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About the conference ICAAM 2022. Preface

This issue is a collection of 13 selected papers. These papers are presented at the Sixth International Conference on Analysis and Applied Mathematics (ICAAM 2022) organized by Bahcesehir University, Turkey, Institute of Mathematics and Mathematical Modelling, Kazakhstan, and Analysis & PDE Center, Ghent University, Belgium. The meeting was held on October 31 – November 6, 2022, in Antalya, Turkey. The conference is organized biannually. Previous conferences were held in Gumushane, Turkey in 2012; in Shymkent, Kazakhstan in 2014; in Almaty, Kazakhstan in 2016; in Cyprus, Turkey in 2018 and 2020; in Antalya, Turkey in 2022. The proceedings of ICAAM 2012, ICAAM 2014, ICAAM 2016, ICAAM 2018, and ICAAM 2020 were published in AIP Conference Proceedings (American Institute of Physics) and in some rating scientific journals. Proceedings of ICAAM 2022 will be published in the world-renowned AIP Conference Proceeding Series. The main aim of the International Conferences on Analysis and Applied Mathematics (ICAAM) is to bring mathematicians working in the area of analysis and applied mathematics together to share new trends of applications of mathematics. In mathematics, the developments in the field of applied mathematics open new research areas in analysis and vice versa. That is why, we planned to found the conference series to provide a forum for researches and scientists to communicate their recent developments and to present their original results in various fields of analysis and applied mathematics. This issue presents papers by authors from different countries: Azerbaijan, Iraq, Russian Federation, Cyprus, Turkey, Kazakhstan, Turkmenistan, Uzbekistan, Kyrgyzstan. Especially we are pleased with the fact that many articles are written by co-authors who work in different countries. We are confident that such international integration provides an opportunity for a significant increase in the quality and quantity of scientific publications. Special thanks to Charyyar Ashyralyyev (Turkey) for their valuable assistance. Finally, but not least, we would like to thank the Editorial board of the «Bulletin of the Karaganda University. Mathematics series», who kindly provided an opportunity for the formation of this special issue.

Keywords: control, partial differential equations, hyperbolic-parabolic equations, integro-differential equations, boundary value problem, Dirichelt problem, well-posedness, regular solutions, numerical methods and solutions, difference scheme, involution, stability.

Guest-Editors: *A. Ashyralyev* and *M. Sadybekov*

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On the solvability of a nonlinear optimization problem with boundary vector control of oscillatory processes

In the paper, the solvability of the nonlinear boundary optimization problem has been investigated for the oscillation processes, described by the integro-differential equation in partial derivatives with Fredholm integral operator. It has been established that the components of the boundary vector control are defined as a solution to a system of nonlinear integral equations of a specific form, and the equations of this system have the property of equal relations. An algorithm for constructing a solution to the problem of nonlinear optimization has been developed.

Keywords: General solution, nonlinear optimization, boundary vector control, functional, optimal conditions, property of equal relations.

Introduction

There are plenty of works [1-12], that are devoted to the study of nonlinear optimization problems described by systems with distributed parameters. However, methods for solving nonlinear optimization problems while minimizing a piecewise linear functional have not been sufficiently developed. This article deals with the solvability of the problem of optimal boundary control of oscillatory processes described by partial integro-differential equations with an integral Fredholm operator, while minimizing a piecewise linear functional.

Consider the following nonlinear optimization problem where it is required to minimize the piecewise linear functional:

$$J\left[u_1(t,x),...u_m(t,x)\right]=\int_0^T\int_Q\left\{\left[V\left(T,x\right)-\xi_1\left(x\right)\right]^2+\left[V_t\left(T,x\right)-\xi_2\left(x\right)\right]^2\right\}dx+ \tag{1}$$

$$+\beta\int_0^T\int_Q\sum_{k=1}^m\left|u_k(t,x)\right|dxdt\to\min,\quad \beta>0,$$

on the set of generalized solutions to the boundary value problem

$$V_{tt}-AV=\lambda\int_0^T K\left(t,\tau\right)V\left(\tau,x\right)d\tau,\ x\in Q\subset R^n,\quad 0<t<T, \tag{2}$$

$$V(0,x)=\psi_1(x),\ V_t(0,x)=\psi_2(x),\ \ x\in Q, \tag{3}$$

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$$GV(t,x) \equiv \sum_{i,j}^{n} a_{ij}(x)V_{x_j}(t,x)\cos(\nu, x_i) + a(x)V(t,x) = \tag{4}$$
$$= f(t,x,u_1(t,x),...,u_m(t,x)), \quad x \in \gamma,\ 0 < t \leq T.$$

Here A is the elliptic operator, ν is a normal vector, emanating from the point $x \in \gamma$; $K(t,\tau)$ is a given function of $H(D), D = \{0 \leq t \leq T, 0 \leq \tau \leq T\}, \psi_1(x) \in H_1(Q), \psi_2(x) \in H(Q), \xi_1(x) \in H(Q), \xi_2(x) \in H(Q)$ are given functions; $f[t,x,u_1(t,x),...,u_m(t,x)] \in H(Q_T)$ is a boundary source function; $f_{u_i}[t,x,u_1(t,x),...,u_m(t,x)] \neq 0, \forall(t,x) \in (Q_T), u_i(t,x) \in H(Q_T), i = 1,2,3,...,m$; is a control function, λ is a parameter, and T is a fixed moment of time. Q is a region of the space R^n bounded by a piecewise smooth surface γ; $Q_t = Q \times (0,T]$.

The boundary value problem (2)–(4) has not a classical solution, with the above conditions imposed on the given functions. Therefore, we use the concept of a generalized solution to the boundary value problem.

With respect to the methodology of work [1], we give a definition.

Definition 1. The generalized solution to the boundary value problem (2)–(4) is called the function $V(t,x) \in H(Q_T)$ that satisfies the integral identity

$$\int_Q [(V_t(t,x)\Phi(t,x)) - (V(t,x)\Phi_t(t,x))]_{t_1}^{t_2} dx \equiv$$

$$\equiv \int_{t_1}^{t_2}\int_Q \left[-V(t,x)\Phi_{tt}(t,x) - \sum_{i,j=1}^{n} a_{ij}(x)V_{x_j}(t,x)\Phi_{x_i}(t,x) - c(x)V(t,x)\Phi(t,x)\right] dxdt +$$
$$+ \int_{t_1}^{t_2}\int_\gamma \left(f[t,x,u_1(t,x),...,u_m(t,x)] - a(x)V(t,x)\right)\Phi(t,x)dxdt +$$
$$+ \int_{t_1}^{t_2}\int_Q \left(\lambda \int_0^T K(t,\tau)V(\tau,x)d\tau\right)\Phi(t,x)dxdt$$

for any t_1 and t_2, $0 < t_1 \leq t \leq t_2 \leq T$, and for any function $\Phi(t,x) \in C^{2,1}(Q_T), C^{2,1}(Q_T)$ is a space of functions defined on the set Q_T and having a second-order derivative with respect to t, and the first order in the variables x_i, and satisfies the initial and boundary conditions in a weak sense, i.e. for any functions $\phi_0(x) \in H(Q)$, $\phi_1(x) \in H(Q)$ the following relations hold

$$\lim_{t\to+0}\int_Q V(t,x)\phi_0(x)dx = \int_Q \psi_1(x)\phi_0(x)dx,$$

$$\lim_{t\to+0}\int_Q V_t(t,x)\phi_1(x)dx = \int_Q \psi_2(x)\phi_1(x)dx.$$

The solution to problem (2)–(4) is sought in the form

$$V(t,x)=\sum_{n=1}^{\infty}V_n(t)z_n(x).$$

Where $V_n(t)=\langle V(t,x),z_n(x)\rangle=\int_Q V(t,x)z_n(x)dx$ are Fourier coefficients, the symbol $<\cdot,\cdot>$ is used for the scalar product in the Hilbert Space $H(Q)$, $z_n(x)$ are eigenfunctions of the boundary value problem [1]

$$D_n\left(\Phi(t,x),z_k(x)\right)\equiv\int_Q\left(\sum_{i,j=1}^{n}a_{ij}(x)\Phi_{x_j}(t,x)z_{kx_i}(x)+c(x)z_k(x)\Phi\left(t,x\right)\right)dx+$$

$$+\int_\gamma a(x)z_k(x)\Phi(t,x)dx=\lambda_k^2\int_Q z_k(x)\Phi\left(t,x\right)dx;$$

$$Gz_k\left(x\right)=0,\quad x\in\gamma,\quad [k=1,2,3,...].$$

Using Liouville method we easily prove that the Fourier coefficients satisfy the relations [2]

$$V_n(t)=\lambda\int_0^T\left(\frac{1}{\lambda_n}\int_0^t\sin\lambda_n\left(t-\tau\right)K\left(\tau,s\right)d\tau\right)V_n\left(s\right)ds+\psi_{1n}\cos\lambda_n t+\frac{\psi_{2n}}{\lambda_n}\sin\lambda_n t+\tag{5}$$

$$+\frac{1}{\lambda_n}\int_0^t\sin\lambda_n\left(t-\tau\right)\cdot\left[f_n\left[\tau,u_1,...,u_m\right]\right]d\tau;\quad n=1,2,3,...,$$

where

$$f_n\left[\tau,u_1,...,u_m\right]=\int_Q f[t,x,u_1(t,x),...,u_m(t,x)]z_n(x)dx.$$

We can rewrite equation (5) as the following equation:

$$V_n(t)=\lambda\int_0^t K_n\left(t,s\right)V_n\left(s\right)ds+a_n(t),\tag{6}$$

where

$$K_n\left(t,s\right)=\frac{1}{\lambda_n}\int_0^t\sin\lambda_n\left(t-\tau\right)K\left(\tau,s\right)d\tau;$$

$$a_n(t)=\psi_{1n}\cos\lambda_n t+\frac{\psi_{2n}}{\lambda_n}\sin\lambda_n t+\frac{1}{\lambda_n}\int_0^t\sin\lambda_n\left(t-\tau\right)\left[f_n\left[\tau,u_1,...,u_m\right]\right]d\tau.$$

The solution of the integral equation (6) is defined by the following formula [2]

$$V_n(t)=\lambda\int_0^T R_n\left(t,s,\lambda\right)a_n\left(s\right)ds+a_n(t),$$

where

$$R_n(t,s,\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} K_{n,i}(t,s), \; n = 1,2,3,\ldots \tag{7}$$

is the resolvent of the kernel $K_n(t,s)$, the iterated kernels are $K_{n,i}(t,s)$ defined by the formulas

$$K_{n,i+1}(t,s) = \int_0^T K_n(t,\eta) K_{n,i}(\eta,s)\, d\eta, i = 1,2,3,\ldots,$$

for each $n = 1,2,3,...$

Now we investigate the convergence of Neumann series (7). According to the following estimates

$$|K_{n,i}(t,s)|^2 \le \left(\frac{T}{\lambda_n^2}\right)^i (K_0 T)^{i-1} \int_0^T K^2(y,s)dy; \quad \forall t \in (0,T);$$

$$\int_0^T K_{n,i}^2(t,s)\, ds \le \left(\frac{T}{\lambda_n^2}\right)^i (K_0 T)^{i-1} \int_0^T \int_0^T K^2(y,s)\, dyds \le \left(\frac{TK_0}{\lambda_n^2}\right)^i (T)^{i-1},$$

we can easily prove that Neumann series converges absolutely for the values of parameter satisfying following condition

$$|\lambda| < \frac{\lambda_n}{T\sqrt{K_0}} \underset{n\to\infty}{\to} \infty, \quad n = 1,2,3,...$$

The radius of the convergence increases when n grows. As the sum of an absolutely convergent series, resolvent $R_n(t,s,\lambda)$ is the continuous function. It is easy to check that the following estimates hold

$$|R_n(t,s,\lambda)| \le \frac{\sqrt{T}\sqrt{\int_0^T K^2(y,s)dy}}{\lambda_n - |\lambda|\sqrt{K_0 T^2}};$$

$$\int_0^T R_n^2(t,s,\lambda)\, ds = \frac{T}{\left(\lambda_n - |\lambda|\sqrt{K_0T^2}\right)^2} \int_0^T \int_0^T K^2(y,s)dyds = \frac{K_0 T}{\left(\lambda_n - |\lambda|\sqrt{K_0T^2}\right)^2}.$$

Note that the Neumann series converges absolutely for values of the parameters satisfying

$$|\lambda| < \frac{\lambda_1}{T\sqrt{K_0}}$$

for each $n = 1,2,3,...$ Thus, we find the solution of problem (2)–(4) by formula

$$V(t,x) = \sum_{n=1}^{\infty} V_n(t) z_n(x) = \sum_{n=1}^{\infty} \left(\lambda \int_0^T R_n(t,s,\lambda)\, a_n(s)\, ds + a_n(t)\right) z_n(x). \tag{8}$$

Lemma 1. The generalized solution of problem (2)–(4) which is defined by (7) and its derivatives are elements of the Hilbert space $H(Q_T)$.

Proof. The proof is carried out by direct calculation and does not present any excessive difficulties.

Optimality conditions and a system of nonlinear integral equations

Since each vector control $\overline{u}\,(t,x) = (u_1(t,x),...,u_m(t,x))$ uniquely defines the solution of boundary value problem (2)–(4), the control $\overline{u}(t,x) + \Delta\overline{u}(t,x)$ corresponds to the solution of the problem (2)–(4) of the form $V(t,x) + \Delta V(t,x)$, where $\Delta V(t,x)$ is the increment corresponding to the increment $\Delta\overline{u}(t,x)$. According to the maximum principle [3–6] we calculate the increment of functional (1)

$$\Delta I\,(\overline{u}) = I\,(\overline{u} + \Delta\overline{u}) - I\,(\overline{u}) =$$

$$= -\int_0^T \Delta\Pi\,(t,x,\omega\,(t,x)\,\overline{u}(t,x))\,dt + \int_Q \left\{\Delta V^2(T,x) + \Delta V_t^2(T,x)\right\} dx,$$

where

$$\Pi\,[t,x,V\,(t,x)\,,\omega\,(t,x)\,,\overline{u}\,(t,x)] = \omega\,(t,x)\cdot f\,(t,x,\overline{u}\,(t,x)) - \beta\sum_{k=1}^{m}|u_k(t,x)|. \tag{9}$$

Function $\omega(t,x)$ is the solution of the following conjugate boundary value problem

$$\omega_{tt} - A\omega = \lambda\int_0^T K(\tau,t)\omega(\tau,x)d\tau, x\in Q, 0<t<T,$$

$$\omega(T,x) + 2[V_t(T,x) - \xi_2(x)] = 0, \omega_t(T,x) - 2[V(T,x) - \xi_1(x)] = 0, x\in Q,$$

$$G\omega(t,x) \equiv \sum_{i,j}^{n} a_{ij}(x)\omega_{x_j}(t,x)\cos(\nu,x_i) + a(x)\omega(t,x) = 0,$$

$$x\in\gamma,\ 0<t<T.$$

We investigate the maximum of the function $\Pi\,[t,x,V\,(t,x)\,,\omega\,(t,x)\,,\overline{u}]$ with respect the variables $u_1, u_2, ..., u_m$, assuming that the set of allowable values of each of them is an open set. Because of the necessary condition of the extremum, we obtain the following relations from (9)

$$\Pi_{u_i}(.) = \omega(t,x)f_{u_i}[t,x,\overline{u}(t,x)] - \beta sign u_i(t,x) = 0, \quad i = 1,2,3,...,m.$$

Further, the second necessary condition of the extremum is determined [5–7] by the inequalities $\Delta_1 < 0,\ \Delta_2 > 0, ...,\ -1^{(k)}\Delta_k > 0,\ k = 1,2,3,...,m$, according to the Sylvester criterion, where $\triangle i$ are the diagonal determinants of the Hess matrix

$$\Gamma(\Pi,\bar{u}) = \begin{pmatrix} \beta\frac{sign(u_1)}{f_{u_1}}f_{u_1u_1} & ... & \beta\frac{sign(u_1)}{f_{u_1}}f_{u_1u_2} \\ ... & ... & ... \\ \beta\frac{sign(u_m)}{f_{u_m}}f_{u_mu_1} & ... & \beta\frac{sign(u_m)}{f_{u_m}}f_{u_mu_m} \end{pmatrix}.$$

Thus, the components of the optimal vector control $\bar{u}^0(t,x)$ should be found from relation (8), taking into account the second necessary optimality condition. We rewrite condition (8) in the form

$$\omega(t,x) = \beta\frac{sign u_1(t,x)}{f_{u_1}[t,x,\bar{u}(t,x)]} = \beta\frac{sign u_2(t,x)}{f_{u_2}[t,x,\bar{u}(t,x)]} = ... = \beta\frac{sign u_m(t,x)}{f_{u_m}[t,x,\bar{u}(t,x)]} = g(t,x).$$

According to the second necessary condition of extremum and the theorem on implicit functions, we have the following relations

$$u_k^0(t,x) = \varphi_k[t,x,g(t,x),\beta]. \tag{10}$$

The unknown function $g(t,x)$ in (9) is defined as a solution to the following equation

$$g(t,x) = \omega[t,x,f(t,x,\varphi_1(t,x,g(t,x),\beta),...,\varphi_m(t,x,g(t,x),\beta))] = W[g(t,x)] \quad (11)$$

which it is a Fredholm nonlinear integral equation of the 2nd kind with respect to $g(t,x)$, where the integral has double non-linearity with respect to the function $g(t,x)$. The solvability of a nonlinear integral equation (11) is studied by the method of contraction operators. Let $g_0(t,x)$ is a solution of the equation (11). Substituting this function into (10) we find the desired optimal controls

$$u_k^0(t,x) = \varphi_k[t,x,g_0(t,x),\beta], \quad k = 1,...,m.$$

Next substituting the found values of these controls into (5), we obtain the value of the optimal processes

$$V^0\left(t,x\right) = \sum_{n=1}^{\infty} V_n\left(t\right) z_n\left(x\right) = \sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n\left(t,s,\lambda\right) a_n^0\left(s\right) ds + a_n^0\left(t\right)\right) z_n\left(x\right),$$

where

$$a_{\ n}^{0}(t) = \psi_{1n}\cos\lambda_n t + \frac{\psi_{2n}}{\lambda_n}\sin\lambda_n t + \frac{1}{\lambda_n}\int_0^t \sin\lambda_n\left(t-\tau\right)\left[f_n\left[\tau, u_{\ 1}^{0},...,u_{\ m}^{0}\right]\right] d\tau.$$

Substituting the found values of the optimal control and optimal processes into the functional (1), we find the minimum value of the functional (1)

$$J\left[u_1^0(t,x),...u_m^0(t,x)\right] = \int_0^T\int_Q\left\{\left[V^0\left(T,x\right)-\xi_1\left(x\right)\right]^2 + \left[V_t^0\left(T,x\right)-\xi_2\left(x\right)\right]^2\right\}dx+$$

$$+\beta\int_0^T\int_Q\sum_{k=1}^{m}\left|u_k^0(t,x)\right| dxdt,$$

The found triple $\left\{\left(\overline{u}^0\left(t,x\right)\right), V^0\left(t,x\right), I\left(\overline{u}^0\left(t,x\right)\right)\right\}$ is determined as the complete solution of the nonlinear optimization problem.

Conclusion

Solving the problem of the minimization of the piece-wise linear functional is a difficult problem. Therefore the results received in the paper have great scientific value. The developed solving procedure of the formulated problem is constructive and useful in applied problems.

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Тербелмелі процестерді шекаралық векторлық басқарумен сызықтыемес оңтайландыру есебінің шешімі туралы

Мақалада Фредгольм интегралдық операторымен интегралды-дифференциалдық дербес теңдеулер арқылы сипатталған тербелмелі процестерді шекаралық векторлық басқарумен сызықтыемес оңтайландыру есебінің шешімділігі зерттелген. Шекаралық векторлық басқарудың құрамдас бөліктері нақты түрдегі сызықтыемес интегралдық теңдеулер жүйесінің шешімі ретінде анықталатыны және бұл жүйенің теңдеулері тең қатынастық қасиетке ие екендігі анықталды. Сызықтыемес оңтайландыру есебінің шешімін құру алгоритмі жасалды.

Кілт сөздер: жалпыланған шешім, сызықтыемес оңтайландыру, шекаралық векторлық бақылау, функционалды, оңтайлылық шарттары, тең қатынастың қасиеті.

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О разрешимости задачи нелинейной оптимизации при граничном векторном управлении колебательными процессами

В статье исследована разрешимость задачи нелинейной оптимизации при граничном векторном управлении колебательными процессами, описываемыми интегро-дифференциальными уравнениями в частных производных с интегральным оператором Фредгольма. Установлено, что компоненты граничного векторного управления определены как решение системы нелинейных интегральных уравнений специфического вида, и уравнения этой системы обладают свойством равных отношений. Разработан алгоритм построения решения задачи нелинейной оптимизации.

Ключевые слова: обобщенное решение, нелинейная оптимизация, граничное векторное управление, функционал, условия оптимальности, свойство равных отношений.

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Asymptotic behavior of solutions of sum-difference equations

In this study, we present an investigation of the asymptotic behavior of solutions of sum-difference equations. Based on some mathematical inequalities, we have obtained our results. The obtained results can apply to some fractional type difference equations as well.

Keywords: asymptotic behavior, oscillation, nonoscillation, difference equation, Caputo fractional difference operator.

Introduction

In alignment with the extensive interest in the research of difference/differential equations which has demonstrated high potential for real life applications, the determination of qualitative behavior of solutions of them have also received significant attention amongst researchers [1–16].

In [15], the authors have investigated the positive solutions of the following equation

$$\begin{aligned} {}_C\Delta^{\alpha} y(t) &= d(t+\alpha) + f(t+\alpha, x\,(t+\alpha)), \\ y(0) &= c_0, \end{aligned}$$

where $0 < \alpha \leq 1$, ${}_C\Delta^{\alpha}$ is Caputo–like delta fractional difference operator, d is a positive sequence. The authors consider the some particular cases of $y(t)$ for the above equation. In [16], the authors have studied the nonoscillatory solutions of the following fractional difference equations

$$\begin{aligned} &{}_C\Delta^{\alpha} y(t) = e(t+\alpha) + f(t+\alpha, x\,(t+\alpha)), \\ &y(0) = c_0, \end{aligned}$$

where $0 < \alpha \leq 1$, ${}_C\Delta^{\alpha}$ is Caputo–like delta fractional difference operator. Considering some particular cases of $y(t)$, they have obtained some nonoscillatory solutions for the equation.

Motivated by the idea in [12–16], in this article, we study the oscillatory behavior of the following difference equations of the form

$$\begin{cases} \Delta y(t) = e(t+\alpha) - \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} k(t+\alpha, s+\alpha) f(s+\alpha, y(s+\alpha)) \\ y(0) = c_0, \end{cases} \tag{1}$$

where $t \in \mathbb{N}_{1-\alpha}$, $0 < \alpha \leq 1$, $\mathbb{N}_t = \{t, t+1, t+2, \ldots\}$, $f : \mathbb{N}_1 \times \mathbb{R} \to \mathbb{R}$, k and e are sequence. By a solution $y(t)$ of Equation (1), we mean a real-valued sequence y satisfying Equation (1) for $t \in \mathbb{N}_{t_0}$ with $t_0 \in \mathbb{N}_1$. A solution y of Equation (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1) is called oscillatory if all of its solutions are oscillatory.

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1 Background materials

In this section, we present some background materials.

Definition 1. [17] The generalized falling function is defined by

$$t^{(r)} = \frac{\Gamma(t+1)}{\Gamma(t-r+1)},$$

for any t, $r \in \mathbb{R}$ for which the right hand-side is defined. Here Γ denotes the Euler's gamma function. We also use the standard extensions of the domain of this rising function by defining it to be zero whenever the numerator is well defined, but the denominator is not defined.

Lemma 1. [18] Assume that $\beta > 1$ and $\gamma > 0$, then

$$\left[t^{(-\gamma)}\right]^{\beta} < \frac{\Gamma(1+\beta\gamma)}{\Gamma^{\beta}(1+\gamma)} t^{(-\beta\gamma)}, \qquad t \in \mathbb{N}_1.$$

Definition 2. [19] Assume $y : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Delta fractional sum of y is defined by

$$\Delta_a^{-\nu} y(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} y(s), \quad t \in \mathbb{N}_{a+\nu}.$$

Lemma 2. [20] Let $\mu \in \mathbb{R}\backslash \{..., -2, -1, 0\}$, $a \in \mathbb{R}$, $\nu > 0$ and $(t-a)^{(\mu)} : \mathbb{N}_{a+\mu} \to \mathbb{R}$. Then,

$$\Delta_{a+\mu}^{-\nu} (t-a)^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} (t-a)^{(\mu+\nu)} \text{ for } t \in \mathbb{N}_{a+\mu+\nu}.$$

Lemma 3. [16] Let $0 < \alpha \leq 1$, $p > 1$, $p(\alpha-1)+1 > 0$ and $\gamma = 2 - \alpha - \frac{1}{p}$. Then one has

$$(t-s-1)^{(p\alpha-p)} \geq (t-s-1+\alpha+p(1-\alpha)-1)^{(p\alpha-p)}$$

and

$$(s)^{(p\gamma-p)} \geq (s+p(\alpha-1)+1)^{(p\gamma-p)},$$

where

$$t \in \mathbb{N}_1 \text{ and } s \in \{1-(p\alpha-p), 2-(p\alpha-p), \cdots, t-2-(p\alpha-p)\}.$$

Lemma 4. [21] Assume that X and Y are nonnegative real numbers, then

$$X^k - (1-k)Y^k - kXY^{k-1} \leq 0, \ \ for \ \ 0 < k < 1,$$

where the equality holds if and only if $X = Y$.

Lemma 5. [22] Assume that m and x be nonnegative sequences and c be a nonnegative constant. If

$$x(t) \leq c + \sum_{s=0}^{t} m(s)\, x(s) \quad \text{for } t \geq 0.$$

Then, the following inequality holds

$$x(t) \leq c \exp\left(\sum_{s=0}^{t} m(s)\right), \quad \text{for } t \geq 0.$$

2 Main results

We assume that there exist positive sequences a, h, m and $\gamma > 0$, $0 < \delta < 1$ are real numbers such that

$$0 \leq k(t,s) \leq a(t)h(s) \text{ for all } t \geq s \geq 0 \tag{2}$$

and

$$0 < yf(t,y) \leq t^{\gamma-1}m(t)|y|^{\delta+1} \text{ for all } y \neq 0 \text{ and } t \geq 0. \tag{3}$$

Furthermore, there exist real numbers $M_1 > 0$ and M_2 such that

$$|a(t)| \leq M_1 \tag{4}$$

and for every $T \geq 0$

$$-M_2 \leq \liminf_{t\to\infty} \frac{1}{t} \sum_{s=t_1}^{t-\alpha} e\left(s+\alpha\right) \leq \limsup_{t\to\infty} \frac{1}{t} \sum_{s=t_1}^{t-\alpha} e\left(s+\alpha\right) \leq M_2. \tag{5}$$

For the sake of convenience, we denote

$$g_1\left(t\right) := \sum_{s=1-\alpha}^{t-1-\alpha} \left(t-s-1\right)^{(\alpha-1)} \left(s+\alpha\right)^{(\gamma-1)} \upsilon^{\delta/1-\delta}\left(s+\alpha\right) m^{1/1-\delta}\left(s+\alpha\right) h^{1/1-\delta}\left(s+\alpha\right),$$

where υ is a postive sequence.

Theorem 1. Assume that q be a conjugate number of $p > 1$, $p < 1/(1-\alpha)$, $\gamma = 1-\alpha+1/q$ and the conditions (2)–(5) and

$$\limsup_{t\to\infty} g_1\left(t\right) < \infty$$

hold. Then every nonoscillatory solution of Equation (1) satisfies

$$\limsup_{t\to\infty} \frac{|y(t)|}{t} < \infty.$$

Proof. Assume that y be a nonoscillatory solution of (1), say $y\left(t\right) > 0$ for all $t \in \mathbb{N}_{t_0}$ where t_0 is a postive integer. Let

$$k_1 := \max\left\{|f(t,y(t))| : t \in \mathbb{N}_{t_0}\right\} \geq 0 \text{ and } k_2 := k_1 \sum_{s=1-\alpha}^{t_1-1-\alpha} \left(t_1-s-1\right)^{(\alpha-1)} h(s+\alpha) \geq 0.$$

From Equation (1) and our conditions, we have

$$\begin{aligned}\Delta y(t) &= e(t+\alpha) - \sum_{s=1-\alpha}^{t_1-1-\alpha}(t-s-1)^{(\alpha-1)}k(t+\alpha,s+\alpha)f\left(s+\alpha,y(s+\alpha)\right)\\ &- \sum_{s=t_1-\alpha}^{t-1-\alpha}(t-s-1)^{(\alpha-1)}k(t+\alpha,s+\alpha)f\left(s+\alpha,y(s+\alpha)\right)\\ &\le e(t+\alpha) + \sum_{s=1-\alpha}^{t_1-1-\alpha}(t_1-s-1)^{(\alpha-1)}k(t+\alpha,s+\alpha)\left|f\left(s+\alpha,y(s+\alpha)\right)\right|\\ &+ \sum_{s=t_1-\alpha}^{t-1-\alpha}(t-s-1)^{(\alpha-1)}k(t+\alpha,s+\alpha)\left|f\left(s+\alpha,y(s+\alpha)\right)\right|\\ &\le e(t+\alpha) + k_1 a\left(t+\alpha\right)\sum_{s=1-\alpha}^{t_1-1-\alpha}(t_1-s-1)^{(\alpha-1)}h\left(s+\alpha\right)\\ &+ a\left(t+\alpha\right)\sum_{s=t_1-\alpha}^{t-1-\alpha}(t-s-1)^{(\alpha-1)}\left(s+\alpha\right)^{(\gamma-1)}h\left(s+\alpha\right)m\left(s+\alpha\right)y^{\delta}\left(s+\alpha\right)\\ &\le e(t+\alpha) + k_2 a\left(t+\alpha\right)\\ &+ a\left(t+\alpha\right)\sum_{s=t_1-\alpha}^{t-1-\alpha}\Big\{(t-s-1)^{(\alpha-1)}\left(s+\alpha\right)^{(\gamma-1)}\\ &\qquad\qquad\times\Big(h\left(s+\alpha\right)m\left(s+\alpha\right)y^{\delta}\left(s+\alpha\right)-\upsilon\left(s+\alpha\right)y\left(s+\alpha\right)\Big)\Big\}\\ &+ a\left(t+\alpha\right)\sum_{s=t_1-\alpha}^{t-1-\alpha}(t-s-1)^{(\alpha-1)}\left(s+\alpha\right)^{(\gamma-1)}\upsilon\left(s+\alpha\right)y\left(s+\alpha\right).\end{aligned}$$

Setting $X = h^{1/\delta}\left(s+\alpha\right)m^{1/\delta}\left(s+\alpha\right)y\left(s\right)$, $Y = \left(\frac{\upsilon(s+\alpha)}{\delta h^{1/\delta}(s+\alpha)m^{1/\delta}(s+\alpha)}\right)^{1/(\delta-1)}$ and $\beta=\delta$, then using the above lemma, we deduce that

$$h\left(s+\alpha\right)m\left(s+\alpha\right)y^{\delta}\left(s+\alpha\right)-\upsilon\left(s+\alpha\right)y\left(s+\alpha\right)\le\lambda_1\upsilon^{\delta/1-\delta}\left(s+\alpha\right)m^{1/1-\delta}\left(s+\alpha\right)h^{1/1-\delta}\left(s+\alpha\right),$$

where $\lambda_1 = (1-\delta)\,\delta^{\delta/(1-\delta)}$. Hence we have

$$\begin{aligned}\Delta y(t) &\le e(t+\alpha)+k_2 M_1\\ &+ \lambda_1 M_1\sum_{s=t_1-\alpha}^{t-1-\alpha}(t-s-1)^{(\alpha-1)}\left(s+\alpha\right)^{(\gamma-1)}\upsilon^{\delta/1-\delta}\left(s+\alpha\right)m^{1/1-\delta}\left(s+\alpha\right)h^{1/1-\delta}\left(s+\alpha\right)\\ &+ M_1\sum_{s=t_1-\alpha}^{t-1-\alpha}(t-s-1)^{(\alpha-1)}\left(s+\alpha\right)^{(\gamma-1)}\upsilon\left(s+\alpha\right)y\left(s+\alpha\right).\end{aligned}$$

Summing both sides from t_1 to $t-1$ and interchanging the order of the summation, we get

$$\begin{aligned}y(t) &\le y\left(t_1\right)+k_2 M_1\left(t-t_1\right)+\sum_{s=t_1}^{t-1}e(s+\alpha)+\frac{\lambda_1 M_1 t}{\alpha}g_1\left(t\right)\\ &+\frac{M_1 t}{\alpha}\sum_{s=t_1-\alpha}^{t-1-\alpha}(t-s-1)^{(\alpha-1)}\left(s+\alpha\right)^{(\gamma-1)}\upsilon\left(s+\alpha\right)y\left(s+\alpha\right).\end{aligned}$$

That is

$$y(t) \leq A_1 + \frac{M_1 t}{\alpha} \sum_{s=t_1-\alpha}^{t-1-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} \upsilon(s+\alpha) y(s+\alpha), \tag{6}$$

where $A_1 = y(t_1) + k_2 M_1 (t-t_1) + \sum_{s=t_1}^{t-1} e(s+\alpha) + \frac{\lambda_1 M_1 t}{\alpha} g_1(t)$. By applying the Holder's inequality and lemmas, we have

$$\sum_{s=t_1-\alpha}^{t-\alpha-1} (t-s-1)^{(\alpha-1)}(s+\alpha)^{(\gamma-1)}\upsilon(s+\alpha)y(s+\alpha)$$

$$\leq \Big[\sum_{s=t_1-\alpha}^{t-\alpha-1} \Big((t-s-1)^{(\alpha-1)}\Big)^p \Big((s+\alpha)^{(\gamma-1)}\Big)^p\Big]^{1/p} \Big[\sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha)y^q(s+\alpha)\Big]^{1/q}$$

$$< \left[\Big(\frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)}\Big)\Big(\frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)}\Big)\right]^{1/p} \Big[\sum_{s=-1-(p\alpha-p)}^{t-(p\alpha-p+1)} \Big((t-s-1)^{(p\alpha-p)}\Big)\Big((s)^{(p\gamma-p)}\Big)\Big]^{1/p}$$

$$\times \Big[\sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha)y^q(s+\alpha)\Big]^{1/q}$$

and then we have

$$\sum_{s=t_1-\alpha}^{t-\alpha-1} (t-s-1)^{(\alpha-1)}(s+\alpha)^{(\gamma-1)}\upsilon(s+\alpha)y(s+\alpha)$$

$$\leq \left[\Big(\frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)}\Big)\left(\frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)}\right)\right]^{1/p} \Big[\Delta_{-1-(p\alpha-p)}^{-(p\alpha-p+1)}(t)^{(p\gamma-p)}\Big] \Big[\sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha)y^q(s+\alpha)\Big]^{1/q}$$

$$= \left[\Big(\frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)}\Big)\left(\frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)}\right)\right]^{1/p} \left[\frac{\Gamma[p(\gamma-1)+1]}{\Gamma[p(\gamma-1)+p(\alpha-1)+2]}(t)^{(p\alpha-p+1+p\gamma-p)}\right]^{1/p}$$

$$\Big[\sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha)y^q(s+\alpha)\Big]^{1/q},$$

or

$$\sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)}(s+\alpha)^{(\gamma-1)}m(s+\alpha)x^\delta(s+\alpha) \leq N_1 \left[\sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha)y^q(s+\alpha)\right]^{1/q},$$

where

$$N_1 = \left[\Big(\frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)}\Big)\left(\frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)}\right)\right]^{1/p} \left[\frac{\Gamma[p(\gamma-1)+1]}{\Gamma[p(\gamma-1)+p(\alpha-1)+2]}\right]^{1/p}.$$

Thus (6) becomes

$$y(t) \leq A_2 t + \frac{K_1 t}{\alpha}\left[\sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha)y^q(s+\alpha)\right]^{1/q},$$

where, in view of the hypothesis of the theorem, A_2 is an upper bound for

$$\frac{y(t_1)}{t} + \frac{k_2 M_1 (t-t_1)}{t} + \frac{1}{t}\sum_{s=t_1}^{t-1} e(s+\alpha) + \frac{\lambda_1 M_1}{\alpha} g_1(t)$$

and $K_1 = M_1 N_1$. Then we have

$$\frac{y(t)}{t} \leq A_2 + \frac{K_1}{\alpha}\left[\sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha) y^q(s+\alpha)\right]^{1/q}.$$

Hence we get

$$\left(\frac{y(t)}{t}\right)^q \leq \left(A_2 + \frac{K_1}{\alpha}\left[\sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha) y^q(s+\alpha)\right]^{1/q}\right)^q.$$

By applying the elementary inequality $(A+B)^q \leq 2^{q-1}(A^q + B^q)$ for $A, B \geq 0$, we have

$$\begin{aligned}
\omega(t) &\leq 2^{q-1} A_2^q + 2^{q-1}\left(\frac{K_1}{\alpha}\right)^q \sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha) y^q(s+\alpha) \\
&= 2^{q-1} A_2^q + 2^{q-1}\left(\frac{K_1}{\alpha}\right)^q \sum_{s=t_1-\alpha}^{t-\alpha-1} (s+\alpha)^q \upsilon^q(s+\alpha)\left(\frac{y(s+\alpha)}{s+\alpha}\right)^q \\
&= 2^{q-1} A_2^q + 2^{q-1}\left(\frac{K_1}{\alpha}\right)^q \sum_{s=t_1-\alpha}^{t-\alpha-1} (s+\alpha)^q \upsilon^q(s+\alpha)\omega(s+\alpha),
\end{aligned}$$

where $\omega(t) = y^q(t)/t^q$. If we apply the Lemma 5, we have the following result

$$\limsup_{t\to\infty} \frac{y(t)}{t} < \infty.$$

This completes the proof.

Theorem 2. In addition to the hypothesis of Theorem 1, suppose that

$$\lim_{t\to\infty} a(t) = 0.$$

If for every $\mu \in (0,1)$ we have

$$\liminf_{t\to\infty}\left[\mu t + \sum_{s=1-\alpha}^{t-1} e(s+\alpha)\right] = -\infty, \quad \limsup_{t\to\infty}\left[\mu t - \sum_{s=1-\alpha}^{t-1} e(s+\alpha)\right] = \infty,$$

then Equation (1) is oscillatory.

Proof. Let y be a nonoscillatory solution of (1). Then we may assume that $y(t)$ is eventually positive for all $t \in \mathbb{N}_{t_0}$ where t_0 is a postive integer. Proceeding as in the proof of the above theorem, we have the following inequality

$$\begin{aligned}
y(t) &\leq y(t_1) + k_2 M_1 (t - t_1) + \sum_{s=t_1}^{t-1} e(s+\alpha) \\
&+ \frac{\lambda_1 M_1 t}{\alpha} g_1(t) + \frac{K_1 t}{\alpha}\left[\sum_{s=t_1-\alpha}^{t-\alpha-1} \upsilon^q(s+\alpha) y^q(s+\alpha)\right]^{1/q}.
\end{aligned}$$

From our conditions, we can make M_1 as small as we please by increasing the size of t_1 if necessary. And also considering Theorem 1, we have

$$y(t) \leq y(t_1) - \sum_{s=1-\alpha}^{t_1-\alpha-1} e(s+\alpha) + \sum_{s=1-\alpha}^{t-\alpha} e(s+\alpha) + \frac{t}{T}, \tag{7}$$

where $T > 1$. Taking infimum and limit as $t \to \infty$ in (7), respectively. Then we obtain a contradiction with the fact that $y(t)$ is eventually positive of Equation (1). The proof when $y(t)$ is eventually negative is similar.

3 Conclusions

In this study, we present an investigation of the asymptotic behavior of solutions of sum-difference equations. Based on some features of the discrete calculus and mathematical inequalities, we have obtained our results. The obtained results can apply to some fractional type difference equations as well. If we consider

$$k(t+\alpha, s+\alpha) = \frac{1}{\Gamma(\alpha)},$$

$$e(t+\alpha) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} g(s+\alpha)$$

we may write from Equation (1) that

$$\Delta y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} [g(s+\alpha) - f(s+\alpha, y(s+\alpha))], \quad 0 < \alpha \leq 1.$$

It is not difficult to see that this equation is equivalent to the fractional difference equation. Here, one can notice that the obtained results can be rewritten for the fractional difference equations.

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Айырымдардың қосындысы теңдеулерінің шешімдерінің асимптотикасы

Мақалада жұмыста айырымдардың қосындысы теңдеулерінің шешімдерінің асимптотикалық өзгеруін зерттеу ұсынылған. Бұл нәтижелер кейбір математикалық теңсіздіктер негізінде алынған. Алынған нәтижелерді бөлшек типтегі кейбір дифференциалдық теңдеулерге де қолдануға болады.

Кілт сөздер: асимптотика, осцилляция, бейосцилляция, айырымдық теңдеу, Капутоның бөлшекті айырымды операторы.

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Асимптотика решений уравнений суммы разностей

В статье мы представлено исследование асимптотического поведения решений уравнений суммы разностей. На основе определенных математических неравенств нами получены результаты, которые можно применить и к некоторым разностным уравнениям дробного типа.

Ключевые слова: асимптотика, осцилляция, неосцилляция, разностное уравнение, оператор дробной разности Капуто.

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On convergence of difference schemes of high accuracy for one pseudo-parabolic Sobolev type equation

Difference schemes of the finite difference method and the finite element method of high-order accuracy in time and space are proposed and investigated for a pseudo-parabolic Sobolev type equation. The order of accuracy in space is improved in two ways using the finite difference method and the finite element method. The order of accuracy of the scheme in time is improved by a special discretization of the time variable. The corresponding a priori estimates are determined and, on their basis, the accuracy estimates of the proposed difference schemes are obtained with sufficient smoothness of the solution to the original differential problem. Algorithms for the implementation of the constructed difference schemes are proposed.

Keywords: pseudo-parabolic equation, difference schemes, finite difference method, finite element method, generalized solutions, a priori estimates, stability, convergence, accuracy.

Introduction

Applied problems of engineering and technology lead to the solution of pseudo-parabolic Sobolev type equations. By pseudo-parabolic equations, we mean all high-order equations with a first-order time derivative of the following form

$$\frac{\partial}{\partial t}\left(A(u)+B(u)\right)=0,$$

$A(u)$ and $B(u)$ are elliptic operators, generally speaking, the nonlinear ones [1]. They refer to constitutive equations. Such problems arise in many fields of modern science. For example, problems in the physics of semiconductors, plasma physics, hydrodynamics of stratified and filterable liquids, the theory of “creep” of structural elements, etc. For example, the equation of waves in thin layers of liquid on the surface of a rotating globe (Rossby waves in oceanology) has the following form [2]

$$\frac{\partial}{\partial t}\Delta_3 u+\beta\frac{\partial}{\partial x_2}u=-f(x,t),\ \ (x,t)\in Q_T, \tag{1}$$

where $\Delta_3=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}$ is the three-dimensional Laplace operator, β is constant, and the equation of pseudo-parabolic type has the following form [1]:

$$\left(\Delta_3 u-u\right)_t+\Delta_3 u+\beta u=-f(x,t),\ \ (x,t)\in Q_T. \tag{2}$$

This equation describes the filtration process in a fractured porous fluid. The equation of moisture transfer in soil can be added to these equations [3]

$$u_t=Lu+f(x,t),\ \ (x,t)\in Q_T, \tag{3}$$

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where $Lu = \sum\limits_{\alpha=1}^{p} L_\alpha u, L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right) + \frac{\partial}{\partial t}\frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right)$. Mathematical models of nonstationary processes in anuniaxial ferroelectric semiconductor lead to initial-boundary value problems for pseudo-parabolic equations of the following form [3]:

$$\frac{\partial}{\partial t}\left(\Delta_3 u - \gamma_1 u\right) + \alpha_1\left(\Delta_2 u - \gamma_1 u\right) + \beta_1 \frac{\partial^2 u}{\partial x_3^2} = -f(x,t), \quad (x,t) \in Q_T. \tag{4}$$

Here $\Delta_2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the two-dimensional Laplace operator, $\gamma_1 = 1/r_d^2$, $\beta_1 = 4\pi\alpha_2 + \alpha_1$, $\alpha_l > 0$ is constant $(l = 1, 2)$, $r_d = \sqrt{T^2/(4\pi e^2 n_0)}$ is the Debye screening effect (Debye radius), e is the absolute value of the electron charge, n_0 is the unperturbed particle density, $Q_T = \{(x,t) : x \in \Omega, \ t \in (0,T]\}$, $\Omega = \{x = (x_1, \, x_2, x_3) : 0 < x_k < l_k, \ k = 1, 2, 3\}$.

The above equations are supplemented with initial and various boundary conditions, for example, local ones - classical boundary conditions and nonlocal ones, where, instead of classical boundary conditions, a certain relationship is specified between the values of the sought-for function on the boundary of the domain and inside it. General questions of unique solvability and analytic properties of such problems were studied in [1–6].

Recently, more attention has been paid to numerical methods for solving the above equations. In particular, in [1, 2], problems of type (1)–(4) were reduced by some transformation to two equations (one contains differentials in time, the other contains differentials in space) and then these equations were solved by the finite difference method using quasi-uniform grids. Difference schemes built on quasi-uniform grids have the second-order of accuracy in time and space variables, with sufficient smoothness of the solution to the original differential problem. Similar problems were studied in [7–10], where high-order Sobolev type equations with a second-order time derivative were considered. High-order accurate schemes of the finite element method were constructed and investigated with minimal requirements for the smoothness of the solution to the original differential problem. Difference schemes for an equation with nonlocal boundary conditions were studied in [11–15], where difference schemes of the first and second orders of accuracy were investigated.

The knowledge of the laws and features of non-stationary processes plays a primary role in the development and improvement of technological processes, technical installations and devices in a number of industries; this determines the relevance of research in the above areas. This implies the need to construct and search for numerical methods of high accuracy (more than the second accuracy) for various non-stationary initial-boundary value problems, including pseudo-parabolic equations. However, numerical methods have their limitations in terms of stability, accuracy, and economy. Therefore, the problem of determining the optimal method is an urgent issue.

In this article, we consider the construction and study of high-accuracy difference schemes of boundary value problems for equation (4). Here, the initial-boundary value problem for this equation is first approximated in spatial variables by the finite difference method and the finite element method; then, for the resulting system of ordinary differential equations, the second-order finite difference method and the fourth-order finite element method (constructed and investigated in [7]) were used.

1 *Statement of the problem*

Consider equation (4) with the following initial and boundary conditions

$$u(x,0) = u_0(x), x \in \overline{\Omega} = \Omega + \Gamma, \tag{5}$$

$$u(x,t) = \mu(t), x \in \Gamma = \partial\overline{\Omega}, t \in (0,T]. \tag{6}$$

As already mentioned, instead of boundary condition (6), one can consider any classical boundary conditions. In addition, nonlocal boundary conditions can be considered. At that, the matrices of difference schemes may turn out to be asymmetric but with the methods of linear algebra, they can be symmetrized, for example, by the bordering method [16].

Let us formulate a generalized statement of problem (4)–(6). Function $u(x,t) \in W_2^1(\Omega)$, is called a generalized solution of the problem, for each $t \in [0,T]$, it has derivative $\frac{\partial u}{\partial t} \in L_2[0,T]$ and satisfies the following relations almost everywhere on $[0,T]$:

$$a_3\left(\frac{du(t)}{dt}, \vartheta\right) + a_2(u(t), \vartheta) + a_1(u(t), \vartheta) = (f(t), \vartheta), \;\; u(0) = u_0, \forall \vartheta(x) \in H, \tag{7}$$

where

$$a_3(u,\vartheta) = -\iint\limits_{\Omega} \left(\sum_{k=1}^{3} u_{x_k}\vartheta_{x_k} + \gamma_1 u\vartheta\right) dx, \;\; a_2(u,\vartheta) = -\alpha_1 \iint\limits_{\Omega} \left(\sum_{k=1}^{2} u_{x_k}\vartheta_{x_k} + \gamma_1 u\vartheta\right) dx,$$

$$a_1(u,\vartheta) = -\beta_1 \iint\limits_{\Omega} u_{x_3}\vartheta_{x_3} dx,$$

$u = u(t)$ is the function of abstract argument $t \in [0,T]$ with values in H. Here $\overset{\circ}{W}{}_2^1(\Omega)$ is the Sobolev space vanishing at the boundaries, where scalar product and norm are defined as follows:

$$(u(x), \vartheta(x)) = \iint\limits_{\Omega} \left(u\vartheta + \sum_{m=1}^{3} \frac{\partial u}{\partial x_m} \cdot \frac{\partial \vartheta}{\partial x_m}\right) dx,$$

$$\|u(x_1,x_2,x_3)\|^2_{W_2^1(\Omega)} = \iint\limits_{\Omega} \left(u^2 + \sum_{m=1}^{3} \left(\frac{\partial u}{\partial x_m}\right)^2\right) dx.$$

It's obvious that

$$c_3 \|u\|_1^2 \le a_3(u,u) \le C_3 \|u\|_1^2, \;\; c_2 \|u\|_1^2 \le a_2(u,u) \le C_2 \|u\|_1^2, \;\; 0 \le a_1(u,u) \le C_1 \|u\|_1^2,$$

where c_2, c_3, C_1, C_2, C_3 are the positive constants. Constant c_1 depends on β_1, c_2 depends on α_1, γ_1, and c_3 depends on γ_1.

The existence and uniqueness of the solution to this problem were studied in [2].

2 Discretization in space

Let us construct the subspace $H_h \subset H$ that approximates H. Consider the following two cases.

The first case corresponds to the approximation of equation (4) in spatial variables by the method of finite differences. Let us introduce a grid uniform in each direction $\overline{\omega}_h = \overline{\omega}_{h_1} \times \overline{\omega}_{h_2} \times \overline{\omega}_{h_3}$, in $\overline{\Omega}$ where $\overline{\omega}_{h_m} = \left\{x_m = i_m h_m, i_m = \overline{0, N}_m, \; h_m = l_m/N_m\right\}$, $m = 1,2,3$. Here $\overline{\omega}_h = \omega_h + \gamma_h$. We define the subspace $H_h = \overset{\circ}{W}{}_2^1(\omega_h)$, the space of grid functions $v(x_1,x_2,x_3)$ with norm $\|v\|_1^2 = \sqrt{\sum\limits_{i_1}^{N_1}\sum\limits_{i_2}^{N_2}\sum\limits_{i_3}^{N_3} h_1h_2h_3 \left[(v_{\overline{x}_1})^2 + (v_{\overline{x}_2})^2 + (v_{\overline{x}_3})^2\right]} \le M$, where the constant M does not depend on h_1, h_2, h_3. Here $v = v(i_1h_1, i_2h_2, i_3h_3)$.

$$v_{\overline{x}_1} = \left[v(i_1h_1, i_2h_2, i_3h_3) - v((i_1-1)h_1, i_2h_2, i_3h_3)\right]/h_1,$$

$$v_{\overline{x}_2} = \left[v(i_1h_1, i_2h_2, i_3h_3) - v(i_1h_1, (i_2-1)h_2, i_3h_3)\right]/h_2,$$

$$v_{\overline{x}_3} = [v(i_1h_1, i_2h_2, i_3h_3) - v(i_1h_1, i_2h_2, (i_3-1)h_3)] / h_3,$$

$\overset{\circ}{W}{}_2^1(\omega_h)$ is the space of grid functions that vanish at the boundaries.

Approximating the expressions for $a_m(u,\vartheta)$ on the grid by the corresponding quadrature formulas $a_m^h(u_h, v_h) = \sum\limits_{i_1=1}^{N_1}\sum\limits_{i_2=1}^{N_2}\sum\limits_{i_3=1}^{N_3} h_1h_2h_3 u_{h\overline{x}_m} v_{h\overline{x}_m}$, we proceed from (7) to the definition of an approximate grid solution:

$$a_3^h\left(\frac{du_h(t)}{dt}, \vartheta\right) + a_2^h(u_h(t), \vartheta) + a_1^h(u_h(t), \vartheta) = (f_h(t), \vartheta), \ \ \forall \vartheta(x) \in H_h, \tag{8}$$

$$u_h(0) = u_{0,h}. \tag{9}$$

Relations (8), (9) correspond to the following Cauchy problem for the function $u_h(t)$:

$$D\frac{du_h(t)}{dt} + Au_h(t) = f_h(t), \quad u_h(0) = u_{0,h}, \tag{10}$$

where

$$D = (\Lambda_1 + \Lambda_2 + \Lambda_3) + \gamma_1 E, \ A = \alpha_1(\Lambda_1 + \Lambda_2 + \gamma_1 E) + \beta_1\Lambda_3, \tag{11}$$

$\Lambda_m y = -y_{x_m\overline{x}_m}$, $m = 1,2,3$, y is the value of the function at a fixed node, $x = (i_1h_1, i_2h_2, i_3h_3)$,

$$y_{x_1\overline{x}_1} = (y((i_1+1)h_1, i_2h_2, i_3h_3) - 2y(i_1h_1, i_2h_2, i_3h_3) + y((i_1-1)h_1, i_2h_2, i_3h_3))/ h_1^2,$$

$$y_{x_2\overline{x}_2} = (y(i_1h_1, (i_2+1)h_2, i_3h_3) - 2y(i_1h_1, i_2h_2, i_3h_3) + y(i_1h_1, (i_2-1)h_2, i_3h_3))/ h_2^2,$$

$$y_{x_3\overline{x}_3} = (y(i_1h_1, i_2h_2, (i_3+1)h_3) - 2y((i_1h_1, i_2h_2, i_3h_3)) + y(i_1h_1, i_2h_2, (i_3-1)h_3))/ h_3^2.$$

Here $u_{h,0} = P_h u_0(x)$ is the interpolant of the initial condition, P_h is the projection operator $P_h : H \to H_h$ and $f_h(t) = P_h f(x,t)$.

Difference operators D and A approximate differential operators $\Delta_3 u - \gamma_1 u$ and $\alpha_1(\Delta_2 u - \gamma_1 u) + \beta_1 \partial^2 u/\partial x_3^2$ with second-order approximation errors.

The second case corresponds to the approximation of equation (4) in spatial variables by the finite element method. Let $H_h \subset H$ be the set of elements of the form $\vartheta_h = \sum\limits_{m=1}^{M} a_m \Phi_m(x)$. Here $\{\Phi_m = \Phi_m(x)\}_{m=1}^M$ is the basis of piecewise polynomial functions that are a polynomial of p degree on each finite element [17]. Let us give an example of a basis based on third degree polynomials. To do this, we introduce a partition of the domain Ω into $N_1 \times N_2 \times N_3$ parallelepipeds

$$\Omega_{ijk} = \{(i-h)h_1 \le x_1 \le ih_1, \ (j-1)h_2 \le x_2 \le jh_2, \ (k-1)h_3 \le x_3 \le kh_3\},$$

$$i = \overline{1, N_1}, \ j = \overline{1, N_2}, \ k = \overline{1, N_3}, \ h_m = l_m/N_m, \ m = 1,2,3.$$

Let us choose the following system of basis functions:

$$\varphi_{ijk}(x_1, x_2, x_3) = \varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3), \ \ i = \overline{1, N_1 - 1}, \ \ j = \overline{1, N_2 - 1}, \ \ k = \overline{1, N_3 - 1},$$

where $\varphi_l(x)$ is the basis function built on the basis of the B_3 - spline [7]. In this case $p = 3$. Then the approximate solution can be represented as a bicubic spline:

$$\vartheta_h(x_1, x_2, x_3, t) = \sum_{k=1}^{N} a_k(t)\varphi_k(x_1, x_2, x_3), \tag{12}$$

where $\varphi_k(x_1,x_2,x_3) = \varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3)$, $i = \overline{1,N_1-1}$, $j = \overline{1,N_2-1}$, $k = \overline{1,N_3-1}$, and $N = (N_1-1)(N_2-1)(N_3-1)$.

The stiffness matrices corresponding to operators D, A are calculated as follows:

$$D = \{a_3(\varphi_l,\varphi_m)\}_{l,m=1}^{M}, \; A = \{a_2(\varphi_l,\varphi_m)\}_{l,m=1}^{M} + \{a_1(\varphi_l,\varphi_m)\}_{l,m=1}^{M}.$$

When choosing a polynomial of a degree no less than three, at each finite element in spatial variables, we have the third order of accuracy in spatial steps.

In both cases

$$D^* = D > 0, \; A^* = A > 0.$$

In what follows, for simplicity of notation, in (10), $u \in H_h$ is used instead of u_h, i.e., problem (10) is written in the following form

$$D\dot{u} + Au = f, \; u(0) = u_0, \tag{13}$$

where $\dot{u} = du/dt$.

3 Time discretization

Here we also consider two cases of approximation. Let discrete function y approximates a continuous function u.

The first case. Let us introduce grid $\omega_\tau = \{t_n = n\tau, \; n = 1,2,..., \; \tau > 0\}$ in time t. Then we approximate problem (13) by the following difference scheme

$$Dy_t + Ay^{(\sigma)} = \varphi, \; y^0 = u_{0,h}, \; y^n \in H_h, \tag{14}$$

where $y_t = (\hat{y} - y)/\tau$, $y = y^n = y(t_n)$, $\hat{y} = y^{n+1} = y(t_n+\tau)$, $y^{(\sigma)} = \sigma\,\hat{y} + (1-\sigma)y$. Here D and A are defined according to (11), and σ is some arbitrary real parameter $\varphi = \overline{f} = f\,(x, t_n + \tau/2\,)$.

It is known from the theory of difference schemes [18] that the approximation error for scheme (14) is:

$$\psi = O\left(\tau^2 + |h|^2\right) \text{ for } \sigma = 0.5, \; \psi = O\left(\tau + |h|^2\right) \text{ for } \sigma \neq 0.5, \; |h|^2 = h_1^2 + h_2^2 + h_3^2.$$

The second case consists in discretizing problem (13) by the finite element method connecting the values $\dot{y}^{n+1}$, $\dot{y}^n$, y^{n+1}, y^n that approximate $\frac{du_h}{dt}(t_n+\tau)$, $\frac{du_h}{dt}(t_n)$, $u_h(t_n+\tau)$, $u_h(t_n)$, respectively. Such a scheme was constructed in [7] and it has the form:

$$Dy_t - \gamma A\dot{y}_t + Ay^{(0.5)} = \varphi_1, \;\; \gamma D\dot{y}_t + \alpha Ay_t - \beta A\dot{y}^{(0.5)} = \varphi_2, \tag{15}$$

where $\varphi_1 = \frac{1}{\tau}\int\limits_{t_n}^{t_{+n1}} f(t)dt$, $\varphi_2 = \frac{1}{\gamma\tau}\int\limits_{t_n}^{t_{n+1}} f(t)(s_1\vartheta_2^{(1)} + s_2\vartheta_2^{(2)})dt$, $s_1 = 15\gamma - 35\alpha/3$, $s_1 = 140\gamma - 350\alpha/3$, $\vartheta_1^{(1)} = 1/2$, $\vartheta_2^{(3)} = \tau\xi(1-\xi)\left(\xi - 1/2\right)$, $\xi = (t - t_n)/\tau$.

The initial conditions for (15) are specified as follows: in addition to the natural condition $y^0 = u_0$, it is necessary to specify $\dot{y}^0$. For this, from the system of equations (13), at $t = 0$, we determine $\dot{u}_0 = D^{-1}(f^0 - \mathrm{A}u_0)$ and set $\dot{y}^0 = \dot{u}_0$, therefore, the initial conditions for (15) have the form:

$$y^0 = u_0, \; \dot{y}^0 = D^{-1}(f^0 - \alpha u_0). \tag{16}$$

From the calculated values of $\dot{y}^{n+1}$, $\dot{y}^n$, y^{n+1}, y^n, it is possible to restore the approximation to $u_h(t)$ for any $t \in [t_n, t_{n+1}]$, $n = 0,1,...$ by the following formula:

$$y(t) = y^n\varphi_{00}^n(t) + \dot{y}^n\varphi_{10}^n(t) + y^{n+1}\varphi_{01}^n(t) + \dot{y}^{n+1}\varphi_{11}^n(t).$$

Here $\varphi_{00}^n(t) = 2\xi^3 - 3\xi^2 + 1$, $\varphi_{01}^n(t) = 3\xi^2 - 2\xi^3$, $\varphi_{10}^n(t) = \tau(\xi^3 - 2\xi^2 + \xi)$, $\varphi_{11}^n(t) = \tau(\xi^3 - \xi^2)$, $\xi = \frac{t-t_n}{\tau}$.

Combining the approximation in space and time, we consider four methods for solving problem (4)–(6):

- *Scheme* 1^0 – difference approximation of the second order of accuracy in space (11) and time (14);
- *Scheme* 2^0 – approximation of the FEM with bicubic elements in space (12) and time (14);
- *Scheme* 3^0 – difference approximation of the second order of accuracy in space (11) and the FEM scheme in time (15), (16);
- *Scheme* 4^0 – approximation of the FEM with bicubic elements in space (12) and the FEM scheme in time (15), (16).

4 Stability and accuracy

Let us analyze the stability and accuracy of the selected schemes. It is known [18], that schemes (14), (15) are stable under the following conditions

$$D > 0, \quad A = A^* > 0, \quad D \geq \tau A/2. \tag{17}$$

Let us check the fulfillment of conditions (17). It is seen from (11) that $D = D^* > 0$, $A = A^* > 0$. The last condition (17) takes the form

$$\Lambda_1 + \Lambda_2 + \Lambda_3 + \gamma_1 E - \frac{\tau}{2}\ [\alpha_1(\Lambda_1 + \Lambda_2 + \gamma_1 E) + \beta_1 \Lambda_3] \geq 0.$$

To satisfy it, it is enough that

$$\tau \leq 2 \max\left(\frac{1}{\alpha_1}, \frac{1}{\beta_1}\right). \tag{18}$$

This condition is interesting because the time step is not related to the space step and is determined by the parameters α_1, β_1 of the problem. Thus, the following theorem holds.

Theorem 1. Under condition (18), the solution of scheme 1^0 converges to a sufficiently smooth solution of problem (4)-(6) and the following accuracy estimate holds

$$\|y(t) - u(t)\|_A + \|y_t(t) - u_t(t)\|_D \leq M(\tau^{m_1} + |h|^{m_2}), \tag{19}$$

where $\|\vartheta\|_D = \sqrt{(D\vartheta, \vartheta)} = \|\vartheta\|_{W_2^1(\omega_h)}$, $\|\vartheta\|_A = \sqrt{(A\vartheta, \vartheta)} = \|\vartheta_{x_1}\|_{L_2(\omega_h)}$ are the norms in the space of grid functions H_h, $y_t = (y^{n+1} - y^n)/\tau$, $m_1 = 1,\ m_2 = 2$ for $\sigma = 0.5$ and $m_1 = 2,\ m_2 = 2$ for $\sigma \neq 0.5$.

Let us formulate a result on the stability and accuracy of scheme 2^0.

Theorem 2. Under condition (18), the solution of scheme 2^0 converges to a sufficiently smooth solution of problem (4)-(6) and the following accuracy estimate (19) holds $m_1 = 1,\ m_2 = 3$ for $\sigma = 0.5$ and $m_1 = 2,\ m_2 = 3$ for $\sigma \neq 0.5$.

Now let us investigate the accuracy of scheme (15), (16). Let $z^n = y^n - u^n$, $\dot{z}^n = \dot{y}^n - \dot{u}^n$, where $u^n = u(t_n)$. Then scheme (15), (16) satisfies the relations

$$Dz_t - \gamma A\dot{z}_t + Az^{(0.5)} = \psi_1,\ \gamma D\dot{z}_t + \alpha A z_t - \beta A\dot{z}^{(0.5)} = \psi_2,\ z^0 =\ 0,\ \dot{z}^0 =\ 0,$$

the approximation error is

$$\psi_1 = \frac{\tau^4}{3840} D\overset{V}{u} + \frac{\tau^4}{720} A\overset{IV}{u} + O(\tau^5),$$

$$\psi_2 = (\alpha + \beta - \gamma)\ddot{\overline{A}} + \frac{\tau^2}{24}\left[(\alpha + 3\beta - \gamma)\dddot{\overline{A}} - (3\gamma - 2\alpha)\dddot{\overline{f}}\right] + O\left(\tau^4\right).$$

Hence, if the following conditions are met

$$\alpha + \beta = \gamma, \ \ \alpha, \ \ \beta, \ \gamma = (\tau^2), \tag{20}$$

then $\psi_1 = \psi_2 = O\left(\tau^4\right)$.

To prove the convergence of the two-layer vector scheme (15), (16), we reduce it to a three-layer scheme separately for y and its derivative $\dot{y}$. When operators D and A are permutable, i.e., $DA = AD$, the following estimate is obtained [7]

$$\|u_h(t) - u(t)\|_A + \|u_{ht}(t) - u_t(t)\|_D \le M\tau^4.$$

Let us $w = D^{1/2}y, \ \dot{w} = D^{1/2}\dot{y}$ instead of $y, \dot{y}$. Note that $\left(D^{1/2}\right)^* = D^{1/2} > 0$ and the inverse operator $D^{-1/2} = \left(D^{-1/2}\right)^* > 0$ exists.

After obvious transformations from (15) we obtain

$$\begin{gathered} \widetilde{D}w_t - \gamma\widetilde{A}\dot{w}_t + \widetilde{A}w^{(0.5)} = \widetilde{\varphi}_1, \ \ \gamma\widetilde{D}\dot{w}_t + \alpha\widetilde{A}w_t - \beta\widetilde{A}\dot{w}^{(0.5)} = \widetilde{\varphi}_2, \\ w^0 = D^{1/2}u_0, \ \ \dot{w}^0 = D^{1/2}(f^0 - Au_0), \end{gathered} \tag{21}$$

where $\widetilde{\varphi}_1 = D^{-1/2}\varphi_1, \ \widetilde{\varphi}_2 = D^{-1/2}\varphi_2, \ \widetilde{D} = E, \ \ \widetilde{A} = D^{-1/2}AD^{-1/2}$. It is clear that $\widetilde{D} = \widetilde{D}^* > 0, \ \ \widetilde{A} = \widetilde{A}^* > 0$ and $\widetilde{D}\ \widetilde{A} = \widetilde{A}\widetilde{D}$. Consequently, there is no need for the permutability of operators D and A. Then, eliminating from (21) first $\dot{w}$, and then $\hat{\dot{w}}$ and adding them, taking into account (16), we obtain the following three-layer difference scheme

$$B_1 w^{n+1} + B_2 w^n + B_3 w^{n-1} = \tau F_n, \ \ n = 1, 2, ..., \text{ where } w^0, w^1 \text{ are given,} \tag{22}$$

$$B_1 = \gamma\widetilde{D}^2 + \frac{\tau}{2}(\gamma - \beta)\widetilde{A}\widetilde{D} - \left(\frac{\tau^2}{4}\beta - \alpha\gamma\right)\widetilde{A}^2,$$

$$B_2 = 2\gamma\widetilde{D}^2 + \left(\frac{\tau^2}{2}\beta + 2\alpha\gamma\right)\widetilde{A}^2,$$

$$B_3 = \gamma\widetilde{D}^2 - \frac{\tau}{2}(\gamma - \beta)\widetilde{A}\widetilde{D} - \left(\frac{\tau^2}{4}\beta - \alpha\gamma\right)\widetilde{A}^2,$$

$$F_n = \left(\gamma\widetilde{D} - \frac{\tau}{2}\beta\widetilde{A}\right)\widetilde{\varphi}_1^n + \gamma\widetilde{A}\widetilde{\varphi}_2^n - \left(\gamma\widetilde{D} + \frac{\tau}{2}\beta\widetilde{A}\right)\widetilde{\varphi}_1^{n-1} - \gamma\widetilde{A}\widetilde{\varphi}_2^{n-1}.$$

Equation (22) can be rewritten in the canonical form:

$$\overline{B}w_{\overset{\circ}{t}} + \tau^2\overline{R}w_{\bar{t}t} + \overline{A}w = \overline{F}, \text{ where } y^0, y^1 \text{ are given,} \tag{23}$$

and operators in (23) have the following form:

$$\begin{aligned} \overline{B} &= \tau(B_1 - B_3) = \tau(\gamma - \beta)\widetilde{A}\widetilde{D} = \tau\alpha\widetilde{A}\widetilde{D}, \\ \overline{R} &= \frac{1}{2}\left(B_1 + B_3\right) = \gamma\widetilde{D}^2 - \left(\frac{\tau^2}{4}\beta - \alpha\gamma\right)\widetilde{A}^2, \\ \overline{A} &= B_1 + B_2 + B_3 = 4\gamma(\widetilde{D}^2 + \alpha\widetilde{A}^2), \\ \overline{F} &= \gamma\widetilde{D}\widetilde{\varphi}^n_{1,\bar{t}} - \beta\widetilde{A}\frac{\widetilde{\varphi}_1^n + \widetilde{\varphi}_1^{n-1}}{2} + \gamma\widetilde{A}\widetilde{\varphi}^n_{2,\bar{t}}. \end{aligned} \tag{24}$$

Hence it is clear that, $\overline{B}^* = \overline{B} > 0$, $\overline{A}^* = \overline{A} > 0$, $\overline{R}^* = \overline{R}$.

Now, based on the results of the theory of difference schemes [18], we check the fulfillment of the stability condition for the three-layer difference scheme (23)

$$\overline{R} \geq \frac{1}{4}\overline{A}. \tag{25}$$

A straightforward computation ensures that (25) holds if the following conditions are met

$$\tau \leq 2/\sqrt{\beta}. \tag{26}$$

Condition (26) always holds if (20) is satisfied. Then, based on the results of the theory of difference schemes [18, 19], we establish the validity of the following theorem.

Theorem 3. Under condition (26), scheme (23) is stable in $H_{\overline{A}}$ by the initial data and by the right-hand side, and its solution satisfies the following estimate

$$\|w^n\|_{\overline{A}}^2 \leq \left\|w^0\right\|_{\overline{A}}^2 + \frac{1}{2}\sum_{k=0}^{n}\tau\left\|\overline{F}_k\right\|_{\overline{B}^{-1}}^2. \tag{27}$$

From inequality (27), returning to the variable y and taking into account the definition of operators $\overline{A}$, $\overline{B}^{-1}$ and $\overline{F}$ in (24), we obtain the estimate

$$\begin{gathered}\|y^n\|_{\widetilde{A}^2} \leq \left\|y^0\right\|_{\widetilde{A}^2} + M\max_k\left(\frac{\gamma}{\sqrt{\alpha\beta}}\left\|\widetilde{\varphi}_{1,\bar{t}}^k\right\|_{\widetilde{A}^{-1}\widetilde{D}}\right. \\ \left. -\frac{\beta}{\alpha}\left\|\frac{\widetilde{\varphi}_1^k + \widetilde{\varphi}_1^{k-1}}{2}\right\|_{\widetilde{A}\widetilde{D}^{-1}} + \frac{\gamma}{\sqrt{\alpha\beta}}\left\|\widetilde{\varphi}_{2,\bar{t}}^k\right\|_{\widetilde{A}\widetilde{D}^{-1}}\right.,\end{gathered} \tag{28}$$

where M is a constant independent of τ and h.

Let us apply the obtained estimate to assess the error of scheme (23). The $z = y - u$ error satisfies the equation $\overline{B}z_{\overset{\circ}{t}} + \tau^2\overline{R}z_{\bar{t}t} + \overline{A}z = \psi$, where $\psi = \overline{F} - (\overline{B}u_{\overset{\circ}{t}} + \tau^2\overline{R}u_{\bar{t}t} + \overline{A}u)$. Hence, the following estimate

$$\begin{gathered}\|z_n\|_{\widetilde{A}^2} \leq \|z_0\|_{\widetilde{A}^2} + M\max_k\left(\frac{\gamma}{\sqrt{\alpha\beta}}\left\|\psi_{1,\bar{t}}^k\right\|_{\widetilde{A}^{-1}\widetilde{D}}\right. \\ \left. +\frac{\beta}{\alpha}\left\|\frac{\psi_1^k + \psi_1^{k-1}}{2}\right\|_{\widetilde{A}\widetilde{D}^{-1}} + \frac{\gamma}{\sqrt{\alpha\beta}}\left\|\psi_{2,\bar{t}}^k\right\|_{\widetilde{A}\widetilde{D}^{-1}}\right.\end{gathered}$$

is valid for z.

Here $\psi_1,\ \psi_2$ are the errors in the approximation of the vector scheme (15).

Eliminating z and $\hat{z}$, from relation (21), we can arrive at an equation of the form (23) for $\dot{z} = \dot{y} - \dot{u}$. Then we obtain $\|z^n\|_{\widetilde{A}^2} = \|u^n - y^n\|_{\widetilde{A}^2} = O(\tau^4)$ and $\|\dot{z}^n\|_{\widetilde{A}^2} = \|\dot{u}^n - \dot{y}^n\|_{\widetilde{A}^2} = O(\tau^4)$ at the point of time $t_n,\ n = 1, 2...$. Therefore, based on estimate (28), under the conditions of (20), we obtain the convergence of scheme (15) to the solution of the original problem $u(t_n) \in C^6[0,T]$ with the fourth order, i.e.,

$$\|y(t_n) - u(t_n)\|_{\widetilde{A}^2} + \|\dot{y}(t_n) - \dot{u}(t_n)\|_{\widetilde{A}^2} \leq M\tau^4.$$

Therefore, for the error $\|y(t) - u(t)\|$, $\forall t \in [t_n, t_{n+1}],\ n = 0, 1, ...$ the following result holds.

Theorem 4. Let the stability conditions (26) be satisfied. Then, if $u(x,t) \in C^6[0,T]$, then scheme (15), (16) converges to the solution of problem (13) and the following accuracy estimates are valid for its solution:

$$\|y(t) - u(t)\|_{\widetilde{A}^2} \leq M\tau^4,\ \|\dot{y}(t) - \dot{u}(t)\|_{\widetilde{A}^2} \leq M\tau^4,\ \forall t \in [0,T].$$

The second estimate of Theorem 4 is obtained using the results of Theorem 3 for the derivative $\dot{z}$.

To estimate the accuracy of schemes 3^0 and 4^0, it is necessary to obtain an estimate of the error $z = u_h - u$. Using the technique of such an estimate in the theory of difference schemes [18] of the theory of the finite element method [17], we formulate the following results.

Theorem 5. Under condition (26), the solution to scheme 3^0 converges to a sufficiently smooth solution of problem (4)-(6) and the following accuracy estimate holds

$$\|y(t) - u(t)\|_1 \leq M(\tau^4 + h^2).$$

Theorem 6. Under condition (26), the solution of scheme 4^0 converges to a sufficiently smooth solution of problem (4)-(6) and the following accuracy estimate holds

$$\|y(t) - u(t)\|_1 + \|\dot{y}(t) - \dot{u}(t)\|_1 \leq M(\tau^4 + h^3).$$

5 *Schemes with skew-symmetric operator*

Let us investigate the stability by the initial data and the right-hand side of scheme (15), (16) with operators $D^* = D > 0$, $A^* = -A$, and write it in the canonical form

$$\widetilde{B}Y_t + \widetilde{A}Y = 0; \quad Y = (y, \dot{y}), \tag{29}$$

where

$$\widetilde{B} = \begin{pmatrix} D + \frac{\tau}{2}A & -\gamma A \\ \alpha A & \gamma D - \frac{\tau}{2}\beta A \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} A & 0 \\ 0 & -\beta A \end{pmatrix}.$$

To prove stability by the initial data of scheme (29), we use the results of [20]. To do this, we take $\alpha = \tau^2/12$ and represent the operator $\widetilde{B}$ in the form $\widetilde{B} = \widetilde{D} + \widetilde{A}C$, where

$$\widetilde{D} = \begin{pmatrix} D & -\gamma A \\ \alpha A & \gamma D \end{pmatrix}, \quad C = \begin{pmatrix} \frac{\tau}{2} & 0 \\ 0 & \frac{\tau}{2} \end{pmatrix}.$$

Then, for the stability of scheme (29), on the basis of the results of the theory of difference schemes [20], it remains to check the fulfillment of condition $C^*\widetilde{D} + \widetilde{D}C \geq \widetilde{D}$. This condition is met if $\alpha > 0$, $\gamma > 0$. Thus, taking into account (18), we arrive at the following statement.

Theorem 7. If the conditions $\alpha > 0$, $\beta > 0$, $\gamma > 0$, are satisfied, then scheme (29) is stable by the initial data and the right-hand side in $H_{\widetilde{D}}$ and the following estimate

$$\|Y^{n+1}\|_{\widetilde{D}} \leq \|Y^0\|_{\widetilde{D}} + \sum_{k=0}^{k} \tau \|\Phi_k\|_{\widetilde{D}}$$

is true.

Based on this estimate, likewise in the previous sections, we obtain the accuracy of scheme (15), (16) with the skew-symmetric operator A, i.e. the results of Theorems 1, 2, 5 and 6 are also valid for scheme (29).

6 *Algorithm for the implementation of the scheme* (21)

Consider one of the possible algorithms for implementing the scheme (21). We rewrite it as

$$m_{11}\hat{w} + m_{12}\overset{\wedge}{\dot{w}} = \phi_1, \; m_{21}\hat{w} + m_{22}\overset{\wedge}{\dot{w}} = \phi_2, \tag{30}$$

where

$$m_{11} = \widetilde{D} + \frac{\tau}{2}\widetilde{A}, \quad m_{12} = -\gamma\widetilde{A}, \quad m_{21} = \alpha\widetilde{A}, \quad m_{22} = \gamma\widetilde{D} - \frac{\tau}{2}\beta\widetilde{A},$$

$$\phi_1 = \tau\widetilde{\varphi}_1 + \left(\widetilde{D} - \frac{\tau}{2}\widetilde{A}\right) w - \gamma\widetilde{A}\dot{w}, \quad \phi_2 = \tau\widetilde{\varphi}_2 + \alpha\widetilde{A}w + \left(\gamma\widetilde{D} + \frac{\tau}{2}\beta\widetilde{A}\right)\dot{w}.$$

To calculate the integrals $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$, we can use the Simpson quadrature formula.

Taking into account the permutable operators $\widetilde{A}$ and $\widetilde{D}$, we exclude $\overset{\wedge}{\dot{w}}$ from equation (30):

$$C\hat{w} = F. \tag{31}$$

Here $C = \gamma\widetilde{D}^2 + \frac{\tau}{2}(\gamma - \beta)\widetilde{A}\widetilde{D} - \left(\frac{\tau^2}{4}\beta - \alpha\gamma\right)\widetilde{A}^2, \quad F = m_{22}\phi_1 - m_{12}\phi_2.$

Equation (31) can be solved either directly by inverting the operator C, or by factoring it

$$C = \gamma\, C_1 C_2 = \gamma\left[\widetilde{D}^2 - (x_1 + x_2)\tau\widetilde{A}\widetilde{D} + x_1 x_2 \tau^2 \widetilde{A}^2\right],\ C_k = \left(\widetilde{D} - x_k\tau\widetilde{A}\right),\ k = 1, 2.$$

Then, equation (31) is solved using the following algorithm:

$$\gamma_1 C_1 \overline{w} = F, \quad C_2\hat{w} = \overline{w}. \tag{32}$$

After determining $\hat{w}$ from (32), the solution of $\overset{\wedge}{\dot{w}}$ is calculated, for example, from the equation $\left(\gamma\widetilde{D} - \frac{\tau}{2}\beta\widetilde{A}\right)\overset{\wedge}{\dot{w}} = \phi_2 - \alpha\widetilde{A}\hat{w}$.

The scheme (14) is implemented as follows:

$$(D + \sigma\tau A)y^{n+1} = \left[D - (1-\sigma)\tau A\right] y^n + \tau\varphi, \quad n = 0, 1, 2, ...,$$

$$y^0 = u_{h0}.$$

Remark. It is possible to prove the stability of scheme (15), (16) with variable operators $A = A_n$, $D = D_n$, for example, in norm A_n. It is required that the operatorA_n be Lipschitz-continuous in t.

Conclusions

The methods of a high degree of accuracy for solving the first boundary value problem for a pseudo-parabolic equation of a special form are developed and investigated in this article. These methods are based on finite-difference and finite-element approximations in space and time. The stability and convergence of the constructed methods are proved, and the accuracy estimates are obtained. An algorithm for the implementation of the finite element method was developed. Other pseudo-parabolic equations given in the introduction, as well as other types of similar equations, are investigated likewise. We can study problems with other local and nonlocal boundary conditions.

The system of ordinary differential equations obtained by spatial approximation may turn out to be rigid. A separate study will be devoted to this issue and numerical modeling, where, based on the algorithm for implementing the method developed here, it will be tested on exact solutions in the form of a Fourier series and the constructed methods will be compared with other methods. In addition, on the basis of a computational experiment, the convergence rates of the method along the spatial and temporal directions will be checked, as well as visualizations, which confirm these theoretical results.

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Соболев типті псевдопараболалық теңдеуі үшін жоғары дәлдіктегі айырмашылық схемаларының жинақтылығы туралы

Соболев типті псевдопараболалық теңдеуі үшін уақыт пен кеңістік бойынша жоғары дәлдіктегі ақырлы айырымдық әдісі мен ақырлы элементтер әдісінің әртүрлі айырымдық схемалары ұсынылған және зерттелген. Кеңістіктегі дәлдік тәртібін арттыру екі жолмен, ақырлы айырымдық схемасы және ақырлы элементтер схемасы мен жүзеге асырылды. Уақыт бойынша тізбектің дәлдігінің жоғары тәртібіне уақыт айнымалысын арнайы іріктеу арқылы қол жеткізілген. Тиісті априорлық бағалаулар және олардың негізінде бастапқы дифференциалдық есепті шешудің жеткілікті тегістігімен ұсынылған айырымдық схемаларының дәлдігінің бағалары алынды. Құрылған айырымдық схемаларын іске асыру алгоритмдері жүзеге асырылды.

Кілт сөздер: псевдопараболалық теңдеу, айырымдық схемалар, ақырлы айырымдар әдісі, ақырлы элементтер әдісі, априорлық бағалаулар, тұрақтылық, жинақтылық, дәлдік.

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О сходимости разностных схем повышенной точности для одного псевдопараболического уравнения соболевского типа

Предложены и исследованы различные разностные схемы метода конечных разностей и метода конечных элементов высокого порядка точности по времени и по пространству для псевдопараболического уравнения соболевского типа. Повышение порядка точности по пространству осуществлено двумя способами: методом конечных разностей и методом конечных элементов. Высокий порядок точности схемы по времени достигнут за счет специальной дискретизации временной переменной. Получены соответствующие априорные оценки, и на их основе оценки точности предложенных разностных схем при достаточной гладкости решения исходной дифференциальной задачи. Реализованы алгоритмы выполнения построенных разностных схем.

Ключевые слова: псевдопараболическое уравнение, разностные схемы, метод конечных разностей, метод конечных элементов, обобщенные решения, априорные оценки, устойчивость, сходимость, точность.

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On the hyperbolic type differential equation with time involution

In the present paper, the initial value problem for the hyperbolic type involutory in t second order linear partial differential equation is studied. The initial value problem for the fourth order partial differential equations equivalent to this problem is obtained. The stability estimates for the solution and its first and second order derivatives of this problem are established.

Keywords: involutory type hyperbolic equation, stability, Banach space.

Introduction

Delay differential equations are universal phenomenon applied their models in engineering systems to behave like a real process [1–6].

Involutary differential equations have been studied in several papers [7–11]. In the paper [10], the boundedness of the solution of the initial value problem

$$y''(t) = f(t, y(t), y(u(t))),\ t \in I = (-\infty, \infty),\ y(t_0) = y_0,\ y'(t_0) = y'_0$$

for the second order ordinary differential equation with involution was investigated. Theorem on stability estimates for the solution of the initial value problem for the second order ordinary linear differential equation with involution was proved. Finally, theorem on existence and uniqueness of bounded solution of initial value problem for the second order nonlinear ordinary differential equation with involution was established. Presently, spectral questions of differential equations with involution were studied in papers [12–20].

Delay hyperbolic differential equations have been investigated in several papers [21–25]. Partial differential equations with involution terms have deeply different properties of solutions then without involution terms [26, 27]. Therefore, it is important to study properties of partial differential equations with involution.

In the present paper, the stability of the solution of the initial value problem for the hyperbolic type time involution partial differential equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - a u_{xx}(t,x) - b u_{xx}(-t,x) = g(t,x),\ t,\ x \in I, \\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),\ x \in I \end{cases} \tag{1a}$$

is investigated. Here, $g(t,x)\,(t, x \in I)$, $\varphi(x)$ and $\psi(x)$ are given smooth functions. The stability estimates for the solution and its first and second order derivatives of this problem are established.

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1 Stability of problem (1a)

Theorem 1.1. Assume that $g(t,x)$ is a continuously differentiable and bounded function and $g(0,x)=0$ and $\varphi(x)$ is a twice continuously differentiable and bounded function and $\psi(x)$ is a continuously differentiable and bounded function and $|b|<a, a\in(0,\infty)$. Then, for solutions of problem (1a) the following stability estimates hold:

$$\sup_{t,x\in I}|u(t,x)|\leq M_1(a,b)\left[\sup_{x\in I}|\varphi(x)|+\int_{-\infty}^{\infty}|\psi(y)|\,dy+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(y,x)|\,dydx\right], \tag{1b}$$

$$\sup_{t,x\in I}|u_t(t,x)|+\sup_{t,x\in I}|u_x(t,x)|\leq M_1(a,b)\left[\sup_{x\in I}|\varphi_x(x)|\right.$$

$$\left.+\sup_{x\in I}|\psi(x)|+\sup_{x\in I}\int_{-\infty}^{\infty}|g(y,x)|\,dy\right], \tag{1c}$$

$$\sup_{t,x\in I}|u_{tt}(t,x)|+\sup_{t,x\in I}|u_{xx}(t,x)|+\sup_{t,x\in I}|u_{tx}(t,x)|$$

$$\leq M_2(a,b)\left[\sup_{x\in I}|\varphi_{xx}(x)|+\sup_{x\in I}|\psi_x(x)|+\sup_{t,x\in I}|g(t,x)|\right]. \tag{1d}$$

Proof. Problem (1a) can be written as abstract initial value problem

$$\begin{cases}\frac{d^2u(t)}{dt^2}+aAu\,(t)+bAu\,(-t)=g(t),\ t\in I,\\ u(0)=\varphi,\ u'(0)=\psi\end{cases} \tag{2a}$$

in a Banach space $C(I)$ of all continuous and bounded functions $f(x)$ defined on I with norm

$$\|f\|_{C(I)}=\sup_{x\in I}|f(x)|.$$

Here, positive operator A defined by the formula

$$Au=-u''(x)$$

with domain $D(A)=\{u:u(x),\ u''(x)\in C(I)\}$, $g(t)=g(t,x)$ and $u(t)=u(t,x)$ are known and unknown abstract functions defined on I with values in $C(I)$ and $\varphi=\varphi(x)$, $\psi=\psi(x)$ are unknown elements of $C(I)$. Now, we will obtain the initial value problem for the fourth order differential equation to problem (2a) under smoothness conditions of solution. Differentiating equation (2a), we get

$$\frac{d^3u\,(t)}{dt^3}+aAu'\,(t)-bAu'\,(-t)=g_t(t),$$

$$\frac{d^4u\,(t)}{dt^4}+aAu''\,(t)+bAu''\,(-t)=g_{tt}(t). \tag{3}$$

Using these equations and initial condition and equation in problem (2a), we get

$$\begin{cases}u(0)=\varphi,\ u'(0)=\psi,\\ u''\,(0)=-\,(a+b)\,A\varphi,\\ u'''\,(0)=(-a+b)\,A\psi+g_t(0).\end{cases}$$

Putting $-t$ instead of t in equation (2a), we get

$$u_{tt}(-t) + aAu(-t) + bAu(t) = g(-t). \tag{4}$$

Applying equations (2a), (3) and (4), we get

$$\frac{d^4u(t)}{dt^4} + aA\frac{d^2u(t)}{dt^2} + bA\left[-aAu(-t) - bAu(t) + g(-t)\right] = g_{tt}(t),$$

$$bAu(-t) = -\frac{d^2u(t)}{dt^2} - aAu(t) + g(t).$$

From these equations it follows equation

$$\frac{d^4u(t)}{dt^4} + aA\frac{d^2u(t)}{dt^2} - aA\left[-\frac{d^2u(t)}{dt^2} - aAu(t) + g(t)\right] - b^2A^2u(t) = -bAg(-t) + g_{tt}(t)$$

or

$$\frac{d^4u(t)}{dt^4} + 2aA\frac{d^2u(t)}{dt^2} + \left(a^2 - b^2\right)A^2u(t) = aAg(t) - bAg(-t) + g_{tt}(t).$$

Then, we have the following initial value problem for the fourth order abstract differential equation

$$\begin{cases} \frac{d^4u(t)}{dt^4} + 2aA\frac{d^2u(t)}{dt^2} + \left(a^2 - b^2\right)A^2u(t) = F(t), \\ F(t) = aAg(t) - bAg(-t) + g_{tt}(t),\ t \in I, \\ u(0) = \varphi,\ u'(0) = \psi,\ u''(0) = -(a+b)A\varphi, \\ u'''(0) = (-a+b)A\psi + g_t(0). \end{cases} \tag{5}$$

Now we will obtain solution of the initial value problem (5). It is easy to see that

$$\frac{d^4u(t)}{dt^4} + 2aA\frac{d^2u(t)}{dt^2} + \left(a^2 - b^2\right)A^2u(t) = \left(\frac{d^2}{dt^2} + (a - |b|)A\right)\left(\frac{d^2}{dt^2} + (a + |b|)A\right)u(t).$$

Therefore, problem (5) can be written as abstract initial value problem

$$\begin{cases} \left(\frac{d^2}{dt^2} + (a+|b|)A\right)u(t) = v(t),\ u(0) = \varphi,\ u'(0) = \psi, \\ \left(\frac{d^2}{dt^2} + (a-|b|)A\right)v(t) = F(t), \\ F(t) = aAg(t) - bAg(-t) + g_{tt}(t),\ t \in I, \\ v(0) = (-b+|b|)A\varphi, \\ v'(0) = (b+|b|)A\psi + g'(0) \end{cases} \tag{6}$$

for the system of second order abstract differential equations in a Banach space $C(I)$. Problem (6) can be written as initial value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - (a+|b|)\, u_{xx}(t,x) = v(t,x), \ t, \ x \in I, \\ u(0,x) = \varphi(x), \ u_t(0,x) = \psi(x), \quad x \in I, \\ \frac{\partial^2 u(t,x)}{\partial t^2} - (a-|b|)\, u_{xx}(t,x) = F(t,x), \\ F(t,x) = -a g_{xx}(t,x) + b g_{xx}(-t,x) + g_{tt}(t,x), \ t, \ x \in I, \\ v(0,x) = (-|b|+b)\, \varphi_{xx}(x), \\ v_t(0,x) = (-|b|-b)\, \psi_{xx}(x) + g'(0,x), \ x \in I \end{cases}$$

for the system of hyperbolic equations. Applying Dalambert's formula, we get

$$u(t,x) = \frac{\varphi(x+\sqrt{a+|b|}t) + \varphi(x-\sqrt{a+|b|}t)}{2} + \frac{1}{2\sqrt{a+|b|}} \int\limits_{x-\sqrt{a+|b|}t}^{x+\sqrt{a+|b|}t} \psi(\xi) d\xi$$

$$+ \int\limits_0^t \frac{-|b|+b}{2\sqrt{a+|b|}} \int\limits_{x-\sqrt{a+|b|}(t-\tau)}^{x+\sqrt{a+|b|}(t-\tau)} \frac{\varphi_{\xi\xi}(\xi+\sqrt{a-|b|}\tau) + \varphi_{\xi\xi}(\xi-\sqrt{a-|b|}\tau)}{2} d\xi d\tau$$

$$- \int\limits_0^t \frac{1}{4\sqrt{a^2-b^2}} \int\limits_{x-\sqrt{a+|b|}(t-\tau)}^{x+\sqrt{a+|b|}(t-\tau)} \int\limits_{\xi-\sqrt{a-|b|}\tau}^{\xi+\sqrt{a-|b|}\tau} (|b|+b)\, \psi_{\lambda\lambda}(\lambda) d\lambda d\xi d\tau$$

$$+ \int\limits_0^t \frac{1}{4\sqrt{a^2-b^2}} \int\limits_{x-\sqrt{a+|b|}(t-\tau)}^{x+\sqrt{a+|b|}(t-\tau)} \int\limits_{\xi-\sqrt{a-|b|}\tau}^{\xi+\sqrt{a-|b|}\tau} g'(0,\lambda) d\lambda d\xi d\tau$$

$$\int\limits_0^t \frac{1}{4\sqrt{a^2-b^2}} \int\limits_{x-\sqrt{a+|b|}(t-\tau)}^{x+\sqrt{a+|b|}(t-\tau)} \int\limits_0^\tau \int\limits_{\xi-\sqrt{a-|b|}(\tau-p)}^{\xi+\sqrt{a-|b|}(\tau-p)} F(p,\lambda) d\lambda dp d\xi d\tau.$$

$$= J_1(t,x) + J_2(t,x) + J_3(t,x) + J_4(t,x),$$

where

$$J_1(t,x) = \frac{\varphi(x+\sqrt{a+|b|}t) + \varphi(x-\sqrt{a+|b|}t)}{2} + \frac{1}{2\sqrt{a+|b|}} \int\limits_{x-\sqrt{a+|b|}t}^{x+\sqrt{a+|b|}t} \psi(\xi) d\xi, \tag{7}$$

$$J_2(t,x) = \int\limits_0^t \frac{-|b|+b}{2\sqrt{a+|b|}} \int\limits_{x-\sqrt{a+|b|}(t-\tau)}^{x+\sqrt{a+|b|}(t-\tau)} \frac{\varphi_{\xi\xi}(\xi+\sqrt{a-|b|}\tau) + \varphi_{\xi\xi}(\xi-\sqrt{a-|b|}\tau)}{2} d\xi d\tau,$$

$$J_3(t,x)=-\int\limits_0^t \frac{1}{4\sqrt{a^2-b^2}} \int\limits_{x-\sqrt{a+|b|}(t-\tau)}^{x+\sqrt{a+|b|}(t-\tau)} \int\limits_{\xi-\sqrt{a-|b|}\tau}^{\xi+\sqrt{a-|b|}\tau} (|b|+b)\,\psi_{\lambda\lambda}(\lambda)d\lambda d\xi d\tau,$$

$$J_4(t,x)=\int\limits_0^t \frac{1}{4\sqrt{a^2-b^2}} \int\limits_{x-\sqrt{a+|b|}(t-\tau)}^{x+\sqrt{a+|b|}(t-\tau)} \int\limits_0^\tau \int\limits_{\xi-\sqrt{a-|b|}(\tau-p)}^{\xi+\sqrt{a-|b|}(\tau-p)} F(p,\lambda)d\lambda dp d\xi d\tau$$

$$+\int\limits_0^t \frac{1}{4\sqrt{a^2-b^2}} \int\limits_{x-\sqrt{a+|b|}(t-\tau)}^{x+\sqrt{a+|b|}(t-\tau)} \int\limits_{\xi-\sqrt{a-|b|}\tau}^{\xi+\sqrt{a-|b|}\tau} g'(0,\lambda)d\lambda d\xi d\tau.$$

Now, we will estimate $J_k(t,x)$, $k=1,2,3,4$, separately. First, we start with estimates for $J_1(t,x)$. Applying the triangle inequality and formula (7), we get

$$|J_1(t,x)|\le M_1(a,b)\left[\sup_{x\in I}|\varphi(x)|+\int\limits_{-\infty}^{\infty}|\psi(y)|\,dy\right],$$

$$|J_{1,t}(t,x)|\,,|J_{1,x}(t,x)|\le M_{11}(a,b)\left[\sup_{x\in I}|\varphi_x(x)|+\sup_{x\in I}|\psi(x)|\right],$$

$$|J_{1,tt}(t,x)|\,,|J_{1,tx}(t,x)|\,,|J_{1,xx}(t,x)|\le M_{111}(a,b)\left[\sup_{x\in I}|\varphi_{xx}(x)|+\sup_{x\in I}|\psi_x(x)|\right]$$

for any t, $x\in I$. Second, we will estimate $J_2(t,x)$. We have that

$$J_2(t,x)=\frac{|b|-b}{2\sqrt{a+|b|}}\left[\varphi\left(x+\sqrt{a+|b|}t\right)\right.$$

$$\left.+\varphi\left(x-\sqrt{a+|b|}t\right)-\varphi\left(x+\sqrt{a-|b|}t\right)-\varphi\left(x-\sqrt{a-|b|}t\right)\right].$$

Applying the triangle inequality and formula (7), we get

$$|J_2(t,x)|\le M_2(a,b)\sup_{x\in I}|\varphi(x)|\,,$$

$$|J_{2,t}(t,x)|\,,|J_{2,x}(t,x)|\le M_2(a,b)\sup_{x\in I}|\varphi_x(x)|\,,$$

$$|J_{2,tt}(t,x)|\,,|J_{2,tx}(t,x)|\,,|J_{2,xx}(t,x)|\le M_2(a,b)\sup_{x\in I}|\varphi_{xx}(x)|$$

for any t, $x\in I$. Third, we will estimate $J_3(t,x)$. We have that

$$J_3(t,x)=\int\limits_0^t \frac{|b|+b}{4\sqrt{a^2-b^2}}[\psi\left(x+\sqrt{a+|b|}(t-\tau)+\sqrt{a+|b|}\tau\right)+\psi\left(x-\sqrt{a+|b|}(t-\tau)-\sqrt{a+|b|}\tau\right)$$

$$-\psi\left(x-\sqrt{a+|b|}(t-\tau)+\sqrt{a-|b|}\tau\right)+\psi\left(x+\sqrt{a+|b|}(t-\tau)-\sqrt{a-|b|}\tau\right)]d\tau. \qquad (8)$$

Applying the triangle inequality and formula (8), we get

$$|J_3(t,x)|\le M_3(a,b)\int\limits_{-\infty}^{\infty}|\psi(y)|\,dy,$$

$$|J_{3,t}(t,x)|\,,|J_{3,x}(t,x)|\leq M_3(a,b)\sup_{x\in I}|\psi(x)|\,,$$

$$|J_{3,tt}(t,x)|\,,|J_{3,tx}(t,x)|\,,|J_{3,xx}(t,x)|\leq M_3(a,b)\sup_{x\in I}|\psi_x(x)|$$

for any t, $x \in I$. Fourth, we will estimate $J_4(t,x)$. We have that

$$J_4(t,x)=\frac{1}{4\sqrt{a^2-b^2}}\int\limits_0^t\int\limits_{x-\sqrt{a+|b|}(t-\tau)}^{x+\sqrt{a+|b|}(t-\tau)}\int\limits_0^\tau\int\limits_{\xi-\sqrt{a-|b|}(\tau-r)}^{\xi+\sqrt{a-|b|}(\tau-r)}\left[-ag_{\lambda\lambda}(r,\lambda)+bg_{\lambda\lambda}(-r,\lambda)\right]d\lambda dr d\xi d\tau$$

$$+\frac{1}{4\sqrt{a^2-b^2}}\int\limits_0^t\int\limits_{x-\sqrt{a+|b|}(t-\tau)}^{x+\sqrt{a+|b|}(t-\tau)}\int\limits_0^\tau\int\limits_{\xi-\sqrt{a-|b|}(\tau-r)}^{\xi+\sqrt{a-|b|}(\tau-r)}g_{rr}(r,\lambda)d\lambda dr d\xi d\tau$$

$$+\frac{1}{4\sqrt{a^2-b^2}}\int\limits_0^t\int\limits_{x-\sqrt{a+|b|}(t-\tau)}^{x+\sqrt{a+|b|}(t-\tau)}\int\limits_{\xi-\sqrt{a-|b|}\tau}^{\xi+\sqrt{a-|b|}\tau}g'(0,\lambda)d\lambda d\xi d\tau.$$

Applying formulas

$$\int\limits_0^\tau\int\limits_{\xi-\sqrt{a-|b|}(\tau-r)}^{\xi+\sqrt{a-|b|}(\tau-r)}\left[-ag_{\lambda\lambda}(r,\lambda)+bg_{\lambda\lambda}(-r,\lambda)\right]d\lambda dr=\frac{2a}{\sqrt{a-|b|}}g(\tau,\xi)-\frac{2b}{\sqrt{a-|b|}}g(-\tau,\xi),$$

$$\int\limits_0^\tau\int\limits_{\xi-\sqrt{a-|b|}(\tau-r)}^{\xi+\sqrt{a-|b|}(\tau-r)}g_{rr}(r,\lambda)d\lambda dr=2\sqrt{a-|b|}g(\tau,\xi)-\int\limits_{\xi-\sqrt{a-|b|}\tau}^{\xi+\sqrt{a-|b|}\tau}g'\left(0,\lambda\right)d\lambda,$$

we get

$$J_4(t,x)=\frac{1}{2\sqrt{a^2-b^2}}\int\limits_0^t\int\limits_{x-\sqrt{a+|b|}(t-\tau)}^{x+\sqrt{a+|b|}(t-\tau)}\left[\frac{a}{\sqrt{a-|b|}}g(\tau,\xi)-\frac{b}{\sqrt{a-|b|}}g(-\tau,\xi)\right]d\xi d\tau$$

$$+\frac{1}{2\sqrt{a^2-b^2}}\int\limits_0^t\int\limits_{x-\sqrt{a+|b|}(t-\tau)}^{x+\sqrt{a+|b|}(t-\tau)}\sqrt{a-|b|}g(\tau,\xi)d\xi d\tau. \tag{9}$$

Applying the triangle inequality and formula (9), we get

$$|J_4(t,x)|\leq M_4(a,b)\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}|g(y,x)|\,dydx,$$

$$|J_{4,t}(t,x)|\,,|J_{4,x}(t,x)|\leq M_4(a,b)\sup_{x\in I}\int\limits_{-\infty}^{\infty}|g(y,x)|\,dy,$$

$$|J_{4,tt}(t,x)|\,,|J_{4,tx}(t,x)|\,,|J_{4,xx}(t,x)|\leq M_4(a,b)\sup_{t,x\in I}|g(t,x)|$$

for any t, $x \in I$. Combining the estimates for $J_k(t,x), k=1,2,3,4$, we obtain estimates (1b)-(1d).

2 Conclusion

In the present paper, the initial value problem for the hyperbolic type time involution linear partial differential equation is investigated. The equivalent initial value problem for the fourth order linear partial differential equations to the initial value problem for this second order linear partial differential equations with involution is presented. The stability estimates for the solution and its first and second order derivatives of this problem are proved.

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Уақытты инволюциялы гиперболалық типті дифференциалдық теңдеу жайында

Мақалада екінші ретті дербес туындылардағы t сызықтық теңдеудегі гиперболалық типтегі инволюциялық теңдеудің бастапқы есебі зерттеледі. Төртінші ретті дербес туындылы теңдеулер үшін осы есептің эквивалентті бастапқы есебі алынды. Жоғарыда аталған есептің шешімінің, бірінші және екінші ретті туындыларының тұрақтылық бағалаулары алынды.

Кілт сөздер: инволюциялық типті гиперболалық теңдеу, тұрақтылық, Банах кеңістігі.

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О дифференциальном уравнении гиперболического типа с инволюцией по времени

В статье изучена начальная задача для инволютивного уравнения гиперболического типа в t линейном уравнении в частных производных второго порядка. Получена эквивалентная этой задаче начальная задача для уравнений в частных производных четвертого порядка. Установлены оценки устойчивости решения и его производных первого и второго порядка указанной выше задачи.

Ключевые слова: гиперболическое уравнение инволютивного типа, устойчивость, банахово пространство.

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Numerical solution of the boundary value problems for the parabolic equation with involution

In this work, we study two boundary value problems for involutary parabolic equation with the first and second kind conditions. We propose absolute stable difference schemes for numerical solutions of these boundary value problems. Actually the stability estimates for solutions of difference schemes are proved. Later error analysis for the numerical solution of both difference schemes are illustrated by test examples.

Keywords: involution, parabolic, finite difference scheme, stability estimate, boundary value problem.

Introduction

It is well known that various models in physics can be reduced to a parabolic equation with delay and involution. Time delay and involutary parabolic equations with local and non local boundary conditions have been investigated by several researchers [1–17].

1 Finite differences for involutary parabolic equation with Dirichlet condition

We consider boundary value problem for parabolic equation with involution and Dirichlet condition as follows

$$\begin{cases} u_t(t,x) - (a(x)u_x(t,x))_x + \delta u(t,x) + q\left(-(a(x)u_t(-t,x))_x + \delta u(-t,x)\right) = f(t,x), \\ t \in I,\ x \in (0,l), \\ u(0,x) = \varphi(x),\ x \in [0,l], \\ u(t,0) = 0,\ u(t,l) = 0,\ t \in I. \end{cases} \tag{1}$$

Here and in future a, φ and f are given smooth functions and δ and q are known numbers such that $a(x) \geq a_0 > 0$, $\forall x \in (0,l)$, $\delta,\ |q| < 1$, $I = (-\infty, \infty)$.

1.1 Stability of differential problem

Denote by $W_2^2(0,l)$, the Sobolev space of all functions $v(x)$ defined on $[0,l]$ equipped with norm

$$\|v\|_{W_2^2(0,l)} = \left(\int_0^l |v(x)|^2\,dx\right)^{\frac{1}{2}} + \left(\int_0^l |v''(x)|^2\,dx\right)^{\frac{1}{2}}.$$

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Theorem 1. Let $\varphi \in W_2^2(0,l)$ and $f(t,x)$ be continuously differentiable on $I \times [0,l]$. Then, for the solution of initial boundary value problem (1) the stability estimates

$$\sup_{t\in I} \|u(t,\cdot)\|_{L_2(0,l)} \le M(\delta)\left[\|\varphi\|_{L_2(0,l)} + \int\limits_{-\infty}^{\infty} \|f(s,\cdot)\|_{L_2(0,l)}\,ds\right],$$

$$\sup_{t\in I} \|u_t(t,\cdot)\|_{L_2(0,l)} + \sup_{t\in I} \|u_t(t,\cdot)\|_{W_2^2(0,l)} \le \tag{2}$$

$$\le M(\delta)\left[\|\varphi\|_{W_2^2(0,l)} + \|f(0,\cdot)\|_{L_2(0,l)} + \int\limits_{-\infty}^{\infty} \|f'(s,\cdot)\|_{L_2(0,l)}\,ds\right]$$

hold, where $M(\delta)$ does not depend on both functions φ and f.

Proof. One can write problem (1) in the abstract initial value problem

$$\begin{cases} u_t(t) + Au(t) + q\ Au(-t) = f(t),\ t \in I, \\ u(0) = \varphi. \end{cases} \tag{3}$$

Here $A = A^x$ is a self adjoint positive definite operator in $H = L_2(0,l)$ which is defined by formula

$$Au(x) = -\left(a(x)u_x(x)\right)_x + \delta u(x) \tag{4}$$

with the domain $D(A) = \left\{u \in W_2^2(0,l)\ \mid u(0) = 0,\ u(l) = 0\ \right\}$, $\varphi = \varphi(x)$ is given element of H and $f(t) = f(t,x)$ is a given abstract function. The proof of Theorem 1 is based on the stability of abstract problem (3) and positiveness and self-adjointness of the abstract operator A defined by (4).

1.2 Stability of difference problems

Let $[0,l]_h = \{x_n = nh,\ 0 \le n \le M\}$ be grid space. Denote by $L_{2h} = L_2\,[0,l]_h$, Hilbert space of grid functions $\rho^h(x) = \{\rho^n\}_0^M$ defined on $[0,l]_h$ equipped with norm

$$\left\|\rho^h\right\|_{L_{2h}} = \left(\sum_{x\in[0,l]_h} \left|\rho^h(x)\right|^2\right)^{\frac{1}{2}}.$$

To the operator (4) we assign the difference operator by formula

$$A_h^x \rho^h(x) = -\left(a(x)\rho_{\bar{x}}^h(x)\right)_x + \delta\rho^h(x)$$

acting in the space of grid functions $\rho^h(x) = \{\rho^n(x)\}_0^M$ and satisfying the conditions $\rho^0 = 0$, $\rho^M = 0$, where

$$\rho_{\bar{x}}^i = \frac{\rho^i - \rho^{i-1}}{h},\ 1 \le i \le M,\ \ \rho_x^k = \frac{\rho^{k+1} - \rho^k}{h},\ 0 \le k \le M-1.$$

In the first step of discreatization we get the following Dirichlet problem

$$\begin{cases} u_t^h(t,x) + A_h^x u^h(t,x) + qA_h^x u^h(-t,x) = f^h(t,x),\ t \in I, \\ u^h(0,x) = \varphi^h(x), x \in\ [0,l]_h\,. \end{cases}$$

In the second step of discreatization one can construct the first order of accuracy difference scheme

$$\begin{cases} \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x u_k^h(x)+qA_h^x u_{-k}^h(x)=f_k^h(x), \quad f_k^h(x)=f^h(t_k,x), \\ t_k=k\tau, \ k\in Z, \ x\in \ [0,l]_h\,, \\ u_0^h(x)=\varphi^h(x), \ x\in \ [0,l]_h \end{cases} \tag{5}$$

and the second order of accuracy difference scheme

$$\begin{cases} \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+\frac{1}{2}\left(A_h^x u_k^h(x)+qA_h^x u_{-k}^h(x)\right)+\frac{1}{2}\left(A_h^x u_{k-1}^h(x)+qA_h^x u_{-k+1}^h(x)\right)= \\ =f_{k+\frac{1}{2}}^h(x)=f^h(t_{k+\frac{1}{2}},x), \ t_{k+\frac{1}{2}}=\left(k+\frac{1}{2}\right)\tau, \\ t_k=k\tau, k\in Z, \quad x\in \ [0,l]_h\,, \\ u_0^h(x)=\varphi^h(x), \quad x\in \ [0,l]_h\,. \end{cases} \tag{6}$$

Theorem 2. Let τ and h be sufficiently small positive numbers. Then, for the solution $\left\{u_k^h(x)\right\}_{-\infty}^{\infty}$of difference schemes (5) and (6) the stability estimates

$$\sup_{k\in Z}\left\|u_k^h\right\|_{L_{2h}} \ \le M(\delta)\left[\left\|\varphi^h\right\|_{L_{2h}}+\sum_{k=-\infty}^{\infty}\left\|f_k^h\right\|_{L_{2h}}\right],$$

$$\sup_{k\in Z}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}+\sup_{k\in Z}\left\|u_k^h\right\|_{W_{2h}^2} \ \le \tag{7}$$

$$\le M(\delta)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|f_0^h\right\|_{L_{2h}}+\sum_{k=-\infty}^{\infty}\left\|\frac{f_k^h-f_{k-1}^h}{\tau}\right\|_{L_{2h}}\right]$$

are valid, where $M(\delta)$ does not depend on τ, h φ^h and f_k^h.

Proof. Difference schemes (5) and (6) can be rewritten as the following abstract difference schemes

$$\begin{cases} \frac{u_k^h-u_{k-1}^h}{\tau}+A_h u_k^h+qA_h u_{-k}^h \ =f_k^h\ , \quad k\in Z, \\ u_0^h \ =\varphi^h \end{cases} \tag{8}$$

and

$$\begin{cases} \frac{u_k^h-u_{k-1}^h}{\tau}+\frac{1}{2}\left(A_h u_k^h \ +qA_h u_{-k}^h\ \right)+\frac{1}{2}\left(A_h u_{k-1}^h \ +qA_h^x u_{-k+1}^h\right)=f_{k+\frac{1}{2}}^h\ , \\ k\in Z, \ u_0^h \ =\varphi^h, \end{cases} \tag{9}$$

correspondingly. So, the proof of Theorem 2 is based on the stability of the difference schemes (8) and (9) on the positive definiteness and self-adjointness of the operator A^h in the Hilbert space L_{2h}.

2 Finite differences for involutary parabolic equation with Neumann condition

Let us take boundary value problem for parabolic equation with involution and Neumann condition as follows

$$\begin{cases} u_t(t,x) - (a(x)u_x(t,x))_x + \delta u(t,x) + q\left(-\left(a(x)u_t(-t,x)\right)_x + \delta u(-t,x)\right) = f(t,x), \\ t \in I,\ x \in (0,l), \\ u(0,x) = \varphi(x),\ x \in [0,l], \\ u_x(t,0) = 0,\ u_x(t,l) = 0,\ t \in I. \end{cases} \tag{10}$$

Theorem 3. Let $\varphi \in W_2^2(0,l)$ and $f(t,x)$ be continuously differentiable on $I \times [0,l]$. Then, for the solution of initial boundary value problem (10) the stability estimates (2) hold.

Proof. One can write problem (10) in the abstract initial value problem (3), where $A = A^x$ is a self adjoint positive definite operator in $H = L_2(0,l)$ which is defined by formula (4) with the domain $D(A) = \left\{u \in W_2^2(0,l) \ \mid u_x(0) = 0,\ u_x(l) = 0\right\}$. So, the proof of Theorem 1 is based on the stability of abstract problem (3) and positiveness and self-adjointness of the abstract operator A defined by (4).

To the operator (4) we assign the difference operator by formula

$$A_h^x \rho^h(x) = -\left(a(x)\rho_{\bar{x}}^h(x)\right)_x + \delta\rho^h(x),$$

acting in the space of grid functions $\rho^h(x) = \{\rho^n(x)\}_0^M$ and satisfying the conditions $\rho^2 = 4\rho^1 - 3\rho^0$, $\rho^{M-2} = 4\rho^{M-1} - 3\rho^M$, where

$$\rho_{\bar{x}}^i = \frac{\rho^i - \rho^{i-1}}{h},\ 1 \le i \le M,\ \rho_x^k = \frac{\rho^{k+1} - \rho^k}{h},\ 0 \le k \le M-1.$$

After discreatization one can construct the following difference schemes

$$\begin{cases} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) + qA_h^x u_{-k}^h(x) = f_k^h(x), \quad f_k^h(x) = f^h(t_k,x), \\ t_k = k\tau,\ k \in Z,\ x \in\ [0,l]_h, \\ u_0^h(x) = \varphi^h(x),\ x \in\ [0,l]_h \end{cases} \tag{11}$$

and

$$\begin{cases} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + \frac{1}{2}\left(A_h^x u_k^h(x) + qA_h^x u_{-k}^h(x)\right) + \frac{1}{2}\left(A_h^x u_{k-1}^h(x) + qA_h^x u_{-k+1}^h(x)\right) = \\ = f_{k+\frac{1}{2}}^h(x) = f^h(t_{k+\frac{1}{2}},x),\ t_{k+\frac{1}{2}} = \left(k+\frac{1}{2}\right)\tau, \\ t_k = k\tau,\ k \in Z,\ x \in\ [0,l]_h, \\ u_0^h(x) = \varphi^h(x), \quad x \in\ [0,l]_h. \end{cases} \tag{12}$$

Theorem 4. Let τ and h be sufficiently small positive numbers. Then, for the solution $\left\{u_k^h(x)\right\}_{-\infty}^{\infty}$of difference schemes (11) and (12) the stability estimates (7) are valid.

Proof. Difference schemes (11) and (12) can be rewritten as the abstract difference schemes (8) and (9), correspondingly. So, the proof of Theorem 2 is based on the stability of the difference schemes (8) and (9) on the positive definiteness and self-adjointness of the operator A^h in the Hilbert space L_{2h}.

3 Numerical implementation

In this section we will consider test examples with the first and second kind boundary conditions.

3.1 Test example for first kind boundary condition

Test example 1. Consider boundary value problem for parabolic equation with involution and Dirichlet condition

$$\begin{cases} u_t(t,x) - ((2+\cos(x))u_x(t,x))_x + \ u(t,x) - ((2+\cos(x))u_x(-t,x))_x + \ u(-t,x) = f(t,x), \\ f(t,x) = \cos t \sin x, \ t \in \ (-\pi,\pi)\,, \ x \in (0,\pi), \\ u(0,x) = 0, \ x \in [0,\pi], \\ u(t,0) = 0, \ u(t,l) = 0, \ t \in [-\pi,\pi]. \end{cases} \tag{13}$$

Here and in future we define sets of grid points as follows

$$[-\pi,\pi]_\tau \times [0, \ \pi]_h = \left\{(t_k,x_i) : t_k = k\tau, \ -N \le k \le N, \ x_i = ih, \ 0 \le k \le M, \ N\tau = \pi, \ Mh = \pi\right\}.$$

By using Taylor decomposition in two points

$$u(t_k) - u(t_{k-1}) = \tau u'(t_k) + o(\tau^2), \tag{14}$$

$$u(t_k) - u(t_{k-1}) = \frac{\tau}{2}u'(t_k) + \frac{\tau}{2}u'(t_{k-1}) + o(\tau^3), \tag{15}$$

$$u''(x_n) = \frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} \ + o(h^2), \tag{16}$$

we present the first order of accuracy difference scheme in t

$$\begin{cases} \frac{u_k^n \ -u_{k-1}^n}{\tau} - (2+\cos(x_n))\frac{u_k^{n+1}-2u_k^n+u_k^{n-1}}{h^2} + \sin(x_n)\frac{u_k^{n+1}-u_k^{n-1}}{2h} + u_k^n - \\ - \ (2+\cos(x_n))\frac{u_{-k}^{n+1}-2u_{-k}^n+u_{-k}^{n-1}}{h^2} + \sin(x_n) \ \frac{u_{-k}^{n+1}-u_{-k}^{n-1}}{2h} + u_{-k}^n = f_k^n, \ f_k^n \ = f \ (t_k,x_n), \\ t_k = k\tau, \ -N+1 \le k \le N, \\ u_k^0 \ = 0, \ u_k^M \ = 0, \ k = 0, \pm1, \pm2, ..., \pm N, \\ u_0^n \ = 0, \ x_n = nh, \ n = 0,1,...,M \end{cases} \tag{17}$$

and the second order of accuracy difference scheme in t

$$\begin{cases} \frac{u_k^n-u_{k-1}^n}{\tau}-\frac{(2+\cos(x_n))}{2}\left(\frac{u_k^{n+1}-2u_k^n+u_k^{n-1}}{h^2}+\frac{u_{-k}^{n+1}-2u_{-k}^n+u_{-k}^{n-1}}{h^2}\right)+\frac{u_k^n+u_{-k}^n}{2}+ \\ +\frac{\sin(x_n)}{2}\left(\frac{u_k^{n+1}-u_k^{n-1}}{2h}+\frac{u_{-k}^{n+1}-u_{-k}^{n-1}}{2h}\right)- \\ -\frac{(2+\cos(x_n))}{2}\left(\frac{u_{k-1}^{n+1}-2u_{k-1}^n+u_k^{n-1}}{h^2}+\frac{u_{-k+1}^{n+1}-2u_{-k+1}^n+u_{-k+1}^{n-1}}{h^2}\right)+ \\ +\frac{\sin(x_n)}{2}\left(\frac{u_{k-1}^{n+1}-u_{k-1}^{n-1}}{2h}+\frac{u_{-k+1}^{n+1}-u_{-k+1}^{n-1}}{2h}\right)+\frac{u_{k-1}^n+u_{-k+1}^n}{2}= \\ = f^h(t_{k+\frac{1}{2}},x),\ t_{k+\frac{1}{2}}=\left(k+\frac{1}{2}\right)\tau,\ t_k=k\tau,\ \ -N+1\le k\le N, \\ u_k^0=0, u_k^M=0,\ k=0,\pm1,\pm2,...,\pm N, \\ u_0^n=0,\ x_n=nh,\ n=0,1,...,M. \end{cases} \tag{18}$$

Later, system of equations (17) and (18) can be rewritten in the matrix form as follows

$$\begin{cases} A_nU_{n-1}+B_nU_n+C_nU_{n+1}=R\varphi_n,\ n=1,..,M-1, \\ U_0=\overrightarrow{O},\ U_M=\overrightarrow{O}. \end{cases} \tag{19}$$

Here R is identity matrix. For solving (19) we apply modified Gauss elimination method by formula

$$U_n=\alpha_nU_{n+1}+\beta_n,\ n=M-1,...,1,0, \tag{20}$$

where α_0 is matrix with zero elements and vector β_0 with zero elements, matrices α_n and vectors β_n are defined recurrently by

$$\begin{aligned} &\alpha_n=(B_n+A_n\alpha_{n-1})^{-1}A_n, \\ &\beta_n=(B_n+A_n\alpha_{n-1})^{-1}(R\varphi_n-A_n\beta_{n-1}), \\ &n=1,...,M-1. \end{aligned} \tag{21}$$

Error is calculated by formula

$$Error(N,M)=\max_{k=0,\pm1,...,\pm N,i=1,...,M}\left|u_k^i-u(t_k,x_i)\right|, \tag{22}$$

where u_k^i and $u(t_k,x_i)$ $(k=0,\pm1,...,\pm N,\ i=1,...,M)$ are values of solution of difference scheme and differential problem at point correspondingly. Table 1 shows that if numbers N and M increase by factor 2 then the values of errors decreases by a factor of approximately $\frac{1}{2}$ for the first order difference scheme (17) and $\frac{1}{4}$ for the second order of accuracy difference scheme (18).

Table 1

Error analysis for test example (13) with Dirichlet condition

$N=M$	1st order difference scheme	2nd order difference scheme
10	0.531	$8{,}35\times10^{-2}$
20	0.351	$2{,}88\times10^{-2}$
40	0.208	$7{,}36\times10^{-3}$
80	0.115	$1{,}92\times10^{-3}$
160	$6{,}09\times10^{-2}$	$5{,}01\times10^{-4}$
320	$3{,}13\times10^{-2}$	$1{,}28\times10^{-4}$

3.2 Test example for second kind boundary condition

Test example 2. Consider boundary problem for involutary parabolic equation with Neumann condition

$$\begin{cases} u_t(t,x)-((2+\cos(x))u_x(t,x))_x+\ u(t,x)-((2+\cos(x))u_x(-t,x))_x+\ u(-t,x)=f(t,x), \\ f(t,x)=\cos t\cos x, t\in\ (-\pi,\pi)\,,\ x\in(0,\pi), \\ u(0,x)=0,\ x\in[0,\pi]\,, \\ u_x(t,0)=0,\ u_x(t,l)=0,\ t\in[-\pi,\pi]\,. \end{cases} \tag{23}$$

By using (14), (15), (16) and

$$u'(0)=\frac{-u(x_2)+4u(x_1)-3u(x_0)}{h^2}\ +o(h^2),$$

$$u'(\pi)=\frac{u(x_{M-2})-4u(x_{M-1})+3u(x_M)}{h^2}\ +o(h^2),$$

one can get the first order of accuracy difference scheme in t

$$\begin{cases} \frac{u_k^n\ -u_{k-1}^n}{\tau}-(2+\cos(x_n))\frac{u_k^{n+1}-2u_k^n+u_k^{n-1}}{h^2}+\sin(x_n)\frac{u_k^{n+1}-u_k^{n-1}}{2h}+u_k^n- \\ -(2+\cos(x_n))\frac{u_{-k}^{n+1}-2u_{-k}^n+u_{-k}^{n-1}}{h^2}+\sin(x_n)\ \frac{u_{-k}^{n+1}-u_{-k}^{n-1}}{2h}+u_{-k}^n=f_k^n, \\ f_k^n\ =f\ (t_k,x_n),\ t_k=k\tau,\ -N+1\le k\le N, \\ u_k^0\ =u_k^1\ ,\ u_k^M\ =u_k^{M-1},\ k=0,\pm1,\pm2,...,\pm N, \\ u_0^n\ =0,\ x_n=nh,\ n=0,1,...,M \end{cases} \tag{24}$$

and the second order of accuracy difference scheme in t

$$\begin{cases} \frac{u_k^n - u_{k-1}^n}{\tau} - \frac{(2+\cos(x_n))}{2}\left(\frac{u_k^{n+1}-2u_k^n+u_k^{n-1}}{h^2} + \frac{u_{-k}^{n+1}-2u_{-k}^n+u_{-k}^{n-1}}{h^2}\right)u_k^n + \\ +\frac{\sin(x_n)}{2}\left(\frac{u_k^{n+1}-u_k^{n-1}}{2h} + \frac{u_{-k}^{n+1}-u_{-k}^{n-1}}{2h}\right) + \frac{u_k^n+u_{-k}^n}{2} + \\ +\frac{(2+\cos(x_n))}{2}\left(\frac{u_{k-1}^{n+1}-2u_{k-1}^n+u_k^{n-1}}{h^2} + \frac{u_{-k+1}^{n+1}-2u_{-k+1}^n+u_{-k+1}^{n-1}}{h^2}\right) + \\ +\frac{(2-\sin(x_n))}{2}\left(\frac{u_{k-1}^{n+1}-u_{k-1}^{n-1}}{2h} + \frac{u_{-k+1}^{n+1}-u_{-k+1}^{n-1}}{2h}\right) + \frac{u_{k-1}^n+u_{-k+1}^n}{2} = \\ = f^h(t_{k+\frac{1}{2}}, x),\ t_{k+\frac{1}{2}} = \left(k+\frac{1}{2}\right)\tau,\ t_k = k\tau,\ -N+1 \le k \le N, \\ 3u_k^0 = 4u_k^1 - u_k^2, 3u_k^M = 4u_k^{M-1} - u_k^{M-2},\ k = 0, \pm 1, \pm 2, ..., \pm N, \\ u_0^n = 0,\ x_n = nh,\ n = 0, 1, ..., M. \end{cases} \tag{25}$$

System of equations (24) and (25) can be written in the following forms

$$\begin{cases} A_n U_{n-1} + B_n U_n + C_n U_{n+1} = R\varphi_n,\ n = 1, .., M-1, \\ U_0 = U_1,\ U_M = U_{M-1}, \end{cases} \tag{26}$$

and

$$\begin{cases} A_n U_{n-1} + B_n U_n + C_n U_{n+1} = R\varphi_n,\ n = 1, .., M-1, \\ 3U_0 = 4U_1 - U_2,\ 3U_M = 4U_{M-1} - U_{M-2}, \end{cases} \tag{27}$$

correspondingly. For solving (26) we use formula (20), where $\alpha_0 = R$ is identity matrix and vector β_0 has only zero elements, matrices α_n and vectors β_n are defined by (21). Errors are computed by formula (22). Let us move to (27). We seek solution (27) in the form (see [18, 19])

$$U_n = \alpha_n U_{n+1} + \beta_n U_{n+2} + \gamma_n,\ n = M-2, M-1, ..., 1, 0.$$

Here auxiliary matrices α_n, β_n and vector γ_n are calculated by formulas

$$\alpha_n = -D_n(A_n + C_n\beta_{n-1}),\ \beta_n = O,$$

$$\gamma_n = D_n\ (R\varphi_n - C_n\ \gamma_{n-1}),$$

$$D_n = (B_n + C_n\ \alpha_{n-1})^{-1},\ n = 0, ..., M-2,$$

$$\alpha_0 = \frac{4}{3}R,\ \beta_0 = -\frac{1}{3}R,\ \alpha_1 = \frac{8}{5}R,\ \beta_1 = -\frac{3}{5}R,\ \gamma_0 = \gamma_1 = \overrightarrow{0}.$$

At the same time formulas for unknown U_M and U_{M-1} are given in [19].

Table 2 shows that if numbers N and M increase by factor 2 then the values of errors decrease by a factor of approximately $\frac{1}{2}$ for difference scheme (24) and $\frac{1}{4}$ for difference scheme (25).

Table 2

Error analysis for test example (23) with Neuman condition

$N = M$	1st order difference scheme	2nd order difference scheme
10	0.619	$8{,}41 \times 10^{-2}$
20	0.414	$2{,}91 \times 10^{-2}$
40	0.253	$7{,}41 \times 10^{-3}$
80	0.144	$1{,}95 \times 10^{-3}$
160	$7{,}77 \times 10^{-2}$	$5{,}03 \times 10^{-4}$
320	$4{,}05 \times 10^{-2}$	$1{,}30 \times 10^{-4}$

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Инволюциялы параболалық теңдеу үшiн шеттiк есептердiң сандық шешiмi

Мақалада бiрiншi және екiншi типтi шарттары бар эволюциялық параболалық теңдеудiң екi шеттiк есептерiн зерттелген. Осы шеттiк есептердi сандық түрде шешу үшiн абсолюттi тұрақты айырымдық схемалары ұсынылған. Айырымдық схемаларының шешiмдерiнiң тұрақтылығын бағалау iс жүзiнде дәлелдендi. Екi айырымдық схемасының сандық шешiмiнiң қателiктерiн одан әрi талдау сынақ мысалдарымен келтiрiлген.

Кiлт сөздер: инволюция, параболалық, ақырлы-айырымдық схемасы, тұрақтылықты бағалау, шеттiк есеп.

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Численное решение краевых задач для параболического уравнения с инволюцией

В статье исследованы две краевые задачи для эвольвентного параболического уравнения с условиями первого и второго рода. Предложены абсолютно устойчивые разностные схемы для численного решения этих краевых задач. Фактически доказаны оценки устойчивости решений разностных схем. Дальнейший анализ погрешностей численного решения обеих разностных схем проиллюстрирован тестовыми примерами.

Ключевые слова: инволюция, парабола, конечно-разностная схема, оценка устойчивости, краевая задача.

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Solution of heat equation by a novel implicit scheme using block hybrid preconditioning of the conjugate gradient method

The main goal of the study is the approximation of the solution to the Dirichlet boundary value problem (DBVP) of the heat equation on a rectangle by developing a new difference method on a grid system of hexagons. It is proved that the given special scheme is unconditionally stable and converges to the exact solution on the grids with fourth order accuracy in space variables and second order accuracy in time variable. Secondly, an incomplete block factorization is given for symmetric positive definite block tridiagonal (SPD-BT) matrices utilizing a conservative iterative method that approximates the inverse of the pivoting diagonal blocks by preserving the symmetric positive definite property. Subsequently, by using this factorization block hybrid preconditioning of the conjugate gradient (BHP-CG) method is applied to solve the obtained algebraic system of equations at each time level.

Keywords: Heat equation, implicit scheme, hexagonal grid, stability analysis, symmetric positive definite matrix, approximate inverse, incomplete block factorization, block hybrid preconditioning, conjugate gradient method.

Introduction

For many mathematical models, especially partial differential equations (PDEs), their analytical solutions are not available. Therefore, for computing the approximate solutions economical and stable numerical algorithms based on effective theoretical results are getting more important as more advanced computers are designed.

Among some numerical methods for approximating the solutions of PDEs, the finite difference method is a widely used approach and the construction of stable and time efficient schemes are essential. Recent advances in finite difference methods for solving PDEs include [1–7].

More then a half century ago, in 1967, the approximation of the pure diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2},$$

on regular hexagonal grids was analyzed by giving two implicit difference schemes, defined on three layers with 21-point and on two layers with 14-point both with fourth order accuracy in space and second order accuracy in time [8].

Since then, the applicability of the hexagonal grids in many branches of science has been investigated. Among them is the research on eligibility of the icosahedral-hexagonal grids in meteorological applications. Finite difference schemes on a spherical geodesic grid were given to integrate the barotropic vorticity equation [9, 10]. Further, the hexagonal grid was extended to the integration of the primitive equations of fluid dynamics [11–13]. Later, an integration scheme of the primitive equation model by using on icosahedral-hexagonal grid system with an application to the shallow water equation was given [14]. Additionally, for the simulations of oscillations in shallow circular basins, finite difference techniques

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on the irregular grids were analyzed [15]. Furthermore, hexagonal grids were used for the simulation of atmospheric processes [16].

Nowadays, the investigation of triangular and hexagonal system of grids has gained more interest in engineering, applied sciences, computer science, natural sciences and in environmental sciences. Such as the numerical solution of boundary value problems of PDEs using finite difference method in convection diffusion equation [17], in the Laplace equation [18], and in the heat equation [5], and derivatives of the solution to the heat equation [6, 7]. Additionally, hexagonal grids were also used in finite volume method [19]. For digital image processing and graph processing, some examples include [20] where digitized rotations of 12 neighbors on the triangular grid were given by considering more general setting especially the midpoint, the corner points and the edge midpoints as rotation centers. Also, in [21] the bijectivity of the digitized rotations for the closest neighbors in rectangular, triangular and hexagonal grids were compared. In addition, the firefighter problem, which is an iterative graph process, was studied on hexagonal grids in [22]. For hydrologic modelling, we mention the study by [23] in which a watershed delineation model using the hexagonal grid spatial discretization method was developed.

The contributions of this work can be summarised as: the DBVP of the heat equation

$$\frac{\partial u}{\partial t} = \omega \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right) - bu + f\left(x_1, x_2, t\right), \tag{1}$$

given on a rectangle D where $\omega > 0$, $b \geq 0$ are constants is considered. A new difference method of order of convergence $O\left(h^4 + \tau^2\right)$ with 14-point on two layers constructed on hexagonal grids is proposed. Here, the increments in the variables x_1 and x_2 are denoted by h and $\frac{\sqrt{3}}{2}h$ accordingly and τ denotes the increment in time. Further, the unconditional stability of the given scheme is shown. Furthermore, for SPD-BT matrices an incomplete block matrix factorization algorithm is developed. At each stage of the recursion for approximating the pivoting diagonal block matrix inverses, the constructed algorithm uses a two step iterative method with very high rate of convergence (order 33 see [24]). It is proven that at each iteration the pivoting diagonal block matrix and its approximate inverse are symmetric positive definite (SPD) matrices. Subsequently this factorization and the pivoting block approximate inverses are used to precondition the conjugate gradient method [25], which we call block hybrid preconditioning of the conjugate gradient (BHP-CG) method.

1 *DBVP of the heat equation and discretization*

We take the rectangle $D = \{x = (x_1, x_2) : 0 < x_1 < a_1, 0 < x_2 < a_2\}$. We denote its sides by υ_j, $j = 1, 2, 3, 4$ and its boundary by $S = \bigcup_{j=1}^{4} \upsilon_j$, so that $\overline{D} = D \cup S$ is the closure of D. Let $Q_T = D \times (0, T)$, and indicate the lateral surface by $S_T = \{(x, t), x \in S, t \in [0, T]\}$ and the closure of Q_T by $\overline{Q}_T$. We consider the DBVP of heat equation in (1)

$$\frac{\partial u}{\partial t} = \omega \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right) - bu + f\left(x_1, x_2, t\right) \text{ on } Q_T, \tag{2}$$

$$u\left(x_1, x_2, 0\right) = \varphi\left(x_1, x_2\right) \text{ on } \overline{D}, \tag{3}$$

$$u\left(x_1, x_2, t\right) = \phi\left(x_1, x_2, t\right) \text{ on } S_T, \tag{4}$$

where $\omega > 0$ and $b \geq 0$ are constant. In this study, further investigations are given with the assumption that DBVP in (2)–(4) has the unique solution u from the Hölder space $C_{x,t}^{6+\alpha, 3+\frac{\alpha}{2}}\left(\overline{Q}_T\right), 0 < \alpha < 1$.

1.1 Implicit scheme on rectangular grids

First we consider the classical rectangular grid approximation of the problem (2)–(4) when the value of the constant $b = 0$ in Equation (2). We take the step sizes $h_1 = \frac{a_1}{M_1}$ and $h_2 = \frac{a_2}{M_2}$ where, M_1 and M_2 are positive integers. Further, the set of rectangular grids on D is defined as

$$D^{h_1,h_2} = \{x = (x_1, x_2) \in D : x_i = l_i h_i,\ l_i = 1, 2, ...M_i - 1, i = 1, 2\}.$$

Let S^{h_1,h_2} be the set of rectangular grid points on S and $\overline{D^{h_1,h_2}} = D^{h_1,h_2} \cup S^{h_1,h_2}$. Further let,

$$\begin{aligned} \gamma_\tau &= \left\{ t_k = k\tau,\ \tau = \frac{T}{M'},\ k = 1, ..., M' \right\}, \\ \overline{\gamma}_\tau &= \left\{ t_k = k\tau,\ \tau = \frac{T}{M'},\ k = 0, ..., M' \right\}. \end{aligned}$$

Also

$$\begin{aligned} D^{h_1,h_2}\gamma_\tau &= D^{h_1,h_2} \times \gamma_\tau = \left\{ (x, t) : x \in D^{h_1,h_2},\ t \in \gamma_\tau \right\}, \\ S_T^{h_1,h_2} &= S^{h_1,h_2} \times \overline{\gamma}_\tau = \left\{ (x, t) : x \in S^{h_1,h_2},\ t \in \overline{\gamma}_\tau \right\}. \end{aligned}$$

The following unconditionally stable 14-point implicit method on rectangular grids is considered [26].

Rectangular Difference Problem (RDP)

$$\begin{aligned} \Gamma u_{h,\tau} &= \omega\sigma_1\Lambda_1 u_{h,\tau}^{k+1} + \omega(1-\sigma_1)\Lambda_1 u_{h,\tau}^{k} + \omega\sigma_2\Lambda_2 u_{h,\tau}^{k+1} + \omega(1-\sigma_2)\Lambda_2 u_{h,\tau}^{k} \\ &\quad + \omega\frac{h_1^2 + h_2^2}{12}\Lambda_1\Lambda_2 u_{h,\tau}^{k} + \beta \text{ on } D^{h_1,h_2}\gamma_\tau, \qquad (5) \\ u_{h,\tau} &= \varphi(x_1, x_2),\ t = 0 \text{ on } \overline{D^{h_1,h_2}}, \qquad (6) \\ u_{h,\tau} &= \phi(x_1, x_2, t) \text{ on } S_T^{h_1,h_2}, \qquad (7) \end{aligned}$$

where

$$\begin{aligned} \sigma_1 &= \frac{1}{2} - \frac{h_1^2}{12\tau},\ \sigma_2 = \frac{1}{2} - \frac{h_2^2}{12\tau}, \\ \Gamma u &= \frac{u(x_1, x_2, t+\tau) - u(x_1, x_2, t)}{\tau}, \\ \Lambda_1 u^k &= \left[u(x_1 + h_1, x_2, t) - 2u(x_1, x_2, t) + u(x_1 - h_1, x_2, t)\right]/h_1^2, \\ \Lambda_2 u^k &= \left[u(x_1, x_2 + h_2, t) - 2u(x_1, x_2, t) + u(x_1, x_2 - h_2, t)\right]/h_2^2, \\ \beta &= f_{P_0}^{k+\frac{1}{2}} + \frac{h_1^2}{12}\Lambda_1 f_{P_0}^{k+\frac{1}{2}} + \frac{h_2^2}{12}\Lambda_2 f_{P_0}^{k+\frac{1}{2}}, \end{aligned}$$

and $f_{P_0}^{k+\frac{1}{2}} = f(x_1, x_2, t + \frac{\tau}{2})$. The scheme has the order of accuracy $O\left(\left|\hat{h}\right|^4 + \tau^2\right)$. Here, $\left|\hat{h}\right| = \sqrt{h_1^2 + h_2^2}$ and we denote the system (5)–(7) by

$$\widetilde{K}_1 U^{k+1} = \widetilde{K}_2 U^k + \tau \widetilde{F}^{k^*}, \qquad (8)$$

where $\widetilde{K}_1, \widetilde{K}_2$ are real block tridiagonal matrices with 5 nonzero and 9 nonzero diagonals, respectively. The vector $\widetilde{F}^{k^*}$ is computed from the initial and boundary function values and the heat source function f.

1.2 Novel implicit scheme on hexagonal grids

Let N_1 be a positive integer and $h = a_1/N_1 > 0$. For the ease of explanation of the new scheme we assume that a_2 is multiple of $\sqrt{3}$. Using the step size h we assign a hexagonal grid on D and denote this set by D^h as

$$D^h = \Big\{x = (x_1, x_2) \in D : x_1 = \frac{p-q}{2}h,\ x_2 = \frac{\sqrt{3}(p+q)}{2}h,$$
$$p = 1, 2, ...,\ q = 0 \pm 1 \pm 2, ...\Big\}.$$

Further $\overline{D^h}$ is the closure of D^h. In addition, P_0 is the center and $P_i, i = 1, ..., 6$ are the neighboring points in the pattern $Patt\,(P_0)$ of the hexagon. The set of interior nodes are categorized as regular and irregular hexagons. Those hexagons with $Patt\,(P_0) \in \overline{D}^h$ are called regular and those with a center P_0 that lies $\frac{h}{2}$ units away from the boundary are called irregular hexagons. The set of irregular hexagons with a left ghost point are denoted by D^{*lh} and those with a right ghost point are presented by D^{*rh}. Also, $D^{*h} = D^{*lh} \cup D^{*rh}$ and $D^{0h} = D^h \backslash D^{*h}$. Table 1 presents the function values of u, f and the second order pure derivatives of f. In this table, if $P_0 \in D^{*lh}\gamma_\tau$ then the value of $\widehat{s} = 0$ and if $P_0 \in D^{*rh}\gamma_\tau$ then $\widehat{s} = a_1$. Besides $k + l, l = 0, \frac{1}{2}, 1$ denote the time levels $t = (k + l)\,\tau$ for $k = 0, 1, ..., M' - 1$. Furthermore, the numerical solution on hexagonal grid system is presented by u^{k+1}_{h,τ,P_i}, $i = 0, ..., 6$, and at boundary points by u^{k+1}_{h,τ,P_A}, when $t = (k + 1)\,\tau$, for $k = 0, 1, ..., M' - 1$. Figure 1 illustrates the irregular hexagons and the exact solution at the center and the neighbouring points of the pattern at $t = k\tau$ and $(k + 1)\,\tau$ time levels.

Table 1

Notations used to denote the function values.

$u^{k+1}_{P_0} = u(x_1, x_2, t + \tau)$	$u^{k+1}_{P_A} = u(\widehat{s}, x_2, t + \tau)$
$u^{k+1}_{P_1} = u(x_1 - \frac{h}{2}, x_2 + \frac{\sqrt{3}}{2}h, t + \tau),$	$f^{k+\frac{1}{2}}_{P_0} = f(x_1, x_2, t + \frac{\tau}{2})$
$u^{k+1}_{P_2} = u(x_1 - h, x_2, t + \tau)$	$f^{k+1}_{P_A} = f(\widehat{s}, x_2, t + \tau)$
$u^{k+1}_{P_3} = u(x_1 - \frac{h}{2}, x_2 - \frac{\sqrt{3}}{2}h, t + \tau)$	$f^{k+\frac{1}{2}}_{P_A} = f(\widehat{s}, x_2, t + \frac{\tau}{2})$
$u^{k+1}_{P_4} = u(x_1 + \frac{h}{2}, x_2 - \frac{\sqrt{3}}{2}h, t + \tau)$	$f^{k}_{P_A} = f(\widehat{s}, x_2, t)$
$u^{k+1}_{P_5} = u(x_1 + h, x_2, t + \tau)$	$\partial^2_{x_1} f^{k+\frac{1}{2}}_{P_0} = \frac{\partial^2 f}{\partial x_1^2}\Big\vert_{(x_1, x_2, t+\frac{\tau}{2})}$
$u^{k+1}_{P_6} = u(x_1 + \frac{h}{2}, x_2 + \frac{\sqrt{3}}{2}h, t + \tau)$	$\partial^2_{x_2} f^{k+\frac{1}{2}}_{P_0} = \frac{\partial^2 f}{\partial x_2^2}\Big\vert_{(x_1, x_2, t+\frac{\tau}{2})}$

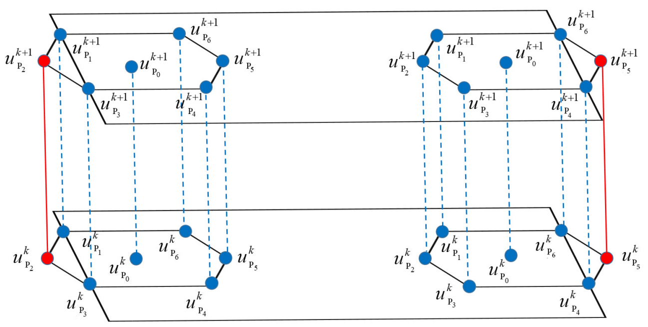

Figure 1. The illustration of the irregular hexagons and the solution for two time echelons.

Also on the hexagon system of grids we present the set of hexagonal grids on S by S^h and the sets

$$D^h\gamma_\tau = D^h \times \gamma_\tau = \left\{(x, t) : x \in D^h,\ t \in \gamma_\tau\right\},$$
$$S^h_T = S^h \times \overline{\gamma}_\tau = \left\{(x, t) : x \in S^h,\ t \in \overline{\gamma}_\tau\right\},$$

present interior, and lateral surface nodes respectively. Let $D^{*lh}\gamma_\tau = D^{*lh} \times \gamma_\tau \subset D^h\gamma_\tau$ and $D^{*rh}\gamma_\tau = D^{*rh} \times \gamma_\tau \subset D^h\gamma_\tau$ and $D^{*h}\gamma_\tau = D^{*lh}\gamma_\tau \cup D^{*rh}\gamma_\tau$, also $D^{0h}\gamma_\tau = D^h\gamma_\tau \backslash D^{*h}\gamma_\tau$. Figure 2 shows the hexagonal grid covering of the rectangle D for three time levels $t-\tau, t$ and $t+\tau$, on which the ghost points are denoted by red colour.

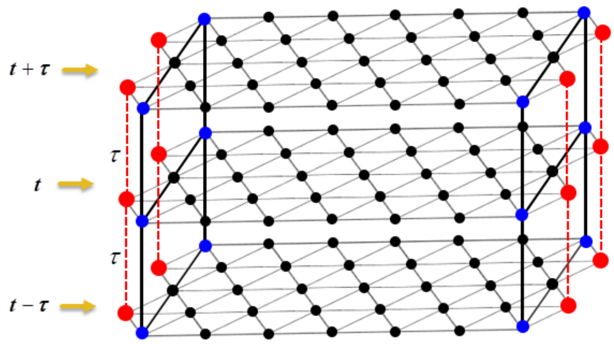

Figure 2. Hexagonal grid covering of the rectangle D for three time echelons $t-\tau, t$ and $t+\tau$.

We propose the next difference problem on hexagon system of grids to approximate the solution of the DBVP in (2)–(4).

Hexagonal Difference Problem (HDP)

$$\begin{aligned} \Theta^1_{h,\tau} u^{k+1}_{h,\tau} &= \Lambda^1_{h,\tau} u^k_{h,\tau} + \psi^1 \text{ on } D^{0h}\gamma_\tau, && (9)\\ \Theta^2_{h,\tau} u^{k+1}_{h,\tau} &= \Lambda^2_{h,\tau} u^k_{h,\tau} + E_{h,\tau}\phi + \psi^2 \text{ on } D^{*h}\gamma_\tau, && (10)\\ u_{h,\tau} &= \varphi(x_1, x_2) \text{ on } t=0,\ \overline{D^h}, && (11)\\ u_{h,\tau} &= \phi(x_1,x_2,t) \text{ on } S^h_T, && (12) \end{aligned}$$

$k = 1, 2, ..., M'-1$, where

$$\psi^1 = f^{k+\frac{1}{2}}_{P_0} + \frac{1}{16}h^2\left(\partial^2_{x_1} f^{k+\frac{1}{2}}_{P_0} + \partial^2_{x_2} f^{k+\frac{1}{2}}_{P_0}\right), \tag{13}$$

$$\begin{aligned} \psi^2 &= \frac{h^2}{96\tau\omega}\left(f^{k+1}_{P_A} - f^k_{P_A}\right) - \left(\frac{1}{6} - \frac{h^2 b}{96\omega}\right) f^{k+\frac{1}{2}}_{P_A} + f^{k+\frac{1}{2}}_{P_0} \\ &+ \frac{1}{16}h^2\left(\partial^2_{x_1} f^{k+\frac{1}{2}}_{P_0} + \partial^2_{x_2} f^{k+\frac{1}{2}}_{P_0}\right), \end{aligned} \tag{14}$$

$$\Theta^1_{h,\tau} u^{k+1} = \left(\frac{3}{4\tau} + \frac{2\omega}{h^2} + \frac{3}{8}b\right) u^{k+1}_{P_0} + \left(\frac{1}{24\tau} - \frac{\omega}{3h^2} + \frac{b}{48}\right)\sum_{i=1}^{6} u^{k+1}_{P_i}, \tag{15}$$

$$\Lambda^1_{h,\tau} u^k = \left(\frac{3}{4\tau} - \frac{2\omega}{h^2} - \frac{3}{8}b\right) u^k_{P_0} + \left(\frac{1}{24\tau} + \frac{\omega}{3h^2} - \frac{b}{48}\right)\sum_{i=1}^{6} u^k_{P_i}, \tag{16}$$

$$\begin{aligned} \Theta^2_{h,\tau} u^{k+1} &= \left(\frac{17}{24\tau} + \frac{7\omega}{3h^2} + \frac{17}{48}b\right) u^{k+1}_{P_0} + \left(\frac{1}{24\tau} - \frac{\omega}{3h^2} + \frac{b}{48}\right)\Big(u(s+\eta, x_2, t+\tau) \\ &+ u(s, x_2 + \frac{\sqrt{3}}{2}h, t+\tau) + u(s, x_2 - \frac{\sqrt{3}}{2}h, t+\tau)\Big), \end{aligned}$$

$$\begin{aligned}
E_{h,\tau}\phi &= \left(-\frac{1}{36\tau}+\frac{2\omega}{9h^2}-\frac{b}{72}\right)\left(\phi(\widehat{s},x_2+\frac{\sqrt{3}}{2}h,t+\tau)+\phi(\widehat{s},x_2-\frac{\sqrt{3}}{2}h,t+\tau)\right)\\
&+\left(\frac{1}{36\tau}+\frac{2\omega}{9h^2}-\frac{b}{72}\right)\left(\phi(\widehat{s},x_2+\frac{\sqrt{3}}{2}h,t)+\phi(\widehat{s},x_2-\frac{\sqrt{3}}{2}h,t)\right)\\
&+\left(\frac{1}{18\tau}+\frac{8\omega}{9h^2}-\frac{h^2b}{48\omega\tau}+\frac{b}{36}-\frac{h^2b^2}{192\omega}\right)\phi(\widehat{s},x_2,t+\tau)\\
&-\left(\frac{1}{18\tau}-\frac{8\omega}{9h^2}-\frac{h^2b}{48\omega\tau}-\frac{b}{36}+\frac{h^2b^2}{192\omega}\right)\phi(\widehat{s},x_2,t),
\end{aligned}$$

$$\begin{aligned}
\Lambda^2_{h,\tau}u^k &= \left(\frac{17}{24\tau}-\frac{7\omega}{3h^2}-\frac{17}{48}b\right)u^k_{P_0}+\left(\frac{1}{24\tau}+\frac{\omega}{3h^2}-\frac{b}{48}\right)\left(u(s,x_2+\frac{\sqrt{3}}{2}h,t)\right.\\
&\left.+u(s,x_2-\frac{\sqrt{3}}{2}h,t)+u(s+\eta,x_2,t)\right),
\end{aligned}$$

and

$$\text{if } P_0\in D^{*lh}\gamma_\tau,\ \text{then } s=h,\ \widehat{s}=0,\eta=\tfrac{h}{2}.$$
$$\text{if } P_0\in D^{*rh}\gamma_\tau,\ \text{then } s=a_1-h,\widehat{s}=a_1,\eta=-\tfrac{h}{2}.$$

2 Analysis of HDP (9)-(12)

First we analyze the approximation order of the special scheme in HDP (9)–(12).

Theorem 1. The scheme HDP (9)–(12) has the approximation order $O\left(h^4+\tau^2\right)$.

Proof. Let $(x_1,x_2,t+\tau)$ and $(x_1,x_2,t)\in D^h\gamma_\tau$ be the centers (P_0) of the hexagons at time moment $(k+1)\tau$ and $k\tau$ respectively for $k=0,...,M'-1$. From Equation (9) and using (13), (15) and (16) for regular hexagonal grids the scheme is

$$\begin{aligned}
&\frac{3}{4}\frac{u^{k+1}_{h,\tau,P_0}-u^k_{h,\tau,P_0}}{\tau}+\frac{1}{24}\sum_{i=1}^{6}\frac{u^{k+1}_{h,\tau,P_i}-u^k_{h,\tau,P_i}}{\tau}\\
=\;&\frac{\omega}{3h^2}\left(\sum_{i=1}^{6}u^{k+1}_{h,\tau,P_i}-6u^{k+1}_{h,\tau,P_0}\right)+\frac{\omega}{3h^2}\left(\sum_{i=1}^{6}u^{k}_{h,\tau,P_i}-6u^{k}_{h,\tau,P_0}\right)\\
&-\frac{b}{48}\sum_{i=1}^{6}u^{k+1}_{h,\tau,P_i}-\frac{3}{8}bu^{k+1}_{h,\tau,P_0}-\frac{b}{48}\sum_{i=1}^{6}u^{k}_{h,\tau,P_i}-\frac{3}{8}bu^{k}_{h,\tau,P_0}\\
&+f^{k+\frac{1}{2}}_{P_0}+\frac{1}{16}h^2\left(\partial^2_{x_1}f^{k+\frac{1}{2}}_{P_0}+\partial^2_{x_2}f^{k+\frac{1}{2}}_{P_0}\right).
\end{aligned}\tag{17}$$

For the irregular hexagons the following approximations are used for $i=2,5$

$$\begin{aligned}
u^{k+1}_{h,\tau,P_i}+u^k_{h,\tau,P_i} &= \frac{h^2}{2\tau\omega}u^{k+1}_{h,\tau,P_A}+\frac{8}{3}u^{k+1}_{h,\tau,P_A}-u^{k+1}_{h,\tau,P_0}-\frac{1}{3}u^{k+1}_{h,\tau,P_{i-1}}\\
&-\frac{1}{3}u^{k+1}_{h,\tau,P_{i+1}}-\frac{h^2}{2\tau\omega}u^{k}_{h,\tau,P_A}+\frac{8}{3}u^{k}_{h,\tau,P_A}-u^{k}_{h,\tau,P_0}\\
&-\frac{1}{3}u^{k}_{h,\tau,P_{i-1}}-\frac{1}{3}u^{k}_{h,\tau,P_{i+1}}+\frac{h^2b}{4\omega}\left(u^{k+1}_{h,\tau,P_A}+u^{k}_{h,\tau,P_A}\right)\\
&-\frac{h^2}{2\omega}f^{k+\frac{1}{2}}_{P_A}+O\left(h^4+h^2\tau^2\right).
\end{aligned}\tag{18}$$

$$\begin{aligned} u^{k+1}_{h,\tau,P_i} - u^{k}_{h,\tau,P_i} &= -u^{k+1}_{h,\tau,P_0} - \frac{1}{3}u^{k+1}_{h,\tau,P_{i+1}} - \frac{1}{3}u^{k+1}_{h,\tau,P_{i-1}} \\ &\quad + \frac{8}{3}u^{k+1}_{h,\tau,P_A} + u^{k}_{h,\tau,P_0} + \frac{1}{3}u^{k}_{h,\tau,P_{i-1}} + \frac{1}{3}u^{k}_{h,\tau,P_{i+1}} \\ &\quad - \frac{8}{3}u^{k}_{h,\tau,P_A} + \frac{h^2 b}{4\omega}\left(u^{k+1}_{h,\tau,P_A} - u^{k}_{h,\tau,P_A}\right) \\ &\quad - \frac{h^2}{4\omega}\left(f^{k+1}_{P_A} - f^{k}_{P_A}\right) + O\left(h^4 + h^2\tau\right). \end{aligned} \tag{19}$$

Hence, the scheme (10) is obtained by substituting (18) and (19) in (17). Consequently, the error function $\varepsilon_{h,\tau} = u_{h,\tau} - u$ satisfies the next difference problem

$$\begin{aligned} \Theta^1_{h,\tau}\varepsilon^{k+1}_{h,\tau} &= \Lambda^1_{h,\tau}\varepsilon^{k}_{h,\tau} + \Psi^k_1 \text{ on } D^{0h}\gamma_\tau, & (20)\\ \Theta^2_{h,\tau}\varepsilon^{k+1}_{h,\tau} &= \Lambda^2_{h,\tau}\varepsilon^{k}_{h,\tau} + \Psi^k_2 \text{ on } D^{*h}\gamma_\tau, & (21)\\ \varepsilon_{h,\tau} &= 0 \text{ on } t = 0,\ \overline{D^h}, & (22)\\ \varepsilon_{h,\tau} &= 0 \text{ on } S^h_T, & (23) \end{aligned}$$

where

$$\begin{aligned} \Psi^k_1 &= \Lambda^1_{h,\tau}u^k - \Theta^1_{h,\tau}u^{k+1} + \psi^1, & (24)\\ \Psi^k_2 &= \Lambda^2_{h,\tau}u^k - \Theta^2_{h,\tau}u^{k+1} + E_{h,\tau}\phi + \psi^2, & (25) \end{aligned}$$

and ψ^1, ψ^2 are as given in (13), (14) respectively. Using Taylor's expansion around the point $\left(x_1, x_2, t + \frac{\tau}{2}\right)$ we obtain $\Psi^k_1 = O\left(h^4 + \tau^2\right)$ and $\Psi^k_2 = O\left(h^4 + \tau^2\right)$.

Next, we analyze the stability for the special scheme in HDP. At every time stage using standard ordering the hexagon points in $D^h\gamma_\tau$ are labeled as E_j, $j = 1, 2, ..., N$. Thus, all hexagon centers have the neighboring topology denoted by the following set

$$S_E = \left\{(i,j) : \text{if the grid } E_i \in Patt\left(E_j\right),\ i \neq j, 1 \leq i,j \leq N\right\}, \tag{26}$$

exhibiting the sparsity structure of $Inc \in R^{N\times N}$ called the incidence matrix with entries

$$[Inc]_{ij} = \begin{cases} 0 \text{ if } (i,j) \notin S_E, \\ 1 \text{ if } (i,j) \in S_E. \end{cases}$$

Further, the scheme in HDP can be put in the subsequent matrix form

$$K_1 U^{k+1} = K_2 U^k + \tau F^{k^*}, \tag{27}$$

where, $K_1, K_2 \in R^{N\times N}$ are given as

$$\begin{aligned} K_1 &= \left(S_1 + \frac{\omega\tau}{h^2}S_2\right),\ K_2 = \left(S_1 - \frac{\omega\tau}{h^2}S_2\right), & (28)\\ S_1 &= D_1 + \frac{1}{24}Inc,\ S_2 = B + \frac{bh^2}{\omega}C, & (29)\\ B &= D_2 - \frac{1}{3}Inc,\ C = D_3 + \frac{1}{48}Inc. & (30) \end{aligned}$$

Also the computed values of f in (13), (14) and the values of φ and ϕ in HDP (9)–(12) are presented by the vector $F^{k^*} \in R^N$. Further, D_1, D_2, D_3 are diagonal matrices with entries

$$[D_1]_{jj} = \begin{cases} \frac{3}{4} \text{ if } E_j \in D^{0h}\gamma_\tau \\ \frac{17}{24} \text{ if } E_j \in D^{*h}\gamma_\tau \end{cases},\ j = 1, 2, ..., N,$$

$$[D_2]_{jj} = \begin{cases} 2 \text{ if } E_j \in D^{0h}\gamma_\tau \\ \frac{7}{3} \text{ if } E_j \in D^{*h}\gamma_\tau \end{cases}, \; j = 1, 2, ..., N,$$

$$[D_3]_{jj} = \begin{cases} \frac{3}{8} \text{ if } E_j \in D^{0h}\gamma_\tau \\ \frac{17}{48} \text{ if } E_j \in D^{*h}\gamma_\tau \end{cases}, \; j = 1, 2, ..., N,$$

accordingly. The stiffness matrix K_1 at the $(k+1)$th time level and the coefficient matrix K_2 at the kth time level both have 7 nonzero diagonals. Next we analyze the properties of the derived matrices.

Lemma 1. a) S_1 in (29) and the matrices B and C in (30) are SPD matrices. b) K_1 in (28) and S_2 in (29) are SPD matrices.

Proof. a) Using (26) if $E_i \in Patt\,(E_j)$ for $i \neq j$, $1 \leq i, j, \leq N$ this implies that $E_j \in Patt\,(E_i)$ giving $Inc^T = Inc$. Thus, S_1, B, and C are real symmetric matrices hence the eigenvalues of S_1, B, and C are real. Hexagonal grid is connected grid in the rectangle D thus, by using (30) it can be easily shown that the matrix B has positive diagonal entries, *i.e.* $b_{ii} > 0$, $i = 1..., N$ and it is irreducibly diagonally dominant matrix. Further, the matrices S_1, and C also have positive diagonal entries and are strictly diagonally dominant matrices [27] therefore, S_1, B and C are SPD matrices. b) From (29), since the sum of two SPD matrices is also an SPD matrix, S_2 and K_1 are SPD matrices.

Theorem 2. The constructed scheme HDP on hexagon system of grids is stable for any $h > 0$ and $\tau > 0$ and the approximate solution $u_{h,\tau}$ converges to the exact solution u with $O\left(h^4 + \tau^2\right)$ of accuracy on the hexagonal grids.

Proof. From Lemma 1, the matrix S_1 is an SPD matrix hence invertible. The linear system (27) can be written as

$$\left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right) U^{k+1} = \left(I - \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right) U^k + \tau\left(S_1\right)^{-1} F^{k^*}, \tag{31}$$

where $I \in R^{N\times N}$ is the identity matrix. On the other hand using (28)–(30) we can express the matrices S_1, C and S_2 as linear combination of the identity matrix I and the matrix B as:

$$S_1 = I - \frac{1}{8}B,\; C = \frac{1}{2}I - \frac{1}{16}B,\; S_2 = \frac{1}{2}\frac{bh^2}{\omega}I + \left(1 - \frac{1}{16}\frac{bh^2}{\omega}\right)B. \tag{32}$$

Because $\left(S_1\right)^{-1} S_2$ commutes and S_1 and S_2 are symmetric implies that $\left(S_1\right)^{-1} S_2$ is also a symmetric matrix. Since the product of two SPD matrices that commute is also an SPD matrix [27, 28] gives $\lambda_s\left(\left(S_1\right)^{-1} S_2\right) > 0$. Let $A = \left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right)$ obviously A is an SPD matrix. Let $\widehat{A} = \left(I - \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right)$.

$$\begin{aligned}
\left(A^{-1}\widehat{A}\right)^T &= \widehat{A}A^{-1} = \left(I - \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right)\left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right)^{-1} \\
&= \frac{1}{\det\left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right)}\left(I - \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right) Adj\left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right) \\
&= \left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right)^{-1}\left[I - \frac{1}{\det\left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right)}\left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right)\right. \\
&\quad \left.\left(\frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right) Adj\left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right)\right] \\
&= \left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right)^{-1}\left(I - \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right) = A^{-1}\widehat{A}.
\end{aligned} \tag{33}$$

Thus $A^{-1}\widehat{A}$ is a symmetric matrix, then there exists an orthogonal matrix $\widetilde{P}$ and a diagonal matrix $\widetilde{D}$ with diagonal entries of eigenvalues $\lambda_s\left(\left(S_1\right)^{-1} S_2\right)$ so that

$$\left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1} S_2\right) = \widetilde{P}^T\left(I + \frac{\omega\tau}{h^2}\widetilde{D}\right)\widetilde{P},$$

and

$$\left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1}S_2\right)^{-1} = \widetilde{P}^T\left(I + \frac{\omega\tau}{h^2}\widetilde{D}\right)^{-1}\widetilde{P}.$$

Thus,

$$\left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1}S_2\right)^{-1}\left(I - \frac{\omega\tau}{h^2}\left(S_1\right)^{-1}S_2\right) = \widetilde{P}^T\left(I + \frac{\omega\tau}{h^2}\widetilde{D}\right)^{-1}\widetilde{P}\widetilde{P}^T\left(I - \frac{\omega\tau}{h^2}\widetilde{D}\right)\widetilde{P},$$

that is, the matrix $A^{-1}\widehat{A}$ is similar to $\left(I + \frac{\omega\tau}{h^2}\widetilde{D}\right)^{-1}\left(I - \frac{\omega\tau}{h^2}\widetilde{D}\right)$. Hence, from (33)

$$\begin{aligned}\left\|A^{-1}\widehat{A}\right\|_2 &= \rho\left(A^{-1}\widehat{A}\right) = \max_{1\leq s\leq N}\left|\lambda_s\left[\left(I + \frac{\omega\tau}{h^2}\widetilde{D}\right)^{-1}\left(I - \frac{\omega\tau}{h^2}\widetilde{D}\right)\right]\right| \\ &\leq \left|\frac{1 - \frac{\omega\tau}{h^2}\min\limits_{1\leq s\leq N}\left(\lambda_s\left(\left(S_1\right)^{-1}S_2\right)\right)}{1 + \frac{\omega\tau}{h^2}\min\limits_{1\leq s\leq N}\left(\lambda_s\left(\left(S_1\right)^{-1}S_2\right)\right)}\right| < 1 \text{ for } \frac{\omega\tau}{h^2} > 0 \qquad (34)\end{aligned}$$

and from Gerchgorin's circle theorem we have

$$0 < \lambda_s\left(B\right) \leq 4. \qquad (35)$$

From (32) and (35) and on the basis of Lemma 1 that $K_1 = S_1 + \frac{\omega\tau}{h^2}S_2$ is an SPD matrix we have

$$\begin{aligned}K_1 &= \left(1 + \frac{1}{2}\tau b\right)I + \left(-\frac{1}{8} + \frac{\omega\tau}{h^2} - \frac{b\tau}{16}\right)B \\ \lambda_s\left(K_1\right) &= \lambda_s\left(S_1 + \frac{\omega\tau}{h^2}S_2\right) \\ &= \left(1 + \frac{1}{2}\tau b\right) + \left(-\frac{1}{8} + \frac{\omega\tau}{h^2} - \frac{b\tau}{16}\right)\lambda_s\left(B\right) \\ \rho\left(\left(K_1\right)^{-1}\right) &= \rho\left(\left(S_1 + \frac{\omega\tau}{h^2}S_2\right)^{-1}\right) = \left\|\left(S_1 + \frac{\omega\tau}{h^2}S_2\right)^{-1}\right\|_2 \leq \frac{1}{\varkappa},\end{aligned}$$

where $\varkappa = \min\left\{1 + \frac{1}{2}\tau b, \frac{1}{2} + \frac{1}{2}b\tau + \frac{4\omega\tau}{h^2}\right\}$, then

$$\left\|\left(K_1\right)^{-1}\right\|_2 \leq \frac{1}{\varkappa} < 2. \qquad (36)$$

Next using (34) and (36) by induction results,

$$\begin{aligned}\left\|U^{k+1}\right\|_2 &\leq \left\|A^{-1}\widehat{A}\right\|_2\left\|U^k\right\|_2 + \tau\left\|\left(K_1\right)^{-1}\right\|_2\left\|F^{k^*}\right\|_2 \\ &\leq \left\|U^0\right\|_2 + 2\sum_{k'=0}^{k}\tau\left\|F^{k'^*}\right\|_2. \qquad (37)\end{aligned}$$

The error function $\varepsilon_{h,\tau}$ satisfying (20)–(23) can also be given in the matrix form (31) as

$$\left(I + \frac{\omega\tau}{h^2}\left(S_1\right)^{-1}S_2\right)\epsilon^{k+1} = \left(I - \frac{\omega\tau}{h^2}\left(S_1\right)^{-1}S_2\right)\epsilon^k + \tau\left(S_1\right)^{-1}\widehat{\Psi}^{k^*}, \qquad (38)$$

where $\epsilon^{k+1}, \epsilon^k$ and $\widehat{\Psi}^{k^*} \in R^N$ and $\widehat{\Psi}^{k^*}$ involves the truncation errors given in (24), (25). Thus, on the basis of Theorem 1 and using (24), (25) and (37), (38) we obtain

$$\left\|\epsilon^{k+1}\right\|_2 \leq 2\sum_{k'=0}^{k}\tau\left\|\widehat{\Psi}^{k'^*}\right\|_2 \leq c_1\left(h^4 + \tau^2\right). \qquad (39)$$

Here, c_1 is a positive constant independent of h and τ. The matrix $A^{-1}\widehat{A}$ is a normal matrix since it is also a symmetric real matrix. The inequality in (34) is sufficient as well as necessary for stability from the Von Neuman condition for stability [29]. Therefore, the unconditional stability of the implicit scheme (9), (10) follows from (37). Let $\left\|\varepsilon_{h,\tau}^{k+1}\right\|_C = \max\limits_{\overline{D^h\gamma_\tau \cap \{t=(k+1)\tau\}}} \left|\varepsilon_{h,\tau}^{k+1}\right| = \left\|\epsilon^{k+1}\right\|_\infty$, then by using (39) and norm concordance we get

$$\left\|\varepsilon_{h,\tau}^{k+1}\right\|_C \leq \left\|\epsilon^{k+1}\right\|_2 \leq c_1\left(h^4+\tau^2\right).$$

Therefore, the order of accuracy of the approximate solution $u_{h,\tau}$ to the exact solution u is $O\left(h^4+\tau^2\right)$.

3 Incomplete block matrix factorization and preconditioning of an SPD-BT matrix

In this section, for a real block matrix $K \in R^{N\times N}$ of block size $n \times n$, the inequality $K \succeq_s 0$ defines that K is symmetric positive semi-definite (SPSD) matrix and $K \succ_s 0$ denotes that K is a symmetric positive definite (SPD) matrix. Analogously, $A \succeq_s B$ ($A \succ_s B$) denotes $A - B \succeq_s 0$ ($A - B \succ_s 0$). Further, for a symmetric matrix K, $\lambda_k(K)$ denotes the kth eigenvalue of K ordered in increasing order and λ_{min} and λ_{max} are the minimum and maximum eigenvalues respectively.

3.1 Block incomplete decomposition algorithm and analysis

We consider symmetric positive definite block tridiagonal (SPD-BT) matrix

$$K = \begin{bmatrix} K_{1,1} & K_{1,2} & \cdots & 0 \\ K_{2,1} & K_{2,2} & K_{2,3} & \vdots \\ \vdots & \ddots & \ddots & K_{n-1,n} \\ 0 & \cdots & K_{n,n-1} & K_{n,n} \end{bmatrix}, \tag{40}$$

of $n \times n$, block size. Additionally, the nonzero blocks may be dense and $K_{p,q}$ is of size $n_p \times n_q$ $(1 \leq n_p, n_q \leq n)$ which includes the case where all or some $K_{p,q}$ are scalar entries of K and main diagonal blocks $K_{p,p}$ are square matrices. We consider approximate factorization of $K = LU - Q$ in block matrix form of a lower block triangular matrix L and an upper block triangular matrix U. We repartition the matrix K into 2×2 block form and initially for $s = 1$, take $K^{(1)} = K$

$$K^{(s)} = \begin{bmatrix} K_{1,1}^{(s)} & K_{1,2}^{(s)} \\ K_{2,1}^{(s)} & K_{2,2}^{(s)} \end{bmatrix}, \tag{41}$$

where $K_{1,1}^{(s)}$ is the current pivot and $K_{i,j}^{(s)}$ is of order $n_i^{(s)} \times n_j^{(s)}$ for $i,j = 1,2$ and $n_2^{(s)} >> n_1^{(s)}$, also $n_1^{(s)} = n_s$ for $s = 1, 2, ..., n$.

For $M-$matrices the two-step iterative method for approximating the pivoting diagonal block inverses with rate of convergence 33 was given in [24]. Algorithm 1 approximates the inverse of the pivoting diagonal block matrix of a block tridiagonal matrix $K \succ_s 0$, analogous to the two-step iterative method in [24] however, is modified in the choice of the initial approximate inverse $Z_0^{(s)}$ at every stage s. Further, Algorithm 2 gives incomplete block factorization of a SPD-BT matrix K, (see also [27] for incomplete block decomposition techniques of matrices with special structure).

Algorithm 1. Modified two-step iterative method (MTSIM) for approximate matrix inversion.
Require: The predescribed accuracy $\varepsilon > 0$.

Ensure: I is the identity matrix and $m_s = 0, 1, ...,$ is the iteration at stage s. Also,

$$\begin{aligned} R^{(s)}_{m_s} &= I - K^{(s)}_{1,1} Z^{(s)}_{m_s} \text{ and } \Omega(R^{(s)}_{m_s}) = R^{(s)}_{m_s} + \left(R^{(s)}_{m_s}\right)^2, \\ \Psi(R^{(s)}_{m_s}) &= \left(R^{(s)}_{m_s}\right)^2 + \left(R^{(s)}_{m_s}\right)^4 \text{ and } \Gamma(R^{(s)}_{m_s}) = \left(R^{(s)}_{m_s}\right)^4. \end{aligned}$$

1. Initial step

$$\begin{aligned} & \text{Step1(I)} \\ \beta^{(s)} &= \lambda_{max}\left(K^{(s)}_{1,1}\left(K^{(s)}_{1,1}\right)^T\right) \\ m_s &= 0,\ Z^{(s)}_0 = \frac{\left(K^{(s)}_{1,1}\right)^T}{\beta^{(s)}},\ \text{and } R^{(s)}_0 = I - K^{(s)}_{1,1} Z^{(s)}_0. \end{aligned}$$

2. Prediction and correction steps:
While $\left\|R^{(s)}_{m_s}\right\|_\infty \leq \varepsilon < 1$ do

$$\begin{aligned} & \text{Step2(P)} \\ Z^{(s)}_{m_s+\frac{1}{2}} &= Z^{(s)}_{m_s}\left[I + \Omega(R^{(s)}_{m_s})\right], \\ R^{(s)}_{m_s+\frac{1}{2}} &= I - K^{(s)}_{1,1} Z^{(s)}_{m_s+\frac{1}{2}}, \\ & \text{Step2(C)} \\ Z^{(s)}_{m_s+1} &= Z^{(s)}_{m_s+\frac{1}{2}}\left[I + \Omega\left(R^{(s)}_{m_s+\frac{1}{2}}\right)\left[I + \Psi\left(R^{(s)}_{m_s+\frac{1}{2}}\right)\left[I + \Gamma\left(R^{(s)}_{m_s+\frac{1}{2}}\right)\right]\right]\right], \\ R^{(s)}_{m_s+1} &= I - K^{(s)}_{1,1} Z^{(s)}_{m_s+1},\ \text{increase } m_s \text{ by one.} \end{aligned}$$

End while.
3. Terminating step: $Z^{(s)}$ denotes the matrix $Z^{(s)}_{m^*_s}$ obtained for performing m^*_s iterations.

Algorithm 2. Incomplete block matrix factorization of $K \succ_s 0$.
Require: $s = 1$ and $K^{(1)} = K$.
1. Partition $K^{(s)}$ as in (41).
2. While $s \leq n$ do
find the approximate inverse $Z^{(s)}$ of $K^{(s)}_{1,1}$ using the Algorithm 1.
Ensure: $\widehat{K}^{(s)}$ is an approximation of $K^{(s)}$ factored as

$$\widehat{K}^{(s)} = L^{(s)} U^{(s)} = \begin{bmatrix} I & 0 \\ K^{(s)}_{2,1} Z^{(s)} & I \end{bmatrix} \begin{bmatrix} K^{(s)}_{1,1} & K^{(s)}_{1,2} \\ 0 & K^{(s+1)} \end{bmatrix},$$

where, $K^{(s+1)} = K^{(s)}_{2,2} - K^{(s)}_{2,1} Z^{(s)} K^{(s)}_{1,2}$.
4. End while.
5. The matrix L in the final approximate factorization of K is block lower triangular matrix with diagonal blocks being identity matrix and the sth column of its lower triangular part is formed by $K^{(s)}_{2,1} Z^{(s)}$.
6. The matrix U is block upper triangular with block diagonal matrix $\left\{K^{(1)}_{1,1}, K^{(2)}_{1,1}, ..., K^{(n)}_{1,1}\right\}$ and the sth row of its upper triangular part is formed by $K^{(s)}_{1,2}$.

Lemma 2. Let K be an SPD-BT matrix and $K_{1,1}^{(s)}$ be the pivoting diagonal block at stage s of the Algorithm 2. If $K_{1,1}^{(s)} \succ_s 0$, then $R_{m_s}^{(s)} \succeq_s 0$ for every $m_s = 0, 1, ...$, and

$$\begin{aligned} \rho\left(R_0^{(s)}\right) &< 1, \\ R_{m_s+1}^{(s)} &= \left(R_0^{(s)}\right)^{33^{m_s+1}}, \end{aligned}$$

where $\rho\left(R_0^{(s)}\right)$ is the spectral radius of $R_0^{(s)} = I - K_{1,1}^{(s)} Z_0^{(s)}$.

Proof. For $m_s = 0$ we have $R_0^{(s)} = I - K_{1,1}^{(s)} Z_0^{(s)} = I - \frac{1}{\beta^{(s)}} K_{1,1}^{(s)} \left(K_{1,1}^{(s)}\right)^T$ and if $K_{1,1}^{(s)} \succ_s 0$, it follows that $R_0^{(s)}$ is symmetric matrix and

$$\begin{aligned} \lambda_k\left(R_0^{(s)}\right) &= 1 - \frac{1}{\beta^{(s)}} \lambda_k\left(K_{1,1}^{(s)} \left(K_{1,1}^{(s)}\right)^T\right), \\ 0 &\leq \lambda_k\left(R_0^{(s)}\right) = 1 - \frac{\lambda_k\left(\left(K_{1,1}^{(s)}\right)^2\right)}{\lambda_{max}\left(\left(K_{1,1}^{(s)}\right)^2\right)} < 1, \end{aligned}$$

which gives $\rho\left(R_0^{(s)}\right) < 1$ and $R_0^{(s)} \succeq_s 0$. Further, from Step2(P) we get

$$\begin{aligned} R_{m_s+\frac{1}{2}}^{(s)} &= I - K_{s,s}^{(s)} Z_{m_s}^{(s)} \left[I + R_{m_s}^{(s)} + \left(R_{m_s}^{(s)}\right)^2\right] \\ &= \left(R_{m_s}^{(s)}\right)^3. \end{aligned} \tag{42}$$

Also from the Step2(C) in Algorithm 1 using (42) we get

$$\Omega(R_{m_s+\frac{1}{2}}^{(s)}) = R_{m_s+\frac{1}{2}}^{(s)} + \left(R_{m_s+\frac{1}{2}}^{(s)}\right)^2 = \left(R_{m_s}^{(s)}\right)^3 + \left(R_{m_s}^{(s)}\right)^6, \tag{43}$$

$$\Psi(R_{m_s+\frac{1}{2}}^{(s)}) = \left(R_{m_s+\frac{1}{2}}^{(s)}\right)^2 + \left(R_{m_s+\frac{1}{2}}^{(s)}\right)^4 = \left(R_{m_s}^{(s)}\right)^6 + \left(R_{m_s}^{(s)}\right)^{12}, \tag{44}$$

$$\Gamma(R_{m_s+\frac{1}{2}}^{(s)}) = \left(R_{m_s+\frac{1}{2}}^{(s)}\right)^4 = \left(R_{m_s}^{(s)}\right)^{12}. \tag{45}$$

Using (42)–(45) at the Step2(C) for the residual error $R_{m_s+1}^{(s)}$ we obtain

$$\begin{aligned} R_{m_s+1}^{(s)} &= I - K_{s,s}^{(s)} Z_{m_s+1}^{(s)} \\ &= I - K_{s,s}^{(s)} Z_{m_s+\frac{1}{2}}^{(s)} \left[I + \Omega\left(R_{m_s+\frac{1}{2}}^{(s)}\right)\left[I + \Psi\left(R_{m_s+\frac{1}{2}}^{(s)}\right)\left[I + \Gamma\left(R_{m_s+\frac{1}{2}}^{(s)}\right)\right]\right]\right] \\ &= \left(R_{m_s}^{(s)}\right)^{33} = \left(R_0^{(s)}\right)^{33^{m_s+1}}. \end{aligned}$$

Thus, $R_{m_s+1}^{(s)} \succeq_s 0$ because $R_0^{(s)} \succeq_s 0$.

Theorem 3. Let K be an SPD-BT matrix and $K^{(s)}$ be the matrix obtained at stage s of the Algorithm 2. If $K_{1,1}^{(s)} \succ_s 0$ then

$$K_{1,1}^{(s)} Z_{m_s+\frac{1}{2}}^{(s)} = Z_{m_s+\frac{1}{2}}^{(s)} K_{1,1}^{(s)}, \text{ and } K_{11}^{(s)} Z_{m_s+1}^{(s)} = Z_{m_s+1}^{(s)} K_{1,1}^{(s)}, \tag{46}$$

and $Z^{(s)}_{m_s+\frac{1}{2}} \succ_s 0$, and $Z^{(s)}_{m_s+1} \succ_s 0$ satisfying

$$\left(K^{(s)}_{1,1}\right)^{-1} \succeq_s Z^{(s)}_{m_s+1} \succeq_s Z^{(s)}_{m_s+\frac{1}{2}} \succeq_s Z^{(s)}_{m_s} \succ_s 0, \tag{47}$$

for every $m_s = 0, 1, ...,$ where $Z^{(s)}_{m_s+\frac{1}{2}}$ and $Z^{(s)}_{m_s+1}$ are the approximate inverse of $K^{(s)}_{1,1}$ obtained by Step2(P) and Step2(C) in Algorithm 1.

Proof. The proof of (46) follows from induction. Using Algorithm 1 for $m_s = 0$ and from Step1(I) gives $Z^{(s)}_0 = \frac{1}{\beta^{(s)}}\left(K^{(s)}_{1,1}\right)^T$. Since $K^{(s)}_{1,1}$ is a symmetric matrix we get

$$K^{(s)}_{1,1}Z^{(s)}_0 = K^{(s)}_{1,1}\frac{1}{\beta^{(s)}}\left(K^{(s)}_{1,1}\right)^T = \frac{1}{\beta^{(s)}}\left(K^{(s)}_{1,1}\right)^T K^{(s)}_{1,1} = Z^{(s)}_0 K^{(s)}_{1,1}.$$

Assume that the proposition is true for m_s that is $K^{(s)}_{1,1}Z^{(s)}_{m_s} = Z^{(s)}_{m_s}K^{(s)}_{1,1}$ then for m_s+1 at the Step2(P) gives

$$\begin{aligned} K^{(s)}_{1,1}Z^{(s)}_{m_s+\frac{1}{2}} &= K^{(s)}_{1,1}Z^{(s)}_{m_s}\left[I + R^{(s)}_{m_s} + \left(R^{(s)}_{m_s}\right)^2\right] \\ &= Z^{(s)}_{m_s}\left[I + R^{(s)}_{m_s} + \left(R^{(s)}_{m_s}\right)^2\right]K^{(s)}_{1,1} \\ &= Z^{(s)}_{m_s+\frac{1}{2}}K^{(s)}_{1,1}. \end{aligned} \tag{48}$$

Also using (42)–(45) and (48) at the Step2(C) gives the second equation in (46).

The proof of (47) also can be given using induction. For $m_s = 0$ from Step1(I) gives $Z^{(s)}_0 = \frac{1}{\beta^{(s)}}\left(K^{(s)}_{1,1}\right)^T$ and from the assumption $K^{(s)}_{1,1} \succ_s 0$ implies that $Z^{(s)}_0 \succ_s 0$. Assume that the proposition is true for m_s that is $Z^{(s)}_{m_s} \succ_s 0$ then from Lemma 2 using that $R^{(s)}_{m_s} \succeq_s 0$ at the Step2(P) and using (46) gives

$$\begin{aligned} \left(Z^{(s)}_{m_s+\frac{1}{2}}\right)^T &= \left[I + R^{(s)}_{m_s} + \left(R^{(s)}_{m_s}\right)^2\right]^T\left(Z^{(s)}_{m_s}\right)^T \\ &= Z^{(s)}_{m_s}\left[I + R^{(s)}_{m_s} + \left(R^{(s)}_{m_s}\right)^2\right] = Z^{(s)}_{m_s+\frac{1}{2}}. \end{aligned} \tag{49}$$

Next using (42)–(45) and (49) at the Step2(C) and from (46) results

$$\begin{aligned} \left(Z^{(s)}_{m_s+1}\right)^T &= \left[I + \Omega\left(R^{(s)}_{m_s+\frac{1}{2}}\right)\left[I + \Psi\left(R^{(s)}_{m_s+\frac{1}{2}}\right)\left[I + \Gamma\left(R^{(s)}_{m_s+\frac{1}{2}}\right)\right]\right]\right]^T\left(Z^{(s)}_{m_s+\frac{1}{2}}\right)^T \\ &= Z^{(s)}_{m_s+1}. \end{aligned} \tag{50}$$

From (49) and (50) we conclude that $Z^{(s)}_{m_s+\frac{1}{2}}$ and $Z^{(s)}_{m_s+1}$ are also symmetric for m_s+1 and from Step2(P) and Step2(C) we get $Z^{(s)}_{m_s+\frac{1}{2}} \succ_s 0$ and $Z^{(s)}_{m_s+1} \succ_s 0$. Further from (46) $Z^{(s)}_{m_s}R^{(s)}_{m_s}$ and $Z^{(s)}_{m_s}\left(R^{(s)}_{m_s}\right)^2$ are symmetric matrices. Thus, yields $Z^{(s)}_{m_s}R^{(s)}_{m_s} \succeq_s 0$ and $Z^{(s)}_{m_s}\left(R^{(s)}_{m_s}\right)^2 \succeq_s 0$. From the Step2(P) results

$$\begin{aligned} \lambda_k\left(Z^{(s)}_{m_s+\frac{1}{2}} - Z^{(s)}_{m_s}\right) &= \lambda_k\left(Z^{(s)}_{m_s}R^{(s)}_{m_s} + Z^{(s)}_{m_s}\left(R^{(s)}_{m_s}\right)^2\right) \\ &\geq \lambda_k\left(Z^{(s)}_{m_s}R^{(s)}_{m_s}\right) + \lambda_{min}\left(Z^{(s)}_{m_s}\left(R^{(s)}_{m_s}\right)^2\right) \geq 0 \end{aligned}$$

giving $Z^{(s)}_{m_s+\frac{1}{2}} \succeq_s Z^{(s)}_{m_s}$. Analogously, using (42)-(45) at the Step2(C) results $Z^{(s)}_{m_s+1} \succeq_s Z^{(s)}_{m_s+\frac{1}{2}}$. Denoting the error by $E^{(s)}_{m_s} = \left(K^{(s)}_{1,1}\right)^{-1} - Z^{(s)}_{m_s}$ at sth stage from $K^{(s)}_{1,1} \succ_s 0$ we get $\left(K^{(s)}_{1,1}\right)^{-1} \succ_s 0$ and using that $Z^{(s)}_0 \succ_s 0$ (for $m_s = 0$) we get $E^{(s)}_0$ is symmetric matrix. Further, it follows that

$$\begin{aligned}\lambda_k\left(E^{(s)}_0\right) &= \lambda_k\left(\left(K^{(s)}_{1,1}\right)^{-1} - Z^{(s)}_0\right) \geq \lambda_{min}\left(\left(K^{(s)}_{1,1}\right)^{-1}\right) + \lambda_k\left(-Z^{(s)}_0\right)\\ &\geq \frac{1}{\sqrt{\beta^{(s)}}} - \frac{\sqrt{\beta^{(s)}}}{\beta^{(s)}} = 0.\end{aligned}$$

Thus $E^{(s)}_0 \succeq_s 0$. Assume that for m_s the proposition $E^{(s)}_{m_s} \succeq_s 0$ is true then using $K^{(s)}_{1,1}E^{(s)}_{m_s} = R^{(s)}_{m_s}$ we obtain

$$\begin{aligned}\left(K^{(s)}_{1,1}\right)^{-1} - E^{(s)}_{m_s+1} &= Z^{(s)}_{m_s+1}\\ &= Z^{(s)}_{m_s+\frac{1}{2}}\left[I + \Omega\left(R^{(s)}_{m_s+\frac{1}{2}}\right)\left[I + \Psi\left(R^{(s)}_{m_s+\frac{1}{2}}\right)\left[I + \Gamma\left(R^{(s)}_{m_s+\frac{1}{2}}\right)\right]\right]\right],\\ &= \left(K^{(s)}_{1,1}\right)^{-1} - E^{(r)}_{m_s}\left(R^{(s)}_{m_s}\right)^{32}.\end{aligned}$$

From Lemma 2, $R^{(s)}_{m_s} \succeq_s 0$ and from (46)

$$E^{(s)}_{m_s}R^{(s)}_{m_s} = R^{(s)}_{m_s}E^{(s)}_{m_s},$$

$$\begin{aligned}E^{(s)}_{m_s}R^{(s)}_{m_s}\left(E^{(s)}_{m_s}R^{(s)}_{m_s}\right)^T &= E^{(s)}_{m_s}R^{(s)}_{m_s}\left(R^{(s)}_{m_s}\right)^T\left(E^{(s)}_{m_s}\right)^T\\ &= E^{(s)}_{m_s}R^{(s)}_{m_s}R^{(s)}_{m_s}E^{(s)}_{m_s} = \left(E^{(s)}_{m_s}R^{(s)}_{m_s}\right)^T E^{(s)}_{m_s}R^{(s)}_{m_s},\end{aligned}$$

that is $E^{(s)}_{m_s}R^{(s)}_{m_s}$ is normal. Thus from Theorem 3 in [30]

$$E^{(s)}_{m_s+1} = E^{(s)}_{m_s}\left(R^{(s)}_{m_s}\right)^{32} \succeq_s 0. \tag{51}$$

Theorem 4. Let K be an SPD-BT matrix. If Algorithm 2 is used then $K^{(s)} \succ_s 0$ and the inequality (47) holds at every stage s of the recursion.

Proof. The proof follows by induction. Assume that $K \succ_s 0$ and is block tridiagonal matrix and Algorithm 2 is used. From the assumption $K^{(1)} = K$ is an SPD matrix and particularly $K^{(1)}_{11} \succ_s 0$, hence Theorem 3 implies that the inequalities in (47) holds true for $s = 1$. Assume that $K^{(s)} \succ_s 0$ then it follows that $K^{(s)}_{i,i} \succ_s 0$ for $i = 1, 2$ and are regular and,

$$S^{(s)}_i = K^{(s)}_{i,i} - K^{(s)}_{i,j}\left(K^{(s)}_{j,j}\right)^{-1}K^{(s)}_{j,i}, i, j = 1, 2, i \neq j, \tag{52}$$

exist and $S^{(s)}_i \succ_s 0$, $i = 1, 2$. Since $K^{(s)} \succ_s 0$ so is $\left(K^{(s)}\right)^{-1}$. Further, from Theorem 3 the approximate inverse $Z^{(s)}$ of $K^{(s)}_{1,1}$ satisfies $\left(K^{(s)}_{1,1}\right)^{-1} \succeq_s Z^{(s)}$ and $Z^{(s)} \succ_s 0$ and from Algorithm 2

$$\left(K^{(s+1)}\right)^T = \left(K^{(s)}_{2,2} - K^{(s)}_{2,1}Z^{(s)}K^{(s)}_{1,2}\right)^T = K^{(s+1)}. \tag{53}$$

Using (52) and (53) follows $K^{(s+1)} \succ_s 0$, and (47) hold true for $s + 1$.

Theorem 5. Let K be an SPD-BT matrix of $n \times n$ block size. If $K_{1,1}^{(s)}$, $s = 1, 2, ..., n$ are the diagonal pivoting blocks of $K^{(s)}$ at stage $s = 1, 2, ..., n$ obtained by the Algorithm 2, then the sequences $\left\{Z_{m_s+1}^{(s)}\right\}$, obtained by Algorithm 1 converge to $\left(K_{1,1}^{(s)}\right)^{-1}$, $s = 1, 2, ..., n$, respectively in Euclidean matrix norm $\|\cdot\|_2$ when $m_s \to \infty$ with 33 order of convergence and the inequality

$$\left\|\left(K_{1,1}^{(s)}\right)^{-1} - Z_{m_s+1}^{(s)}\right\|_2 \leq \frac{\left\|R_0^{(s)}\right\|_2^{33^{m_s+1}} \left\|\left(K_{1,1}^{(s)}\right)^T\right\|_2}{\beta^{(s)}\left(1 - \left\|R_0^{(s)}\right\|_2\right)},$$

holds true at the sth stage.

Proof. By taking the initial approximate inverse $Z_0^{(s)} = \frac{1}{\beta^{(s)}}\left(K_{1,1}^{(s)}\right)^T$ the proof is analogous to the proof of Theorem 4 in [24].

3.2 Block hybrid preconditioning of the Conjugate Gradient method

We consider the linear system $K\widetilde{u} = \widetilde{b}$ where, $K \succ_s 0$ is a block tridiagonal matrix of the form (40).

Theorem 6. Let K be an SPD-BT matrix of $n \times n$ block size. If $K_{1,1}^{(s)}$, $s = 1, 2, ..., n$ are the diagonal pivoting blocks of $K^{(s)}$ at stage $s = 1, 2, ..., n$ obtained by the Algorithm 2, and $Z^{(s)}$ are the corresponding approximate inverses obtained by Algorithm 1 by performing m_s^* iterations, then $Z^{(s)}K_{1,1}^{(s)}$ are SPD matrices and

$$\kappa\left(Z^{(s)}K_{1,1}^{(s)}\right) \leq \frac{1+\varepsilon}{1-\varepsilon}, \tag{54}$$

where, $\kappa\left(Z^{(s)}K_{1,1}^{(s)}\right) = \left\|\left(Z^{(s)}K_{1,1}^{(s)}\right)^{-1}\right\|_2 \left\|Z^{(s)}K_{1,1}^{(s)}\right\|_2$ is the condition number of $Z^{(s)}K_{1,1}^{(s)}$ and $0 < \varepsilon < 1$ is the predescribed accuracy in Algorithm 1.

Proof. On the basis of Theorem 3, we have $K_{1,1}^{(s)}Z^{(s)} = Z^{(s)}K_{1,1}^{(s)}$ for every $s = 1, 2, ..., n$ and $Z^{(s)} \succ_s 0$. Theorem 4 implies that $K_{1,1}^{(s)} \succ_s 0$ thus the product of two commuting symmetric positive definite matrices is also symmetric positive definite we get $Z^{(s)}K_{1,1}^{(s)} \succ_s 0$. Next, since $I - K_{1,1}^{(s)}Z^{(s)}$ is symmetric matrix and Algorithm 1 gives $\left\|I - K_{1,1}^{(s)}Z^{(s)}\right\|_\infty \leq \varepsilon$, yielding

$$\rho\left(I - K_{1,1}^{(s)}Z^{(s)}\right) = \left\|I - K_{1,1}^{(s)}Z^{(s)}\right\|_2 \leq \left\|I - K_{1,1}^{(s)}Z^{(s)}\right\|_\infty \leq \varepsilon < 1.$$

Therefore,

$$\left|\left\|K_{1,1}^{(s)}Z^{(s)}\right\|_2 - \|I\|_2\right| \leq \varepsilon,$$

giving

$$1 - \varepsilon \leq \left\|K_{1,1}^{(s)}Z^{(s)}\right\|_2 \leq 1 + \varepsilon. \tag{55}$$

Also

$$\left\|\left(Z^{(s)}K_{1,1}^{(s)}\right)^{-1}\right\|_2 = \left\|\left(I - I - Z^{(s)}K_{1,1}^{(s)}\right)^{-1}\right\|_2 \leq \frac{1}{1-\varepsilon} \tag{56}$$

so from (55) and (56) follows (54). (57)

Theorem 6 shows that $Z^{(s)}$ may be used as approximate inverse preconditioners for $K_{1,1}^{(s)}$ for $s = 1, 2, ..., n$. Algorithm 3 gives the BHP-CG method for solving $K\widetilde{u} = \widetilde{b}$ based on the CG method in [25]. In this algorithm incomplete block factorization LU of K is used as implicit preconditioner while the approximate inverses $Z^{(s)}$ are used as explicit preconditioners for $K_{1,1}^{(s)}$ for $s = 1, 2, ..., n$.

Algorithm 3. BHP-CG method.
Ensure: the construction of L and U by using the Algorithm 2.
Require: $l=0$ and $\widetilde{u}_0$ as an initial guess, $r_0=\widetilde{b}-K\widetilde{u}_0$.
Require: p_{-1} arbitrary and $\sigma_0=0$.
1. While $\frac{\|r_l\|_\infty}{\|\widetilde{b}\|_\infty}\leq\eta<1$ do
2. Solve the system $LUz_l=r_l$. For the solution of the block lower triangular system $L\omega_l=r_l$ where $\omega_l=Uz_l$ forward substitution works since the diagonal blocks of L are identity matrices. Then for the solution of the block upper triangular system $Uz_l=\omega_l$, the preconditioned CG method is used to solve the block subsystems with the explicit preconditioners $Z^{(s)}$ for the matrices $K_{1,1}^{(s)}$.
3. If $l\geq 1$ then compute $\sigma_l=\langle z_l,LUz_l\rangle/\langle z_{l-1},LUz_{l-1}\rangle$.
4. Else $\sigma_0=0$.
5. End if.
6. $p_l=z_l+\sigma_l p_{l-1}$ and $\alpha_l=\langle z_l,LUz_l\rangle/\langle p_l,Kp_l\rangle$,
7. $\widetilde{u}_{l+1}=\widetilde{u}_l+\alpha_l p_l$ and $r_{l+1}=r_l-\alpha_l Kp_l$.
8. End while.
9. Let l^* be the iteration number performed, in 1–8 then $\widetilde{u}_{l^*}$ is the approximate solution satisfying $\frac{\|r_{l^*}\|_\infty}{\|\widetilde{b}\|_\infty}\leq\eta$.

4 Numerical investigation

We take $D=\left\{(x_1,x_2):0<x_1<1,0<x_2<\frac{\sqrt{3}}{2}\right\}$, for $t\in[0,1]$ and the prediscribed accuracy ε in Algorithm 1 is taken as 5×10^{-5}. Also in all tables $CPUs$ stands for Central Processing Unit time in seconds and ptl stands for per time level wherever they appear. Let in addition, the following notations be used in this section where K_1 is the matrix in (27) and $\widetilde{K}_1$ is as given in (8).

M_{14P}^{H},M_{14P}^{R} denote the newly developed HDP and classical RDP.

$N^{h,\tau}(M_{14P}^{H}),N^{h,\tau}(M_{14P}^{R})$ denote the size of the matrices K_1 and $\widetilde{K}_1$.

$Pre^{h,\tau}(M_{14P}^{H}),Pre^{h,\tau}(M_{14P}^{R})$ are the preconditioning time of K_1 and $\widetilde{K}_1$.

$Con^{h,\tau}(M_{14P}^{H}),Con^{h,\tau}(M_{14P}^{R})$ are the condition number of K_1 and $\widetilde{K}_1$.

$CT^{M_{14P}^{H}},CT^{M_{14P}^{R}}$ denote the $CPUs$ ptl for the method M_{14P}^{H} and M_{14P}^{R}.

$TCT^{M_{14P}^{H}},TCT^{M_{14P}^{R}}$ denote the total $CPUs$ required by the method M_{14P}^{H} and M_{14P}^{R} for solving the problem on $t\in[0,1]$.

neg means that $CPUs$ is less than one millisecond.

We present the function $\varepsilon_{h,\tau}$ defining the error on the grid points $\overline{D^h\gamma_\tau}$, by $\varepsilon^{M_{14P}^{H}(h,\tau)}$ obtained from the application of the method M_{14P}^{H} . Similarly we use $\varepsilon^{M_{14P}^{R}(h,\tau)}$ to show the error function $\varepsilon_{h,\tau}$ obtained by the method M_{14P}^{R} on the grid points $\overline{D^{h_1,h_2}\gamma_\tau}$. In addition, the convergence order of the methods M_{14P}^{H} and M_{14P}^{R} are

$$\begin{aligned}\Re^{M_{14P}^{H}} &= \log_2\left(\frac{\left\|\varepsilon^{M_{14P}^{H}(2^{-\mu},2^{-\lambda})}\right\|_{\infty}}{\left\|\varepsilon^{M_{14P}^{H}(2^{-(\mu+1)},2^{-(\lambda+2)})}\right\|_{\infty}}\right),\\ \Re^{M_{14P}^{R}} &= \log_2\left(\frac{\left\|\varepsilon^{M_{14P}^{R}(2^{-\mu},2^{-\lambda})}\right\|_{\infty}}{\left\|\varepsilon^{M_{14P}^{R}(2^{-(\mu+1)},2^{-(\lambda+2)})}\right\|_{\infty}}\right),\end{aligned}$$

respectively, where μ, λ are positive integers.

4.1 Test problem: Example 1

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+f(x_1,x_2,t) \text{ on } Q_T,\\ u(x_1,x_2,0) &= 0.07x_1^{6+\alpha}+0.3x_2^{6+\alpha}+1 \text{ on } \overline{D},\\ u(x_1,x_2,t) &= v(x_1,x_2,t) \text{ on } S_T,\\ f(x_1,x_2,t) &= \left(3+\frac{\alpha}{2}\right)t^{2+\frac{\alpha}{2}}\cos\left(t^{3+\frac{\alpha}{2}}\right)-e^{-t}\\ &\quad -(6+\alpha)(5+\alpha)\left(0.07x_1^{4+\alpha}+0.3x_2^{4+\alpha}\right),\\ v(x_1,x_2,t) &= 0.07x_1^{6+\alpha}+0.3x_2^{6+\alpha}+\sin(t^{3+\frac{\alpha}{2}})+e^{-t},\end{aligned}$$

where v is the exact solution. Table 2 shows the $CT^{M_{14P}^{H}}$, $CT^{M_{14P}^{R}}$ and the error norms $\left\|\varepsilon^{M_{14P}^{H}(h,\tau)}\right\|_{\infty}$, $\left\|\varepsilon^{M_{14P}^{R}(h,\tau)}\right\|_{\infty}$ for $h = 2^{-\mu}, \mu = 4,5,6,7,8$ when $\tau = 2^{-\lambda}, \lambda = 6,8,10,12,14$ and the order of convergence $\Re^{M_{14P}^{H}}, \Re^{M_{14P}^{R}}$ for Example 1 when $\alpha = 0.8$. Table 3 shows the same quantities by using the methods M_{14P}^{H} and M_{14P}^{R} when $\alpha = 0.01$. These tables indicate that both methods have fourth order convergence in spatial variables and second order convergence in time variable.

On the other hand the second and fifth columns of these tables show the computational time $CT^{M_{14P}^{H}}$ and $CT^{M_{14P}^{R}}$ required for the method M_{14P}^{H} and M_{14P}^{R} respectively. By analyzing the values of $CT^{M_{14P}^{H}}$ and $CT^{M_{14P}^{R}}$ we conclude that the proposed method is more economical in computational time per time level when the BHP-CG method given in Algorithm 3 is applied to solve the derived systems. This conclusion is also supported by the results given in Table 4 which demonstrates the number of grid points in the stiffness matrices $N^{h,\tau}(M_{14P}^{H})$ and $N^{h,\tau}(M_{14P}^{R})$, the preconditioning times $Pre^{h,\tau}(M_{14P}^{H})$ and $Pre^{h,\tau}(M_{14P}^{R})$, the condition numbers of the preconditioned matrices $Con^{h,\tau}(M_{14P}^{H})$ and $Con^{h,\tau}(M_{14P}^{R})$ and the total computational time required in seconds $TCT^{M_{14P}^{H}}$ and $TCT^{M_{14P}^{R}}$ of the methods M_{14P}^{H} and M_{14P}^{R} respectively for Example 1 when $\alpha = 0.8$.

Further, when $h = 2^{-6}$ and $\tau = 2^{-10}$ for $\alpha = 0.8$, the grid function $\left|\varepsilon^{M_{14P}^{H}(2^{-6},2^{-10})}\right|$ presenting the errors in absolute values at four time stages $t = 0.25, 0.5, 0.75, 1$ by the method M_{14P}^{H} are shown in Figure 3 for Example 1. Analogously, Figure 4 demonstrate the function $\left|\varepsilon^{M_{14P}^{R}(2^{-6},2^{-10})}\right|$ at the same time levels and (h,τ) pair and α value obtained by the method M_{14P}^{R}.

Table 2

Results by the methods M_{14P}^{H} and M_{14P}^{R} for Example 1 when $\alpha = 0.8$

(h,τ)	$CT^{M_{14P}^{H}}$	$\left\lVert \varepsilon^{M_{14P}^{H}(h,\tau)} \right\rVert_{\infty}$	$\Re^{M_{14P}^{H}}$	$CT^{M_{14P}^{R}}$	$\left\lVert \varepsilon^{M_{14P}^{R}(h,\tau)} \right\rVert_{\infty}$	$\Re^{M_{14P}^{R}}$
$(2^{-4},2^{-6})$	*neg*	$4.19389E-5$		*neg*	$4.26584E-5$	
$(2^{-5},2^{-8})$	0.047	$2.62266E-6$	3.9992	0.047	$2.66787E-6$	3.9991
$(2^{-6},2^{-10})$	0.156	$1.63922E-7$	3.9999	0.234	$1.66749E-7$	3.9999
$(2^{-7},2^{-12})$	0.641	$1.02449E-8$	4.0000	1.016	$1.04224E-8$	3.9999
$(2^{-8},2^{-14})$	2.578	$6.40304E-10$	4.0000	4.312	$6.51384E-10$	4.0000

Table 3

Results by the methods M_{14P}^{H} and M_{14P}^{R} for Example 1 when $\alpha = 0.01$

(h,τ)	$CT^{M_{14P}^{H}}$	$\left\lVert \varepsilon^{M_{14P}^{H}(h,\tau)} \right\rVert_{\infty}$	$\Re^{M_{14P}^{H}}$	$CT^{M_{14P}^{R}}$	$\left\lVert \varepsilon^{M_{14P}^{R}(h,\tau)} \right\rVert_{\infty}$	$\Re^{M_{14P}^{R}}$
$(2^{-4},2^{-6})$	*neg*	$4.19389E-5$		*neg*	$2.98695E-5$	
$(2^{-5},2^{-8})$	0.047	$2.62266E-6$	3.9994	0.047	$1.86757E-6$	3.9994
$(2^{-6},2^{-10})$	0.188	$1.63922E-7$	3.9999	0.219	$1.16726E-7$	3.9999
$(2^{-7},2^{-12})$	0.64	$1.02449E-8$	4.0000	1.016	$7.29597E-9$	3.9999
$(2^{-8},2^{-14})$	2.5	$6.40298E-10$	4.0000	4.25	$4.56001E-10$	3.9999

Table 4

Computational efficiency comparison of M_{14P}^{H}, M_{14P}^{R} for Example 1 when $\alpha = 0.8$

(h,τ)	$(2^{-4},2^{-6})$	$(2^{-5},2^{-8})$	$(2^{-6},2^{-10})$	$(2^{-7},2^{-12})$	$(2^{-8},2^{-14})$
$N^{h,\tau}(M_{14P}^{H})$	233	977	4001	16193	65153
$N^{h,\tau}(M_{14P}^{R})$	225	961	3969	16129	65025
$Pre^{h,\tau}(M_{14P}^{H})$	*neg*	*neg*	0.063	0.36	2.797
$Pre^{h,\tau}(M_{14P}^{R})$	*neg*	*neg*	0.062	0.359	2.625
$Con^{h,\tau}(M_{14P}^{H})$	0.99997	0.99993	0.99989	0.99986	0.99983
$Con^{h,\tau}(M_{14P}^{R})$	0.99991	0.99988	0.99987	0.99985	0.99981
$TCT^{M_{14P}^{H}}$	0.61	9.09	194.84	2659.03	42582.52
$TCT^{M_{14P}^{R}}$	0.70	11.83	272.91	4258.53	71073.79

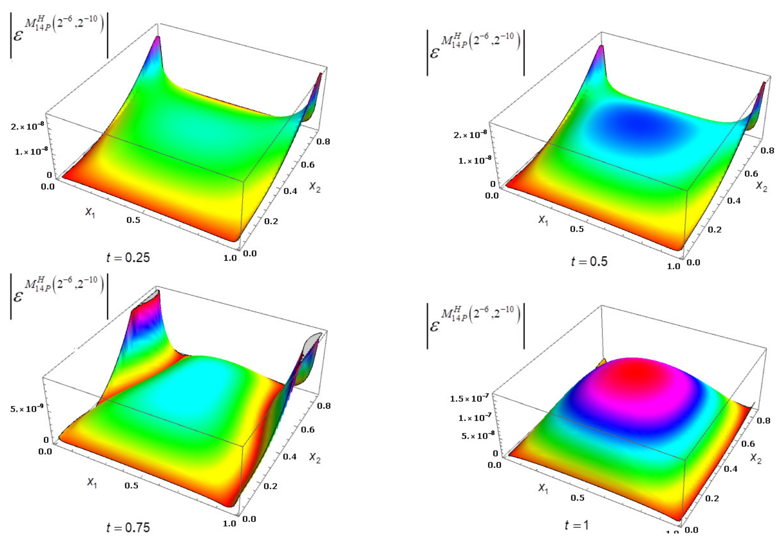

Figure 3. The grid function $\left| \varepsilon^{M_{14P}^{H}(2^{-6},2^{-10})} \right|$ when $t = 0.25, 0.5, 0.75, 1$ by M_{14P}^{H} for Example 1.

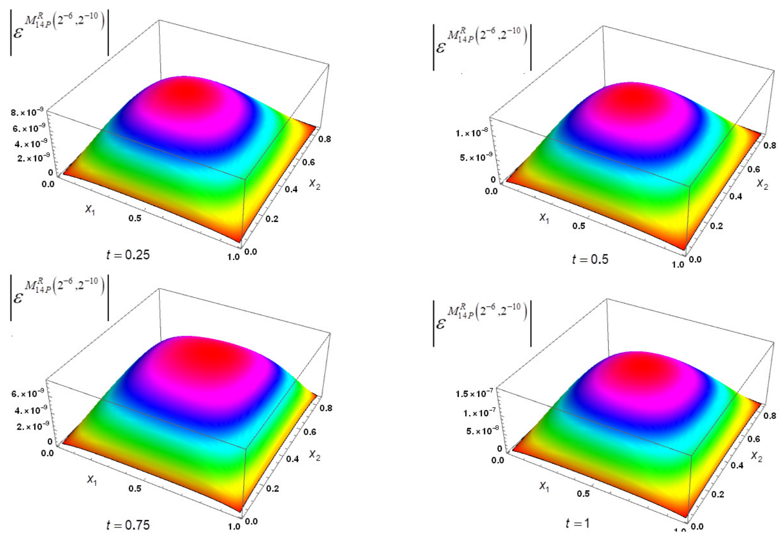

Figure 4. The grid function $\left|\varepsilon^{M_{14P}^{R}(2^{-6},2^{-10})}\right|$ when $t = 0.25, 0.5, 0.75, 1$ by M_{14P}^{R} for Example 1.

4.2 Test problem: Example 2

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - 0.5u + f(x_1, x_2, t) \text{ on } Q_T, \\
u(x_1, x_2, 0) &= \frac{1}{2}x_1^{\frac{37}{6}} + x_2^{\frac{37}{6}} + 1 \text{ on } \overline{D}, \\
u(x_1, x_2, t) &= v(x_1, x_2, t) \text{ on } S_T, \\
f(x_1, x_2, t) &= -\left(\frac{37}{12}t^{\frac{25}{12}} \sin\left(t^{\frac{37}{12}}\right) + \frac{1147}{72}x_1^{\frac{25}{6}} + \frac{1147}{36}x_2^{\frac{25}{6}}\right) \\
&\quad +0.5\left(\frac{1}{2}x_1^{\frac{37}{6}} + x_2^{\frac{37}{6}} + \cos\left(t^{\frac{37}{12}}\right)\right), \\
v(x_1, x_2, t) &= \frac{1}{2}x_1^{\frac{37}{6}} + x_2^{\frac{37}{6}} + \cos\left(t^{\frac{37}{12}}\right),
\end{aligned}$$

where, v is the exact solution. Table 5 demonstrates the $CT^{M_{14P}^{H}}$, $TCT^{M_{14P}^{H}}$ and the error norms for $h = 2^{-\mu}, \mu = 4, 5, 6, 7, 8$ when $\tau = 2^{-\lambda}, \lambda = 6, 8, 10, 12, 14$ respectively, and the order of convergence $\Re^{M_{14P}^{H}}$ for Example 2. Figure 5 shows the absolute error function $\left|\varepsilon^{M_{14P}^{H}(2^{-6},2^{-10})}\right|$ for time values $t = 0.25, 0.5, 0.75, 1$ obtained by the given method M_{14P}^{H} for Example 2.

Table 5

Results by the method M_{14P}^{H} for Example 2

(h, τ)	$CT^{M_{14P}^{H}}$	$TCT^{M_{14P}^{H}}$	$\left\Vert \varepsilon^{M_{14P}^{H}(h,\tau)} \right\Vert_{\infty}$	$\Re^{M_{14P}^{H}}$
$(2^{-4}, 2^{-6})$	*neg*	0.61	$2.378442E-5$	
$(2^{-5}, 2^{-8})$	0.047	9.907	$1.543029E-6$	3.9462
$(2^{-6}, 2^{-10})$	0.172	207.547	$1.015411E-7$	3.9256
$(2^{-7}, 2^{-12})$	0.735	2904.99	$6.623985E-9$	3.9382
$(2^{-8}, 2^{-14})$	2.829	50743	$4.251592E-10$	3.9616

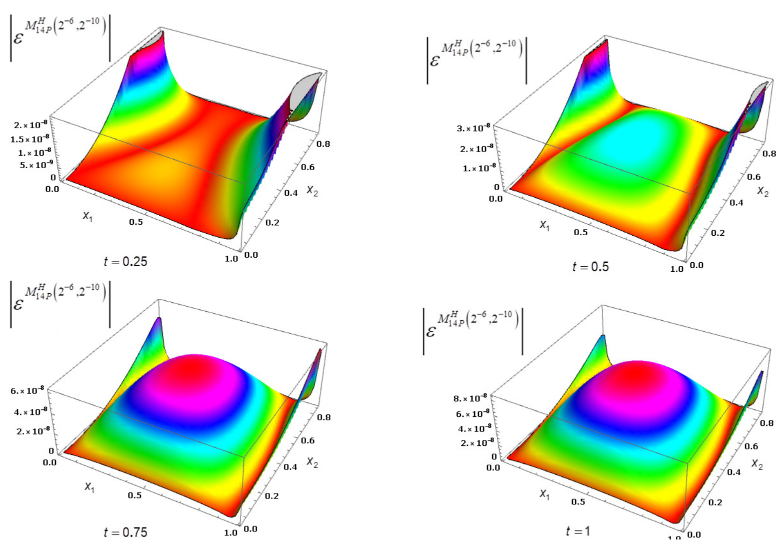

Figure 5. The grid function $\left|\varepsilon^{M^H_{14P}(2^{-6},2^{-10})}\right|$ when $t = 0.25, 0.5, 0.75, 1$ by M^H_{14P} for Example 2.

5 *Conclusion*

On a hexagonal system of grids, a novel implicit method is developed for approximating the solution to the DBVP of the heat equation (2)–(4) on rectangle. Further, by using the modified two-step iterative method, block hybrid preconditioning of the conjugate gradient method is given. The obtained theoretical and numerical results demonstrate that the given implicit method is economical since it is computationally time efficient. We remark that in Section 2, the given implicit scheme on hexagonal grids was studied in the dissertation [31].

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Түйіндес градиенттер әдісін блокты-гибридті қайта шарттауға қолдана отырып, жаңа айқынемес схема бойынша жылуөткізгіштік теңдеуін шешу

Зерттеудің негізгі мақсаты – алтыбұрыштардың тор жүйесінде жаңа айырымдық әдісін жасау арқылы тіктөртбұрыштағы жылуөткізгіштік теңдеуінің Дирихле шеттік есептердің шешімін жуықтау. Бұл арнайы схема сөзсіз тұрақты және кеңістіктік айнымалылар бойынша төртінші дәлдік реті және уақыт айнымалысы бойынша екінші дәлдік реті бар торлардағы нақты шешімге жақындайтыны дәлелденді. Екіншіден, толық емес блоктық факторландыру симметриялы оң анықталған блоктық үшбұрышты матрицалар үшін симметриялы оң анықталған қасиетті сақтай отырып, айналмалы диагональды блоктардың кері жағына жуықтайтын консервативті итерациялық әдісті қолдана отырып берілген. Болашақта факторландыру блогының көмегімен алынған алгебралық теңдеулер жүйесін әр уақыт деңгейінде шешу үшін түйіндес градиенттер әдісінің гибридті қайта шарттауы қолданылады.

Кілт сөздер: жылуөткізгіштік теңдеуі, айқынемес схема, алтыбұрышты тор, тұрақтылықты талдау, симметриялы оң анықталған матрица, жуықталған кері, толықемес блокты факторландыру, блокты-гибридті қайта шарт қою, түйіндес градиенттер әдісі.

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Решение уравнения теплопроводности по новой неявной схеме с использованием блочно-гибридного предобусловливания метода сопряженных градиентов

Основной целью исследования является аппроксимация решения краевой задачи Дирихле уравнения теплопроводности на прямоугольнике путем разработки нового разностного метода на сеточной системе шестиугольников. Доказано, что данная специальная схема безусловно устойчива и сходится к точному решению на сетках с четвертым порядком точности по пространственным переменным и вторым порядком точности по временной переменной. Во-вторых, неполная блочная факторизация дана для симметричных положительно определенных блочных трехдиагональных матриц с использованием консервативного итеративного метода, который аппроксимирует обратную сторону поворотных диагональных блоков, сохраняя симметричное положительно определенное свойство. В дальнейшем с помощью этого блока факторизации применено гибридное предобусловливание метода сопряженных градиентов для решения полученной алгебраической системы уравнений на каждом временном уровне.

Ключевые слова: уравнение теплопроводности, неявная схема, гексагональная сетка, анализ устойчивости, симметричная положительно определенная матрица, приближенная обратная, неполная блочная факторизация, блочно-гибридное предобусловливание, метод сопряженных градиентов.

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A remark on Schottky representations and Reidemeister torsion

The present paper establishes a formula of Reidemeister torsion for Schottky representations. The theoretical results are applied to 3−manifolds with boundary consisting orientable surfaces with genus at least 2.

Keywords: Schottky representations, Reidemeister torsion, representation varieties, Atiyah-Bott-Goldman symplectic form, Thurston symplectic form.

Introduction

It is well-known that the representation varieties are important in many branches of mathematics and physics. For instance, let Σ be a compact Riemann surface of genus at least 2, Teichmüller space $\text{Teich}(\Sigma)$ of Σ is the space of deformation classes of complex structures on it. By the uniformization Theorem, it is the space of hyperbolic metrics, namely Riemannian metrics on Σ with Gaussian curvature constant (-1). Furthermore, Teichmüller space of Σ can be interpreted as discrete faithful representations of the fundamental group $\pi_1(\Sigma)$ of the surface to $\text{PSL}(2, \mathbb{R})$. It is well-known that some certain geometric structures on Σ can also be identified as certain surface group variety [1–6] and the references therein.

Representation varieties have a large number of applications in many branches of mathematics and physics such as in 3−manifold topology (in Bass-Culler-Shalen theory [7, 8], in A-polynomial [9], in hyperbolic geometry [10], in Casson invariant theory [11]), in Yang-Mills and Chern-Simons quantum field theories [12, 13], in skein theory of quantum invariants of 3-manifolds [14, 15], in the moduli spaces of flat connections, holomorphic bundles, and Higgs bundles [16].

Reidemeister torsion(R-torsion) is a topological invariant and was introduced by K. Reidemeister [17]. Using this invariant, he classified 3−dimensional lens spaces. W. Franz extended the R-torsion and classified the higher dimensional lens spaces [18]. R-torsion has many applications in several branches of mathematics and theoretical physics such as topology [19], differential geometry [20], representation spaces [21] dynamical systems [22], 3-dimensional Seiberg-Witten theory [23], algebraic K-theory [24], Chern-Simon theory [13], knot theory [24], theoretical physics and quantum field theory [13]. See Refs. [25] and [26] and the references therein for further information.

Real symplectic chain complex is a algebraic topological instrument and was introduced by E. Witten [21]. Combining this and R-torsion, he evaluated the volume of several moduli space of $\text{Rep}(\Sigma, G)$, which is the set of all conjugacy classes of homomorphisms from the fundamental group $\pi_1(\Sigma)$ of a Riemann surface Σ to the compact gauge group $G \in \{\text{SU}(2), \text{SO}(3)\}$.

In paper [27], we considered the set $\text{Rep}(\Sigma, G)$ of $G-$valued representations from the fundamental group $\pi_1(\Sigma)$ of the surface Σ to the exceptional groups G_2, F_4, and E_6. We proved the well-definiteness of R-torsion of such representations. We also established a formula for computing R-torsion of such representations in terms of the well known symplectic structure on $\text{Rep}(\Sigma, G)$, namely, Atiyah-Bott-Goldman symplectic form for the Lie group G. Then, we applied to $G-$valued Hitchin representations.

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In paper [28], we investigated $G-$valued representations of free or surface group with genus >1 for $G \in \{\mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C})\}$. We also established a formula for computing R-torsion of such representations in terms of Atiyah-Bott-Goldman symplectic form for G. Moreover, we applied the obtained results to hyperbolic $3-$manifolds.

In the present paper, we prove a formula of R-torsion for Schottky representations. The theoretical results are applied to $3-$manifolds with boundary consisting orientable surfaces with genus at least 2.

1 Preliminaries

In this section, we provide the necessary definition and basic facts about the topological invariant R-torsion and the symplectic chain complex. For further information the reader is referred to [21,25,26,29] and the references therein.

Let $C_* = \left(0 \to C_n \stackrel{\partial_n}{\to} C_{n-1} \to \cdots \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0\right)$ be a chain complex of finite dimensional vector spaces over the field $\mathbb{C}$ of complex numbers. For $p = 0, \ldots, n$, we denote the kernel of ∂_p, the image of ∂_{p+1}, and the pth homology group of the chain complex C_* by $Z_p(C_*)$, $B_p(C_*)$, and $H_p(C_*)$, respectively. From the definition of $Z_p(C_*)$, $B_p(C_*)$, and $H_p(C_*)$ it follows

$$0 \longrightarrow Z_p(C_*) \hookrightarrow C_p \twoheadrightarrow B_{p-1}(C_*) \longrightarrow 0$$

and

$$0 \longrightarrow B_p(C_*) \hookrightarrow Z_p(C_*) \twoheadrightarrow H_p(C_*) \longrightarrow 0.$$

For $p = 0, \ldots, n$, if $\mathbf{c}_p$, $\mathbf{b}_p$, and $\mathbf{h}_p$ are bases of C_p, $B_p(C_*)$, and $H_p(C_*)$, respectively and if $\ell_p : H_p(C_*) \to Z_p(C_*)$, $s_p : B_{p-1}(C_*) \to C_p$ are sections of $Z_p(C_*) \to H_p(C_*)$, $C_p \to B_{p-1}(C_*)$, respectively, then with the help of above short-exact sequences we have the basis $\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1})$ of C_p. Here, $\sqcup$ denotes the disjoint union.

Let $\mathbf{c}_p$, $\mathbf{b}_p$, $\mathbf{h}_p$, ℓ_p, and s_p be as above. Then, *R-torsion* of the chain complex C_* with respect to bases $\{\mathbf{c}_p\}_{p=0}^n$, $\{\mathbf{h}_p\}_{p=0}^n$ is defined by

$$\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n\right) = \prod_{p=0}^{n} \left[\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p\right]^{(-1)^{(p+1)}},$$

where $[\mathbf{e}_p, \mathbf{f}_p]$ denotes determinant of the change-base-matrix from basis $\mathbf{f}_p$ to $\mathbf{e}_p$ of C_p.

R-torsion does not depend on the bases $\mathbf{b}_p$ and sections s_p, ℓ_p [24].

Let $\mathbf{c}'_p$, $\mathbf{h}'_p$ be also bases of C_p, $H_p(C_*)$, respectively. Then, the following change-base-formula is valid [24]:

$$\mathbb{T}\left(C_*, \{\mathbf{c}'_p\}_0^n, \{\mathbf{h}'_p\}_0^n\right) = \prod_{p=0}^{n} \left(\frac{[\mathbf{c}'_p, \mathbf{c}_p]}{[\mathbf{h}'_p, \mathbf{h}_p]}\right)^{(-1)^p} \mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n\right).$$

Let

$$0 \longrightarrow A_* \stackrel{\imath}{\longrightarrow} B_* \stackrel{j}{\longrightarrow} D_* \longrightarrow 0 \tag{1}$$

be a short-exact sequence of chain complexes, and let $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, $\mathbf{h}_p^B$, and $\mathbf{h}_p^D$ are bases of A_p, B_p, D_p, $H_p(A_*)$, $H_p(B_*)$, and $H_p(D_*)$, respectively. Let us consider the corresponding Mayer-Vietoris long-exact sequence of vector spaces

$$\mathcal{C}_* : \ \cdots \longrightarrow H_p(A_*) \stackrel{\imath_p}{\longrightarrow} H_p(B_*) \stackrel{j_p}{\longrightarrow} H_p(D_*) \stackrel{\delta_p}{\longrightarrow} H_{p-1}(A_*) \longrightarrow \cdots$$

associated to short-exact sequence (1). Note that $\mathcal{C}_{3p} = H_p(D_*)$, $\mathcal{C}_{3p+1} = H_p(A_*)$, and $\mathcal{C}_{3p+2} = H_p(B_*)$ then we can consider the bases $\mathbf{h}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^B$ for $\mathcal{C}_{3p}$, $\mathcal{C}_{3p+1}$, and $\mathcal{C}_{3p+2}$, respectively.

Theorem 1. [24] Suppose $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, $\mathbf{h}_p^B$, and $\mathbf{h}_p^D$ are as above. Suppose also $\left[\mathbf{c}_p^B, \mathbf{c}_p^A \oplus \widetilde{\mathbf{c}_p^D}\right] = \pm 1$, where $j\left(\widetilde{\mathbf{c}_p^D}\right) = \mathbf{c}_p^D$. Then, it follows

$$\begin{aligned} \mathbb{T}\left(B_*, \{\mathbf{c}_p^B\}_0^n, \{\mathbf{h}_p^B\}_0^n\right) &= \mathbb{T}\left(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n\right) \\ \times \quad \mathbb{T}\left(D_*, \{\mathbf{c}_p^D\}_{p=0}^n, \{\mathbf{h}_p^D\}_0^n\right) &\mathbb{T}\left(C_*, \{\mathbf{c}_{3p}\}_0^{3n+2}, \{0\}_0^{3n+2}\right). \end{aligned}$$

Theorem 1 yields the sum-lemma.

Lemma 1. Assume A_*, D_* are chain complexes of vector spaces and $\mathbf{c}_p^A$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^D$ are bases of A_p, D_p, $H_p(A_*)$, and $H_p(D_*)$, respectively. Then, the following equality

$$\mathbb{T}(A_* \oplus D_*, \{\mathbf{c}_p^A \sqcup \mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^A \sqcup \mathbf{h}_p^D\}_0^n) = \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n)\mathbb{T}(D_*, \{\mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^D\}_0^n)$$

is valid.

The proof of Lemma 1 can also be found in [30].

$(C_*, \partial_*, \{\omega_{*,q-*}\})$ is said to be $\mathbb{C}-$*symplectic chain complex* of length q, if

1 $C_* : 0 \to C_q \stackrel{\partial_q}{\to} C_{q-1} \to \cdots \to C_{q/2} \to \cdots \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0$ is a chain complex of length q, where $q \equiv 2 \pmod 4$,

2 for $p = 0, \ldots, q$, $\omega_{p,q-p} : C_p \times C_{q-p} \to \mathbb{C}$ is a $\partial-$compatible non-degenerate anti-symmetric bilinear form. Namely,

$$\omega_{p,q-p}\left(\partial_{p+1}a, b\right) = (-1)^{p+1}\omega_{p+1,q-(p+1)}\left(a, \partial_{q-p}b\right)$$

and

$$\omega_{p,q-p}(a, b) = (-1)^{p(q-p)}\omega_{q-p,p}(b, a).$$

From the fact that $q \equiv 2 \pmod 4$ we have $\omega_{p,q-p}(a, b)$ is $(-1)^p\omega_{q-p,p}(b, a)$. From $\partial-$compatibility of $\omega_{p,q-p}$ we obtain the non-degenerate pairing $[\omega_{p,q-p}] : H_p\left(C_*\right) \times H_{q-p}\left(C_*\right) \to \mathbb{C}$.

For the rest of the paper, if the $\mathbb{C}-$*symplectic chain complex* $(C_*, \partial_*, \{\omega_{*,q-*}\})$ is clear, then $\Delta\left(\mathbf{h}_p, \mathbf{h}_{q-p}\right)$ is the determinant of the matrix of the non-degenerate pairing

$$[\omega_{p,q-p}] : H_p\left(C_*\right) \times H_{q-p}\left(C_*\right) \to \mathbb{C}$$

in the bases $\mathbf{h}_p$, $\mathbf{h}_{q-p}$.

Assume C_* is a $\mathbb{C}-$symplectic chain complex of length q and $\mathbf{c}_p$, $\mathbf{c}_{q-p}$ are bases of C_p, C_{q-p}, respectively. We say $\omega-$*compatible*, if the matrix of $\omega_{p,q-p}$ in $\mathbf{c}_p$, $\mathbf{c}_{q-p}$ is equal to the $k \times k$ identity matrix $\mathrm{Id}_{k\times k}$ when $p \neq q/2$ and $\begin{pmatrix} 0_{l\times l} & \mathrm{Id}_{l\times l} \\ -\mathrm{Id}_{l\times l} & 0_{l\times l} \end{pmatrix}$ when $p = q/2$, where $k = \dim C_p = \dim C_{q-p}$ and $2l = \dim C_{q/2}$.

For computing R-torsion in terms of intersections pairings, we have the following result suggests a formula. Namely,

Theorem 2. [31] If $(C_*, \partial_*, \{\omega_{*,q-*}\})$ is a $\mathbb{C}-$symplectic chain complex with the $\omega-$compatible bases $\mathbf{c}_p$, $p = 0, \ldots, q$ and if $\mathbf{h}_p$ is a basis of $H_p\left(C_*\right)$, $p = 0, \ldots, q$, then the following formula holds:

$$\left|\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^q, \{\mathbf{h}_p\}_0^q\right)\right| = \prod_{p=0}^{(q/2)-1} |\Delta(\mathbf{h}_p, \mathbf{h}_{q-p})|^{(-1)^p} \sqrt{\left|\Delta\left(\mathbf{h}_{q/2}, \mathbf{h}_{q/2}\right)\right|}^{(-1)^{q/2}}. \tag{2}$$

In case $\mathbf{h}_p = \mathbf{h}_{q-p} = 0$, the convention $0 = 1.0$ is used and hence $\Delta(\mathbf{h}_p, \mathbf{h}_{q-p}) = 1$. Let us also note that equation (2) can be improved as:

$$\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^q, \{\mathbf{h}_p\}_0^q\right) = \prod_{p=0}^{(q/2)-1} \Delta(\mathbf{h}_p, \mathbf{h}_{q-p})^{(-1)^p} \sqrt{\Delta\left(\mathbf{h}_{q/2}, \mathbf{h}_{q/2}\right)}^{(-1)^{q/2}}. \tag{3}$$

For details of (3), we refer the reader to [28; Remark 2.4]. See [27, 28, 30], for further applications of Theorem 2.

2 Main results

Let Σ be a closed orientable surface of genus at least 2 with the universal covering $\widetilde{\Sigma}$. Let G be the Lie group $\mathrm{PSL}(2,\mathbb{C})$ and $\mathcal{G}$ be the Lie algebra of G with the non-degenerate symmetric bilinear form B. Here, B is the Killing form.

Assume $\varrho : \pi_1(\Sigma) \to G$ is a homomorphism from the fundamental group $\pi_1(\Sigma)$ of Σ to G. Let $E_\varrho = \widetilde{\Sigma} \times \mathcal{G}/\sim$ be the corresponding adjoint bundle over Σ. Here, $(x_1, t_1) \sim (x_2, t_2)$, if $(x_2, t_2) = (\gamma \cdot x_1, \gamma \cdot t_1)$ for some $\gamma \in \pi_1(\Sigma)$, the action of γ in the first component by deck transformation $(\gamma \cdot x_1 = \gamma(x_1))$ and in the second component by the adjoint action $(\gamma \cdot t_1 = \mathrm{Ad}_{\varrho(\gamma)}(t_1) = \varrho(\gamma)\, t_1 \varrho(\gamma)^{-1})$.

Let K be a cell-decomposition of Σ for which the adjoint bundle E_ϱ is trivial over each cell and $\widetilde{K}$ be the lift of K to the $\widetilde{\Sigma}$. Denote by $\mathbb{Z}[\pi_1(\Sigma)]$ the integral group ring. Let $C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) = C_*\left(\widetilde{K}; \mathbb{Z}\right) \otimes \mathcal{G}/\sim$, where for all $\gamma \in \pi_1(\Sigma)$, $\sigma \otimes t \sim \gamma \cdot \sigma \otimes \gamma \cdot t$, the action of γ by the first component is by deck transformation and in the second is by adjoint action. We have the following chain complex:

$$0 \longrightarrow C_2\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \overset{\partial_2 \otimes \mathrm{id}}{\longrightarrow} C_1\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \overset{\partial_1 \otimes \mathrm{id}}{\longrightarrow} C_0\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow 0. \tag{4}$$

Here, ∂_p denotes the usual boundary operator. Denote by $H_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $H^*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ the homologies and cohomologies of the chain complex (4), respectively, where $C^*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ denotes the set of $\mathbb{Z}[\pi_1(\Sigma)]$-module homomorphisms from $C_*\left(\widetilde{K}; \mathbb{Z}\right)$ to $\mathcal{G}$. See [25] for details and unexplained subjects.

Clearly, for conjugate $\varrho, \varrho' : \pi_1(\Sigma) \to G$ i.e. $\varrho'(.) = A\varrho(.)A^{-1}$ for some $A \in G$, we have isomorphic $C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $C_*\left(K;\ \mathcal{G}_{\mathrm{Ad}_{\varrho'}}\right)$. Similarly, the corresponding cochains $C^*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $C^*\left(K;\ \mathcal{G}_{\mathrm{Ad}_{\varrho'}}\right)$ are isomorphic.

Consider chain complex (4). Assume $\left\{e_j^p\right\}_{j=1}^{m_p}$ is a basis of $C_p(K; \mathbb{Z})$. For $j = 1, \ldots, m_p$, fix a lift $\widetilde{e}_j^p$ of e_j^p. Then, $c_p = \left\{\widetilde{e}_j^p\right\}_{j=1}^{m_p}$ of $C_p\left(\widetilde{K}; \mathbb{Z}\right)$ is a $\mathbb{Z}[\pi_1(\Sigma)]-$basis. Assume $\mathcal{A} = \{a_k\}_{k=1}^{\dim \mathcal{G}}$ is a $B-$*orthonormal basis* of the Lie algebra $\mathcal{G}$. Namely, the matrix of the form B equals to the identity matrix of size $\dim \mathcal{G}$. Hence, we obtain a $\mathbb{C}-$basis $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$ of $C_p\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$. We call such a basis a *geometric basis* for $C_p\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$.

If $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$ and $\mathbf{h_p}$ are respectively the geometric basis of $C_p\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and a basis of $H_p\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$, then $\mathbb{T}\left(C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right), \{c_p \otimes_\varrho \mathcal{A}\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2\right)$ is said to be the *R-torsion* of the triple K, Ad_ϱ, and $\{\mathbf{h}_p\}_{p=0}^2$.

Theorem 3. [28; Theorem 3.1] If Σ, K, ϱ, $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$, and $\mathbf{h}_p$, $p = 0, 1, 2$, are as above, then $\mathbb{T}\left(C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right), \{c_p \otimes_\varrho \mathcal{A}\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2\right)$ does not depend on the basis $\mathcal{A}$, lifts $\widetilde{e}_j^p$, conjugacy class of ϱ, and the cell-decomposition K.

From Theorem 3, we have the well-definiteness of R-torsion of such representations, and hence we write $\mathbb{T}(\Sigma, \{\mathbf{h}_p\}_{p=0}^2)$ rather than $\mathbb{T}\left(C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right), \{c_p \otimes_\varrho \mathcal{A}\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2\right)$.

Assume Σ, K, G, $\mathcal{G}$, ϱ, $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$ are as above. Let us consider the dual cell-decomposition K' of Σ corresponding to the cell-decomposition K. Consider the lifts $\widetilde{K}$ and $\widetilde{K'}$ of K and K', respectively. For $i = 0, 1, 2$, we have the intersection form

$$(\cdot, \cdot)_{i,2-i} : C_i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C_{2-i}\left(K'; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow \mathbb{C} \tag{5}$$

defined by $(\sigma_1 \otimes t_1, \sigma_2 \otimes t_2)_{i,2-i} = \sum_{\gamma\in\pi_1(\Sigma)} \sigma_1 . (\gamma \bullet \sigma_2) \; B\left(t_1, \gamma \bullet t_2\right)$. Here, "." denotes the intersection number pairing, the action of γ on σ_2 by deck transformation and on t_2 is by the adjoint action.

Using the anti-symmetric, $\partial-$compatible $(\cdot, \cdot)_{i,2-i}$, we have the non-degenerate anti-symmetric form

$$[\cdot, \cdot]_{i,2-i} : H_i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times H_{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow \mathbb{C}. \tag{6}$$

Note that if $D_i = C_i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \oplus C_i\left(K'; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$, and if we consider the bilinear form $\omega_{i,2-i} : D_i \times D_{2-i} \to \mathbb{C}$ defined by extending the intersection form (5) zero on $C_i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C_{2-i}\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $C_i\left(K'; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C_{2-i}\left(K'; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$, then D_* becomes a $\mathbb{C}-$symplectic chain complex. Note also that the bases c_i of $C_i\left(\widetilde{K}; \mathbb{Z}\right)$ and c_i' of $C_i\left(\widetilde{K'}; \mathbb{Z}\right)$ corresponding to c_i result an $\omega-$compatible basis for D_*.

Kronecker pairing $\langle \cdot, \cdot \rangle : C^i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C_i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \to \mathbb{C}$ is defined by $\langle \theta, \sigma \otimes_\varrho t \rangle = B\left(t, \theta\left(\sigma\right)\right)$. It has natural extended to $\langle \cdot, \cdot \rangle : H^i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times H_i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \to \mathbb{C}$.

Recall the cup product $\cup : C^i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C^j\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \to C^{i+j}\left(\widetilde{\Sigma}; \mathbb{C}\right)$ is defined by $\left(\theta_i \cup \theta_j\right)\left(\sigma_{i+j}\right) = B\left(\theta_i\left(\left(\sigma_{i+j}\right)_{\text{front}}\right), \theta_j\left(\left(\sigma_{i+j}\right)_{\text{back}}\right)\right)$. Here, σ_{i+j} is in $C_{i+j}(\widetilde{K}; \mathbb{Z})$ and $\widetilde{K}$ denotes the lift of K to $\widetilde{\Sigma}$ $\theta_i : C_i\left(\widetilde{K}; \mathbb{Z}\right) \to \mathcal{G}$, $\theta_j : C_j\left(\widetilde{K}; \mathbb{Z}\right) \to \mathcal{G}$ are $\mathbb{Z}[\pi_1(\Sigma)]$-module homomorphisms. This yields the cup product

$$\smile_B : C^i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C^j\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow C^{i+j}\left(K; \mathbb{C}\right)$$

with natural extension

$$\smile_B : H^i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times H^j\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow H^{i+j}\left(\Sigma; \mathbb{C}\right),$$

where $[\theta_i] \smile_B [\theta_j] = [\theta_i \smile_B \theta_j]$.

Using the isomorphisms by (6) and the Kronecker pairing, we get the Poincare duality isomorphisms

$$\mathrm{PD} : H_i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \cong H_{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)^* \cong H^{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right).$$

For $i = 0, 1, 2$ we have the

$$\begin{array}{ccccc} H^{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) & \times & H^i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) & \overset{\smile_B}{\longrightarrow} & H^2\left(\Sigma; \mathbb{C}\right) \\ \uparrow \mathrm{PD} & & \uparrow \mathrm{PD} & \circlearrowleft & \uparrow \\ H_i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) & \times & H_{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) & \overset{[\cdot,\cdot]_{i,2-i}}{\longrightarrow} & \mathbb{C}. \end{array}$$

Here, $\mathbb{C} \to H^2\left(\Sigma; \mathbb{C}\right)$ sends $1 \in \mathbb{C}$ to the fundamental class of $H^2(\Sigma; \mathbb{C})$ and the inverse of this is integration over Σ.

Clearly, we have the following pairing

$$\Omega_{i,2-i} : H^i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times H^{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \overset{\smile_B}{\longrightarrow} H^2\left(\Sigma; \mathbb{C}\right) \overset{\int_\Sigma}{\longrightarrow} \mathbb{C}. \tag{7}$$

$\Omega_{1,1}$ is called Atiyah-Bott-Goldman symplectic form for G on the representation variety $\mathrm{Rep}(\Sigma, G)$.

In [28], we established a formula for computing Reidemeister torsion of representations in terms of $\Omega_{1,1}$ Atiyah-Bott-Goldman symplectic form for the Lie group G. More precisely, *Theorem 4.* [28; Theorem 3.2] Let Σ, K, K', ϱ be as above. Let $\mathbf{c}_p$ and $\mathbf{c}'_p$ be the corresponding geometric bases of $C_p\left(K;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $C_p\left(K';\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$, respectively, $p = 0,1,2$. If $\mathbf{h}_p$ is a basis of $H_p\left(\Sigma;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$, $p = 0,1,2$, then the following formulas are valid

i. $\mathbb{T}\left(\Sigma,\{\mathbf{h}_p\}_{p=0}^2\right) = ie^{\frac{i\theta}{2}}\dfrac{\Delta\left(\mathbf{h}_0,\mathbf{h}_2\right)}{\sqrt{\Delta\left(\mathbf{h}_1,\mathbf{h}_1\right)}}$,

ii. $\mathbb{T}\left(\Sigma,\{\mathbf{h}_p\}_{p=0}^2\right) = ie^{\frac{i\theta}{2}}\dfrac{\sqrt{\delta\left(\mathbf{h}^1,\mathbf{h}^1\right)}}{\delta\left(\mathbf{h}^2,\mathbf{h}^0\right)}$.

Here, $\Delta\left(\mathbf{h}_p,\mathbf{h}_{2-p}\right)$ is the determinant of the matrix of (6) in $\mathbf{h}_p$ and $\mathbf{h}_{2-p}$, $\Delta\left(\mathbf{h}_0,\mathbf{h}_2\right) = \left|\Delta\left(\mathbf{h}_0,\mathbf{h}_2\right)\right| e^{i\theta}$, where $i = \sqrt{-1}$ and $-\pi < \theta \le \pi$. $\delta\left(\mathbf{h}^{2-p},\mathbf{h}^p\right)$ is the determinant of the matrix of (7) in $\mathbf{h}^p$ and $\mathbf{h}^{2-p}$, and $\mathbf{h}^p$ denotes the Poincare dual basis of $H^p(\Sigma;\mathcal{G}_{\mathrm{Ad}_\varrho})$ corresponding to $\mathbf{h}_p$ of $H_p(\Sigma;\mathcal{G}_{\mathrm{Ad}_\varrho})$, $p = 0,1,2$.

Note that in case $H_0\left(\Sigma;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and thus $H_2\left(\Sigma;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ are zero, by Theorem 4 we get

$$\mathbb{T}\left(\Sigma,\{0,\mathbf{h}_1,0\}\right) = i\ \sqrt{\Delta\left(\mathbf{h}_1,\mathbf{h}_1\right)}^{\,(-1)} = i\ \sqrt{\delta\left(\mathbf{h}^1,\mathbf{h}^1\right)}.$$

3 *Applications*

Schottky representation and Thurston symplectic form

Before stating our application, let us recall Thurston symplectic form. For more information and unexplained subjects, we refer [32] and the references therein.

Let Σ_g, $g \ge 2$, be a closed orientable surface. We say that $\lambda \subset \Sigma_g$ is a *geodesic lamination*, if it is closed and also consists of disjoint complete geodesics without any self-intersection points, called *leaves* of λ (see Figure 1 (a)). We say that the geodesic lamination λ is *maximal*, if the complement $\Sigma_g - \lambda$ consists of finitely many ideal triangles, that is, triangles with vertices at infinity (see Figure 1 (b)).

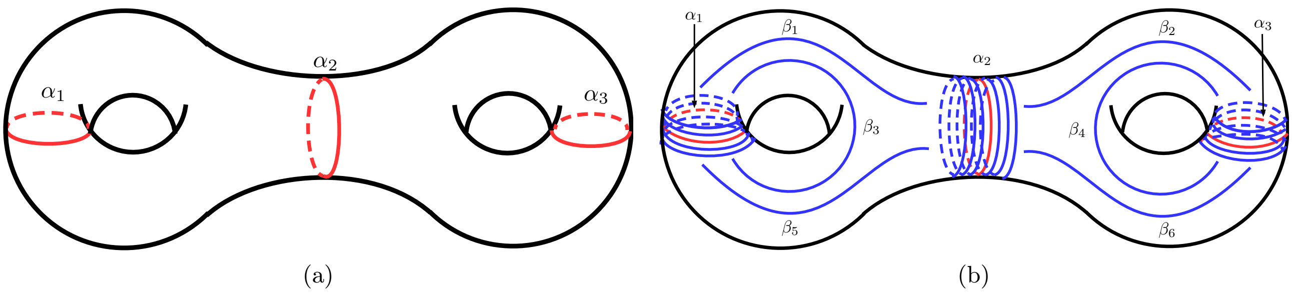

Figure 1. (a) Geodesic lamination with 3 closed leaves (b) Maximal geodesic lamination with 3 closed leaves and 6 infinite leaves spiraling towards closed leaves.

Let $\lambda \subset \Sigma_g$ be a geodesic lamination and G be an abelian group. A $G-$valued *transverse cocycle* σ for λ is a function from the set of all transverse arcs to the leaves of λ to G so that σ is finitely additive and invariant under the homotopy of arcs transverse to λ. To be more precise, $\sigma(k) = \sigma(k_1) + \sigma(k_2)$, when the arc k transverse to leaves of λ is decomposed into two subarcs k_1, k_2 with disjoint interiors, and $\sigma(k) = \sigma(k')$ when the transverse arc k is deformed to arc k' through arcs transverse to the leaves of the geodesic lamination λ (Fig. 2). Let us denote the group of $G-$valued transverse cocycles for λ by $\mathcal{H}(\lambda;G)$. In the case λ is a maximal geodesic lamination and $G = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{R}/2\pi\mathbb{Z}$, $\mathcal{H}(\lambda;G)$ is isomorphic to G^{6g-6} [33]. For example, by using a (fattened) train-track $\Phi \subset \Sigma_g$ carrying the lamination λ, one gets the isomorphism $\mathcal{H}(\lambda;\mathbb{R}) \cong \mathbb{R}^{6g-6}$.

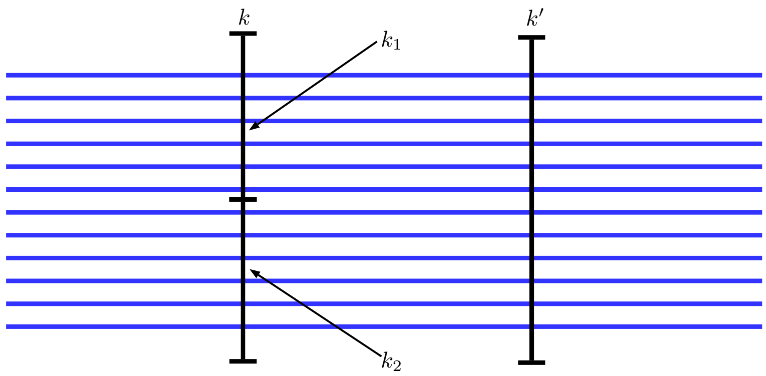

Figure 2. The arcs k and k' are transverse to the leaves of lamination λ. The arc k is deformed to k' through arcs transverse to the leaves of the geodesic lamination. Moreover, k is splitted into two transverse subarcs k_1, k_2 with disjoint interiors.

Recall that a *train-track* $\Phi \subset \Sigma_g$ is composed of finitely many "long" rectangles $e_1, \ldots, e_n$, called *edges* of Φ, foliated by arcs parallel to the "short" sides and meeting only along arcs (possibly reduced to a point) lying in their short sides. Furthermore, each point of the "short" side of a rectangle is also contained in another rectangle, each component of the union of the short sides of all rectangles is an arc, as opposed to a closed curve, and finally since the closure $\overline{\Sigma_g - \Phi}$ of the complement $\Sigma_g - \Phi$ has a certain number of "spikes", corresponding to the points where at least 3 rectangles meet, it is also required that no component of $\overline{\Sigma_g - \Phi}$ be a disc with $0, 1$ or 2 spikes or an annulus with no spike.

Note that foliating the edges of the train-track Φ by using the short sides, we get a foliation of Φ, and the leaves are called the *ties* of Φ. The finitely many ties where several edges meet are said to be the *switches* of Φ. If a tie is not a switch, then it is called a *generic* tie. If λ lies entirely in the interior of Φ and if, moreover, the leaves of λ are transverse to the ties of Φ, then λ is said to be *carried* by Φ (Fig. 3). We refer [34] for constructions of a train-track.

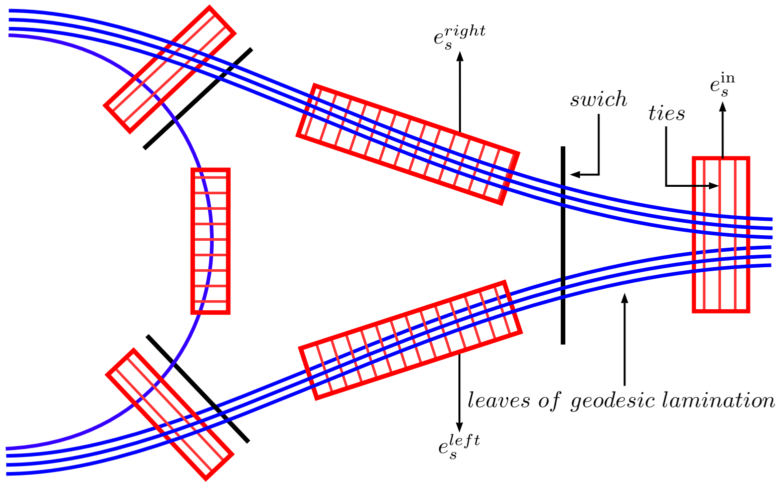

Figure 3. Locally a train-track carries a geodesic lamination.

Suppose $\Phi \subset \Sigma_g$ is a train-track. A real-valued function from the set of edges of Φ is called an *edge* *weight system* for Φ, if it satisfies the *switch relation*. Namely, for each switch s of Φ, let $e_1, \ldots, e_p$ be the edges adjacent to one side of s and let $e_{p+1}, \ldots, e_{p+q}$ be the edges adjacent to the other side, we have $\sum_{i=1}^{p} a(e_i) = \sum_{j=p+1}^{p+q} a(e_j)$. Let us denote the real vector space of all edge weight systems for Φ by $\mathcal{W}(\Phi;\mathbb{R})$.

Let $\lambda \subset \Sigma_g$ be a geodesic lamination carried by the train-track Φ. Consider the injective map associating each transverse cocycle $\sigma \in \mathcal{H}(\lambda;\mathbb{R})$ to the edge weight system $a_\sigma \in \mathcal{W}(\Phi;\mathbb{R})$ defined by $a_\sigma(e) = \sigma(k_e)$. Here, k_e is a tie of e. In the case of maximal lamination λ, the map is an isomorphism $\mathcal{H}(\lambda;\mathbb{R}) \cong \mathcal{W}(\Phi;\mathbb{R})$ [33].

One can arrange the train-track Φ so that at each switch s of Φ, there are one incoming edge e_s^{in} touching the switch s on one side and two outgoing edges e_s^{left}, e_s^{right} touching s on the other side, where as seen from the incoming edge e_s^{in} and for the orientation of the surface Σ_g, e_s^{left} branches out to the left and e_s^{right} branches out to the right. *Thurston symplectic form* on $\mathcal{W}(\Phi)$ is the anti-symmetric bilinear form $\omega_{\text{Thurston}} : \mathcal{W}(\Phi;\mathbb{R}) \times \mathcal{W}(\Phi,\mathbb{R}) \to \mathbb{R}$ defined by

$$\omega_{\text{Thurston}}(a,b) = \frac{1}{2}\sum_s \det \begin{bmatrix} a\left(e_s^{\text{left}}\right) & a\left(e_s^{\text{right}}\right) \\ b\left(e_s^{\text{left}}\right) & b\left(e_s^{\text{right}}\right) \end{bmatrix},$$

where the summation is over all switches of Φ.

By using the isomorphism $\mathcal{H}(\lambda;\mathbb{R}) \cong \mathcal{W}(\Phi;\mathbb{R})$, we have the *Thurston symplectic form* $\omega_{\text{Thurston}} : \mathcal{H}(\lambda;\mathbb{R}) \times \mathcal{H}(\lambda;\mathbb{R}) \to \mathbb{R}$. As is well known that ω_{Thurston} is an algebraic intersection number and is independent of Φ [32, 34].

Recall that Teichmüller space $\text{Teich}(\Sigma_g)$ of the surface Σ_g is the space of isotopy classes of complex structures on Σ_g. By The Uniformization Theorem, it is the space of isotopy classes of Riemannian metrics with constant Gaussian curvature (-1), that is, hyperbolic metrics on Σ_g. One can also identify it with the space of conjugacy classes of all discrete faithful homomorphisms from the fundamental group $\pi_1(\Sigma_g)$ to $\text{PSL}(2,\mathbb{R})$. With the help of a maximal geodesic lamination $\lambda \subset \Sigma_g$ and sending to each hyperbolic metric $m \in \text{Teich}(\Sigma_g)$ the corresponding *shearing cocycle* $\sigma_m \in \mathcal{H}(\lambda;\mathbb{R})$, F. Bonahon embedded $\text{Teich}(\Sigma_g)$ as an open cone $\mathcal{C}(\lambda) \subset \mathcal{H}(\lambda;\mathbb{R})$ [32]. If k is an arc transverse to λ, the shearing cocycle $\sigma_m(k)$ measures the "shift to the left" between the two ideal triangles in $\mathbb{H}^2/\varrho_m(\pi_1(\Sigma_g))$ corresponding to the components of $\Sigma_g - \lambda$ containing the endpoints of k. Here, $\varrho_m : \pi_1(\Sigma_g) \to \text{PSL}(2,\mathbb{R})$ is the discrete faithful representation associated to m.

Recall that for a homorphism $\varrho : \pi_1(\Sigma_g) \to \text{PSL}(2,\mathbb{C})$, there is the following commutative diagram

$$\begin{array}{ccccc} H^1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{\text{Ad}_\varrho}\right) & \times & H^1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{\text{Ad}_\varrho}\right) & \overset{\smile_B}{\longrightarrow} & H^2(\Sigma_g;\mathbb{C}) \\ \uparrow\text{PD} & & \uparrow\text{PD} & \circlearrowleft & \uparrow \\ H_1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{\text{Ad}_\varrho}\right) & \times & H_1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{\text{Ad}_\varrho}\right) & \overset{[\cdot,\cdot]_{1,1}}{\longrightarrow} & \mathbb{C}. \end{array} \tag{8}$$

Here, $\mathbb{C} \to H^2(\Sigma_g;\mathbb{C})$ is the isomorphism sending $1 \in \mathbb{C}$ to the fundamental class of $H^2(\Sigma_g;\mathbb{C})$.

Recall also that

$$\omega_{\text{PSL}(2,\mathbb{C})} : H^1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{\text{Ad}_\varrho}\right) \times H^1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{\text{Ad}_\varrho}\right) \overset{\smile_B}{\to} H^2(\Sigma_g;\mathbb{C}) \overset{\int_{\Sigma_g}}{\to} \mathbb{C}$$

is called Atiyah-Bott-Goldman symplectic form for $\text{PSL}(2,\mathbb{C})$ [35]. It is known that $\omega_{\text{PSL}(2,\mathbb{C})}$ is related with the Goldman symplectic form on $\text{Teich}(\Sigma_g)$

$$\omega_{\text{Goldman}} : H^1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{R})_{\text{Ad}_\varrho}\right) \times H^1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{R})_{\text{Ad}_\varrho}\right) \overset{\smile_{B_\mathbb{R}}}{\to} H^2(\Sigma_g;\mathbb{R}) \overset{\int_{\Sigma_g}}{\to} \mathbb{R}.$$

Here, $B_{\mathbb{R}}$ is the Killing form of the set $\mathfrak{sl}(2,\mathbb{R})$, which is 2×2 trace zero matrices over $\mathbb{R}$.

In [31], considering the isomorphism $T_{\varrho}\mathrm{Teich}(\Sigma_g) \cong \mathcal{H}(\lambda;\mathbb{R})$, which is obtained by the real-analytical parameterization of F. Bonahon [32] and complexfying $\omega_{\mathrm{Thurston}}$, it was proved that

$$\omega_{\mathrm{PSL}(2,\mathbb{C})} = 2\omega_{\mathrm{T}}. \tag{9}$$

Here,

$$\omega_{\mathrm{T}} : \mathcal{H}(\lambda;\mathbb{C}) \times \mathcal{H}(\lambda;\mathbb{C}) \to \mathbb{C} \tag{10}$$

is the complexfied Thurston symplectic form.

For more information and unexplained subjects, we refer the reader to [31] and the references therein.

For a fixed $g \geq 2$, let us consider the free group $\mathrm{F_g}$ with generators $X = \{x_1,\ldots,x_g\}$. The set $\mathrm{Hom}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ of all homomorphisms from $\mathrm{F_g}$ to $\mathrm{PSL}(2,\mathbb{C})$ can be identified with $\mathrm{PSL}(2,\mathbb{C})^g$ by considering the map $\varrho \mapsto (\varrho(x_1),\ldots,\varrho(x_g))$.

Let $\chi(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ be the quotient $\mathrm{Hom}(\mathrm{F_g},G)//G$. As is well known that $\chi(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ naturally has the structure of an algebraic variety and it differs from the set theoretical quotient $\mathrm{Hom}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$ only at reducible points, namely, representations whose images fix a point on $\widehat{\mathbb{C}}$ [36]. Let $\mathcal{D}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ and $\mathcal{E}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ denote respectively the set of all discrete, faithful representations and those of representations with dense image in $\mathrm{PSL}(2,\mathbb{C})$. It is well known $\mathcal{E}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ is not empty and open, $\mathcal{D}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ is closed and outside of these representations in $\chi(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ has measure zero [37] and the references therein.

Let $A_i, B_i,\ i=1,\ldots,g$, be $2g$ disjoint closed (topological) disks in $\partial\mathbb{H}^3$ and let $\gamma_1,\ldots,\gamma_g \in \mathrm{PSL}(2,\mathbb{C})$ be the Möbiüs transformations of the Riemann sphere $\widehat{\mathbb{C}}$ so that $\gamma_i(A_i)$ is the closure of the complement of B_i. The set $\{\gamma_1,\ldots,\gamma_g\}$ generate a free discrete group of rank g, called a *Schottky group*. The representation ϱ obtained by $x_i \mapsto \gamma_i$ is in $\mathcal{D}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$. Let $\mathcal{S}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ be the set of Schottky representations. As is well known that $\mathcal{S}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ lies in the interior of $\mathcal{D}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ [38].

In [39], Y. Minsky proved the existence of an open set $\mathcal{M}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ of $\chi(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ which is strictly larger than $\mathcal{S}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ and on which $Out(\mathrm{F_g})$ acts properly discontinuously.
We have

Theorem 5. Let $\mathrm{F_g}$ denote the fundamental group $\pi_1(H_g)$ of handle body $\mathrm{H_g}$ of genus $g \geq 2$ with boundary Σ_g, and let M denote the double of $\mathrm{H_g}$. Suppose $\lambda \subset \Sigma_g$ is a fixed maximal geodesic lamination and $\varrho \in \mathcal{M}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ is such that $\varrho\circ r \in \mathrm{Teich}(\Sigma_g)$. Let $\mathbf{h}_i^{\mathrm{F}_g}$ be bases for $H_i\left(\mathrm{F}_g;\mathfrak{sl}(2,\mathbb{C})_{Ad_\varrho}\right)$, $i=0,1,2,3$. Then, there exist basis $\mathbf{h}_j^{\mathrm{M}}$ and $\mathbf{h}_k^{\Sigma_g}$ of $H_j\left(\mathrm{M};\mathfrak{sl}(2,\mathbb{C})_{Ad_\varrho}\right)$ and $H_k\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{Ad_{\varrho\circ r}}\right)$, $j=0,1,2,3$, $k=0,1,2$, respectively so that Reidemeister torsion of the corresponding Mayer-Vietoris long exact sequence $\mathcal{H}_\star$ in these bases is 1. In addition, the following formula holds:

$$\mathbb{T}\left(\mathrm{F}_g,\left\{\mathbf{h}_i^{\mathrm{F}_g}\right\}_0^3\right) = e^{\frac{\sqrt{-1}}{4}(-\beta_0+\pi-\theta_1)}\, 2^{\frac{\chi(\Sigma_g;\mathfrak{sl}(2,\mathbb{C}))}{4}}\sqrt[4]{\Omega_{\mathrm{T}}}.$$

Here, $\beta_0 = \dim H_0\left(\mathrm{M};\mathfrak{sl}(2,\mathbb{C})_{Ad_\varrho}\right)$, $\mathbf{h}_{1,1}^0$ is a basis of $H_1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{Ad_{\varrho\circ r}}\right)\oplus H_1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{Ad_{\varrho\circ r}}\right)$ such that $\mathbb{T}\left(C_*\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{Ad_{\varrho\circ r}}\right)\oplus C_*\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{Ad_{\varrho\circ r}}\right),\left\{\mathbf{h}_{1,1}^0\right\}\right)$ is equal to 1, $\left[\mathbf{h}_{1,1}^0,\mathbf{h}_1^{\Sigma_g}\oplus\mathbf{h}_1^{\Sigma_g}\right]$ $=\left|\left[\mathbf{h}_{1,1}^0,\mathbf{h}_1^{\Sigma_g}\oplus\mathbf{h}_1^{\Sigma_g}\right]\right| e^{\sqrt{-1}\theta_1}$. Here, $\chi(\Sigma_g;\mathfrak{sl}(2,\mathbb{C}))$ is $\chi(\Sigma_g)\dim_{\mathbb{C}}\mathfrak{sl}(2,\mathbb{C})$, Ω_{T} is determinant of the matrix of the symplectic form (10) in the basis $\mathfrak{h}\oplus\sqrt{-1}\,\mathfrak{h}$, $\mathfrak{h}$ is the basis of $\mathcal{H}(\lambda;\mathbb{R})$ associated with the isomorphism obtained by the embedding $\mathrm{Teich}(\Sigma_g)\hookrightarrow\mathcal{H}(\lambda;\mathbb{R})$[32], and $\mathbf{h}^1$ is the Poincare dual basis of $H^1\left(\Sigma_g;\mathfrak{sl}\left(2,\mathbb{C}\right)_{Ad_{\varrho\circ r}}\right)$ corresponding to $\mathbf{h}_1^{\Sigma_g}$. Here, $r:\pi_1\left(\Sigma_g\right)\to\pi_1\left(\mathrm{F}_g\right)$ is the homomorphism obtained by the embedding $\Sigma_g\hookrightarrow\mathrm{F}_g$.

The proof of Theorem 5 is based on combining Theorem 4 and [28; Theorem 4.2], and the above results, using the commutative diagram (8), Eq. (9), and the definition of $\omega_{\mathrm{PSL}(2,\mathbb{C})}$.

Let us now apply [28; Theorem 4.3]. As is well known that for a compact orientable 3−manifold H, the holonomy representation of the complete hyperbolic structure $\mathrm{Hol}:\pi_1(\mathrm{H})\to\mathrm{Isom}^+\mathbb{H}^3\cong\mathrm{PSL}(2,\mathbb{C})$ can be lifted to a representation $\widetilde{\mathrm{Hol}}:\pi_1(\mathrm{H})\to\mathrm{SL}(2,\mathbb{C})$ [40]. It is also well known that there is a one-to-one correspondence between the lifts and spin structures on H. Considering one of the lifts and composing one of a finite dimensional representation V of $\mathrm{SL}(2,\mathbb{C})$, we get a representation $\varrho:\pi_1(\mathrm{H})\to\mathrm{SL}(V)$. Recall that for every positive integer n there is a unique irreducible representation V_n of $\mathrm{SL}(2,\mathbb{C})$ of dimension n, namely, $(n-1)$-th symmetric power of the standard representation $V_2=\mathbb{C}^2$. Considering V_n and all above, we get $\varrho_n:\pi_1(\mathrm{H})\to\mathrm{SL}(n,\mathbb{C})$.

Let H be a compact orientable non-elementary hyperbolic 3−manifold with a boundary consisting of ℓ surfaces $\Sigma_{g_1},\ldots,\Sigma_{g_\ell}$ of genus at least 2, and $n\geq 2$. Recall that H is non-elementary if its holonomy is an irreducible representation in $\mathrm{PSL}(2,\mathbb{C})$.

In [40; Theorem 0.1], P. Menal-Ferrer and J. Porti prove that the inclusion $\partial\mathrm{H}\subset\mathrm{H}$ induces an injection, $H^1\left(\mathrm{H};\mathfrak{sl}\left(n,\mathbb{C}\right)_{Ad_{\varrho n}}\right)\hookrightarrow H^1\left(\partial\mathrm{H};\mathfrak{sl}\left(n,\mathbb{C}\right)_{Ad_{\varrho n}}\right)$ with $\dim H^1\left(\mathrm{H};\mathfrak{sl}\left(n,\mathbb{C}\right)_{Ad_{\varrho n}}\right)=(1/2)$ $\dim H^1\left(\partial\mathrm{H};\mathfrak{sl}\left(n,\mathbb{C}\right)_{Ad_{\varrho n}}\right)$, and an isomorphism $H^2\left(\mathrm{H};\mathfrak{sl}\left(n,\mathbb{C}\right)_{Ad_{\varrho n}}\right)\cong H^2\left(\partial\mathrm{H};\mathfrak{sl}\left(n,\mathbb{C}\right)_{Ad_{\varrho n}}\right)$.

Theorem 6. Assume Σ_{g_i}, H, M, G, $\mathcal{G}$, ϱ, $\mathbf{h}_k^{\mathrm{H}}$, $\mathbf{h}_k^{\mathrm{M}}$, and $\mathbf{h}_j^{\Sigma_{g_i}}$ are as above. Then, the following formula is valid:

$$\mathbb{T}\left(\mathrm{H},\left\{\mathbf{h}_k^{\mathrm{H}}\right\}_0^3\right) = e^{\frac{\sqrt{-1}}{4}\left(-\beta_0+\ell\pi-\sum_{i=1}^{\ell}\theta_1^{\Sigma_{g_i}}\right)}\prod_{i=1}^{\ell}\Delta\left(\mathbf{h}_1^{\Sigma_{g_i}},\mathbf{h}_1^{\Sigma_{g_i}}\right)^{-1/4}$$
$$= e^{\frac{\sqrt{-1}}{4}\left(-\beta_0+\ell\pi-\sum_{i=1}^{\ell}\theta_1^{\Sigma_{g_i}}\right)}\prod_{i=1}^{\ell}\sqrt[4]{\delta\left(\mathbf{h}^{1,i},\mathbf{h}^{1,i}\right)}.$$

Here, $\left[\mathbf{h}_{1,1}^{0,\Sigma_{g_i}},\mathbf{h}_1^{\Sigma_{g_i}}\oplus\mathbf{h}_1^{\Sigma_{g_i}}\right]=\left|\left[\mathbf{h}_{1,1}^{0,\Sigma_{g_i}},\mathbf{h}_1^{\Sigma_{g_i}}\oplus\mathbf{h}_1^{\Sigma_{g_i}}\right]\right|e^{\sqrt{-1}\theta_1^{\Sigma_{g_i}}}$, $r_i:\pi_1\left(\Sigma_{g_i}\right)\to\pi_1(\mathrm{H})$ denotes the homomorphism obtained by the embedding $\Sigma_{g_i}\hookrightarrow\mathrm{H}$, $\beta_0=\dim H_0\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right)$ and $\mathbf{h}_{1,1}^{0,\Sigma_{g_i}}$ is a basis of $H_1\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)\oplus H_1\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)$ so that $\mathbb{T}\left(C_*\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)\oplus C_*\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right),\left\{\mathbf{h}_{1,1}^{0,\Sigma_{g_i}}\right\}\right)=1$, $\mathbf{h}^{j,i}$ is the Poincare dual basis of $H^j\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)$ corresponding to the basis $\mathbf{h}_j^{\Sigma_{g_i}}$ of $H_j\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)$.

The proof of Theorem 6 is based on considering the short-exact sequence

$$0\to\bigoplus_{i=1}^{\ell}C_*\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)\to C_*\left(\mathrm{H};\mathcal{G}_{Ad_\varrho}\right)\oplus C_*\left(\mathrm{H};\mathcal{G}_{Ad_\varrho}\right)\to C_*\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right)\to 0$$

and combining [28; Theorem 4.1] and [28; Theorem 4.3].

Combining these and Theorem 6, we have

Theorem 7. Considering $n=2$ and for $i=1,\ldots,\ell$, fixing a maximal geodesic lamination $\lambda_i\subset\Sigma_{g_i}$, if $\varrho_2:\pi_1\left(\mathrm{H}\right)\to\mathrm{SL}(2,\mathbb{C})$ is such that $\varrho_2\circ r_i\in\mathrm{Teich}\left(\Sigma_{g_i}\right)$, $i=1,\ldots,\ell$, applying (ii) of Theorem 5, and using the notation there, we get

$$\mathbb{T}\left(\mathrm{H},\left\{\mathbf{h}_k^{\mathrm{H}}\right\}_0^3\right)=e^{\frac{\sqrt{-1}}{4}\left(-\beta_0+\ell\pi-\sum_{i=1}^{\ell}\theta_1^{\Sigma_{g_i}}\right)}\,2^{\frac{1}{4}\sum_{i=1}^{\ell}\chi\left(\Sigma_{g_i};\mathfrak{sl}(2,\mathbb{C})\right)}\prod_{i=1}^{\ell}\sqrt[4]{\Omega_{\mathrm{T},i}}.$$

Here, $\Omega_{\mathrm{T},i}$ is the matrix of the complex Thurston symplectic form $\omega_{\mathrm{T}} : \mathcal{H}(\lambda_i;\mathbb{C}) \times \mathcal{H}(\lambda_i;\mathbb{C}) \to \mathbb{C}$ in the basis $\mathfrak{h}^i \oplus \sqrt{-1}\,\mathfrak{h}^i$, and $\mathbf{h}^{j,i}$ is the Poincare dual basis of $H^j\left(\Sigma_{g_i};\mathfrak{sl}(2,\mathbb{C})_{Ad_{\varrho_2\circ r_i}}\right)$ corresponding to $\mathbf{h}_j^{\Sigma_{g_i}}$, and $\mathfrak{h}^i$ is the basis of $\mathcal{H}(\lambda_i;\mathbb{R})$ associated with the isomorphism obtained by the real analytical embedding $\mathrm{Teich}(\Sigma_{g_i}) \hookrightarrow \mathcal{H}(\lambda_i;\mathbb{R})$ [32]. Here, $r_i : \pi_1(\Sigma_{g_i}) \to \pi_1(\mathrm{H})$ is the homomorphism obtained by the embedding $\Sigma_{g_i} \hookrightarrow \mathrm{H}$.

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Шоттки көрсетілімі мен Рейдмейстер бұралуы жайында ескерту

Мақалада Шоттки көрсетілімі үшін Рейдмейстердің бұралу формуласы анықталған. Теориялық нәтижелер 2-ден кем емес текті бағдарланған беттерден тұратын жиекті 3−көпбейнелерге қолданылады.

Кілт сөздер: Шоттки көрсетілімі, Рейдмейстер бұралуы, көрсетілімнің көпбейнелері, Атьи-Ботта-Голдман симплектикалық формасы, Терстонның симплектикалық формасы.

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Замечание о представлениях Шоттки и кручении Рейдемейстера

В статье установлена формула кручения Рейдемейстера для представлений Шоттки. Теоретические результаты применены к 3−многообразиям с краем, состоящим из ориентируемых поверхностей рода не менее 2.

Ключевые слова: представления Шоттки, кручение Рейдемейстера, многообразие представлений, симплектическая форма Атьи-Ботта-Голдмана, симплектическая форма Терстона.

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Minimizing sequences for a linear-quadratic control problem with three-tempo variables under weak nonlinear perturbations

The paper deals with the construction of minimizing sequences for the problem of minimizing a weakly nonlinearly perturbed quadratic performance index on trajectories of a weakly nonlinear system with three-tempo state variables. For this purpose, the so-called direct scheme for constructing an asymptotic solution is used, which consists in immediate substituting the postulated asymptotic expansion of the solution into the problem conditions and constructing a series of optimal control problems (in the case under consideration, linear-quadratic ones), the solutions of which are terms of the asymptotic expansion of the solution of the original nonlinear control problem. An estimate is obtained for the proximity of the optimal trajectory to the trajectory of the equation of state when some asymptotic approximation to the optimal control is used as a control. An example is given that illustrates in detail the proposed scheme for constructing minimizing sequences.

Keywords: three-tempo variables, nonlinear optimal control problems, asymptotic estimates, minimizing sequences.

Introduction

Mathematical models of many real processes contain multi-tempo fast variables. In review [1], there are 74 links to publications devoted to the study of such models.

Difficulties of using numerical methods for solving differential equations with quickly changing variables are well known. Therefore the employment of asymptotic methods is sometimes more preferable. The most popular method for asymptotic solving optimal control problems is constructing an asymptotic solution of problem following from control optimality conditions [2–4]. Another method, the so called direct scheme, consists of immediate substituting a postulated asymptotic solution into the problem condition and receiving a series of problems for finding asymptotic terms. The second approach allows to establish non-increasing of performance index values, if a next optimal control approximation is used, and gives the possibility to use standard programs for solving optimal control problems for finding asymptotics terms. For two-tempo systems, it is presented, for example, in [5, 6].

The direct scheme was applied in [7,8] for asymptotic solving an optimal control problem with weak nonlinear perturbations in a quadratic performance index and a linear state equation of the following form:

$$P_\varepsilon : \; J_\varepsilon(u) = \int_0^T (1/2(w(t,\varepsilon)'W(t)w(t,\varepsilon) + u(t,\varepsilon)'R(t)u(t,\varepsilon)) + \varepsilon F(w(t,\varepsilon),u(t,\varepsilon),t,\varepsilon))\,dt \to \min_u, \tag{1}$$

$$\mathcal{E}(\varepsilon)\frac{dw(t,\varepsilon)}{dt} = A(t)w(t,\varepsilon) + B(t)u(t,\varepsilon) + \varepsilon f(w(t,\varepsilon),u(t,\varepsilon),t,\varepsilon), \; t \in [0,T], \tag{2}$$

$$w(0,\varepsilon) = w^0. \tag{3}$$

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Here ε is a non-negative small parameter, $T > 0$ is fixed, the prime means transposition; $w(t,\varepsilon) = (x(t,\varepsilon)', y(t,\varepsilon)', z(t,\varepsilon)')'$, $x(t,\varepsilon) \in \mathbb{R}^{n_1}$, $y(t,\varepsilon) \in \mathbb{R}^{n_2}$, $z(t,\varepsilon) \in \mathbb{R}^{n_3}$, $u(t,\varepsilon) \in \mathbb{R}^m$; $\mathcal{E}(\varepsilon) = diag(I_{n_1}, \varepsilon I_{n_2}, \varepsilon^2 I_{n_3})$, I_{n_i} is the identity matrix of order n_i, $f = (\overset{(1)}{f}{}', \overset{(2)}{f}{}', \overset{(3)}{f}{}')'$, $\overset{(i)}{f} \in \mathbb{R}^{n_i}$, $B = (\overset{(1)}{B}{}', \overset{(2)}{B}{}', \overset{(3)}{B}{}')'$, $\overset{(i)}{B} : \mathbb{R}^m \to \mathbb{R}^{n_i}$, $i = \overline{1,3}$; all functions in (1), (2) are sufficiently smooth with respect to their arguments; for all $t \in [0,T]$ matrices $W(t)$, $R(t)$ are symmetric, moreover, $W(t)$, $R(t)$ and $S(t) = B(t)R(t)^{-1}B(t)'$ are positive definite.

The matrices A_{33} and $A_{22} - A_{23}A_{33}^{-1}A_{32}$ are assumed to be stable. Here and further A_{ij}, $i,j = \overline{1,3}$, mean matrices from a block representation of a matrix A with number of rows and columns n_1, n_2, n_3.

The rigorous justification of applying the direct scheme method to problem (1)–(2) is presented in [8]. The proof of estimates of the proximity between the exact solution and asymptotic one for the control, state trajectory and performance index value is also given. Moreover, this paper contains the proof of non-increasing performance index values when some new asymptotic approximations to the optimal control are used.

The construction of minimizing sequences is very important for approximate solving optimal control problems. Some facts concerning such sequences are given, for instance, in [9; 18, 22].

It should be noted that any illustrative examples are absent in [7, 8], though any example is very useful for understanding, in general, not simple algorithm of constructing minimizing sequences for problem (1)–(3). Such example is given in the present paper. A statement on estimate of the proximity between the optimal trajectory and a trajectory of system (2), (3), when some asymptotic approximation to the optimal control is used as control, is also proved here. In comparison with [8], some additional minimizing sequences are considered.

Some results of this paper were presented at the ICAAM 2022 [10].

This paper is organized as follows. For convenience, when considering an illustrative example, we present in the next section the algorithm of the direct scheme applied to problem (1)–(3) and give explicit formulas from [8] for linear-quadratic optimal control problems, solutions of which are asymptotic terms for a solution of problem (1)–(3). In section 2, we give some theorems on estimates from [8] and the proof of one theorem on an estimate of the proximity of the optimal trajectory to a trajectory of system (2), (3) under a special choice of the control. The last section is devoted to the detailed study of the first order approximation for an asymptotic solution of an illustrative example. A table containing values of the performance index for terms of constructed minimizing sequences is given.

1 Formalism of direct scheme method with explicit forms of problems for finding asymptotics terms

Following to the A.B. Vasil'eva's boundary function method [11], a solution of problem (1)–(3) is sought in the form

$$\vartheta(t,\varepsilon) = \overline{\vartheta}(t,\varepsilon) + \sum_{i=0}^{1} (\Pi_i \vartheta(\tau_i,\varepsilon) + Q_i \vartheta(\sigma_i,\varepsilon)), \tag{4}$$

where $\vartheta(t,\varepsilon) = (w(t,\varepsilon)', u(t,\varepsilon)')'$, $\tau_i = t/\varepsilon^{i+1}$, $\sigma_i = (t-T)/\varepsilon^{i+1}$, $i = 0,1$. Each term from (4) has an asymptotic expansion according to non-negative integer powers of the small parameter ε, i.e. $\overline{\vartheta}(t,\varepsilon) = \sum_{j\geq 0} \varepsilon^j \overline{\vartheta}_j(t)$, $\Pi_i\vartheta(\tau_i,\varepsilon) = \sum_{j\geq 0} \varepsilon^j \Pi_{ij}\vartheta(\tau_i)$, $Q_i\vartheta(\sigma_i,\varepsilon) = \sum_{j\geq 0} \varepsilon^j Q_{ij}\vartheta(\sigma_i)$. Here, $\overline{\vartheta}_j(t)$ are regular functions and $\Pi_{ij}\vartheta(\tau_i)$, $Q_{ij}\vartheta(\sigma_i)$ are boundary functions of exponential type in neighborhoods $t = 0$ and $t = T$ respectively.

For any sufficiently smooth function $G(w(t,\varepsilon), u(t,\varepsilon), t, \varepsilon)$ we will use the notation $G(\vartheta(t,\varepsilon), t, \varepsilon)$

and the asymptotic representation

$$G(\vartheta,t,\varepsilon)=\overline{G}(t,\varepsilon)+\sum_{i=0}^{1}(\Pi_iG(\tau_i,\varepsilon)+Q_iG(\sigma_i,\varepsilon)), \tag{5}$$

$\overline{G}(t,\varepsilon)=G(\overline{\vartheta}(t,\varepsilon),t,\varepsilon)=\sum_{j\geq 0}\varepsilon^j\overline{G}_j(t)$, $\Pi_0G(\tau_0,\varepsilon)=G(\overline{\vartheta}(\varepsilon\tau_0,\varepsilon)+\Pi_0\vartheta(\tau_0,\varepsilon),\varepsilon\tau_0,\varepsilon)-$ $-\,G(\overline{\vartheta}(\varepsilon\tau_0,\varepsilon),\varepsilon\tau_0,\varepsilon)=\sum_{j\geq 0}\varepsilon^j\Pi_{0j}G(\tau_0)$, $\Pi_1G(\tau_1,\varepsilon)=G(\overline{\vartheta}(\varepsilon^2\tau_1,\varepsilon)+\Pi_0\vartheta(\varepsilon\tau_1,\varepsilon)+$ $+\,\Pi_1\vartheta(\tau_1,\varepsilon),\varepsilon^2\tau_1,\varepsilon)-G(\overline{\vartheta}(\varepsilon^2\tau_1,\varepsilon)+\Pi_0\vartheta(\varepsilon\tau_1,\varepsilon),\varepsilon^2\tau_1,\varepsilon)=\sum_{j\geq 0}\varepsilon^j\Pi_{1j}G(\tau_1)$, $Q_0G(\sigma_0,\varepsilon)=G(\overline{\vartheta}(T+\varepsilon\sigma_0,\varepsilon)+Q_0\vartheta(\sigma_0,\varepsilon),T+\varepsilon\sigma_0,\varepsilon)-G(\overline{\vartheta}(T+\varepsilon\sigma_0,\varepsilon),T+\varepsilon\sigma_0,\varepsilon)=\sum_{j\geq 0}\varepsilon^jQ_{0j}G(\sigma_0)$, $Q_1G(\sigma_1,\varepsilon)=G(\overline{\vartheta}(T+\varepsilon^2\sigma_1,\varepsilon)+Q_0\vartheta(\varepsilon\sigma_1,\varepsilon)+Q_1\vartheta(\sigma_1,\varepsilon),T+\varepsilon^2\sigma_1,\varepsilon)-G(\overline{\vartheta}(T+\varepsilon^2\sigma_1,\varepsilon)+Q_0\vartheta(\varepsilon\sigma_1,\varepsilon),T+\varepsilon^2\sigma_1,\varepsilon)=\sum_{j\geq 0}\varepsilon^jQ_{1j}G(\sigma_1)$.

The first step of the algorithm of the direct scheme method consists of the substitution of expansion (4) into problem condition (1)–(3) taking into account (5). Equating in the transformed expressions for (2),(3) terms of the same powers of ε, separately depending on t, τ_i, σ_i, $i=0,1$, we obtain relations for defining asymptotics terms. Whence, in particular, it follows that

$$E_1\Pi_{00}w(\tau_0)=0,\ E_1\Pi_{10}w(\tau_1)=E_1\Pi_{11}w(\tau_1)=0,\ E_1Q_{00}w(\sigma_0)=0,$$
$$E_1Q_{10}w(\sigma_1)=E_1Q_{11}w(\sigma_1)=0,\ E_2\Pi_{10}w(\tau_1)=0,\ E_2Q_{10}w(\sigma_1)=0.$$

With the help of passing in the integrals from the expressions depending on τ_i, σ_i, $i=0,1$, to integrals over the corresponding intervals $[0,+\infty)$ and $(-\infty,0]$, in the transformed integrand from (1) the functional $J_\varepsilon(u)$ is written in the form

$$J_\varepsilon(u)=\sum_{j\geq 0}\varepsilon^jJ_j. \tag{6}$$

Analyzing the structure of coefficients J_j with even and odd indices separately, five linear-quadratic optimal control problems $\overline{P}_j$, $\Pi_{ij}P$, $Q_{ij}P$, $i=0,1$, solutions of which are terms of asymptotic solution of problem (1)–(3), are formulated in [8]. Further, the explicit formulas for these problems will be given.

Let us introduce the following notation:

$$E_1=diag(I_{n_1},0,0),\ E_2=diag(0,I_{n_2},0),\ E_3=diag(0,0,I_{n_3}),$$
$$\phi(\vartheta,t,\varepsilon)=A(t)w(t,\varepsilon)+B(t)u(t,\varepsilon)+\varepsilon f(w(t,\varepsilon),u(t,\varepsilon),t,\varepsilon),$$
$$\rho(\vartheta,\psi,t,\varepsilon)=W(t)w(t,\varepsilon)-A(t)'\psi(t,\varepsilon)+\varepsilon(F_w(\vartheta,t,\varepsilon)'-f_w(\vartheta,t,\varepsilon)'\psi(t,\varepsilon)),$$
$$\chi(\vartheta,\psi,t,\varepsilon)=R(t)u(t,\varepsilon)-B(t)'\psi(t,\varepsilon)+\varepsilon(F_u(\vartheta,t,\varepsilon)'-f_u(\vartheta,t,\varepsilon)'\psi(t,\varepsilon)).$$

The coefficient with ε^j in an expansion of a function $\omega=\omega(\varepsilon)$ in a series in powers of ε will be denoted by w_j or $[w]_j$. The k-th partial sum of a series will denoted by upper wave and the low index k, i.e. $\widetilde{\omega}_k=\sum_{j=0}^{k}\varepsilon^j\omega_j$. The hat and the low index k in a function notation will be mean that the function is calculated with the functional argument equal to the k-th partial sum of the corresponding expansion, e.g., $\widehat{\overline{f}}_k(t,\varepsilon)=f(\widetilde{\overline{\vartheta}}_k(t,\varepsilon),t,\varepsilon)$. Functions with negative indices will be considered equal to zero.

In the following expressions with ρ and χ in the performance indices of the formulated linear-quadratic optimal control problems we take $\psi(t,\varepsilon)=\sum_{j=0}^{\infty}\varepsilon^j(\overline{\psi}_j(t)+(\varepsilon E_1+E_2+E_3)(\Pi_{0j}\psi(\tau_0)+Q_{0j}\psi(\sigma_0))+(\varepsilon^2E_1+\varepsilon E_2+E_3)(\Pi_{1j}\psi(\tau_1)+Q_{1j}\psi(\sigma_1)))$, where $\overline{\psi}_j$, $\Pi_{ij}\psi(\tau_i)$, $Q_{ij}\psi(\sigma_i)$, $i=0,1$, are costate variables in problems $\overline{P}_j$, $\Pi_{ij}P$, $Q_{ij}P$, $i=0,1$, respectively.

Regular functions $\overline{\vartheta}_j(t)$, $t \in [0,T]$, are solutions of the following problems

$$\overline{P}_j: \ \overline{J}_j(\overline{u}_j) = \overline{w}_j(T)' E_1 (Q_{0(j-1)}\psi(0) + Q_{1(j-2)}\psi(0)) + \int\limits_0^T (\overline{w}_j(t)'(\frac{1}{2}W(t)\overline{w}_j(t) + $$
$$+ [\widehat{\overline{\rho}}_{j-1}(t,\varepsilon)]_j - E_2\frac{d\overline{\psi}_{j-1}(t)}{dt} - E_3\frac{d\overline{\psi}_{j-2}(t)}{dt}) + \overline{u}_j(t)'(\frac{1}{2}R(t)\overline{u}_j(t) + [\widehat{\overline{\chi}}_{j-1}(t,\varepsilon)]_j))\, dt \to \min_{\overline{u}_j}, \tag{7}$$

$$E_1\frac{d\overline{w}_j(t)}{dt} + E_2\frac{d\overline{w}_{j-1}(t)}{dt} + E_3\frac{d\overline{w}_{j-2}(t)}{dt} = [\overline{\phi}(t,\varepsilon)]_j, \tag{8}$$

$$E_1\overline{w}_j(0) = E_1 w^0, \ j=0, \ E_1(\overline{w}_j(0) + \Pi_{0j}w(0)) = 0, \ j=1,$$
$$E_1(\overline{w}_j(0) + \Pi_{0j}w(0) + \Pi_{1j}w(0)) = 0, \ j \geq 2. \tag{9}$$

The boundary functions $\Pi_{0j}\vartheta(\tau_0)$, $\tau_0 \in [0,+\infty)$, are solutions of the problems

$$\Pi_{0j}P: \ \Pi_{0j}J(\Pi_{0j}u) = \int\limits_0^{+\infty} (\Pi_{0j}w(\tau_0)'(\frac{1}{2}W(0)\Pi_{0j}w(\tau_0) + [\widehat{\Pi}_{0(j-1)}\rho(\tau_0,\varepsilon)]_j - E_3\frac{d\Pi_{0(j-1)}\psi(\tau_0)}{d\tau_0}) +$$
$$+ \Pi_{0j}u(\tau_0)'(\frac{1}{2}R(0)\Pi_{0j}u(\tau_0) + [\widehat{\Pi}_{0(j-1)}\chi(\tau_0,\varepsilon)]_j))\, d\tau_0 \to \min_{\Pi_{0j}u}, \tag{10}$$

$$(E_1+E_2)\frac{d\Pi_{0j}w(\tau_0)}{d\tau_0} + E_3\frac{d\Pi_{0(j-1)}w(\tau_0)}{d\tau_0} = E_1[\Pi_0\phi(\tau_0,\varepsilon)]_{j-1} + (E_2+E_3)[\Pi_0\phi(\tau_0,\varepsilon)]_j, \tag{11}$$

$$\Pi_{0j}x(+\infty) = 0, \ E_2(\overline{w}_j(0) + \Pi_{0j}w(0)) = E_2 w^0, \ j=0,$$
$$E_2(\overline{w}_j(0) + \Pi_{0j}w(0) + \Pi_{1j}w(0)) = 0, \ j \geq 1. \tag{12}$$

The boundary functions $Q_{0j}\vartheta(\sigma_0)$, $\sigma_0 \in (-\infty, 0]$, are solutions of the problems

$$Q_{0j}P: \ Q_{0j}J(Q_{0j}u) = Q_{0j}w(0)'E_2(\overline{\psi}_j(T) + Q_{1(j-1)}\psi(0)) +$$
$$+ \int\limits_{-\infty}^0 (Q_{0j}w(\sigma_0)'(\frac{1}{2}W(T)Q_{0j}w(\sigma_0) + [\widehat{Q}_{0(j-1)}\rho(\sigma_0,\varepsilon)]_j - E_3\frac{dQ_{0(j-1)}\psi(\sigma_0)}{d\sigma_0}) +$$
$$+ Q_{0j}u(\sigma_0)'(\frac{1}{2}R(T)Q_{0j}u(\sigma_0) + [\widehat{Q}_{0(j-1)}\chi(\sigma_0,\varepsilon)]_j))\, d\sigma_0 \to \min_{Q_{0j}u}, \tag{13}$$

$$(E_1+E_2)\frac{dQ_{0j}w(\sigma_0)}{d\sigma_0} + E_3\frac{dQ_{0(j-1)}w(\sigma_0)}{d\sigma_0} = E_1[Q_0\phi(\sigma_0,\varepsilon)]_{j-1} + (E_2+E_3)[Q_0\phi(\sigma_0,\varepsilon)]_j, \tag{14}$$

$$(E_1+E_2)Q_{0j}w(-\infty) = 0. \tag{15}$$

The boundary functions $\Pi_{1j}\vartheta(\tau_1)$, $\tau_1 \in [0,+\infty)$ are solutions of the problems

$$\Pi_{1j}P: \ \Pi_{1j}J(\Pi_{1j}u) = \int\limits_0^{+\infty} (\Pi_{1j}w(\tau_1)'(\frac{1}{2}W(0)\Pi_{1j}w(\tau_1) + [\widehat{\Pi}_{1(j-1)}\rho(\tau_1,\varepsilon)]_j) +$$
$$+ \Pi_{1j}u(\tau_1)'(\frac{1}{2}R(0)\Pi_{1j}u(\tau_1) + [\widehat{\Pi}_{1(j-1)}\chi(\tau_1,\varepsilon)]_j))\, d\tau_1 \to \min_{\Pi_{1j}u}, \tag{16}$$

$$\frac{d\Pi_{1j}w(\tau_1)}{d\tau_1} = E_1[\Pi_1\phi(\tau_1,\varepsilon)]_{j-2} + E_2[\Pi_1\phi(\tau_1,\varepsilon)]_{j-1} + E_3[\Pi_1\phi(\tau_1,\varepsilon)]_j, \tag{17}$$

$$(E_1+E_2)\Pi_{1j}w(+\infty)=0,\ E_3(\overline{w}_j(0)+\Pi_{0j}w(0)+\Pi_{1j}w(0))=\begin{cases}E_3w^0,\ j=0,\\ 0,\ j\geq 1.\end{cases} \tag{18}$$

The boundary functions $Q_{1j}\vartheta(\sigma_1)$, $\sigma_1\in(-\infty,0]$ are solutions of the problems

$$Q_{1j}P:\ Q_{1j}J(Q_{1j}u)=Q_{1j}w(0)'E_3(\overline{\psi}_j(T)+Q_{0j}\psi(0))+\int\limits_{-\infty}^{0}(Q_{1j}w(\sigma_1)'(\frac{1}{2}W(T)Q_{1j}w(\sigma_1)+$$
$$+[\widehat{Q}_{1(j-1)}\rho(\sigma_1,\varepsilon)]_j)+Q_{1j}u(\sigma_1)'(\frac{1}{2}R(T)Q_{1j}u(\sigma_1)+[\widehat{Q}_{1(j-1)}\chi(\sigma_1,\varepsilon)]_j))\,d\sigma_1\to\min_{Q_{1j}u}, \tag{19}$$

$$\frac{dQ_{1j}w(\sigma_1)}{d\sigma_1}=E_1[Q_1\phi(\sigma_1,\varepsilon)]_{j-2}+E_2[Q_1\phi(\sigma_1,\varepsilon)]_{j-1}+E_3[Q_1\phi(\sigma_1,\varepsilon)]_j, \tag{20}$$

$$Q_{1j}w(-\infty)=0. \tag{21}$$

2 Asymptotic estimates

Let eigenvalues of the matrix $\begin{pmatrix}A_{33} & S_{33}\\ W_{33} & -A_{33}'\end{pmatrix}$ are different for all $t\in[0,T]$ (condition I from [8]) and the same condition is satisfied for the matrix
$\begin{pmatrix}A_{22} & S_{22}\\ W_{22} & -A_{22}'\end{pmatrix}-\begin{pmatrix}A_{23} & S_{23}\\ W_{23} & -A_{32}'\end{pmatrix}\begin{pmatrix}A_{33} & S_{33}\\ W_{33} & -A_{33}'\end{pmatrix}^{-1}\begin{pmatrix}A_{32} & S_{23}'\\ W_{23}' & -A_{23}'\end{pmatrix}$ (condition II from [8]).

Under these conditions, the following Theorems 1–3 have been proved in [8].

Theorem 1. Solution $\vartheta_*(t,\varepsilon)$ of problem P_ε for sufficiently small $\varepsilon>0$, $t\in[0,T]$, satisfies the inequality

$$\|\vartheta_*(t,\varepsilon)-\widetilde{\vartheta}_n(t,\varepsilon)\|\leq c\varepsilon^{n+1}.$$

Here and further c is a positive constant independent of t, ε.

Note a slip of the presentation of Theorem 1 in [8], namely, the condition II for this theorem has been formulated inside of the theorem proof.

Theorem 2. For sufficiently small $\varepsilon>0$, the following inequality for the performance index is valid

$$J_\varepsilon(\widetilde{u}_n)-J_\varepsilon(u_*)\leq c\varepsilon^{2n+2}.$$

Theorem 3. For sufficiently small $\varepsilon>0$, the following inequalities are valid

$$J_\varepsilon(\widetilde{u}_{*(n-1)})\geqslant J_\varepsilon(\widetilde{u}_{*(n-1)}+\varepsilon^n\overline{u}_{*n})\geqslant$$
$$\geqslant J_\varepsilon(\widetilde{u}_{*(n-1)}+\varepsilon^n(\overline{u}_{*n}+\Pi_{0n}u_*+Q_{0n}u_*))\geqslant J_\varepsilon(\widetilde{u}_{*n}),\ n\geqslant 1.$$

If an addition to $\widetilde{u}_{*(n-1)}$ is non-zero, then the corresponding inequality is strict.

Here the notation $\widetilde{u}_*$, $j=n,n-1$ is used for the j-th order approximation for the optimal control u_*.

Denote by $\widetilde{w}=\widetilde{w}(t,\varepsilon)$ a solution of problem (2)–(3) at $u=\widetilde{u}_{*n}$ and $\delta w=\delta w(t,\varepsilon)=w_*(t,\varepsilon)-\widetilde{w}(t,\varepsilon)$, $\delta u=\delta u(t,\varepsilon)=u_*(t,\varepsilon)-\widetilde{u}_{*n}(t,\varepsilon)$.

Under proving Theorem 2 in [8], the estimate for $\delta w(t,\varepsilon)$ has been used without the rigorous proof. The proof of it will be given below, i.e. we will prove the following.

Theorem 4. For sufficiently small $\varepsilon>0$, the inequality

$$\|\delta w(t,\varepsilon)\|\leqslant c\varepsilon^{n+1} \tag{22}$$

is fulfilled.

Proof. It follows from (2), (3), that δw satisfies the system

$$\mathcal{E}(\varepsilon)\frac{d\delta w(t,\varepsilon)}{dt} = A(t)\delta w(t,\varepsilon) + B(t)\delta u(t,\varepsilon) + \varepsilon(f(w_*(t,\varepsilon),u_*(t,\varepsilon),t,\varepsilon) -$$
$$-f(w_*(t,\varepsilon)-\delta w(t,\varepsilon),u_*(t,\varepsilon)-\delta u(t,\varepsilon),t,\varepsilon)),$$
$$\delta w(0,\varepsilon)=0.$$

In view of Theorem 1

$$\|\delta u(t,\varepsilon)\| \leqslant c\varepsilon^{n+1}. \tag{23}$$

Write out the problem for $\delta w = (\delta x', \delta y', \delta z')'$ in the form

$$\frac{d\delta x}{dt} = A_{11}(t)\delta x + A_{12}(t)\delta y + A_{13}(t)\delta z + \overset{(1)}{B}(t)\delta u + \varepsilon\overset{(1)}{g}(\delta w,\delta u,t,\varepsilon),\ \delta x(0,\varepsilon)=0, \tag{24}$$

$$\varepsilon\frac{d\delta y}{dt} = A_{21}(t)\delta x + A_{22}(t)\delta y + A_{23}(t)\delta z + \overset{(2)}{B}(t)\delta u + \varepsilon\overset{(2)}{g}(\delta w,\delta u,t,\varepsilon),\ \delta y(0,\varepsilon)=0, \tag{25}$$

$$\varepsilon^2\frac{d\delta z}{dt} = A_{31}(t)\delta x + A_{32}(t)\delta y + A_{33}(t)\delta z + \overset{(3)}{B}(t)\delta u + \varepsilon\overset{(3)}{g}(\delta w,\delta u,t,\varepsilon),\ \delta z(0,\varepsilon)=0, \tag{26}$$

where $\overset{(i)}{g}(\delta w,\delta u,t,\varepsilon) = \varepsilon(\overset{(i)}{f}(w_*,u_*,t,\varepsilon) - \overset{(i)}{f}(w_*-\delta w,u_*-\delta u,t,\varepsilon))$, $i=\overline{1,3}$.

For brevity, the arguments t, ε are dropped in some of the last relations.

Using the fundamental matrix $\overset{(3)}{U}(t,s,\varepsilon)$ of the system

$$\varepsilon^2\frac{dZ}{dt} = A_{33}(t)Z, \tag{27}$$

we write out the problem (26) in the integral form

$$\delta z(t,\varepsilon) = \frac{1}{\varepsilon^2}\int_0^t \overset{(3)}{U}(t,s,\varepsilon)(A_{31}(s)\delta x(s,\varepsilon) + A_{32}(s)\delta y(s,\varepsilon) + \overset{(3)}{B}(s)\delta u(s,\varepsilon) + \\ +\varepsilon\overset{(3)}{g}(\delta w(s,\varepsilon),\delta u(s,\varepsilon),s,\varepsilon))\,ds. \tag{28}$$

Due to stability of the matrix $A_{33}(t)$ the matrix $\overset{(3)}{U}(t,s,\varepsilon)$ has the estimate [10; 69]

$$\|\overset{(3)}{U}(t,s,\varepsilon)\| \leqslant c\exp(-\frac{æ(t-s)}{\varepsilon^2}), \tag{29}$$

where $0 \leqslant s \leqslant t \leqslant T$ and here and further æ is a positive constant independent of t, ε.

In the following, we will denote functions, appearing under transformations of problems (24)–(26) and satisfying the next two conditions 1) and 2) by $\overset{(i)}{h}(\delta w,t,\varepsilon)$, $i=\overline{1,3}$. Specific forms of these functions are omitted since they are insignificant for the proof.

1) For any $q>0$, there exist such constants $\Delta=\Delta(q)$ and $\varepsilon_0=\varepsilon_0(q)$ that, for $\|\delta w_i\|_{C_{[0,T]}} \leqslant \Delta$, $i=\overline{1,2}$, $0<\varepsilon\leqslant\varepsilon_0$

$$\|\overset{(i)}{h}(\delta w_1,t,\varepsilon) - \overset{(i)}{h}(\delta w_2,t,\varepsilon)\| \leqslant q\|\delta w_1-\delta w_2\|_{C_{[0,T]}},$$

2) $\|\overset{(i)}{h}(0,t,\varepsilon)\| \leqslant c\varepsilon^{n+1}$.

In view of (27), taking into account that $\overset{(3)}{U}(t,s,\varepsilon) = \overset{(3)}{U}(t,\varepsilon)\overset{(3)}{U}(s,\varepsilon)^{-1}$, we have

$$\varepsilon^2 \frac{\partial \overset{(3)}{U}(t,s,\varepsilon)}{\partial s} = -\varepsilon^2 \overset{(3)}{U}(t,\varepsilon)\overset{(3)}{U}(s,\varepsilon)^{-1}\frac{d\overset{(3)}{U}(s,\varepsilon)}{ds}\overset{(3)}{U}(s,\varepsilon)^{-1} =$$
$$= -\overset{(3)}{U}(t,s,\varepsilon)A_{33}(s)\overset{(3)}{U}(s,\varepsilon)\overset{(3)}{U}(s,\varepsilon)^{-1} = -\overset{(3)}{U}(t,s,\varepsilon)A_{33}(s).$$

It follows from here that

$$\overset{(3)}{U}(t,s,\varepsilon) = -\varepsilon^2 \frac{\partial \overset{(3)}{U}(t,s,\varepsilon)}{\partial s} A_{33}(s)^{-1}. \tag{30}$$

Substituting this expression into (28), then integrating by parts the terms, containing δx and δy, due to the initial values $\delta y(0,\varepsilon)$, $\delta x(0,\varepsilon)$ and the estimates (23), (29) we obtain

$$\delta z(t,\varepsilon) = -A_{33}(t)^{-1}A_{31}(t)\delta x(t,\varepsilon) - A_{33}(t)^{-1}A_{32}(t)\delta y(t,\varepsilon) + \overset{(3)}{h}(\delta w,t,\varepsilon). \tag{31}$$

In view of the last expression, we get from (25) the problem

$$\varepsilon\frac{d\delta y}{dt} = (A_{21}(t) - A_{23}(t)A_{33}(t)^{-1}A_{31}(t))\delta x + (A_{22}(t) - A_{23}(t)A_{33}(t)^{-1}A_{32}(t))\delta y +$$
$$+ \overset{(2)}{h}(\delta w,t,\varepsilon),\ \ \delta y(0,\varepsilon) = 0. \tag{32}$$

Using the fundamental matrix $\overset{(2)}{U}(t,s,\varepsilon)$ of the system

$$\varepsilon\frac{dY}{dt} = (A_{22}(t) - A_{23}(t)A_{33}(t)^{-1}A_{32}(t))Y,$$

we write out the problem (32) in the integral form

$$\delta y(t,\varepsilon) = \frac{1}{\varepsilon}\int_0^t \overset{(2)}{U}(t,s,\varepsilon)((A_{21}(s) - A_{23}(s)A_{33}(s)^{-1}A_{31}(s))\delta x(s,\varepsilon) + \overset{(2)}{h}(\delta w,s,\varepsilon))\,ds. \tag{33}$$

Due to stability of the matrix $A_{22}(t) - A_{23}(t)A_{33}(t)^{-1}A_{32}(t)$ the matrix $\overset{(2)}{U}(t,s,\varepsilon)$ has the estimate [10; 69].

$$\|\overset{(2)}{U}(t,s,\varepsilon)\| \leqslant c\exp(-\frac{æ(t-s)}{\varepsilon}),\ \ 0 \leqslant s \leqslant t \leqslant T. \tag{34}$$

Analogously to (30) we get the relation

$$\overset{(2)}{U}(t,s,\varepsilon) = -\varepsilon\frac{\partial \overset{(2)}{U}(t,s,\varepsilon)}{\partial s}(A_{22}(s) - A_{23}(s)A_{33}(s)^{-1}A_{32}(s))^{-1}.$$

Substituting this expression into (33), then integrating by parts the term containing δx, due to the initial value $\delta x(0,\varepsilon)$ and the estimates (23), (34) we obtain

$$\delta y(t,\varepsilon) = -(A_{22}(t) - A_{23}(t)A_{33}(t)^{-1}A_{32}(t))^{-1}(A_{21}(t) - A_{23}(t)A_{33}(t)^{-1}A_{31}(t))\delta x(t,\varepsilon) +$$
$$+ \overset{(2)}{h}(\delta w,t,\varepsilon). \tag{35}$$

Substituting the expressions (31), (35) into equation (24), we obtain the problem of the form

$$\frac{d\delta x}{dt} = C(t)\delta x + \overset{(1)}{h}(\delta w, t, \varepsilon),\ \delta x(0,\varepsilon) = 0. \tag{36}$$

The explicit expression for $C(t)$ can be easy written taking into account (24), (31), (35).

The fundamental matrix $\overset{(1)}{U}(t,s)$ of the system $\frac{dX}{dt} = C(t)X$ is bounded, i.e. $\|\overset{(1)}{U}(t,s)\| \leqslant c$. Therefore, writing the problem (36) in an integral form, we get

$$\delta x(t,\varepsilon) = \overset{(1)}{h}(\delta w, t, \varepsilon).$$

So, from the last relation and (31), (35) it follows that system (24)–(26) can be written in the form

$$\delta w = h(\delta w, t, \varepsilon), \tag{37}$$

where h satisfies to the conditions 1) and 2).

If we take in condition 2) $q < 1$ then h willbe a contraction mapping in $C_{[0,T]}$. According to the contractive mapping principle, equation (37) has a unique solution, and this solution can be found by the method of successive approximations.

In view of condition 2) and Lemma 3 in [8] we obtain the provable estimate (22).

The proof of Theorem 3 in [8] is based on the following theorem on the structure of coefficients J_j in (6) proven in [8] .

Theorem 5. The sum $\overline{J}_j + \Pi_{1(j-1)}J + Q_{1(j-1)}J$ of the performance indices in problems $\overline{P}_j$, $\Pi_{1(j-1)}P$, $Q_{1(j-1)}P$ is obtained by transforming the coefficient J_{2j} in expansion (6) and dropping terms, which are known after solving problems $\overline{P}_k$, $\Pi_{0k}P$, $Q_{0k}P$, $k = \overline{0, j-1}$, $\Pi_{1k}P$, $Q_{1k}P$, $k = \overline{0, j-2}$. The sum $\Pi_{0j}J + Q_{0j}J$ of the performance indices in problems $\Pi_{0j}P$, $Q_{0j}P$ is obtained by transforming the coefficient $J_{2(j+1)}$ in expansion (6) and dropping terms, which are known after solving problems $\overline{P}_k$, $k = \overline{0, j}$, $\Pi_{ik}P$, $Q_{ik}P$, $i = 0, 1$, $k = \overline{0, j-1}$.

Similarly, using Theorem 5, we can establish some generalization of Theorem 3.

Theorem 6. For sufficiently small $\varepsilon > 0$, $\{u_{*(n-1)}\}$ and the sequence with terms, obtained by supplementing to $\widetilde{u}_{*(n-1)}$ one or several terms from the expansions (4) for the optimal control u_* with ε^n are minimizing.

Detailing, $\{\widetilde{u}_{*(n-1)}\}$ and the sequence $\{\widetilde{u}_{*(n-1)} + \varepsilon^n \overline{u}_{*n}\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n \Pi_{0n}u_*\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n Q_{0n}u_*\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n \Pi_{1n}u_*\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n Q_{1n}u_*\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{0n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + Q_{0n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{1n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + Q_{1n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\Pi_{0n}u_* + Q_{0n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\Pi_{0n}u_* + \Pi_{1n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\Pi_{0n}u_* + Q_{1n}u_*)\}$, ..., $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{0n}u_* + Q_{0n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{0n}u_* + \Pi_{1n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{0n}u_* + Q_{1n}u_*)\}$,..., $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{0n}u_* + Q_{0n}u_* + \Pi_{1n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{0n}u_* + Q_{0n}u_* + Q_{1n}u_*)\}$ are minimizing.

3 *Illustrative example*

Let us consider the problem P_ε with $n_i = 1$, $u = (\overset{(1)}{u}, \overset{(2)}{u}, \overset{(3)}{u})'$, $\overset{(i)}{u} \in \mathbb{R}$, $i = \overline{1,3}$, of the form

$$\begin{gathered} J_\varepsilon(u) = \int\limits_0^1 (\frac{1}{2}(x(t,\varepsilon)^2 + y(t,\varepsilon)^2 + z(t,\varepsilon)^2 + \overset{(1)}{u}(t,\varepsilon)^2 + \overset{(2)}{u}(t,\varepsilon)^2 + \overset{(3)}{u}(t,\varepsilon)^2) + \\ + \varepsilon x(t,\varepsilon)\overset{(1)}{u}(t,\varepsilon) + \varepsilon y(t,\varepsilon)\overset{(3)}{u}(t,\varepsilon))\, dt \to \min_u, \end{gathered} \tag{38}$$

$$\frac{dx(t,\varepsilon)}{dt} = x(t,\varepsilon) + \overset{(1)}{u}(t,\varepsilon),$$
$$\varepsilon\frac{dy(t,\varepsilon)}{dt} = -y(t,\varepsilon) + \overset{(2)}{u}(t,\varepsilon) + \varepsilon\overset{(3)}{u}(t,\varepsilon), \tag{39}$$
$$\varepsilon^2\frac{dz(t,\varepsilon)}{dt} = -z(t,\varepsilon) + \overset{(3)}{u}(t,\varepsilon) + \varepsilon y(t,\varepsilon),$$
$$x(0,\varepsilon) = y(0,\varepsilon) = z(0,\varepsilon) = 10. \tag{40}$$

By setting $\varepsilon = 0$ in (38)–(40), taking into account the equalities $\Pi_{i0}x(\tau_i) = 0$, $i = 0, 1$, we obtain degenerate problem $\overline{P}_0$:

$$\overline{J}_0(\overline{u}_0) = \frac{1}{2}\int_0^1 (\overline{x}_0(t)^2 + \overline{y}_0(t)^2 + \overline{z}_0(t)^2 + \overset{(1)}{\overline{u}}_0(t)^2 + \overset{(2)}{\overline{u}}_0(t)^2 + \overset{(3)}{\overline{u}}_0(t)^2)\,dt \to \min_{\overline{u}_0},$$

$$\frac{d\overline{x}_0(t)}{dt} = \overline{x}_0(t) + \overset{(1)}{\overline{u}}_0(t),\ \overline{x}_0(0) = 10,$$
$$0 = -\overline{y}_0(t) + \overset{(2)}{\overline{u}}_0(t),$$
$$0 = -\overline{z}_0(t) + \overset{(3)}{\overline{u}}_0(t).$$

The form of this problem follows also from (7)–(9) with $j = 0$. It is not difficult to find the solution

$$\overline{x}_0(t) = 2a((\sqrt{2}+1)e^{\sqrt{2}t} + (\sqrt{2}-1)e^{-\sqrt{2}(t-2)}),\ \overset{(1)}{\overline{u}}_0(t) = 2a(e^{\sqrt{2}t} - e^{-\sqrt{2}(t-2)}),$$
$$\text{where } a = 5/((\sqrt{2}-1)e^{2\sqrt{2}} + \sqrt{2} + 1),$$
$$\overline{y}_0(t) = \overline{z}_0(t) = \overset{(2)}{\overline{u}}_0(t) = \overset{(3)}{\overline{u}}_0(t) = 0.$$

Using (10)–(12) at $j = 0$ and taking into account that $\Pi_{00}x(\tau_0) = 0$, we write out problem $\Pi_{00}P$ in the following form:

$$\Pi_{00}J(\Pi_{00}u) = \frac{1}{2}\int_0^{+\infty} (\Pi_{00}y(\tau_0)^2 + \Pi_{00}z(\tau_0)^2 + \Pi_{00}\overset{(1)}{u}(\tau_0)^2 + \Pi_{00}\overset{(2)}{u}(\tau_0)^2 + \Pi_{00}\overset{(3)}{u}(\tau_0)^2)\,d\tau_0 \to \min_{\Pi_{00}u},$$

$$\frac{d\Pi_{00}y(\tau_0)}{d\tau_0} = -\Pi_{00}y(\tau_0) + \Pi_{00}\overset{(2)}{u}(\tau_0),\ \overline{y}_0(0) + \Pi_{00}y(0) = 10,$$
$$0 = -\Pi_{00}z(\tau_0) + \Pi_{00}\overset{(3)}{u}(\tau_0).$$

Using (13)–(15) at $j = 0$ and taking into account that $Q_{00}x(\sigma_0) = 0$, we write out problem $Q_{00}P$ in the following form:

$$Q_{00}J(Q_{00}u) = Q_{00}y(0)\overline{\eta}_0(1) + \frac{1}{2}\int_{-\infty}^{0} (Q_{00}y(\sigma_0)^2 + Q_{00}z(\sigma_0)^2 + Q_{00}\overset{(1)}{u}(\sigma_0)^2 +$$
$$+ Q_{00}\overset{(2)}{u}(\sigma_0)^2 + Q_{00}\overset{(3)}{u}(\sigma_0)^2)\,d\sigma_0 \to \min_{Q_{00}u},$$

$$\frac{dQ_{00}y(\sigma_0)}{d\sigma_0} = -Q_{00}y(\sigma_0) + Q_{00}\overset{(2)}{u}(\sigma_0),$$
$$0 = -Q_{00}z(\sigma_0) + Q_{00}\overset{(3)}{u}(\sigma_0),$$

$$Q_{00}y(-\infty) = 0.$$

In this example, $\overline{\psi}_j(t) = (\overline{\xi}_j(t), \overline{\eta}_j(t), \overline{\zeta}_j(t))'$ means a costate variable for the problem $\overline{P}_j$, $\Pi_{ij}\psi(\tau_i) = (\Pi_{ij}\xi(\tau_i), \Pi_{ij}\eta(\tau_i), \Pi_{ij}\zeta(\tau_i))'$ is a costate variable for the problem $\Pi_{ij}P$, and $Q_{ij}\psi(\sigma_i) = (Q_{ij}\xi(\sigma_i), Q_{ij}\eta(\sigma_i), Q_{ij}\zeta(\sigma_i))'$ is a costate variable for the problem $Q_{ij}P$, $i = 0, 1$.

Using (16)–(18) at $j = 0$ and taking into account that $\Pi_{10}x(\tau_1) = \Pi_{10}y(\tau_1) = 0$, we write out problem $\Pi_{10}P$ in the following form:

$$\Pi_{10}J(\Pi_{10}u) = \frac{1}{2}\int\limits_0^{+\infty} (\Pi_{10}z(\tau_1)^2 + \Pi_{10}\overset{(1)}{u}(\tau_1)^2 + \Pi_{10}\overset{(2)}{u}(\tau_1)^2 + \Pi_{10}\overset{(3)}{u}(\tau_1)^2)\, d\tau_1 \to \min_{\Pi_{10}u},$$

$$\frac{d\Pi_{10}z(\tau_1)}{d\tau_1} = -\Pi_{10}z(\tau_1) + \Pi_{10}\overset{(3)}{u}(\tau_1),$$

$$\overline{z}_0(0) + \Pi_{00}z(0) + \Pi_{10}z(0) = 10.$$

Using (19)–(21) at $j = 0$ and taking into account that $Q_{10}x(\sigma_1) = Q_{10}y(\sigma_1) = 0$, we write out problem $Q_{10}P$ in the following form:

$$Q_{10}J(Q_{10}u) = Q_{10}z(0)(\overline{\zeta}_0(1) + Q_{00}\zeta(0)) + \frac{1}{2}\int\limits_{-\infty}^{0} (Q_{10}z(\sigma_1)^2 + Q_{10}\overset{(1)}{u}(\sigma_1)^2 +$$

$$+ Q_{10}\overset{(2)}{u}(\sigma_1)^2 + Q_{10}\overset{(3)}{u}(\sigma_1)^2)\, d\sigma_1 \to \min_{Q_{10}u},$$

$$\frac{dQ_{10}z(\sigma_1)}{d\sigma_1} = -Q_{10}z(\sigma_1) + Q_{10}\overset{(3)}{u}(\sigma_1),$$

$$Q_{10}z(-\infty) = 0.$$

Taking into account the solution of problem $\overline{P}_0$ and solving problems $\Pi_{i0}P$, $Q_{i0}P$, $i = 0, 1$, we get the zero order approximation of an asymptotic solution of problem (38)–(40) of the form (4).

$$\widetilde{x}_0(t,\varepsilon) = \overline{x}_0(t),\ \overset{(1)}{\widetilde{u}}_0(t,\varepsilon) = \overset{(1)}{\overline{u}}_0(t),\ \widetilde{y}_0(t,\varepsilon) = 10e^{-\sqrt{2}t/\varepsilon},$$

$$\overset{(2)}{\widetilde{u}}_0(t,\varepsilon) = 10(1-\sqrt{2})e^{-\sqrt{2}t/\varepsilon},\ \widetilde{z}_0(t,\varepsilon) = 10e^{-\sqrt{2}t/\varepsilon^2},\ \overset{(3)}{\widetilde{u}}_0(t,\varepsilon) = 10(1-\sqrt{2})e^{-\sqrt{2}t/\varepsilon^2}.$$

Further, in the expressions of problems for finding asymptotics terms of the first order approximation we take into account the found asymptotics terms of the zero order approximation. We omit zero terms, in particular, $\overset{(2)}{\overline{u}}_0(t)$, $\overset{(3)}{\overline{u}}_0(t)$, $\overline{y}_0(t)$, $\overline{z}_0(t)$, $\Pi_{00}\overset{(1)}{u}(\tau_0)$, $\Pi_{00}\overset{(3)}{u}(\tau_0)$, $\Pi_{00}z(\tau_0)$, $\Pi_{10}\overset{(1)}{u}(\tau_1)$, $\Pi_{10}\overset{(2)}{u}(\tau_1)$. Note, that problems $Q_{00}P$ and $Q_{10}P$ have the zero solution.

Using (7)–(9) at $j = 1$, we write out problem $\overline{P}_1$ in the following form:

$$\overline{J}_1(\overline{u}_1) = \int\limits_0^1 (\frac{1}{2}(\overline{x}_1(t)^2 + \overline{y}_1(t)^2 + \overline{z}_1(t)^2 + \overset{(1)}{\overline{u}_1}(t)^2 + \overset{(2)}{\overline{u}_1}(t)^2 + \overset{(3)}{\overline{u}_1}(t)^2) + \overline{x}_0(t)\overset{(1)}{\overline{u}_1}(t) + \overline{x}_1(t)\overset{(1)}{\overline{u}_0}(t))\, dt \to \min_{\overline{u}_1},$$

$$\frac{d\overline{x}_1(t)}{dt} = \overline{x}_1(t) + \overset{(1)}{\overline{u}_1}(t),\ \overline{x}_1(0) + \Pi_{01}x(0) = 0,$$

$$0 = -\overline{y}_1(t) + \overset{(2)}{\overline{u}_1}(t),$$

$$0 = -\overline{z}_1(t) + \overset{(3)}{\overline{u}_1}(t).$$

Using (10)–(12) at $j = 1$ and taking into account that $\Pi_{00}x(\tau_0) = 0$, we write out problem $\Pi_{01}P$ in the following form:

$$\Pi_{01}J(\Pi_{01}u) = \int\limits_0^{+\infty} (\frac{1}{2}(\Pi_{01}x(\tau_0)^2 + \Pi_{01}y(\tau_0)^2 + \Pi_{01}z(\tau_0)^2 + \Pi_{01}\overset{(1)}{u}(\tau_0)^2 + \Pi_{01}\overset{(2)}{u}(\tau_0)^2 + \Pi_{01}\overset{(3)}{u}(\tau_0)^2) +$$

$$+\Pi_{01}x(\tau_0)\Pi_{00}\overset{(1)}{u}(\tau_0) + \Pi_{01}\overset{(3)}{u}(\tau_0)(\Pi_{00}y(\tau_0) - \Pi_{00}\eta(\tau_0)))\, d\tau_0 \to \min_{\Pi_{01}u},$$

$$\frac{d\Pi_{01}x(\tau_0)}{d\tau_0} = 0,\ \Pi_{01}x(+\infty) = 0,$$

$$\frac{d\Pi_{01}y(\tau_0)}{d\tau_0} = -\Pi_{01}y(\tau_0) + \Pi_{01}\overset{(2)}{u}(\tau_0),\ \overline{y}_1(0) + \Pi_{01}y(0) + \Pi_{11}y(0) = 0,$$

$$0 = -\Pi_{01}z(\tau_0) + \Pi_{01}\overset{(3)}{u}(\tau_0) + \Pi_{00}y(\tau_0).$$

In view of (13)–(15) at $j = 1$, problem $Q_{01}P$ is defined by the relations

$$Q_{01}J(Q_{01}u) = Q_{01}y(0)\overline{\eta}_1(1) + \frac{1}{2}\int\limits_{-\infty}^{0} (Q_{01}x(\sigma_0)^2 + Q_{01}y(\sigma_0)^2 + Q_{01}z(\sigma_0)^2 + Q_{01}\overset{(1)}{u}(\sigma_0)^2 +$$

$$+Q_{01}\overset{(2)}{u}(\sigma_0)^2 + Q_{01}\overset{(3)}{u}(\sigma_0)^2)\, d\tau_0 \to \min_{Q_{01}u},$$

$$\frac{dQ_{01}x(\sigma_0)}{d\sigma_0} = 0,\ \frac{dQ_{01}y(\sigma_0)}{d\sigma_0} = -Q_{01}y(\sigma_0) + Q_{01}\overset{(2)}{u}(\sigma_0),$$

$$0 = -Q_{01}z(\sigma_0) + Q_{01}\overset{(3)}{u}(\sigma_0),$$

$$Q_{01}x(-\infty) = Q_{01}y(-\infty) = 0.$$

Taking into account that $\Pi_{11}x(\tau_1) = 0$, in view of (16)-(18) at $j = 1$, problem $\Pi_{11}P$ is defined by the following way:

$$\Pi_{11}J(\Pi_{11}u) = \frac{1}{2}\int\limits_0^{+\infty} (\Pi_{11}y(\tau_1)^2 + \Pi_{11}z(\tau_1)^2 + \Pi_{11}\overset{(1)}{u}(\tau_1)^2 + \Pi_{11}\overset{(2)}{u}(\tau_1)^2 + \Pi_{11}\overset{(3)}{u}(\tau_1)^2)\, d\tau_1 \to \min_{\Pi_{11}u},$$

$$\frac{d\Pi_{11}y(\tau_1)}{d\tau_1} = 0,\ \Pi_{11}y(+\infty) = 0,$$

$$\frac{d\Pi_{11}z(\tau_1)}{d\tau_1} = -\Pi_{11}z(\tau_1) + \Pi_{11}\overset{(3)}{u}(\tau_1),\ \overline{z}_1(0) + \Pi_{01}z(0) + \Pi_{11}z(0) = 0.$$

Using (19)–(21) at $j = 1$ and taking into account that $Q_{11}x(\sigma_1) = 0$, we obtain problem $Q_{11}P$ in the form:

$$Q_{11}J(Q_{11}u) = Q_{11}z(0)(\overline{\zeta}_1(1) + Q_{01}\zeta(0)) + \frac{1}{2}\int\limits_{-\infty}^{0} (Q_{11}y(\sigma_1)^2 + Q_{11}z(\sigma_1)^2 + Q_{11}\overset{(1)}{u}(\sigma_1)^2 +$$

$$+Q_{11}\overset{(2)}{u}(\sigma_1)^2 + Q_{11}\overset{(3)}{u}(\sigma_1)^2)\, d\sigma_1 \to \min_{Q_{11}u},$$

$$\frac{dQ_{11}y(\sigma_1)}{d\sigma_1} = 0,\ Q_{11}y(-\infty) = 0,$$
$$\frac{dQ_{11}z(\sigma_1)}{d\sigma_1} = -Q_{11}z(\sigma_1) + Q_{11}\overset{(3)}{u}(\sigma_1),\ Q_{11}z(-\infty) = 0.$$

Solving problems $\overline{P}_1$, $\Pi_{i1}P$, $Q_{i1}P$, $i = 0,1$, we get the first order approximation of asymptotic solution of problem (38)–(40):

$$\widetilde{x}_1(t,\varepsilon) = \widetilde{x}_0(t,\varepsilon) - \varepsilon a((2+\sqrt{2})te^{\sqrt{2}t} - (2-\sqrt{2})te^{-\sqrt{2}(t-2)}),$$
$$\overset{(1)}{\widetilde{u}}_1(t,\varepsilon) = \overset{(1)}{\widetilde{u}}_0(t,\varepsilon) - \varepsilon a((2+\sqrt{2}+\sqrt{2}t)e^{\sqrt{2}t} - (2-\sqrt{2}-\sqrt{2}t)e^{-\sqrt{2}(t-2)}),$$
$$\widetilde{y}_1(t,\varepsilon) = \widetilde{y}_0(t,\varepsilon),\ \overset{(2)}{\widetilde{u}}_1(t,\varepsilon) = \overset{(2)}{\widetilde{u}}_0(t,\varepsilon),$$
$$\widetilde{z}_1(t,\varepsilon) = \widetilde{z}_0(t,\varepsilon) + \varepsilon 5((-\sqrt{2}+1)e^{-\sqrt{2}t/\varepsilon} + (\sqrt{2}-1)e^{-\sqrt{2}t/\varepsilon^2}),$$
$$\overset{(3)}{\widetilde{u}}_1(t,\varepsilon) = \overset{(3)}{\widetilde{u}}_0(t,\varepsilon) + \varepsilon 5(-(\sqrt{2}+1)e^{-\sqrt{2}t/\varepsilon} + (2\sqrt{2}-3)e^{-\sqrt{2}t/\varepsilon^2}).$$

The exact solution of problem (38)–(40) was calculated by means of Maple 2022.

The exact solution and asymptotic approximations to the solution of problem (38)–(40) at $\varepsilon = 0.25$ are presented in Figures 1–6, where the black line denotes the exact solution, the yellow line means the solution of the degenerate problem, the red line – the zero order approximation and the green one – the first order approximation. Please note that the degenerate and the zero order asymptotics solutions for the trajectory $x(t,\varepsilon)$ and the control $\overset{(1)}{u}(t,\varepsilon)$ are equal, the zero and the first order approximations for the trajectory $y(t,\varepsilon)$ and the control $\overset{(2)}{u}(t,\varepsilon)$ are the same.

Values of the performance index $J_\varepsilon(u)$ corresponding to the optimal control u_* and it's approximations $\overline{u}_0$, $\widetilde{u}_0$, $\widetilde{u}_1$ are presented in Table. We give here three decimal points using ordinary rules of approximating. From this table it is seen that for a less values of ε there is a better proximity between values of the performance index for asymptotic approximations to the optimal control and it's minimal value and $J_\varepsilon(\overline{u}_0) > J_\varepsilon(\widetilde{u}_0) > J_\varepsilon(\widetilde{u}_1) > J_\varepsilon(u_*)$, this corresponds to Theorems 2, 3.

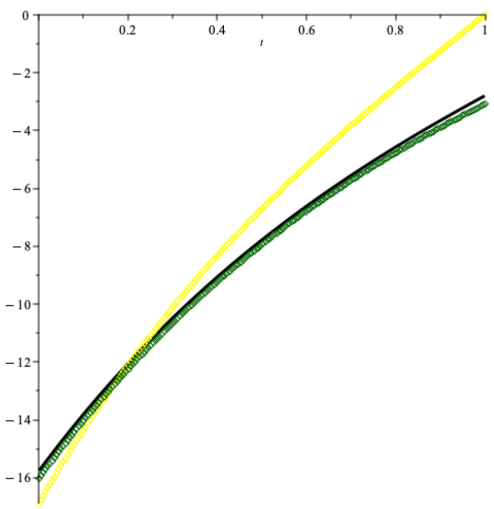

Figure 1. Control $\overset{(1)}{u}(t,\varepsilon)$.

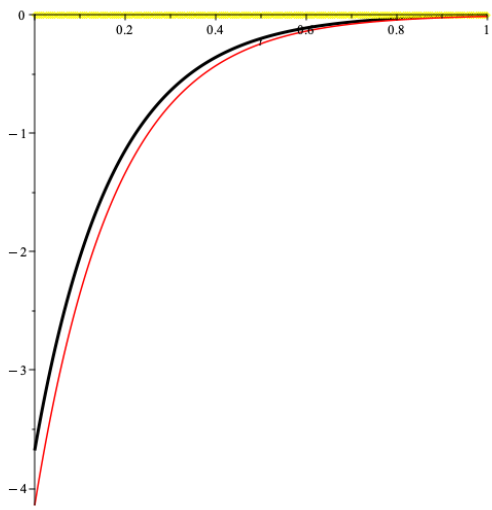

Figure 2. Control $\overset{(2)}{u}(t,\varepsilon)$.

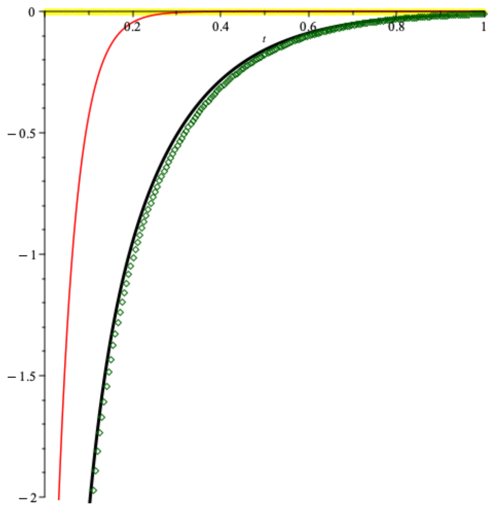

Figure 3. Control $\overset{(3)}{u}(t,\varepsilon)$.

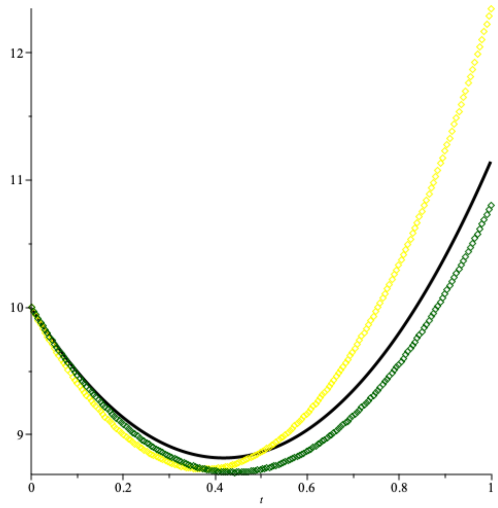

Figure 4. Trajectory $x(t,\varepsilon)$.

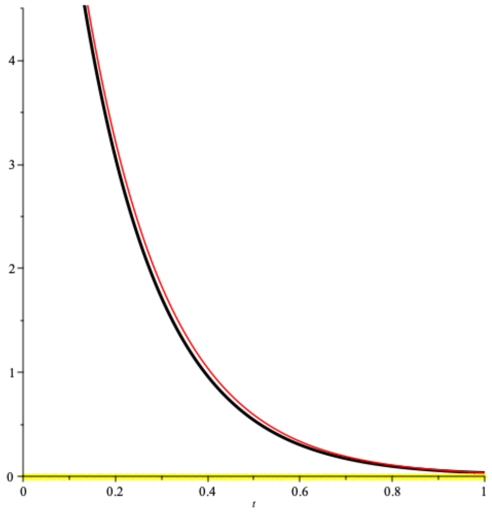

Figure 5. Trajectory $y(t,\varepsilon)$.

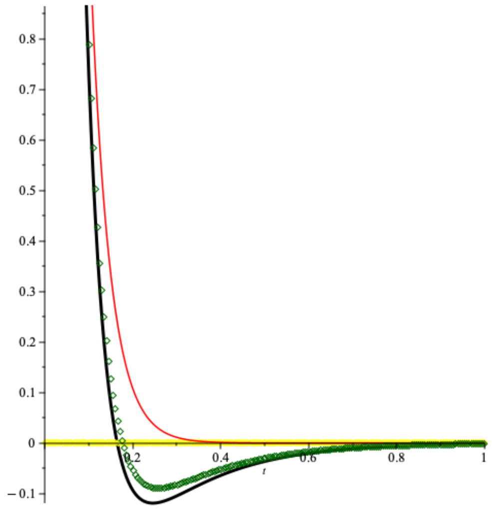

Figure 6. Trajectory $z(t,\varepsilon)$.

T a b l e

Values of the performance index

ε	$J_\varepsilon(\overline{u}_0)$	$J_\varepsilon(\widetilde{u}_0)$	$J_\varepsilon(\widetilde{u}_1)$	$J_\varepsilon(u_*)$
0.25	76.413	74.184	72.268	72.200
0.125	79.716	78.991	78.559	78.555

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Үшқарқынды айнымалылар мен әлсіз бейсызықты ауытқуы бар сызықтық-квадраттық басқару есептері үшін минимумдаушы тізбектер

Мақала үшқарқынды күй айнымалылары бар әлсіз бейсызықты жүйенің траекторияларында әлсіз бейсызықты ауытқу квадраттық сапа критерийін азайту есебі үшін минимизациялық тізбектерді құруға арналған. Бұл жағдайда шешімнің асимптотикалық ыдырауын есептің шарттарына тікелей ауыстырудан және шешімдері бастапқы бейсызықты басқару есебінің шешімінің асимптотикалық ыдырауының мүшелері болып табылатын тиімді басқару есептерінің үйірін (қарастырылып отырған жағдайда сызықты-квадраттық) құрудан тұратын шешімнің асимптотикалық құрылысының тікелей схемасы қолданылады. Тиімді басқаруға кейбір асимптотикалық жуықтауды басқару ретінде пайдаланған кезде тиімді траекторияның күй теңдеуінің траекториясына жақындығы бағаланады. Минимумдаушы тізбектерді құрудың схемасын егжей-тегжейлі көрсететін мысал келтірілген.

Кілт сөздер: үшқарқынды айнымалылар, тиімді басқарудың бейсызықты есептері, асимптотикалық бағалаулар, минимумдаушы тізбектер.

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Минимизирующие последовательности для линейно-квадратичной задачи управления с трехтемповыми переменными и слабыми нелинейными возмущениями

Статья посвящена построению минимизирующих последовательностей для задачи минимизации слабо нелинейно возмущенного квадратичного критерия качества на траекториях слабо нелинейной системы с трехтемповыми переменными состояния. При этом использована так называемая прямая схема построения асимптотики решения, заключающаяся в непосредственной подстановке постулируемого асимптотического разложения решения в условия задачи и построении серии задач оптимального управления (в рассматриваемом случае линейно-квадратичных), решения которых являются членами асимптотического разложения решения исходной нелинейной задачи управления. Получена оценка близости оптимальной траектории к траектории уравнения состояния при использовании в качестве управления некоторого асимптотического приближения к оптимальному управлению. Приведен пример, детально иллюстрирующий предложенную схему построения минимизирующих последовательностей.

Ключевые слова: трехтемповые переменные, нелинейные задачи оптимального управления, асимптотические оценки, минимизирующие последовательности.

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An optimal control problem for the systems with integral boundary conditions

In this paper, we consider an optimal control problem with a «pure», integral boundary condition. The Green's function is constructed. Using contracting Banach mappings, a sufficient condition for the existence and uniqueness of a solution to one class of integral boundary value problems for fixed admissible controls is established. Using the functional increment method, the Pontryagin's maximum principle is proved. The first and second variations of the functional are calculated. Further, various necessary conditions for optimality of the second order are obtained by using variations of controls.

Key words: integral boundary conditions, singular control, optimal control problem, existence and uniqueness of the solution.

Introduction

Boundary value problems with integral conditions last few decades became one of the intensively studied classes of the problems of mathematical physics. These problems included different problems with two-, three-, multiple and non-local boundary value problems [1–3]. One of the reasons that make these problems so actual is that they have a strong relation with various fields of applications (see, for example [4, 5] and references therein).

There exist many works devoted to investigation of the systems with local conditions and finding necessary optimality conditions of first and second orders [6–10]. For such problems with integral conditions we refer to [11–15].

Various type optimal control problems for the systems with boundary conditions are considered in [16–22] and with integral boundary condition in [16, 17], where the first order necessary conditions are obtained. In some cases, when the first order optimality conditions are "degenerated", i.e. are fulfilled trivially one has to try to obtain second order conditions.

Another direction in investigation of the optimal control problems with multipoint and integral boundary conditions is developing the numerical methods. For the first-order ordinary differential equations such problems are studied in [23, 24].

In this paper, optimal control problem is investigated, when the state of the system is described by differential equations with integral boundary conditions. The existence and uniqueness of solutions to the boundary value problem is investigated. The first and second variations of the corresponding functional are calculated. Optimality conditions of first and second order are obtained applying the method of variations of the controls.

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Problem Statement

Consider the following system of differential equations with an integral boundary condition

$$\frac{dx}{dt} = f(t, x, u\,(t)), 0 \le t \le T, \tag{1}$$

$$\int_0^T m\,(t)\, x\,(t)\, dt = C, \tag{2}$$

$$u\,(t) \in U, t \in [0, T]\,, \tag{3}$$

where $x(t) \in R^n$; $f(t, x, u)$ is n-dimensional continuous function; $C \in R^n$ is a given constant vector and $m\,(t) \in R^{n\times n}$ is $n \times n$ matrix function; u is a control parameter; $U \in R^r$ is bounded set.

The problem is: to minimize the functional

$$J\,(u) = \varphi\,(x\,(0)\,, x\,(T)) + \int_0^T F\,(t, x, u)\, dt \tag{4}$$

on the solutions of problem (1)–(3).

The following assumption is accepted: the scalar functions $\varphi\,(x, y)$ and $F\,(t, x, u)$ are continuous with respect to their own arguments and have continuous and bounded first order partial derivatives with respect to x, y. As a solution of problem (1)-(3) corresponding to the fixed control $u\,(t)$ we consider absolutely continuous on $[0, T]$ function $x(t) : [0, T] \to R^n$. The space of such functions is denoted as $AC\,([0, T]\,, R^n)$. $C\,([0, T]\,, R^n)$ stands for the space of continuous functions on $[0, T]$ which gets values from R^n. It is obvious that this is a Banach with the norm $\|x\|_{C[0,T]} = \max\limits_{[0,T]} |x(t)|\,,$ where $|\cdot|$ is the norm in space R^n.

As admissible controls we consider the functions from the class of bounded measurable functions with the values from the set $U \subset R^r$. We call the pair consisting of admissible control and the corresponding solution of (1), (2) an admissible process.

Thus the admissible process $\{u\,(t)\,, x\,(t, u)\}$ that is a solution to (1)-(4), subject to (1)-(3), is said to be an optimal process, and $u\,(t)$ – an optimal control.

The existence of an optimal control in problem (1)–(4) is also assumed.

Existence of solutions of boundary value problem (1)–(3)

Let's set the following conditions.

H1) Let $\det B \ne 0$, where $B = \int_0^T m\,(t)\, dt$.

H2) $f : [0, T] \times R^n \times R^r \to R^n$ is a continuous function and there exists the constant $K \ge 0$

$$|f(t, x, u) - f(t, y, u)| \le K\,|x - y|\,, \quad t \in [0, T], \quad x, y \in R^n, u \in U.$$

H3) $L = KTM < 1$,

where $M = \max\limits_{0\le t,s\le T} \|M\,(t, s)\|$, $M\,(t, s) = \begin{cases} B^{-1}\int_0^s m\,(\tau)\, d\tau, 0 \le t \le s \\ -B^{-1}\int_s^T m\,(\tau)\, d\tau, s \le t \le T. \end{cases}$

Theorem 1. Under the condition H1) the function $x(\cdot) \in AC\,([0, T], R^n)$ is an absolutely continuous solution to problem (1)-(3) iff

$$x(t) = B^{-1}C + \int_0^T M(t, \tau) f(\tau, x(\tau), u\,(\tau)) d\tau. \tag{5}$$

Here $M\,(t, s) = \begin{cases} B^{-1}\int_0^s m\,(\tau)\, d\tau, & 0 \le t \le s, \\ -B^{-1}\int_s^T m\,(\tau)\, d\tau, & s < t \le T. \end{cases}$

Proof. It is obvious that if $x = x(\cdot)$ is a solution to (1), then

$$x(t) = x(0) + \int_0^t f(s, x(s), u(s))ds, \tag{6}$$

for $t \in (0, T)$, where $x(0)$ is an arbitrary constant. To determine $x(0)$ we suppose that the function given by (6) satisfy (2), i.e.

$$Bx(0) = C - \int_0^T m(t) \int_0^t f(\tau, x(\tau), u(\tau))d\tau dt.$$

Since $\det B \neq 0$ we have

$$x(0) = B^{-1}C - B^{-1} \int_0^T m(t) \int_0^t f(\tau, x(\tau), u(\tau))d\tau dt. \tag{7}$$

Considering in (6) the value of $x(0)$ determined by equality (7) we get

$$x(t) = B^{-1}C + \int_0^T M(t, \tau) f(\tau, x(\tau), u(\tau))d\tau.$$

By this way we reduced boundary value problem (1)–(3) to the integral equation (5). It is easy to check that the solution of integral equation (5) also satisfies (1)–(3). Theorem 1 is proved.

Introduce the operator $P : C([0, T], R^n) \to C([0, T], R^n)$ as

$$(Px)(t) = B^{-1}C + \int_0^T M(t, \tau) f(\tau, x(\tau), u(\tau))d\tau. \tag{8}$$

Theorem 2. Within the conditions H1)-H3) for any $C \in R^n$ and for each fixed admissible control, problem (1)–(3) has a unique solution that satisfies the following relation

$$x(t) = B^{-1}C + \int_0^T M(t, \tau) f(\tau, x(\tau), u(\tau))d\tau. \tag{9}$$

Proof. Let $C \in R^n$ and $u(\cdot) \in U$ be fixed. Consider the mapping $P : C([0, T], R^n) \to C([0, T], R^n)$ defined by (8). It is obvious that the fixed points of the operator $(Px)(t)$ are the solutions of (1)–(2). To prove that the mapping P has a fixed point we apply the Banach contraction principle. For any $v, w \in C([0, T], R^n)$ we have

$$|(Pv)(t) - (Pw)(t)| \leq \int_0^T |M(t, s)| \cdot |f(s, v(s), u(s)) - f(s, w(s), u(s))| \, ds \leq$$

$$\leq KTN \, \|v(\cdot) - w(\cdot)\|_{C[0,T]}, \quad t \in [0, T],$$

or

$$\|Pv - Pw\|_{C[0,T]} \leq L \, \|v - w\|_{C[0,T]}.$$

The last relation shows that P is the contraction in the space $C([0, T], R^n)$. Thus, based on the principle of contraction operators one can state that P has a unique fixed point at $C([0, T], R^n)$. It means that integral equation (9) or boundary value problem (1)–(3) has a unique solution.

Theorem 2 is proved.

Derivation of Pontryagin's maximum principle

Here we assume that U is closed set in R^r. To obtain the necessary conditions for optimality one should analyze the variation of the objective functional caused by some control impulse [7] i.e. one must calculate the increment formula that obtained from Taylor's series expansion. It is important to give a definition of the conjugate system that allows one to determine the dominant term that leads to the necessary condition for optimality. For the sake of simplicity, it is expedient to construct a linearized model of system (8), (9) in some small neighborhood.

Let $\{u, x = x(t,u)\}$ and $\{\tilde{u} = u + \Delta u,\ \tilde{x} = x + \Delta x = x(t,\tilde{u})\}$ be two admissible processes. Introduce the boundary value problem for problem (1)–(3):

$$\Delta\dot{x} = \Delta f(t,x,u), \quad t \in [0,T],$$

$$\int_0^T m(t)\,\Delta x(t)\,dt = 0,$$

where $\Delta f(t,x,u) = f(t,\tilde{x},\tilde{u}) - f(t,x,u)$ stands for the total increment of the function $f(t,x,u)$. Then we can represent the increment of the functional as

$$\Delta J(u) = J(\tilde{u}) - J(u) = \Delta\varphi(x(0),x(T)) + \int_0^T \Delta F(x,u,t)\,dt.$$

Consider the non-trivial vector-function $\psi(t)$, $\psi(t) \in R^n$, and numerical vector $\lambda \in R^n$. Then the increment of the functional (4) can be written as

$$\Delta J(u) = J(\tilde{u}) - J(u) = \Delta\varphi(x(0),x(T)) + \int_0^T \Delta F(x,u,t)\,dt +$$
$$+ \int_0^T \langle \psi(t), \Delta\dot{x}(t) - \Delta f(t,x,u)\rangle\,dt + \left\langle \lambda, \int_0^T m(t)\,\Delta x(t)\,dt \right\rangle.$$

Making standard operations for the increment of the functional we obtain the formula

$$\begin{aligned}\Delta J(u) = &-\int_0^T \Delta_{\tilde{u}} H(t,\psi,x,u)\,dt - \int_0^T \left\langle \Delta_{\tilde{u}} \frac{\partial H(t,\psi,x,u)}{\partial x}, \Delta x(t) \right\rangle dt + \\ &+ \int_0^T \left\langle \frac{\partial H(t,\psi,x,u)}{\partial x} + m'(t)\,\lambda + \dot{\psi}(t), \Delta x(t) \right\rangle dt + \left\langle \left[\frac{\partial\varphi}{\partial x(0)} - \psi(0)\right], \Delta x(0) \right\rangle + \\ &+ \left\langle \left[\frac{\partial\varphi}{\partial x(T)} + \psi(T)\right], \Delta x(T) \right\rangle + \eta_u,\end{aligned} \tag{10}$$

where

$$H(t,\psi,x,u) = \langle \psi, f(t,x,u)\rangle - F(t,x,u),$$

$$\eta_{\tilde{u}} = -\int_0^T o_H(\|\Delta x(t)\|)\,dt + o_\varphi(\|\Delta x(t_0)\|, \|\Delta x(t_1)\|).$$

Let the vector function $\psi(t) \in R^n$ and vector $\lambda \in R^n$ be a solution of the following conjugate problem

$$\begin{cases} \dot{\psi}(t) = -\frac{\partial H(t,\psi,x,u)}{\partial x} - m'(t)\,\lambda, \quad t \in [0,T], \\ \frac{\partial\varphi}{\partial x(0)} - \psi(0) = 0, \ \frac{\partial\varphi}{\partial x(T)} + \psi(T) = 0. \end{cases} \tag{11}$$

Then, formula (10) takes the form

$$\Delta J(u) = -\int_0^T \Delta_{\tilde{u}} H(t,\psi,x,u)\,dt - \int_0^T \left\langle \Delta_{\tilde{u}} \frac{\partial H(t,\psi,x,u)}{\partial x}, \Delta x(t) \right\rangle dt + \eta_{\tilde{u}}. \tag{12}$$

Taking as parameters the point $\tau \in (0,T]$, number $\varepsilon \in (0,\tau]$, vector $v \in U$ and variation interval $(\tau-\varepsilon,\tau)$ from $[0,T]$ we consider needle-shaped variation of the admissible control. Then needle-shaped variation of the control $u = u(t)$ may be given by the relation

$$\tilde{u} = u_\varepsilon(t) = \begin{cases} v \in U, \ t \in (\tau-\varepsilon,\tau] \subset [0,T], \ \varepsilon > 0, \\ u(t), \quad t \notin (\tau-\varepsilon,\tau]. \end{cases} \tag{13}$$

To obtain the necessary optimality condition from the increment formula (12) one have to show that on the needle-shaped variation $\tilde{u}(t) = u_\varepsilon(t)$ the state increment $\Delta_\varepsilon x(t)$ has the order ε.
Since,

$$\Delta x(t) = \int_0^T M(t,s)\left[f(s,x(s)+\Delta x(s),\tilde{u}(s)) - f(s,x(s),\tilde{u}(s))\right] ds +$$

$$+\int_0^T M(t,s)\,\Delta_{\overleftrightarrow{u}} f(s,x(s),u(s))\,ds.$$

The last implies that

$$\|\Delta x(t)\| \le (1-L)^{-1}\int_0^T \|\Delta_{\tilde{u}} f(t,x(t),u(t))\|\,dt,$$

which proves the hypothesis on response of the state increment caused by the needle-shaped variation given by (13)

$$\|\Delta_\varepsilon x(t)\| \le \tilde{L}\varepsilon, \ \ t \in [0,T], \ \tilde{L} = const > 0.$$

This also implies that for $\tilde{u}(t) = u_\varepsilon(t)$ the relation

$$\int_{\tau-\varepsilon}^{\tau} \left\langle \Delta_v \frac{\partial H(t,\psi,x,u)}{\partial x}, \Delta_\varepsilon x(t) \right\rangle dt + \eta_{u_\varepsilon}(\|\Delta_\varepsilon x(t)\|) = o(\varepsilon)$$

holds true, where

$$\Delta_\varepsilon x(t) = x(t,u_\varepsilon) - x(t,u) \sim \varepsilon.$$

It means that according to (12) the variation of the functional caused by the needle-shaped variation (13) can be written

$$\Delta_\varepsilon J(u) = J(u_\varepsilon) - J(u) = -\Delta_v H(s,\psi,x,u).\varepsilon + o(\varepsilon), \ v \in U, \ s \in [0,T]. \tag{14}$$

Note that in the last expression, the mean value theorem was used.
Formula (14) with respect to the estimate for $\|\Delta_\varepsilon x\|$ implies the necessary optimality condition in the form of the maximum principle for the needle-shaped variation of optimal process $\{u^0, x^0 = x(t,u^0)\}$.

Theorem 3. (Pontryagin‘s maximum principle). Assume that the admissible process $\{u^0, x^0 = x(t,u^0)$ is optimal for problem (1)–(4) and $\psi^0(t)$ is a solution to problem (11) calculated on the optimal process. Then, inequality

$$\Delta_v H\left(s,\psi^0,x^0,u^0\right) \le 0, \ \ for\, every\, v \in U, \tag{15}$$

is valid for all $s \in [0,T]$.

Remark. If the function f is linear with respect to (x, u) and the functions, F, φ are convex with respect to $x(0)$, $x(T)$, and $x(t)$, respectively, then maximum principle (15) is both necessary and sufficient optimality condition. This fact can be easily obtained from the formula

$$\Delta J(u) = -\int_0^T \Delta_{\tilde{u}} H(t, \psi, x, u)\, dt + o_\varphi(\|\Delta x(0)\|, \|\Delta x(T)\|) + \int_0^T o_F(\|x(t)\|)\, dt,$$

where $o_\varphi \geq 0,\ o_F \geq 0$.

The second order formula for the increment of the functional and variation of the functional

Let us suppose that the scalar functions $\varphi(x, y)$ and $F(t, x, u)$ are continuous over their own arguments and have continuous and bounded partial derivatives with respect to x, y and u up to second order, inclusively. Let U be an open set in R^r and $\{u, x = x(t, u)\}$, $\{\tilde{u} = u + \Delta u,\ \tilde{x} = x + \Delta x = x(t, \tilde{u})\}$ be two admissible processes.

Under the above assumptions increment formula (12) turns to

$$\begin{aligned}
\Delta J(u) = &-\textstyle\int_0^T \left\langle \frac{\partial H(t,\psi,x,u)}{\partial u}, \Delta u(t) \right\rangle dt - \frac{1}{2}\int_0^T \left\langle \Delta u(t)' \frac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \Delta u(t) \right\rangle dt - \\
&-\textstyle\int_0^T \left\langle \Delta u(t)' \frac{\partial H^2(t,\psi,x,u)}{\partial x \partial u} + \frac{1}{2}\Delta x'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial x^2}, \Delta x(t) \right\rangle dt + \\
&+\tfrac{1}{2}\left\langle \Delta x(0)' \frac{\partial^2 \varphi}{\partial x(0)^2} + \Delta x(T)' \frac{\partial^2 \varphi}{\partial x(0) \partial x(T)}, \Delta x(0) \right\rangle + \\
&+\tfrac{1}{2}\left\langle \Delta x(0)' \frac{\partial^2 \varphi}{\partial x(T) \partial x(0)} + \Delta x(T)' \frac{\partial^2 \varphi}{\partial x(T)^2}, \Delta x(T) \right\rangle + \eta_{\tilde{u}}.
\end{aligned} \tag{16}$$

Take $\Delta u(t) = \varepsilon \delta u(t)$, where $\varepsilon > 0$ is small enough number, $\delta u(t)$ is some piecewise continuous function. Then the expression $\Delta J(u) = J(\tilde{u}) - J(u)$ for the fixed functions $u(t)$, $\Delta u(t)$ will be a function of the parameter ε. If the representation

$$\Delta J(u) = \varepsilon \delta J(u) + \frac{1}{2}\varepsilon^2 \delta^2 J(u) + o\left(\varepsilon^2\right) \tag{17}$$

holds true, then $\delta J(u)$ is called the first, $\delta^2 J(u)$ the second variation of the functional. To get an obvious expression for the first and second variations we have to select in $\Delta x(t)$ the principal term with respect to ε.

Let

$$\Delta x(t) = \varepsilon \delta x(t) + o(\varepsilon, t), \tag{18}$$

where $\delta x(t)$ is the variation of the trajectory. Obviously, such a representation exists and for the function $\delta x(t)$ one can obtain an equation in variations. Using the definition of $\Delta x(t)$ we get

$$\Delta x(t) = \int_0^T M(t, \tau) \Delta f(\tau, x(\tau), u(\tau)) d\tau.$$

Using the Taylor formula, we get:

$$\varepsilon \delta x(t) + o(\varepsilon, t) = \int_0^T M(t, \tau) \left\{ \frac{\partial f(\tau, x, u)}{\partial x} [\varepsilon \delta x(\tau) + o(\varepsilon, \tau)] + \varepsilon \frac{\partial f(\tau, x, u)}{\partial u} \delta u + o_1(\varepsilon, \tau) \right\} d\tau.$$

If to consider that the last formula is true for any ε we have

$$\delta x(t) = \int_0^T M(t, \tau) \left\{ \frac{\partial f(\tau, x, u)}{\partial x} \delta x(\tau) + \frac{\partial f(\tau, x, u)}{\partial u} \delta u(t) \right\} d\tau. \tag{19}$$

Equation (19) is called the equation in variations. Obviously, (19) is equivalent to the following nonlocal boundary value problem

$$\delta\dot{x}(t) = \frac{\partial f(t,x,u)}{\partial x}\delta x(t) + \frac{\partial f(t,x,u)}{\partial u}\delta u(t), \tag{20}$$

$$\int_0^T m(t)\,\delta x(t)\,dt = 0. \tag{21}$$

Any solution of (20) may be written in the form

$$\delta x(t) = \Phi(t)\,\delta x(0) + \Phi(t)\int_0^t \Phi^{-1}(\tau)\,\frac{\partial f(\tau,x,u)}{\partial u}\delta u(\tau)\,d\tau, \tag{22}$$

where $\Phi(t)$ is a solution of the equation

$$\frac{d\Phi(t)}{dt} = \frac{\partial f(t,x,u)}{\partial x}\Phi(t),$$

$$\Phi(0) = E.$$

Let the solution of (20) determined by equality (22) satisfiy (21). Then for the solutions of problem (20), (21) we obtain the explicit formula

$$\delta x(t) = \int_0^T G(t,\tau)\,\frac{\partial f(\tau,x,u)}{\partial u}\delta(\tau)\,d\tau, \tag{23}$$

where

$$G(t,s) = \begin{cases} \Phi(t)\,B_1{}^{-1}\int_0^s m(\tau)\,\Phi(\tau)\,d\tau\Phi^{-1}(\tau), & 0 \le s \le t \\ -\Phi(t)\,B_1{}^{-1}\int_s^T m(\tau)\,\Phi(\tau)\,d\tau\Phi^{-1}(\tau), & t \le s \le T \end{cases},$$

$$B_1 = \int_0^T m(t)\,\Phi(t)\,dt.$$

Considering (18) in (16), we obtain

$$\Delta J(u) = -\varepsilon\int_0^T \left\langle \frac{\partial H(t,\psi,x,u)}{\partial u}, \delta u(t)\right\rangle dt - \frac{\varepsilon^2}{2}\left\{\int_0^T \left[\left\langle \delta x'(t)\frac{\partial^2 H(t,\psi,x,u)}{\partial x^2}, \delta x(t)\right\rangle + \right.\right.$$

$$\left. +2\left\langle \delta u'(t)\frac{\partial^2 H(t,\psi,x,u)}{\partial x\partial u}, \delta x(t)\right\rangle + \left\langle \delta u'(t)\frac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \delta u(t)\right\rangle\right] dt -$$

$$-\left\langle \delta x'(0)\frac{\partial^2\varphi}{\partial x(0)^2} + \Delta x'(T)\frac{\partial^2\varphi}{\partial x(0)\partial x(T)}, \delta x(0)\right\rangle -$$

$$\left. -\left\langle \delta x'(0)\frac{\partial^2\varphi}{\partial x(T)\partial x(0)} + \delta x'(T)\frac{\partial^2\varphi}{\partial x(T)^2}, \delta x(T)\right\rangle\right\} + o(\varepsilon^2).$$

Using (17) from the last we get

$$\delta J(u) = -\int_0^T \left\langle \frac{\partial H(t,\psi,x,u)}{\partial u}, \delta u(t)\right\rangle dt$$

$$\delta^2 J(u) = -\int_0^T \left[\left\langle \delta x'(t)\tfrac{\partial^2 H(t,\psi,x,u)}{\partial x^2}, \delta x(t)\right\rangle + \right.$$

$$\left. +2\left\langle \delta u'(t)\tfrac{\partial^2 H(t,\psi,x,u)}{\partial x\partial u}, \delta x(t)\right\rangle + \left\langle \delta u'(t)\tfrac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \delta u(t)\right\rangle\right] dt+$$

$$+\left\langle \delta x'(0)\tfrac{\partial^2\varphi}{\partial x(0)^2} + \Delta x'(T)\tfrac{\partial^2\varphi}{\partial x(0)\partial x(T)}, \delta x(0)\right\rangle +$$

$$+\left\langle \delta x'(0)\tfrac{\partial^2\varphi}{\partial x(T)\partial x(0)} + \delta x'(T)\tfrac{\partial^2\varphi}{\partial x(T)^2}, \delta x(T)\right\rangle.$$

Derivation of Legendre-Klebsh conditions

It follows from (17) that the conditions

$$\delta J(u^0) = 0, \;\; \delta^2 J(u^0) \geq 0 \tag{24}$$

are fulfilled on the optimal control $u^0(t)$.

From (24) it follows that

$$\int_0^T \left\langle \frac{\partial H(t,\psi^0,x^0,u^0)}{\partial u}, \delta u(t)\right\rangle dt = 0.$$

Hence the validity of the equality

$$\frac{\partial H(t,\psi^0,x^0,u^0)}{\partial u} = 0, \quad t\in[0,T] \tag{25}$$

can be proved along the optimal control that indeed is the Euler equation. From (24) we obtain the validity of the following inequality along the optimal control

$$\begin{aligned}\delta^2 J(u) = &-\int_0^T \left[\left\langle \delta x'(t)\tfrac{\partial^2 H(t,\psi,x,u)}{\partial x^2}, \delta x(t)\right\rangle + \right.\\ &\left. +2\left\langle \delta u'(t)\tfrac{\partial^2 H(t,\psi,x,u)}{\partial x\partial u}, \delta x(t)\right\rangle + \left\langle \delta u'(t)\tfrac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \delta u(t)\right\rangle\right] dt+\\ &+\left\langle \delta x'(0)\tfrac{\partial^2\varphi}{\partial x(0)^2} + \Delta x'(T)\tfrac{\partial^2\varphi}{\partial x(0)\partial x(T)}, \delta x(0)\right\rangle +\\ &+\left\langle \delta x'(0)\tfrac{\partial^2\varphi}{\partial x(T)\partial x(0)} + \delta x'(T)\tfrac{\partial^2\varphi}{\partial x(T)^2}, \delta x(T)\right\rangle \geq 0.\end{aligned} \tag{26}$$

Inequality (26) is an implicit necessary optimality condition of first order. Since the verification of the last conditions require heavy calculations their application meets difficulties.

To obtain more effective optimality conditions of the second order, we use (23) in (26) and introduce the matrix function

$$R(\tau,s) = -G'(0,\tau)\tfrac{\partial^2\varphi}{\partial x(0)^2}G(0,s) - G'(T,\tau)\tfrac{\partial^2\varphi}{\partial x(T)\partial x(0)}G(0,s) -$$

$$-G'(0,\tau)\tfrac{\partial^2\varphi}{\partial x(0)\partial x(T)}G(T,s) - G'(T,\tau)\tfrac{\partial^2\varphi}{\partial x(T)^2}G(T,s) + \int_0^T G'(t,\tau)\tfrac{\partial^2 H}{\partial x^2}G(t,s)\,dt.$$

It allows us to obtain the following terminal formula for the second variation of the functional

$$\delta^2 J(u) = -\left\{\int_0^T\int_0^T \left\langle \delta' u(\tau)\tfrac{\partial' f(\tau,x,u)}{\partial u}R(\tau,s)\tfrac{\partial f(s,x,u)}{\partial u}, \delta u(s)\right\rangle dtds\right.$$

$$+\int_0^T \left\langle \delta' u(t)\tfrac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \delta u(t)\right\rangle dt$$

$$\left. +\, 2\int_0^T\int_0^T \left\langle \delta u'(t)\tfrac{\partial^2 H(t,\psi,x,u)}{\partial x\partial u}G(t,s)\tfrac{\partial f(s,x,u)}{\partial u}, \delta u(s)\right\rangle dtds\right\}.$$

Theorem 4. Let the admissible control $u(t)$ satisfy condition (25). Then in order to this function be optimal in problem (1)-(4), the inequality

$$\begin{aligned}\delta^2 J(u) = -\Big\{\int_0^T\int_0^T \left\langle \delta' u(\tau)\tfrac{\partial' f(\tau,x,u)}{\partial u}R(\tau,s)\tfrac{\partial f(s,x,u)}{\partial u}, \delta u(s)\right\rangle d\tau ds \\ +\int_0^T \left\langle \delta' u(\tau)\tfrac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \delta u(t)\right\rangle dt+ \\ +2\int_0^T\int_0^T \left\langle \delta u'(t)\tfrac{\partial^2 H(t,\psi,x,u)}{\partial x\partial u}G(t,s)\tfrac{\partial f(s,x,u)}{\partial u}, \delta u(s)\right\rangle dtds\Big\} \geq 0\end{aligned} \tag{27}$$

should be fulfilled for all $\delta u(t) \in L_\infty[0,T]$.

The analogy of the Legandre-Klebsh condition for the considered problem follows from condition (28).

Theorem 5. The inequality holds true

$$\nu'\frac{\partial^2 H(\theta,\psi(\theta),x(\theta),u(\theta))}{\partial u^2}v \leq 0 \tag{28}$$

over the optimal process $(u(t),x(t))$ for all $\nu \in R^r$ and $\theta \in [0,T]$.

Proof. To prove the theorem, we calculate the variation of the control

$$\delta u(t) = \begin{cases} v & t\in[\theta,\theta+\varepsilon) \\ 0 & t\notin[\theta,\theta+\varepsilon) \end{cases}, \tag{29}$$

where $\varepsilon > 0$, v is some r-dimensional vector.

By virtue of (23) the variation of the corresponding trajectory is

$$\delta x(t) = a(t)\varepsilon + o(\varepsilon,t), \quad t\in[0,T], \tag{30}$$

where $a(t)$ is a continuous bounded function.

Substituting variation (29) into (27) and selecting the principal term with respect to ε we obtain

$$\delta^2 J(u) = -\int_\theta^{\theta+\varepsilon} v'\frac{\partial^2 H(t,\psi(t),x(t),u(t))}{\partial u^2}vdt + o(\varepsilon) =$$

$$= -\varepsilon v'\frac{\partial^2 H(\theta,\psi(\theta),x(\theta),u(\theta))}{\partial u^2}v + o_1(\varepsilon).$$

From this using condition of (24) the Legandre-Klebsh criterion (28) is obtained.

Condition (30) is the second order optimality condition. It is obvious that when the right hand side of system (1) is linear with respect to control parameters, condition (28) also degenerates, i.e. is fulfilled trivially.

If for all $\theta \in (0,T)$, $v \in R^r$ the relations

$$\frac{\partial H(\theta,\psi(\theta),x(\theta),u(\theta))}{\partial u} = 0, \quad v'\frac{\partial^2 H(\theta,\psi(\theta),x(\theta),u(\theta))}{\partial u^2}v = 0,$$

hold true then the admissible control $u(t)$ is said be a singular control in the classic sense.

Theorem 6. Assume that the control $u(t)$ is the singular in the classic sense. Then for optimality of $u(t)$

$$\begin{aligned}\nu'\Big\{\int_0^T\int_0^T \left\langle \tfrac{\partial f(t,x,u)}{\partial u}R(t,s), \tfrac{\partial f(s,x,u)}{\partial u}\right\rangle dtds + \\ +2\int_0^T\int_0^T \left\langle \tfrac{\partial^2 H(t,\psi,x,u)}{\partial x\partial u}G(t,s), \tfrac{\partial f(s,x,u)}{\partial u}\right\rangle dtds\Big\} v \leq 0\end{aligned} \tag{31}$$

should be fulfilled for all $v \in R^n$.

Condition (31) is an integral necessary condition of optimality of the controls singular in the classic sense. One can obtain various type necessary optimality conditions by taking the special variation of various forms in formula (30).

Conclusion

In this paper, the optimal control problem is considered when the considered system is described by the differential equations with integral boundary conditions. The existence and uniqueness of the solution is proved for the corresponding boundary value problem. The first and second order increment formulas of the functional are obtained. Various necessary conditions of optimality of the first and second order are obtained. Of course, such type existence and uniqueness results and necessary conditions of optimality hold under the same sufficient conditions on nonlinear terms of the system of nonlinear differential equations (1), subject to multi-point nonlocal and integral boundary conditions type of

$$\int_0^T m(t)\,x(t)\,dt + \sum_{j=1}^{J} B_j x(t_j) = C,$$

where $B_j \in R^{n\times n}$ are given matrices and

$$\det\left(B + \sum_{j=1}^{J} B_j\right) \neq 0,$$

here, $0 < t_1 < t_2 < ... < t_J \leq T$ for controls singular in the classic sense. Selecting special variation in different way in formula (30) we can get various necessary optimality conditions.

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Интегралды шекаралық шарттары бар жүйелер үшін тиімді басқару есебі

Мақалада «таза» интегралды шекаралық шартпен тиімді басқару есебі қарастырылған. Грин функциясы құрылған. Банахтың қысып бейнелеу принципін қолдана отырып, бекітілген рұқсат етілген басқару кезінде интегралды шеттік есептердің бір класының шешімнің бар болуының жеткілікті шарты мен жалғыздығы анықталды. Функционалдың ауытқуы әдісімен Понтрягиннің максимум принципі дәлелденді. Функционалдың бірінші және екінші вариациялары есептелген. Басқарудың вариацияларының көмегімен екінші ретті тиімділіктің әртүрлі қажетті шарттары алынды.

Кілт сөздер: интегралды шекаралық шарттар, ерекше басқару, тиімді басқару есебі, шешімнің бар және жалғыз болуы.

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Задача оптимального управления для систем с интегральными граничными условиями

В статье рассмотрена задача оптимального управления с «чистым» интегральным граничным условием. Построена функция Грина. С помощью принципа сжимающих отображений Банаха установлено достаточное условие существования и единственности решения одного класса интегральных краевых задач при фиксированных допустимых управлениях. Методом приращений функционала доказан принцип максимума Понтрягина. Вычислены первая и вторая вариации функционала. С помощью вариаций управлений получены различные необходимые условия оптимальности второго порядка.

Ключевые слова: интегральные граничные условия, особые управления, задача оптимального управления, существование и единственность решения.

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Solving Volterra-Fredholm integral equations by natural cubic spline function

Using the natural cubic spline function, this paper finds the numerical solution of Volterra-Fredholm integral equations of the second kind. The proposed method is based on employing the natural cubic spline function of the unknown function at an arbitrary point and using the integration method to turn the Volterra-Fredholm integral equation into a system of linear equations concerning to the unknown function. An approximate solution can be easily established by solving the given system. This is accomplished with the help of a computer program that runs on Python 3.9.

Keywords: Volterra integral equation, Fredholm integral equation, spline function.

Introduction

Integral equations can be used to express a variety of mathematical physics topics. Some of these will be examined and treated explicitly as examples. It would be nearly impossible to compile a list of such applications. To say that integral equations play a role in practically every area of applied mathematics and mathematical physics is an understatement; because, the literature on integral equations and their applications is extensive.
Many researches have been conducted in recent years, with the results revealing the interaction of Fredholm integral equation, Volterra integral equation, Volterra-Fredholm integral equation, and numerical solutions of these three types of the integral equation.

The linear Volterra-Fredholm integral equations (VFIEs) of the following type were taken into consideration in this work:

$$u(x) = f(x) + \lambda_1 \int_a^x K(x,t)u(t)dt + \lambda_2 \int_a^b L(x,t)u(t)dt, \tag{1}$$

where the functions $f(x)$, and the kernels $K(x,t)$ and $L(x,t)$ are known L^2 analytic functions and λ_1, λ_2 are arbitrary constants, x is variable and $u(x)$ is the unknown continuous function to be determined. Numerous applications in the fields of physics, fluid dynamics, electrodynamics, and biology include the use of these equations. These integral equations are a reduction of several boundary value formulations using Neumann, Dirichlet, or boundary conditions. Additionally, they offer mathematical models for the spread of an epidemic as well as a host of other physical and biological problems.

Since the analytical solution to VFIEs generally does not exist outside of special cases, the numerical method is the most successful and efficient way to solve these issues. In order to solve VFIEs, a number of numerical and approximative techniques have been developed, including the linear and quadratic spline functions by Salim, et. al. [1, 2], Taylor polynomial by Yalcinbacs and Sezer [3]; Yalcinbas [4], the least square method and Chebyshev polynomials by Dastjerdi and Ghaini [5].

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Also, Lagrange collocation method by Wang [6], Series solution, successive approximation method and method of successive substitutions by Saeed and Berdawood [7], trigonometric Functions and Laguerre Polynomials by Hasan and Sulaiman [8], Touchard Polynomials (T-Ps) method by Al-Miah and Taie [9]. Some iterative numerical methods by Micula [10], Taylor polynomial by Didgara and Vahidi [11]. For additional information, the reader might turn to the following references and the references given there: (Jerry [12], Atkinson [13], Lange and Herbert [14], Kaminaka and Wadati [15], Ladopoulos [16], Corduneanu [17], Saeed and Aziz [18] and Jaber and Alrammahi [19]).

Equation (1) will be studied in this work using the natural cubic spline function. The rest of this paper is structured as follows. Our approach is introduced in Section 2 for solving Equation (1). We examine various numerical examples proving the viability of our method in Section 3. Some conclusions will be made in Section 4.

1 *Description of the method*

In this section, we solve Equation (1) by using the quadratic spline function (Cheney and Kincaid [20], Saeed *et. al.* [21]). The unknown function $u(x)$ in (1) is approximated by the quadratic spline function $S(x)$. In the interval $[x_i, x_{i+1}]$, the quadratic spline function is defined by the following formula:

$$S_i(x) = A_i(x)S_i + B_i(x)S_{i+1} + C_i(x)S'_i + D_i(x)S'_{i+1}, \tag{2}$$

where $A_i(x) = 1-3\frac{(x-t_i)^2}{h^2}+2\frac{(x-t_i)^3}{h^3}$, $B_i(x) = 1-A_i(x)$, $C_i(x) = \frac{(x-t_i)(x-t_{i+1})^2}{h^2}$, $D_i(x) = \frac{(x-t_{i+1})(x-t_i)^2}{h^2}$, and $h = x_{i+1} - x_i$ for all $i = 0, 1, \cdots, n-1$. Now substituting (2) in (1) and letting $x = x_i$, we get

$$\begin{aligned}
S_i &= f_i + \lambda_1 \int_a^{x_i} K(x_i,t)S(t) + \lambda_2 \int_a^b L(x_i,t)S(t)dt \\
&= f(x_i) + \lambda_1 \left[\sum_{j=0}^{j=i-2} \int_{x_j}^{x_{j+1}} K(x_i,t)[A_j(t)S_j + B_j(t)S_{j+1} + C_j(t)S'_j + D_j(t)S'_{j+1}]dt \right. \\
&\left. + \int_{x_{i-1}}^{x_i} K(x_i,t)[A_{i-1}(t)S_{i-1} + B_{i-1}(t)S_i + C_iS'_{i-1} + D_i(t)S'_i]dt \right] \\
&+ \lambda_2 \left[\int_{x_0=a}^{x_1} L(x_i,t)S_0(t)dt + \int_{x_1}^{x_2} L(x_i,t)S_1(t)dt + \cdots + \int_{x_{n-1}}^{x_n=b} L(x_i,t)S_{n-1}(t)dt \right] \\
&= f(x_i) + \lambda_1 \left[\sum_{j=0}^{j=i-2} \int_{x_j}^{x_{j+1}} K(x_i,t)[A_j(t)S_j + B_j(t)S_{j+1} + C_j(t)S'_j + D_j(t)S'_{j+1}]dt \right. \\
&\left. + \int_{x_{i-1}}^{x_i} K(x_i,t)[A_{i-1}(t)S_{i-1} + B_{i-1}(t)S_i + C_iS'_{i-1} + D_i(t)S'_i]dt \right] \\
&+ \lambda_2 \left[\int_{x_0=a}^{x_1} L(x_i,t)[A_0(t)S_0 + B_0(t)S_1 + C_0(t)S'_0 + D_0(t)S'_1]dt \right. \\
&+ \int_{x_1}^{x_2} L(x_i,t)[A_1(t)S_1 + B_1(t)S_2 + C_1(t)S'_1 + D_1(t)S'_2]dt + \cdots \\
&\left. + \int_{x_{n-1}}^{x_n=b} L(x_i,t)[A_{n-1}(t)S_{n-1} + B_{n-1}(t)S_n + C_{n-1}(t)C_j(t){S'}_{n-1} + D_n - 1(t)S'_n]dt \right].
\end{aligned}$$

By computing the integrals in the above equation using the trapezoidal rule, we get

$$S_i = f_i + \frac{h}{2}(\lambda_1 K_{i0} + \lambda_2 L_{i0})S_0 + h\sum_{j=1}^{i-1}(\lambda_1 K_{ij} + \lambda_2 L_{ij})S_j + \frac{h}{2}(\lambda_1 K_{ii} + \lambda_2 L_{ii})S_i, \tag{3}$$

for $i = 0, 1, \cdots, n$.

In this way, Equation (3) constructs a system of linear equations concerning to the unknown function S_i. Briefly, this system can be rewritten as follows:

$$CS = F, \tag{4}$$

where

$$S = \begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_n \end{bmatrix}, \quad F = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}, \quad C = \begin{bmatrix} C_0 & C_1 & C_2 & \dots & C_{n-1} & C_n \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 1 - \frac{\lambda_2 h}{2} L_{00} \\ -\frac{h}{2}(2\lambda_1 K_{10} + \lambda_2 L_{10}) \\ -\frac{h}{2}(\lambda_1 K_{20} + \lambda_2 L_{20}) \\ -\frac{h}{2}(\lambda_1 K_{30} + \lambda_2 L_{30}) \\ \vdots \\ -\frac{h}{2}(\lambda_1 K_{n0} + \lambda_1 L_{n0}) \end{bmatrix}, \quad C_1 = \begin{bmatrix} -\lambda_2 h L_{01} \\ 1 - (\frac{h}{2}(\lambda_1 K_{11} + 2\lambda_2 L_{11}) \\ -\frac{h}{2}(3\lambda_1 K_{21} + 2\lambda_2 L_{21}) \\ -h(\lambda_1 K_{31} + \lambda_2 L_{31}) \\ \vdots \\ -h(\lambda_1 K_{n1} + \lambda_2 L_{n1}) \end{bmatrix},$$

$$C_2 = \begin{bmatrix} -\lambda_2 h L_{02} \\ -\frac{h}{2}(2\lambda_2 L_{12} - \lambda_1 K_{10}) \\ 1 - \frac{h}{2}(\lambda_1 K_{22} + 2\lambda_2 L_{22}) \\ -\frac{h}{2}(3\lambda_1 K_{32} + 2\lambda_2 L_{32}) \\ \vdots \\ -h(\lambda_1 K_{n2} + \lambda_2 L_{n2}) \end{bmatrix}, \quad \dots, \quad C_{n-1} = \begin{bmatrix} -\lambda_2 h L_{0(n-1)} \\ -\lambda_2 h L_{1(n-1)} \\ -\lambda_2 h L_{2(n-1)} \\ \vdots \\ 1 - \frac{h}{2}\big(\lambda_1 K_{(n-1)(n-1)} + 2\lambda_2 L_{(n-1)(n-1)}\big) \\ -\frac{h}{2}\big(\lambda_1 K_{n(n-1)} + 2\lambda_2 L_{n(n-1)}\big) \end{bmatrix},$$

and

$$C_n = \begin{bmatrix} -\frac{\lambda_2 h}{2} L_{0n} \\ -\frac{\lambda_2 h}{2} L_{1n} \\ -\frac{\lambda_2 h}{2} L_{2n} \\ \vdots \\ -\frac{\lambda_2 h}{2} L_{(n-1)(n-1)} \\ 1 - \frac{h}{2}(\lambda_1 (K_{nn} + \lambda_2 L_{nn}) \end{bmatrix}.$$

In the sequel, using a standard rule to the resulting system yields an approximate solution of Equation (1) as $S_i(x)$ given by Equation (2).

2 Numerical examples

In this section, we present three examples to illustrate the efficiency and accuracy of the proposed method. The computed errors e_i are defined by $e_i = |u_i - S_i|$, where u_i is the exact solution of Equation

(1) and S_i is an approximate solution of the same equation. Also we compute Least square error (LSE) which is defined by formula $= \sum_{i=0}^{n}(u_i - S_i)^2$ and all computations are performed using the Python program.

Example 1. Consider the linear Volterra-Fredholm integral equation

$$u(x) = -\frac{x^2}{2} - \frac{7x}{2} + 2 + \int_0^x u(t)dt + \int_0^1 xu(t)dt.$$

The exact solution to this equation is given by $u(x) = x + 2$.
Table (1) demonstrates LSE obtained from applying our method to Example (1) for $n = 5$.

Table 1

The Numerical Results for Example (1) with $n = 5$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	2	2	0	0
0.2	2.2	2.167670	0.323298	0.00104522
0.4	2.4	2.369779	0.302205	0.00091328
0.6	2.6	2.533617	0.06738252	0.00454041
0.8	2.8	3.0875551	0.2875513	0.08268795
1	3	3.0788111	0.0788111	0.00621119
LSE				9.5398050×10^{-2}

Example 2. Consider the linear Volterra-Fredholm integral equation

$$u(x) = 2cos(x) - 1 + \int_0^x (x-t)u(t)dt + \int_0^{\pi} u(t)dt.$$

The exact solution to this equation is given by $u(x) = cos(x)$.
Table (2) demonstrates LSE obtained from applying our method to Example (2) for $n = 5$.

Table 2

The Numerical Results for Example (2) with $n = 5$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	1	0.9020653	0.0979346	$9.59119752 \times 10^{-3}$
$\frac{\pi}{5}$	0.8090169	0.80093654	0.00808045	$6.52937193 \times 10^{-5}$
$\frac{2\pi}{5}$	0.3090169	0.38139274	0.07237575	$5.23824911 \times 10^{-3}$
$\frac{3\pi}{5}$	-0.3090169	-0.15642311	0.15259389	$2.32848948 \times 10^{-2}$
$\frac{4\pi}{5}$	-0.8090169	0.8466296	0.0376126	$1.41470781 \times 10^{-3}$
π	$-1.$	-0.9488828	0.0511172	$2.61296767 \times 10^{-3}$
LSE				$4.22073106838 \times 10^{-2}$

Example 3. Consider the linear Volterra-Fredholm integral equation

$$u(x) = -\frac{x^5}{10} + 2x^3 - \frac{x^2}{2} - \frac{3x}{2} + \frac{1}{10} + \int_0^x (x+t)u(t)dt + \int_0^1 (x-t)u(t)dt.$$

The exact solution to this equation is given by $u(x) = 2x^3 + 1$.
Table (3) demonstrates LSE obtained from applying our method to Example (3) for $n = 5$.

Table 3

The Numerical Results for Example (3) with $n = 5$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	0.9320698	0.0679302	0.00461451
0.2	1.016	0.91835798	0.09764202	0.00953396
0.4	1.128	0.99225104	0.13574896	0.01842778
0.6	1.432	1.24719269	0.18480731	0.03415374
0.8	2.024	1.83381163	0.19018837	0.03617162
1	3	2.72882412	0.27117588	0.07353636
LSE				$1.76437975899 \times 10^{-1}$

Table 4

LSE for different values of n for Examples (1)–(3)

LSEn	10	20	30	40	50
Example 1	$2.02193681 \times 10^{-2}$	$5.21316981 \times 10^{-3}$	$2.36837446 \times 10^{-3}$	$1.348919901 \times 10^{-3}$	8.7012926×10^{-4}
Example 2	$3.03911660 \times 10^{-3}$	$1.87835246 \times 10^{-4}$	$3.79028932 \times 10^{-5}$	$1.24158506 \times 10^{-5}$	$5.28945050 \times 10^{-6}$
Example 3	$1.95064939 \times 10^{-2}$	$2.31413450 \times 10^{-3}$	$6.752544397 \times 10^{-4}$	$2.82826976 \times 10^{-4}$	$1.44203795 \times 10^{-4}$

3 Conclusion

In this paper the cubic spline function is used to solve linear Volterra-Fredholm integral equations, and it is a powerful numerical approach. The numerical results in the present section demonstrate that the proposed method can successfully tackle the Volterra-Fredholm type problem. Table (4) shows that the proposed method has extremely good stability; as n increases, the error decreases at first and then stabilizes. We also conclude that we have high accuracy when the exact solution is a trigonometric function. The present method can be easily extended to systems of Volterra-Fredholm integral equations and systems of Volterra-Fredholm integro-differential equations. The current method may be simply extended to Volterra-Fredholm integral equations and Volterra-Fredholm integro-differential equations.

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Вольтерра-Фредгольм интегралдық теңдеулерін кубтық сплайн-функциясымен шешу

Мақалада табиғи кубтық сплайн-функциясын қолданып екінші текті Вольтерра-Фредгольм аралас интегралдық теңдеулерінің сандық шешімі табылған. Ұсынылған әдіс еркін нүктеде белгісіз функцияның табиғи кубтық сплайн-функциясын қолдануға және Вольтерра-Фредгольм интегралдық теңдеуін белгісіз функцияға қатысты сызықтық теңдеулер жүйесіне түрлендіру үшін интегралдау әдісін қолдануға негізделген. Бұл жүйені шешу арқылы жуық шешімді табу оңай. Бұған Python 3.9-да жұмыс істейтін компьютерлік бағдарлама арқылы қол жеткізіледі.

Кілт сөздер: Вольтерра интегралдық теңдеуі, Фредгольм интегралдық теңдеуі, сплайн-функциясы.

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Решение интегральных уравнений Вольтерра-Фредгольма с помощью естественной кубической сплайн-функции

В статье с использованием функции натурального кубического сплайна найдено численное решение смешанных интегральных уравнений Вольтерра-Фредгольма второго рода. Предлагаемый метод основан на применении естественной кубической сплайн-функции неизвестной функции в произвольной точке и метода интегрирования для преобразования интегрального уравнения Вольтерра-Фредгольма в систему линейных уравнений относительно неизвестной функции. Приближенное решение легко получить, решив данную систему. Это достигается с помощью компьютерной программы, работающей на Python 3.9.

Ключевые слова: интегральное уравнение Вольтерра, интегральное уравнение Фредгольма, сплайн-функция.

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Numerical method to solution of generalized model Buckley-Leverett in a class of discontinuous functions

A new numerical method is proposed for solving the generalized Buckley-Leverett problem, which describes the movement of two-phase mixtures of Bazhenov bed sediments in a class of discontinuous functions. To this end, we introduce an auxiliary problem that has advantages over the main problem, and using these advantages, an original finite difference method to solve of the auxiliary problem is developed. Using the suggested auxiliary problem, a solution which expresses exactly all physical characteristics of the problem is obtained.

Keywords: generalized Buckley-Leverett problem, auxiliary problem, finite differences method in a class of discontinuous functions.

Introduction

We consider the following problem in the upper half of the Euclidean $R^2_+(x,t)$ space

$$\frac{\partial u(x,t)}{\partial t}+\frac{\partial \varphi(u(x,t))}{\partial x}-\psi\big(u(x,t)\big)=0, \tag{1}$$

$$u(x,0)=u_0(x), x\geq 0, \tag{2}$$

$$u(0,t)=u_1(x), t\geq 0, \tag{3}$$

where $\varphi(u)$ and $\psi(u)$ are known functions according to argument u and have the following properties:

- $\varphi(u)$, $\psi(u)$ and $\varphi'(u)$, $\psi'(u)$ are continuous functions, and they are bounded for bounded u, and $\varphi''(u)$ does not change its sign,
- $\varphi(u)\geq 0$ and $\varphi'(u)\geq 0$ for $u\geq 0$, and the argument u has values such that the function $\psi(u)$ becomes zero at these points,
- $\psi'(u)$ is bounded function for $u\geq 0$.

Here, $u_0(x)$ and $u_1(x)$ are given functions satisfying the condition $u_0(0)\neq u_1(0)$.

In the case of $\psi\big(u(x,t)\big)\equiv 0$, the problem (1)–(3) is used to solve many problems in hydrodynamics, including the qualitative characteristics of the mechanism of compression of oil with gas or water in a porous medium, which is called the Buckley-Leverett model in the literature [1]. It has been proven that when the initial and boundary functions are incompatible (for the initial-boundary problem) or the initial profile has a decreasing part with respect to the spatial coordinate (for the initial value problem), the jump points, locations of which are not known beforehand, occur in the solution of the problem [2–9]. In other words, there is no classical solution for the problem under consideration, and the question of the uniqueness of the solution remains open. For this purpose, criteria for the uniqueness

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of the solution and robustness of the jump are proposed in [4, 6, 8, 10, 11]. In the theory of hyperbolic equations, the stable jump motion in a problem is called a discontinuity disintegration problem and has been widely studied in the literature, as specified in [2, 3, 5, 8, 12, 13], etc.

There are conservative finite volume methods of practical importance, which are based on dividing the spatial domain into intervals (also called «finite volumes» or grid cells), and establishing certain approximations to the integral of the flow over each of these volumes in [10, 11, 14–18], etc. Also, Godunov-type finite difference algorithms were developed considering the properties of the analytical solution in [13].

In [19], a method in the class of discontinuous functions was proposed to find an analytical solution of the problem (1)–(3), and using this method, a finite difference algorithm was established that accurately expresses all the properties of the physical processes of the problem in [20].

In the case of $\psi\big(u(x,t)\big) \neq 0$, the problem (1)–(3) is called generalized Buckley-Leverett problem in a physical sense and differs from the classic Buckley-Leverett problem in that the trajectory of the jump does not coincide with the characteristics, and the discontinuity jump approaches zero as time values increase [21].

In [21], the dynamics of chemical and physico-chemical changes in a multi-phase and multi-component oilfield after exposure to thermogas was investigated by means of a mathematical model, where the process of injecting hot water into the reservoir containing hydrocarbons was specifically discussed. Usually, this type of impact method is applied to an oilfield (Bazhenov-type deposit) including kerogen containing oil in a bound state. Such deposits have a layered structure in which oil is located in the pores as well. Permeable non-productive strata alternate with productive impermeable strata. Mathematical modeling of deposits with such a structure in the non-isothermal mode becomes even more complicated.

The purpose of the thermal impact mechanism is to inject a certain amount of hot fluid such as hot water into the reservoir in order to increase the reservoir temperature, and then to displace oil by water at the common contact interface. This can also release trapped oil and isolated pores. During treatment with hot water injected into the reservoir, some additional amount of oil is released into the pore volume, which affects the regime of the displacement of oil by water. Ultimately, it leads to an increase in the flow rate of light crude oil trapped in the reservoir.

Since the filtration process is slow, the deformation of the bed can be neglected. On the other hand, the movement happens so quickly that it is possible to ignore the conductive heat transfer as the essential mechanism of heat transfer is convection.

More interesting problems arise in investigating the role of spatial structures in the creation and evolution of living organisms in molecular biology [22]. When studying this type of problem, $\varphi(u(x,t))$ in Eq. (1) represents the convective flow function of the reaction component, and $\psi(u(x,t))$ represents the kinetics of the reaction. If $\partial\varphi(u(x,t))/\partial x$ is a strictly non-linear function, then jumps occur in the solution of the problem, in which case such solutions are understood as weak solutions.

In this study, we consider problems with source terms that do not include delta functions that typically converge to zero throughout most of the region, ignoring the existence of very thin reaction zones that occur dynamically as part of the solution. Such source terms are often expressed in delta functions, but their positions and strengths are often not known in advance.

In this article, in order to show what behaviors are expected from the process, problem (1)–(3) is handled only mathematically, with respect to wave propagation, without considering the mechanism of any chemical reaction. In general, soft solutions found by the characteristics method do not enable us to explore the dynamics from the beginning of the process to the end.

As it is known that, the solutions of problem (1)–(3) has the points of discontinuities locations of which are unknown beforehand. Existence of the points of discontinuities in the solutions involves difficulties in applying the classical numerical methods to that equations [19]. The necessity to work with discontinuous functions and to find a solution that can accurately express the dynamics of the process require the creation of sensitive numerical methods in the class of discontinuous functions.

1 Finding the analytical solution

For the sake of simplicity, we will first consider the Cauchy problem for Eq. (1). We can easily get the solution of problem (1), (2) by the method of characteristics. For this, if we search for the solution of problem (1), (2) in the closed form $V(t,x,u)=0$, we reach the following quasi linear equation in accordance with the V function

$$\frac{\partial V}{\partial t}+\varphi'(u)\frac{\partial V}{\partial x}+\psi(u)\frac{\partial V}{\partial u}=0. \tag{4}$$

The system of characteristic equations for (4) is

$$\frac{dt}{1}=\frac{dx}{\varphi'(u)}=\frac{du}{\psi(u)}.$$

From here, the following system of equations is obtained

$$\begin{cases} \frac{dx}{dt}=\varphi'(u), \\ \frac{du}{dt}=\psi(u). \end{cases} \tag{5}$$

The first intermediate integrals for the system (5) are

$$c_1=x-\int\frac{\varphi'(u)du}{\psi(u)}, \quad c_2=t-\int\frac{du}{\psi(u)}. \tag{6}$$

According to the general theory, for an arbitrary continuously differentiable function F, the general solution of problem (1), (2) is written as $F(c_1,c_2)=0$ or

$$x-\int\psi^{-1}(u)\varphi'(u)du=f\Big(t-\int\psi^{-1}(u)du\Big), \tag{7}$$

where the function f is any continuously differentiable function. Expression in the form of (7) is called soft solution.

To check the effectiveness of the proposed method, and to find a clear expression of the analytical solution, instead of Eq. (1), the following equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-u(1-u)=0 \tag{8}$$

is considered in the special case of $\varphi(u)=\frac{u^2}{2}$ and $\psi(u)=u(1-u)$. In this case, Eq. (8) is called the Burgers equation with a robust source in hydrodynamics. For Eq. (8), the expression in (6) and (7) take the following form

$$c_1=x+\ln(1-u), \quad c_2=\frac{e^t}{u}-e^t$$

and

$$x+\ln(1-u)=f\Big(\frac{e^t}{u}-e^t\Big)$$

respectively. Here, f is any continuously differentiable function. Considering the initial condition (2), the soft solution of problem (1), (2) is obtained as

$$u(x,t)=e^t(1-u+ue^{-t})u_0\big(\ln e^x(1-u+ue^{-t}\big). \tag{9}$$

By a simple calculation, it is verified that the function $u(x,t)$ given by (9) is a soft solution of problem (1), (2). In a special case if it is assumed $u_0(x) = e^{-x}$, then expression (9) takes the following form

$$u(x,t) = e^{-x+t}.$$

Also, it is easily shown that function (9) satisfies Eq. (8). In other words, function (9) is a soft solution of Eq. (8).

When the second equation of system (5) is considered

$$\frac{du}{dt} = u(1-u), \tag{10}$$

it is seen that the constant functions $u = 0$ and $u = 1$ are equilibrium solutions. This equation, called the logistic equation, describes the growth of population and is also applied to the growth of bacteria, fruit flies and flour beetles, etc. [22]. It can be shown that $u = 0$ is an unstable equilibrium point, and $u = 1$ is a stable one. The initial condition $u(x,0)$ just indicates a non-regular spatial distribution.

Solution of Eq. (10) is

$$u(\xi) = \frac{1}{2}\Big(1 + \tanh\frac{\xi}{2D}\Big), \quad \xi = x - Dt$$

and this becomes a piece-wise continuous function from zero to one rapidly, regardless of the initial profile. To see the subsequent evolution of the solution, it is sufficient to consider the Riemann problem with a jump from 0 to 1 or from 1 to 0. In the Burgers equation without source function, the jump line moves with velocity $D = 1/2$ for $u_{left} = 1$ and $u_{right} = 0$. Then the source term is identically equal to zero, and therefore, has no effect on the solution of the problem. An interesting situation occurs if $u_{left} = 0$ and $u_{right} = 1$, in which case the Burgers equation converts the jump in the initial profile into a rarefaction wave.

2 Finite differences in a class of continuous functions

Firstly, let's divide the interval $[a,b]$ into n equal parts by means of the points x_j, $(j = 0,1,2,...,n)$ and by setting $h_x = (b-a)/n$, $(i = 0,1,2,...,n)$ that is, $x_j = a + jh_x$. In a similar way, let's divide the interval $[0,T)$ into time layers by means of points $t_k = kh_\tau$, $(k = 0,1,2,...)$, where $h_\tau > 0$. Here a, b, and T are given real numbers. Thus, we have constructed two one-dimensional grids over the intervals $[a,b]$ and $[0,T)$, respectively

$$\omega_{h_x} = \{x_j = a + jh_x, \quad h_x = (b-a)/n, \quad (j = 0,1,2,...,n)\},$$

$$\omega_{h_\tau} = \{t_k = kh_\tau, \quad h_\tau > 0, \quad (k = 0,1,2,...)\}.$$

Eventually, we cover the region by a uniform grid $\Omega_{h_x h_\tau} = \omega_{h_x} \times \omega_{h_\tau}$.

The need to work with discontinuous functions and find a solution that can accurately express the dynamics of the process requires the creation of a sensitive numerical method in the class of discontinuous functions. Now, we can study the techniques in discretizing the differential problem.

To find the numerical solution of problem (8), (2), let us include the following operator

$$A(\centerdot) = \frac{\partial(\centerdot)}{\partial x}.$$

It is clear that this operator has an inverse denoting by $A^{-1}(\centerdot)$, which differs from it by a constant. Applying the operator $A^{-1}(\centerdot)$ to both sides of the Eq. (3) we get

$$A^{-1}\left(\frac{\partial u}{\partial t}\right) + A^{-1}\left(\frac{1}{2}\frac{\partial u^2}{\partial x}\right) - A^{-1}\big(u(1-u)\big) = A^{-1}(0).$$

Let $A^{-1}(0) = h(t)$, from here we have $Ah = 0$. The last equation is written as

$$\frac{\partial A^{-1}u}{\partial t} + \frac{u^2}{2} - A^{-1}\big(u(1-u)\big) = h(t), \tag{11}$$

where $h(t) \in A^{-1}(0) = \ker A = \{h(t) \in C[0,\infty) \ : \ Ah = 0\}$ is any function. We introduce the following transformation

$$A^{-1}u + h(t) = v(x,t). \tag{12}$$

From (12) we obtain

$$u(x,t) = A\big(v(x,t)\big). \tag{13}$$

Substituting the relations (12) and (13) in Eq. (11), we get

$$\frac{\partial v(x,t)}{\partial t} + \frac{1}{2}\big(u(x,t)\big)^2 - \alpha\int_{-\infty}^{x} u(\xi,t)\big(1-u(\xi,t)\big)d\xi = 0. \tag{14}$$

The initial condition for Eq. (14) is

$$v(x,0) = v_0(x). \tag{15}$$

Here the function $v_0(x)$ is any continuously differentiable solution of equation $A\big(v(x,0)\big) = u(x,0)$, which is

$$\frac{dv_0(x)}{dx} = u_0(x). \tag{16}$$

We will call the problem (14)–(15) as an auxiliary problem. In accordance with [19] and [23] we consider this special auxiliary problem in order to determine the weak solution of the problem (8), (2).

The auxiliary problem has the following advantages:

- The differentiability property of the function $v(x,t)$ is of a higher order than the differentiability property of the function $u(x,t)$,
- The function $u(x,t)$ may be a discontinuous function, as long as it is an integrable one,
- Algorithms built to calculate the function $u(x,t)$ do not require the derivative of $u(x,t)$ with respect to any variables.

Theorem 1. If the function $v(x,t)$ is a solution of the auxiliary problem (14), (15), then the function $u(x,t) = A\big(v(x,t)\big)$ is a weak solution of the main problem (8), (2).

Proof. To prove the theorem, it is sufficient to apply the operator A directly to the Eq. (14) and consider the expression in (13).

The construction of finite difference algorithms

We will apply two finite difference schemes for Eq. (14) using explicit and implicit schemes.

Explicit scheme : Firstly, let us discretize Eq. (14) as follows

$$\frac{V_{i,k+1} - V_{i,k}}{\tau} + \frac{1}{2}U_{i,k}^2 - \alpha h\sum\nolimits_{j=1}^{i} U_{j,k}(1-U_{j,k}) = 0$$

or

$$V_{i,k+1} = V_{i,k} - \tau U_{i,k}\Big[U_{i,k}\Big(\frac{1}{2}+\alpha h\Big)\Big] + \tau\alpha h\sum\nolimits_{j=1}^{i-1} U_{j,k}(1-U_{j,k}), \quad j = 1,2,...,n-1; \quad k = 0,1,2,... \tag{17}$$

where h and τ are steps of the grid for x and t variables, respectively.

The initial condition for (17) is

$$V_{j,0} = v_0(x_j), \quad j = 0, 1, 2, ..., n.$$

Here, $v_0(x)$ is the grid function corresponding to the continuous function found from Eq. (16). The validity of the following equality can be easily shown as

$$U_{i,k+1} = \frac{V_{i,k+1} - V_{i-1,k+1}}{h}.$$

Implicit scheme : Now let's write an implicit scheme for problem (14), (15). For this, let's write equation (14) as a finite difference equivalent as follows

$$V_{i,k+1} = V_{i,k} - \tau U_{i,k+1}\left[U_{i,k+1}\left(\frac{1}{2} + \alpha h\right) + \alpha h\right] + \tau\alpha h \sum_{j=1}^{i-1} U_{j,k+1}(1 - U_{j,k+1}),$$

$$j = 1, 2, ..., n; k = 0, 1, 2, ...$$

We can obtain the solution of the last system of nonlinear algebraic equations by applying Newton's method.

Computer tests

In order to compare the solutions found by the finite difference algorithm we proposed solutions in the literature, as $u_0(x)$ function e^{-x} is accepted. The calculation results are shown in Figures 1-3 accordingly.

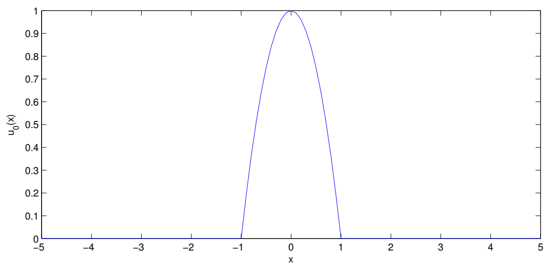

Figure 1. The source function.

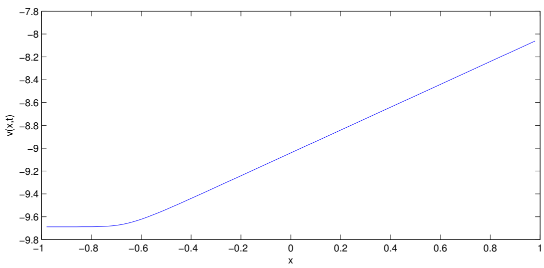

Figure 2. Graph of the solution of problem (14), (15) at $T = 1.5$.

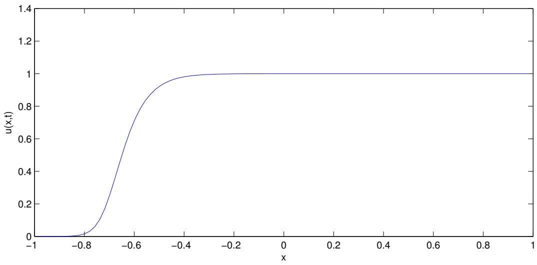

Figure 3. Graph of the function $u(x,t) = A(v(x,t))$.

3 Conclusion

The results obtained in this paper can be listed as follows:

- An original method in the class of discontinuous functions is proposed to find the numerical solution of the Cauchy problem for the first order nonlinear partial differential equation with a nonlinear source function.

- The special auxiliary problem of which the differentiable properties of the solution one order higher than the differentiable properties of the main problem is introduced.

- Using the advantages of the auxiliary problem the efficient numerical algorithms are suggested in a class of discontinuous functions. The obtained solutions express the all physical properties accurately.

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Үзілісті функциялар класындағы Бакли-Левереттің жалпыланған моделін шешудің сандық әдісі

Бакли-Левереттің жалпыланған есебін шешудің жаңа сандық әдісі ұсынылды, ол үзілісті функциялар класындағы Бажен қабатының екі фазалы қоспаларының қозғалысын сипаттайды. Ол үшін негізгі есептен артықшылығы бар көмекші есеп енгізілді және осы басымдықтардың көмегімен көмекші есепті шешу үшін ақырлы айырымдық түпнұсқа әдісі әзірленді. Ұсынылған көмекші есептің көмегімен есептің барлық физикалық сипаттамаларын дәл көрсететін шешім алынды.

Кілт сөздер: Бакли-Левереттің жалпыланған есебі, көмекші есеп, үзіліс функциялары класындағы ақырлы айырымдық әдіс.

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Численный метод решения обобщенной модели Бакли-Леверетта в классе разрывных функций

Предложен новый численный метод решения обобщенной задачи Бакли-Леверетта, описывающий движение двухфазных смесей отложений баженовской толщи в классе разрывных функций. Для этого введена вспомогательная задача, имеющая преимущества перед основной, и с помощью данных приоритетов разработана оригинальный метод конечных разностей для решения вспомогательной задачи. С помощью предложенной вспомогательной задачи получено решение, точно выражающее все физические характеристики задачи.

Ключевые слова: обобщенная задача Бакли-Леверетта, вспомогательная задача, метод конечных разностей в классе разрывных функций.

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On one solution of a periodic boundary value problem for a hyperbolic equations

In a rectangular domain, we consider a boundary value problem periodic in one variable for a system of partial differential equations of hyperbolic type. Introducing a new unknown function, this problem is reduced to an equivalent boundary value problem for an ordinary differential equation with an integral condition. Based on the parametrization method, new approaches to finding an approximate solution to an equivalent problem are proposed and its convergence is proved. This made it possible to establish conditions for the existence of a unique solution of a semiperiodic boundary value problem for a system of second-order hyperbolic equations.

Keywords: boundary value problem, hyperbolic equations, algorithm, parametrization method, approximate solution.

Introduction

Boundary value problems for hyperbolic equations arise when studying the processes of transverse vibrations of a string, longitudinal vibrations of a rod, electrical vibrations in a wire, torsional vibrations of a shaft, gas vibrations, etc. [1–3].

To date, well-known methods are used to solve the problems under consideration, such as the Fourier method, the method of successive approximations, methods of function theory, variational methods, numerical methods, etc. [4–9]. This makes it possible to obtain various solvability conditions for boundary value problems for hyperbolic equations and construct analytical or approximate solutions [10–16].

In [17–20], such problems were solved by introducing functional parameters. Using this method, sufficient conditions were obtained for the correct solvability of nonlocal boundary value problems for systems of hyperbolic equations with a mixed derivative in terms of the initial data, and algorithms for finding their solutions were proposed. Based on the equivalence of the correct solvability of a boundary value problem with data on the characteristics for systems of linear hyperbolic equations and the correct solvability of a two-point boundary value problem for a family of systems of ordinary differential equations, a criterion for the correct solvability of the problem under study is established.

In this paper, we propose an algorithm where, in contrast to works [18–21], there is no need to find the Goursat or Cauchy problem at each step of the algorithm. In addition, when compared with the algorithm proposed in [22–23], this approach is more simplified. But despite this, the approximate solution is more accurate. The main characteristic of this algorithm is the effective verifiability of the conditions for their applicability and the ability to use it to find solutions with a given accuracy. This approach can be applied to problems of the third and fourth orders [24, 25] and obtain verifiable conditions.

On $\Omega = [0, X] \times [0, Y]$ the semiperiodic boundary value problem is considered

$$\frac{\partial^2 z}{\partial x \partial y} = A(x, y)\frac{\partial z}{\partial x} + B(x, y)z + f(x, y), \quad (x, y) \in \Omega, \tag{1}$$

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$$z(0,y) = \varphi(y), \quad y \in [0,Y], \tag{2}$$

$$z(x,0) = z(x,Y), \quad x \in [0,X], \tag{3}$$

where $(n \times n)$ - matrices $A(x,y), B(x,y)$, n-vector function $f(x,y)$ are continuous on Ω, n-vector function $\varphi(y)$, is continuously differentiable on $[0,Y]$, there is a condition of agreement $\varphi(0) = \varphi(Y)$, $\|z(x,y)\| = \max\limits_{i=\overline{1,n}} |z_i(x,y)|, \quad \|A(x,y)\| = \max\limits_{i=\overline{1,n}} \sum\limits_{j=1}^{n} |a_{ij}(x,y)|.$

Let $C(\Omega, R^n)$ be the spaces of functions $z : \Omega \to R^n$ which are continuous on Ω.

The function $z(x,y) \in C(\Omega, R^n)$, with partial derivatives $\frac{\partial^2 z(x,y)}{\partial x \partial y} \in C(\Omega, R^n)$, $\frac{\partial z(x,y)}{\partial x} \in C(\Omega, R^n)$, $\frac{\partial z(x,y)}{\partial y} \in C(\Omega, R^n)$ is called the classical solution to the problem (1)–(3), if it satisfies the system (1) for all $(x,y) \in \Omega$ and boundary conditions (2), (3).

2 Main results

We introduce the functions $u(x,y) = \frac{\partial z(x,y)}{\partial x}$, to find a solution and the problem (1)–(3) we write as

$$\frac{\partial u}{\partial y} = A(x,y)u + B(x,y)z(x,y) + f(x,y), \quad (x,y) \in \Omega, \tag{4}$$

$$u(x,0) = u(x,Y), \quad x \in [0,X], \tag{5}$$

$$z(x,y) = \varphi(y) + \int\limits_0^x u(\xi,y)d\xi. \tag{6}$$

Problems (1)–(3) and (4)–(6) are equivalent in the sense that if the function $z(x,y)$ is a solution to problem (1)–(3), then the pair $(u(x,y), z(x,y))$ will be a solution to problem (4)–(6), and vice versa, if the pair $\widehat{u}(x,y), \widehat{z}(x,y)$ is a solution to problem (4)–(6), then $\widehat{z}(x,y)$ will be a solution to problem (1)–(3).

To solve problem (4)–(6) we will apply the parameterization method. For the step $h > 0 : Nh = Y$ we partition $[0,Y) = \bigcup\limits_{r=1}^{N} [(r-1)h, rh)$, $N = 1, 2, \ldots$. In this case Ω is divided into N parts. By $u_r(x,y)$ we denote the restrictions of the functions $u(x,y)$ on $\Omega_r = [0,X] \times [(r-1)h, rh), \quad r = \overline{1,N}$. Then problem (4), (5) will be equivalent to the boundary value problem

$$\frac{\partial u_r}{\partial y} = A(x,y)u_r(x,y) + B(x,y)z_r(x,y) + f(x,y), \quad (x,y) \in \Omega_r, \quad r = \overline{1,N}, \tag{7}$$

$$u_1(x,y) - \lim_{y \to Y-0} u_N(x,y) = 0, \quad x \in [0,X], \tag{8}$$

$$\lim_{y \to sh-0} u_s(x,y) = u_{s+1}(x,y), \quad x \in [0,X], \quad s = \overline{1, N-1}, \tag{9}$$

$$z_r(x,y) = \varphi(y) + \int\limits_0^x u_r(\xi,y)d\xi, \quad (x,y) \in \Omega_r, \quad r = \overline{1,N}. \tag{10}$$

where (9) is the condition for the continuity of functions in the internal partition lines. Problems (1)–(3) and (7)–(10) are equivalent. If $z(x,y)$ - solution of problem (1)–(3), then the system of its restrictions $z(x,[y]) = (z_1(x,y), z_2(x,y), ..., z_N(x,y))'$, $u(x,[y]) = (u_1(x,y), u_2(x,y), ..., u_N(x,y))'$, where $u_r(x,y) = \frac{\partial z_r(x,y)}{\partial x}, r = \overline{1,N}$ will be a solution to problem (7)–(10).

By $\lambda_r(x)$ we denote the function $u_r(x,y)$ for $y=(r-1)h$, i.e. $\lambda_r(x)=u_r(x,(r-1)h)$ and make a replacement $v_r(x,y)=u_r(x,y)-\lambda_r(x), r=\overline{1,N}$. We get an equivalent boundary value problem with unknown functions $\lambda_r(x)$:

$$\frac{\partial v_r}{\partial y}=A(x,y)v_r(x,y)+A(x,y)\lambda_r(x)+B(x,y)z_r(x,y)+f(x,y),(x,y)\in\Omega_r, r=\overline{1,N}, \tag{11}$$

$$v_r(x,(r-1)h)=0, \quad x\in[0,X], \quad r=\overline{1,N}, \tag{12}$$

$$\lambda_1(x)-\lambda_N(x)-\lim_{y\to Y-0}v_N(x,y)=0, \quad x\in[0,X], \tag{13}$$

$$\lambda_s(x)+\lim_{y\to sh-0}v_s(x,y)-\lambda_{s+1}(x)=0, \quad x\in[0,X], \quad s=\overline{1,N-1}. \tag{14}$$

$$z_r(x,y)=\varphi(y)+\int\limits_0^x v_r(\xi,y)d\xi+\int\limits_0^x\lambda_r(\xi)d\xi, \quad (x,y)\in\Omega_r, \quad r=\overline{1,N}. \tag{15}$$

Problems (7)–(10) and (11)–(15) are equivalent in the sense that if the system of pairs $\{u_r(x,y), z_r(x,y)\}, r=\overline{1,N}$, is a solution to the problem (7)–(10), then the system $\{\lambda_r(x)=u_r(x,(r-1)h), v_r(x,y)=u_r(x,y)-u_r(x,(r-1)h), z_r(x,y)\}$, $r=\overline{1,N}$, is a solution to the problem (11)–(15), and vice versa, if the pair $\{\lambda_r(x),v_r(x,y),z_r(x,y)\}, r=\overline{1,N}$, is a solution to problem (11)–(15) , then $\{\lambda_r(x)+v_r(x,y),z_r(x,y)\}, r=\overline{1,N}$, will be a solution to problem (7)–(10).

Problem (10), (11) at fixed $\lambda_r(x), v_r(x,y), z_r(x,y)$ is a family of Cauchy problems for ordinary differential equations, where $x\in[0,X]$, and is equivalent to the integral equation

$$v_r(x,y)=\int\limits_{(r-1)h}^{y}A(x,\eta)v_r(x,\eta)d\eta+\int\limits_{(r-1)h}^{y}A(x,\eta)d\eta\cdot\lambda_r(x)+\int\limits_{(r-1)h}^{y}[B(x,\eta)z_r(x,\eta)+f(x,\eta)]d\eta. \tag{16}$$

Passing to the limit at $y\to rh-0$ in (16) and substituting into (13), (14) instead of $\lim\limits_{t\to rh-0}v_r(x,y)$, $r=\overline{1,N}$, their corresponding right-hand sides for unknown functions $\lambda_r(x), r=\overline{1,N}$, we get a system of functional equations:

$$\lambda_1(x)-\lambda_N(x)-\int\limits_{(N-1)h}^{Y}A(x,\eta)v_N(x,\eta)d\eta+\int\limits_{(N-1)h}^{Y}A(x,\eta)d\eta\cdot\lambda_N(x)+$$

$$+\int\limits_{(N-1)h}^{Y}[B(x,\eta)z_N(x,\eta)+f(x,\eta)]d\eta=0,$$

$$\lambda_s(x)+\int\limits_{(s-1)h}^{sh}A(x,\eta)v_s(x,\eta)d\eta+\int\limits_{(s-1)h}^{sh}A(x,\eta)d\eta\cdot\lambda_s(x)+$$

$$+\int\limits_{(s-1)h}^{sh}[B(x,\eta)z_s(x,\eta)+f(x,\eta)]d\eta-\lambda_{s+1}(x)=0, \quad s=\overline{1,N-1}.$$

We write the resulting system of equations in the following form

$$Q(x,h)\lambda(x)=-G(x,h,v)-F(x,h,z), \tag{17}$$

where

$$Q(x,h)=$$

$$=\begin{pmatrix} I & 0 & \dots & 0 & -I-\int\limits_{(N-1)h}^{Nh} A(x,\eta)d\eta \\ I+\int\limits_0^h A(x,\eta)d\eta & -I & \dots & 0 & 0 \\ 0 & I+\int\limits_h^{2h} A(x,\eta)d\eta & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I+\int\limits_{(N-2)h}^{(N-1)h} A(x,\eta)d\eta & -I \end{pmatrix},$$

$$G(x,h,v)=\begin{pmatrix} -\int\limits_{(N-1)h}^{Nh} A(x,\eta)v_N(x,\eta)d\eta \\ \int\limits_0^h A(x,\eta)v_1(x,\eta)d\eta \\ \int\limits_h^{2h} A(x,\eta)v_2(x,\eta)d\eta \\ \dots \\ \int\limits_{(N-2)h}^{(N-1)h} A(x,\eta)v_{N-1}(x,\eta)d\eta \end{pmatrix},$$

$$F(x,h,z)=\begin{pmatrix} -\int\limits_{(N-1)h}^{Nh} [B(x,\eta)z_N(x,\eta)+f(x,\eta)]d\eta \\ \int\limits_0^h [B(x,\eta)z_1(x,\eta)+f(x,\eta)]d\eta \\ \int\limits_h^{2h} [B(x,\eta)z_2(x,\eta)+f(x,\eta)]d\eta \\ \dots \\ \int\limits_{(N-2)h}^{(N-1)h} [B(x,\eta)z_{N-1}(x,\eta)+f(x,\eta)]d\eta \end{pmatrix}.$$

I is an identity matrix of dimension n.

To find a solution to a system of three functions $\{\lambda_r(x), v_r(x,y), z_r(x,y)\}$, $r=\overline{1,N}$, we have a closed system consisting of equations (17), (16), (15).

Suppose that the matrix $Q(x,h)$ is invertible for all $x\in[0,X]$.

Taking $z_r^{(0)}(x,y)=\varphi(y)$, $r=\overline{1,N}$, as the initial approximation, we find the solution of the boundary value problem (11)–(15) as the limit triple sequences $\{\lambda_r^{(k)}(x), v_r^{(k)}(x,y), z_r^{(k)}(x,y)\}$, $k=1,2,\dots$, determined by the following algorithm:

A) Assuming that $z_r(x,y)=z_r^{(k-1)}(x,y)$, $r=\overline{1,N}$, we find $k-$th approximations $\lambda_r^{(k)}(x), v_r^{(k)}(x,y)$ $r=\overline{1,N}$, as the limit of sequences $\lambda_r^{(k,m)}(x), v_r^{(k,m)}(x,y)$ $r=\overline{1,N}$, $\quad m=0,1,2,\dots$, defined as follows:

$$\lambda_r^{(k,0)}(x)=\lambda_r^{(k-1)}(x), \quad v_r^{(k,0)}(x,y)=v_r^{(k-1)}(x,y),$$

$$\lambda^{(k,m+1)}(x)=-[Q(x,h)]^{-1,}\Big(G(x,h,v^{(k,m)})+F(x,h,z^{(k-1)})\Big),$$

$$v_r^{(k,m+1)}(x,y) = \int\limits_{(r-1)h}^{y} A(x,\eta)v_r^{(k,m)}(x,\eta)d\eta + \int\limits_{(r-1)h}^{y} A(x,\eta)d\eta \cdot \lambda_r^{(k,m+1)}(x)+$$

$$+\int\limits_{(r-1)h}^{y} [B(x,\eta)z_r^{(k-1)}(x,\eta) + f(x,\eta)]d\eta,$$

those. pair system sequence $\{\lambda_r^{(k,m+1)}(x), v_r^{(k,m+1)}(x,y)\}$, for $m\to\infty$ converges to $\{\lambda_r^{(k)}(x), v_r^{(k)}(x,y)\}$, $r=\overline{1,N}$,

B) The functions $z_r^{(k)}(x,y), r=\overline{1,N}$, are determined from the relations

$$z_r^{(k)}(x,y) = \varphi(y) + \int\limits_0^x v_r^{(k)}(\xi,y)d\xi + \int\limits_0^x \lambda_r^{(k)}(\xi)d\xi.$$

The conditions of the following statement ensure the feasibility and convergence of the proposed algorithm, as well as the unique solvability of problem (11)–(15).

Theorem 1. Let $(nN\times nN)$ matrix $Q(x,h)$ be invertible for all $x\in[0,X]$ and the inequalities

1) $\|[Q(x,h)]^{-1}\| \le \gamma(x,h)$; 2) $q(x,h) = h\alpha(x)\Big(1+\gamma(x,h)h\alpha(x)\Big) \le \mu < 1$,

where $\alpha(x) = \sup\limits_{y\in[0,Y]} \|A(x,y)\|$. Then there is a unique solution to problem (11)–(15) and fair assessments

$$a)\ \max_{r=\overline{1,N}} \|\lambda_r^*(x) - \lambda_r^{(1)}(x)\| + \max_{r=\overline{1,N}} \sup_{(x,y)\in\Omega_r} \|v_r^*(x,y) - v_r^{(1)}(x,y)\| \le$$

$$\le \theta(x,h)\beta(x)\exp\left(\int\limits_0^x \theta(\xi,h)\beta(\xi)d\xi\right)\int\limits_0^x \theta(\xi,h)d\xi \max\left\{\max_{y\in[0,Y]}\|\varphi(y)\|, \max_{(x,y)\in\Omega}\|f(x,y)\|\right\},$$

$$b)\ \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^*(x,y) - z_r^{(1)}(x,y)\| \le$$

$$\le \int\limits_0^x \max_{r=\overline{1,N}} \|\lambda^*(\xi) - \lambda^{(1)}(\xi)\|d\xi + \int\limits_0^x \max_{r=\overline{1,N}} \sup_{(x,y)\in\Omega_r} \|v_r^*(\xi,y) - v_r^{(1)}(\xi,y)\|d\xi,$$

where $\beta(x) = \sup\limits_{y\in[0,Y]} \|B(x,y)\|, \quad \theta(x,h) = \frac{1}{1-q(x,h)}h\Big(1+\gamma(x,h)+\alpha(x)\gamma(x,h)h\Big)$.

Proof. Under assumptions about the data of the problem, the inequalities take place

$$\|G(x,h,v)\| \le \alpha(x)h \max_{l=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r(x,y)\|,$$

$$\|F_\nu(x,h,\varphi)\| \le h\beta(x) \max_{l=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r(x,y)\| + h \max_{(x,y)\in\Omega} \|f(x,y)\|.$$

The following estimates follow from the algorithm:

$$\max_{r=\overline{1,N}} \|\lambda_r^{(1,1)}(x)\| \le \gamma(x,h)h(\beta(x)+1)\max\left\{\max_{y\in[0,Y]}\|\varphi(y)\|, \max_{(x,y)\in\Omega}\|f(x,y)\|\right\},$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,1)}(x,y)\| \le h\Big(1+\alpha(x)\gamma(x,h)h\Big)(\beta(x)+1)\max\left\{\max_{y\in[0,Y]}\|\varphi(y)\|, \max_{(x,y)\in\Omega}\|f(x,y)\|\right\}.$$

The following estimates follow from the algorithm:

$$\max_{r=\overline{1,N}} \|\lambda_r^{(1,2)}(x) - \lambda_r^{(1,1)}(x)\| \le \alpha(x)\gamma(x,h)h \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,1)}(x,y)\|,$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,2)}(x,y) - v_r^{(1,1)}(x,y)\| \le q(x,h) \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,1)}(x,y)\|.$$

Let's establish the inequality

$$\max_{r=\overline{1,N}} \|\lambda_r^{(1,m+2)}(x) - \lambda_r^{(1,m+1)}(x)\| \le \alpha(x)\gamma(x,h)h \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,m+1)}(x,y) - v_r^{(1,m)}(x,y)\|,$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,m+2)}(x,y) - v_r^{(1,m+1)}(x,y)\| \le$$

$$\le h\alpha(x)[1+\alpha(x)\gamma(x,h)h] \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,m+1)}(x,y) - v_r^{(1,m)}(x,y)\| \le$$

$$\le q(x,h) \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,m+1)}(x,y) - v_r^{(1,m)}(x,y)\|.$$

By virtue of the inequality $q(x,h) < 1$, the sequences $v_r^{(1,m+2)}(x,y)$ converge uniformly as $(x,y) \in \Omega_r$ to $v_r^{(1)}(x,y)$ and the convergence of the sequence of systems of functions $\lambda_r^{(1,m+2)}(x)$ to functions $\lambda_r^{(1)}(x)$ continuous on $x \in [0,X]$ for all $r = \overline{1,N}$.

$$\max_{r=\overline{1,N}} \|\lambda_r^{(1,m+2)}(x) - \lambda_r^{(1,1)}(x)\| \le$$

$$\max_{r=\overline{1,N}} \|\lambda_r^{(1,m+2)}(x) - \lambda_r^{(1,m+1)}(x)\| + ... + \max_{r=\overline{1,N}} \|\lambda_r^{(1,2)}(x) - \lambda_r^{(1,1)}(x)\| \le$$

$$\le \sum_{j=0}^{m} [q(x,h)]^j \alpha(x)\gamma(x,h)h \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,1)}(x,y)\|,$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,m+2)}(x,y) - v_r^{(1,1)}(x,y)\| \le$$

$$\le \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,m+2)}(x,y) - v_r^{(1,m+1)}(x,y)\| + ... + \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,2)}(x,y) - v_r^{(1,1)}(x,y)\| \le$$

$$\le \sum_{j=1}^{m+1} [q(x,h)]^j \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,1)}(x,y)\|.$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1,m+2)}(x,y)\| \le$$

$$\le \sum_{j=0}^{m+1} [q(x,h)]^j h\Big(1+\alpha(x)\gamma(x,h)h\Big)(\beta(x)+1) \max\left\{ \max_{y\in[0,Y]} \|\varphi(y)\|, \max_{(x,y)\in\Omega} \|f(x,y)\| \right\},$$

$$\max_{r=\overline{1,N}} \|\lambda_r^{(1,m+2)}(x)\| \le \sum_{j=0}^{m+1} [q(x,h)]^j \gamma(x,h)h(\beta(x)+1) \max\left\{ \max_{y\in[0,Y]} \|\varphi(y)\|, \max_{(x,y)\in\Omega} \|f(x,y)\| \right\}.$$

Passing to the limit as $m \to \infty$ we obtain the estimates:

$$\max_{r=\overline{1,N}} \|\lambda_r^{(1)}(x)\| \leq \frac{\gamma(x,h)h(\beta(x)+1)}{1-q(x,h)} \max\left\{ \max_{y\in[0,Y]} \|\varphi(y)\|, \max_{(x,y)\in\Omega} \|f(x,y)\| \right\},$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1)}(x,y)\| \leq$$

$$\leq \frac{1}{1-q(x,h)} h\Big(1+\alpha(x)\gamma(x,h)h\Big)(\beta(x)+1) \max\left\{ \max_{y\in[0,Y]} \|\varphi(y)\|, \max_{(x,y)\in\Omega} \|f(x,y)\| \right\},$$

$$\max_{r=\overline{1,N}} \|\lambda_r^{(1)}(x)\| + \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1)}(x,y)\| \leq \theta(x,h)(\beta(x)+1) \max\left\{ \max_{y\in[0,Y]} \|\varphi(y)\|, \max_{(x,y)\in\Omega} \|f(x,y)\| \right\},$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(1)}(x,y)-\varphi(y)\| \leq \int_0^x \max_{r=\overline{1,N}} \|\lambda_r^{(1)}(\xi)\| d\xi + \int_0^x \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1)}(\xi,y)\| d\xi \leq$$

$$\leq \int_0^x \theta(\xi,h)(\beta(\xi)+1) d\xi \max\left\{ \max_{y\in[0,Y]} \|\varphi(y)\|, \max_{(x,y)\in\Omega} \|f(x,y)\| \right\}.$$

The following estimates follow from the algorithm:

$$\max_{r=\overline{1,N}} \|\lambda_r^{(2,1)}(x)-\lambda_r^{(2,0)}(x)\| \leq \gamma(x,h)h\beta(x) \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(1)}(x,y)-\varphi(y)\|,$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2,1)}(x,y)-v_r^{(2,0)}(x,y)\| \leq h\beta(x)[1+\gamma(x,h)h\alpha(x)] \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(1)}(x,y)-\varphi(y)\|$$

The following estimates follow from the algorithm:

$$\max_{r=\overline{1,N}} \|\lambda_r^{(2,2)}(x)-\lambda_r^{(2,1)}(x)\| \leq \alpha(x)\gamma(x,h)h \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2,1)}(x,y)-v_r^{(2,0)}(x,y)\|,$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2,2)}(x,y)-v_r^{(2,1)}(x,y)\| \leq q(x,h) \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2,1)}(x,y)-v_r^{(2,0)}(x,y)\|$$

Let's establish the inequality

$$\max_{r=\overline{1,N}} \|\lambda_r^{(2,m+2)}(x)-\lambda_r^{(2,m+1)}(x)\| \leq \gamma(x,h)h\alpha(x) \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2,m+1)}(x,y)-v_r^{(2,m)}(x,y)\|,$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2,m+2)}(x,y)-v_r^{(2,m+1)}(x,y)\| \leq q(x,h) \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2,m+1)}(x,y)-v_r^{(2,m)}(x,y)\|.$$

By virtue of the inequality $q(x,h) < 1$, the sequences $v_r^{(2,m+1)}(x,y)$ converge uniformly as $(x,y) \in \Omega_r$ to $v_r^{(2)}(x,y)$ and the convergence of a sequence of systems of functions $\lambda_r^{(2,m+1)}(x)$ to functions $\lambda_r^{(2)}(x)$ continuous on $x \in [0,X]$ for all $r = \overline{1,N}$.

$$\max_{r=\overline{1,N}} \|\lambda_r^{(2,m+2)}(x) - \lambda_r^{(2,0)}(x)\| \leq$$

$$\leq \sum_{j=0}^{m} [q(x,h)]^j \alpha(x)\gamma(x,h)h \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2,1)}(x,y) - v_r^{(2,0)}(x,y)\| + \max_{r=\overline{1,N}} \|\lambda_r^{(2,1)}(x) - \lambda_r^{(2,0)}(x)\|,$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2,m+1)}(x,y) - v_r^{(2,0)}(x,y)\| \leq \sum_{j=0}^{m} [q(x,h)]^j \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2,1)}(x,y) - v_r^{(2,0)}(x,y)\|.$$

Passing to the limit as $m \to \infty$ we obtain the estimates:

$$\max_{r=\overline{1,N}} \|\lambda_r^{(2)}(x) - \lambda_r^{(1)}(x)\| \leq$$

$$\leq \frac{1}{1-q(x,h)} \alpha(x)\gamma(x,h)h \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2,1)}(x,y) - v_r^{(2,0)}(x,y)\| + \max_{r=\overline{1,N}} \|\lambda_r^{(2,1)}(x) - \lambda_r^{(2,0)}(x)\| \leq$$

$$\leq \frac{1}{1-q(x,h)} \alpha(x)\gamma(x,h)hh\beta(x)[1+\gamma(x,h)h\alpha(x)] \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(1)}(x,y) - \varphi(y)\| +$$

$$+\gamma(x,h)h\beta(x) \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(1)}(x,y) - \varphi(y)\| \leq$$

$$\leq \frac{1}{1-q(x,h)} \gamma(x,h)h\beta(x) \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(1)}(x,y) - \varphi(y)\|,$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2)}(x,y) - v_r^{(1)}(x,y)\| \leq \frac{1}{1-q(x,h)} \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2,1)}(x,y) - v_r^{(2,0)}(x,y)\| \leq$$

$$\leq \frac{1}{1-q(x,h)} h\beta(x)[1+\gamma(x,h)h\alpha(x)] \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(1)}(x,y) - \varphi(y)\|,$$

$$\max_{r=\overline{1,N}} \|\lambda_r^{(2)}(x) - \lambda_r^{(1)}(x)\| + \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(2)}(x,y) - v_r^{(1)}(x,y)\| \leq$$

$$\leq \theta(x,h)\beta(x) \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(1)}(x,y) - \varphi(y)\|,$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(2)}(x,y) - z_r^{(1)}(x,y)\| \leq \int_0^x \theta(\xi,h)\beta(\xi) \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(1)}(\xi,y) - \varphi(y)\| d\xi.$$

At the k -th step, we obtain the estimates:

$$\max_{r=\overline{1,N}} \|\lambda_r^{(k+1)}(x) - \lambda_r^{(k)}(x)\| \leq$$

$$\leq \frac{q(x,h)}{1-q(x,h)} \gamma(x,h)h\beta(x) \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(k)}(x,y) - z_r^{(k-1)}(x,y)\|,$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(k+1)}(x,y) - v_r^{(k)}(x,y)\| \leq$$

$$\leq \frac{1}{1-q(x,h)} h\beta(x)[1+\gamma(x,h)h\alpha(x)] \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(k)}(x,y) - z_r^{(k-1)}(x,y)\|,$$

$$\max_{r=\overline{1,N}} \|\lambda_r^{(k+1)}(x) - \lambda_r^{(k)}(x)\| + \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(k+1)}(x,y) - v_r^{(k)}(x,y)\| \leq$$

$$\leq \theta(x,h)\beta(x)\int_0^x \left(\max_{r=\overline{1,N}} \|\lambda_r^{(k)}(\xi) - \lambda_r^{(k-1)}(\xi)\| + \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(k)}(\xi,y) - v_r^{(k-1)}(\xi,y)\| \right) d\xi,$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(k+1)}(x,y) - z_r^{(k)}(x,y)\| \leq$$

$$\leq \int_0^x \left(\max_{r=\overline{1,N}} \|\lambda_r^{(k+1)}(\xi) - \lambda_r^{(k)}(\xi)\| + \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(k+1)}(\xi,y) - v_r^{(k)}(\xi,y)\| \right) d\xi.$$

Let's establish the inequalities

$$\max_{r=\overline{1,N}} \|\lambda_r^{(k+1)}(x) - \lambda_r^{(k)}(x)\| + \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(k+1)}(x,y) - v_r^{(k)}(x,y)\| \leq$$

$$\leq \frac{\theta(x,h)\beta(x)}{k!} \left(\int_0^x \theta(\xi,h)\beta(\xi)d\xi \right)^k \int_0^x \left(\max_{r=\overline{1,N}} \|\lambda_r^{(1)}(\xi)\| + \max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|v_r^{(1)}(\xi,y)\| \right) d\xi.$$

$$\max_{r=\overline{1,N}} \|\lambda^{(k+1)}(x) - \lambda^{(1)}(x)\| + \max_{r=\overline{1,N}} \sup_{(x,y)\in\Omega_r} \|v_r^{(k)}(x,y) - v_r^{(1)}(x,y)\| \leq$$

$$\leq \theta(x,h)\beta(x) \sum_{j=0}^{k-1} \frac{1}{j!} \left(\int_0^x \theta(\xi,h)\beta(\xi)d\xi \right)^j \int_0^x \theta(\xi,h)d\xi \max\left\{ \max_{y\in[0,Y]} \|\varphi(y)\|, \max_{(x,y)\in\Omega} \|f(x,y)\| \right\}.$$

$$\max_{r=\overline{1,N}} \sup_{y\in[0,Y]} \|z_r^{(k+1)}(x,y) - z_r^{(1)}(x,y)\| \leq$$

$$\leq \int_0^x \max_{r=\overline{1,N}} \|\lambda^{(k+1)}(\xi) - \lambda^{(1)}(\xi)\| d\xi + \int_0^x \max_{r=\overline{1,N}} \sup_{(x,y)\in\Omega_r} \|v_r^{(k+1)}(\xi,y) - v_r^{(1)}(\xi,y)\| d\xi.$$

Passing to the limit as $k \to \infty$, we obtain the estimates of Theorem 1. The uniqueness of the solution of this problem is proved by contradiction. Theorem 1 is proved.

The proof is complete.

If instead of $v_r(x,y)$ we substitute the corresponding right side of equality (16) and repeating this process $\nu(\nu = 1, 2, ...)$ times we get

$$v_r(x,y) = G_{\nu r}(x,y,v_r) + D_{\nu r}(x,y)\lambda_r(\xi)d\xi + F_{\nu r}(x,y,z_r), \tag{18}$$

where

$$G_{\nu r}(x,y,v_r) = \int_{(r-1)h}^{y} A(x,\eta_1) ... \int_{(r-1)h}^{\eta_{\nu-2}} A(x,\eta_{\nu-1}) \int_{(r-1)h}^{\eta_{\nu-1}} A(x,\eta_\nu) v_r(x,\eta_\nu) d\eta_\nu d\eta_{\nu-1} ... d\eta_1,$$

$$D_{\nu r}(x,y) = \sum_{j=0}^{\nu-1} \int_{(r-1)h}^{y} A(x,\eta_1) ... \int_{(r-1)h}^{\eta_j} A(x,\eta_{j+1}) d\eta_{j+1} ... d\eta_1,$$

$$F_{\nu r}(x,y,z_r) = \int\limits_{(r-1)h}^{y} [B(x,\eta_1)z_r(x,\eta_1) + f(x,\eta_1)]d\eta_1 +$$

$$+\sum_{j=1}^{\nu-1} \int\limits_{(r-1)h}^{y} A(x,\eta_1)... \int\limits_{(r-1)h}^{\eta_{j-1}} A(x,\eta_j) \int\limits_{(r-1)h}^{\eta_j} [B(x,\eta_{j+1})z_r(x,\eta_{j+1}) + f(x,\eta_{j+1}))]d\eta_{j+1}d\eta_j...d\eta_1.$$

Passing to the limit at $y \to rh-0$ in (18) and substituting in (13), (14) instead of $\lim\limits_{t\to rh-0} v_r(x,y)$, $r=\overline{1,N}$, the corresponding right-hand sides for the unknown functions $\lambda_r(x)$, $r=\overline{1,N}$, we obtain the system of functional equations:

$$Q_\nu(x,h)\lambda(x) = -G_\nu(x,h,v) - F(x,h,z), \tag{19}$$

where

$$Q_\nu(x,h) =$$

$$= \begin{pmatrix} I & 0 & \dots & 0 & -I - D_{\nu N}(x,Nh) \\ I + D_{\nu 1}(x,h) & -I & \dots & 0 & 0 \\ 0 & I + D_{\nu 2}(x,2h) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I + D_{\nu(N-1)}(x,(N-1)h) & -I \end{pmatrix},$$

$$G_\nu(x,h,v) = \begin{pmatrix} -G_{\nu N}(x,Nh,v_N) \\ G_{\nu 1}(x,h,v_1) \\ G_{\nu 2}(x,2h,v_2) \\ ... \\ G_{\nu(N-1)}(x,(N-1)h,v_{N-1}) \end{pmatrix}, \quad F_\nu(x,h,z) = \begin{pmatrix} -F_{\nu N}(x,Nh,z_N) \\ F_{\nu 1}(x,h,z_1) \\ F_{\nu 2}(x,2h,z_2) \\ ... \\ F_{\nu(N-1)}(x,(N-1)h,z_{N-1}) \end{pmatrix}.$$

I is an identity matrix of dimension n.

To find a solution to a system of three functions $\{\lambda_r(x), v_r(x,y), z_r(x,y)\}$, $r=\overline{1,N}$, we have a closed system consisting of from equations (19), (18), (15).

Suppose that the matrix $Q_\nu(x,h)$ is invertible for all $x \in [0,X]$.

Taking $z_r^{(0)}(x,y) = \varphi(y)$, $r=\overline{1,N}$, as an initial approximation we find the solution of the boundary value problem (11)–(15) as the limit of the sequence of the system of triplets $\{\lambda_r^{(k)}(x), v_r^{(k)}(x,y), z_r^{(k)}(x,y)\}$, $k=1,2,\dots$, determined by the following algorithm: A) Assuming that $z_r(x,y) = z_r^{(k-1)}(x,y)$, $r=\overline{1,N}$, we find $k-$th approximations $\lambda_r^{(k)}(x), v_r^{(k)}(x,y)$ $r=\overline{1,N}$, as the limit of sequences $\lambda_r^{(k,m)}(x), v_r^{(k,m)}(x,y)$ $r=\overline{1,N}$, $\quad m=0,1,2,\dots$, defined as follows:

$$\lambda_r^{(k,0)}(x) = \lambda_r^{(k-1)}(x), \quad v_r^{(k,0)}(x,y) = v_r^{(k-1)}(x,y),$$

$$\lambda^{(k,m+1)}(x) = -[Q_\nu(x,h)]^{-1}\Big(G_\nu(x,h,v^{(k,m)}) + F_\nu(x,h,z^{(k-1)})\Big),$$

$$v_r^{(k,m+1)}(x,y) = G_{\nu r}(x,y,v_r^{(k,m)}) + D_{\nu r}(x,y)\lambda_r^{(k,m+1)}(\xi)d\xi + F_{\nu r}(x,y,z_r^{(k-1)}),$$

those pair system sequence $\{\lambda_r^{(k,m+1)}(x), v_r^{(k,m+1)}(x,y)\}$, for $m\to\infty$ converges to $\{\lambda_r^{(k)}(x), v_r^{(k)}(x,y)\}$, $r=\overline{1,N}$.

B) Functions $z_r^{(k)}(x,y), r=\overline{1,N}$, are determined from the relations

$$z_r^{(k)}(x,y) = \varphi(y) + \int\limits_0^x v_r^{(k)}(\xi,y)d\xi + \int\limits_0^x \lambda_r^{(k)}(\xi)d\xi.$$

The conditions of the following statement ensure the feasibility and convergence of the proposed algorithm, as well as the unique solvability of problem (11)–(15).

Theorem 2. Let for some choice of step $h > 0 : Nh = Y$, $N = 1, 2, ...$, and the number of substitutions ν, $\nu \in \mathbb{N}$, matrix $Q_\nu(x,h)$ of dimension $(nN \times nN)$ is invertible for all $x \in [0, X]$ and the inequalities hold:

1) $\|[Q_\nu(x,h)]^{-1}\| \leq \gamma_\nu(x,h)$;

2) $q_\nu(x,h) = \frac{(h\alpha(x))^\nu}{\nu!}\left(1 + \gamma_\nu(x,h)\sum\limits_{j=1}^{\nu}\frac{(h\alpha(x))^j}{j!}\right) \leq \mu < 1$, where $\alpha(x) = \sup\limits_{y\in[0,Y]}\|A(x,y)\|$.

Then there is a unique solution to problem (11)–(15) and fair assessments

$$a)\ \max_{r=\overline{1,N}}\|\lambda^*(x) - \lambda^{(1)}(x)\| + \max_{r=\overline{1,N}}\sup_{(x,y)\in\Omega_r}\|v_r^*(x,y) - v_r^{(1)}(x,y)\| \leq$$

$$\leq \theta_\nu(x,h)\beta(x)\exp\left(\int_0^x \theta_\nu(\xi,h)\beta(\xi)d\xi\right)\int_0^x \theta_\nu(\xi,h)d\xi \max\left\{\max_{y\in[0,Y]}\|\varphi(y)\|, \max_{(x,y)\in\Omega}\|f(x,y)\|\right\},$$

$$b)\ \max_{r=\overline{1,N}}\sup_{y\in[0,Y]}\|z_r^*(x,y) - z_r^{(1)}(x,y)\| \leq$$

$$\leq \int_0^x \max_{r=\overline{1,N}}\|\lambda^*(\xi) - \lambda^{(1)}(\xi)\|d\xi + \int_0^x \max_{r=\overline{1,N}}\sup_{(x,y)\in\Omega_r}\|v_r^*(\xi,y) - v_r^{(1)}(\xi,y)\|d\xi,$$

where $\beta(x) = \sup\limits_{y\in[0,Y]}\|B(x,y)\|$, $\quad \theta_\nu(x,h) = \frac{h}{1-q_\nu(x,h)}\left(1 + \gamma_\nu(x,h)\sum\limits_{j=0}^{\nu}\frac{(h\alpha(x))^j}{j!}\right)$.

The proof of Theorem 2 is similar to the proof of Theorem 1.

By virtue of the equivalence of problems (1)–(3) and (11)–(15), Theorem 1 implies

Theorem 3. Let the conditions of Theorem 1 be satisfied. Then problem (1)–(3) has a unique solution $z^*(x,t)$ and the estimate

$$\max\left\{\max_{(x,y)\in\Omega}\|z^*(x,y)\|, \max_{(x,y)\in\Omega}\left\|\frac{\partial z^*(x,y)}{\partial x}\right\|\right\} \leq$$

$$\leq \max\{1 + XM(h)(\beta+1), M(h)(\beta+1)\}\max\left\{\max_{y\in[0,Y]}\|\varphi(y)\|, \max_{(x,y)\in\Omega}\|f(x,y)\|\right\},$$

where

$$M(h) = \frac{1 - q(h) + [q(h)]^2}{1 - q(h)}\gamma_1(h)hXe^{\gamma_1(h)h\beta X} + \frac{Xh}{1-q(h)}\left(1 + \gamma_1(h)h(\alpha + \beta X)e^{\gamma_1(h)h\beta X}\right).$$

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Гиперболалық теңдеулер үшін периодтық шеттік есептің бір шешімі туралы

Тікбұрышты облыста гиперболалық типті дербес туындылы дифференциалдық теңдеулер жүйесі үшін бір айнымалыдан тәуелді периодты шекаралық есеп қарастырылды. Авторлар жаңа белгісіз функцияны енгізе отырып, бұл есепті интегралдық шарты бар қарапайым дифференциалдық теңдеу үшін эквивалентті шекаралық есепке келтіреді. Параметрлеу әдісі негізінде эквивалентті есептің жуық шешімін табудың жаңа тәсілдері ұсынылып, оның жинақтылығы дәлелденеді. Бұл екінші ретті гиперболалық теңдеулер жүйесі үшін жартылай периодтық шекаралық есептің бірегей шешімі бар жағдайларды анықтауға мүмкіндік берді.

Кілт сөздер: шеттік есеп, гиперболалық теңдеулер, алгоритм, параметрлеу әдісі, жуық шешім.

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Об одном решении периодической краевой задачи для гиперболического уравнения

В прямоугольной области рассмотрена периодическая по одной переменной краевая задача для системы дифференциальных уравнений в частных производных гиперболического типа. Авторы вводя новую неизвестную функцию, данную задачу сводят к эквивалентной краевой задаче для обыкновенного дифференциального уравнения с интегральным условием. На основе метода параметризации предложены новые подходы нахождения приближенного решения эквивалентной задачи и доказана его сходимость. Это позволило установить условия существования единственного решения полупериодической краевой задачи для системы гиперболических уравнений второго порядка.

Ключевые слова: краевая задача, гиперболические уравнения, алгоритм, метод параметризации, приближенное решение.

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Extensions of some differential inequalities for fractional integro-differential equations via upper and lower solutions

This paper deals with some differential inequalities for generalized fractional integro-differential equations by using the technique of upper and lower solutions. The fractional differential operator is taken in Caputo's sense and the nonlinear term divided into two parts depends on the fractional integrals of an unknown function with two different fractional orders. The results are studied by employing a variety of coupled upper and lower solutions. These theorems have some potential for extending the iterative techniques to fractional order integro-differential equations and to coupled systems of integro-differential fractional equations to obtain the existence of solutions as well as approximate solutions for the considered problem.

Keywords: fractional differential equations, differential inequalities, upper and lower solutions, boundary value problem.

Introduction

Although fractional calculus has existed for as long as «conventional» calculus, it was not until recent decades that the study of fractional differential equations became popular. This is because fractional operators commonly offer better accurate models than those with integer derivatives. See [1,2] for the recent developments and further information. Among the different definitions for fractional order derivatives, the Caputo fractional derivative stands out and has been intensely utilized since it is best suited for describing many events and the initial conditions for fractional differential equations are the same form as that of ordinary differential equations with integer derivatives. Due to the fact that it is far more extensive than the theory of classical ordinary differential equations, the theory of fractional differential equations has drawn a lot of attention. Although there has been tremendous recent progress in the study of fractional differential equations, there is still a significant potential in this area. After reviewing the literature, we find a number of publications on basic arguments, such as existence, uniqueness and stability results for fractional differential equations. See [3–10] and the references therein.

Differential and integral inequalities are crucial in the qualitative study of differential and integral equations. They are used to investigate the concepts of existence, uniqueness, boundedness, stability, continuous dependence, and so on. The method of upper and lower solutions is a quite effective concept in the theory of nonlinear differential equations with initial or boundary conditions. Recently, these methods have been applied to fractional differential equations as well as differential inequalities [11–20]. We give some comparison results for several types of coupled upper and lower solutions for a given boundary value problems of fractional integro-differential equations. The results here can be viewed as expansions and generalizations of corresponding analogous results from the integer order case to the fractional order case.

The purpose of this paper is to refine some previously published results for a given boundary value problems of fractional integro-differential equations by employing the method of upper and lower solutions together with strict and non-strict inequalities.

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1 Mathematical Preliminaries

This section provides background knowledge on fractional calculus and fractional differential equations in order to improve understanding.

Definition 1. [1] Let $[a,b] \subset \mathbb{R}$, $Re(q) > 0$ and $f \in L_1[a,b]$. Then the left and right Riemann-Liouville fractional integrals I_{a+}^{q} and I_{b-}^{q} of order α are defined as

$$I_{a+}^{q} f(x) := \frac{1}{\Gamma(q)} \int_{a}^{x} \frac{f(t)\,dt}{(x-t)^{1-q}}, \; x \in (a,b]$$

and

$$I_{b-}^{q} f(x) := \frac{1}{\Gamma(q)} \int_{x}^{b} \frac{f(t)\,dt}{(t-x)^{1-q}}, x \in [a,b)$$

respectively.

Definition 2. [1] Let $[a,b] \subset \mathbb{R}$, $Re(q) \in (0,1)$ and $f \in L_1[a,b]$. The left and right Caputo fractional derivatives of order q are given by

$${}^{c}D_{a+}^{q} f(x) := I_{a+}^{1-q} Df(x), \forall x \in (a,b]$$

and

$${}^{c}D_{b-}^{q} f(x) := -I_{b-}^{1-q} Df(x), \forall x \in [a,b)$$

respectively.

Let $F, G \in C[J \times \mathbb{R} \times \mathbb{R}_{+}, \mathbb{R}]$, $u \in C^{1}[J, \mathbb{R}]$, $J = [0,T]$. We consider the following fractional boundary value problem.

$${}^{C}D^{q_1} u(t) = F(t, u(t), I^{q_2} u(t)) + G(t, u(t), I^{q_3} u(t)), \quad g(u(0), u(T)) = 0, \tag{1}$$

where $0 < q_3 \leq q_2 \leq q_1 < 1$ and $g \in C[\mathbb{R}^2, \mathbb{R}]$. From now on, the fractional operator ${}^{C}D^{q}$ stands for the left Caputo fractional derivative as well as I^{q} represents the left Riemann Liouville fractional integral operator.

Definition 3. Let $\alpha, \beta \in C^{1}[J, \mathbb{R}]$. Then α and β are said to be
(i) natural lower and upper solutions of (1) respectively if

$${}^{C}D^{q_1}\alpha(t) \leq F(t, \alpha(t), I^{q_2}\alpha(t)) + G(t, \alpha(t), I^{q_3}\alpha(t)), \quad g(\alpha(0), \alpha(T)) \leq 0, \tag{2}$$
$${}^{C}D^{q_1}\beta(t) \geq F(t, \beta(t), I^{q_2}\beta(t)) + G(t, \beta(t), I^{q_3}\beta(t)), \; g(\beta(0), \beta(T)) \geq 0, \tag{3}$$

(ii) coupled lower and upper solutions of type I of (1) respectively if

$${}^{C}D^{q_1}\alpha(t) \leq F(t, \alpha(t), I^{q_2}\beta(t)) + G(t, \alpha(t), I^{q_3}\beta(t)), \quad g(\alpha(0), \alpha(T)) \leq 0, \tag{4}$$
$${}^{C}D^{q_1}\beta(t) \geq F(t, \beta(t), I^{q_2}\alpha(t)) + G(t, \beta(t), I^{q_3}\alpha(t)), \; g(\beta(0), \beta(T)) \geq 0, \tag{5}$$

(iii) coupled lower and upper solutions of type II of (1) respectively if

$${}^{C}D^{q_1}\alpha(t) \leq F(t, \beta(t), I^{q_2}\alpha(t)) + G(t, \beta(t), I^{q_3}\alpha(t)), \quad g(\alpha(0), \alpha(T)) \leq 0,$$
$${}^{C}D^{q_1}\beta(t) \geq F(t, \alpha(t), I^{q_2}\beta(t)) + G(t, \alpha(t), I^{q_3}\beta(t)), \; g(\beta(0), \beta(T)) \geq 0,$$

(iv) coupled lower and upper solutions of type III of (1) respectively if

$$\begin{aligned} {}^{C}D^{q_1}\alpha(t) &\leq F(t,\beta(t),I^{q_2}\beta(t)) + G(t,\beta(t),I^{q_3}\beta(t)), \quad g(\alpha(0),\alpha(T)) \leq 0, \\ {}^{C}D^{q_1}\beta(t) &\geq F(t,\alpha(t),I^{q_2}\alpha(t)) + G(t,\alpha(t),I^{q_3}\alpha(t)), \quad g(\beta(0),\beta(T)) \geq 0, \end{aligned}$$

(v) coupled lower and upper solutions of type IV of (1) respectively if

$$\begin{aligned} {}^{C}D^{q_1}\alpha(t) &\leq F(t,\alpha(t),I^{q_2}\alpha(t)) + G(t,\beta(t),I^{q_3}\beta(t)), \quad g(\alpha(0),\alpha(T)) \leq 0, \\ {}^{C}D^{q_1}\beta(t) &\geq F(t,\beta(t),I^{q_2}\beta(t)) + G(t,\alpha(t),I^{q_3}\alpha(t)), \quad g(\beta(0),\beta(T)) \geq 0, \end{aligned}$$

(vi) coupled lower and upper solutions of type V of (1) respectively if

$$\begin{aligned} {}^{C}D^{q_1}\alpha(t) &\leq F(t,\beta(t),I^{q_2}\beta(t)) + G(t,\alpha(t),I^{q_3}\alpha(t)), \quad g(\alpha(0),\alpha(T)) \leq 0, \\ {}^{C}D^{q_1}\beta(t) &\geq F(t,\alpha(t),I^{q_2}\alpha(t)) + G(t,\beta(t),I^{q_3}\beta(t)), \quad g(\beta(0),\beta(T)) \geq 0. \end{aligned}$$

Lemma 1. [3] Let $m \in C^1[J,\mathbb{R}]$ and assume that $m(t_1) = 0$ for $t_1 \in (0,T]$ and $m(t) \leq 0$ for $0 \leq t \leq t_1$. Then we have ${}^{C}D^{q}m(t_1) \geq 0$.

The Laplace transform technique, as is well known, is a beneficial tool for solving initial value problems. Using this method, the stated problem is turned to an algebraic expression. The next lemma, which is about the inverse Laplace transform of the given function, is critical in this case.

Lemma 2. [21] Let $\alpha \geq \beta > 0$, $\alpha > \gamma$, $a,b \in \mathbb{R}$, $s^{\alpha-\beta} > |a|$ and $\left|s^{\alpha} + as^{\beta}\right| > |b|$. Then we get

$$\mathcal{L}^{-1}\left\{\frac{s^{\gamma}}{(s^{\alpha} + as^{\beta} + b)}\right\} = t^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^n(-a)^k\binom{n+k}{k}t^{k(\alpha-\beta)+n\alpha}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma).}$$

We prove the following lemma in order to solve the given linear fractional initial value problem. It allows the corresponding result in [16] to be a specific case of this lemma.

Lemma 3. Assume that $\lambda \in C^1[J,\mathbb{R}]$, $0 < q_3 \leq q_2 \leq q_1 < 1$ and $L_1, M_1, M_2 \in \mathbb{R}$. The explicit solution of the following linear fractional integro-differential equation,

$$ {}^{C}D^{q_1}\lambda(t) = L_1\lambda(t) + M_1 I^{q_2}\lambda(t) + M_2 I^{q_3}\lambda(t), \quad \lambda(0) = \lambda_0 \tag{6}$$

is given by

$$\lambda(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(M_1)^n(L_1)^k(M_2)^i\binom{n+k}{k}\binom{n+k+i}{i}t^{q_1(n+k+i)+nq_2+iq_3}}{\Gamma(q_1(n+k+i)+nq_2+iq_3+1)}\lambda_0$$

provided that $|s^{q_1+q_3}| > |M_2|$, $|s^{q_1} - M_2 s^{-q_3}| > |L_1|$ and $|s^{q_1+q_2} - M_2 s^{q_2-q_3} - L_1 s^{q_2}| > |M_1|$.

Proof. If we apply the Laplace transform on both side of the equation (6), we find the following relations

$$\begin{aligned}
\mathcal{L}\left\{{}^{C}D^{q_1}\lambda(t)\right\} &= L_1\mathcal{L}\{\lambda(t)\}+M_1\mathcal{L}\{I^{q_2}\lambda(t)\}+M_2\mathcal{L}\{I^{q_3}\lambda(t)\} \\
\frac{s\lambda(s)-\lambda_0}{s^{1-q_1}} &= L_1\lambda(s)+M_1\frac{\lambda(s)}{s^{q_2}}+M_2\frac{\lambda(s)}{s^{q_3}} \\
s^{q_1}\lambda(s)-s^{q_1-1}\lambda_0 &= L_1\lambda(s)+M_1\lambda(s)s^{-q_2}+M_2\lambda(s)s^{-q_3} \\
\lambda(s) &= \frac{s^{q_1+q_2-1}}{s^{q_1+q_2}-M_2s^{q_2-q_3}-L_1s^{q_2}-M_1}\lambda_0 \\
&= \frac{s^{q_1+q_2-1}}{\left(s^{q_1+q_2}-M_2s^{q_2-q_3}-L_1s^{q_2}\right)\left(1-\frac{M_1}{s^{q_1+q_2}-M_2s^{q_2-q_3}-L_1s^{q_2}}\right)}\lambda_0 \\
&= \frac{s^{q_1+q_2-1}}{s^{q_1+q_2}-M_2s^{q_2-q_3}-L_1s^{q_2}}\sum_{n=0}^{\infty}\frac{(M_1)^n}{\left(s^{q_1+q_2}-M_2s^{q_2-q_3}-L_1s^{q_2}\right)^n}\lambda_0 \\
&= s^{q_1+q_2-1}\sum_{n=0}^{\infty}\frac{(M_1)^n}{\left(s^{q_1+q_2}-M_2s^{q_2-q_3}-L_1s^{q_2}\right)^{n+1}}\lambda_0 \\
&= s^{q_1+q_2-1}\sum_{n=0}^{\infty}\frac{(M_1)^n}{\left(s^{q_1+q_2}-M_2s^{q_2-q_3}\right)^{n+1}\left(1-\frac{L_1s^{q_2}}{s^{q_1+q_2}-M_2s^{q_2-q_3}}\right)^{n+1}}\lambda_0 \\
&= s^{q_1+q_2-1}\sum_{n=0}^{\infty}\frac{(M_1)^n}{\left(s^{q_1+q_2}-M_2s^{q_2-q_3}\right)^{n+1}}\sum_{k=0}^{\infty}\frac{(L_1)^k\left(s^{q_2}\right)^k\binom{n+k}{k}}{\left(s^{q_1+q_2}-M_2s^{q_2-q_3}\right)^k}\lambda_0 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(M_1)^n(L_1)^k\binom{n+k}{k}}{s^{-q_1-q_2(k+1)+1}\left(s^{q_1+q_2}-M_2s^{q_2-q_3}\right)^{n+k+1}}\lambda_0 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(M_1)^n(L_1)^k\binom{n+k}{k}}{s^{-q_1-q_2(k+1)+1}\left(s^{q_1+q_2}\right)^{n+k+1}\left(1-\frac{M_2s^{q_2-q_3}}{s^{q_1+q_2}}\right)^{n+k+1}}\lambda_0 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(M_1)^n(L_1)^k\binom{n+k}{k}}{s^{-q_1-q_2(k+1)+1}\left(s^{q_1+q_2}\right)^{n+k+1}}\sum_{i=0}^{\infty}(M_2)^i\binom{n+k+i}{i}s^{(-q_3-q_1)i}\lambda_0 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(M_1)^n(L_1)^k(M_2)^i\binom{n+k}{k}\binom{n+k+i}{i}}{s^{-q_1-q_2(k+1)+1+(q_1+q_2)(n+k+1)+i(q_1+q_3)}}\lambda_0 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(M_1)^n(L_1)^k(M_2)^i\binom{n+k}{k}\binom{n+k+i}{i}}{s^{q_1(n+k+i)+nq_2+iq_3}}\lambda_0
\end{aligned}$$

provided that $|s^{q_1+q_3}|>|M_2|$, $|s^{q_1}-M_2s^{-q_3}|>|L_1|$ and $|s^{q_1+q_2}-M_2s^{q_2-q_3}-L_1s^{q_2}|>|M_1|$. At this stage, we arrive at by implementing the inverse Laplace transform

$$\begin{aligned}
\mathcal{L}^{-1}\{\lambda(s)\} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}(M_1)^n(L_1)^k(M_2)^i\binom{n+k}{k}\binom{n+k+i}{i}\mathcal{L}^{-1}\left\{\frac{1}{s^{q_1(n+k+i)+nq_2+iq_3}}\right\}\lambda_0 \\
\lambda(t) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(M_1)^n(L_1)^k(M_2)^i\binom{n+k}{k}\binom{n+k+i}{i}t^{q_1(n+k+i)+nq_2+iq_3}}{\Gamma\left(q_1(n+k+i)+nq_2+iq_3+1\right)}\lambda_0.
\end{aligned}$$

2 Formulas and theorems

Depending on the selection of upper and lower solutions of (1), we will assume the suitable conditions to establish some differential inequalities.

Theorem 1. Let α and β be natural lower and upper solutions of (1). $F(t,u,v)$ and $G(t,u,v)$ is non-decreasing in v and following Lipschitz-like conditions are also satisfied for $L_1, L_2, M_1, M_2 > 0$

$$F(t,u_1(t),v_1(t)) - F(t,u_2(t),v_2(t)) \leq L_1(u_1-u_2) + M_1(v_1-v_2), \quad (7)$$
$$G(t,u_1(t),\bar{v}_1(t)) - G(t,u_2(t),\bar{v}_2(t)) \leq L_2(u_1-u_2) + M_2(\bar{v}_1-\bar{v}_2) \quad (8)$$

whenever $u_1 \geq u_2$. Then we have $\alpha(t) \leq \beta(t)$ provided $\alpha(0) \leq \beta(0)$.

Proof. In order to make it compatible with the problem (1), the functions $v_i, \bar{v}_i$ must be evaluated as follows $v_i = I^{q_2}u_i$ and $\bar{v}_i = I^{q_3}u_i$, $i = 1,2$. Clearly, $u_1 \leq u_2$ implies that $v_1 \leq v_2$ and $\bar{v}_1 \leq \bar{v}_2$.

We now set $\alpha_\epsilon(t) = \alpha(t) - \epsilon\lambda(t)$ for arbitrary small number $\epsilon > 0$, where

$$\lambda(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \frac{(L)^k (M_1)^n (N_1)^i \binom{n+k}{k}\binom{n+k+i}{i} t^{q_1(n+k+i)+nq_2+iq_3}}{\Gamma(q_1(n+k+i)+nq_2+iq_3+1)}$$

is unique positive solution of the equation

$${}^C D^{q_1}\lambda(t) = L\lambda(t) + M_1 I^{q_2}\lambda(t) + M_2 I^{q_3}\lambda(t), \quad \lambda(0) = 1, \quad (9)$$

where L is a positive number such that $L > L_1 + L_2$. Notice that $\alpha_\epsilon(0) = \alpha(0) - \epsilon\lambda(0) < \alpha(0)$, $\alpha_\epsilon(t) < \alpha(t)$ for $0 \leq t \leq T$. If we differentiate $\alpha_\epsilon(t)$ in terms of Caputo's sense, and using (2) we get

$$\begin{aligned} {}^C D^{q_1}\alpha_\epsilon(t) &= {}^C D^{q_1}\alpha(t) - \epsilon {}^C D^{q_1}\lambda(t) \\ &\leq F(t,\alpha(t), I^{q_2}\alpha(t)) + G(t,\alpha(t), I^{q_3}\alpha(t)) \\ &\quad - L\epsilon\lambda(t) - M_1\epsilon I^{q_2}\lambda(t) - M_2\epsilon I^{q_3}\lambda(t). \end{aligned}$$

We observe that $\alpha_\epsilon(t) < \alpha(t)$ on J yields $I^{q_2}\alpha_\epsilon(t) < I^{q_2}\alpha(t)$ and $I^{q_3}\alpha_\epsilon(t) < I^{q_3}\alpha(t)$ on J by the definition of R-L fractional integral. We then employ the Lipschitz-like inequalities in (7) and (8) to obtain

$$\begin{aligned} {}^C D^{q_1}\alpha_\epsilon(t) &\leq F(t,\alpha(t), I^{q_2}\alpha(t)) - F(t,\alpha_\epsilon(t), I^{q_2}\alpha_\epsilon(t)) + G(t,\alpha(t), I^{q_3}\alpha(t)) \\ &\quad - G(t,\alpha_\epsilon(t), I^{q_3}\alpha_\epsilon(t)) - L\epsilon\lambda(t) - M_1\epsilon I^{q_2}\lambda(t) - M_2\epsilon I^{q_3}\lambda(t) \\ &\quad + F(t,\alpha_\epsilon(t), I^{q_2}\alpha_\epsilon(t)) + G(t,\alpha_\epsilon(t), I^{q_3}\alpha_\epsilon(t)) \\ &\leq L_1\epsilon(\alpha(t)-\alpha_\epsilon(t)) + M_1\epsilon I^{q_2}(\alpha(t)-\alpha_\epsilon(t)) + L_2\epsilon(\alpha(t)-\alpha_\epsilon(t)) \\ &\quad + M_2\epsilon I^{q_3}(\alpha(t)-\alpha_\epsilon(t)) - L\epsilon\lambda(t) - M_1\epsilon I^{q_2}\lambda(t) - M_2\epsilon I^{q_3}\lambda(t) \\ &\quad + F(t,\alpha_\epsilon(t), I^{q_2}\alpha_\epsilon(t)) + G(t,\alpha_\epsilon(t), I^{q_3}\alpha_\epsilon(t)) \\ &= F(t,\alpha_\epsilon(t), I^{q_2}\alpha_\epsilon(t)) + G(t,\alpha_\epsilon(t), I^{q_3}\alpha_\epsilon(t)) + \epsilon\lambda(t)(L_1+L_2-L) \\ &< F(t,\alpha_\epsilon(t), I^{q_2}\alpha_\epsilon(t)) + G(t,\alpha_\epsilon(t), I^{q_3}\alpha_\epsilon(t)). \end{aligned}$$

We intend to demonstrate $\alpha_\epsilon(t) < \beta(t)$ for $t \in [0,T]$, which concludes the proof by letting $\epsilon \longrightarrow 0$. Suppose that $\alpha_\epsilon(t) < \beta(t)$ on $t \in [0,T]$ is false. Then the set $A = \{t : t \in [0,T], \alpha_\epsilon(t) \geq \beta(t)\}$ is nonempty. Let t_* be the greatest lower bound of A, then $\alpha_\epsilon(t_*) = \beta(t_*)$ and $\alpha_\epsilon(t) < \beta(t)$ for $0 \leq t < t_*$.

By generating $m(t) = \alpha_\epsilon(t) - \beta(t)$, it is written that $m(t) \leq 0$ for $0 \leq t < t_*$ and $m(t_*) = 0$. Because of Lemma 1, it leads to ${}^C D^{q_1}m(t_*) \geq 0$.

Since $\alpha_\epsilon(s) \leq \beta(s)$ for $0 \leq s \leq t_*$, we immediately get

$$\begin{aligned} I^{q_2}\alpha_\epsilon(t_*) &= \frac{1}{\Gamma(q_2)}\int_0^{t_*}(t_*-s)^{1-q_2}\alpha_\epsilon(s)\,ds \\ &\leq \frac{1}{\Gamma(q_2)}\int_0^{t_*}(t_*-s)^{1-q_2}\beta(s)\,ds \\ &= I^{q_2}\beta(t_*). \end{aligned}$$

A similar discussion offers $I^{q_3}\alpha_\epsilon(t_*) \leq I^{q_3}\beta(t_*)$. By recalling the non-decreasing features of F and G, we follow that

$$\begin{aligned} F(t_*,\alpha_\epsilon(t_*),I^{q_2}\alpha_\epsilon(t_*))+G(t_*,\alpha_\epsilon(t_*),I^{q_3}\alpha_\epsilon(t_*)) &> {}^{C}D^{q_1}\alpha_\epsilon(t_*) \\ &\geq {}^{C}D^{q_1}\beta(t_*) \\ &\geq F(t_*,\beta(t_*),I^{q_2}\beta(t_*))+G(t_*,\beta(t_*),I^{q_3}\beta(t_*)) \\ &\geq F(t_*,\beta(t_*),I^{q_2}\alpha_\epsilon(t_*))+G(t_*,\beta(t_*),I^{q_3}\alpha_\epsilon(t_*)) \end{aligned}$$

giving rise to a contradiction because of the fact that $\alpha_\epsilon(t_*) = \beta(t_*)$. Then the inequality

$$\alpha_\epsilon(t) < \beta(t), \forall t \in J$$

holds, which proves $\alpha(t) \leq \beta(t)$ on J.

Corollary 1. This result includes the Theorem 2 in [11] as a special case when $F \equiv 0$ and $q_1 = q_2$ or $G \equiv 0$ and $q_1 = q_3$.

Theorem 2. Let α and β be coupled lower and upper solutions of type I of (1). $F(t,u,v)$ and $G(t,u,v)$ is both non-increasing in v and they hold the following inequalities for $u_1 \geq u_2, v_1 \geq v_2$ and L_1, L_2, M_1, M_2 positive constants such that

$$F(t,u_1(t),v(t)) - F(t,u_2(t),v(t)) \leq L_1(u_1-u_2), \tag{10}$$

$$G(t,u_1(t),\bar{v}(t)) - G(t,u_2(t),\bar{v}(t)) \leq L_2(u_1-u_2), \tag{11}$$

$$F(t,u(t),v_1(t)) - F(t,u(t),v_2(t)) \geq -M_1(v_1-v_2), \tag{12}$$

$$G(t,u(t),\bar{v}_1(t)) - G(t,u(t),\bar{v}_2(t)) \geq -M_2(\bar{v}_1-\bar{v}_2). \tag{13}$$

If $\alpha(0) \leq \beta(0)$, then it yields that $\alpha(t) \leq \beta(t)$ on J.

Proof. We begin by constructing $\alpha_\epsilon(t) = \alpha(t) - \epsilon\lambda(t)$ and $\beta_\epsilon(t) = \beta(t) + \epsilon\lambda(t)$ for $\epsilon > 0$. The function $\lambda(t)$ is also supposed to be unique positive solution of (9) with $L_1 + L_2 > L > 0$. It is clear that $\beta_\epsilon(0) = \beta(0) + \epsilon\lambda(0) > \beta(0)$ and $\alpha_\epsilon(0) = \alpha(0) - \epsilon\lambda(0) < \alpha(0)$ that imply $\alpha_\epsilon(0) < \beta_\epsilon(0)$ and for $0 \leq t \leq T$, we get $\beta_\epsilon(t) > \beta(t)$ and $\alpha_\epsilon(t) < \alpha(t)$.

Differentiating both sides of $\beta_\epsilon(t) = \beta(t) + \epsilon\lambda(t)$ leads to

$$\begin{aligned} {}^{C}D^{q_1}\beta_\epsilon(t) &= {}^{C}D^{q_1}\beta(t) + \epsilon^{C}D^{q_1}\lambda(t) \\ &\geq F(t,\beta(t),I^{q_2}\alpha(t)) + G(t,\beta(t),I^{q_3}\alpha(t)) \\ &\quad + L\epsilon\lambda(t) + M_1\epsilon I^{q_2}\lambda(t) + M_2\epsilon I^{q_3}\lambda(t). \end{aligned}$$

Since $\beta_\epsilon(t) > \beta(t)$, we can utilize the inequalities (10) and (11) in hypothesis to get

$$\begin{aligned} F(t,\beta_\epsilon(t),I^{q_2}\alpha(t)) - F(t,\beta(t),I^{q_2}\alpha(t)) &\leq L_1(\beta_\epsilon(t)-\beta(t)), \\ F(t,\beta(t),I^{q_2}\alpha(t)) &\geq F(t,\beta_\epsilon(t),I^{q_2}\alpha(t)) - L_1(\beta_\epsilon(t)-\beta(t)) \end{aligned}$$

and

$$\begin{aligned} G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right)-G\left(t,\beta\left(t\right),I^{q_3}\alpha\left(t\right)\right) &\leq L_2\left(\beta_\epsilon\left(t\right)-\beta\left(t\right)\right), \\ G\left(t,\beta\left(t\right),I^{q_3}\alpha\left(t\right)\right) &\geq G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right)-L_2\left(\beta_\epsilon\left(t\right)-\beta\left(t\right)\right). \end{aligned}$$

Putting these results into the foregoing inequality, we write

$$\begin{aligned} {}^CD^{q_1}\beta_\epsilon\left(t\right) &\geq F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha\left(t\right)\right)-L_1\left(\beta_\epsilon\left(t\right)-\beta\left(t\right)\right) \\ &\quad +G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right)-L_2\left(\beta_\epsilon\left(t\right)-\beta\left(t\right)\right) \\ &\quad +L\epsilon\lambda\left(t\right)+M_1\epsilon I^{q_2}\lambda\left(t\right)+M_2\epsilon I^{q_3}\lambda\left(t\right) \\ &= F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha\left(t\right)\right)+G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right)+(L-L_1-L_2)\epsilon\lambda\left(t\right) \\ &\quad +M_1\epsilon I^{q_2}\lambda\left(t\right)+M_2\epsilon I^{q_3}\lambda\left(t\right) \\ &> F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha\left(t\right)\right)+G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right)+M_1\epsilon I^{q_2}\lambda\left(t\right)+M_2\epsilon I^{q_3}\lambda\left(t\right). \end{aligned}$$

Since the fact that $\alpha\left(t\right)>\alpha_\epsilon\left(t\right)$, we can have $I^{q_2}\alpha\left(t\right)>I^{q_2}\alpha_\epsilon\left(t\right)$ and $I^{q_3}\alpha\left(t\right)>I^{q_3}\alpha_\epsilon\left(t\right)$. Therefore, the following inequalities can be found by considering inequalities (12) and (13)

$$\begin{aligned} F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha\left(t\right)\right)-F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right) &\geq -M_1I^{q_2}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right), \\ F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha\left(t\right)\right) &\geq F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)-M_1I^{q_2}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right) \end{aligned}$$

and

$$\begin{aligned} G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right)-G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right) &\geq -M_2I^{q_3}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right), \\ G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right) &\geq G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)-M_2I^{q_3}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right). \end{aligned}$$

Combining these results with previous inequality, we arrive at

$$\begin{aligned} {}^CD^{q_1}\beta_\epsilon\left(t\right) &> F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)-M_1I^{q_2}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right) \\ &\quad +G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)-M_2I^{q_3}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right) \\ &\quad +M_1\epsilon I^{q_2}\lambda\left(t\right)+M_2\epsilon I^{q_3}\lambda\left(t\right) \\ &= F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)+G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right). \end{aligned}$$

We intend to demonstrate $\alpha\left(t\right)<\beta_\epsilon\left(t\right)$ on J. If we use the similar technique as before we first assume that assertion is false which gives a contradiction itself. Therefore, when $\epsilon\longrightarrow 0$ gives the desired result.

Remark 1. Notice that if $F(t,u,v)$ and $G(t,u,v)$ are non-decreasing in v for each (t,u) whenever $\alpha\leq\beta$, then natural lower and upper solutions given by (2) and (3) imply the coupled lower and upper solutions of type I given by (4) and (5). Conversely, if $F(t,u,v)$ and $G(t,u,v)$ are non-increasing in v for each (t,u) whenever $\alpha\leq\beta$, then coupled lower and upper solutions of type I reduce to natural lower and upper solutions respectively.

Theorem 3. Let α and β be coupled lower and upper solutions of type II of (1) as well as $F(t,u,v)$ and $G(t,u,v)$ is non-decreasing in v. We also assume that

$$\begin{aligned} F\left(t,u_1\left(t\right),v\left(t\right)\right)-F\left(t,u_2\left(t\right),v\left(t\right)\right) &\geq -L_1\left(u_1-u_2\right) \\ G\left(t,u_1\left(t\right),\bar{v}\left(t\right)\right)-G\left(t,u_2\left(t\right),\bar{v}\left(t\right)\right) &\geq -L_2\left(u_1-u_2\right) \\ F\left(t,u\left(t\right),v_1\left(t\right)\right)-F\left(t,u\left(t\right),v_2\left(t\right)\right) &\leq M_1\left(v_1-v_2\right) \\ G\left(t,u\left(t\right),\bar{v}_1\left(t\right)\right)-G\left(t,u\left(t\right),\bar{v}_2\left(t\right)\right) &\leq M_2\left(\bar{v}_1-\bar{v}_2\right) \end{aligned}$$

whenever $u_1\geq u_2, v_1\geq v_2$, where $L_1,L_2>0, M_1,M_2\geq 0$. Then $\alpha\left(0\right)\leq\beta\left(0\right)$ implies that $\alpha\left(t\right)\leq\beta\left(t\right)$ on J.

Proof. For the proof, we recall the previous definitions of functions $\alpha_\epsilon(t), \beta_\epsilon(t)$ on J such that for $\epsilon > 0$

$$\alpha_\epsilon(t) = \alpha(t) - \epsilon\lambda(t), \beta_\epsilon(t) = \beta(t) + \epsilon\lambda(t).$$

The function $\lambda(t)$ is the unique positive solution of (9) with $L_1 + L_2 < L$. We can achieve the desired conclusion by using a similar process as described above.

Theorem 4. Let α and β be coupled lower and upper solutions of type III of (1) as well as both $F(t,u,v)$ and $G(t,u,v)$ is non-increasing in v. We also assume that

$$F(t, u_1(t), v_1(t)) - F(t, u_2(t), v_2(t)) \geq -L_1(u_1 - u_2) - M_1(v_1 - v_2),$$

$$G(t, u_1(t), \bar{v}_1(t)) - G(t, u_2(t), \bar{v}_2(t)) \geq -L_2(u_1 - u_2) - M_2(\bar{v}_1 - \bar{v}_2)$$

whenever $u_1 \geq u_2, v_1 \geq v_2$ and $L_1, L_2 > 0, M_1,\ M_2 \geq 0$. Then $\alpha(t) \leq \beta(t)$ on J provided that $\alpha(0) \leq \beta(0)$.

Proof. By using analogous considerations as mentioned previously, we can gain the conclusion of theorem directly. For space-saving, we omit the details here.

Corollary 2. If we take $G \equiv 0$ in the problem (1), then the results in Theorems 1–4 are reduced to the results in [16].

Remark 2. It is worthwhile to note that if $\alpha \leq \beta$ on J, then the monotonicity assumption of F and G in Theorem 3 combined with allowing α, β to be the coupled lower and upper solutions of type II respectively is equivalent to the case in which the monotonicity assumption of F and G in Theorem 4 combined with α, β being the coupled lower and upper solutions of type III respectively.

Theorem 5. Let α and β be coupled lower and upper solutions of type IV of (1). $F(t,u,v)$ is non-decreasing in v while $G(t,u,v)$ is non-increasing in v. Assume further that following inequalities are satisfied:

$$\begin{aligned} F(t, u_1(t), v_1(t)) - F(t, u_2(t), v_2(t)) &\leq L_1(u_1 - u_2) + M_1(v_1 - v_2), && (14)\\ G(t, u_1(t), \bar{v}_1(t)) - G(t, u_2(t), \bar{v}_2(t)) &\geq -L_2(u_1 - u_2) - M_2(\bar{v}_1 - \bar{v}_2), && (15)\end{aligned}$$

where $L_1, L_2 > 0, M_1, M_2 \geq 0$, whenever $u_1 \geq u_2, v_1 \geq v_2$. Then $\alpha(0) \leq \beta(0)$ implies that $\alpha(t) \leq \beta(t)$ on J.

Proof. We begin by constructing $\beta_\epsilon(t) = \beta(t) + \epsilon\lambda(t)$ and $\alpha_\epsilon(t) = \alpha(t) - \epsilon\lambda(t)$ for $\epsilon > 0$. The function $\lambda(t)$ is also supposed to be unique positive solution of (9) such that $L > L_1 + L_2$. It is clear that $\beta_\epsilon(0) = \beta(0) + \epsilon\lambda(0) > \beta(0)$ and $\alpha_\epsilon(0) = \alpha(0) - \epsilon\lambda(0) < \alpha(0)$ imply $\alpha_\epsilon(0) < \beta_\epsilon(0)$. In addition to that for $0 \leq t \leq T$ we get $\beta_\epsilon(t) > \beta(t)$ and $\alpha_\epsilon(t) < \alpha(t)$.

Differentiating both sides of $\beta_\epsilon(t) = \beta(t) + \epsilon\lambda(t)$ leads to

$$\begin{aligned} {}^CD^{q_1}\beta_\epsilon(t) &= {}^CD^{q_1}\beta(t) + {}^CD^{q_1}\epsilon\lambda(t)\\ &\geq F(t, \beta(t), I^{q_2}\beta(t)) + G(t, \alpha(t), I^{q_3}\alpha(t))\\ &\quad + L\epsilon\lambda(t) + M_1\epsilon I^{q_2}\lambda(t) + M_2\epsilon I^{q_3}\lambda(t). \qquad (16)\end{aligned}$$

Since $\beta_\epsilon(t) > \beta(t)$ for $0 \leq t \leq T$, we can employ the inequality (14) and (15) then it yields

$$F(t, \beta_\epsilon(t), I^{q_2}\beta_\epsilon(t)) - F(t, \beta(t), I^{q_2}\beta(t)) \leq L_1(\beta_\epsilon - \beta) + M_1 I^{q_2}(\beta_\epsilon - \beta),$$

$$F(t, \beta(t), I^{q_2}\beta(t)) \geq F(t, \beta_\epsilon(t), I^{q_2}\beta_\epsilon(t)) - L_1\epsilon\lambda(t) - M_1\epsilon I^{q_2}\lambda(t) \qquad (17)$$

and

$$G\left(t,\alpha\left(t\right),I^{q_3}\alpha\left(t\right)\right)-G\left(t,\alpha_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)\geq -L_2\left(\alpha-\alpha_\epsilon\right)-M_2I^{q_3}\left(\alpha-\alpha_\epsilon\right),$$
$$G\left(t,\alpha\left(t\right),I^{q_3}\alpha\left(t\right)\right)\geq G\left(t,\alpha_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)-L_2\epsilon\lambda\left(t\right)-M_2\epsilon I^{q_3}\lambda\left(t\right). \tag{18}$$

If we substitute (17) and (18) into (16), we get

$$\begin{aligned}{}^CD^{q_1}\beta_\epsilon\left(t\right) &\geq F\left(t,\beta\left(t\right),I^{q_2}\beta\left(t\right)\right)+G\left(t,\alpha\left(t\right),I^{q_3}\alpha\left(t\right)\right)\\ &\quad +L\epsilon\lambda\left(t\right)+M_1\epsilon I^{q_2}\lambda\left(t\right)+M_2\epsilon I^{q_3}\lambda\left(t\right)\\ &\geq F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\beta_\epsilon\left(t\right)\right)-L_1\epsilon\lambda\left(t\right)-M_1\epsilon I^{q_2}\lambda\left(t\right)\\ &\quad +G\left(t,\alpha_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)-L_2\epsilon\lambda\left(t\right)-M_2\epsilon I^{q_3}\lambda\left(t\right)\\ &\quad +L\epsilon\lambda\left(t\right)+M_1\epsilon I^{q_2}\lambda\left(t\right)+M_2\epsilon I^{q_3}\lambda\left(t\right)\\ &> F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\beta_\epsilon\left(t\right)\right)+G\left(t,\alpha_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right).\end{aligned}$$

A similar procedure can be applied to $\alpha_\epsilon\left(t\right)=\alpha\left(t\right)-\epsilon\lambda\left(t\right)$ to achieve the following result

$${}^CD^{q_1}\alpha_\epsilon\left(t\right)<F\left(t,\alpha_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)+G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\beta_\epsilon\left(t\right)\right)$$

on $[0,T]$.

We next prove that $\alpha_\epsilon\left(t\right)<\beta_\epsilon\left(t\right)$ on $[0,T]$. Contrary to this claim, we presume for a moment that the inequality is not true and, setting $m\left(t\right)=\alpha_\epsilon\left(t\right)-\beta_\epsilon\left(t\right)$ there would exist a point t_* such that $m\left(t_*\right)=0$ and $m\left(t\right)\leq 0$ for $0\leq t<t_*$. We get at once ${}^CD^{q_1}m\left(t_*\right)\geq 0$ by Lemma 1. Obviously, it causes a contradiction. Then, it has to be

$$\alpha_\epsilon\left(t\right)<\beta_\epsilon\left(t\right)$$

on J. Finally, letting $\epsilon\longrightarrow 0$, we reach at

$$\begin{aligned}\lim_{\epsilon\longrightarrow 0}\left(\alpha\left(t\right)-\epsilon\lambda\left(t\right)\right) &\leq \lim_{\epsilon\longrightarrow 0}\left(\beta\left(t\right)+\epsilon\lambda\left(t\right)\right),\\ \alpha\left(t\right) &\leq \beta\left(t\right),\end{aligned}$$

for $t\in J$, ending the proof.

Corollary 3. This result is evaluated as the generalization of Theorem 2.10 in [17] to fractional orders by simple modifications.

Theorem 6. Let α and β be coupled lower and upper solutions of type V of (1). $F(t,u,v)$ is non-increasing and $G(t,u,v)$ is non-decreasing in v. Additionally, following inequalities hold:

$$F\left(t,u_1\left(t\right),v_1\left(t\right)\right)-F\left(t,u_2\left(t\right),v_2\left(t\right)\right) \geq -L_1\left(u_1-u_2\right)-M_1\left(v_1-v_2\right), \tag{19}$$
$$G\left(t,u_1\left(t\right),\bar{v}_1\left(t\right)\right)-G\left(t,u_2\left(t\right),\bar{v}_2\left(t\right)\right) \leq L_2\left(u_1-u_2\right)+M_2\left(\bar{v}_1-\bar{v}_2\right), \tag{20}$$

where $L_1,L_2,M_1,M_2>0$, whenever $u_1\geq u_2,v_1\geq v_2$. Then $\alpha\left(0\right)\leq\beta\left(0\right)$ implies that $\alpha\left(t\right)\leq\beta\left(t\right)$ on J.

Proof. In that case, for some $\epsilon>0$, we compose $\beta_\epsilon\left(t\right)=\beta\left(t\right)+\epsilon\lambda\left(t\right)$ and $\alpha_\epsilon\left(t\right)=\alpha\left(t\right)-\epsilon\lambda\left(t\right)$ where the function $\lambda\left(t\right)$ is taken as the nonnegative unique solution of the following linear equation

$${}^CD^{q_1}\lambda\left(t\right)=L\lambda\left(t\right)+M_1I^{q_2}\lambda\left(t\right)+M_2I^{q_3}\lambda\left(t\right),\quad \lambda\left(0\right)=1.$$

Taking derivatives in Caputo's sense on both sides of constructed functions and using (19) and (20), we have the following strict inequalities

$${}^CD^{q_1}\beta_\epsilon\left(t\right)>F\left(t,\alpha_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)+G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\beta_\epsilon\left(t\right)\right)$$

and

$$ {}^{C}D^{q_1}\alpha_\epsilon(t) < F(t,\beta_\epsilon(t), I^{q_2}\beta_\epsilon(t)) + G(t,\alpha_\epsilon(t), I^{q_3}\alpha_\epsilon(t)). $$

At this stage we apply proof by contradiction with the help of Lemma 1 to show $\alpha_\epsilon(t) < \beta_\epsilon(t)$ on J. As a final step, performing $\epsilon \longrightarrow 0$, we get the desired result

$$ \alpha(t) \le \beta(t), $$

for $t \in J$, which completes the proof.

3 Conclusion

Using the method of upper and lower solutions, this research discusses some differential inequalities for generalized fractional integro-differential equations. Multiple coupled upper and lower solutions are used to examine the results. These theorems provide some possibilities for stretching iterative techniques to fractional order integro-differential equations and coupled systems of integro-differential fractional equations in order to determine the existence of solutions as well as approximations for the problem under consideration.

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Жоғарғы және төменгі шешімдер арқылы бөлшек ретті интегралды дифференциалдық теңдеулер үшін кейбір дифференциалдық теңсіздіктердің кеңеюі

Мақалада жоғарғы және төменгі шешімдер техникасын қолдана отырып, бөлшек ретті жалпыланған интегралды-дифференциалдық теңдеулер үшін кейбір дифференциалдық теңсіздіктер қарастырылған. Бөлшек дифференциалдық оператор Капуто мағынасында түсініледі, ал екіге бөлінген сызықтық емес термин екі түрлі бөлшек реті бар белгісіз функцияның бөлшек интегралдарына тәуелді. Нәтижелер әртүрлі байланысты жоғарғы және төменгі шешімдерді қолдану арқылы зерттелген. Бұл теоремалар қайталанатын әдістерді бөлшек ретті интегралды-дифференциалдық теңдеулерге және шешімдердің болуын, сондай-ақ қарастырылып отырған мәселе үшін жуықталған шешімдерді алу үшін бөлшек ретті интегралды-дифференциалдық теңдеулердің байланысты жүйелеріне тарату үшін белгілі бір әлеуетке ие.

Кілт сөздер: бөлшек дифференциалдық теңдеулер, дифференциалдық теңсіздіктер, жоғарғы және төменгі шешімдер, шеткі есеп.

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Расширения некоторых дифференциальных неравенств для интегро-дифференциальных уравнений дробного порядка через верхние и нижние решения

В статье рассмотрены некоторые дифференциальные неравенства для обобщенных интегро-дифференциальных уравнений дробного порядка с использованием техники верхних и нижних решений. Дробно-дифференциальный оператор понимается в смысле Капуто, а нелинейный член, разделенный на две части, зависит от дробных интегралов неизвестной функции с двумя различными дробными порядками. Результаты изучены с использованием различных связанных верхних и нижних решений. Эти теоремы имеют некоторый потенциал для распространения итерационных методов на интегро-дифференциальные уравнения дробного порядка и на связанные системы интегро-дифференциальных уравнений дробного порядка для получения существования решений, а также приближенных решений для рассматриваемой задачи.

Ключевые слова: дробные дифференциальные уравнения, дифференциальные неравенства, верхние и нижние решения, краевая задача.

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Generalized differential transformation method for solving two-interval Weber equation subject to transmission conditions

The main goal of this study is to adapt the classical differential transformation method to solve new types of boundary value problems. The advantage of this method lies in its simplicity, since there is no need for discretization, perturbation or linearization of the differential equation being solved. It is an efficient technique for obtaining series solution for both linear and nonlinear differential equations and differs from the classical Taylor's series method, which requires the calculation of the values of higher derivatives of given function. It is known that the differential transformation method is designed for solving single interval problems and it is not clear how to apply it to many-interval problems. In this paper we have adapted the classical differential transformation method for solving boundary value problems for two-interval differential equations. To substantiate the proposed new technique, a boundary value problem was solved for the Weber equation given on two non-intersecting segments with a common end, on which the left and right solutions were connected by two additional transmission conditions.

Keywords: two-interval problems, the differential transformation method, Weber equation, transmission conditions.

Introduction

It is well known that two-dimensional elliptic equations often occur as a mathematical model of steady-state or equilibrium problems. For example, for a stationary flow of an incompressible inviscid fluid, the velocity potential satisfies the two-dimensional elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

the so-called Laplace's equation. Separation of variables method applied to the Laplace equation in parabolic coordinates leads to the Weber equation

$$y'' + (n + \frac{1}{2} - \frac{x^2}{4})y = 0,$$

where n is a constant. This equation was first studied by H. Weber in connection with the parabolic cylinder in the potential theory [1]. The Weber equation converts to the equation

$$u'' - xu' + nu = 0 \tag{1}$$

via the substitution $y = ue^{\frac{-x^2}{4}}$. Note that the solutions of the Weber equation are known as Weber-Hermite functions or parabolic cylinder functions. In the case when the constant n is a non-negative integer, the Weber equation (1) has the solution

$$u = e^{\frac{-x^2}{4}} H_n(x),$$

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where $H_n(x)$ is the Hermite polynomial defined by the equality

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

In recent years, there has been increased interest in boundary value problems for many-interval differential equations with additional transmission conditions. [2–6]. Such type of transmission problems are motivated by the emergence of new and interesting applications in physics.

In this article, the Weber equation given on two non-intersecting intervals and satisfying supplementary transmission conditions between left and right solutions, will be solved by the differential transformation method (DTM, for short). The main idea of this method was first proposed by Zhou in connection with some problems of electrical circuits [7]. Using differential transform, the given differential equation and related initial and/or boundary conditions can be replaced by linear algebraic equations. Therefore this method is of great interest in physics, engineering and other natural sciences ([8–14]). For example, Sepasgozar et al. used DTM to solve the momentum and the heat transfer problems of non-Newtonian fluid flow in an axis-symmetric channel with porous wall [15]. Usman et al. applied differential transformation technique to investigate unsteady two phases on non-fluid flow and the heat transfer between moving parallel plates in the presence of the magnetic field [16].

In recent years, various modifications of the DTM have been used to solve many interesting problems that arise not only in theoretical mathematics, but also in applied sciences (see, for example [17–20] and references cited therein)

1 Differential transformation and Differential inverse transformation

Let $f = f(x)$ be an infinitely differentiable function on the real axis $R = (-\infty, \infty)$ and let $x_0 \in R$ be any point. Denote by $Y_{x_0}(f, n), \quad n = 0, 1, 2, ...$ the coefficient at the n. term of the Taylor series of the function f in the neighborhood of the point x_0, that is $Y_{x_0}(f, n) := \frac{1}{n!} f^{(n)}(x_0)$.

Definition 1. The sequence $Y_{x_0}(f) := (Y_{x_0}(f, 1), Y_{x_0}(f, 2), ...)$ is said to be differential transformation of the function f at the point x_0.

Definition 2. Let $A := (a_n)$ be any sequence, such that the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is convergent on the whole R. Then the function

$$Y_{x_0}^{-1}(A, x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is said to be the differential inverse transformation of the sequence $A := (a_n)$ at the point $x = x_0$.

It is obvious that any analytic function $f(x)$ satisfies the following equality

$$Y_{x_0}^{-1}(Y_{x_0}(f), x) = f(x).$$

Let $C^\infty(R)$ be the set of all infinitely differentiable functions defined on the real axis R. It is easy to verify that the following properties are valid

(i) $Y_{x_0}(f + g, n) = Y_{x_0}(f, n) + Y_{x_0}(g, n), \quad f, g \in C^\infty(R), \quad n = 0, 1, 2, ...;$
(ii) $Y_{x_0}(\lambda f, n) = \lambda Y_{x_0}(f, n), \quad \lambda \in R, \quad f \in C^\infty(R);$
(iii) $Y_{x_0}(\frac{d^s f}{dx^s}, n) = \frac{(s+n)!}{n!} Y_{x_0}(f, s + n), \quad s, n = 0, 1, 2, ..., \quad f \in C^\infty(R);$
(iv) $Y_{x_0}(fg, n) = \sum_{k=0}^{n} Y_{x_0}(f, k) Y_{x_0}(g, n-k)$, that is $Y_{x_0}(fg) = Y_{x_0}(f) * Y_{x_0}(g)$, where $Y_{x_0}(f) * Y_{x_0}(g)$ denotes the convolution of the sequences $Y_{x_0}(f)$ and $Y_{x_0}(g)$.

Remark 1. Let $A = (a_n)$ be any real sequence. If we denote the sequence $(a_1, a_2, ..., a_k, 0, 0, 0...)$. by A^k, then we have

$$\lim_{k\to\infty} Y_{x_0}^{-1}((Y_{x_0}(f))^k, x) = f(x)$$

provided that $f(x)$ is an analytic function on the real axis R. Hence for sufficiently large k we can take

$$f(x) \approx Y_{x_0}^{-1}((Y_{x_0}(f))^k, x)$$

in real applications.

2 *Solution of two-interval Weber equation using the modified differential transformation technique*

Example 1. Consider the two-interval Weber equation

$$y''(x) - xy'(x) + 2y(x) = 0, \quad x \in [1, \frac{1}{2}) \cup (\frac{1}{2}, 1] \tag{2}$$

subject to the boundary conditions, given by

$$y(0) = 0, \quad y(1) = 1$$

and additional transmission conditions at the interior singular point $x = \frac{1}{2}$, given by

$$y\left(\frac{1}{2} - 0\right) - \gamma_1 y\left(\frac{1}{2} + 0\right) = 0, \tag{3}$$

$$y'\left(\frac{1}{2} - 0\right) - \gamma_2 y'\left(\frac{1}{2} + 0\right) = 0, \tag{4}$$

where γ_1 and γ_2 are real numbers that will be specified later. We will consider the equation (2) on the left side $[0, \frac{1}{2})$ and the right side $(\frac{1}{2}, 1]$ of the domain $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, separately.

We will denote by $Y_0(y*, k)$ and $Y_1(y**, k)$ the differential transformation of $y(x)$ at the left end-point $x = 0$ and the right end-point $x = 1$, respectively. Applying the differential transformation to the differential equation (2) in the left interval $[0, \frac{1}{2})$, we have the following linear algebraic equations

$$Y_0(y*, k+2) = \frac{1}{(k+2)(k+1)}\left[\sum_{r=0}^{k}(k - n + 1)Y_0(y*, k - r + 1)\delta(r - 1) - 2Y_0(y*, k)\right], \tag{5}$$

where $Y_0(y*, k) = \frac{1}{k!}\frac{d^k y*(x)}{dx^k}|_{x=0}$. The differential inverse transformation in the left interval has the following form:

$$y*(x) = Y_0(y*, 0) + xY_0(y*, 1) + ... + x^n Y_0(y*, n) + \dots$$

The first boundary condition $y(0) = 0$ becomes $Y_0(y*, 0) = 0$. Denoting $Y_0(y*, 1) = A$, (5) we have where A is unknown number that will be calculated later, and then substituting in the recursive relation $Y_0(y*, 3) = \frac{-A}{6}$, $Y_0(y*, 4) = 0$, $Y_0(y*, 5) = \frac{-A}{120}$, $Y_0(y*, 6) = 0$, $Y_0(y*, 7) = \frac{-A}{1680}$, ...

Thus we have the following series expansion of the left solution:

$$y*(x) \quad = \quad Ax - \frac{A}{6}x^3 - \frac{A}{120}x^5 - \frac{A}{1680}x^7 + ... \tag{6}$$

Applying differential transformation in the neighborhood of the right end-point $x_0 = 1$ we have

$$Y_1(y**, k+2) = \frac{1}{(k+2)(k+1)}[(k+1)Y_1(y**, k+1) + kY_1(y**, k) - 2Y_1(y**, k)]. \tag{7}$$

Applying the differential inverse transformation in the right interval $(\frac{1}{2}, 1]$ gives

$$y**(x) = Y_1(y**, 0) + (x-1)Y_1(y**, 1) + ... + (x-1)^n Y_1(y**, n) + ...$$

The boundary condition $y(1) = 1$ becomes $Y_1(y**, 0) = 1$. Let $Y_1(y**, 1) = B$. Here B is unknown parameter that will be calculated later. Using the recursive relation (7) we have
$Y_1(y**, 2) = \frac{1}{2}(B-2)$, $Y_1(y**, 3) = \frac{-1}{3}$, $Y_1(y**, 4) = \frac{-1}{12}$, $Y_1(y**, 5) = \frac{-1}{30}$, $Y_1(y**, 6) = \frac{-1}{90}$, $Y_1(y**, 7) = \frac{-1}{252}$, ...

Then we have the following series expansion of the right solution

$$\begin{aligned} y**(x) &= 1 + B(x-1) + \frac{1}{2}(B-2)(x-1)^2 - \frac{1}{3}(x-1)^3 - \\ &- \frac{1}{12}(x-1)^4 - \frac{1}{30}(x-1)^5 - \frac{1}{90}(x-1)^6 - \frac{1}{252}(x-1)^7 + ... \end{aligned} \tag{8}$$

To find the unknown parameters A and B, we put the relations (6) and (8) into the transmission conditions (3)–(4). Then using "Mathematica"8, we can calculate approximate values of the unknown numbers A and B as $A = 1.21302$, $B = 0.550509$. Here we continued iterating up to the 7 th term in the series expansion for DTM-solutions $y*(x)$ and $y**(x)$. Below, Figure 1 shows the graph of the DTM-solution

$$y(x) = \begin{cases} y*(x) & for \quad x \in [0, \frac{1}{2}), \\ y**(x) & for \quad x \in (\frac{1}{2}, 1]. \end{cases}$$

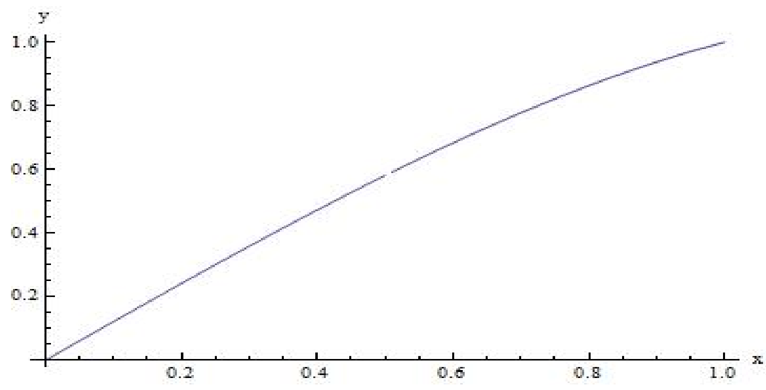

Figure 1. Approximate DTM- solution of the problem (2)–(4) for $\gamma_1 = \gamma_2 = 1$.

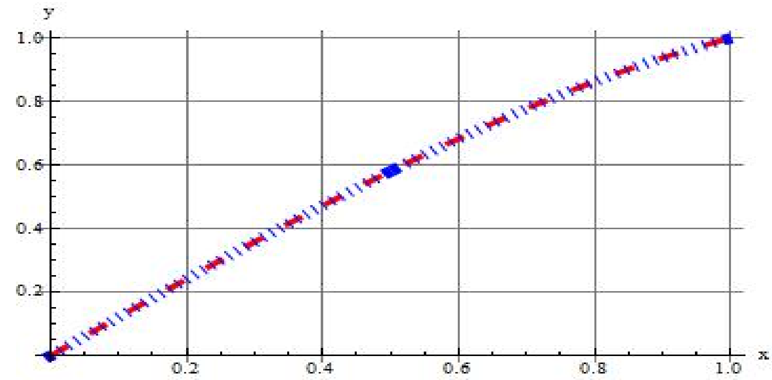

Figure 2. Comparison of the exact solution (red line) and the DTM-solution (blue line) of the problem (2)–(4) for $\gamma_1 = \gamma_2 = 1$.

Example 2. Now we shall consider a two-interval Weber equation with negative n, given by

$$y''(x) - xy'(x) - 4y(x) = 0, \qquad x \in [1, \frac{1}{2}) \cup (\frac{1}{2}, 1] \tag{9}$$

together with the boundary conditions at the end points $x = 0$ and $x = 1$, given by

$$y(0) = 1, \quad y(1) = 0 \tag{10}$$

subject to the additional transmission conditions, given by

$$c_1 y\left(\frac{1}{2} - 0\right) - c_2 y\left(\frac{1}{2} + 0\right) = 0, \tag{11}$$

$$c_3 y'\left(\frac{1}{2} - 0\right) - c_4 y'\left(\frac{1}{2} + 0\right) = 0, \tag{12}$$

where c_1, c_2, c_3 and c_4 are real numbers that will be specified later. As above, we shall consider the differential equation (9) on the left side $[0, \frac{1}{2})$ and the right side $(\frac{1}{2}, 1]$ of the domain $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, separately.

As in above, $Y_0(y*, k)$ and $Y_1(y*, k)$ denotes the Y- transforms of $y(x)$ at the left end-point $x = 0$ and the right end-point $x = 1$, respectively. Using differential transformation in the left interval, i.e. in the neighborhood of the point $x_0 = 0$, we have

$$Y_0(y*, k+2) = \frac{1}{(k+2)(k+1)}\left[\sum_{r=0}^{k}(k-n+1)Y_0(y*, k-r+1)\delta(r-1) + 4Y_0(y*, k)\right], \tag{13}$$

where $Y_0(y*, k) = \frac{1}{k!}\frac{d^k y*(x)}{dx^k}|_{x=0}$. The differential inverse transformation in the left interval has the following form:

$$y*(x) = Y_0(y*, 0) + xY_0(y*, 1) + ... + x^n Y_0(y*, n) + ...$$

The first boundary condition $y(0) = 1$ becomes $Y_0(y*, 0) = 1$. Denoting $Y_0(y*, 1) = K$, where K is unknown parameter that will be calculated later, and then substituting in the recursive relation (13), we have

$$Y_0(y*,2)=2,\ \ Y_0(y*,3)=\tfrac{5K}{6},\ \ Y_0(y*,4)=1,\quad Y_0(y*,5)=\tfrac{7K}{24},$$
$$Y_0(y*,6)=\tfrac{4}{15},\quad Y_0(y*,7)=\tfrac{K}{16},\quad Y_0(y*,8)=\tfrac{1}{21},\ ...$$

Thus we have the series expansion of the left solution $y*(x)$ in the form

$$y*(x)\ \ =\ \ 1+Kx+2x^2+\frac{5K}{6}x^3+x^4+\frac{7K}{24}x^5+\frac{4}{15}x^6+\frac{K}{16}x^7+\frac{1}{21}x^8+...\qquad(14)$$

Now applying differential transformation to the equation (9) in the right interval, we have

$$Y_1(y**,k+2)=\frac{1}{(k+2)(k+1)}[(k+1)Y_1(y**,k+1)+kY_1(y**,k)-4Y_1(y**,k)].\qquad(15)$$

The differential inverse transformation in the right interval $(\frac{1}{2},1]$ has the following form:

$$y**(x)=Y_1(y**,0)+(x-1)Y_1(y**,1)+...+(x-1)^nY_1(y**,n)+...$$

The second boundary condition $y(1)=0$ becomes $Y_1(y**,0)=0$. Putting $Y_1(y**,1)=M$, where M is unknown parameter that will be calculated later, and using the recursive relation (15) we have

$$Y_1(y**,2)=\tfrac{M}{2},\ \ Y_1(y**,3)=M,\ \ Y_1(y**,4)=\tfrac{M}{2},$$
$$Y_1(y**,5)=\tfrac{9M}{20},\ \ Y_1(y**,6)=\tfrac{5M}{24},\quad Y_1(y**,7)=\tfrac{53M}{420},\ ...$$

Consequently we have the series expansion of the right solution $y**(x)$ in the form

$$y**(x)\ \ =\ \ M(x-1)+\frac{M}{2}(x-1)^2+M(x-1)^3+\frac{M}{2}(x-1)^4+\frac{9M}{20}(x-1)^5\frac{5M}{24}(x-1)^6-$$
$$-\ \ \frac{53M}{420}(x-1)^7+...\qquad(16)$$

Substituting (14)–(16) in the transmission conditions (11)–(12) we obtain two algebraic equation with respect to the variables K,M.

Finally, using «Mathematica 8», we can calculate approximate values of the parameters K and M as $K=-1.93316$, $M=-0.70003$. Here we were continued iterating up to the 7 th term in the series expansion for the DTM-solutions $y*(x)$ and $y**(x)$. The approximate DTM-solution of the problem (9)–(10) is presented graphically in Figure 3 and Figure 4.

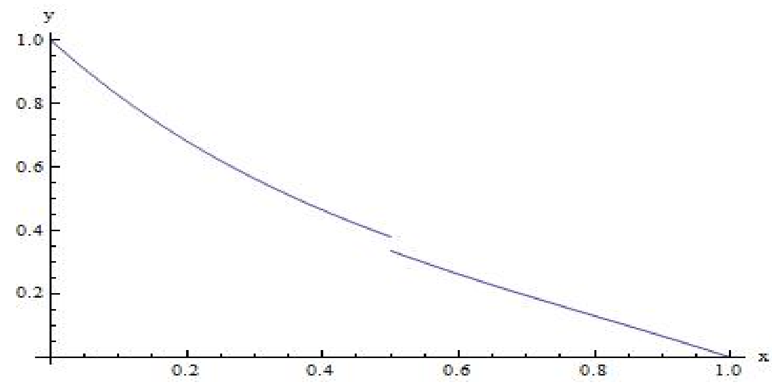

Figure 3. Approximate solution of the problem (9)–(12) for $c_1=3$, $c_2=4$, $c_3=5$, $c_4=6$.

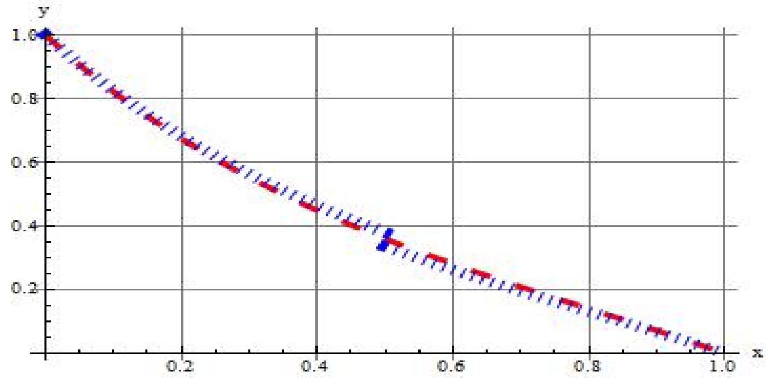

Figure 4. Comparison of the exact solution (red line) and the classical DTM-solution (blue line).

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Тарату шарттарын ескере отырып, Вебердің екіинтервалды теңдеуін шешуге арналған жалпыланған дифференциалдық түрлендіру әдісі

Зерттеудің негізгі мақсаты – классикалық дифференциалдық түрлендіру әдісін жаңа шеттік есептердің шешуге бейімдеу. Бұл әдістің артықшылығы оның қарапайымдылығында, өйткені шешілетін дифференциалдық теңдеуді іріктеу, ауытқу немесе сызықтық ету қажет емес. Осы сызықтық және бейсызықты дифференциалдық теңдеулер үшін қатарлар түрінде шешімдер алудың тиімді әдісі және берілген функцияның жоғары туындыларының мәндерін есептеуді қажет ететін Тейлор қатарларының классикалық әдісінен ерекшеленеді. Дифференциалды түрлендіру әдісі бір интервалды есептерді шешуге арналғаны белгілі және оны көп интервалды есептерге қалай қолдану керектігі белгісіз. Осы мақалада біз екі интервалды дифференциалдық теңдеулер үшін шеттік есептерді шешу үшін классикалық дифференциалдық түрлендіру әдісін бейімдедік. Ұсынылған жаңа әдістемені негіздеу үшін сол және оң жақты шешімдер екі қосымша берілу шарттарымен байланысты болатын ортақ ұшы бар екі қиылыспайтын кесінділер бойынша берілген Вебер теңдеуінің шеттік есебі шешілді.

Кілт сөздер: екіинтервалды есептер, дифференциалды түрлендіру әдісі, Вебер теңдеуі, тарату шарттары.

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Метод обобщенного дифференциального преобразования для решения двухинтервального уравнения Вебера с учетом условий передачи

Основная цель данного исследования состоит в том, чтобы адаптировать классический дифференциальный метод преобразования для решения новых типов краевых задач. Преимущество этого метода заключается в его простоте, так как нет необходимости в дискретизации, возмущении или линеаризации решаемого дифференциального уравнения. Это эффективный метод получения решений в виде рядов как для линейных, так и нелинейных дифференциальных уравнений, и он отличается от классического метода рядов Тейлора, который требует вычисления значений высших производных заданной функции. Известно, что метод дифференциального преобразования предназначен для решения одноинтервальных задач и не ясно, как его применять к многоинтервальным задачам. В настоящей статье мы адаптировали классический метод дифференциального преобразования для решения краевых задач для двухинтервальных дифференциальных уравнений. Для обоснования предложенной новой методики решалась краевая задача для уравнения Вебера, заданного на двух непересекающихся отрезках с общим концом, на которых левое и правое решения были связаны двумя дополнительными условиями передачи.

Ключевые слова: двухинтервальные задачи, метод дифференциального преобразования, уравнение Вебера, условия передачи.

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