson quasivarieties of universal unars and undirected graphs. The first section of the article consists of basic necessary concepts from Jonsson model theory. The following two sections are results of using new notions of semantic Jonsson quasivariety of Robinson unars $J\mathbb{C}_{\mathfrak{U}}$ and semantic Jonsson quasivariety of Robinson undirected graphs $J\mathbb{C}_{\mathfrak{G}}$, its elementary theory and semantic model. In order to prove two main results of the paper, Robinson spectra $RSp(J\mathbb{C}_{\mathfrak{U}})$ and $RSp(J\mathbb{C}_{\mathfrak{G}})$ and their partition onto equivalence classes $[\Delta]_{\mathfrak{U}}$ and $[\Delta]_{\mathfrak{G}}$ by cosemanticness relation were considered. The main results are presented in the form of theorems 11 and 13 and imply following useful corollaries: countably categorical Robinson theories of unars are totally categorical; countably categorical Robinson theories of undirected graphs are totally categorical. The obtained results can be useful for continuation of the various Jonsson algebras' research, particularly semantic Jonsson quasivariety of S-acts over cyclic monoid.

Keywords: Jonsson theory, unar, graph, undirected graph, universal theory, Robinson theory, quasivariety, semantic Jonsson quasivariety, Jonsson spectrum, Robinson spectrum, cosemanticness, categoricity, countable categoricity.

Introduction

This paper and focuses on the study of model-theoretic properties of well-known and sufficiently simple classes in the sense of the signature of algebras, namely unars and undirected graphs. One can note that this paper is a continuation of works [1–4].

At one time, the famous mathematician-logician H.J. Keisler, in his review article "Fundamentals of Model Theory" in the four-volume monograph "Reference Book on Mathematical Logic" (edited by J. Barwise), defined the basic concepts and directions of the development of model theory. H.J. Keisler identified two historical trends in the development of model theory. They are called "western" and "eastern" model theory. This division is due to the fact that A. Tarski lived on the west coast from 1940, and A. Robinson lived on the east coast from 1967 until his premature death in 1975. This distinction has long lost its geographical significance, but it is useful from a mathematical point of view.

"Western" model theory develops in the traditions of Skulem and A. Tarski. It was mostly motivated by problems in number theory, calculus and set theory, it uses all the formulas of first-order logic.

"Eastern" model theory develops in the traditions of A.I. Mal'tsev and A. Robinson. It was motivated by problems in abstract algebra, where the formulas of theories usually have at most two blocks of quantifiers. It emphasizes a set of quantifier-free and existential formulas.

Jonsson theories as an object of research were first considered in the works of Jonsson [5] and Morley, Voot [6]. In the mid-80s of the twentieth century, the works of T.G. Mustafin identified a new direction in the study of Jonsson theories. In particular, he defined a natural subclass of Jonsson theories, which

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he called perfect. The main method of his research was the following: the study of the properties of arbitrary Jonsson theories by transferring the properties of the central completion of this Jonsson theory. In the early 90s of the twentieth century A.R. Yeshkeyev obtained a criterion for the perfection of the Jonsson theory [7]. In particular, there was obtained a complete description of the Jonsson universal unars in the works [2,3] by A.R. Yeshkeyev, T.G. Mustafin, as well as the relationship between the theory of unars and their center in the language of stability. On the other hand, one of the weak points in the study of Jonsson theories within the framework of the method proposed by T.G. Mustafin was the presence of an additional axiom about the existence of a strongly inaccesible cardinal to the axioms of Zermelo-Frenkel set theory in the definition of a semantic model. It should be noted that during the talk of R.M. Ospanov at the "5th Kazakh-French colloquium on model theory well-known experts in the field of model theory Ye.A. Palyutin and B. Poizat pointed out the need to change this definition. The realization of this remark was the output of the work of Ye.T. Mustafin [8], in which he redefines the concept of k-homogeneity and semantic model. Accordingly, the modified definition of the perfection of the Jonsson theory appeared in [9], in which the main results obtained earlier in [10] were re-demonstrated within the framework of the new definition.

The results discussed here relate in their content to the "eastern" model theory. Various properties of unars from the perspective of "western" model theory (the case when the complete theory of some unar is considered) were obtained in the works of Yu.E. Shishmarev [11], A.N. Ryaskin [12].

The notion of countable categoricity in "western" model theory distinguishes probably the narrowest class of theories, and it is well studied. In the case of "eastern" model theory (meaning studies of Jonsson theories), it should be noted that Vought theorem on the relationship between completeness and categoricity of the theory does not hold, since Jonsson theories, generally speaking, are not complete and have finite models. The following question of Ye.A. Palyutin is well-known: is there an ω-categorical universal K that is not ω_1-categorical? If this question is projected into the framework of research on the Jonsson theories, then one can notice some interesting connections between the Jonsson theory itself and its center appear.

In this regard, A.R. Yeshkeyev [13] obtained the following results:

Theorem 1. If the Jonsson theory T is ω-categorical, then T is perfect.

Theorem 2. If the Jonsson theory T is k-categorical, then the $\#$-companion of the theory T is k-categorical, where $k \geq \omega$.

Theorem 3. In the case of a negative answer to question of Ye.A. Palyutin for a Jonsson theory that satisfies the conditions of the question, the center of the Jonsson theory cannot be finitely axiomatized.

There is considered a class of existentially closed models of an arbitrary universal theory in the work of A. Pillai [14], and for this class he develops a forking theory with a suitable concept of the simplicity of the theory. S. Shelah [15], E. Hrushovski [16] studied classes of existentially closed models of Robinson theory. A theory is called a Robinson theory if it is universal and admits AP and JEP. From here it is easy to see that any Robinson theory is a special case of the Jonsson theory. And if we take into account that unars and undirected graphs are Jonsson universals, then obtaining a description of their existentially closed models within the framework of the above topic is an urgent task. This article discusses the description of an existentially closed model of a countably categorical universal of unars, as well as undirected graphs.

All definitions that were not given in the current article can be extracted from [7, 17–27].

1 *Necessary concepts of Jonsson model theory*

Let us recall the conditions, that should be satisfied in order for a theory to be Jonsson.

Definition 1. [5] A theory T is said to be Jonsson, if:

1) T has at least one infinite model;
2) T is $\forall\exists$-axiomatising;
3) T has JEP property;
4) T has AP property.

For example, the following theories are Jonsson: unars, graphs and their various subclasses, groups, abelian groups, Boolean algebras, linear order, fields of characteristic p (p is a prime number or zero), ordered fields. In addition to these natural examples and rather broad classes of algebras, we may also notice that for an arbitrary theory T its scolemization and morleization are also examples of Jonsson theories.

$\forall$-axiomatizing Jonsson theory is called the Robinson theory.

By virtue of theorem of Morley and Vaught [6] an arbitrary Jonsson theory T has T-universal, T-homogeneous model $\mathfrak{C}$ in some inaccessible cardinality. Let us consider elementary theory $Th(\mathfrak{C})$. We denote it as T^*, i.e. $T^* = Th(\mathfrak{C})$.

The next definitions belong to T.G. Mustafin.

Definition 2. [7] 1) Let T be a Jonsson theory. A model $\mathfrak{C}_T$ of power $2^{|T|}$ is called to be a semantic model of the theory T if $\mathfrak{C}_T$ is a $|T|^+$-homogeneous $|T|^+$-universal model of the theory T.

2) The elementary theory of a semantic model of the Jonsson theory T is called the central completion or center of this theory. The center is denoted by T^*, i.e. $Th(C) = T^*$.

In the "west" model theory, when isomorphic embedding in the definitions of universal and homogeneous model changes to elementary embedding, and also the definition of the homogeneous model changes, then the following theorem is true:

Theorem 4. [7] A system $\mathfrak{A}$ is saturated iff it is homogeneous and universal.

Unfortunately, in the "east" model theory T-universal, T-homogeneous model does not have to be saturated model. The following notions are required for proofing the main theorems of this paper.

Definition 3. [7] Jonsson theory T is called perfect theory, if its semantic model $\mathfrak{C}_T$ is saturated.

Theorem 5. [7] Let T be arbitrary Jonsson theory, then the following conditions are equivalent:
1) Theory T is perfect,
2) T^* is model completion of theory T.

The following criterium is nedded for clarification of constructing semantic Jonsson quasivariety.

Theorem 6. [7] T is Jonsson iff it has a semantic model $\mathfrak{C}_T$.

Since we will work with Robinson theories of unars and undirected graphs, let us recall the definition of universal.

Definition 4. [2] If $T = T_\forall$, then T is said to be universal.

The next two notions of κ-categorical Jonsson theory and existentially closed model of theory T are nedded for obtaining main theorems of this paper.

Definition 5. [7] Let $\kappa \geq \omega$. Jonsson theory T is called κ-categorical, if any two models of power κ of theory T are isomorphic to each other.

Definition 6. [7] Model A of theory T is called existentially closed model of theory T, if for any model B of theory T such that $A \subseteq$ B, for any $\exists$ - formula $\exists x\varphi(x, \overline{y})$, for any $\overline{a}$ from A $(l(\overline{a})) = (l(\overline{y}))$ from $B \models \exists x\varphi(x, \overline{a})$ follows that $A \models \exists x\varphi(x, \overline{a})$

We will denote a class of existentially closed models of theory T as E_T.

Since the current research is connected with consideration of Robinson spectrum for classes of algebras, let us give the following conditions of Jonsson theories cosemanticness.

Definition 7. [7] Let T_1 and T_2 be Jonsson theories, $\mathfrak{C}_{T_1}$ and $\mathfrak{C}_{T_2}$ be their semantic models, respectively. T_1 and T_2 are said to be cosemantic Jonsson theories (denoted by $T_1 \bowtie T_2$), if $\mathfrak{C}_{T_1} = \mathfrak{C}_{T_2}$.

Theorem 7. [7] Let T_1 and T_2 be Jonsson theories, $\mathfrak{C}_{T_1}$ and $\mathfrak{C}_{T_2}$ be their semantic models, respectively. Then the next conditions are equivalent:
1) $\mathfrak{C}_{T_1} \bowtie \mathfrak{C}_{T_2}$;
2) $\mathfrak{C}_{T_1} \equiv_J \mathfrak{C}_{T_2}$;
3) $\mathfrak{C}_{T_1} = \mathfrak{C}_{T_2}$.

Let K be a class of models of fixed signature σ. Then we can consider Jonsson spectrum for K, which can be defined as follows.

Definition 8. [28] A set $JSp(K)$ of Jonsson theories of signature σ, where

$$JSp(K) = \{T\ |T \text{ is Jonsson theory and } K \subseteq Mod(T)\},$$

is called the Jonsson spectrum for class K.

Hence, in the particular case, when the Jonsson theory is $\forall$-axiomatising we get the concept of the Robinson theory, respectively, the notion of the Jonsson spectrum allows us to consider the Robinson spectrum.

Definition 9. [4] A set $RSp(K)$ of Robinson theories of signature σ, where

$$RSp(K) = \{T\ |T \text{ is Robinson theory and } \forall A \in K, A \models T\},$$

is called the Robinson spectrum for class K.

Based on theorem 7, we can consider the cosemanticity relation on Jonsson spectrum $JSp(K)$ and obtain a partition of $JSp(K)$ onto equivalence classes. we get a factor-set, denoted as $JSp(K)_{/\bowtie}$. The factor-set $RSp(K)_{/\bowtie}$ will be obtained correspondingly.

Let K be a class of quasivariety in the sense of [29] of first-order language L, $L_0 \subset L$, where L_0 is the set of sentences of language L. Let us consider the elementary theory $Th(K)$ of such class K. By adding to $Th(K)$ $\forall\exists$ sentences of language L, that are not contained in the $Th(K)$, we can consider the set of Jonsson theories $J(Th(K))$ defined as follows.

Denotation 1. [4] A set $J(Th(K)) = \{\Delta \mid \Delta - \text{Jonsson theory},\ \Delta = Th(K) \cup \{\varphi^i\}\}$, where $\varphi^i \in \forall\exists(L_0)$ and $\varphi^i \notin Th(K)$, $i \in \{0,1\}$, $Th(K)$ is elementary theory of class of quasivariety K, $\forall\exists(L_0)$ is a set of all $\forall\exists$ sentences of language L.

Let us consider the set of such semantic models and denote it as $J\mathbb{C}$.

Denotation 2. [4] A set $J\mathbb{C} = \{\mathfrak{C}_\Delta \mid \Delta \in J(Th(K)), \mathfrak{C}_\Delta \text{ is semantic model of } \Delta\}$.

We will call the set $J\mathbb{C}$ semantic Jonsson quasivariety of class K if its elementary theory $Th(J\mathbb{C})$ is Jonsson theory.

2 *Countable categoricity of semantic Jonsson quasivarieties of universal unars*

Let $\mathfrak{A}$ be some unar, i.e. the model of signature $\sigma = \{f\}$, where f is a unary functional symbol. Let $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$, $n \in \omega$. Elements $a, b \in \mathfrak{A}$ are called $\mathfrak{A}$-connected in X if there exist natural numbers m and n such that $(f^m(a) = f^n(b))$ and $f^0(a) = f^m(a), f^0(b), ..., f^n(b) \in X$.

A set $X \subseteq \mathfrak{A}$ is called $\mathfrak{A}$-connected if any two elements from X are $\mathfrak{A}$-connected. A subsystem $\mathfrak{B} \subseteq \mathfrak{A}$ carrier of which is the maximal $\mathfrak{A}$-connected subset of carrier $\mathfrak{A}$ is called a component in $\mathfrak{A}$. If $\mathfrak{B}$ is a component in system $\mathfrak{A}$, then the set $\{a \in \mathfrak{B} : \mathfrak{A} \models (f^n(a) = a)$ for some $n \in \omega$ is called a cycle of component. By $K(a, \mathfrak{A})$ we denote the restriction of $\mathfrak{A}$ to the set $\{b \in \mathfrak{A} : \mathfrak{A} \models (f^n(b) = a)$ for some

$n \in \omega\}$ and we call it the root of the element a in the unar $\mathfrak{A}$, while the element a is called the vertex of the root $K(a, \mathfrak{A})$.

We will write down the special connections between the elements of the unar in the form of $\exists$-formulas:

1) the property of the elements to be at "the beginning of the cycle":

$\Phi_0^n(z) = \Phi^n(z) \& \exists y \neg \Phi^(y) \& f(y) = z$, where $\Phi^n(z) = (f^n(z) = z) \& (f(z) \neq z) ... (f^{n-1}(z) \neq z)$;

2) "x has no less than k different immediate representatives":

$\Theta(x) = \exists x_1, ..., \exists k(\wedge_{i \neq j < x} x_i \neq x_j \wedge \wedge_{i=1}^{k} f(x_i) = x)$;

3) "there are exactly k different elements between x and the beginning of the cycle":

$\Psi_k(x) = \exists z \exists y_1 ... \exists y_k (\wedge_{i \neq j < x} (y_i \neq y_j) \wedge f^i(x) = y_i \wedge \wedge_{i=1}^{k-1} f(y_i) \neq f(y_{i+1}) \wedge \Phi_0^n(z) \wedge f(y_k) = z)$.

By virtue of works [2, 4] we can use the conclusion that $\forall$-axiomatisability of elemantary theory of unars, $Th_{\forall}(\mathfrak{U})$ is the Robinson theory of unars.

Thus, we consider a set $J\mathbb{C}_{\mathfrak{U}} = \{\mathfrak{C}_{\Delta_{\mathfrak{U}}} \mid \Delta_{\mathfrak{U}} \in J(Th(K)), \mathfrak{C}_{\Delta_{\mathfrak{U}}}$is a semantic model $\Delta_{\mathfrak{U}}\}$ of signature $\sigma_{\mathfrak{U}} =< f >$, where $\Delta_{\mathfrak{U}}$ is a Robinson theory of unars, f is unary functional symbol. Such $J\mathbb{C}_{\mathfrak{U}}$ defines semantic Jonsson quasivariety of Robinson unars as in [4].

We are using the definition of the Robinson spectrum of the set $J\mathbb{C}_{\mathfrak{U}}$ [4].

Definition 10. [4] A set $RSp(J\mathbb{C}_{\mathfrak{U}})$ of Robinson theories of signature $\sigma_{\mathfrak{U}}$, where

$$RSp(J\mathbb{C}_{\mathfrak{U}}) = \{\Delta_{\mathfrak{U}} \mid \Delta_{\mathfrak{U}} \text{ is Robinson theory of unars and } \forall \mathfrak{C}_{\Delta_{\mathfrak{U}}} \in J\mathbb{C}_{\mathfrak{U}}, \mathfrak{C}_{\Delta_{\mathfrak{U}}} \models \Delta_{\mathfrak{U}}\},$$

is called the Robinson spectrum for class $J\mathbb{C}_{\mathfrak{U}}$, where $J\mathbb{C}_{\mathfrak{U}}$ is semantic Jonsson quasivariety of Robinson unars.

Further we obtain a factor-set, denoted as $RSp(J\mathbb{C}_{\mathfrak{U}})_{/\bowtie}$ and consisted of equivalence classes parted by cosemanticness relation $[\Delta_{\mathfrak{U}}] \in RSp(J\mathbb{C}_{\mathfrak{U}})_{/\bowtie}$.

Remark 1. Everywhere in this section $[\Delta_{\mathfrak{U}}]$ denotes an equivalence class of Robinson theories of unars parted by cosemanticness relation on Robinson spectrum $RSp(J\mathbb{C}_{\mathfrak{U}})$. $\mathfrak{C}_{\Delta_{\mathfrak{U}}}$ denotes semantic model and $E_{\Delta_{\mathfrak{U}}}$ denotes a class of existentially closed models of class $[\Delta_{\mathfrak{U}}]$.

Further we obtained two useful theorems, concerning the equivalence class $[\Delta_{\mathfrak{U}}]$ of Robinson theories of unars parted by cosemanticness relation on Robinson spectrum $RSp(J\mathbb{C}_{\mathfrak{U}})$.

We will use the denotations from [2–4].

Theorem 8. Let $[\Delta_{\mathfrak{U}}]$ be a class of Robinson theories of unars, $[\Delta_{\mathfrak{U}}^*]$ its center. Then

1) $[\Delta_{\mathfrak{U}}^*]$ is model completion of $[\Delta_{\mathfrak{U}}]$;

2) $[\Delta_{\mathfrak{U}}^*]$ allows quantifier elimination (i.e. submodel complete);

3) $[\Delta_{\mathfrak{U}}^*]$ is ω-stable.

Proof. 1) Let $\mathfrak{C}$ be semantic model of $[\Delta_{\mathfrak{U}}]$. Then $[\Delta_{\mathfrak{U}}^*] = Th(\mathfrak{C})$. Let $\mathfrak{C}^*$ be saturated model of $[\Delta_{\mathfrak{U}}^*]$. We can assume that $\mathfrak{C}^* \subseteq \mathfrak{C}$. It easy to understand that if $a \in \mathfrak{C}^*$, then $tp^{\mathfrak{C}^*}(a, \varnothing) = tp^{\mathfrak{C}}(a, \varnothing) = \chi(a)$. Hence $C^{\mathfrak{C}}(a) \simeq C^{\mathfrak{C}^*}(a)$, whene $C^{\mathfrak{A}}(b)$ by definition is $\{c \in \mathfrak{A} : \exists n, k < \omega \ \ f^n(c) = f^k(b)\}$. The quantity of pairwise isomorpic components is uniquely defined by $char[\Delta_{\mathfrak{U}}]$. Hence $\mathfrak{C}^* \simeq \mathfrak{C}$. It means that $[\Delta_{\mathfrak{U}}]$ is perfect Jonsson theory and $[\Delta_{\mathfrak{U}}^*]$ is its model completion.

2) follows from 1) and Robinson theorem [3].

3) Let H be arbitrary subunar of $\mathfrak{C}$. From lemma 5 [2] we have

$$|S^{\mathfrak{C}}(H)| \leq (1 + \omega^2) + (1 + \omega) + |H|, \text{ because}$$
$$|\{f(a) : a \in \mathfrak{C}\}| - |\Omega| \leq 1 + \omega^2,$$
$$|\{\rho(a, H) : a \in \mathfrak{C}\}| \leq 1 + \omega,$$
$$|\{\text{enter}(a, H) : a \in \mathfrak{C}\}| \leq |H|.$$

From this, if $|H| \leq$, then $|S^{\mathfrak{C}}(H)| \leq \omega$.

The theorem is proven.

Theorem 9. 1) The quantity of pairwise different $[\Delta_{\mathfrak{U}}]$ classes of Robinson theories of unars is equal to 2^{ω}.

2) The quantity of pairwise different maximal $[\Delta_{\mathfrak{U}}]$ classes of primitive Robinson theories is equal to 2^{ω}.

3) The quantity of pairwise different maximal $[\Delta_{\mathfrak{U}}]$ classes of Robinson theories of unars is equal to ω. Moreover, these are precisely the classes of theories, that have following characteristics: $\pi_{\omega}, \{\pi_{0,m} \;\; 1 \leq m < \omega\}, \{\pi_{n,m} \;\; 1 \leq n, m < \omega\}$, where

$\pi_{\omega} : \Omega = \{\omega\}, \nu(m) = 0 \;\; \forall m < \omega, \mu(\omega) = 1, \varepsilon = \infty;$

$\pi_{0,m} : \Omega = \{(0,m)\}, \nu(m) = \begin{cases} 0, \text{if } \; k \neq m, \\ \infty, \text{if } \; k = m; \end{cases} \quad \mu(0,m) = 0, \varepsilon = 0;$

$\pi_{n,m} : \Omega = \{(0,m), .., (n,m)\}, \nu(k) = \begin{cases} 0, \text{if } \; k \neq m, \\ 1, \text{if } \; k = m, \end{cases}$

$\mu(k,m) = \begin{cases} 1, \text{if } \; k < n-1, \\ \infty, \text{if } \; k = n-1, \\ 0, \text{if } \; k = n, \end{cases} \;\; \varepsilon = 0.$

4) Maximal ∇-complete $[\Delta_{\mathfrak{U}}]$ classes of Robinson theories of unars is the only class, that has characterstic π_{ω}.

Proof. 1) It is easy to note that the quantity of pairwise different characteristics is equal to 2^{ω}. By theorem 3 [4] the quantity of $[\Delta_{\mathfrak{U}}]$ classes of Robinson theories of unars is equal to 2^{ω}.

2) Let $[\Delta_{\mathfrak{U}}]'_{\pi} = (Th(\mathfrak{C}_{\pi}))_{\nabla}$ where $\mathfrak{C}_{\pi}$ is semantic model of class of Robinson theories of unars of characteristic π. Obviously $[\Delta_{\mathfrak{U}}]'_{\pi}$ is ∇-complete primitive. By lemma 1 [3] $[\Delta_{\mathfrak{U}}]'_{\pi}$ is class of Robinson theories of unars. By Proposition 3 [3] $[\Delta_{\mathfrak{U}'_{\pi}}]$ is maximal class of primitive Robinson theories. If $\pi_1 \neq \pi_2$, then $[\Delta_{\mathfrak{U}'_{\pi_1}}] \neq [\Delta_{\mathfrak{U}'_{\pi_2}}]$, since $([\Delta_{\mathfrak{U}'_{\pi_1}}])_{\forall} \neq ([\Delta_{\mathfrak{U}'_{\pi_2}}])_{\forall}$, hence, the quantity of maximal $[\Delta_{\mathfrak{U}}]$ classes of primitive Robinson theories is equal to 2^{ω}.

3) Let us consider partial order on set of all characteristics in following form. Let $\pi_i = (\Omega_i, \nu_i, \mu_i, \varepsilon_i)$, $i = 1, 2$. Then suppose $\pi_1 \leq \pi_2 \Leftrightarrow \Omega_1 \subseteq \Omega_2 \;\; \& \;\; \forall m < \omega(\nu_1(m)) \leq \nu_2(m)) \;\; \& \;\; \forall \alpha \in \Omega_1(\mu_1(\alpha) \leq \mu_2(\alpha)) \;\; \& \;\; \varepsilon_1 \subseteq \varepsilon_2$. From definition of class $[\Delta_{\mathfrak{U}]_{\pi}}$ in the proof for theorem 3 [4] it easy to see that

$$[\Delta_{\mathfrak{U}]_{\pi_1}} \supseteq [\Delta_{\mathfrak{U}]_{\pi_2}} \Leftrightarrow \pi_1 \leq \pi_2.$$

Case 1. $\varepsilon = \infty$.

Among such characteristics the minimal is the only characteristic π_{ω}.

Case 2. $\varepsilon = 0$.

In this case $\omega \notin \Omega$ and $|\Omega| < \omega$. By condition 10) from definition of characteristic [3] either $\exists 0 < k < \omega \;\; (\nu(k) = \infty)$, either $\exists (k,l) \in \Omega, \;\; (\mu(k,l)) = \infty)$.

Case 2.1. $\exists 1 \leq k < \omega \;\; (\nu(k)) = \infty))$.

Among such characteristics the minimal are characteristics $\pi_{0,m}, 1 \leq n < \omega, 1 \leq m < \omega$.

Case 2.2. $\exists 1 \leq k < \omega, 1 \leq l < \omega \;\; (\mu(k,1) = \infty)$.

In the set of such characteristics the minimal are characteristics $\pi_{n,m}, 1 \leq n < \omega, 1 \leq m < \omega$.

4) Note that the class $[\Delta_{\mathfrak{U}]_{\pi}}$, that has characteristic π_{ω} is complete, in particular ∇-complete. Therefore it is maximal among classes of Robinson theories of unars. Classes $[\Delta_{\mathfrak{U}]_{n,m}}, 0 \leq n < \omega, 1 \leq m < \omega$ are not ∇-complete, since $[\Delta_{\mathfrak{U}]_{n,m}} \cup \exists x_1, ..., x_{m+1}(\wedge_{1 \leq i < j \leq m+1}(x_i \neq x_j))$ and $[\Delta_{\mathfrak{U}]_{n,m}} \cup \forall x_1, ..., x_{m+1}(\vee_{1 \leq i < j \leq m+1}(x_i = x_j))$ are consistent. The theorem is proven.

By consideration of theorems 9 and 10, we can obtain the following result:

Theorem 10. Let $[\Delta_{\mathfrak{U}}]$ be a class of ω-categorical Robinson theories of unars. Then the following conditions are equivalent:

1) $\mathfrak{A} \in E_{\Delta_{\mathfrak{U}}}$, where $\mathfrak{A}$ is a model of class $[\Delta_{\mathfrak{U}}]$;

2) $\mathfrak{A}$ is disjoint union of components with cycles of the same length.

Proof. The proof of this theorem is based on the following theorem, three facts and three lemmas.

Theorem 11. [30] In order for the algebraic system $\mathfrak{A}$ to be some ω-categorical universal, it is necessary and sufficient that the following conditions will be satisfied:

1) $\mathfrak{A}$ is locally finite;

2) there is a function $g : \omega \to \omega$ such that for every $a \in \mathfrak{A}$ and for every finite subset $X \subseteq \mathfrak{A}$ the type $tp(a, X, \mathfrak{A})$ is realized in every subsystem $\mathfrak{B} \subseteq \mathfrak{A}$ that contains X and has a power $\geq g(|X|)$.

Fact 1. [13] If the Jonsson theory T is ω - categorical, then T is perfect.

Fact 2. [30] Let T be a Jonsson theory. Then the following conditions are equivalent:

1) T is perfect;

2) $E(T) = ModT^*$;

3) T^* is a model companion of the theory T.

Fact 3. [31] Let T be $\forall\exists$-complete Jonsson theory. Then the following conditions are equivalent:

1) T is ω-categorical;

2) T^* is ω-categorical.

We get as a consequence of these facts (1 - 3) that, since $[\Delta_{\mathfrak{U}}]$ is ω-categorical, $[\Delta_{\mathfrak{U}}]$ is an equivalence class of perfect Robinson theories, and $E_{\Delta_{\mathfrak{U}}} = Mod(\Delta^*_{\mathfrak{U}})$ is ω-categorical universal. Thus, if $\mathfrak{A} \in E_{\Delta_{\mathfrak{U}}}$, then $\mathfrak{A} \in Mod(\Delta^*_{\mathfrak{U}})$. Consequently, $\mathfrak{A}$ satisfies the conditions of E.A. Palyutin criterion (Theorem 11).

By virtue of these arguments, it is sufficient to prove the following lemmas to prove Theorem 10.

Lemma 1. Let $\mathfrak{A} \in \omega$-categorical Jonsson universal, $x \in \mathfrak{A}$. Then $\exists n, k\ \omega : f^n(x) = f^k(x)$.

Proof. By virtue of E.A. Palyutin criterion, $\mathfrak{A}$ is locally finite. Now suppose that $\forall n, k \in \omega$: $f^n(x) \neq f^k(x)$. This means that there is a set $Y = \{y_1, y_2, ..., y_n, ...\} \subseteq \mathfrak{A}$, where $f(y_i) = y_{i+1}$, and $y_i \neq y_j$ if $i \neq j$, where $i \in \{1, 2, ...\}$. But then an element, for example y_1, generates an infinite (countable) set Y. And this contradicts the local finiteness of unar $\mathfrak{A}$.

Lemma 2. Let $\mathfrak{A} \in \omega$-categorical Jonsson universal. Then for any element $a \in \mathfrak{A}$, the root $K(a, \mathfrak{A})$ is finite.

Proof. Let us assume the opposite. Let there be an element $a \in \mathfrak{A}$ such that the root $K(a, \mathfrak{A})$ is infinite. Then there are two possible cases:

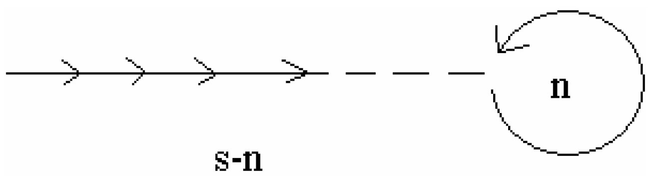
1) $\Psi_k(x) : k \in \omega$ is realized in unar $\mathfrak{A}$;

2) $\Theta_k(x) : k \in \omega$ is realized in unar $\mathfrak{A}$.

By virtue of the Palyutin criterion, there exists a function $\phi : \omega \to \omega$ such that for any $n \in \omega$, for any subunar B [3] of an unar $\mathfrak{A}$ with a power of at least $\phi(n)$, for any type $p \in S^j(\bar{b})$ $(\bar{b} \in B)$ from the fact that $\mathfrak{A} \models p(a)$, it follows that there exists a $b \in B$ such that $\mathfrak{A} \models p(b)$. Let $\phi(0) = s$. Then according to the criterion for any subunar B of a unar $\mathfrak{A}$ with a power of at least s for any type $p \in S^j(\bar{b})$ $(\bar{b} \in B)$ $\mathfrak{A} \models p(a) \Rightarrow \exists b \in B : \mathfrak{A} \models p(b)$(i.e. any type of element of unar $\mathfrak{A}$ is realized in B).

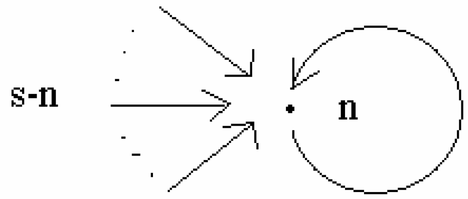
1) Consider the chain Γ. Let Γ_s be a subchain Γ with a cycle, and the number of elements in Γ_s is equal to s.

It has the form:



A type containing the formula $\Psi_{s-n+1}(x)$ cannot be implemented in Γ_s (i.e., there are exactly $s - n + 1$ different elements between x and the beginning of the cycle).

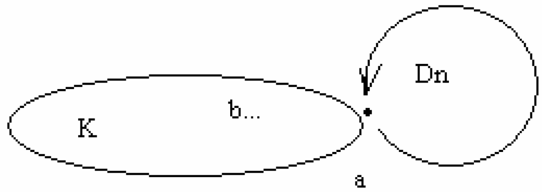
2) Consider a subset where the number of elements of the preimages with a cycle is s:



It is clear that no finite unar realizes the set of formulas $\{\Theta_k(x) : k \in \omega\}$. We get a contradiction.

Lemma 3. Let $\mathfrak{A} \in \omega$-categorical Jonsson universal. Then:
1) each element of $\mathfrak{A}$ enters some cycle;
2) all cycles of unar $\mathfrak{A}$ have the same length.

Proof. By virtue of the previous lemmas, each component of the unar $\mathfrak{A}$ is finite and has the form $D_n \oplus^n K$, where D_n is a cycle of length n, a is an element of the cycle, K is the finite root of a.



Let $b \in K$ and $b \neq a$. b is not included in any cycle. Then there exists k such that $f^k(b) = a$ and $f^s(b) \neq a$ for $s < k$. Consider the formula $\exists y(f^k(y) = a\&_{i<k}f^i(y) \neq a)\&f^k(a) = a\&\&_{i<k}f^i(a) \neq a, k > 1$. It is clear that in the infinite subunar $A' \subseteq \mathfrak{A}$, obtained by combining only the elements included in some cycles, this formula is not realized. Which contradicts condition 2) of the criterion. Thus point 1) of the lemma is proved.

2) Let us assume the opposite: there are at least two cycles of different lengths. Then there are two possible cases:

2.1) For some n there is a finite number of cycles of length n. Then for some n_0 (with a non-empty set of cycles of length n_0), we remove all cycles of length n_0 from unar $\mathfrak{A}$. We get an infinite subunar in which the formula $f^{n_0}(x) = x\&\&_{i<n_0}f^i(x) \neq x$ is not realized.

2.2) (Negation of the first case) Let n_0 be a number for which there is an infinite set of length n_0 in $\mathfrak{A}$. By assumption, there is at least one cycle $k \neq n_0$ in $\mathfrak{A}$. Remove all cycles of length k from $\mathfrak{A}$. We get an infinite subunar in which the formula $f^k(x) = x\&\&_{i<k}f^i(x) = Z$ is not realized.

There is obtained a contradiction to condition 2) of the criterion in each of the two cases.

Let us prove sufficiency. If unar $\mathfrak{A}$ is a disjunctive union of an infinite number of components that are a cycle of the same length, then $\mathfrak{A}$ is ω-categorical universal.

We will show the satisfaction of points 1) and 2) of the criterion.

1) Consider a finite subset of $\{a_1, ..., a_n\} \subseteq \mathfrak{A}$. Each of the elements generates a cycle of length n. Therefore, a subsystem generated by a finite subset of $\{a_1, ..., a_n\}$ contains no more than nk elements.

2) Find the function g, the existence of which is required by the criterion. Consider a finite subset of elements $X_k = \{a_1, ..., a_k\} \subseteq \mathfrak{A}$. It is not difficult to understand that the total number of different types over X_k does not exceed the number $n(k+1)$. Then any submodel contains cycles "connected" with elements from X_k, and one cycle independent of them realizes all $n(k+1)$ types. Therefore, $g(k)$ will be equal to $n(k+1)$.

In connection with the above question by Ye.A. Palyutin, from the description of the existentially closed unar model (Theorem 11.), it can be noted that

Corollary 1. Countably categorical Robinson theories of unars are totally categorical.

3 Countable categoricity of semantic Jonsson quasivarieties of undirected graphs

$\mathfrak{G}$ graph is further understood as an algebraic system of the signature $< R >$, where R is binary symmetric relation, i.e. an undirected graph. Further, the elements of the graph are called vertices, and pairs $< x, y >$ such that $R(x, y)$ are called edges. A graph set of edges of which is empty is called a quite disconnected graph. A path in graph G is an alternating sequence of vertices and edges: $x_i, < x_i, x_{i+1} >, x_{i+1}, < x_{i+1}, x_{i+2} >, \ldots$. A path is called a chain if all its edges are different, and a simple chain if all vertices (and therefore edges) are different. A graph $\mathfrak{G}$ is called connected if any pair of its vertices is connected by a simple chain. A graph is called acyclic if there are no cycles in it. A tree is a connected acyclic graph. The maximal connected subgraph of a graph $\mathfrak{G}$ is called a connectivity component, or simply a graph component. A subgraph of a graph $\mathfrak{G}$ is a graph in which all vertices and edges belong to $\mathfrak{G}$. The degree of a vertex in a graph $\mathfrak{G}$ is the number of edges incident to this vertex. A vertex of degree I is called a pendant (or end point) vertex.

Countably categorical graphs were studied in [32]. The main result of this work is the following theorem:

Theorem 12. Let $\mathfrak{G}$ be an arbitrary countable graph in which each component contains a finite number of cycles. Then $\mathfrak{G}$ is ω-categorical if and only if $\mathfrak{G}$ is bounded and a finite number of I-types is realized in it.

By virtue of works [3, 4] we can use the conclusion that $\forall$-axiomatisability of elemantary theory of graphs, $Th_{\forall}(\mathfrak{G})$ is the Robinson theory of graphs.

Thus, we consider a set $J\mathbb{C}_{\mathfrak{G}} = \{\mathfrak{C}_{\Delta_{\mathfrak{G}}} \mid \Delta_{\mathfrak{G}} \in J(Th(K)), \mathfrak{C}_{\Delta_{\mathfrak{G}}}$ is a semantic model $\Delta_{\mathfrak{G}}\}$ of signature $< R >$, where $\Delta_{\mathfrak{G}}$ is a Robinson theory of unars, R is binary symmetric relation. Such $J\mathbb{C}_{\mathfrak{G}}$ defines semantic Jonsson quasivariety of Robinson undirected graphs as in [4].

We are using the definition of the Robinson spectrum of the set $J\mathbb{C}_{\mathfrak{G}}$ as in [4].

Definition 11. A set $RSp(J\mathbb{C}_{\mathfrak{G}})$ of Robinson theories of signature $< R >$, where

$$RSp(J\mathbb{C}_{\mathfrak{G}}) = \{\Delta_{\mathfrak{G}} \mid \Delta_{\mathfrak{G}} \text{ is Robinson theory of graphs and } \forall \mathfrak{C}_{\Delta_{\mathfrak{G}}} \in J\mathbb{C}_{\mathfrak{G}}, \mathfrak{C}_{\Delta_{\mathfrak{G}}} \models \Delta_{\mathfrak{G}}\},$$

is called the Robinson spectrum for class $J\mathbb{C}_{\mathfrak{G}}$, where $J\mathbb{C}_{\mathfrak{G}}$ is semantic Jonsson quasivariety of Robinson undirected graphs.

Further we obtain a factor-set, denoted as $RSp(J\mathbb{C}_{\mathfrak{G}})_{/\bowtie}$ and consisted of equivalence classes parted by cosemanticness relation $[\Delta_{\mathfrak{G}}] \in RSp(J\mathbb{C}_{\mathfrak{G}})_{/\bowtie}$.

Remark 2. Everywhere in this section $[\Delta_{\mathfrak{G}}]$ denotes an equivalence class of Robinson theories of undirected graphs parted by cosemanticness relation on Robinson spectrum $RSp(J\mathbb{C}_{\mathfrak{G}})$. $\mathfrak{C}_{\Delta_{\mathfrak{G}}}$ denotes semantic model and $E_{\Delta_{\mathfrak{G}}}$ denotes a class of existentially closed models of class $[\Delta_{\mathfrak{G}}]$.

Let us compare theorem 12 with the following theorem.

Theorem 13. Let $[\Delta_{\mathfrak{G}}]$ be a class of ω-categorical Robinson theories of undirected graphs. Then the following conditions are equivalent:
1) $\mathfrak{B} \in E_{\Delta_{\mathfrak{G}}}$, where $\mathfrak{B}$ is a model of class $[\Delta_{\mathfrak{G}}]$;
2) $\mathfrak{B}$ is infinite quite disconnected graph.

Proof. To prove this theorem, the same scheme is used as in the proof of Theorem 10 of the previous paragraph, i.e. it is enough for us to prove the following lemmas.

Lemma 4. The following conditions are equivalent:
1) $\mathfrak{G}$ is a countably categorical universal graph;
2) $\mathfrak{G}$ is infinite quite disconnected graph.

Let us prove the necessity.

Let us assume the opposite. Suppose that there is a pair $< x, y >$ in graph $\mathfrak{G}$ such that xRy.

The following statement is known: If $\mathfrak{G}$ is a countably categorical universal graph, then from the fact that $\mathfrak{G}$ has an infinite number of disconnected components follows that $\mathfrak{G}$ is quite disconnected.

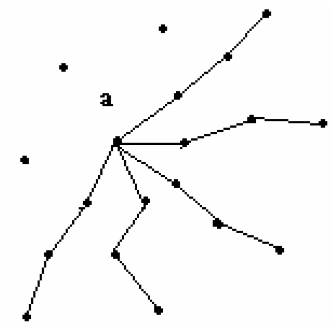
Thus, $\mathfrak{G}$ consists of a finite number of components, but then, due to the infinity of the graph $\mathfrak{G}$, there must be at least one infinite component. Possible cases:

1. There is a bound for the lengths of the chains.

2. There are chains of any given length.

Consider the first case.

Let us take an arbitrary point a from this component. Consider the set of all paths passing through a. The set of all points included in these paths coincides with the component, therefore, is infinite. Since the lengths of the paths are limited, an infinite number of paths pass through a. The ends of these paths are pendant vertices:



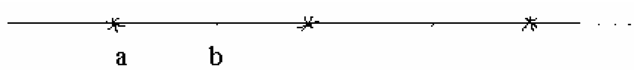
Consider a subgraph Γ consisting only of these pendant vertices.

Obviously, if there are $a \in \mathfrak{G}$ and $b \in \mathfrak{G}$ such that $R(a, b)$, then the type $tp(a, b/\varnothing)$ is not realized in Γ. Which contradicts the criterion of Ye.A. Palyutin.

Consider the second case. To do this, we will prove the following lemma.

Lemma 5. Let $[\Delta_{\mathfrak{G}}]$ be a class of ω-categorical Robinson theories of undirected graphs. If $\mathfrak{G} \models [\Delta_{\mathfrak{G}}]$ and without cycles, then there are no infinite chains in $\mathfrak{G}$.

Proof. Let $\{x_i\}_{i\in\omega}$ be a chain. Consider the subgraph $\{x_i\}_{i\in\omega} \backslash \{x_{3k}\}_{k\in\omega}$, which has the form:



We select a disconnected subgraph Γ in the chain, then the type $tp(a, b/\varnothing)$ is not realized in Γ. By virtue of infinity, Γ contradicts universal categoricity (Palyutin criterion).

The lemma is proved.

Let Γ be a connected component, B_Γ be a set of pendant vertices.

Lemma 6. B_Γ is an infinite set.

Proof. Suppose the opposite: B_Γ is finite. Since the component is infinite, and the set of B_Γ is finite, therefore, there is an infinite set E of Γ vertices that are not pendant. Let $E = \{e_1, e_2, ...\}$. But Γ is a connected component, which means that the set of non-pendant vertices forms an infinite chain, which contradicts the last Lemma 5.

So, we have obtained that if a graph $\mathfrak{G}$ has a pair $< x, y >$ such that xRy, then the graph does not satisfy the assumption condition of Lemma 4 on the countably categorical universality of the graph.

Therefore, if the graph $\mathfrak{G}$ is a countably categorical universal graph, then $\mathfrak{G}$ is a quite disconnected graph.

Let us prove sufficiency.

If the graph $\mathfrak{G}$ is an infinite quite disconnected graph, then $\mathfrak{G}$ is a countably categorical universal graph.

Let us show the satisfaction of the conditions:

1) universality and 2) categoricity.

1) The universality of the class of quite disconnected graphs follows from the fact that it is axiomatized by the universal formula $\forall x \forall y \neg R(x,y)$.

2) Take two subgraphs Γ_1, Γ_2 such that $|\Gamma_1| = |\Gamma_2|$. The set-theoretic mapping of Γ_1 to Γ_2 gives us an isomorphism of Γ_1 and Γ_2 as graphs. The theorem is proved.

Just as in the case of unars with respect to the question of Palyutin, from the description of an existentially closed graph, the following obviously takes place

Corollary 2. Countably categorical Robinson theories of graphs are totally categorical.

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Семантикалық йонсондық квазикөптүрлілік аясында универсал унарлар мен бағытталмаған графтар үшін категориялық мәселелер туралы

Мақала универсалды унарлар мен бағытталмаған графтардың семантикалық йонсондық квазикөптүрліліктерін зерттеуге арналған. Мақаланың бірінші бөлімі йонсондық модельдер теориясының негізгі қажетті ұғымдарынан тұрады. Келесі екі бөлімде робинсондық унарлардың семантикалық йонсондық квазикөптүрліліктерінің $j_{\mathfrak{U}}$ және робинсондық бағытталмаған графтардың семантикалық йонсондық квазикөптүрліліктерінің $J\mathbb{C}_{\mathfrak{G}}$, оның элементар теориясы мен семантикалық моделінің жаңа ұғымдарын қолданудың нәтижелері берілген. Мақаланың екі негізгі нәтижесін дәлелдеу үшін $RSp(J\mathbb{C}_{\mathfrak{U}})$ және $RSp(J\mathbb{C}_{\mathfrak{G}})$ робинсондық спектрлері және олардың косемантты қатынас арқылы $[\Delta]_{\mathfrak{U}}$ және $[\Delta]_{\mathfrak{G}}$ эквиваленттік кластарға бөлінуі қарастырылды. Негізгі нәтижелер 11 және 13 теоремалар ретінде ұсынылған және келесі пайдалы салдарлар туындайды: унарлардың саналымды категориялық робинсондық теориялары тоталды категориялық; бағытталмаған графтардың саналымды категориялық робинсондық теориялары тоталды категориялық. Алынған нәтижелер әртүрлі йонсондық алгебраларды, атап айтқанда циклді моноид арқылы анықталған полигондардың семантикалық йонсондық квазикөптүрліліктерді зерттеуді жалғастыру үшін пайдалы болуы мүмкін.

Кілт сөздер: йонсондық теория, унарлар, графтар, бағытталмаған графтар, универсалды теория, робинсондық теория, квазикөптүрлілік, семантикалық йонсондық квазикөптүрлілік, йонсондық спектр, робинсондық спектр, косеманттылық, категориялық, саналымды категориялық.

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О категоричности универсальных унаров и неориентированных графов с позиции семантического йонсоновского квазимногообразия

Статья посвящена изучению семантических йонсоновских квазимногообразий универсальных унаров и неориентированных графов. Первый раздел статьи состоит из базовых необходимых понятий из йонсоновской теории моделей. Следующие два–это результаты использования новых понятий семантического йонсоновского квазимногообразия робинсоновских унаров $J\mathbb{C}_{\mathfrak{U}}$ и семантического йонсоновского квазимногообразия робинсоновских неориентированных графов $J\mathbb{C}_{\mathfrak{G}}$, их элементарной теории и семантической модели. Для того чтобы доказать главные результаты статьи, были рассмотрены робинсоновские спектры $RSp(J\mathbb{C}_{\mathfrak{U}})$ и $RSp(J\mathbb{C}_{\mathfrak{G}})$ и их разбиение на классы эквивалентности $[\Delta]_{\mathfrak{U}}$ и

$[\Delta]_{\mathfrak{S}}$ с помощью отношения косемантичности. Были проанализированы особенности таких классов эквивалентности $[\Delta] \in RSp(J\mathbb{C}_U)$. Основные результаты представлены в виде теорем 11 и 13 и влекут за собой следующие полезные следствия: счетно категоричные робинсоновские теории унаров — тотально категоричные; счетно категоричные робинсоновские теории неориентированных графов — тотально категоричные. Полученные результаты могут быть полезны в продолжении исследования различных йонсоновских алгебр, в частности, семантического йонсоновского квазимногообразия полигонов над циклическим моноидом.

Ключевые слова: йонсоновская теория, унар, граф, неориентированный граф, универсальная теория, робинсоновская теория, квазимногообразие, семантическое йонсоновское квазимногообразие, йонсоновский спектр, робинсоновский спектр, косемантичность, категоричность, счетная категоричность.

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On the function approximation by trigonometric polynomials and the properties of families of function classes over harmonic intervals

The article is devoted to research on approximation theory. When approximating functions by trigonometric polynomials, the spectrum is chosen from various sets. In this paper, the spectrum consists of harmonic intervals. Devices, various processes, perception of the senses have a limited range. In the mathematical modeling of numerous practical problems and in the further study of such mathematical models, it is sufficient to find a solution in this range. It is possible to study such models to some extent with the help of harmonic intervals. To prove the main theorem, an auxiliary lemma was proved, and elements of the theory of approximations with respect to harmonic intervals were used. For the constructed families of function classes associated with the best approximations by trigonometric polynomials with a spectrum of harmonic intervals, their relationship with classical Besov spaces is shown.

Keywords: harmonic interval, spectrum, the best approximation of a function by trigonometric polynomials with a spectrum of harmonic intervals, Dirichlet kernel, family of function classes.

Introduction

In recent decades, the penetration of ideas and methods of the approximation theory into various branches of mathematical science has been observed. According to a certain rule, the approximation of a function is understood as the replacement of one function by another, close to the original in one sense or another. In the study of periodic functions, trigonometric polynomials occupy a central position as approximating objects. The fundamental results in this theory were obtained in classical works [1, 2]. Further development of the theory is connected with the works of [3, 4] and with the works of other mathematicians. The results obtained are also described in detail in books [5, 6] and others.

When choosing an approximating functions, the spectrum is essential. The spectrum of approximating functions can have the most diverse configuration and consist of the most diverse sets. For example, the spectrum can be a hyperbolic cross [7, 8] or the spectrum is a ball [9], etc.

Devices, various processes, perception of the human senses have a limited finite range. In the mathematical modeling of numerous practical and applied problems and in the subsequent study of the compiled mathematical models, it is enough to find a solution in this range. The study of such models [10, 11] can be carried out to some extent using harmonic intervals.

Harmonic intervals are defined as sets I_k^N [12] of a special form, where the parameter characterizes the specified limited range to some extent. The definitions of harmonic segments and harmonic intervals were given by E.D. Nursultanov in [13, 14]. These sets built according to a certain rule, and their accompanying elements have found wide application in harmonic analysis.

The lemma and the main theorem are presented in the second section. The theorem is proved using an auxiliary lemma and using the properties of harmonic intervals and the mathematical objects associated with them.

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As an auxiliary problem, families of function classes $\left\{B^r_{p,q,N}\right\}_N$ connected by the best approximations over harmonic intervals are introduced. Section 3 is based on the study of the properties of these families of function classes $\left\{B^r_{p,q,N}\right\}_N$. The constructed families of function classes are related to the classical Besov spaces, and this is shown in the third section.

1 Definitions and auxiliary results

Definition 1. [12] If $k, \nu, d, N \in \mathbb{N}$, $k < N$, then the sets of the following types

$$I_k^{N,d} = \bigcup_{\nu=-d}^{d} \left([-k,k] + 2\nu N\right),$$

$$I_k^N = \bigcup_{\nu=-\infty}^{\infty} \left([-k,k] + 2\nu N\right) = \bigcup_{\nu=-\infty}^{\infty} \left(m + 2\nu N : m \in [-k,k]\right)$$

are called harmonic segment and harmonic interval in $\mathbb{Z}$, respectively.

Let T_k^N be the set of trigonometric polynomials in the harmonic interval, defined by the formula [12]

$$T_k^N = \left\{ \sum_{\nu=-s}^{s} a_\nu \cdot e^{i\nu x} : a_\nu = 0 \;\; if \;\; \nu \notin I_k^N, s \in \mathbb{N} \right\}.$$

We have

$$E_k^N(f)_p = \inf_{t \in T_k^N} \left\| f - t \right\|_p,$$

where $E_k^N(f)_p$ is the best approximation over the harmonic interval I_k^N of the function $f \in L_p[0, 2\pi)$, $1 \le p \le \infty$ by trigonometric polynomials from T_k^N of order less than or equal to k [12].

If $f \in L_p[0, 2\pi)$, $1 \le p \le \infty$, then we will consider the following sums as partial sums of the Fourier series of the function f over the harmonic segment $I_k^{N,d}$ and the harmonic interval I_k^N, respectively [12]

$$S_k^{N,d}(f) = \sum_{\nu \in I_k^{N,d}} a_\nu \cdot e^{i\nu x}, \quad S_k^N(f) = \sum_{\nu \in I_k^N} a_\nu \cdot e^{i\nu x}.$$

Lemma 1. Let the functions f and g belong to the space $L_{2k}[0, 2\pi)$, where $k \in \mathbb{N}$. If the functions f and g satisfy the condition

$$\int_0^{2\pi} f^k \cdot \bar{g}^k dx = 0, \tag{1}$$

then we have an inequality of the form

$$\left(\int_0^{2\pi} \left(|f|^{2k} + |g|^{2k}\right) dx \right)^{\frac{1}{2k}} \leqslant k \left(\int_0^{2\pi} |f+g|^{2k} dx \right)^{\frac{1}{2k}}.$$

Lemma 2. [14] Let $B = [-k, k]$ be a segment in $\mathbb{Z}$. $k, d, h \in \mathbb{N}$, $k < h$. $\left\{I_B^{h,d}\right\}_{d=0}^{\infty}$ be a sequence of harmonic segments in $\mathbb{Z}$, converging to a harmonic integral I_B^h, and

$$I_B^h = \bigcup_{\nu=-\infty}^{\infty} \left(B + \nu h\right).$$

If $f \in L_p[0, 2\pi)$, $1 \leq p \leq \infty$, $\sum_{\nu\in\mathbb{Z}} a_\nu \cdot e^{i\nu x}$ is its Fourier series, then the sequence of partial sums of the Fourier series of the function f over the harmonic segment

$$S_B^{h,d}(f) = \sum_{\nu \in I_B^{h,d}} a_\nu \cdot e^{i\nu x}$$

converges in $L_p[0, 2\pi)$ as $d \to \infty$ to the function

$$S_B^h(f) = \frac{1}{h} \sum_{\nu=0}^{h-1} f\left(x + \frac{2\pi r}{h}\right) D_B\left(\frac{2\pi r}{h}\right),$$

where

$$D_B(x) = \sum_{m \in B} a_m \cdot e^{imx}$$

is the Dirichlet kernel corresponding to the segment B from $\mathbb{Z}$, and its Fourier series will be the function $\sum_{\nu\in I_B^h} a_\nu \cdot e^{i\nu x}$.

Theorem 1. [15] Let $f \in L_p[0, 2\pi)$, $1 < p < \infty$, $m \in \mathbb{N}$, $S_m^N(f)$ be the partial sum of the Fourier series and $E_m^N(f)$ be the best approximation of the function f over the harmonic interval I_m^N, then the following correspondence is fulfilled

$$E_m^N(f)_p \sim \left\|f - S_m^N(f)\right\|_p.$$

Let $1 \leq p,\ q \leq \infty$, $r > 0$, $f \in L_p[0, 2\pi)$. Let's construct a family of function classes $\left\{B_{p,q.N}^r\right\}_N$ satisfying the condition

$$B_{p,q.N}^r = \left\{f : \left\|f\right\|_{B_{p,q.N}^r} < \infty\right\},\ \ N \in \mathbb{N},$$

where

$$\left\|f\right\|_{B_{p,q.N}^r} = \left(\sum_{k=1}^{N} k^{rq-1}\left(E_{k-1}^N(f)_p\right)^q\right)^{\frac{1}{q}}.$$

2 Properties of partial sums of the Fourier series over harmonic intervals

Lemma 3. Let f be a function from the space $L_{2k}[0, 2\pi)$, where $k \in \mathbb{N}$. $\sum_{n\in\mathbb{Z}} a_n \cdot e^{inx}$ is its Fourier series, $d \in \mathbb{N}$, $(m+1)k < d-1$.

$$I_1 = I_m^d = \bigcup_{\nu=-\infty}^{\infty}\left([0, m] + \nu d\right),$$

$$I_2 = \bigcup_{\nu=-\infty}^{\infty}\left\{\left[m+1, \left[\frac{d-1}{k}\right]\right] + \nu d\right\},$$

$$I_{[\frac{d-1}{k}]}^d = \bigcup_{\nu=-\infty}^{\infty}\left\{\left[0, \left[\frac{d-1}{k}\right]\right] + \nu d\right\}$$

are harmonic intervals in $\mathbb{Z}$; $S_m^d(f)$ and $S_{[\frac{d-1}{k}]}^d(f)$ are partial sums of the Fourier series of the function $f(x)$ over harmonic intervals I_m^d and $I_{[\frac{d-1}{k}]}^d$, respectively, then the inequality holds

$$\left\|S_m^d(f)\right\|_{2k} \leq k\left\|S_{[\frac{d-1}{k}]}^d(f)\right\|_{2k}.$$

Proof. We introduce the following notation

$$u(x)=\sum_{n\in I_1}a_n\cdot e^{inx},\quad v(x)=\sum_{n\in I_2}a_n\cdot e^{inx}.$$

The functions $u(x)$ and $v(x)$ are partial sums of the Fourier series of the function $f(x)$ over harmonic intervals I_1 and I_2, respectively, and therefore belongs to the space $L_{2k}[0,2\pi)$.

Let's prove that

$$\int_0^{2\pi}u^k\cdot\bar{v}^k dx=0$$

or

$$\int_0^{2\pi}\left(\sum_{n\in I_1}a_n\cdot e^{inx}\right)^k\cdot\left(\sum_{n\in I_2}\bar{a}_n\cdot e^{inx}\right)^k dx=0.$$

Taking into account the values of the integral $\int_o^{2\pi}e^{inx}dx$ when $n=0$ and $n\neq 0$ we conclude that the last condition will be satisfied if there are no identical numbers among the numbers $n\in I_1$ and $n\in I_2$ when raising the partial sums $\sum_{n\in I_1}a_n\cdot e^{inx}$ and $\sum_{n\in I_2}a_n\cdot e^{inx}$ to the power of k.

Note that when $u(x)$ is raised to the power of k, the numbers n fall into the set, which is a harmonic interval, which we denote by I_{1k} and

$$I_{1k}=\bigcup_{\nu=-\infty}^{\infty}\left([0,mk]+\nu d\right).$$

Indeed, by definition, we have

$$I_1+I_2+...+I_r=\left\{n_1+n_2+...n_r,\ \ n_i\in I_i,\ \ n=1,2,...r\right\}$$

so

$$I_{1k}=\underbrace{I_1+...+I_1}_{k}=\left\{n_1+n_2+...n_k,\ \ n_i\in I_1,\ \ n=1,2,...k\right\}.$$

Since $n_i\in I_1$, then $n_i=l_i+\nu d$, where $l_i\in[0,m]$, $\nu\in\mathbb{Z}$, $i=1,...,k$. Therefore,

$$\sum_{i=1}^k n_i=\sum_{i=1}^k l_i+\nu d.$$

Thus,

$$\nu d\leqslant\sum_{i=1}^k n_i\leqslant mk+\nu d.$$

It means that $\sum_{i=1}^k n_i\in I_k$.

Applying the same reasoning, we get that the numbers n, when the partial sum $\bar{v}(x)$ is raised to the power k, fall into the harmonic interval I_{2k}, and

$$I_{2k}=\bigcup_{\nu=-\infty}^{\infty}\left\{\left[(m+1)k,\ d-1\right]+\nu d\right\}.$$

It is obvious that

$$I_{1k}\cap I_{2k}=\varnothing.$$

This equality ensures the fulfilment of the condition (1) for $u(x)$ and $v(x)$. The fulfilment of this condition guarantees the application of Lemma 1, namely

$$\left\{\int_0^{2\pi} \left|u\right|^{2k} dx\right\}^{\frac{1}{2k}} \leqslant \left\{\int_0^{2\pi} \left(\left|u\right|^{2k} + \left|v\right|^{2k}\right) dx\right\}^{\frac{1}{2k}} \leqslant \left\{\int_0^{2\pi} \left|u+v\right|^{2k} dx\right\}^{\frac{1}{2k}},$$

$$\left(\int_0^{2\pi} \left|\sum_{n\in I_m^d} a_n \cdot e^{inx}\right|^{2k} dx\right)^{\frac{1}{2k}} \leqslant k \left(\int_0^{2\pi} \left|\sum_{n\in I^d_{\left[\frac{d-1}{k}\right]}} a_n \cdot e^{inx}\right|^{2k} dx\right)^{\frac{1}{2k}}$$

or

$$\left\|S_m^d(f)\right\|_{2k} \leq k \left\|S^d_{\left[\frac{d-1}{k}\right]}(f)\right\|_{2k}.$$

Lemma 3 is proved.

Theorem 2. Let $f \in L_p[0,2\pi)$, $1 < p < \infty$, $\sum_{\nu\in\mathbb{Z}} a_\nu \cdot e^{i\nu x}$ be its trigonometric Fourier series, then the following inequality

$$\left\|f - \frac{1}{2N}\sum_{r=0}^{2N-1} f\left(x + \frac{\pi r}{N}\right) D_m\left(\frac{\pi r}{N}\right)\right\|_p \leqslant C\left\|f - S_m(f)\right\|_p, \tag{2}$$

is true, where $D_m(y)$ is Dirichlet kernel corresponding to the segment$[-m;m]$, C is a constant that depends only on the parameter p.

Proof. According to Lemma 2, we have

$$\left\|f - \frac{1}{2N}\sum_{r=0}^{2N-1} f\left(x + \frac{\pi r}{N}\right) D_m\left(\frac{\pi r}{N}\right)\right\|_p = \left\|f - S_m^N(f)\right\|_p =$$

$$= \left\|\sum_{\nu\in\mathbb{Z}\setminus I_m^N} a_\nu \cdot e^{i\nu x}\right\|_p = \left\|\sum_{\nu\in Q_{m+1}^N} a_\nu \cdot e^{i\nu x}\right\|_p = \left\|S_{Q_{m+1}^N}(f)\right\|_p,$$

where Q_{m+1}^N are harmonic intervals in $\mathbb{Z}$, and

$$Q_{m+1}^N = \bigcup_{\nu=-\infty}^{\infty} \left\{[-N,\ -m-1] \cup [m+1,\ N] + 2\nu N\right\}.$$

Then we have

$$\left\|S_{Q_{m+1}^N}\right\|_p = \left\|S_{Q_{m+1}^N}(f - S_m)\right\|_p.$$

Since $S_{Q_{m+1}^N}(f - S_m)$ is a partial sum of the Fourier series of the function

$$f - S_m(f) = \sum_{\nu\in\mathbb{Z}\setminus[-m,\ m]} a_\nu \cdot e^{i\nu x}$$

then, by the theorem [14] on the boundedness of partial sums of Fourier series over the harmonic interval, we obtain the necessary inequality

$$\left\|f - S_m^N(f)\right\|_p = \left\|S_{Q_{m+1}^N}(f - S_m)\right\|_p \leqslant C\left\|f - S_m(f)\right\|_p.$$

Thereby, Theorem 2 is proved.

Note 1. According to Theorem 1 and Lemma 9.3 [16] the relation (2) can be presented in the equivalent form

$$E_m^N(f)_p \leqslant E_m(f)_p,$$

3 Properties of the family of function classes $\left\{B^r_{p,q.N}\right\}_N$

Definition 2. [12] Let two classes of functions A^N and B^N depending on the parameter N be given. We will say that the class of functions A^N is embedded in the class of functions B^N and denote it by $A^N \hookrightarrow B^N$ if the following conditions are satisfied:

1) $A^N \subset B^N$;

2) there is a parameter C such that for any $f \in A^N$ the relation

$$\|f\|_{B^N} \leqslant C\|f\|_{A^N}$$

is true, moreover, the parameter C does not depend on f and N.

Definition 3. [15] Function classes $\left\{A^N\right\}_N$ and $\left\{B^N\right\}_N$, where $N \in \mathbb{N}$, are equivalent

$$\|f\|_{A^N} \sim \|f\|_{B^N},$$

if there are parameters C_1, C_2 such that for any $f \in A^N$ there is a correspondence

$$C_1\|f\|_{B^N} \leq \|f\|_{A^N} \leq C_2\|f\|_{B^N},$$

moreover, the parameters C_1, C_2 do not depend on f and N.

In this case, the families of function classes $\left\{A^N\right\}_N$ and $\left\{B^N\right\}_N$ coincide, namely

$$\left\{A^N\right\}_N = \left\{B^N\right\}_N.$$

Theorem 3 relates families of function classes $\left\{B^r_{p,q.N}\right\}_N$ to classical Besov spaces [17].

Theorem 3. Let $N \in \mathbb{N}$, $1 \leqslant p,\ q \leqslant \infty$, $r > 0$ then the following relationship is performed

$$\bigcap_{N=1}^{\infty} B^r_{p,q,N} = B^r_{p,q}.$$

Proof By definition, we have

$$\|f\|_{\bigcap\limits_{N=1}^{\infty} B^r_{p,q,N}} = \sup_N \|f\|_{B^r_{p,q,N}}$$

Since the following inequality

$$\|f\|_{B^r_{p,q,N}} \leqslant C\|f\|_{B^r_{p,q}}$$

holds for any $N \in \mathbb{N}$ then we obtain the accordance

$$\sup_N \|f\|_{B^r_{p,q,N}} = \|f\|_{\bigcap\limits_{N=1}^{\infty} B^r_{p,q,N}} \leqslant \|f\|_{B^r_{p,q}}.$$

This correspondence follows from the last inequality

$$B^r_{p,q} \hookrightarrow \bigcap_{N=1}^{\infty} B^r_{p,q,N}.$$

From other side, for a partial sum $S_{2^m(f)}$, where $m \in \mathbb{N}$, we get the ratio

$$\|S_{2^m}(f)\|_{B^r_{p,q}} = \|S_{2^m}(f)\|_{B^r_{p,q,2^m}} \leqslant C(p,q,r)\|f\|_{B^r_{p,q,2^m}} \leqslant$$

$$\leqslant C(p,q,r) \sup_N \|f\|_{B^r_{p,q,N}} = C(p,q,r)\|f\|_{\bigcap\limits_{N=1}^{\infty} B^r_{p,q,N}}.$$

Further, from the last relation, according to the Banach-Steinhaus theorem [18], we obtain the desired inequality

$$\|f\|_{B^r_{p,q}} \leqslant C(p,q,r)\|f\|_{\bigcap\limits_{N=1}^{\infty} B^r_{p,q,N}}$$

or

$$\bigcap_{N=1}^{\infty} B^r_{p,q,N} \hookrightarrow B^r_{p,q}.$$

Thus, Theorem 3 is proved.

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Функцияларды тригонометриялық көпмүшелер арқылы жуықтау және гармониялық интервалдарға қатысты функциялар кластарының үйірлерінің қасиеттері туралы

Мақала жуықтау теориясы саласындағы зерттеулерге арналған. Тригонометриялық көпмүшелер арқылы функцияларды жуықтау кезінде спектр әртүрлі жиындардан таңдалады. Бұл жұмыста спектр гармоникалық интервалдардан тұрады. Құрылғылар, әртүрлі процестер, сезімдерді қабылдау мүшелері шектеулі ауқымға ие. Көптеген практикалық есептерді математикалық модельдеу кезінде және берілген математикалық модельдерді одан әрі зерттеу кезінде осындай диапазонда шешім табу жеткілікті. Мұндай модельдерді зерттеу белгілі бір дәрежеде гармоникалық интервалдардың көмегімен мүмкін болады. Негізгі теореманы дәлелдеу үшін көмекші лемма дәлелденді және гармоникалық интервалдар бойынша жуықтау теориясының элементтері қолданылды. Гармоникалық интервалдардың спектрі бар тригонометриялық көпмүшеліктермен функцияның ең жақсы жуықтауымен байланысқан функциялар кластарының құрылған үйірі үшін олардың классикалық Бесов кеңістіктері мен байланысы көрсетілген.

Кілт сөздер: гармоникалық интервал, спектр, гармоникалық интервалдардың спектрі бар тригонометриялық көпмүшеліктермен функцияның ең жақсы жуықтауы, Дирихле өзегі, функция кластарының үйірі.

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О приближении функций тригонометрическими полиномами и свойствах семейств классов функций по гармоническим интервалам

Статья посвящена исследованию по теории приближений. При приближении функций тригонометрическими полиномами спектр выбирается из различных множеств. В работе спектр состоит из гармонических интервалов. Приборы, различные процессы, восприятие органов чувств имеют ограниченный диапазон. При математическом моделировании многочисленных практических задач и дальнейшем исследовании таких математических моделей достаточно найти решение в заданном диапазоне. Проведение исследований таких моделей возможно в некоторой степени с помощью гармонических интервалов. Для доказательства основной теоремы была приведена вспомогательная лемма и использовались элементы теории приближений по гармоническим интервалам. Для построенных семейств классов функций, связанных с наилучшими приближениями тригонометрическими полиномами со спектром из гармонических интервалов, показана их связь с классическими пространствами Бесова.

Ключевые слова: гармонический интервал, спектр, наилучшее приближение функции тригонометрическими полиномами со спектром из гармонических интервалов, ядро Дирихле, семейство классов функций.

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