ISSN 2518-7929 (Print)
ISSN 2663-5011 (Online)

BULLETIN
OF THE KARAGANDA UNIVERSITY

MATHEMATICS
Series

№ 4(108)/2022

ISSN 2518-7929 (Print)
ISSN 2663–5011(Online)
Индексі 74618
Индекс 74618

ҚАРАҒАНДЫ УНИВЕРСИТЕТІНІҢ **ХАБАРШЫСЫ**

ВЕСТНИК КАРАГАНДИНСКОГО УНИВЕРСИТЕТА

BULLETIN OF THE KARAGANDA UNIVERSITY

МАТЕМАТИКА сериясы

Серия МАТЕМАТИКА

MATHEMATICS Series

№ 4(108)/2022

Қазан–қараша–желтоқсан
30 желтоқсан 2022 ж.

Октябрь–ноябрь–декабрь
30 декабря 2022 г.

October–November–December
December, 30th, 2022

1996 жылдан бастап шығады
Издается с 1996 года
Founded in 1996

Жылына 4 рет шығады
Выходит 4 раза в год
Published 4 times a year

Қарағанды, 2022
Караганда, 2022
Karaganda 2022

Postal address: 28, University Str., Karaganda, 100024, Kazakhstan
Tel.: (7212) 77-04-38 (add. 1026); fax: (7212) 35-63-98.

E-mail: vestnikku@gmail.com. Web-site: mathematics-vestnik.ksu.kz

Bulletin of the Karaganda University. Mathematics series.
ISSN 2518-7929 (Print). ISSN 2663–5011 (Online).

Proprietary: NLC «Karagandy University of the name of academician E.A. Buketov».

Registered by the Ministry of Information and Social Development of the Republic of Kazakhstan. Rediscount certificate No. KZ43VPY00027385 dated 30.09.2020.

Signed in print 29.12.2022. Format 60×84 1/8. Offset paper. Volume 16,375 p.sh. Circulation 200 copies. Price upon request. Order № 117.

Printed in the Publishing house of NLC «Karagandy University of the name of acad. E.A. Buketov».

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CONTENT

MATHEMATICS

DOI 10.31489/2022M4/4-19

UDC 517.95:519.6

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On the convergence of difference schemes of high accuracy for the equation of ion-acoustic waves in a magnetized plasma

Multiparametric difference schemes of the finite element method of a high order of accuracy for the Sobolev-type equation of the fourth-order in time are studied. In particular, the first boundary value problem for the equation of ion-acoustic waves in a magnetized plasma is considered. A high-order accuracy of the scheme is achieved due to the special discretization of time and space variables. The presence of parameters in the scheme makes it possible to regularize the accuracy of the schemes and optimize the implementation algorithm. An a priori estimate in a weak norm is obtained by the method of energy inequality. Based on this estimate and the Bramble-Hilbert lemma, the convergence of the constructed algorithms in classes of generalized solutions is proved. An algorithm for implementing the difference scheme is proposed.

Keywords: Sobolev type equation, difference schemes, finite difference method, finite element method, stability, convergence, accuracy.

Introduction

As is known, the solution of complex applied problems requires the creation of more accurate numerical algorithms or the improvement of existing ones. This is especially seen in the study of complex non-stationary processes, for example, in boundary value problems for high-order partial differential equations. The study of such equations began with the research works of S.L. Sobolev. They are applied in solving problems of geophysics, oceanology, atmospheric physics, physics of magnetically ordered structures related to the propagation of waves in media with a strong dispersion, and many other problems [1–3]. For example, the equation of ion-acoustic waves in a magnetized plasma [3]

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial t^2}+\omega_{B_i}^2\right)\left(\Delta_3 u - r_D^{-2}u\right)+\omega_{p_i}^2\frac{\partial^2}{\partial t^2}\Delta_3 u+\omega_{p_i}^2\omega_{B_i}^2\frac{\partial^2 u}{\partial x_3^2}=f(x,t), \tag{1}$$

$$(x,t)\in Q_T=\Omega\cup\partial\Omega,\ \Omega=\left\{x \mid x=(x_1,x_2,x_3),\ 0<x_\alpha<l,\ \alpha=\overline{1,3}\right\},$$

refers to such equations. Here $u=(x,t)$ is the motion velocity, $\Delta_3=\partial^2u/\partial x_1^2+\partial^2u/\partial x_2^2+\partial^2u/\partial x_3^2$, $r_D^2=T_e^2/\left(4\pi e^2 n_0\right)$ is the Debye radius, $\omega_{B_i}=eB_0/(Mc)$ is the ion gyrofrequency, $\omega_{p_i}^2=4\pi e^2 n_0/M$ is the Langmuir frequency for ions, M is the mass, c is the speed of light in vacuum, B_0 is the external constant magnetic field, n_0 is the unperturbed particle density, e is the absolute value of the electron charge, T_e is the temperature of the electrons. In addition, similar equations appear in the mathematical modeling of internal waves in the ocean and atmosphere [4–6].

The study in [3] is devoted to analytical methods for solving problems of this type, where the problems of global and local solvability of initial-boundary value problems for linear and nonlinear equations are considered. Numerical methods for solving equations unresolved with respect to the time derivative are also considered. Non-stationary equations of the second order in time and pseudo-parabolic equations are considered. Here and in [7], these equations are reduced by some substitution

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to two equations (one contains differentiation with respect to time, the other - with respect to space only); then, these equations are solved by the finite difference method on quasi-uniform grids. The second order of approximation in both variables is proved.

The studies in [8, 9] are devoted to numerical methods for solving initial-boundary value problems for equation (1). In [8], a mathematical model of ion-acoustic waves in plasma is considered in an external magnetic field. Issues of unique solvability of the Cauchy-Dirichlet problem are considered. Based on the theoretical results, an algorithm was developed for the numerical solution of the problem based on the modified Galerkin method. An implementation algorithm is given. A problem similar to an optimal control problem for the mathematical model (1), was considered in [9], where an algorithm for a numerical solution based on the modified Galerkin method and the Ritz method was developed.

In this article, the authors consider the issues of constructing and investigating difference schemes of high accuracy of initial-boundary value problems for the non-stationary equation of ion-acoustic waves in a magnetized plasma (1). First, we approximate the space variables, and the time variable is stored in differential form. As a result, we obtain a system of ordinary differential equations of large dimensions, solved by the difference scheme of the finite element method of the fourth-order accuracy. To obtain an accuracy estimate, a special technique for obtaining a priori estimates was used since the classical approach to studying the convergence of difference schemes based on the Taylor formula places high demands on the smoothness of the sought-for solution. Therefore, a number of results have recently been obtained on estimating the rate of convergence of difference schemes for equations of mathematical physics based on the Bramble-Hilbert lemma [10]. Such studies for various stationary and nonstationary problems were conducted in [11–15]. The notation from [16] is used in this article.

1 Statement of the problem

Let us rewrite equation (1) in the following form:

$$\frac{\partial^4}{\partial t^4}\left(\Delta_3 u - r_D^{-2} u\right) + \frac{\partial^2}{\partial t^2}[(\omega_{B_i}^2 + \omega_{p_i}^2)\Delta_3 u - \omega_{B_i}^2 r_D^{-2} u] + \omega_{p_i}^2 \omega_{B_i}^2 \frac{\partial^2 u}{\partial x_3^2} = f(x,t), \tag{2}$$

$$(x,t) \in \Omega = \{x = (x_1, x_2, x_3) : \, 0 < x_k < l_k, \, k = 1,2,3\}.$$

The initial and boundary conditions have the following form:

$$\left.\frac{\partial^\nu}{\partial t^\nu} u(x,t)\right|_{t=0} = u_{0,\nu}, \quad \nu = \overline{0,3}, \quad x \in \overline{\Omega} = \Omega \cup \partial\Omega, \tag{3}$$

$$u(x,t)|_{\partial\Omega} = 0, \;\; t \in (0,T]. \tag{4}$$

The existence and uniqueness of solutions to such problems are considered in [1–3].

In our case, we will assume that $r_D^2 \notin \sigma(\Delta) = \lambda_k$ is the set of eigenvalues of the homogeneous Dirichlet problem for the Laplace operator in domain Ω.

Let us formulate a generalized statement of problem (2)–(4). Function $u(x,t)$, which for each $t \in (0,T]$ belongs to $H = \left\{ u \in W_2^1(\Omega), \; u = 0, \; x \in \partial\Omega \right\}$ is called the generalized solution to the problem; it has derivative $\frac{\partial^4 u}{\partial t^4} \in W_2^1(\Omega)$, and satisfies the following relations almost everywhere for all $t \in (0,T]$:

$$a_3\left(\frac{d^4 u(t)}{dt^4}, \vartheta\right) + a_2\left(\frac{d^2 u(t)}{dt^2}, \vartheta\right) + a_1(u(t), \vartheta) = (f(t), \vartheta), \tag{5}$$

$$\left(\frac{d^k u}{dt^k}(0) - u_{0,k}, \vartheta\right) = 0, \;\; k = \overline{0,3}, \;\; \forall \vartheta(x) \in H. \tag{6}$$

Here

$$a_1(u,\vartheta)=\omega_{p_i}^2\omega_{B_i}^2\int\limits_{\Omega}(u_{x_3}\vartheta_{x_3})dx,\ a_2(u,\vartheta)=\int\limits_{\Omega}\left[\sum_{k=1}^{3}(\omega_{B_i}^2+\omega_{p_i}^2)u_{x_k}\vartheta_{x_k}-\omega_{B_i}^2 r_D^{-2}u\vartheta\right]dx,$$

$$a_3(u,\vartheta)=\int\limits_{\Omega}\left[\sum_{k=1}^{3}u_{x_k}\vartheta_{x_k}-r_D^{-2}u\vartheta\right]dx.$$

We denote $|u|_m=\sqrt{a_m(u,u)}$, $m=\overline{1,3}$, the energy seminorms in H, corresponding to bilinear forms $a_m(u,\vartheta)$. The energy space H_{A_m}, generated by seminorm $|u|_m$, is equivalent to space $H=\overset{\circ}{W}{}_2^1(\Omega)$ [17], therefore, the following estimates $0\le a_m(u,u)\le C_m\|u\|_1^2$, $m=\overline{1,3}$, are true, where C_m are the positive constants depending on ω, r_D.

2 *Discretization in space*

We discretize the problem in terms of space variables using the finite element method. Let $H_h\subset H$ be the set of elements of the form $\vartheta_h=\sum\limits_{m=1}^{M}a_m\Phi_m(x)$. Here $\{\Phi_m=\Phi_m(x)\}_{m=1}^{M}$ is the basis of piecewise polynomial functions that are a degree p polynomial on each finite element [18, 19].

Let us give an example of a basis based on third degree polynomials. Let us introduce a partition of domain Ω into $M=N_1*N_2*N_3$ parallelepipeds:

$$\Omega_{ijk}=\{(i-1)h_1\le x_1\le ih_1,\ (j-1)h_2\le x_2\le jh_2,\ \ (k-1)h_3\le x_3\le kh_3\},$$

$$i=\overline{1,N_1},\ j=\overline{1,N_2},\ k=\overline{1,N_3},\ h_s=l_s/N_s,\ s=1,2,3.$$

We choose a system of basis functions:

$$\Phi_{ijk}(x_1,x_2,x_3)=\varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3),\ i=\overline{1,N_1},\ j=\overline{1,N_2},\ k=\overline{1,N_3},$$

where $\varphi_l(x)$ is the basis function built on the basis of the B_3-spline [18]. In this case $p=3$.

Let us put the semidiscrete problem for $t\in[0,T]$ in correspondence with (5), (6):

$$a_3\left(\frac{d^4u_h(t)}{dt^4},\vartheta_h\right)+a_2\left(\frac{d^2u_h(t)}{dt^2},\vartheta_h\right)+\ a_1\left(u_h,\vartheta_h\right)=\left(f(t),\vartheta_h\right),\tag{7}$$

$$\left(\frac{d^\nu u_h}{dt^\nu}(0)-u_{0,\nu},\vartheta_h\right)=0,\nu=\overline{0,3},\forall\vartheta_h(x)\in H_h.\tag{8}$$

Problem (7), (8) corresponds to the following Cauchy problem:

$$D\frac{d^4u_h(t)}{dt^4}+B\frac{d^2u_h(t)}{dt^2}+Au_h(t)=f_h(t),\ \frac{d^\nu u_h}{dt^\nu}(0)=u_{1,\nu,h},\ \ \nu=\overline{0,3}.\tag{9}$$

Operators $D,\ B,\ A$ operate from H_h to H_h. They correspond to stiffness matrices $D=a_3(\varphi_l,\varphi_m)_{l,m=1}^{M}$, $B=a_2(\varphi_l,\varphi_m)_{l,m=1}^{M}$, $A=a_1(\varphi_l,\varphi_m)_{l,m=1}^{M}$. Besides, $u_{k,h}=P_hu_k(x)$, $k=\overline{0,3}$, where P_h is the projection operator $P_hH=H_h$.

The boundary conditions are approximated exactly.

3 Discretization in time

Following [20], problem (9) is approximated by the finite element method. Its generalized solution is defined as a continuous function $u(t) \in C^2[0,T]$ satisfying the following integral identity for arbitrary function $\vartheta(t) \in C^2(t_b, t_f)$

$$\int_{t_b}^{t_f} (D\ddot{u}\ddot{\vartheta} - B\dot{u}\dot{\vartheta} + Au\vartheta)dt + \left[D\dddot{u}\vartheta - D\ddot{u}\dot{\vartheta} + B\dot{u}\vartheta\right]\Big|_{t_b}^{t_f} = \int_{t_b}^{t_f} (f, \vartheta)\, dt, \tag{10}$$

where $0 \le t_b \le t_f \le T,\ \dot{u} = du/dt,\ \ddot{u} = d^2u/dt^2,\ \dddot{u} = d^3u/dt^3$.

On the segment $[0,T]$, we introduce uniform grid $\overline{\omega}_\tau = \{t_n = n\tau,\ n = 0,1,...;\ \tau > 0\}$. On each of intervals (t_n, t_{n+1}), we seek an approximate solution to problem (9) in the form of fifth degree polynomials

$$y(t) = \varphi_{00}^n(t)y^n + \varphi_{01}^n(t)y^{n+1} + \varphi_{10}^n(t)\dot{y}^n + \varphi_{11}^n(t)\dot{y}^{n+1} + \varphi_{20}^n(t)\ddot{y}^n + \varphi_{21}^n(t)\ddot{y}^{n+1}, \tag{11}$$

where $y^n = y(t_n),\ y^{n+1} = y(t_{n+1}),\ \dot{y}^n = dy(t_n)/dt,\ \dot{y}^{n+1} = dy(t_{n+1})/dt,\ \ddot{y}^n = d^2y(t_n)/dt^2,$ $\ddot{y}^{n+1} = d^2y(t_{n+1})/dt^2,\ \varphi_{00}^n(t) = -6\xi^5 + 15\xi^4 + 6\xi^5 - 10\xi^3 + 1,\ \varphi_{01}^n(t) = 6\xi^5 - 15\xi^4 + 10\xi^3,$ $\varphi_{10}^n(t) = \tau(-3\xi^5 + 8\xi^4 - 6\xi^3 + \xi),\ \varphi_{11}^n(t) = \tau(-3\xi^5 + 7\xi^4 - 4\xi^3),\ \varphi_{20}^n(t) = \tau^2(-\xi^5/2 + 3\xi^4/2$ $-3\xi^3/2 + \xi^2/2),\ \varphi_{21}^n(t) = \tau(\xi^5/2 - \xi^4 + \xi^3/2),\ \xi = (t - t_n)/\tau$.

Choosing weight functions $\vartheta(t)$, in the form of linear combinations of interpolation functions and substituting them into (10), we obtain the following parametric difference scheme

$$\begin{aligned} &D_\eta \dot{y}_t - \eta\tau^2 A y^{(0.5)} - D\ddot{y}^{(0.5)} = \varphi_1, \\ &D_\gamma y_t - D_\gamma \dot{y}^{(0.5)} + \eta\tau^2 D\ddot{y}_t = \varphi_2, \\ &D_\alpha \dot{y}_t - D_\beta \ddot{y}^{(0.5)} - \eta\tau^2 A y^{(0.5)} = \varphi_3, \end{aligned} \tag{12}$$

where

$$D_m = D - m\tau^2 B,\ m = \alpha, \beta, \gamma, \eta,\ \varphi_1 = -\frac{\tau}{6}\int_{t_n}^{t_{n+1}} f(t)dt = -\frac{\tau^2}{6}\int_0^1 f(t_n + \tau\xi)d\xi,$$

$$\varphi_2 = -\frac{7\tau}{60}\int_{t_n}^{t_{n+1}} f(t)\vartheta_2^{(\gamma,\eta)}(t)dt = -\frac{7\tau^2}{60}\int_0^1 f(t_n + \tau\xi)[s_1\vartheta_2^{(1)}(\xi) + s_2\vartheta_2^{(5)}(\xi)]d\xi,$$

$$\varphi_3 = -\frac{10}{\tau}\int_{t_n}^{t_{n+1}} f(t)\vartheta_3^{(\alpha,\beta,\eta)}(t)dt = -10\int_0^1 f(t_n + \tau\xi)[s_3\vartheta_3^{(2)} + s_4\vartheta_3^{(4)}]d\xi,\ \vartheta_2^{(\gamma,\eta)} = s_1\vartheta_2^{(1)} + s_2\vartheta_2^{(5)},$$

$$\vartheta_2^{(1)} = \tau(\xi - 1/2), \vartheta_2^{(5)} = \tau(3\xi^5 + 15\xi^4/2 - 5\xi^3 + \xi/2), s_1 = 3 - 120\gamma, s_2 = 14 - 840\gamma,$$

$$\vartheta_3^{(\alpha,\beta,\eta)} = s_3\vartheta_3^{(2)} + s_4\vartheta_3^{(4)}, \vartheta_3^{(2)} = \tau^2\xi(\xi-1)/2, \vartheta_3^{(4)} = \tau^2\xi^2(\xi-1)^2/4, s_3 = 140\alpha + 15,\ s_4 = 1400\alpha + 140,$$

here $\alpha,\ \beta,\ \gamma,\ \eta$ - are some constants.

The first initial condition is approximated exactly. The remaining initial conditions are approximated as in [16], by the fourth-order approximation, using the Taylor series and initial equations:

$$\dot{y}(0) = u_{0,1} + \frac{\tau}{2}\left(E - \frac{\tau^2}{12}D^{-1}B\right)u_{0,2} + \frac{\tau^2}{6}u_{0,3} + \frac{\tau^3}{24}D^{-1}[f(0) - Au_{0,0}],$$

$$\ddot{y}(0) = u_{0,2} + \tau u_{0,3} + \frac{\tau^2}{2} D^{-1}[f(0) - Bu_{0,2} - Au_{0,0}] + \frac{\tau^3}{4} D^{-1}[\dot{f}(0) - B\dot{u}_{0,2} - A\dot{u}_{0,0}].$$

It is easy to check that the scheme has the fourth order of approximation error on smooth solutions, i.e. $\psi_1 = O(\tau^4)$, $\psi_2 = O(\tau^4)$, $\psi_3 = O(\tau^4)$ if the following conditions are met

$$\alpha - \beta = 1/12, \ \eta = 1/12, \tag{13}$$

γ is an arbitrary constant.

4 Estimation of accuracy in space

Theorem 1. Let $u(x,t)$, $\frac{\partial u}{\partial t}(x,t) \in L_2\{[0,T]; W_2^{k+1}(\Omega) \cap \overset{\circ}{W}{}_2^1\}$. If the narrowing of space H_h to a separate finite element is a k degree polynomial, then for solving problem $u_h(t) \in H_h$ (9) approximating problem (2)–(4), the following accuracy estimate holds

$$\left\| \frac{\partial u}{\partial t}(x,t) - \frac{\partial u_h}{\partial t}(x,t) \right\|_1 + \|u(x,t) - u_h(x,t)\|_1 + \int_0^t \left\| \frac{\partial^2 u}{\partial t^2}(x,t') - \frac{\partial^2 u_h}{\partial t^2}(x,t') \right\|_1 dt' +$$

$$+ \int_0^t \left\| \frac{\partial u}{\partial t}(x,t') - \frac{\partial u_h}{\partial t}(x,t') \right\|_1 dt' \le Mh^k \left(\sqrt{\int_0^t \|u(x,t')\|_{k+1}^2 \, dt'} + \sqrt{\int_0^t \left\| \frac{\partial u}{\partial t}(x,t') \right\|_{k+1}^2 dt'} \right),$$

$$\forall t \in [0,T], \ M = M(r_D, \omega) > 0.$$

Proof. We integrate identity (5) over t from t_n to $t_{n+1} = t_n + \tau$, and applying the integration-by-parts formula, we obtain:

$$\int_{t_n}^{t_{n+1}} \Big[a_3(\ddot{u}(t), \ddot{\vartheta}) - a_2(\dot{u}(t), \dot{\vartheta}) + a_1(u(t), \vartheta) \Big](t)dt + a_3(\dddot{u}(t), \vartheta)|_{t_n}^{t_{n+1}} - a_3 \, (\ddot{u}(t), \dot{\vartheta})\Big|_{t_n}^{t_{n+1}} +$$

$$+ a_2(\dot{u}(t), \vartheta)|_{t_n}^{t_{n+1}} = \int_{t_n}^{t_{n+1}} (f(t), \vartheta)dt, \ \forall \vartheta(x) \in H_h. \tag{14}$$

Likewise, from (7) we obtain

$$\int_{t_n}^{t_{n+1}} \Big[a_3(\ddot{u}_h, \ddot{\vartheta}_h) - a_2(\dot{u}_h, \dot{\vartheta}_h) + a_1(u_h, \vartheta_h) \Big](t)dt + a_3(\dddot{u}_h(t), \vartheta_h)|_{t_n}^{t_{n+1}} - a_3 \, (\ddot{u}_h, \dot{\vartheta}_h)\Big|_{t_n}^{t_{n+1}} +$$

$$+ a_2(\dot{u}_h, \vartheta_h)|_{t_n}^{t_{n+1}} = \int_{t_n}^{t_{n+1}} (f(t), \vartheta_h)dt, \ \forall \vartheta_h(x) \in H_h.$$

Choosing $\vartheta = \vartheta_h \in H_h \subset H$ from (14) and subtracting both obtained identities, we have:

$$\int_{t_n}^{t_{n+1}} \Big[a_3(\ddot{z}_h, \ddot{\vartheta}_h) - a_2(\dot{z}_h, \dot{\vartheta}_h) + a_1(z_h, \vartheta_h) \Big](t)dt + a_3(\dddot{z}_h, \vartheta_h)|_{t_n}^{t_{n+1}} - a_3 \, (\ddot{z}_h, \dot{\vartheta}_h)\Big|_{t_n}^{t_{n+1}} +$$

$$+ a_2(\dot{z}_h, \vartheta_h)|_{t_n}^{t_{n+1}} = 0, \ \forall \vartheta_h(x) \in H_h, \tag{15}$$

where $z_h = u - u_h,\ e_h = u - u_I,\ \xi_h = u_I - u_h,\ u_I = u_I(x,t)$ is the solution interpolant $u(x,t)$ in x [19]. Let us choose a test function

$$\vartheta_h(t) = -\int\limits_t^s \xi_h(t')dt' \in H_h,\ t < s;\ \vartheta_h(t) = 0,\ t \geq s,\ \dot{\vartheta}_h(t) = \xi_h(t),\ \vartheta_h(s) = \dot{\vartheta}_h(s) = 0.$$

Then, with $z_h = \xi_h + e_h$, identity (15) can be written in the following form:

$$\int\limits_{t_n}^{t_{n+1}} \left[a_3(\ddot{\xi}_h, \dot{\xi}_h) + a_2(\dot{\xi}_h, \xi_h) + a_1(\dot{\vartheta}_h, \vartheta_h)\right](t)dt + [a_3(\dddot{z}_h, \vartheta_h) - a_3(\ddot{z}_h, \dot{\vartheta}_h) + a_2(\dot{z}_h, \vartheta_h)]\Big|_{t_n}^{t_{n+1}} =$$

$$= -\int\limits_{t_n}^{t_{n+1}} \left[a_3(\ddot{e}_h, \dot{\xi}_h) + a_2(\dot{e}_h, \xi_h) + a_1(e_h, \vartheta_h)\right](t)dt.$$

Hence, given the following relations:

$$a_3(\ddot{\xi}_h,\, \dot{\xi}_h) = \frac{1}{2}\frac{d}{dt}a_3(\dot{\xi}_h,\, \dot{\xi}_h),\ a_2(\dot{\xi}_h,\, \xi_h) = \frac{1}{2}\frac{d}{dt}a_2(\xi_h,\, \xi_h),\ a_1(\dot{\vartheta}_h,\, \vartheta_h) = \frac{1}{2}\frac{d}{dt}a_1(\vartheta_h,\, \vartheta_h),$$

$$a_3(\ddot{e}_h,\, \dot{\xi}_h) = \frac{d}{dt}a_3(\dot{e}_h,\, \dot{\xi}_h) - a_3(\dot{e}_h,\, \ddot{\xi}_h),\ a_2(\dot{e}_h,\, \xi_h) = \frac{d}{dt}a_2(e_h,\, \xi_h) - a_2(e_h,\, \dot{\xi}_h),$$

we obtain

$$E_h(t_{n+1}) + 0.5a_1(\vartheta_h, \vartheta_h)(t_{n+1}) + [a_3(\dddot{z}_h, \vartheta_h) - a_3(\ddot{z}_h, \dot{\vartheta}_h) + a_2(\dot{z}_h, \vartheta_h)]\Big|_{t_n}^{t_{n+1}} =$$

$$= E_h(t_n) + 0.5a_1(\vartheta_h, \vartheta_h)(t_n) - [a_3(\dot{e}_h, \dot{\xi}_h)(t_{n+1}) - a_3(\dot{e}_h, \dot{\xi}_h)(t_n) + a_2(e_h, \xi_h)(t_{n+1}) -$$

$$-a_2(e_h, \xi_h)(t_n)] + \int\limits_{t_n}^{t_{n+1}} \left[a_3(\dot{e}_h, \ddot{\xi}_h) + a_2(e_h, \dot{\xi}_h) + a_1(e_h, \vartheta_h)\right](t)dt,$$

where $E_h(t) = 0.5[a_3(\dot{\xi}_h, \dot{\xi}_h) + a_2(\xi_h, \xi_h)]$. Now let us sum this equation over $n = \overline{1, m-1}$, where m corresponds to the time point $s = m\tau$:

$$\begin{gathered} E_h(s) + 0.5a_1(\vartheta_h, \vartheta_h)(s) + [a_3(\dddot{z}_h, \vartheta_h) - a_3(\ddot{z}_h, \dot{\vartheta}_h) + a_2(\dot{z}_h, \vartheta_h)]\,|_0^s = \\ = E_h(0) + 0.5a_1(\vartheta_h, \vartheta_h)(0) - [a_3(\dot{e}_h, \dot{\xi}_h)(s) - a_3(\dot{e}_h, \dot{\xi}_h)(0) + a_2(e_h, \xi_h)(s) - \\ -a_2(e_h, \xi_h)(0)] + \int\limits_0^s [a_3(\dot{e}_h, \ddot{\xi}_h) + a_2(e_h, \dot{\xi}_h) + a_1(e_h, \vartheta_h)](t)dt. \end{gathered} \tag{16}$$

Taking into account the properties of functions $\vartheta_h(t)$ and initial conditions $z_h(0) = \dot{z}_h(0) = \ddot{z}_h(0) = \dddot{z}_h(0) = 0,\ \xi_h(0) = \dot{\xi}_h(0) = \ddot{\xi}_h(0) = \dddot{\xi}_h(0) = 0$, from (16) we obtain

$$E_h(s) + 0.5a_1(\vartheta_h, \vartheta_h)(0) = \int\limits_{t_n}^{t_{n+1}} [a_3(\dot{e}_h, \ddot{\xi}_h) + a_2(e_h, \dot{\xi}_h) + a_1(e_h, \vartheta_h)](t)dt. \tag{17}$$

Let us introduce one more function

$$w_h(t) = \int_0^t \xi_h(t')dt' \in H_h,\ t < s;\ w_h(t) = 0,\ t \geq s.$$

Then, $\vartheta_h(t) = w_h(t) - w_h(s)$ and from (17) we have the energy identity:

$$E_h(s) + 0.5a_1(w_h, w_h)(s) = \int_0^s [a_3(\dot{e}_h, \ddot{\xi}_h) + a_2(e_h, \dot{\xi}_h) + a_1(e_h, w_h(t) - w_h(s))]dt. \tag{18}$$

Let us estimate the terms on the right-hand side of (18):

$$\int_0^s a_3\,(\dot{e}_h, \ddot{\xi}_h)dt \leq \varepsilon_1 \int_0^s a_3(\ddot{\xi}_h, \ddot{\xi}_h)dt + \frac{1}{4\varepsilon_1}\int_0^s a_3(\dot{e}_h, \dot{e}_h)dt,$$

$$\int_0^s a_2(e_h, \dot{\xi}_h)dt \leq \varepsilon_2 \int_0^s a_2(\dot{\xi}_h, \dot{\xi}_h)dt + \frac{1}{4\varepsilon_2}\int_0^s a_2(e_h, e_h)dt,$$

$$\int_0^s a_1(e_h, w_h(t) - w_h(s))dt \leq \varepsilon_3 \int_0^s a_1(w_h(t), w_h(t))dt + s\varepsilon_3 a_1(w_h(s), w_h(s)) + \frac{1}{2\varepsilon_3}\int_0^s a_1(e_h, e_h)dt.$$

Choosing $\varepsilon_1 = \varepsilon_2 = 1/2$, and ε_3 from condition $\frac{\varepsilon_1}{2} + \varepsilon_3 T \leq \frac{3}{4}$, from (18) we have the following estimate:

$$\begin{gathered} E_h(s) + \int_0^s [a_3(\ddot{\xi}_h, \ddot{\xi}_h) + a_2(\dot{\xi}_h, \dot{\xi}_h)\,](t)dt + a_1(w_h,\, w_h)(s) \leq \\ \leq M\left(\int_0^s [a_1(w_h,\, w_h)(t)dt + \int_0^s [a_3(\dot{e}_h, \dot{e}_h) + a_2(e_h, e_h)\ + a_1(e_h, e_h)](t)dt\right), \end{gathered} \tag{19}$$

where $M - const$. Applying the Gronwall lemma for inequality (19), we obtain the error estimate

$$\begin{gathered} E_h(s) + \int_0^s [a_3(\ddot{\xi}_h, \ddot{\xi}_h) + a_2(\dot{\xi}_h, \dot{\xi}_h)\,](t)dt + a_1(w_h,\, w_h)(s) \leq \\ \leq \int_0^s [a_3(\dot{e}_h, \dot{e}_h) + a_2(e_h, e_h) + a_1(e_h, e_h)](t)dt, \end{gathered}$$

It is evident that $k_0 \left\|w_h(s)\right\|_1^2 \leq a(w_h, w_h)(s) \leq k_1 \left\|w_h(s)\right\|_1^2$, $a(\xi_h, \xi_h)(s) = \|\xi_h(s)\|_1^2$, $a(\dot{\xi}_h, \dot{\xi}_h)(s) = \left\|\dot{\xi}_h(s)\right\|_1^2$, $a(e_h, e_h)(s) = \|e_h(s)\|_1^2$, $a(\dot{e}_h, \dot{e}_h)(s) = \|\dot{e}_h(s)\|_1^2$, so for the error we have the final estimate:

$$\left\|\dot{\xi}_h(s)\right\|_1^2 + \|\xi_h(s)\|_1^2 + \int_0^s \left[\left\|\ddot{\xi}_h(t)\right\|_1^2 + \left\|\dot{\xi}_h(t)\right\|_1^2\right] dt \leq M \int_0^s \left[\|\dot{e}_h(t)\|_1^2 + \|e_h(t)\|_1^2\right] dt. \tag{20}$$

The following estimates hold for solution $u(x,t) \in W_2^{k+1}(\Omega),\ \forall t \in [0,T]$ [18], [19]:

$$\|\dot{e}_h(t)\|_1 \leq Mh^k \|\dot{u}(t)\|_{k+1}, \|e_h(t)\|_1 \leq Mh^k \|u(t)\|_{k+1}.$$

Therefore, based on (20) and triangle inequality $\|z_h\| \leq \|e_h\| + \|\xi_h\|$, the assertion of the theorem holds.

5 Estimation of accuracy in time

Let us now proceed to estimate the discretization error of problem (9) with respect to time. To approximate problem (9), scheme (12) is used, and to estimate the accuracy with respect to time variable, the Bramble-Hilbert lemma is used. Note that solution $u_h(t)$ of the semidiscrete problem (9) for each t is an element of the discrete subspace $u_h(t) \in H_h$.

Let us denote subspace H_τ of functions of argument t, which are Hermitian splines of the form (11) on interval $[t_n, t_{n+1}]$, $n = 0, 1, 2, \ldots$. Solution of scheme (12) is $y(t) \in H_\tau$. $y(t)$ is an element of subspace H_h for each t simultaneously. Actually $y(x,t) \in H_h^\tau = H_h \otimes H_\tau$.

The following theorem holds.

Theorem 2. Let $D^* = D > 0$, $B^* = B \geq 0$, $A^* = A > 0$. In addition, let the approximation conditions (13) and stability conditions be met

$$D - \mu\tau^2 A \geq \varepsilon D, \quad \forall\varepsilon \in (0,1), \quad \mu = \max\{\alpha, \beta, \gamma, \eta\}. \tag{21}$$

Then, for the solution of scheme (12) approximating the solution to problem (9) such that $\frac{d^4 u_h}{dt^4}(t) \in C[0,T]$, the following accuracy estimate holds

$$\|\dot{u}_h(t) - \dot{y}(t)\|_1 + \|u_h(t) - y(t)\|_1 + \int\limits_0^s \|\ddot{u}_h(t) - \ddot{y}(t)\|_1^2 \, dt +$$

$$+ \int\limits_0^s \|\dot{u}_h(t) - \dot{y}(t)\|_1^2 \, dt \leq M\tau^3 \sqrt{\int\limits_0^t \left\| \frac{d^4 u_h}{dt^4}(t') \right\|_1^2 dt'}, \ M - const.$$

Proof. Difference scheme (12) corresponds to the weak statement

$$\int\limits_{t_n}^{t_{n+1}} \Big[a_3(\ddot{y}(t), \ddot{\vartheta}_\tau) - a_2(\dot{y}(t), \dot{\vartheta}_\tau) + a_1(y(t), \vartheta_\tau)\Big] dt + a_3(\dddot{y}(t), \vartheta_\tau)\big|_{t_n}^{t_{n+1}} - a_3(\ddot{y}(t), \dot{\vartheta}_h)\Big|_{t_n}^{t_{n+1}} +$$

$$+ a_2(\dot{y}(t), \vartheta_\tau)\big|_{t_n}^{t_{n+1}} = \int\limits_{t_n}^{t_{n+1}} (f(t), \vartheta_\tau) dt, \ \forall\vartheta_\tau(x) \in H_h^\tau, \tag{22}$$

where $y(t)$ is the Hermitian spline (11). Choosing $\vartheta = \vartheta_\tau$ in (14) and subtracting the identity (22), we have the following identity for error: $\zeta_\tau(t) = u_h(t) - y(t)$:

$$\int\limits_{t_n}^{t_{n+1}} \Big[a_3(\ddot{\zeta}_\tau, \ddot{\vartheta}_\tau) - a_2(\dot{\zeta}_\tau, \dot{\vartheta}_\tau) + a_1(\zeta_\tau, \vartheta_\tau)\Big](t) dt + a_3\left(\dddot{\zeta}_\tau(t), \vartheta_\tau\right)\big|_{t_n}^{t_{n+1}} - a_3(\ddot{\zeta}_\tau, \dot{\vartheta}_\tau)\Big|_{t_n}^{t_{n+1}} +$$

$$+ a_2(\dot{\zeta}_\tau, \vartheta_\tau)\Big|_{t_n}^{t_{n+1}} = 0, \ \forall\vartheta_\tau \in H_h^\tau. \tag{23}$$

Let us represent $\zeta_\tau(t)$ as $\zeta_\tau(t) = u_h(t) - y(t) = u_h(t) - u_I^\tau(t) + u_I^\tau(t) - y(t)$, where $u_I^\tau(t)$ is interpolant $u_h(t)$, i.e. $u_I^\tau(t)$, as well as $y(t)$, is the Hermitian spline, such that $u_I^\tau(t_n) = u_h(t_n)$, $\dot{u}_I^\tau(t_n) = \dot{u}_h(t_n)$, $n = 0, 1, \ldots$. The scheme error is $\zeta_\tau(t) = \xi_\tau(t) + e_\tau(t)$, where $e_\tau = u_h - u_I^\tau$, $\xi_\tau = u_I^\tau - y$. We choose test function $\vartheta_\tau(t) = -\int\limits_t^s \xi_\tau(t) dt'$, $t < s$; $\vartheta_\tau(t) = 0$, $t \geq s$. Then identity (23) can be written as:

$$\int\limits_{t_n}^{t_{n+1}} \Big[a_3(\ddot{\xi}_\tau, \dot{\xi}_\tau) + a_2(\dot{\xi}_\tau, \xi_\tau) + a_1(\dot{\vartheta}_\tau, \vartheta_\tau)\Big] dt + \left[a_3(\dddot{\xi}_\tau, \dot{\vartheta}_\tau) - a_3(\ddot{\zeta}_\tau, \dot{\vartheta}_\tau) + a_2(\dot{\zeta}_\tau, \vartheta_\tau)\right]\Big|_{t_n}^{t_{n+1}} =$$

$$= -\int\limits_{t_n}^{t_{n+1}} [a_3(\ddot{e}_\tau, \xi_\tau) + a_2(\dot{e}_\tau, \xi_\tau) + a_1(e_\tau, \vartheta_\tau)]dt.$$

Hence, given the following relations:

$$a_3(\ddot{\xi}_\tau,\, \dot{\xi}_\tau) = \frac{1}{2}\frac{d}{dt}a_3(\dot{\xi}_\tau,\, \dot{\xi}_\tau),\;\; a_2(\dot{\xi}_\tau,\, \xi_\tau) = \frac{1}{2}\frac{d}{dt}a_2(\xi_\tau,\, \xi_\tau),\;\; a_1(\dot{\vartheta}_\tau,\, \vartheta_\tau) = \frac{1}{2}\frac{d}{dt}a_1(\vartheta_\tau,\, \vartheta_\tau),$$

$$a_3(\ddot{e}_\tau,\, \dot{\xi}_\tau) = \frac{d}{dt}a_3(\dot{e}_\tau,\, \dot{\xi}_\tau) - a_3(\dot{e}_\tau,\, \ddot{\xi}_\tau),\;\; a_2(\dot{e}_\tau,\, \xi_\tau) = \frac{d}{dt}a_2(e_\tau,\, \xi_\tau) - a_2(e_\tau,\, \dot{\xi}_\tau),$$

from the last identity we obtain

$$E_\tau(t_{n+1}) + 0.5a_1(\vartheta_\tau, \vartheta_\tau)(t_{n+1}) + [a_3(\dddot{\zeta}_\tau, \dot{\vartheta}_\tau) - a_3(\ddot{\zeta}_\tau, \dot{\vartheta}_\tau) + a_2(\dot{\zeta}_\tau, \vartheta_\tau)]\Big|_{t_n}^{t_{n+1}} =$$

$$= E_\tau(t_n) + 0.5a_1(\vartheta_\tau, \vartheta_\tau)(t_n) - [a_3(\dot{e}_\tau, \dot{\xi}_\tau)(t_{n+1}) - a_3(\dot{e}_\tau, \dot{\xi}_\tau)(t_n) + a_2(e_\tau, \xi_\tau)(t_{n+1}) - a_2(e_\tau, \xi_\tau)(t_n)] +$$
$$+ \int\limits_{t_n}^{t_{n+1}} \Big[a_3(\dot{e}_\tau, \ddot{\xi}_\tau) + a_2(e_\tau, \dot{\xi}_\tau) + a_1(e_\tau, \vartheta_\tau)\Big](t)dt,$$

where $E_\tau(t) = 0.5[a_3(\dot{\xi}_\tau, \dot{\xi}_\tau) + a_2(\xi_\tau, \xi_\tau)]$. Now let us sum this equation over $n = \overline{1, m-1}$, where m corresponds to the time point $s = m\tau$:

$$E_\tau(s) + 0.5a_1(\vartheta_\tau, \vartheta_\tau)(s) + [a_3(\dddot{\zeta}_\tau, \dot{\vartheta}_\tau) - a_3(\ddot{\zeta}_\tau, \dot{\vartheta}_\tau) + a_2(\dot{\zeta}_\tau, \vartheta_\tau)]\,|_0^s =$$

$$= E_\tau(0) + 0.5a_1(\vartheta_\tau, \vartheta_\tau)(0) - [a_3(\dot{e}_\tau, \dot{\xi}_\tau)(s) - a_3(\dot{e}_\tau, \dot{\xi}_\tau)(0) + a_2(e_\tau, \xi_\tau)(s) - a_2(e_\tau, \xi_\tau)(0)] +$$
$$+ \int\limits_0^s \Big[a_3(\dot{e}_\tau, \ddot{\xi}_\tau) + a_2(e_\tau, \dot{\xi}_\tau) + a_1(e_\tau, \vartheta_\tau)\Big](t)dt, \qquad (24)$$

Taking into account the properties of functions $\vartheta_\tau(t)$ and initial conditions $\zeta_\tau(0) = \dot{\zeta}_\tau(0) = \ddot{\zeta}_\tau(0) = \dddot{\zeta}_\tau(0) = 0$, $\xi_\tau(0) = \dot{\xi}_\tau(0) = \ddot{\xi}_\tau(0) = \dddot{\xi}_\tau(0) = 0$, we obtain from (24)

$$E_\tau(s) + 0.5a_1(\vartheta_\tau, \vartheta_\tau)(0) = \int\limits_{t_n}^{t_{n+1}} \Big[a_3(\dot{e}_\tau, \ddot{\xi}_\tau) + a_2(e_\tau, \dot{\xi}_\tau) + a_1(e_\tau, \vartheta_\tau)\Big](t)dt. \qquad (25)$$

We introduce one more function

$$w_\tau(t) = \int\limits_0^t \xi_\tau(t')dt' \in H_\tau,\;\; t < s;\;\; w_\tau(t) = 0,\;\; t \geq s.$$

Then, $\vartheta_\tau(t) = w_\tau(t) - w_\tau(s)$ and finally, from (25) we have the energy identity:

$$E_\tau(s) + 0.5a_1(w_\tau, w_\tau)(s) = \int\limits_0^s [a_3(\dot{e}_\tau, \ddot{\xi}_\tau) + a_2(e_\tau, \dot{\xi}_\tau) + a_1(e_\tau, w_\tau(t) - w_\tau(s))](t)dt. \qquad (26)$$

Let us estimate the terms on the right-hand side of (26):

$$\int\limits_0^s a_3(\dot{e}_\tau, \ddot{\xi}_\tau)dt \leq \varepsilon_1 \int\limits_0^s a_3\,(\ddot{\xi}_\tau, \ddot{\xi}_\tau)dt + \frac{1}{4\varepsilon_1}\int\limits_0^s a_3(\dot{e}_\tau, \dot{e}_\tau)dt,$$

$$\int_0^s a_2(e_\tau, \dot{\xi}_\tau)dt \le \varepsilon_2 \int_0^s a_2(\dot{\xi}_\tau, \dot{\xi}_\tau)dt + \frac{1}{4\varepsilon_2}\int_0^s a_2(e_\tau, e_\tau)dt,$$

$$\int_0^s a_1(e_\tau, w_\tau(t) - w_\tau(s))dt \le$$

$$\le \varepsilon_3 \int_0^s a_1(w_\tau(t), w_\tau(t))dt + s\varepsilon_3 a_1(w_\tau(s), w_\tau(s)) + \frac{1}{2\varepsilon_3}\int_0^s a_1(e_\tau, e_\tau)dt.$$

Choosing $\varepsilon_1 = \ \varepsilon_2 = 1/2$, and ε_3 from condition $\frac{\varepsilon_1}{2} + \varepsilon_3 T \le \frac{3}{4}$, we have the following estimate from (26):

$$E_\tau(s) + \int_0^s \Big[a_3(\ddot{\xi}_\tau, \ddot{\xi}_\tau) + a_2(\dot{\xi}_\tau, \dot{\xi}_\tau)\Big](t)dt + a_1(w_\tau,\, w_\tau)(s) \le$$

$$\le M\left(\int_0^s [a_1(w_\tau,\, w_\tau)(t)dt + \int_0^s \,[a_3(\dot{e}_\tau, \dot{e}_\tau) + a_2(e_\tau, e_\tau) + a_1(e_\tau, e_\tau)](t)dt\right),$$

where $M - const$. Hence, applying the Gronwall lemma, we obtain the error estimate

$$E_\tau(s) + \int_0^s \Big[a_3(\dot{e}_\tau, \ddot{\xi}_\tau) + a_2(e_\tau, \dot{\xi}_\tau)\Big](t)dt + a_1(w_\tau,\, w_\tau)(s) \le$$

$$\le \int_0^s [a_3(\dot{e}_\tau, \dot{e}_\tau) + a_2(e_\tau, e_\tau) + a_1(e_\tau, e_\tau)]dt.$$

Obviously, $k_0 \|w_\tau(s)\|_1^2 \le a(w_\tau, w_\tau)(s) \le k_1 \|w_\tau(s)\|_1^2$, $a(\xi_\tau, \xi_\tau)(s) = \|\xi_\tau(s)\|_1^2$, $a(\dot{\xi}_\tau, \dot{\xi}_\tau)(s) = \|\dot{\xi}_\tau(s)\|_1^2$, $a(e_\tau, e_\tau)(s) = \|e_\tau(s)\|_1^2$, $a(\dot{e}_\tau, \dot{e}_\tau)(s) = \|\dot{e}_\tau(s)\|_1^2$, so, we have the final estimate for the error:

$$\left\|\dot{\xi}_\tau(s)\right\|_1^2 + \|\xi_\tau(s)\|_1^2 + \int_0^s \left[\left\|\ddot{\xi}_\tau(t)\right\|_1^2 + \left\|\dot{\xi}_\tau(t)\right\|_1^2\right] dt \le M\left(\int_0^s \,[\|\dot{e}_\tau(t)\|_1^2 + \|e_\tau(t)\|_1^2]dt\right). \tag{27}$$

Linear bounded functionals $e_\tau(t)$, $\dot{e}_\tau(t)$ vanish on polynomials up to the third degree inclusive with respect to variable t. Then, based on the Bramble-Hilbert lemma, the following estimate holds [10], [13]:

$$\int_0^s \|e_\tau(t')\|_1^2\, dt' = \overline{M}^2\tau^8 \int_0^s \left\|\frac{d^4 u_h}{dt^4}(t)\right\|_1^2 dt, \int_0^s \|\dot{e}_\tau(t)\|_1^2\, dt \le \overline{M}^2\tau^6 \int_0^s \left\|\frac{d^4 u_h}{dt^4}(t)\right\|_1^2 dt. \tag{28}$$

Consequently, estimates (27), (28) imply the assertion of the theorem.

6 On convergence of the scheme

Note that in the estimate of Theorem 2, the error depends on solution $u_h(t)$ of the semidiscrete problem (9), while it is desirable to have smoothness conditions for the solution of original problem (2)–(4). To do this, we use the following estimate [18], [19]:

$$\|u_h\|_k = \|u - u + u_h\|_k \le \|u\|_k + \|u - u_h\|_k \le \|u\|_k + Ch|u|_{k+1} \le \bar{C}\|u\|_{k+1},\ k = 0, 1.$$

Constant $\bar{C}$ does not depend on h.

Consequently, the estimate in Theorem 2 takes the following form

$$\left\|\dot{\xi}_\tau(s)\right\|_1^2 + \|\xi_\tau(s)\|_1^2 + \int\limits_0^s \left[\left\|\ddot{\xi}_\tau(t)\right\|_1^2 + \left\|\dot{\xi}_\tau(t)\right\|_1^2\right] dt \le M\tau^3 \sqrt{\int\limits_0^t \left\|\frac{\partial^4 u}{\partial t^4}(x,t')\right\|_2^2 dt'}.$$

On the basis of Theorems 1 and 2, the following assertion holds.

Theorem 3. Let the conditions of Theorem 2 be satisfied. Then for the solution of scheme (12) approximating the solution of problem (2)–(4) such that $u(x,t),\ \frac{\partial u}{\partial t}(x,t) \in L_2\left\{[0,T];\ W_2^{k+1}(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)\right\}$, $\frac{\partial^4 u}{\partial t^4}(x,t) \in C\left\{[0,T]\ ;\ W_2^2(\Omega)\right\}$, the following accuracy estimate is true:

$$\left\|\frac{\partial u}{\partial t}(x,t) - \frac{\partial u_h}{\partial t}(x,t)\right\|_1 + \|u(x,t) - u_h(x,t)\|_1 + \int\limits_0^t \left\|\frac{\partial^2 u}{\partial t^2}(x,t') - \frac{\partial^2 u_h}{\partial t^2}(x,t')\right\|_1 dt' +$$

$$+ \int\limits_0^t \left\|\frac{\partial u}{\partial t}(x,t') - \frac{\partial u_h}{\partial t}(x,t')\right\|_1 dt' \le M\left\{h^k\left(\sqrt{\int\limits_0^t \|u(x,t')\|_{k+1}^2 dt'} + \sqrt{\int\limits_0^t \left\|\frac{\partial u}{\partial t}(x,t')\right\|_{k+1}^2 dt'}\right) + \right.$$

$$\left. + \tau^3 \sqrt{\int\limits_0^t \left\|\frac{\partial^4 u}{\partial t^4}(x,t')\right\|_2^2 dt'}\right\},\ \forall t \in [0,T], M = M(r_D, \omega) > 0.$$

When choosing a degree $k = 3$ polynomial on each finite element in space, we have the third-order accuracy in space steps h.

Let us verify the stability condition (21). We represent the operators of scheme (12) in the following form

$$D = A_1 + A_2 + A_3 - r_D^{-2}E,\ A = (\omega_{B_i}^2 + \omega_{p_i}^2)(A_1 + A_2 + A_3) - \omega_{B_i}^2 r_D^{-2}E + \omega_{p_i}^2\omega_{B_i}^2 A_3,$$

where operators $A_k \ge 0$ correspond to stiffness matrices $A_k = (b_k(\varphi_l, \varphi_m))_{l,m=1}^M$ with bilinear form $b_k(u, \vartheta) = \int\limits_\Omega (u_{x_k}\vartheta_{x_k})\, dx$. Condition (21) takes the following form

$$(1-\varepsilon)(A_1 + A_2 + A_3 - r_D^{-2})E \ge \mu\tau^2(\omega_{B_i}^2 + \omega_{p_i}^2)(A_1 + A_2 + A_3) - \omega_{B_i}^2 r_D^{-2}E + \omega_{p_i}^2\omega_{B_i}^2 A_3,$$

or with

$$\left\|A_1 + A_2 + A_3 - r_D^{-2}E\right\| / \left\|(\omega_{B_i}^2 + \omega_{p_i}^2)(A_1 + A_2 + A_3) - \omega_{B_i}^2 r_D^{-2}E + \omega_{p_i}^2\omega_{B_i}^2 A_3\right\| < 1,$$

we obtain $\tau^2 \le \frac{1-\varepsilon}{\mu}$, where $0 < \varepsilon < 1$.

This condition is interesting because the time step is not related to the space step and is determined by the scheme parameters. For the parameters of scheme (12), for example, for $\alpha = 1/10,\ \beta = 1/60,\ \gamma = 1/40,\ \eta = 1/12$ we have $\mu = 1/10$. So finally $\tau \le \sqrt{10(1-\varepsilon)}$.

7 Algorithm for implementing the scheme

Consider one of the possible algorithms for implementing scheme (12). We rewrite it in the following form

$$\begin{cases} m_{11}\hat{y} + m_{12}\hat{\dot{y}} + m_{13}\hat{\ddot{y}} = \Phi_1, \\ m_{21}\hat{y} + m_{22}\hat{\dot{y}} + m_{23}\hat{\ddot{y}} = \Phi_2, \\ m_{31}\hat{y} + m_{32}\hat{\dot{y}} + m_{33}\hat{\ddot{y}} = \Phi_3. \end{cases} \tag{29}$$

Here

$$m_{11} = -\eta\frac{\tau^3}{2}A,\ m_{12} = D_\eta,\ m_{13} = -\frac{\tau}{2}D,\ m_{21} = D_\gamma,\ m_{22} = -\frac{\tau}{2}D_\gamma,\ m_{23} = \eta\tau^2 D,$$

$$m_{31} = -\eta\frac{\tau^3}{2}A,\ m_{32} = D_\alpha,\ m_{33} = -\frac{\tau}{2}D_\beta,\ \Phi_1 = \tau\varphi_1 + \eta\frac{\tau^3}{2}Ay + D_\eta\dot{y} + \frac{\tau}{2}D\ddot{y},$$

$$\Phi_2 = \tau\varphi_2 + D_\gamma y + \frac{\tau}{2}D_\gamma\dot{y} + \eta\tau^2 D\ddot{y},\ \Phi_3 = \tau\varphi_3 + \eta\frac{\tau^3}{2}Ay + D_\alpha\ddot{y} + \frac{\tau}{2}D_\beta\ddot{y}.$$

Assuming the mutual commutability of operators D, B and A, we exclude $\hat{\ddot{y}}$ from the system of equations (29). As a result, we obtain the following system of equations

$$\begin{cases} g_{11}\hat{y} + g_{12}\hat{\dot{y}} = \widetilde{\Phi}_1, \\ g_{21}\hat{y} + g_{22}\hat{\dot{y}} = \widetilde{\Phi}_2, \end{cases} \tag{30}$$

where

$$g_{11} = m_{23}m_{11} - m_{13}m_{21},\ g_{12} = m_{23}m_{12} - m_{13}m_{22},\ g_{21} = m_{33}m_{11} - m_{13}m_{31},$$

$$g_{22} = m_{33}m_{12} - m_{13}m_{32},\ \tilde{\Phi}_1 = m_{23}\Phi_1 - m_{13}\Phi_2,\ \tilde{\Phi}_2 = m_{33}\Phi_1 - m_{13}\Phi_3.$$

Further, excluding $\hat{\dot{y}}$ from (30), we obtain

$$C\hat{y} = F \tag{31}$$

where $C = g_{22}g_{11} - g_{12}g_{21}$, $F = g_{21}\tilde{\Phi}_1 - g_{12}\tilde{\Phi}_2$.

After determining $\hat{y}$ from (31), we find $\hat{\dot{y}}$ from one of equations (30), for example, from the first equation

$$C_1\hat{\dot{y}} = F_1,$$

where $C_1 = g_{22}g_{12}$, $F_1 = g_{22}\tilde{\Phi}_1 - g_{22}g_{11}\hat{y}$. Then, the value of $\hat{\ddot{y}}$ is found from system (29), for example, also from the first equation $C_2\hat{\ddot{y}} = F_2$, where $C_2 = m_{13}$, $F_2 = \Phi_1 - m_{11}\hat{y} - m_{12}\hat{\dot{y}}$.

As is known, problems (5), (6) were obtained as a result of approximation of space variables, so, the matrices corresponding to operators $D,\ B,\ A$ are ill-conditioned and sparse. Then, the conditionality of matrix C also worsens. Therefore, the implementation of the scheme by directly solving equation (31) is not desirable, so, in the numerical modeling of problems with specific data, it is better to factorize operator C. In addition, operators $D,\ B,\ A$ may turn out to be degenerate. Then, to eliminate the problem of operator degeneracy, the regularization principle is applied, which allows applying the spectrum of shift-operators: $\widetilde{D} = D + \varepsilon E,\ \widetilde{B} = B + \varepsilon E,\ \widetilde{A} = A + \varepsilon E$ for self-adjoint operators. Here, $\varepsilon > 0$ is a small parameter setting the value of the spectrum of shift-operators. As a result, scheme (12) is replaced by a regularized scheme with operators $\widetilde{D},\ \widetilde{B},\ \widetilde{A}$.

8 Conclusions

A boundary value problem for the equation of ion-acoustic waves in a magnetized plasma was considered. On the basis of the finite element method, parametric difference schemes of high-order accuracy were constructed and investigated. A high-order accuracy of the scheme was achieved due to the special discretization of time and space variables. In addition, the presence of parameters in the scheme makes it possible to regularize the schemes in order to optimize the implementation algorithm and the accuracy of the scheme. The corresponding a priori estimates were obtained and, on their basis, theorems on the rate of convergence and accuracy of the constructed algorithms on the smoothness of solutions to the original differential problem were proved under weak assumptions. An algorithm for the implementation of these schemes was proposed. These schemes have certain advantages over other schemes – they are two-layer schemes of high-order accuracy, except the solution itself, its derivative (velocity) is determined with the same accuracy; using the interpolation representation (11), if necessary, a solution can be obtained at any time. In addition, to achieve a certain accuracy, it allows us to select large time steps, etc.

Based on these advantages, it is possible to study other boundary value problems, including nonlocal boundary value problems. Besides, these results can be transferred to loaded equations with local and nonlocal boundary conditions.

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Магниттелген плазмадағы ионды-акустикалық толқындардың теңдеуі үшін жоғары дәлдіктегі айырмашылық схемаларының жинақтылығы туралы

Уақыт бойынша төртінші ретті Соболев типті теңдеу үшін дәлдігі жоғары ақырлы элементтер әдісінің көппараметрлі айырымдық схемалары зерттелген. Атап айтқанда, магниттелген плазмадағы ионды-акустикалық толқындардың теңдеуіне арналған бірінші шекаралық есеп қарастырылған. Схеманың жоғары ретті дәлдігі уақыт пен кеңістік айнымалыларының арнайы дискретизациясының арқасында қол жеткізіледі. Схемада параметрлердің болуы схемалардың дәлдігін жоғарғы ретке келтіруге және іске асыру алгоритмін оңтайландыруға мүмкіндік береді. Әлсіз нормадағы априорлық бағалау энергетикалық теңсіздік әдісімен алынады. Осы бағалаудың және Брамбл-Гильберт леммасының негізінде жалпыланған шешімдер кластарында құрастырылған алгоритмдердің жинақтылығы дәлелденді. Айырымдық схеманы жүзеге асыру алгоритмі ұсынылған.

Кілт сөздер: Соболев типті теңдеу, айырымдық схемалар, ақырлы айырымдар әдісі, ақырлы элементтер әдісі, тұрақтылық, жинақтылық, дәлдік.

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О сходимости разностных схем повышенной точности для уравнения ионно-звуковых волн в замагниченной плазме

Исследованы многопараметрические разностные схемы метода конечных элементов высокого порядка точности для уравнения соболевского типа четвертого порядка по времени. В частности, рассмотрены первая краевая задача для уравнения ионно-звуковых волн в замагниченной плазме. Высокий порядок точности схемы достигается за счет специальной дискретизации временной и пространственных переменных. Наличие параметров в схеме позволяет произвести регуляризацию точности схем, а также оптимизацию алгоритма реализации. Методом энергетических неравенств получена априорная оценка в некоторой слабой норме. На основе этой оценки и леммы Брэмбла-Гильберта доказана сходимость построенных алгоритмов в классах обобщенных решений. Предложен алгоритм реализации разностной схемы.

Ключевые слова: уравнение соболевского типа, разностные схемы, метод конечных разностей, метод конечных элементов, устойчивость, сходимость, точность.

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DOI 10.31489/2022M4/20-33

UDC 517.948.34

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Asymptotic estimations of the solution for a singularly perturbed equation with unbounded boundary conditions

The paper studies a two-point boundary value problem with unbounded boundary conditions for a linear singularly perturbed differential equation. Asymptotic estimates are given for a linearly independent system of solutions of a homogeneous perturbed equation. Auxiliary, so-called boundary functions, the Cauchy function are defined. For sufficiently small values of the parameter, estimates for the Cauchy function and boundary functions are found. An algorithm for constructing the desired solution of the boundary value problem has been developed. A theorem on the solvability of a solution to a boundary value problem is proved. For sufficiently small values of the parameter, an asymptotic estimate for the solution of the inhomogeneous boundary value problem is established. The initial conditions for the degenerate equation are determined. The formula is determined; the phenomena of the initial jump are studied.

Keywords: two-point boundary value problem, initial jumps, degenerate problem, small parameter, initial function, boundary functions.

Introduction

Researchers [1–14] have developed efficient asymptotic methods for singularly perturbed problems. For sufficiently small values of the parameter, these methods make it possible to construct uniform asymptotic approximations. However, for some singularly perturbed two-point boundary value problems with initial jumps, the choice of an appropriate method for constructing asymptotic approximations without a preliminary study turns out to be almost impossible. The first studies devoted to the phenomena of initial jumps were the works of Vishik, Lyusternik [15] and Kasymov [16]. In [17–19] these studies were summarized and continued. The jump phenomenon in many real problems of practice is a significant component, which is taken into account when building a model of these processes. In this case, the value of the jump is the condition for the perturbed problem to be replaced by a degenerate problem. For example, a new justification for the Painlev paradox, the existence of contrast structures, and the jump phenomenon were established by Neimark and Smirnova [20]. Asymptotic behavior, jump phenomena of the solution of a general two-point perturbed boundary value problem with finite boundary conditions were considered in [21–23]. In these papers, using the formula for solving a two-point boundary value problem for sufficiently small values of the parameter, asymptotic estimates are established, a theorem on the solvability of a solution to a two-point boundary value problem is formulated and proved, and the phenomena of initial and boundary jumps are revealed.

1 *Statement of the problem*

The next natural continuation in this direction is the study of the asymptotic behavior of solutions to perturbed two-point boundary value problems with unbounded boundary conditions. This work is devoted to the consideration of such problems.

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Consider the following inhomogeneous differential equation:

$$L_\varepsilon y \equiv \varepsilon y''' + A(t)y'' + B(t)y' + C(t)y = F(t) \tag{1}$$

with unbounded boundary conditions of the form:

$$y(0,\varepsilon) = a_1, \quad y'(0,\varepsilon) = \frac{a_2}{\varepsilon}, \quad y(1,\varepsilon) = a_3 \tag{2}$$

where $\varepsilon > 0$ is a small positive parameter, $a_2 \neq 0$, a_i, $i = 1,2,3$ are known constants, $A(t)$, $B(t)$, $C(t)$, $F(t)$ are functions defined on the interval $[0,1]$.

In this paper, based on the analytical representation of the solution to problem (1), (2), the existence and uniqueness of the sought solution is proved.

Assume that the following conditions hold:

C1) $A(t), B(t), C(t), F(t)$ *are sufficiently smooth functions defined on the interval [0,1];*

C2) $A(t) \geq \gamma = const > 0, \ 0 \leq t \leq 1;$

C3) $\overline{J} = \begin{vmatrix} y_{10}(0) & y_{20}(0) \\ y_{10}(1) & y_{20}(1) \end{vmatrix} \neq 0;$

C4) Let $a_1 + \frac{a_2}{A(0)} \neq 0$.

2 The fundamental set of solutions to the homogeneous perturbed equation

Consider the following homogeneous equation associated with (1)

$$L_\varepsilon y(t,\varepsilon) \equiv \varepsilon y''' + A(t)y'' + B(t)y' + C(t)y = 0, \tag{3}$$

corresponding to the inhomogeneous equation (1). For the fundamental system of solutions to equation (3), the following lemma [1] is valid.

Lemma 1. If conditions (C1) and (C2) are satisfied, then the fundamental set of solutions $y_i(t,\varepsilon)$, $i = 1,2,3$ of (3) in the interval $0 \leq t \leq 1$ has the following asymptotic representation as $\varepsilon \to 0$:

$$\begin{cases} y_i^{(j)}(t,\varepsilon) = y_{i0}^{(j)}(t) + O(\varepsilon), \quad i=1,2, \quad j = 0,1,2, \\ y_3^{(j)}(t,\varepsilon) = \dfrac{1}{\varepsilon^j} exp\left(\dfrac{1}{\varepsilon}\displaystyle\int_0^t \mu(x)dx\right)\mu^j(t)y_{30}(t)[1+O(\varepsilon)], \quad j=0,1,2, \end{cases} \tag{4}$$

where $\mu(t) = -A(t) < 0, \ y_{i0}(t), \ i = 1,2,$ are solutions of the problem

$$L_0 y_{i0}(t) \equiv A(t)y_{i0}'' + B(t)y_{i0}' + C(t)y_{i0} = 0, \quad i = 1,2, \tag{5}$$

with initial conditions:

$$y_{10}(0) = 1, \ \ y_{10}'(0) = 0, \ \ y_{20}(0) = 0, \ \ y_{20}'(0) = 1,$$

Functions $y_{30}(t)$ has the form

$$y_{30}(t) = (A(0)/A(t))^2 exp\left(\int_0^t (B(x)/A(x))dx\right) \neq 0. \tag{6}$$

By applying asymptotic representation (4), for the $W[y_1(t,\varepsilon), y_2(t,\varepsilon), y_3(t,\varepsilon)]$ with sufficiently small ε we get

$$W(t,\varepsilon) = \frac{1}{\varepsilon^2} exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx\right) y_{30}(t)\mu^2(t)\overline{W}(t)[1+O(\varepsilon)] \neq 0, \tag{7}$$

where $\overline{W}(t)$ is the Wronsky determinant of the fundamental system of solutions $y_{i0}(t)$, i=1, 2, 3 of equations (5),(6), and $\overline{W}(t) \neq 0$.

3 Constructing the initial and boundary functions

Also as in previous works [17, 18], we introduce the initial function

$$K(t,s,\varepsilon) = \frac{W(t,s,\varepsilon)}{W(s,\varepsilon)}, \tag{8}$$

determined from the next problem:

$$L_\varepsilon K(t,s,\varepsilon) = 0, \quad K(s,s,\varepsilon) = 0, \quad K_t^{'}(s,s,\varepsilon) = 0, \quad K_t^{''}(s,s,\varepsilon) = 1, \tag{9}$$

where $W(t,s,\varepsilon)$ is the determinant obtained from the Wronskian $W(s,\varepsilon)$ by replacing the third row with the fundamental set of solutions $y_1(t,\varepsilon), y_2(t,\varepsilon), y_3(t,\varepsilon)$.

Obviously, the initial function $K(t,s,\varepsilon)$ satisfies equation (3) and initial conditions (9), where the function $K(t,s,\varepsilon)$ does not depend on the choice of the solution fundamental set for equation (3). Therefore, the initial function for equation (3) exists, it can be expressed by formula (8) and it is determined by a unique form.

Let us consider the determinant

$$J(\varepsilon) = \begin{vmatrix} y_1(0,\varepsilon) & y_2(0,\varepsilon) & y_3(0,\varepsilon) \\ y_1^{'}(0,\varepsilon) & y_2^{'}(0,\varepsilon) & y_3^{'}(0,\varepsilon) \\ y_1(1,\varepsilon) & y_2(1,\varepsilon) & y_3(1,\varepsilon) \end{vmatrix}. \tag{10}$$

Due to asymptotic estimation (4), elements of determinant (10) has next form $\varepsilon \to 0$:

$$y_i^j(0,\varepsilon) = y_{io}^j(0) + O(\varepsilon), \; i = 1,2, \; j = 0,1, \; y_i(1,\varepsilon) = y_{i0}(1) + O(\varepsilon), \; i = 1,2, \tag{11}$$

$$y_3(0,\varepsilon) = y_{30}[1+O(\varepsilon)], \; y_3^{'}(0,\varepsilon) = \frac{1}{\varepsilon} y_{30}(0)\mu(0)[1+O(\varepsilon)],$$

$$y_3(1,\varepsilon) = exp\left(\frac{1}{\varepsilon}\int\limits_0^1 \mu(x)dx\right)[y_{30}(1) + O(\varepsilon)].$$

Then the determinant $J(\varepsilon)$ taking into account (11) has the following representation as $\varepsilon \to 0$

$$J(\varepsilon) = -\frac{1}{\varepsilon}\mu(0)\overline{J}(1+O(\varepsilon)). \tag{12}$$

Definition 1. The functions $\Phi_i(t,\varepsilon)$, $i = 1,2,3$, are called boundary functions for boundary value problem (1) and (2), if they satisfy homogeneous equation (3) and boundary conditions

$$\Phi_i^{(j)}(0,\varepsilon) = \begin{cases} 1, & j = i-1, \; i\text{=}1,2, \\ 0, & j \neq i-1, \; i = 1,2,3, \; j = 0,1, \end{cases} \tag{13}$$

$$\Phi_i(1,\varepsilon) = \begin{cases} 1, & i=3, \\ 0, & i=1,2. \end{cases}$$

The following theorem is valid.

Theorem 1. If conditions (C1)–(C3) are satisfied, then the boundary functions $\Phi_i(t,\varepsilon)$, $i=1,2,3$, on the interval [0,1] exist, unique and can be expressed by formula:

$$\Phi_i(t,\varepsilon) = \frac{J_i(t,\varepsilon)}{J(\varepsilon)}, \quad i=1,2,3, \tag{14}$$

where $J_i(t,\varepsilon)$, $i=1,2,3$ is the determinant obtained from $J(\varepsilon)$ by replacing the i-th row with the fundamental set of solutions $y_1(t,\varepsilon)$, $y_2(t,\varepsilon)$, $y_3(t,\varepsilon)$.

Proof. We seek the boundary functions $\Phi_i(t,\varepsilon)$, $i=1,2,3$ in the next form which satisfy the condition (13)

$$\Phi_i(t,\varepsilon) = c_1^i y_1(t,\varepsilon) + c_2^i y_2(t,\varepsilon) + c_3^i y_3(t,\varepsilon), \ i=1,2,3. \tag{15}$$

where c_1^i, c_2^i, c_3^i are unknown constants which are defined from the function (15), that function satisfies boundary condition (13). Obviously, the function (15) depending on one variable t satisfies the homogeneous equation (3). By substituting (15) into (13), we obtain

$$\Phi_i^k(0,\varepsilon) = \begin{cases} 1, & k=i-1, \ i=1,2, \\ 0, & k\neq i-1, \ i=1,2,3, \ k=0,1, \end{cases} \tag{16}$$

$$\Phi_i(1,\varepsilon) = c_1^i y_1(1,\varepsilon) + c_2^i y_2(1,\varepsilon) + c_3^i y_3(1,\varepsilon) = \begin{cases} 1, & i=3, \\ 0, & i=1,2. \end{cases}$$

With a fixed value i system (16) has a linear algebraic system of equations for determining c_1^i, c_2^i, c_3^i, which determinant is $J(\varepsilon)$. Then, by means of (12) for a sufficiently small ε the system (16) is uniquely solvable. Solving (16), we have

$$c_k^i = \frac{J_{ik}}{J(\varepsilon)}, \quad i=1,2,3, \tag{17}$$

where J_{ik} is the algebraic complement of the determinant element $J(\varepsilon)$, at the intersection of the $i-th$ row and $k-th$ column. Substituting (17) into (16) and comparing the decomposition obtained with the determinant decomposition $J_i(t,\varepsilon)$ by elements $i-th$ row, we get formula (14). Hereby, the functions $\Phi_i(t,\varepsilon)$, $i=1,2,3$, defined by the formula (14) satisfy the equation (3) and boundary condition (13). Consequently, functions $\Phi_i(t,\varepsilon)$, $i=1,2,3$, are defined on the interval $0\le t\le 1$, are boundary functions of the perturbed problem (1), (2). The theorem is proved.

Lemma 2. The initial function $K(t,s,\varepsilon)$ and its derivatives by variable t to the second order are defined on the interval [0,1] at $s\le t$ have following asymptotic representation as $\varepsilon\to 0$:

$$K^{(j)}(t,s,\varepsilon) = \frac{\varepsilon}{\mu(s)\overline{W}(s)}\left[-\overline{W}^j(t,s) + \varepsilon^{1-j}\frac{y_{30}(t)\mu^j(t)}{y_{30}(s)\mu(s)} exp\left(\frac{1}{\varepsilon}\int_s^t \mu(x)dx\right)\cdot\right.$$
$$\left.\cdot\,\overline{W}(s) + O\left(\varepsilon + \varepsilon^{2-j} exp\left(\frac{1}{\varepsilon}\int_s^t \mu(x)dx\right)\right)\right], \quad j=0,1,2, \tag{18}$$

where $\overline{W}(t,s) = \begin{vmatrix} y_{10}(s) & y_{20}(s) \\ y_{10}(t) & y_{20}(t) \end{vmatrix}$.

Proof . We expand $W(t,s,\varepsilon)$ by the elements of the third column:

$$W^{(j)}(t,s,\varepsilon) = y_3(s,\varepsilon)W_{13}(t,s) - y_3^{'}(s,\varepsilon)W_{23}(t,s) + y_3^{(j)}(t,s)W_{33}(s,\varepsilon), \tag{19}$$

where the minors $W_{13}(t,s)$ $(i=1,2,3)$ by virtue of (11) as $\varepsilon \to 0$ can be represented in the form

$$W_{13}(t,s,\varepsilon) = y_{10}^{'}(s)y_{20}^{(j)}(t) - y_{20}^{'}(s)y_{10}^{(j)}(t) + O(\varepsilon), \tag{20}$$

$$W_{23}(t,\varepsilon) = \overline{W}^{(j)}(t,s) + O(\varepsilon),$$

$$W_{33}(s,\varepsilon) = \overline{W}(s) + O(\varepsilon)$$

where $\overline{W}^{j}(t,s)$ is determinant obtained from the $\overline{W}t,s$ after deleting second row to $y_{10}^{(j)}(t)$, $y_{20}^{(j)}(t)$. Then, taking into account estimates (20) and (4), from (19) for the function $W^{(j)}(t,s,\varepsilon)$ we have following representation as $\varepsilon \to 0$

$$W^{(j)}(t,s,\varepsilon) = \frac{1}{\varepsilon}exp\left(\frac{1}{\varepsilon}\int\limits_0^s \mu(x)dx\right) y_{30}(s)\mu(s)\left[-\overline{W}^{(j)}(t,s) + \right.$$
$$+\varepsilon^{1-j}exp\left(\frac{1}{\varepsilon}\int\limits_s^t \mu(x)dx\right)\cdot\frac{y_{30}(t)\mu^{(j)}(t)}{y_{30}(s)\mu^{(j)}(s)}\overline{W}(s) +$$
$$\left. +O\left(\varepsilon + \varepsilon^{2-j}exp\left(\frac{1}{\varepsilon}\int\limits_s^t \mu(x)dx\right)\right)\right]. \tag{21}$$

Use (21) and (7) in (8) we obtain estimate (18). Lemma is proved.

Lemma 3. Under conditions (C1)–(C3), on the interval for the boundary functions $\Phi_i(t,\varepsilon)$, $i = 1,2,3$, the following asymptotic representation holds as $\varepsilon \to 0$

$$\Phi_1^{(j)}(t,\varepsilon) = \frac{\overline{J}_1^{(j)}(t)}{\overline{J}} - \frac{\varepsilon}{\varepsilon^j}exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx\right)\frac{y_{30}(t)\mu^j(t)}{\overline{J}y_{30}(0)\mu(0)}\frac{\overline{J}_1^{(2)}(0)}{\overline{J}} +$$
$$+O\left(\varepsilon + \frac{\varepsilon^2}{\varepsilon^j}exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx\right)\right), \tag{22}$$

$$\Phi_2^{(j)}(t,\varepsilon) = -\varepsilon\frac{\overline{J}_1^{(j)}(t)}{\mu(0)\overline{J}} + \frac{\varepsilon}{\varepsilon^j}exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx\right)\frac{y_{30}(t)\mu^j(t)}{y_{30}(0)\mu(0)} +$$
$$+O\left(\varepsilon^2 + \frac{\varepsilon^2}{\varepsilon^j}exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx\right)\right),$$

$$\Phi_3^{(j)}(t,\varepsilon) = \frac{\overline{J}_2^{(j)}(t)}{\overline{J}} - \frac{\varepsilon}{\varepsilon^j}exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx\right)\frac{y_{30}(t)\mu^j(t)}{y_{30}(0)\mu(0)}\frac{\overline{J}_2^{(2)}(0)}{\overline{J}} +$$

$$+O\left(\varepsilon+\frac{\varepsilon^2}{\varepsilon^j}exp\left(\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx\right)\right), \quad j=0,1,2,$$

where $\overline{J}_i^{(j)}(t)$ is the determinant obtained from $\overline{J}$ by replacing the $i-th$ row with the fundamental set of solutions $y_{10}^{(j)}(t)$, $y_{20}^{(j)}(t)$.

Proof. By spreading $J_i^{(j)}(t,\varepsilon)$ the element of the third column and taking into account the estimation (4), we have

$$J_1^{(j)}(t,\varepsilon)=-\frac{1}{\varepsilon}y_{30}(0)\mu(0)\left[\overline{J}_1^{(j)}(t)+\varepsilon^{1-j}\frac{y_{30}(t)\mu^j(t)}{y_{30}(0)\mu(0)}exp\left(\frac{1}{s}\int\limits_0^t\mu(x)dx\right)\overline{J}_1^{(2)}(0)+\right.$$

$$\left.+O\left(\varepsilon+\varepsilon^{2-j}exp\left(\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx\right)\right)\right],$$

$$J_2^{(j)}(t,\varepsilon)=y_{30}(0)\left[\overline{J}_1^{(j)}(t)-\varepsilon^{-j}\frac{y_{30}(t)\mu^j(t)}{y_{30}(0)}exp\left(\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx\right)\overline{J}+\right.$$

$$\left.+O\left(\varepsilon+\varepsilon^{1-j}exp\left(\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx\right)\right)\right], \tag{23}$$

$$J_3^{(j)}(t,\varepsilon)=-\frac{1}{\varepsilon}y_{30}(0)\mu(0)\left[\overline{J}_2^{(j)}(t)-\varepsilon^{1-j}\frac{y_{30}(t)\mu^j(t)}{y_{30}(0)\mu(0)}exp\left(\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx\right)\cdot\right.$$

$$\left.\cdot\,\overline{J}_2^{(2)}(0)+O\left(\varepsilon+\varepsilon^{2-j}exp\left(\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx\right)\right)\right].$$

Then, using (12) and (23) in (14), we get (22). Lemma is proved.

4 Constructing the solution of the boundary value problem

Theorem 2. If conditions (C1)–(C3) are satisfied then for sufficiently small $\varepsilon>0$ boundary value problem (1), (2) on the interval [0,1] has a unique solution $y(t,\varepsilon)$, which can be presented in the following form

$$y(t,\varepsilon)=a_1\Phi_1(t,\varepsilon)+\frac{a_2}{\varepsilon}\Phi_2(t,\varepsilon)+a_3\Phi_3(t,\varepsilon)-$$

$$-\Phi_3(t,\varepsilon)\frac{1}{\varepsilon}\int\limits_0^1K(1,s,\varepsilon)F(s)ds+\int\limits_0^tK(t,s,\varepsilon)F(s)ds. \tag{24}$$

Proof. We seek the solution $y(t,\varepsilon)$ of BVP (1), (2) in the form:

$$y(t,\varepsilon)=C_1\Phi_1(t,\varepsilon)+C_2\Phi_2(t,\varepsilon)+C_3\Phi_3(t,\varepsilon)+\frac{1}{\varepsilon}\int\limits_0^tK(t,s,\varepsilon)F(s)ds, \tag{25}$$

where $C_i,\ \ i = 1, 2, 3$, are unknown constants. By directly substituting (25) in (1) we make sure that the function $y(t,\varepsilon)$ is defined by formula (25) is a solution of equation (1). For determination $C_i,\ i = 1, 2, 3$, we use (25) in (2). Then we will have:

$$C_1 = a_1,\quad C_2 = \frac{a}{\varepsilon},\quad C_3 = a_3 - \frac{1}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds. \tag{26}$$

Substituting found values (26) into (25), we obtain (24). From here and from that, the boundary function $\Phi_i(t,\varepsilon)$ does not depend on the choice of the fundamental solution system of equation (3) it follows that solutions of the boundary value problem (1), (2) exist, are unique and are expressed by formula (25). The theorem is proved.

Theorem 3. Under conditions (C1)–(C3), for the solution $y(t,\varepsilon)$ of the boundary value problem (1) and (2) in the interval [0,1] the following asymptotic estimations hold uniformly by variable t and as $\varepsilon \to 0$

$$|y(t,\varepsilon)| \le \Big[|a_1 - \frac{a_2}{\mu(0)}||\overline{\Phi}_1(t)| + |a_3||\overline{\Phi}_2(t)| +$$
$$+|a_2|\left(\exp\left(-\gamma\frac{t}{\varepsilon}\right)\right) + \max_{0\le t\le 1}|F(t)|\Big], \tag{27}$$

where $C \ge 0$ are constants independent of t and ε, functions $\overline{\Phi}_1(t)=\frac{\overline{J}_1(t)}{\overline{J}}$, $\overline{\Phi}_2(t)=\frac{\overline{J}_2(t)}{\overline{J}}$ satisfy the degenerate homogeneous equation

$$A(t)\overline{y}^{''} + B(t)\overline{y}^{'} + C(t)\overline{y} = 0 \tag{28}$$

and boundary conditions

$$\begin{aligned} \overline{\Phi}_1(0) = 1, \quad & \overline{\Phi}_2(0) = 0, \\ \overline{\Phi}_1(1) = 0, \quad & \overline{\Phi}_2(1) = 1. \end{aligned} \tag{29}$$

Proof . In (24) the expression $\frac{1}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds$ can be expressed in the next form

$$\frac{1}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds = -\int\limits_0^1 \overline{K}(1,s)\frac{F(s)}{\mu(s)}ds + O(\varepsilon) =$$
$$= \int\limits_0^1 \overline{K}(1,s)\frac{F(s)}{A(s)}ds + O(\varepsilon), \tag{30}$$

where the function

$$\overline{K}(t,s) = \frac{\overline{W}(t,s)}{\overline{W}(s)}, \tag{31}$$

is an initial function of equation (28), by means of (31) and by variable t satisfies equation (28) and initial conditions

$$\overline{K}(s,s) = 0, \overline{K'}(s,s) = 1$$

and, the function $\overline{K}(t,s)$ does not depend on the choice of the fundamental set of solution $y_{10}(t)$, $y_{20}(t)$ to equation (28). The function

$$\overline{\Phi}_k(t) = \frac{\overline{J}_k(t)}{\overline{J}}, \quad k = 1, 2, \tag{32}$$

satisfies the degenerate homogeneous equation (28) and the boundary conditions (29).

Consequently, functions (32) are boundary functions of the unknown degenerate problem. The function $\overline{\Phi}_k(t)$ does not depend on the choice of the fundamental set of solution $y_{10}(t)$, $y_{20}(t)$ of equation (28) too. From (24), and by means of (30),(18), (22), we have

$$y(t,\varepsilon) = a_1\frac{\overline{J}_1(t)}{\overline{J}} + a_2\left[-\frac{\overline{J}_1(t)}{\mu(0)\overline{J}} + exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)\frac{y_{30}(t)}{y_{30}(t)\mu(0)}\right] - \tag{33}$$

$$-a_3\frac{\overline{J}_2(t)}{\overline{J}} - \frac{\overline{J}_2(t)}{\overline{J}}\int_0^1 \overline{K}(1,s)\frac{F(s)}{A(s)}ds + \int_0^t \overline{K}(t,s)\frac{F(s)}{A(s)}ds + O(\varepsilon).$$

Hence, we obtain the desired estimate (27). The theorem is proved.

5 *Constructing a solution of the unperturbed problem*

Now we formulate a degenerate problem. Let's consider the degenerate equation

$$L_0\overline{y} \equiv A(t)\overline{y}^{''} + B(t)\overline{y}^{'} + C(t)\overline{y} = F(t). \tag{34}$$

In order to select the boundary conditions of the degenerate problem, we turn to the estimate (27). In the first approximation, in these estimates, for the boundary functions of the degenerate problem, there are constants $a_1 - \frac{a_2}{\mu(0)}$, a_3 Taking into account this consideration, we construct a solution to equation (34) under the conditions:

$$\overline{y}(0) = a_1 - \frac{a_2}{\mu(0)}, \quad \overline{y}(1) = a_3. \tag{35}$$

Theorem 4. Under the conditions (C1)–(C4), the solution $\overline{y}(t)$ of the boundary value problem (34), (35) on the interval [0,1] is unique and can be presented in the following form

$$\overline{y}(t) = \left(a_1 - \frac{a_2}{\mu(0)}\right)\overline{\Phi}_1(t) + a_3\overline{\Phi}_2(t) -$$

$$-\overline{\Phi}_2(t)\int_0^1 \overline{K}(1,s)\frac{F(s)}{A(s)}ds + \int_0^t \overline{K}(1,s)\frac{F(s)}{A(s)}ds, \tag{36}$$

where $\overline{\Phi}_k(t),\ k = 1, 2,\ \overline{K}(t,s)$ are functions defined in (31), (32).

The proof of Theorem 3 is carried out similarly to the proof of Theorem 2.

6 *About limit transition and initial jump*

Theorem 5. Under the conditions (C1)–(C4), for a sufficiently small $\varepsilon \geq 0$ the difference between solution $y(t,\varepsilon)$ of BVP (1), (2) and solution $\overline{y}(t)$ of problem (34), (35) on the interval [0,1] satisfies the following inequality:

$$|y(t,\varepsilon) - \overline{y}(t)| \leq C\left(\varepsilon + exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)\right). \tag{37}$$

Proof . We introduce a function $u(t,\varepsilon) = y(t,\varepsilon) - \overline{y}(t)$. The problem (1), (2) have the next form:

$$L_\varepsilon u(t,\varepsilon) = -\varepsilon \overline{y}^{'''}(t),\ u(0,\varepsilon) = \frac{a_2}{\mu(0)},\ u^{'}(0,\varepsilon) = \frac{a_2}{\varepsilon} - \overline{y}^{'}(0),\ u(1,\varepsilon) = 0. \tag{38}$$

By applying Theorem 3 to boundary value problem (38), taking into account (37), we obtain the following inequality:

$$|u(t,\varepsilon)| \leq C \left(\varepsilon + exp \left(\frac{1}{\varepsilon} \int\limits_0^t \mu(x)dx \right) \right),$$

which proves the estimation (38).

Thus, from Theorem 5 it follows that

$$\lim_{\varepsilon \to 0} y(t,\varepsilon) = \overline{y}(t),\ 0 < t \leq 1. \tag{39}$$

Now we determine the magnitude of the initial jump. For this, we turn to the estimate (33). From (33) taking into account (36), we obtain

$$\lim_{\varepsilon \to 0} y(0,\varepsilon) - \overline{y}(0) = \frac{a_2}{\mu(0)},\quad y^{'}(0,\varepsilon) = \frac{a_2}{\varepsilon}. \tag{40}$$

Based on (39) and (40), we conclude, that the solution $y(t,\varepsilon)$ of a singularly perturbed equation (1) with unbounded boundary conditions (2) has the zero order of initial jump at point $t = 0$, which is one of the features of the studied problem.

Thus, we conclude that the established algorithm for studying the solution of a boundary value problem with unbounded boundary conditions allows us to investigate the asymptotic behavior of the solution of a general boundary value problem with unlimited boundary conditions for higher order linear equations. However, the proposed algorithm does not allow to construct the asymptotic solution of substantially nonlinear boundary value problems with unbounded boundary conditions that possess the phenomena of initial jumps. A natural direction for further research is the construction linear asymptotic solution, and nonlinear singularly perturbed boundary problems with unbounded boundary conditions possessing initial jumps. Therefore, the study of the asymptotic behavior and the construction asymptotic solution of the singularly perturbed boundary problems with unbounded boundary conditions that possess the phenomena of initial jumps are still relevant, of particular theoretical interest and important in applications. The obtained results provide opportunities for further research and development of the theory of boundary value problems for ordinary differential equations with a small parameter at the highest derivatives. The constructed initial approximations can be used when considering various problems of chemical kinetics.

7 Conclusion

Thus, the initial and boundary functions for perturbed and unperturbed problems are introduced and constructed, and their asymptotic estimations are found. Using these functions, we constructed an analytical representation of the solution to a singularly perturbed boundary value problem (1), (2) with unbounded boundary conditions. The unperturbed boundary value problem is formulated. The difference between the solutions of the degenerate and initial boundary value problems is estimated for sufficiently small $\varepsilon > 0$, and thus it is proved that the solution of the perturbed problem tends to solve the degenerate problem as the small parameter tends to zero. The growth of the derivative with respect to a small parameter is established. The class of boundary problems with unbounded boundary conditions with the phenomenon of initial jumps is distinguished.

The obtained results give the opportunity for further research in the theory of singularly perturbed boundary value problems, to reduce the boundary value problem (1), (2) to the Cauchy problem with unbounded initial conditions, which in turn can be considered as the basis for constructing the asymptotic expansions of some singularly perturbed boundary value problems with unbounded boundary conditions.

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DOI 10.32014/2020.2518-1726.23.

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Сингулярлы ауытқыған дифференциалдық теңдеу үшін шектелмеген шекаралық шарттары бар есептің асимптотикалық бағалаулары

Мақалада сызықты сингулярлы ауытқыған дифференциалдық теңдеу үшін шектелмеген шекаралық шарттары бар екі нүктелі шекаралық есеп зерттелген. Ауытқыған біртекті теңдеу шешімдерінің сызықты тәуелсіз жүйесі үшін асимптотикалық бағалаулар берілген. Шекаралық функциялар, Коши функциясы деп аталатын көмекші функциялар анықталған. Параметрдің жеткілікті аз мәндері үшін

Коши функциясы мен шекаралық функциялардың бағалаулары табылған. Зерттелетін шекаралық есептің қажетті шешімін құру алгоритмі құрастырылды. Шеттік есептің шешімінің шешілетіндігі туралы теорема дәлелденді. Параметрдің жеткілікті аз мәндері үшін қарастырылып отырған біртекті емес шекаралық есептің шешімі үшін асимптотикалық бағалау берілді. Өзгертілген теңдеудің бастапқы шарттары анықталған. Формула анықталды, бастапқы секіріс құбылысы зерттелген.

Кілт сөздер: екі нүктелі шекаралық есеп, бастапқы секіріс, ауытқыған есеп, кіші параметр, бастапқы функция, шекаралық функциялар.

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Асимптотические оценки решения сингулярно возмущенной краевой задачи с неограниченными граничными условиями

В статье исследована двухточечная краевая задача с неограниченными краевыми условиями для линейного сингулярно возмущенного дифференциального уравнения. Даны асимптотические оценки для линейно независимой системы решений однородного возмущенного уравнения. Определены вспомогательные, так называемые граничные функции, функция Коши. При достаточно малых значениях параметра найдены оценки для функции Коши и граничных функций. Разработан алгоритм построения искомого решения исследуемой краевой задачи. Доказана теорема о разрешимости решения краевой задачи. При достаточно малых значениях параметра установлена асимптотическая оценка решения рассматриваемой неоднородной краевой задачи. Определены начальные условия для вырождающегося уравнения. Формула определена, изучены явления начального скачка.

Ключевые слова: двухточечная краевая задача, начальные скачки, вырожденная задача, малый параметр, начальная функция, граничные функции.

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DOI 10.31489/2022M/34-41
UDC 517.9

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On the Correctness of Boundary Value Problems for the Two-Dimensional Loaded Parabolic Equation

The paper studies the problems of the correctness of setting boundary value problems for a loaded parabolic equation. The feature of the problems is that the order of the derivative in the loaded term is less than or equal to the order of the differential part of the equation, and the load point moves according to a nonlinear law. At the same time, the distinctive characteristic is that the line, on which the loaded term is set is at the zero point. On the basis of the study the authors proved the theorems about correctness of the studied boundary value problems.

Keywords: loaded differential equations, parabolic type equations, uniqueness, existence, boundary problem, loading, perturbation.

Introduction

The steadily growing interest in the study of loaded differential equations is explained by the expanding scope of their applications and the fact that loaded equations constitute a special class of equations with their own specific problems. The main questions that arise in the theory of boundary value problems for partial differential equations remain the same for boundary value problems for loaded equations. However, the presence of a loaded operator does not always allow one to apply the well-known theory of boundary value problems for loaded equations without changes. For example, the question of the functional spaces correct choice for solving problems is relevant.

Loaded differential equations are differential equations containing values of the unknown function and its derivatives at some fixed points of the domain or on some manifolds of nonzero measure. General boundary value problems consisting of general boundary conditions and so-called differential boundary equations (loaded differential equations) were studied by many researchers in the last century, for example, the review article by Kraal [1] and the literature cited therein. Recently, there has been renewed interest in the study of these kinds of problems, for instance, [2–6]. Because of their complexity, numerical methods and, in particular, finite difference methods are mainly used to solve these general boundary value problems [7–10].

Loaded differential equations also arise in applied mathematics, where mathematical problems are modeled by simpler ones that are easier to solve. As such an example, let us mention the case of the Fredholm integro-differential equations, where the integral term is replaced by an approximate quadrature rule, leading to loaded differential equations [11]. Then these equations are solved directly or, in most cases, they are discretized using various difference schemes for the derivatives, leading to loaded difference equations or systems of loaded difference equations. This procedure has recently been implemented to solve linear boundary value problems for first order integro-differential Fredholm equations [12].

Boundary value problems for loaded differential equations, in some cases, are correct in natural classes of functions, that is, in this case, the loaded term is interpreted as a weak perturbation. If

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the uniqueness of the solution of the boundary value problem is violated, then the loading can be interpreted as a strong perturbation. It turns out here that the character of the load is a perturbation (weak or strong perturbation) [13–20].

1 Problem setting

In the domain

$$\Omega = \{(r,t)\,, r>0, t>0\}$$

consider the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1-2\beta}{r}\dfrac{\partial u}{\partial r} - \lambda \left.\dfrac{\partial^k u}{\partial r^k}\right|_{r=t^\alpha} + f(r,t) & (1) \\ u(r,0)=0; u(0,t)=0, & (2) \end{cases}$$

where $0<\beta<1$, $\lambda \in R$ is a spectral parameter, $f(r,t) \in M(\Omega)$ is a given function,

$$f_k(t) = \left.\left(\frac{\partial^k}{\partial r^k}\int_0^t\int_0^\infty \left[\frac{r^\beta \xi^{1-\beta}}{2(t-\tau)}\exp\left[-\frac{r^2+\xi^2}{4(t-\tau)}\right] I_\beta\left(\frac{r\xi}{2(t-\tau)}\right)\right]\cdot f(\xi,\tau)d\xi d\tau\right)\right|_{r=t^\alpha} \in M(0,\infty)$$

$$M(\Omega) = L_\infty(\Omega)\cap C(\Omega), M(0,\infty) = L_\infty(0,\infty)\cap C(0,\infty).$$

The purpose is to determine for which integer values $k=0,1,2$ and for which values $\alpha>0,\ 0<\beta<1$ the problem (1)–(2) will be correct in other words have a unique solution.

2 Main part

Remark 1. Obviously, this problem at $\lambda = 0$ has a unique solution due to the lack of a loaded term

$$u(r,t) = \int_0^t\int_0^\infty \left[\frac{r^\beta \xi^{1-\beta}}{2(t-\tau)}\exp\left[-\frac{r^2+\xi^2}{4(t-\tau)}\right] I_\beta\left(\frac{r\xi}{2(t-\tau)}\right)\right]\cdot f(\xi,\tau)d\xi d\tau.$$

By means of this solution, we invert the differentiated part of the problem. In order to invert it, we transfer the loaded term $\lambda \left.\frac{\partial^k u}{\partial r^k}\right|_{r=t^\alpha}$ to the right-hand side and consider it temporarily known, we obtain

$$u(r,t) = \lambda\int_0^t \left.\frac{\partial^k u}{\partial \xi^k}\right|_{\xi=\tau^\alpha} d\tau \int_0^\infty \left[\frac{r^\beta \xi^{1-\beta}}{2(t-\tau)}\exp\left[-\frac{r^2+\xi^2}{4(t-\tau)}\right] I_\beta\left(\frac{r\xi}{2(t-\tau)}\right)\right] d\xi + f_0(r,t), \tag{3}$$

where

$$f_0(r,t) = \int_0^t\int_0^\infty \left[\frac{r^\beta \xi^{1-\beta}}{2(t-\tau)}\exp\left[-\frac{r^2+\xi^2}{4(t-\tau)}\right] I_\beta\left(\frac{r\xi}{2(t-\tau)}\right)\right]\cdot f(\xi,\tau)d\xi d\tau.$$

In equality (3) we calculate the inner integral:

$$Q(r,t-\tau) = \int_0^\infty \left[\frac{r^\beta \xi^{1-\beta}}{2(t-\tau)}\exp\left[-\frac{r^2+\xi^2}{4(t-\tau)}\right] I_\beta\left(\frac{r\xi}{2(t-\tau)}\right)\right] d\xi =$$

$$= \frac{r^\beta}{2(t-\tau)}\exp\left[-\frac{r^2}{4(t-\tau)}\right]\int_0^\infty \xi^{1-\beta}\exp\left[-\frac{\xi^2}{4(t-\tau)}\right].$$

Using the following substitution

$$\frac{r}{2(t-\tau)}\xi = \eta; \qquad \xi = \frac{2(t-\tau)}{r}\eta; \qquad d\xi = \frac{2(t-\tau)}{r}d\eta$$

we obtain

$$Q(r,t-\tau)=\frac{r^{\beta}}{2(t-\tau)}\exp\left[-\frac{r^2}{4(t-\tau)}\right]\int_0^{\infty}\frac{2^{1-\beta}(t-\tau)^{1-\beta}}{r^{1-\beta}}\eta^{1-\beta}\exp\left[-\frac{t-\tau}{r^2}\eta^2\right]I_{\beta}(\eta)\cdot\frac{2(t-\tau)}{r}d\eta=$$
$$=\exp\left[-\frac{r^2}{4(t-\tau)}\right]\frac{2^{1-\beta}(t-\tau)^{1-\beta}}{r^{2-2\beta}}\int_0^{\infty}\exp\left[-\frac{t-\tau}{r^2}\eta^2\right]\eta^{1-\beta}I_{\beta}(\eta)d\eta=$$
$$=\exp\left[-\frac{r^2}{4(t-\tau)}\right]\frac{2^{1-\beta}(t-\tau)^{1-\beta}}{r^{2-2\beta}}\cdot\exp\left[\frac{r^2}{4(t-\tau)}\right]\frac{2^{\beta-1}(t-\tau)^{\beta-1}}{\Gamma(\beta)r^{2\beta-2}}\cdot\gamma\left(\beta,\frac{r^2}{4(t-\tau)}\right)=$$
$$=\frac{1}{\Gamma(\beta)}\cdot\gamma\left(\beta,\frac{r^2}{4(t-\tau)}\right),$$

where $\gamma(\beta,z)$ is an incomplete gamma function.

Therefore, we are able to express $Q(r,t-\tau)$ in the following way:

$$Q(r,t-\tau)=\frac{1}{\Gamma(\beta)}\cdot\gamma\left(\beta,\frac{r^2}{4(t-\tau)}\right).$$

Then the integral representation of solution (3) takes the form

$$u(r,t)=\lambda\int_0^t\frac{1}{\Gamma(\beta)}\cdot\gamma\left(\beta,\frac{r^2}{4(t-\tau)}\right)\cdot\left.\frac{\partial^k u}{\partial\xi^k}\right|_{\xi=\tau^{\alpha}}d\tau+f_0(r,t)\tag{4}$$

it is apparent from (4) that in order to find a solution to problem (1)–(2), it is sufficient to find the value of the loaded term $\left.\frac{\partial^k u}{\partial r^k}\right|_{r=t^{\alpha}}$.

I. Let $k=0$ then relation (4) takes the following form

$$u(r,t)=\lambda\int_0^t\frac{1}{\Gamma(\beta)}\cdot\gamma\left(\beta,\frac{r^2}{4(t-\tau)}\right)\cdot u(\xi,\tau)|_{\xi=\tau^{\alpha}}d\tau+f_0(r,t).\tag{5}$$

Assuming $r=t^{\alpha}$ in both parts of the equality (5) and introducing the notation $\mu_0(t)=u(r,t)|_{r=t^{\alpha}}$ we obtain the following integral equation with respect to the unknown function:

$$\mu_0(t)=\lambda\int_0^t K_0(t,\tau)\cdot\mu_0(\tau)d\tau+f_0(t),$$

where

$$K_0(t,\tau)=\frac{1}{\Gamma(\beta)}\cdot\gamma\left(\beta,\frac{r^2}{4(t-\tau)}\right),f_0(t)=\left.f_0(r,t)\right|_{r=t^{\alpha}},$$

which solution $\forall\lambda\in R$, $\forall\alpha>0$, $\forall f_0(t)\in M(0,\infty)$ can be found by the method of successive approximations. Here we take into account that $K_0(t,\tau)\le 1$, and is continuous $\forall(t,\tau)$, $0<\tau<t$. This implies that problem (1)–(2) has a unique solution.

Theorem 1. For $k=0$ and $\forall\lambda\in R$, $\forall\alpha>0$, $\forall f_0(t)\in M(0,\infty)$ the boundary value problem (1)–(2) has a unique solution.

II. Let us assume that $k=1$. Then (5) takes the following form:

$$u(r,t)=\lambda\int_0^t\frac{1}{\Gamma(\beta)}\cdot\gamma\left(\beta,\frac{r^2}{4(t-\tau)}\right)\cdot\left.\frac{\partial u}{\partial\xi}\right|_{\xi=\tau^{\alpha}}d\tau+f_1(r,t).$$

In order to determine the loaded term $\frac{\partial u}{\partial \xi}\Big|_{\xi=\tau^\alpha}$ considering it is possible, we differentiate both parts of this equality by the variable r and take following consideration $r = t^\alpha$. Yet it would be convenient to calculate beforehand following:

$$\frac{\partial}{\partial r}\gamma\left(\beta, \frac{r^2}{4(t-\tau)}\right)\Big|_{r=t^\alpha} = \frac{1}{\Gamma(\beta)}\cdot\left[\frac{r^2}{4(t-\tau)}\right]^{\beta-1}\cdot\frac{r}{2(t-\tau)}\exp\left[-\frac{r^2}{4(t-\tau)}\right]\Big|_{r=t^\alpha} =$$
$$= \frac{1}{\Gamma(\beta)}\frac{1}{2^{2\beta-1}}\cdot\frac{t^{(2\beta-1)\alpha}}{(t-\tau)^\beta}\cdot\exp\left[-\frac{t^{2\alpha}}{4(t-\tau)}\right].$$

By denoting $\mu_1(t) = \frac{\partial u}{\partial r}\big|_r = t^\alpha$, we obtain the following integral equation with respect to the unknown function $\mu_1(t)$

$$\mu_1(t) = \lambda\int_0^t K_1(t,\tau)\cdot\mu_1(\tau) + f_1(t), \tag{6}$$

where

$$K_1(t,\tau) = \frac{1}{\Gamma(\beta)}\frac{1}{2^{2\beta-1}}\cdot\frac{t^{(2\beta-1)\alpha}}{(t-\tau)^\beta}\cdot\exp\left[-\frac{t^{2\alpha}}{4(t-\tau)}\right],$$
$$f_1(t) = \frac{\partial}{\partial r}\, f_1(r,t)\big|_{r=t^\alpha}.$$

If $0 < \beta \le 1/2$, then the kernel $K_1(t,\tau)$ has a weak singularity $\forall\alpha > 0$, yet if $1/2 < \beta < 1$, then in order to have a unique solution the condition must be satisfied for the integral equation (6):

$$0 < \alpha < \frac{2-\beta}{1-2\beta}.$$

Consequently, the theorem is valid.

Theorem 2. If $k = 1$, then for $0 < \beta \le 1/2$ and $\forall\lambda \in R$, $\forall\alpha > 0$, $\forall f_1(t) \in M(0,\infty)$ the boundary value problem (1)–(2) has a unique solution $u(r,t) \in M(\Omega)$. As for $1/2 < \beta < 1$ in order for the boundary value problem (1)–(2) to have a unique solution $u(r,t) \in M(\Omega)$ the following condition must be satisfied:

$$0 < \alpha < \frac{2-\beta}{1-2\beta}.$$

Remark 2. Thus, for $k = 0$ and for $k = 1$ under the conditions of Theorem 2 the loaded term $\frac{\partial u}{\partial \xi}\Big|_{\xi=\tau^\alpha}$ in equation (1) of problem (1)–(2) can be interpreted as weak perturbation.

III. Let us assume that $k = 2$. Then (5) takes the following form:

$$u(r,t) = \lambda\int_0^t \frac{1}{\Gamma(\beta)}\cdot\gamma\left(\beta, \frac{r^2}{4(t-\tau)}\right)\cdot\frac{\partial^2 u}{\partial\xi^2}\Big|_{\xi=\tau^\alpha} d\tau + f_2(r,t). \tag{7}$$

In order to determine the loaded term $\frac{\partial^2 u}{\partial \xi^2}\Big|_{\xi=\tau^\alpha}$ let us differentiate both parts of equality (7) twice with respect to the variable r and take following consideration $r = t^\alpha$. Yet it would be convenient to calculate beforehand next expression:

$$\frac{\partial^2}{\partial r^2}\gamma\left(\beta, \frac{r^2}{4(t-\tau)}\right)\Big|_{r=t^\alpha} = \frac{2\beta-1}{2^{2\beta-1}\Gamma(\beta)}\cdot\frac{r^{2\beta-2}}{(t-\tau)^\beta}\cdot\exp\left[-\frac{r^2}{4(t-\tau)}\right]\Big|_{r=t^\alpha} -$$
$$- \frac{1}{2^{2\beta}\Gamma\beta}\cdot\frac{r^{2\beta}}{(t-\tau)^{\beta+1}}\exp\left[-\frac{r^2}{4(t-\tau)}\right]\Big|_{r=t^\alpha} =$$
$$= \frac{1}{\Gamma(\beta)}\left\{\frac{2\beta-1}{2^{2\beta-1}}\cdot\frac{1}{2^{2\alpha(1-\beta)}(t-\tau)^{\beta+1}} - \frac{1}{2^{2\beta}}\cdot\frac{t^{2\alpha\beta}}{(t-\tau)^{\beta+1}}\right\}\cdot\exp\left[-\frac{t^{2\alpha}}{4(t-\tau)}\right].$$

By denoting

$$\mu_1(t) = \left.\frac{\partial^2 u}{\partial r^2}\right|_r = t^\alpha,$$

we obtain the following integral equation with respect to the unknown function $\mu_2(t)$:

$$\mu_2(t) - \lambda \int_0^t K_2(t,\tau) \cdot \mu_2(\tau) = f_2(t), \tag{8}$$

where

$$K_2(t,\tau) = \frac{1}{\Gamma(\beta)} \left\{ \frac{1}{2^{2\beta}} \cdot \frac{t^{2\alpha\beta}}{(t-\tau)^{\beta+1}} - \frac{2\beta-1}{2^{2\beta-1}} \cdot \frac{1}{2^{2\alpha(1-\beta)}(t-\tau)^{\beta+1}} \right\} \cdot \exp\left[-\frac{t^{2\alpha}}{4(t-\tau)}\right].$$

$$f_2(t) = \left.\frac{\partial^2}{\partial r^2} f_0(r,t)\right|_{r=t^\alpha}.$$

Let us study the kernel $K_2(t,\tau)$ of this equation. Initially, we calculate the following integral:

$$\int_0^t K_2(t,\tau) = \frac{1}{\Gamma(\beta)}\Gamma\left(\beta, \frac{t^{2\alpha-1}}{4}\right) - \frac{2\beta-1}{2\Gamma(\beta)}\Gamma\left(\beta-1, \frac{t^{2\alpha-1}}{4}\right). \tag{9}$$

Further, using the following equality:

$$\Gamma(\alpha+1, x) = \alpha\Gamma(\alpha, x) + x^\alpha e^{-x},$$

we obtain

$$\Gamma\left(\beta-1, \frac{t^{2\alpha-1}}{4}\right) = -\frac{1}{1-\beta}\Gamma\left(\beta, \frac{t^{2\alpha-1}}{4}\right) + \frac{1}{1-\beta}\frac{4^{1-\beta}t^{1-\beta}}{t^{2\alpha(1-\beta)}} \cdot \exp\left[-\frac{t^{2\alpha-1}}{4}\right].$$

Hence it follows that

$$\begin{aligned} \int_0^t K_2(t,\tau) =& \frac{1}{\Gamma(\beta)}\Gamma\left(\beta, \frac{t^{2\alpha-1}}{4}\right) + \\ &+ \frac{2\beta-1}{2\Gamma(\beta)}\frac{1}{1-\beta}\Gamma\left(\beta, \frac{t^{2\alpha-1}}{4}\right) - \\ &- \frac{2\beta-1}{2\Gamma(\beta)}\frac{4^{1-\beta}}{1-\beta} \cdot t^{(1-2\alpha)(1-\beta)} \cdot \exp\left[-\frac{t^{2\alpha-1}}{4}\right] = \\ &= \frac{\Gamma\left(\beta, \frac{t^{2\alpha-1}}{4}\right)}{\Gamma(\beta)}\left[1 + \frac{2\beta-1}{2(1-\beta)}\right] - \frac{2\beta-1}{2^{2\beta-2}(1-\beta)}\frac{1}{\Gamma(\beta)} \cdot t^{(1-2\alpha)(1-\beta)} \exp\left[-\frac{t^{2\alpha-1}}{4}\right]. \end{aligned} \tag{10}$$

This implies that the inhomogeneous integral equation (8) has a unique solution $\forall\beta \in (0,1)$ if the condition $\alpha \in \left(0, \frac{1}{2}\right)$ is satisfied.

Theorem 3. If $k = 2$, then for the condition $0 < \alpha < 1/2$ the boundary value problem (1)–(2) has a unique solution $u(r,t) \in M(\Omega)$ for $\forall\lambda \in R,\ \forall\beta \in (0,1),\ \forall f_2(t) \in M(0,\infty)$.

3 Conclusion

From (9)–(10) it follows that for $\beta = 1/2, \alpha > 1/2$ we have the following equality:

$$\lim_{t\to 0} \int_0^t K_2(t,\tau)d\tau = 1.$$

In summation, this implies that the Volterra type integral equation of the second kind (8) cannot be solved by the method of successive approximations. Moreover, the corresponding homogeneous integral equation for $\lambda \geq 1$ will have nonzero solutions, thus the inhomogeneous integral equation has a non-unique solution. Then from relation (3) it will follow that the boundary value problem (1)–(2) will be incorrect, since it has a non-unique solution.

As noted in [21–24], the corresponding boundary problems may turn out to be Noetherians with both positive and negative indices. Further investigations of boundary problems of type (1)–(2) for different laws of motion of the load point will be continued.

Acknowledgments

The research is funded by the Science Committee of the Ministry of Higher Education and Science of the Republic of Kazakhstan (Grants no. AP09259780, 2021-2023, Grants no. AP09258892, 2021-2023).

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Екі өлшемді жүктелген параболалық теңдеу үшін шекаралық есептердің дұрыс қойылуы

Мақалада жүктелген параболалық теңдеу үшін шекаралық есептерді дұрыс қоюдың сұрақтары зерттелген. Есептердің ерекшелігі - жүктелген мүшедегі туындының реті теңдеудің дифференциалдық бөлігінің ретінен кіші және оған тең, ал жүктеме нүктесі сызықты емес заң бойынша қозғалады. Бұл жағдайда ерекшеленетін белгі — жүктелген термин көрсетілген қарастырылатын сызық нөлдік нүктеде орналасқан. Зерттеу негізінде авторлар зерттелетін шекаралық есептердің дұрыс қойылғандығы туралы теоремаларды дәлелдеді.

Кілт сөздер: жүктелген дифференциалдық теңдеулер, параболалық типті теңдеулер, бірегейлік, бар болу, шекаралық есеп, жүктеме, ауытқу.

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О корректности краевых задач для двумерного нагруженного параболического уравнения

В статье исследованы вопросы корректности постановок краевых задач для нагруженного параболического уравнения. Особенностью задач является то, что порядок производной в нагруженном слагаемом меньше и равен порядку дифференциальной части уравнения, и при этом точка нагрузки движется по нелинейному закону. Кроме того, отличительной чертой является то, что рассматриваемая линия, на которой задается нагруженное слагаемое, расположена в точке нуль. На основе исследования авторы доказали теоремы о корректности исследуемых краевых задач.

Ключевые слова: нагруженные дифференциальные уравнения, уравнения параболического типа, единственность, существование, граничная задача, нагрузка, возмущение.

DOI 10.31489/2022M4/42-50
UDC 517.5

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The theorems about traces and extensions for functions from Nikol'skii-Besov spaces with generalized mixed smoothness

The theory of embedding of spaces of differentiable functions studies important relations of differential (smoothness) properties of functions in various metrics and has wide application in the theory of boundary value problems of mathematical physics, approximation theory and other fields of mathematics.
In this article, we prove the theorems about traces and extensions for functions from Nikol'skii-Besov spaces with generalized mixed smoothness and mixed metrics. The proofs of the obtained results is based on the inequality of different dimensions for trigonometric polynomials in Lebesgue spaces with mixed metrics and the embedding theorem of classical Nikol'skii-Besov spaces in the space of continuous functions.

Keywords: Nikol'skii-Besov spaces, generalized mixed smoothness, mixed metrics, a trace of function, an extension of function.

Introduction

One of the first results related to the theory of embedding of spaces of differentiable functions was a result of S.L. Sobolev [1]. This theory studies important relations of differential (smoothness) properties of functions in various metrics. Further development of this theory is associated with new classes of function spaces defined and studied in the works of S.M. Nikol'skii [2, 3], O.V. Besov [4, 5], P.I. Lizorkin [6], H. Triebel [7, 8] and many others. The development of this research was determined both by its internal problems and by its applications in the theory of boundary value problems of mathematical physics and approximation theory [9]–[17].

This paper continues our investigations of Nikol'skii-Besov spaces with generalized mixed smoothness and mixed metrics, which began in the works [18, 19]. In this article, we prove theorems on traces and continuations for functions from the above-mentioned spaces. The proof of these results is based on applying the inequality of different dimensions for trigonometric polynomials in mixed-metric Lebesgue spaces and the Nikol'skii-Besov classical spaces embedding theorem into the space of continuous functions.

1 Definitions and auxiliary results

Let $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$, $\mathbb{T}^{\mathbf{d}} = \{\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n) : \mathbf{x}_i \in \mathbb{T}^{d_i} = [0, 2\pi)^{d_i}, i = 1, \dots, n\}$ and $f(\mathbf{x}) = f(\mathbf{x}_1, \dots, \mathbf{x}_n)$ be measurable function on $\mathbb{T}^{\mathbf{d}}$.

Let $\mathbf{1} \le \mathbf{p} = (p_1, \dots, p_n) \le \infty$. We say that the function f belongs to the Lebesgue space with mixed metrics $L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})$ if

$$\|f\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} = \left(\int_{\mathbb{T}^{d_n}} \left(\dots \left(\int_{\mathbb{T}^{d_1}} |f(\mathbf{x}_1, \dots, \mathbf{x}_n|^{p_1} d\mathbf{x}_1\right)^{p_2/p_1} \dots\right)^{p_n/p_{n-1}} d\mathbf{x}_n\right)^{1/p_n} < \infty.$$

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In a case when $p_i = \infty$ the expression $\left(\int_{\mathbb{T}^{d_i}} |f(\mathbf{x}_i)|^{p_i}\, d\mathbf{x}_i\right)^{1/p_i}$ means that $\operatorname{ess\,sup}_{\mathbf{x}_i\in\mathbb{T}^{d_i}} |f(\mathbf{x}_i)|$.

Let us denote by

$$\Delta_{\mathbf{s}}(f,\mathbf{x}) = \sum_{\mathbf{k}\in\rho(\mathbf{s})} a_{\mathbf{k}} e^{i\langle \mathbf{k},\mathbf{x}\rangle_{\mathbf{d}}}$$

the trigonometric series of $f \sim \sum_{\mathbf{k}\in\mathbb{Z}^{\mathbf{d}}} a_{\mathbf{k}} e^{i\langle \mathbf{k},\mathbf{x}\rangle_{\mathbf{d}}}$, where $\langle \mathbf{k},\mathbf{x}\rangle_{\mathbf{d}} = \sum_{i=1}^{n}\sum_{j=1}^{d_i} k_j^i x_j^i$ is the (modified) inner product, $\rho(\mathbf{s}) = \{\mathbf{k} = (\mathbf{k}_1,\ldots,\mathbf{k}_n) \in \mathbb{Z}^{\mathbf{d}} : \left[2^{s_i-1}\right] \le \max_{j=1,\ldots,d_i} |k_j^i| < 2^{s_i}, i = 1,\ldots,n\}$ and $[a]$ is the integer part of the number a.

Let $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{R}^n$, $\mathbf{1} \le \mathbf{q} = (\mathbf{q_1},\ldots,\mathbf{q_n}) \le \infty$ and $\mathbf{1} < \mathbf{p} = (p_1,\ldots,p_n) < \infty$.

The anisotropic Nikol'skii-Besov space with generalized mixed smoothness and mixed metrics $B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ is a set of the series $f \sim \sum_{\mathbf{k}\in\mathbb{Z}^{\mathbf{d}}} a_{\mathbf{k}} e^{i\langle \mathbf{k},\mathbf{x}\rangle_{\mathbf{d}}}$ such that

$$\|f\|_{B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})} = \left\|\left\{2^{(\alpha,\mathbf{s})}\|\Delta_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})}\right\}\right\|_{l_{\mathbf{q}}} < \infty,$$

where $(\alpha,\mathbf{s}) = \sum_{i=1}^{n} \alpha_i s_i$ is the inner product and $\|\cdot\|_{l_{\mathbf{q}}}$ is the norm of a discrete Lebesgue space with mixed metrics $l_{\mathbf{q}}$.

Here $B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ is a version of spaces, which was introduced and studied in [20].

Remark 1. The anisotropic Nikol'skii-Besov space with generalized mixed smoothness $B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ mentioned above is a hybrid structure of Nikol'skii-Besov space (concerning to variables included in one multi-variable) [2, 4] and spaces with dominant mixed derivative (concerning to variables included in different multi-variables) [21, 22]. In the isotropic case, when p and q are scalars, analogs of these spaces were studied by D.B. Bazarkhanov [23].

Let us denote by $\bar{\mathbf{p}} = (p_1,\ldots,p_m)$ for the multi-index $\mathbf{p} = (p_1,\ldots,p_m,p_{m+1},\ldots,p_n)$.

Lemma 1 (Inequality of different dimensions, [3]). Let $T_{\mathbf{s}}(\mathbf{x})$ be a trigonometric polynomial of order not higher than $\mathbf{s} = (\mathbf{s_1},\ldots,\mathbf{s_m},\mathbf{s_{m+1}},\ldots,\mathbf{s_n})$ by variables $\mathbf{x} = (\mathbf{x_1},\ldots,\mathbf{x_m},\mathbf{x_{m+1}},\ldots,\mathbf{x_n})$ and $\mathbf{1} < \mathbf{p} = (p_1,\ldots,p_m,p_{m+1},\ldots,p_n) < \infty$, then for an arbitrary fixed point $(\mathbf{x_{m+1}},\ldots,\mathbf{x_n}) \in \mathbb{T}^{d_{m+1}} \times \ldots \times \mathbb{T}^{d_n}$ the following inequality holds

$$\|T_{\mathbf{s}}(\cdot,\ldots,\cdot,\mathbf{x_{m+1}},\ldots,\mathbf{x_n})\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \le C \prod_{i=m+1}^{n} s_i^{d_i/p_i} \|T_{\mathbf{s}}\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})},$$

where C is a positive constant independent on $\mathbf{s}$.

2 Main results

In this section, we prove the trace and continuation theorems for functions from Nikol'skii-Besov spaces with generalized mixed smoothness and anisotropic Lorentz spaces are proved.

Theorem 1. Let $\mathbf{0} < \alpha = (\alpha_1,\ldots,\alpha_m,\alpha_{\mathbf{m+1}},\ldots,\alpha_{\mathbf{n}}) < \infty$, $\mathbf{1} \le \mathbf{q} = (q_1,\ldots,q_m,q_{m+1},\ldots,q_n) \le \infty$, $\mathbf{1} < \mathbf{p} = (p_1,\ldots,p_m,p_{m+1},\ldots,p_n) < \infty$, then for $\alpha_i = d_i/p_i$, $q_i = 1$ where $i = m+1,\ldots,n$ the following embedding holds

$$B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}}) \hookrightarrow B_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\mathbf{q}}}(\mathbb{T}^{\bar{\mathbf{d}}}).$$

Proof. According to the inequality of different dimensions (Lemma 1) and Minkowski inequality, we obtain

$$\|f(\cdot,\ldots,\cdot,\mathbf{x}_{m+1},\ldots,\mathbf{x}_n)\|_{B^{\bar{\alpha}\bar{\mathbf{q}}}_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} =$$

$$= \left\| \left\{ 2^{(\bar{\alpha},\bar{\mathbf{s}})} \left\| \Delta_{\bar{\mathbf{s}}}\left(f(\cdot,\ldots,\cdot,\mathbf{x}_{m+1},\ldots,\mathbf{x}_n)\right)\right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \right\} \right\|_{l_{\bar{\mathbf{q}}}} \le$$

$$\le \left\| \left\{ 2^{(\bar{\alpha},\bar{\mathbf{s}})} \left\| \sum_{s_n=0}^{\infty} \cdots \sum_{s_{m+1}=0}^{\infty} \Delta_{s_{m+1}}\left(\ldots \Delta_{s_n}\left(\Delta_{\bar{\mathbf{s}}}(f)\right)\right) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \right\} \right\|_{l_{\bar{\mathbf{q}}}} \le$$

$$\le \left\| \left\{ 2^{(\bar{\alpha},\bar{\mathbf{s}})} \sum_{s_n=0}^{\infty} \cdots \sum_{s_{m+1}=0}^{\infty} \left\| \Delta_{s_{m+1}}\left(\ldots \Delta_{s_n}\left(\Delta_{\bar{\mathbf{s}}}(f)\right)\right) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \right\} \right\|_{l_{\bar{\mathbf{q}}}} =$$

$$= \left\| \left\{ 2^{(\bar{\alpha},\bar{\mathbf{s}})} \sum_{s_n=0}^{\infty} \cdots \sum_{s_{m+1}=0}^{\infty} \left\| \Delta_{\mathbf{s}}(f) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \right\} \right\|_{l_{\bar{\mathbf{q}}}} \le$$

$$\le C_1 \left\| \left\{ 2^{(\bar{\alpha},\bar{\mathbf{s}})} \sum_{s_n=0}^{\infty} 2^{(s_n d_n)/p_n} \cdots \sum_{s_{m+1}=0}^{\infty} 2^{(s_{m+1} d_{m+1})/p_{m+1}} \left\| \Delta_{\mathbf{s}}(f) \right\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \right\} \right\|_{l_{\bar{\mathbf{q}}}} \le$$

$$\le C_1 \sum_{s_n=0}^{\infty} \cdots \sum_{s_{m+1}=0}^{\infty} \left\| \left\{ 2^{(\alpha,\mathbf{s})} \left\| \Delta_{\mathbf{s}}(f) \right\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \right\} \right\|_{l_{\bar{\mathbf{q}}}} = C_1 \left\| \left\{ 2^{(\alpha,\mathbf{s})} \left\| \Delta_{\mathbf{s}}(f) \right\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \right\} \right\|_{l_{\mathbf{q}}} = C_1 \|f\|_{B^{\alpha\mathbf{q}}_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})},$$

here $\alpha_i = d_i/p_i$, $q_i = 1$ where $i = m+1,\ldots,n$.

Let us show that the conditions $\alpha_i = d_i/p_i$, $q_i = 1$ where $i = m+1,\ldots,n$ ensure that the following property holds

$$\|f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - \varphi(\mathbf{x}_1,\ldots,\mathbf{x}_m)\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \to 0$$

for $\max\limits_{i=m+1,\ldots,n} |\mathbf{h}_i| \to \mathbf{0}$, here $|\mathbf{h}_i| = \sqrt{\sum\limits_{j=1}^{d_i} (h^i_j)^2}$, $(i = m+1,\ldots,n)$.

Indeed, let $N \in \mathbb{N}$ and

$$\Gamma_N = \left\{ \mathbf{s} = (\mathbf{s_1},\ldots,\mathbf{s_n}) \in \mathbb{Z}^n : \prod_{i=1}^{n} \max(1, \max_{j=1,\ldots,d_i} |s^i_j|) \le N \right\},$$

then

$$\|f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - \varphi(\mathbf{x}_1,\ldots,\mathbf{x}_m)\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} =$$

$$= \|f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0})\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} =$$

$$= \|f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) +$$

$$+ S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0}) +$$

$$+ S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0}) - f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0})\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \le$$

$$\le \|f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n)\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} +$$

$$+ \|S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0})\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} +$$

$$+ \|S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0}) - f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0})\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} = I_1 + I_2 + I_3,$$

where $S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{x}_{m+1},\ldots,\mathbf{x}_n)$ is the partial sum of the Fourier series of the function $f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{x}_{m+1},\ldots,\mathbf{x}_n)$, corresponding to the hyperbolic cross Γ_N.

We will use Minkowski, different dimensions (Lemma 1) and Hölder inequalities to estimate I_1 and I_3. For $k=1$ or $k=3$ we have

$$I_k \le \sup_{\mathbf{x}_{m+1},\ldots,\mathbf{x}_n} \|f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{x}_{m+1},\ldots,\mathbf{x}_n)-$$

$$-S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{x}_{m+1},\ldots,\mathbf{x}_n)\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} =$$

$$= \sup_{\mathbf{x}_{m+1},\ldots,\mathbf{x}_n} \left\| \sum_{\mathbf{s}\notin\Gamma_N} \Delta_{\mathbf{s}}(f;\mathbf{x}) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \le \sum_{\mathbf{s}\notin\Gamma_N} \sup_{x_{m+1},\ldots,x_n} \|\Delta_{\mathbf{s}}(f;\mathbf{x})\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \le$$

$$\le C_1 \sum_{\mathbf{s}\notin\Gamma_N} 2^{-\langle\bar{\alpha},\bar{\mathbf{s}}\rangle} 2^{\langle\alpha,\mathbf{s}\rangle} \|\Delta_{\mathbf{s}}(f;\mathbf{x})\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \le$$

$$\le C_1 \left\| \left\{ 2^{\langle\alpha,\mathbf{s}\rangle} \|\Delta_{\mathbf{s}}(f;\mathbf{x})\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \right\}_{\mathbf{s}\notin\Gamma_N} \right\|_{l_{\mathbf{q}}} \cdot \left\| \left\{ 2^{-\langle\bar{\alpha},\bar{\mathbf{s}}\rangle} \right\}_{\mathbf{s}\notin\Gamma_N} \right\|_{l_{\mathbf{q}'}} \le$$

$$\le C_2 \|f - S_{\Gamma_N}(f)\|_{B^{\alpha\mathbf{q}}_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \to 0 \text{ при } N\to\infty.$$

Moreover, to estimate I_2, we use the fact that the trigonometric polynomial $S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{x}_{m+1},\ldots,\mathbf{x}_n)$ is a continuous function, then we receive

$$\|S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0})\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \to 0$$

for $\max\limits_{i=m+1,\ldots,n} |\mathbf{h}_i| \to 0$.

This completes the proof.

Remark 2. In contrast to the trace theorem for functions from Nikol'skii-Besov spaces with dominating mixed derivative [21, 22], proved for $\alpha_i > d_i/p_i$ where $i=m+1,\ldots,n$, in Theorem 1, the limiting case $\alpha_i = d_i/p_i$ is considered under the condition $q_i = 1$ for $i=m+1,\ldots,n$ (this effect was previously seen for example in [24, 25]).

Theorem 2. Let $\mathbf{0} < \alpha = (\alpha_1,\ldots,\alpha_m,\alpha_{m+1},\ldots,\alpha_n) < \infty$, $\mathbf{1} \le \mathbf{q} = (q_1,\ldots,q_m,q_{m+1},\ldots,q_n) \le \infty$, $\mathbf{1} < \mathbf{p} = (p_1,\ldots,p_m,p_{m+1},\ldots,p_n) < \infty$. Then for $\alpha_i = d_i/p_i$ and $q_i = 1$, $i=m+1,\ldots,n$, for the function $\varphi(\mathbf{x}_1,\ldots,\mathbf{x}_m) \in B^{\bar{\alpha}\bar{\mathbf{q}}}_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})$ it is possible to construct a function $f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{x}_{m+1},\ldots,\mathbf{x}_n)$ having the following properties

$$f \in B^{\alpha\mathbf{q}}_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}});$$

$$\|f\|_{B^{\alpha\mathbf{q}}_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \le C \|\varphi\|_{B^{\bar{\alpha}\bar{\mathbf{q}}}_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})};$$

$$f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0}) = \varphi(\mathbf{x}_1,\ldots,\mathbf{x}_m).$$

Proof. Let $\varphi \in B^{\bar{\alpha}\bar{\mathbf{q}}}_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})$. This function can be represented as a series converging to it in the sense of $L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})$

$$\varphi(\mathbf{x}_1,\ldots,\mathbf{x}_m) = \sum_{\bar{\mathbf{s}}=\mathbf{0}}^{\infty} \Delta_{\bar{\mathbf{s}}}(\varphi(\mathbf{x}_1,\ldots,\mathbf{x}_m))$$

and

$$\|\varphi\|_{B^{\bar{\alpha}\bar{\mathbf{q}}}_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} = \left\| \left\{ 2^{(\bar{\alpha},\bar{\mathbf{s}})} \|\Delta_{\bar{\mathbf{s}}}(\varphi)\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \right\} \right\|_{l_{\bar{\mathbf{q}}}}.$$

Let us choose the functions $f_i(\mathbf{x}_i)$ from the $B_{p_i}^{\alpha_i 1}(\mathbb{T}^{d_i})$, where $\alpha_i = d_i/p_i$, such that $f_i(\mathbf{0}) = 1$ for $i = m+1, \ldots, n$. Let us introduce a new function

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{x}_{m+1}, \ldots, \mathbf{x}_n) = \sum_{\mathbf{s}=0}^{\infty} \Delta_{\bar{\mathbf{s}}} \left(\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_m)\right) \prod_{i=m+1}^{n} \Delta_{s_i} \left(f_i(\mathbf{x}_i)\right).$$

Consequently, for this function, we get

$$\|f\|_{B_{\mathbf{p}}^{\alpha \mathbf{q}}(\mathbb{T}^{\mathbf{d}})} = \left\| \left\{ 2^{(\alpha, \mathbf{s})} \left\| \Delta_{\mathbf{s}}(\varphi) \right\|_{L_{\mathbf{p}}(\mathbb{T}^{\bar{\mathbf{d}}})} \right\} \right\|_{l_{\mathbf{q}}} =$$

$$= \left\| \left\{ 2^{(\bar{\alpha}, \bar{\mathbf{s}})} \left\| \Delta_{\bar{\mathbf{s}}}(\varphi) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \right\} \right\|_{l_{\bar{\mathbf{q}}}} \prod_{i=m+1}^{n} \|f_i\|_{B_{p_i}^{\alpha_i 1}(\mathbb{T}^{d_i})} = C_6 \|\varphi\|_{B_{\bar{\mathbf{p}}}^{\bar{\alpha} \bar{\mathbf{q}}}(\mathbb{T}^{\bar{\mathbf{d}}})}.$$

According to the condition $f_i(\mathbf{0}) = 1$ for $i = m+1, \ldots, n$ we have

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{0}, \ldots, \mathbf{0}) = \varphi(\mathbf{x}_1, \ldots, \mathbf{x}_m).$$

Note that the conditions $\alpha_i = d_i/p_i$ and $q_i = 1$ for $i = m+1, \ldots, n$ ensure the continuity of the functions $f_i(\mathbf{x}_i)$. Therefore we obtain

$$\|f(\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{x}_{m+1}, \ldots, \mathbf{x}_n) - \varphi(\mathbf{x}_1, \ldots, \mathbf{x}_m)\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} =$$

$$= \left\| \varphi(\mathbf{x}_1, \ldots, \mathbf{x}_m) \left(\prod_{i=m+1}^{n} f_i(\mathbf{x}_i) - 1 \right) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \leq$$

$$\leq \|\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_m)\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \left| \prod_{i=m+1}^{n} f_i(\mathbf{x}_i) - 1 \right| \to 0$$

here $\max\limits_{i=m+1,\ldots,n} |\mathbf{x}_i| \to 0$.

These arguments show that φ is the trace of the function f.

The proof is complete.

Remark 3. Note that the continuation operator constructed in the proof of Theorem 2 is linear. We should note here that in the work of V.I. Burenkov and M.L. Goldman [26], where it is shown that in the limiting case for classical anisotropic Nikol'skii-Besov spaces it is possible to construct only a nonlinear continuation operator, but this effect is not observed for Nikol'skii-Besov spaces with a dominant mixed derivative.

Acknowledgments

Research was partially supported by the grant of the Science Committee of Ministry of Education and Science of the Republic of Kazakhstan (grant No. AP14869553).

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Жалпыланған аралас тегістілігі бар Никольский-Бесов кеңістіктеріндегі функциялар үшін іздер және жалғасулар теоремалары

Дифференциалданатын функциялар кеңістіктерінің енгізу теориясы әртүрлі метрикаларда функциялардың маңызды байланыстары мен функцияның дифференциалдық (тегістілік) қасиеттерінің қатынастарын зерттейді және де математикалық физиканың шектік есептер теориясында, жуықтау теориясында және математиканың басқа да салаларында кеңінен қолданысқа ие. Мақалада жалпыланған аралас тегістігі және аралас метрикасы бар Никольский—Бесов кеңістіктері функцияларының іздері мен жалғасы туралы теоремалар дәлелденген. Алынған нәтижелердің дәлелдеуі аралас метрикасы бар Лебег кеңістіктеріндегі тригонометриялық полиномдар үшін әр түрлі өлшемді теңсіздіктерін және классикалық Никольский—Бесов кеңістіктерінің үзіліссіз функциялар кеңістігіне ену теоремасын қолдануға негізделген.

Кілт сөздер: Никольский-Бесовтың кеңістігі, жалпыланған аралас тегістік, аралас метрика, функцияның ізі, функцияның жалғасы.

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Теоремы о следах и продолжениях для функций из пространств Никольского-Бесова с обобщенной смешанной гладкостью

Теория вложения пространств дифференцируемых функций изучает важные связи и соотношения дифференциальных (гладкостных) свойств функций в различных метриках и имеет широкое применение в теории краевых задач математической физики, теории приближений и других разделах математики. В данной статье мы доказываем теоремы о следах и продолжениях для функций из пространств Никольского-Бесова с обобщенной смешанной гладкостью и со смешанной метрикой. Доказательства полученных результатов основаны на использовании неравенства разных измерений для тригонометрических полиномов в пространствах Лебега со смешанной метрикой и теореме вложения классических пространств Никольского-Бесова в пространство непрерывных функций.

Ключевые слова: пространства Никольского-Бесова, обобщенная смешанная гладкость, смешанная метрика, след функции, продолжение функции.

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DOI 10.31489/2022M4/51-59
UDC 517.95

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Inverse coefficient problem for differential equation in partial derivatives of a fourth order in time with integral over-determination

Derivatives in time of higher order (more than two) arise in various fields such as acoustics, medical ultrasound, viscoelasticity and thermoelasticity. The inverse problems for higher order derivatives in time equations connected with recovery of the coefficient are scarce and need additional consideration. In this article the inverse problem of determination is considered, which depends on time, lowest term coefficient in differential equation in partial derivatives of fourth order in time with initial and boundary conditions from an additional integral observation. Under some conditions of regularity, consistency and orthogonality of data by using of the contraction principle the unique solvability of the solution of the coefficient identification problem on a sufficiently small time interval has been proved.

Keywords: Inverse problems for PDEs, fourth order in time PDE, existence and uniqueness.

Introduction

Fourth order derivative in time arises in various fields. For instance, in the Taylor series expansion of the Hubble law [1], in the study of chaotic hyper jerk systems [2] and in the kinematic performance of long-dwell mechanisms of linkage type [3]. The fourth order in time equation, that is our motivation point, was introduced and first studied by Dell'Oro and Pata [4]

$$\partial_{\tau\tau\tau\tau}u(x,\tau) + \alpha\partial_{\tau\tau\tau}u(x,\tau) + \beta\partial_{\tau\tau}u(x,\tau) - \gamma \triangle \partial_{\tau\tau}u(x,\tau) - \rho \triangle u(x,\tau) = 0,$$

where $\alpha, \beta, \gamma, \rho$ are real numbers. More recently, this model has attracted the attention of many authors [5–9].

We consider an inverse problem of recovering the time-dependent lowest term in the fourth order in time partial differential equation in the following type

$$\partial_{\tau\tau\tau\tau}u(x,\tau) + \partial_{\tau\tau}u(x,\tau) - \triangle\partial_{\tau\tau}u(x,\tau) - \triangle u(x,\tau) = a(\tau)u(x,\tau) + f(x,\tau) \quad (1)$$

subject to the initial conditions

$$u(x,0) = \xi_0(x),\ u_\tau(x,0) = \xi_1(x),\ u_{\tau\tau}(x,0) = \xi_2(x), u_{\tau\tau\tau}(x,0) = \xi_3(x) \quad (2)$$

and the boundary conditions

$$u(0,\tau) = u_x(1,\tau) = 0, \quad (3)$$

and the additional condition

$$\int_0^1 u(x,\tau)dx = E(\tau), \quad (4)$$

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where $D_T = \{(x,\tau) : 0 \le x \le 1,\ 0 \le \tau \le T\}$ for some fixed $T > 0$, $f(x,\tau)$ is the force function, $\xi_i(x)$, $i = 0,1,2,3$ are initial displacements, and $E(\tau)$ is the extra integral measurement to obtain the solution of the inverse problem.

The inverse coefficient problems for the first or second order in time (i.e. parabolic and hyperbolic equations, respectively) PDEs are studied satisfactorily. The inverse problems of the parabolic and hyperbolic PDEs investigated numerically and/or theoretically in [10–12] and [13, 14], respectively. The inverse problems of determining time or space dependent coefficients for the higher order in time (more than 2) PDEs attract many scientists. The inverse problem of recovering the solely space dependent and solely time dependent coefficients for the third order in time PDEs are studied by [15, 16], respectively. More recently, in [17] authors studied the inverse problem of determining time dependent potential and time dependent force terms from the third order in time partial differential equation theoretically and numerically by considering the critical parameter equal to zero.

Main purpose of this paper is the simultaneous identification of the time-dependent lowest coefficient $a(\tau)$, and $u(x,\tau)$, for the first time, from the equation (1), initial conditions (2), homogeneous boundary conditions (3) and additional condition (4) under some regularity and consistency conditions.

The article is organized as following: in Section 2, we first present the eigenvalues and eigenfunctions of the corresponding Sturm-Liouville spectral problem for equation (1), and two Banach spaces, which are related to the eigenvalues and eigenfunctions of the auxiliary Sturm-Liouville spectral problem, are introduced. Then, we transform the inverse problem into a system of Volterra integral equations by using the eigenfunction expansion method. Under some consistency and regularity conditions on data, the existence and uniqueness theorem of the solution of the inverse problem is proved via Banach fixed point theorem for sufficiently small times.

1 *Existence and Uniqueness*

In this section, we will set and prove the existence and uniqueness theorem of the solution of the inverse initial-boundary value problem for the fourth order in time equation by using Banach fixed point theorem.

The auxiliary spectral problem of the inverse problem (1)–(4) is

$$\begin{cases} Y''(x) + \lambda Y(x) = 0, \quad 0 \le x \le 1, \\ Y(0) = Y'(1) = 0. \end{cases} \tag{5}$$

The eigenvalues and corresponding eigenfunctions of these eigenvalues of the spectral problem (5) are $\mu_n = \left(\frac{2n+1}{2}\pi\right)^2$ and $Y_n(x) = \sqrt{2}\sin(\sqrt{\mu_n}x)$, $n = 0,1,2,...$, respectively. The system of eigenfunctions $Y_n(x)$ are biorthonormal on $[0,1]$, i.e.

$$\int_0^1 Y_n(x)Y_m(x)dx = \begin{cases} 1, & m = n \\ 0, & m \ne n \end{cases}.$$

Also the system of eigenfunctions $Y_n(x) = \sqrt{2}\sin(\sqrt{\mu_n}x)$, $n = 0,1,2,...$ forms a Riesz basis in $L_2[0,1]$.

Definition 1. Let the pair of functions $\{u(x,\tau), a(\tau)\}$ be from the class $C^{2,4}(D_T) \times C[0,T]$ and satisfies the equation (1) and conditions (2)–(4). Then the pair $\{u(x,\tau), a(\tau)\}$ is called the classical solution of the inverse problem (1)–(4).

Now, let us introduce two Banach spaces that are connected with the eigenvalues and eigenfunctions of the auxiliary spectral problem (5):

I.

$$B_T = \left\{ u(x,\tau) = \sum_{n=0}^{\infty} u_n(\tau) Y_n(x) : \ u_n(\tau) \in C[0,T], \right.$$

$$\left. J_T(u) = \left(\sum_{n=0}^{\infty} (\mu_n \left\| u_n(\tau) \right\|_{C[0,T]})^2 \right)^{1/2} < +\infty \right\},$$

where $u_n(\tau) = \sqrt{2} \int_0^1 u(x,\tau) \sin(\sqrt{\mu_n} x) dx$, and $J_T(u) := \left\| u(x,\tau) \right\|_{B_T}$ is the norm of the function $u(x,\tau)$.

II. $E_T = B_T \times C[0,T]$ is a Banach space with the norm

$$\left\| w(x,\tau) \right\|_{E_T} = \left\| u(x,\tau) \right\|_{B_T} + \left\| a(\tau) \right\|_{C[0,T]},$$

where $w(x,\tau) = \{u(x,\tau), a(\tau)\}$ is a vector function.

These spaces are suitable to investigate the solution of the inverse problem (1)–(4).

After giving these preliminary results, we can set and prove the existence and uniqueness theorem for the inverse problem (1)–(4):

Theorem 1. Let the assumptions

A$_1$ $\xi_0(x) \in C^1[0,1]$, $\xi_0''(x) \in L_2[0,1]$, $\xi_0(0) = \xi_0'(1) = 0$,

A$_2$ $\xi_1(x) \in C^1[0,1]$, $\xi_1''(x) \in L_2[0,1]$, $\xi_1(0) = \xi_1'(1) = 0$,

A$_3$ $\xi_2(x) \in C^1[0,1]$, $\xi_2''(x) \in L_2[0,1]$, $\xi_2(0) = \xi_2'(1) = 0$,

A$_4$ $\xi_3(x) \in C^1[0,1]$, $\xi_3''(x) \in L_2[0,1]$, $\xi_3(0) = \xi_3'(1) = 0$,

A$_5$ $E(\tau) \in C^4[0,T]$, $E(\tau) \neq 0 \ \forall \tau \in [0,T]$, $E^{(i)}(0) = \int_0^1 \xi_i(x) dx$, $i = 0,1,2,3$ and $E^{(i)}(\tau) = \frac{d^i}{d\tau^i} E(\tau)$,

A$_6$ $f(x,\tau) \in C(\overline{D}_T)$, $f_x, f_{xx} \in C[0,1]$, $\forall \tau \in [0,T]$, $f(0,\tau) = f_x(1,\tau) = 0$,

be satisfied, and $\Delta = (1+\mu_n)^2 - 4\mu_n > 0$. Then, the inverse problem (1)–(4) has a unique solution for small T.

Proof. For arbitrary $a(\tau) \in C[0,T]$, to construct the formal solution of the inverse problem (1)–(4), we will use the Fourier (Eigenfunction expansion) method. In accordance with this, let us consider

$$u(x,\tau) = \sum_{n=0}^{\infty} u_n(\tau) Y_n(x), \tag{6}$$

is a solution of the inverse problem (1)–(4), where $Y_n(x) = \sqrt{2} \sin(\sqrt{\mu_n} x)$, $n = 0,1,2,...$ are the eigenfunctions and $\mu_n = \left(\frac{2n+1}{2} \pi \right)^2$, $n = 0,1,2,...$ are the eigenvalues of the corresponding spectral problem.

Since $u(x,\tau)$ is the formal solution of the inverse problem (1)-(4), we get the following Cauchy problems with respect to $u_n(\tau)$ from the equation (1) and initial conditions (2);

$$\begin{cases} u_n^{(4)}(\tau) + (1+\mu_n) u_n''(\tau) + \mu_n u_n(\tau) = F_n(\tau; a, u), \\ u_n(0) = \xi_{0n}, \ u_n'(0) = \xi_{1n}, \ u_n''(0) = \xi_{2n}, u_n'''(0) = \xi_{3n}, \ n = 0,1,2,.... \end{cases} \tag{7}$$

Here $F_n(\tau; a, u) = a(\tau)u_n(\tau) + f_n(\tau)$, $u_n(\tau) = \sqrt{2}\int_0^1 u(x,\tau)\sin(\sqrt{\mu_n}x)dx$, $f_n(\tau) = \sqrt{2}\int_0^1 f(x,\tau)\sin(\sqrt{\mu_n}x)dx$, and $\xi_{in} = \sqrt{2}\int_0^1 \xi_i(x)\sin(\sqrt{\mu_n}x)dx$, $i = 0,1,2,3,\ n = 0,1,2,....$

These Cauchy problems have the quartic characteristic polynomial

$$P_4(k) = k^4 + (1+\mu_n)k^2 + \mu_n.$$

If we convert this quartic equation to a quadratic equation by changing the variable $s = k^2$, we obtain

$$P_2(s) = s^2 + (1+\mu_n)s + \mu_n.$$

It is easy to see that $\Delta = (1+\mu_n)^2 - 4\mu_n = (\mu_n - 1)^2$ and that is always positive. Then $P_2(s) = 0$ has two real roots

$$s_1 = -1,\ s_2 = -\mu_n.$$

Thus $P_4(k) = 0$ has four complex conjugate roots

$$k_{1,2} = \pm i,\ k_{3,4} = \pm i\sqrt{\mu_n}.$$

Solving (7) by using the these roots of the characteristic polynomial, we obtain

$$u_n(\tau) = \frac{\mu_n\cos(\tau) - \cos(\sqrt{\mu_n}\tau)}{\mu_n - 1}\xi_{0n} + \frac{\mu_n^{3/2}\sin(\tau) - \sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2} - \sqrt{\mu_n}}\xi_{1n} +$$
$$+\frac{\cos(\tau) - \cos(\sqrt{\mu_n}\tau)}{\mu_n - 1}\xi_{2n} + \frac{\sqrt{\mu_n}\sin(\tau) - \sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2} - \sqrt{\mu_n}}\xi_{3n} +$$
$$+\int_0^\tau \frac{\sqrt{\mu_n}\sin((\tau - s)) - \sin(\sqrt{\mu_n}(\tau - s))}{\mu_n^{3/2} - \sqrt{\mu_n}}F_n(s; a, u)ds. \quad (8)$$

Substitute the expression (8) into (6) to determine $u(x,\tau)$. Then we get

$$u(x,\tau) = \sum_{n=0}^\infty \left[\frac{\mu_n\cos(\tau)-\cos(\sqrt{\mu_n}\tau)}{\mu_n-1}\xi_{0n} + \frac{\mu_n^{3/2}\sin(\tau)-\sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2}-\sqrt{\mu_n}}\xi_{1n} + \right.$$
$$+\frac{\cos(\tau)-\cos(\sqrt{\mu_n}\tau)}{\mu_n-1}\xi_{2n} + \frac{\sqrt{\mu_n}\sin(\tau)-\sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2}-\sqrt{\mu_n}}\xi_{3n} + \quad (9)$$
$$\left. +\int_0^\tau \frac{\sqrt{\mu_n}\sin((\tau-s))-\sin(\sqrt{\mu_n}(\tau-s))}{\mu_n^{3/2}-\sqrt{\mu_n}}F_n(s;a,u)ds\right]Y_n(x).$$

Let us derive the equation of $a(\tau)$. If we integrate the equation (1) from $x = 0$ to $x = 1$ with respect to x, and consider the additional condition (4), then we have:

$$a(\tau) = \frac{1}{E(\tau)}\left[E^{(4)}(\tau) + E''(\tau) - f_{int}(\tau) + \sum_{n=0}^\infty \sqrt{\mu_n}\left(u_n''(\tau) + u_n(\tau)\right)\right], \quad (10)$$

where $f_{int}(\tau) = \int_0^1 f(x,\tau)dx$. If we consider $u_n(\tau)$ which is defined in (8) and its second derivative into the last equation, we get

$$a(\tau) = \frac{1}{E(\tau)}\left[E^{(4)}(\tau) + E''(\tau) - f_{int}(\tau) + \sum_{n=0}^\infty \sqrt{\mu_n}\left(\cos(\sqrt{\mu_n}\tau)\xi_{0n} + \frac{\sin(\sqrt{\mu_n}\tau)}{\sqrt{\mu_n}}\xi_{1n} + \right.\right.$$
$$\left.\left. +\cos(\sqrt{\mu_n}\tau)\xi_{2n} + \frac{\sin(\sqrt{\mu_n}\tau)}{\sqrt{\mu_n}}\xi_{3n} + \int_0^\tau \frac{1}{\sqrt{\mu_n}}\sin(\sqrt{\mu_n}(\tau - s))F_n(s;a,u)ds\right)\right]. \quad (11)$$

We convert the inverse problem (1)–(4) into the system of Volterra integral equations (9)–(10) with respect to $u(x,\tau)$ and $a(\tau)$ by considering

$$u_n(\tau)=\int_0^1 u(x,\tau)Y_n(x)dx,\ n=0,1,2,...$$

is the solution of the system of differential equations (7). Analogously, we can prove that if $\{u(x,\tau),a(\tau)\}$ is a solution of the inverse problem (1)–(4), then $u_n(\tau),\ n=0,1,2,...$satisfy the system of differential equations (7). For proof of this assertion please see [18]. From this assertion we can conclude that proving the uniqueness of the solution of the inverse problem (1)–(4), it suffices to prove the uniqueness of the solution of the system (9) and (11).

To prove the existence of a unique solution of the system (9) and (11) we need to rewrite this system into operator form and to show that this operator a contraction operator. To this end let us denote $w(x,\tau)=[u(x,\tau),a(\tau)]^T$ is a 2×1 vector function and rewrite the system of equations (9) and (11) in the following operator equation

$$w=\mathbf{\Pi}(w), \tag{12}$$

where $\mathbf{\Pi}(w)\equiv[\mathit{\Pi}_1,\mathit{\Pi}_2]^T$ and $\mathit{\Pi}_1$ and $\mathit{\Pi}_1$ are equal to the right hand sides of (9) and (11), respectively.

Using integration by parts under the assumptions $(A_1)-(A_6)$, we obtain following equalities

$$\xi_{0n}=\tfrac{1}{\mu_n}\alpha_{0n},\ \xi_{1n}=\tfrac{1}{\mu_n}\alpha_{1n},\ \xi_{2n}=\tfrac{1}{\mu_n}\alpha_{2n},\ \xi_{3n}=\tfrac{1}{\mu_n}\alpha_{3n},\ f_n(\tau)=\tfrac{1}{\mu_n}\omega_n(\tau),$$

where $\omega_n(\tau)=-\sqrt{2}\int_0^1 f_{xx}(x,\tau)\sin(\sqrt{\mu_n}x)dx,\ \alpha_{in}=-\sqrt{2}\int_0^1\xi_i''(x)\sin(\sqrt{\mu_n}x)dx,\ i=0,1,2,3.$ Since $\sqrt{2}\sin(\sqrt{\mu_n}x)$ forms a biorthonormal system of functions on $[0,1]$, by using Bessel's inequality we get

$$\sum_{n=0}^{\infty}|\alpha_{in}|^2\le\left\|\xi_i''\right\|^2_{L_2[0,1]},\ i=0,1,2,3,\ \sum_{n=0}^{\infty}|\omega_n(\tau)|^2\le\|f_{xx}(\cdot,\tau)\|^2_{L_2[0,1]}. \tag{13}$$

Before showing that Φ is a contraction operator, let us find the estimates for the coefficients arising in the operator equations (9) and (11):

$$\begin{aligned}\left|\frac{\mu_n\cos(\tau)-\cos(\sqrt{\mu_n}\tau)}{\mu_n-1}\right| &\le \frac{\mu_n+1}{\mu_n-1}=d_n^1,\left|\frac{\mu_n^{3/2}\sin(\tau)-\sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2}-\sqrt{\mu_n}}\right|\le\frac{\mu_n^{3/2}+1}{\mu_n^{3/2}-\sqrt{\mu_n}}=d_n^2,\\ \left|\frac{\cos(\tau)-\cos(\sqrt{\mu_n}\tau)}{\mu_n-1}\right| &\le \frac{2}{\mu_n-1}=d_n^3,\left|\frac{\sqrt{\mu_n}\sin(\tau)-\sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2}-\sqrt{\mu_n}}\right|\le\frac{\sqrt{\mu_n}+1}{\mu_n^{3/2}-\sqrt{\mu_n}}=d_n^4.\end{aligned}$$

Since the sequences $d_n^i,\ i=1,2,3,4$ are convergent, they are bounded. Consider that

$$d_n^i\le m_i,\ \text{for each}\ i=1,2,3,4, \tag{14}$$

where m_i are real constants.

Now we can show in two steps that $\mathbf{\Pi}$ is a contraction operator by considering the assumptions and estimates are given above.

I) First let us demonstrate that $\mathbf{\Pi}$ is a continuous map which maps the space E_T onto itself continuously. That is to say, our aim is to show $\mathit{\Pi}_1(z)\in B_T$ and $\mathit{\Pi}_2(z)\in C[0,T]$ for arbitrary $w=[u(x,\tau),a(\tau)]^T$ such that $u(x,\tau)\in B_T$, $a(\tau)\in C[0,T]$.

Let us start with showing that $\mathit{\Pi}_1(z)\in B_T$, i.e. we need to verify

$$J_T(\mathit{\Pi}_1)=\left(\sum_{n=0}^{\infty}(\mu_n\left\|\mathit{\Pi}_{1,n}(\tau)\right\|_{C[0,T]})^2\right)^{1/2}<+\infty,$$

where

$$\mathit{\Pi}_{1,n}(\tau)=\frac{\mu_n\cos(\tau)-\cos(\sqrt{\mu_n}\tau)}{\mu_n-1}\xi_{0n}+\frac{\mu_n^{3/2}\sin(\tau)-\sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2}-\sqrt{\mu_n}}\xi_{1n}+$$

$$+\frac{\cos(\tau)-\cos(\sqrt{\mu_n}\tau)}{\mu_n-1}\xi_{2n}+\frac{\sqrt{\mu_n}\sin(\tau)-\sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2}-\sqrt{\mu_n}}\xi_{3n}+$$

$$+\int_0^\tau\frac{\sqrt{\mu_n}\sin((\tau-s))-\sin(\sqrt{\mu_n}(\tau-s))}{\mu_n^{3/2}-\sqrt{\mu_n}}F_n(s;a,u)ds.$$

After some manipulations under the assumptions $(A_1)-(A_6)$, using the estimates (14) we obtain

$$(J_T(\mathit{\Pi}_1))^2=\sum_{n=0}^{\infty}(\mu_n\left\|\mathit{\Pi}_{1,n}(\tau)\right\|_{C[0,T]})^2\le$$

$$\le 6\sum_{i=0}^{3}m_{i+1}^2\sum_{n=0}^{\infty}|\alpha_{in}|^2+6m_4^2T^2\sum_{n=0}^{\infty}\left(\max_{0\le\tau\le T}|\omega_n(\tau)|\right)^2+6T^2\left\|a(\tau)\right\|_{C[0,T]}^2\sum_{n=0}^{\infty}\left(\mu_n\left\|u_n(\tau)\right\|_{C[0,T]}\right)^2.$$

Since $u(x,\tau),a(\tau)$ belong to the spaces B_T, and $C[0,T]$, respectively, the series at the right hand side of $(J_T(\mathit{\Pi}_1))^2$ are convergent from the Bessel's inequality (considering the estimates (13)). $J_T(\mathit{\Pi}_1)$ is convergent (i.e. $J_T(\mathit{\Pi}_1)<+\infty$) because $(J_T(\mathit{\Pi}_1))^2$ is bounded above. Thus we can conclude that $\mathit{\Pi}_1(z)$ belongs to the space B_T.

Now let us verify $\mathit{\Pi}_2(w)\in C[0,T]$. From the equation (10) we have

$$|\mathit{\Pi}_2(w)|\le\frac{1}{\min\limits_{0\le\tau\le T}|E(\tau)|}\left[\left|E^{(4)}(\tau)\right|+\left|E''(\tau)\right|+|f_{int}(\tau)|+\sum_{n=0}^{\infty}\sqrt{\lambda_n}\left(\left|u_n''(\tau)\right|+|u_n(\tau)|\right)\right].$$

Using the Cauchy-Schwartz inequality and the estimates are given in (13) and (14) we obtain

$$\max_{0\le\tau\le T}|\mathit{\Pi}_2(z)|\le\frac{1}{\min\limits_{0\le\tau\le T}|E(\tau)|}\left[\max_{0\le\tau\le T}\left|E^{(4)}(\tau)\right|+\max_{0\le\tau\le T}\left|E''(\tau)\right|+\max_{0\le\tau\le T}|f_{int}(\tau)|+\right.$$

$$+\left(\sum_{n=0}^{\infty}\frac{1}{\mu_n}\right)^{1/2}\left\{\left(\sum_{n=0}^{\infty}|\alpha_{0n}|^2\right)^{1/2}+\left(\sum_{n=0}^{\infty}|\alpha_{2n}|^2\right)^{1/2}\right\}+\left(\sum_{n=0}^{\infty}\frac{1}{\mu_n^2}\right)^{1/2}\left\{\left(\sum_{n=0}^{\infty}|\alpha_{1n}|^2\right)^{1/2}+\right.$$

$$\left.+\left(\sum_{n=0}^{\infty}|\alpha_{3n}|^2\right)^{1/2}+T\left(\sum_{n=0}^{\infty}\left(\max_{0\le\tau\le T}|\omega_n(\tau)|\right)^2\right)^{1/2}+T\left\|a(\tau)\right\|_{C[0,T]}^2\left(\sum_{n=0}^{\infty}\left(\mu_n\left\|u_n(\tau)\right\|_{C[0,T]}\right)^2\right)^{1/2}\right].\qquad(15)$$

Considering the estimates (13) and $\sum_{n=0}^{\infty}\frac{1}{\mu_n}$, $\sum_{n=0}^{\infty}\frac{1}{\mu_n^2}$ are convergent, the majorizing series (15) are also convergent. According to Weierstrass M-test, $\mathit{\Pi}_2(z)$ is continuous and belongs to the space $C[0,T]$.

Therefore, we show that $\mathbf{\Pi}$ maps E_T onto itself continuously.

II) Since $\mathbf{\Pi}$ maps E_T onto itself continuously, let us show that $\mathbf{\Pi}$ is contraction mapping operator. Assume that let w_1 and w_2 be any two elements of E_T such that $w_i(x,\tau)=\left[u^{(i)}(x,\tau),a^{(i)}(\tau)\right]^T$, $i=1,2$. From the definition of the space E_T, we have $\left\|\mathbf{\Pi}(w_1)-\mathbf{\Pi}(w_2)\right\|_{E_T}=\left\|\mathit{\Pi}_1(w_1)-\mathit{\Pi}_1(w_2)\right\|_{B_T}+\left\|\mathit{\Pi}_2(w_1)-\mathit{\Pi}_2(w_2)\right\|_{C[0,T]}$. For the convenience of this norm, let us consider the following differences:

$$\mathit{\Pi}_1(w_1)-\mathit{\Pi}_1(w_2)=$$
$$=\sum_{n=0}^{\infty}\int_0^\tau\frac{\sqrt{\mu_n}\sin((\tau-s))-\sin(\sqrt{\mu_n}(\tau-s))}{\mu_n^{3/2}-\sqrt{\mu_n}}\left(F_n(s;a^1,u^1)-F_n(s;a^2,u^2)\right)dsY_n(x)$$

$$\varPi_2(w_1) - \varPi_2(w_2) = \frac{1}{E(t)} \sum_{n=0}^{\infty} \int_0^{\tau} \frac{1}{\sqrt{\mu_n}} \sin(\sqrt{\mu_n}(\tau - s)) \left(F_n(s; a^1, u^1) - F_n(s; a^2, u^2)\right) ds.$$

After some manipulations in last equations under the assumptions (A_1)–(A_6) and using the estimates (13)–(14), we obtain

$$\|\varPi_1(z_1) - \varPi_1(z_2)\|_{B_T} \le \sqrt{2} m_4 T \left[\left\|a^{(1)}\right\|_{C[0,T]} \left\|u^{(1)} - u^{(2)}\right\|_{B_T} + \left\|u^{(2)}\right\|_{B_T} \left\|a^{(1)} - a^{(2)}\right\|_{C[0,T]} \right],$$

$$\|\varPi_2(z_1) - \varPi_2(z_2)\|_{C[0,T]} \le$$

$$\le \frac{T}{\min\limits_{0 \le t \le T} |E(t)|} \left(\sum_{n=0}^{\infty} \frac{1}{\mu_n} \right)^{1/2} \left[\left\|a^{(1)}\right\|_{C[0,T]} \left\|u^{(1)} - u^{(2)}\right\|_{B_T} + \left\|u^{(2)}\right\|_{B_T} \left\|a^{(1)} - a^{(2)}\right\|_{C[0,T]} \right].$$

From the last inequalities it follows that

$$\|\mathbf{\Pi}(z_1) - \mathbf{\Pi}(z_2)\|_{E_T} \le C(T, a^{(1)}, u^{(2)}) \, \|z_1 - z_2\|_{E_T},$$

where $C(T, a^{(1)}, u^{(2)}) = T \left(\|a^{(1)}\|_{C[0,T]} + \|u^{(2)}\|_{B_T} \right) \left(\sqrt{2} m_4 + \frac{1}{\min\limits_{0 \le \tau \le T} |E(\tau)|} \left(\sum_{n=0}^{\infty} \frac{1}{\mu_n} \right)^{1/2} \right)$.

Since $E(\tau) \in C^4[0,T]$, $E(\tau) \ne 0 \ \forall \tau \in [0,T]$, $a^{(1)}(\tau) \in C[0,T]$, $u^{(2)}(x,\tau) \in B_T$ and m_4 is a finite constant, $\left(\|a^{(1)}\|_{C[0,T]} + \|u^{(2)}\|_{B_T} \right) \left(\sqrt{2} m_4 + \frac{1}{\min\limits_{0 \le \tau \le T} |E(\tau)|} \left(\sum_{n=0}^{\infty} \frac{1}{\mu_n} \right)^{1/2} \right)$ is bounded above. Thus $C(T, a^{(1)}, u^{(2)})$ tends to zero as $T \to 0$. In other words, for sufficiently small T we have $0 < C(T, a^{(1)}, u^{(2)}) < 1$. This means that the operator $\mathbf{\Pi}$ is a contraction mapping operator.

From the first and second steps, the operator $\mathbf{\Pi}$ is contraction mapping operator that is a continuous and onto map on E_T. Then according to Banach fixed point theorem the solution of the operator equation (12) exists and it is unique.

2 Conclusion

The paper considers the inverse problem of determining the time dependent lowest term coefficient in fourth order in time partial differential equation with initial and boundary conditions from an additional observation. The unique solvability of the solution of the inverse problem on a sufficiently small time interval has been proved by using of the contraction principle. The proposed work is a novel and has never been solved theoretically nor numerically before. Our results shed light on the methodology for the existence and uniqueness of the inverse problem for the fourth order in time PDEs in two dimensions.

Acknowledgments

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Интегралдық түрлендіруі бар уақыт бойынша дербес туындылы төртінші ретті дифференциалдық теңдеу үшін коэффициентті кері есебі

Уақыт бойынша жоғары ретті (екіден көп) туындылар акустика, медициналық ультрадыбыста, тұтқырлық және жылу серпімділігі сияқты әртүрлі салаларда пайда болады. Коэффициентті қалпына келтіруге байланысты уақыт бойынша теңдеулердегі жоғары туындылар үшін кері есептер аз және қосымша қарауды қажет етеді. Мақалада дифференциалдық теңдеудегі уақытқа тәуелді кіші коэффициентке қосымша интегралды бақылау жүргізіп, уақыт бойынша бастапқы және шекаралық шарттары бар төртінші ретті дербес туындылы анықтаудың кері есебі қарастырылған. Сығымдау принципін қолдана отырып, шарттардың регулярлығы, қарама-қайшы болмауы және ортогоналдылығының кейбір жағдайларында коэффициенттерді жеткілікті аз уақыт аралығында анықтау есебін шешудің бір мәнді шешімділігі дәлелденді.

Кілт сөздер: дербес туындылы дифференциалдық теңдеулер үшін кері есептер, уақыт бойынша төртінші ретті дербес туындылы дифференциалдық теңдеулер, бар болуы және жалғыздығы.

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Обратная коэффициентная задача для дифференциального уравнения в частных производных четвертого порядка по времени с интегральным переопределением

Производные по времени более высокого порядка (больше двух) возникают в различных областях, таких как акустика, медицинский ультразвук, вязкоупругость и термоупругость. Обратные задачи для высших производных в уравнениях по времени, связанные с восстановлением коэффициента, немногочисленны и требуют дополнительного рассмотрения. В статье рассмотрена обратная задача определения, зависящая от времени, младшего коэффициента в дифференциальном уравнении в частных производных четвертого порядка по времени с начальными и граничными условиями по дополнительному интегральному наблюдению. При некоторых условиях регулярности, непротиворечивости и ортогональности данных с использованием принципа сжатия доказана однозначная разрешимость решения задачи определения коэффициентов на достаточно малом интервале времени.

Ключевые слова: обратные задачи для УрЧП, УрЧП четвертого порядка по времени, существование и единственность.

DOI 10.31489/2022M4/60-75

UDC 517.928

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Systems of integral equations with a degenerate kernel and an algorithm for their solution using the Maple program

In the mathematical literature, a scalar integral equation with a degenerate kernel is well described (see below (1)), where all the written functions are scalar quantities). The authors are not aware of publications where systems of integral equations of (1) type with kernels in the form of a product of matrices would be considered in detail. It is usually said that the technique for solving such systems is easily transferred from the scalar case to the vector one (for example, in the monograph A.L. Kalashnikov "Methods for the approximate solution of integral equations of the second kind" (Nizhny Novgorod: Nizhny Novgorod State University, 2017), a brief description of systems of equations with degenerate kernels is given, where the role of degenerate kernels is played by products of scalar rather than matrix functions). However, as the simplest examples show, the generalization of the ideas of the scalar case to the case of integral systems with kernels in the form of a sum of products of matrix functions is rather unclear, although in this case the idea of reducing an integral equation to an algebraic system is also used. At the same time, the process of obtaining the conditions for the solvability of an integral system in the form of orthogonality conditions, based on the conditions for the solvability of the corresponding algebraic system, as it seems to us, has not been previously described. Bearing in mind the wide applications of the theory of integral equations in applied problems, the authors considered it necessary to give a detailed scheme for solving integral systems with degenerate kernels in the multidimensional case and to implement this scheme in the Maple program. Note that only scalar integral equations are solved in Maple using the *intsolve* procedure. The authors did not find a similar procedure for solving systems of integral equations, so they developed their own procedure.

Keywords: integral operator, degenerate kernel, Maple program procedure, scalar integral equation.

1 Fredholm integral equations with a degenerate kernel (general theory)

Consider the integral system

$$y(t) = \lambda \sum_{j=1}^{m} A_j(t) \int_0^T B_j(s) y(s) ds + h(t). \tag{1}$$

Let the expressions $A_j(t)$ and $B_j(s)$, forming the kernel of the integral operator in it be matrix functions (their smoothness and dimensions are specified below). Just as in the one-dimensional case [1–4], such systems can be reduced to algebraic systems using the following operations. Denote

$$w_j = \int_0^T B_j(s) y(s) ds, \;\; j = \overline{1, m}. \tag{2}$$

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Then instead of (1) we get the equality

$$y(t) = \lambda \sum_{j=1}^{m} A_j(t) w_j + h(t). \tag{3}$$

Multiplying in turn equality (3) on the left by matrices $B_1(t), ..., B_m(t)$ and integrating the resulting equalities with respect to $t \in [0, T]$, we get

$$\int_0^T B_i(t) y(t) dt = \lambda \sum_{j=1}^{m} \left(\int_0^T B_i(t) A_j(t) dt \right) w_j + \int_0^T B_i(t) h(t) dt, \;\; i = \overline{1, m}.$$

Using (2), we obtain the algebraic system of equations

$$w_i = \lambda \sum_{j=1}^{m} c_{ij} w_j + H_i \;\; (i = \overline{1, m}), \tag{4}$$

where indicated: $c_{ij} = \int_0^T B_i(t) A_j(t) dt$, $H_i = \int_0^T B_i(t) h(t) dt$, $i, j = \overline{1, m}$. Now let us refine the conditions on the matrices $A_j(t)$, $B_j(t), j = \overline{1, m}$. It is clear that these matrices must be integrable on the $[0, T]$. We assume that all their elements are continuous on the segment $[0, T]$. In addition, there must be matrices $A_j B_j, B_i A_j, \sum_{i=1}^{m} B_i A_j, B_j y, B_j h$, so their sizes must be consistent for all $i, j = \overline{1, m}$. This can be achieved if we take all matrices $A_i(t)$ of the same size $n \times p$ and all matrices $B_j(t)$ of the same size $p \times n$, where p is any natural number. Then the vector w_i will be a column of the size $p \times 1$, c_{ij} *is* $(p \times p)$-matrix, H_j *is* $(p \times 1)$-vector, $i, j = \overline{1, n}$. Introduce the vectors $w = \{w_1, \ldots, w_m\}$, $H = \{H_1, \ldots, H_m\}$ of the size $(pm) \times 1$ and the matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1m} \\ c_{21} & c_{22} & \ldots & c_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \ldots & c_{mm} \end{pmatrix}.$$

This matrix is square in size $(mp) \times (mp)$. Now system (4) can be written as follows:

$$w = \lambda C w + H \Leftrightarrow (I - \lambda C) w = H. \tag{5}$$

For $\lambda = 0$ system (5) has an obvious solution $w = H$, so we will assume that $\lambda \neq 0$. In this case, system (5) can be rewritten as

$$(\mu I - C)\, w = \mu H \quad \left(\mu = \frac{1}{\lambda}\right). \tag{6}$$

Now let's establish a connection between system (6) and system (1). These systems are equivalent in the following sense: *if there exists the solution $y = y(t) \in C([0, T], \mathbb{C}^n)$ of the system (1), then there exists the corresponding solution*

$$w = \{ \int_0^T B_1(s) y(s) ds, \; \ldots, \int_0^T B_m(s) y(s) ds \}$$

of the system (6). Conversely, if there exists the solution $w = \{w_1, \ldots, w_m\} \in \mathbb{C}^{mp}$ of the system (6), then there is the solution $y(t) = \lambda \sum_{j=1}^{m} A_j(t) w_j + h(t)$of the original system (1).

The last statement needs proof, but we will not carry it out. Let us find out in which case different solutions of the system (6) generate different solutions of the integral system (1). So, let $w = \{w_1, \ldots, w_m\}$ and $\tilde{w} = \{\tilde{w}_1, \ldots, \tilde{w}_m\}$ there be different solutions of the system (6). Then the solutions $y(t)$ and $\tilde{y}(t)$ of the integral system (1), corresponding to them, will coincide, if

$$\sum_{j=1}^{m} A_j(t)w_j \equiv \sum_{j=1}^{m} A_j(t)\tilde{w}_j \Leftrightarrow \sum_{j=1}^{m} A_j(t)(w_j - \tilde{w}_j) \equiv 0 \ \ (\forall t \in [0,T]). \tag{7}$$

If we denote by $A_j^{(k)}$ the k-th column of the matrix A_j, and by $w_j^{(k)}$, $\tilde{w}_j^{(k)}$ the k-th components of the vectors w_j and $\tilde{w}_j$ respectively, then identity (7) can be written in the form

$$\sum_{j=1}^{m}\sum_{k=1}^{p} A_j^{(k)}(t) \cdot (w_j^{(k)} - \tilde{w}_j^{(k)}) \equiv 0 \ (\forall t \in [0,T]). \tag{8}$$

Since $w \neq \tilde{w}$, then at least one of the differences $w_j^{(k)} - \tilde{w}_j^{(k)}$ is not equal to zero, therefore the identity (8) means that the columns of the matrix

$$S(t) = (A_1^{(1)}(t), \ldots, A_1^{(p)}(t); A_2^{(1)}(t), \ldots, A_2^{(p)}(t); \ldots; A_m^{(1)}(t), \ldots, A_m^{(p)}(t))$$

are linearly dependent on the segment $[0,T]$. Hence, if the columns of the matrix $S(t)$ are linearly independent on the segment $[0,T]$, then it follows from the identity (8) that everything $w_j^{(k)} \equiv \tilde{w}_j^{(k)}$, and therefore $y(t) \equiv \tilde{y}(t)$. So, in the case of linear independence on the segment $[0,T]$ of the columns of the matrix $S(t)$, the correspondence $w \to y(t)$ will be one-to-one ($w \leftrightarrow y(t)$), therefore, in this case, we can replace the study of the solvability of the system (1) with the study of the solvability of the algebraic system of equations (6) (or what is the same system (5)). Henceforth, we will assume that *the columns of the matrix* $S(t)$ *are linearly independent on a segment* $[0,T]$. Systems of type (6) are well studied in linear algebra. It is known that if $\mu = \frac{1}{\lambda}$ is not an eigenvalue of the matrix C, then the homogeneous system $(\mu I - C)w = 0$ has only a trivial solution $w = 0$. This means that the corresponding integral system (1) has a solution for any right side $h(t) \in \mathrm{C}([0,T], \mathbb{C}^n)$, which can be written as

$$y(t) = \frac{1}{\mu}\sum_{j=1}^{m} A_j(t)w_j + h(t) \ \ (w \equiv \{w_1, \ldots, w_m\}).$$

If $\mu = \frac{1}{\lambda}$ $(\lambda \neq 0)$ is an eigenvalue of the geometric multiplicity r of the matrix C, then the homogeneous system $(\mu I - C)w = 0$ has the basic system $w^{(1)}, \ldots, w^{(r)}$ of solutions, and its general solution can be written as

$$w = \alpha_1 w^{(1)} + \ldots + \alpha_r w^{(r)},$$

where $\alpha_1, \ldots, \alpha_r$ are arbitrary constants. In this case, the conjugate homogeneous system $(\bar{\mu}I - C^*)z = 0$ also has a basic system $z^{(1)}, \ldots, z^{(r)}$ of solutions, consisting of r vectors. In order for the inhomogeneous system (6) to have a solution, it is necessary and sufficient that its right side be orthogonal to all vectors of the basis system of solutions of the adjoint homogeneous system:

$$\left(\mu H, z^{(j)}\right) = 0 \Leftrightarrow \left(H, z^{(j)}\right) = 0, \, j = \overline{1,r}. \tag{9}$$

In this case, the inhomogeneous system (6) has the following solution:

$$w = \alpha_1 w^{(1)} + \ldots + \alpha_r w^{(r)} + \tilde{w}, \tag{10}$$

where $\alpha_1, \ldots, \alpha_r$ are arbitrary constants, $w = \tilde{w}$ is a particular solution of the system (6) (or, what is the same, of the system (5)). Let's see which condition for the original integral system (1) is equivalent to condition (9). For this, we write a homogeneous adjoint equation for (1):

$$\hat{y}(t) = \bar{\lambda} \sum_{j=1}^{m} \int_0^T B_j^*(t) A_j^*(s) \hat{y}(s) ds \Leftrightarrow$$

$$\Leftrightarrow \hat{y}(t) = \bar{\lambda} \textstyle\sum_{j=1}^{m} B_j^*(t) \int_0^T A_j^*(s) \hat{y}(s) ds. \tag{11}$$

Denoting $z_j = \int_0^T A_j^*(s)\hat{y}(s)ds$, $j = \overline{1,m}$, we rewrite system (11) as

$$\hat{y}(t) = \bar{\lambda} \sum_{j=1}^{m} B_j^*(t) z_j. \tag{12}$$

Multiplying both parts of (12) on the left by $A_i^*(t)$ and integrating over $t \in [0,T]$, we obtain

$$\begin{aligned} &\int_0^T A_i^*(t)\hat{y}(t)dt = \bar{\lambda} \textstyle\sum_{j=1}^{m} (\int_0^T A_i^*(t) B_j^*(t) dt) z_j \Leftrightarrow \\ &\Leftrightarrow z_i = \bar{\lambda} \textstyle\sum_{j=1}^{m} d_{ij} z_j, \; i = \overline{1,m}, \end{aligned} \tag{12_1}$$

where indicated: $d_{ij} = \int_0^T A_i^*(t) B_j^*(t) dt$, $i, j = \overline{1,m}$. It is easy to see that $d_{ij} = c_{ji}^*$, where c_{ij} are the matrices involved in system (4). The matrix of the system (12_1) has the form

$$C^* = \begin{pmatrix} c_{11}^* & c_{21}^* & \cdots & c_{m1}^* \\ c_{12}^* & c_{22}^* & \cdots & c_{m2}^* \\ \cdots & \cdots & \cdots & \cdots \\ c_{1m}^* & c_{2m}^* & \cdots & c_{mm}^* \end{pmatrix},$$

therefore, the algebraic system corresponding to the homogeneous conjugate integral equation (11) will be as follows:

$$\begin{aligned} &z = \bar{\lambda} C^* z \Leftrightarrow (I - \bar{\lambda} C^*) z = 0 \Leftrightarrow \\ &\Leftrightarrow (\bar{\mu} I - C^*) z = 0 \; (\mu = \tfrac{1}{\lambda}, \lambda \neq 0). \end{aligned} \tag{13}$$

All solutions of the adjoint equation (11) are found from (12), where $z = \{z_1, \ldots, z_m\}$ is the solution of the system (13). Orthogonality (9) means that (take into account that

$$H = \{\int_0^T B_1(t)h(t)dt, \ldots, \int_0^T B_m(t)h(t)dt, \; z^{(j)} = \{z_1^{(j)}, \ldots, z_m^{(j)}\})$$

$$\sum_{i=1}^{m} \int_0^T (B_i(t)h(t)dt, z_i^{(j)})dt = 0 \Leftrightarrow \sum_{i=1}^{m} \int_0^T (h(t), B_i^*(t) z_i^{(j)})dt = 0 \Leftrightarrow$$

$$\Leftrightarrow \int\limits_0^T (h(t), \bar{\lambda} \sum_{i=1}^{m} B_i^*(t) z_i^{(j)}) dt = 0, j = \overline{1, r}.$$

According to (12), we obtain from this that the orthogonality conditions (9) are equivalent to the conditions

$$\int\limits_0^T (h(t), \hat{y}^{(j)}(t)) dt = 0, j = \overline{1, r}, \tag{14}$$

where $\hat{y}^{(1)}(t), \ldots, \hat{y}^{(m)}(t)$ is the basic system of solutions of the conjugate homogeneous integral system (11). Thus, if $\mu = \frac{1}{\lambda}(\lambda \neq 0)$ is an eigenvalue of the geometric multiplicity r of the kernel of equation (1), then for the solvability of the integral system (1) in the space $\mathrm{C}([0,T], \mathbb{C}^n)$, it is necessary and sufficient that orthogonality conditions (14) hold. In this case, the general solution of the equation (1) can be written as

$$y(t) = \alpha_1 y^{(1)}(t) + \ldots + \alpha_r y^{(r)}(t) + \tilde{y}(t),$$

where $\alpha_1, \ldots, \alpha_r$ are arbitrary constants (the same as in (10)), $y^{(1)}(t), \ldots, y^{(r)}(t)$ is the basic system of solutions of the corresponding homogeneous equation, and $y(t) = \lambda \int\limits_0^T K(t,s) y(s) ds$ $(\lambda = \frac{1}{\mu})$ is a particular solution of the inhomogeneous system (1).

2 Computational implementation of finding solutions to the integral system (1) with a degenerate kernel

It was shown above that in order to obtain a solution to the integral system (1), it is necessary to find vectors $w = \{w_1, ..., w_m\}$ from system (5) and substitute its components into formula (3). However, despite the simplicity of this scheme, its implementation is associated with considerable computational difficulties. Let's list them:

1) calculation of integrals $c_{ij} = \int\limits_0^T B_i(t) A_j(t) dt \left(i, j = \overline{1, m}\right)$ and compilation of matrices $C = (c_{ij})$, $C^* = (d_{ij})$ of the size $(mp) \times (mp)$;

2) calculation of integrals $H_i = \int\limits_0^T B_i(t) h(t) dt \left(i = \overline{1, m}\right)$ and composing the vector $H = \{H_1, ..., H_m\}$ of the size $(pm) \times 1$;

3) find the solution of the adjoint system $\left(I - \bar{\lambda} C^*\right) z = 0$;

4) verification of the orthogonality conditions $\left(H, z^{(j)}\right) = 0 \ \left(j = \overline{1, m}\right)$, where $z^{(1)}, ..., z^{(m)}$ are the basic solutions of the adjoint system;

5) when the orthogonality conditions are met, the calculation of the solution $w = w_1, ..., w_m$ of the algebraic system $(I - \lambda C) w = H$;

6) constructing the solution to the original integral system (1): $y(t) = \lambda \sum_{j=1}^{m} A_j(t) w_j + h(t)$.

Overcoming these difficulties manually will take a long time, so there is a need to overcome them with the help of some program on the computer. The *intsolve* program in Maple allows you to quickly and efficiently solve scalar integral equations with a degenerate kernel [5–9]. We do not know an analogue of such a program for systems of integral equations, so we considered it necessary to develop it ourselves. For the sake of simplicity of presentation of such a program, consider the case of a second-order system

$$\left[\begin{array}{c} y(t) \\ z(t) \end{array}\right] = \int\limits_0^1 \left[\begin{array}{cc} a_1(t) b_1(s) & a_2(t) b_2(s) \\ a_3(t) b_3(s) & a_4(t) b_4(s) \end{array}\right] \cdot \left[\begin{array}{c} y(s) \\ z(s) \end{array}\right] ds + \left[\begin{array}{c} m(t) \\ n(t) \end{array}\right] \tag{15}$$

(the unit in the upper limit of the integral is not essential here; it can be replaced by an arbitrary number T). There is no doubt that this system is a system with a degenerate kernel, but it is not so easy to represent it in the form (1), i.e., to write the kernel as a sum of products of matrices $A_i(t)$ and $B_i(s)$. Therefore, below we choose a way to represent the kernel as a sum of products of matrices with separated variables, based on the expansion of any matrix in a standard basis:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The easiest way to do this is with Maple. First, note that in Maple, indexes can be written both in square brackets and directly in the usual form. For example, a with an index j can be written both in the form a_j and in the form $a[j]$. If we denote by $e1$ and $e2$ unit vectors $e1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then this decomposition can be written as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \cdot e[1] \,.\, (e[1])^{\%T} + b \cdot e[1] \,.\, (e[2])^{\%T} + c \cdot e[2] \,.\, (e[1])^{\%T} + d \cdot e[2] \,.\, (e[2])^{\%T},$$

where $\%T$ is the sign of the transposition, the dot in the middle means the multiplication of a scalar by a vector, and the dot below is the matrix multiplication of vectors. For example,

$$\alpha \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} . \begin{bmatrix} z & u \end{bmatrix} = \begin{bmatrix} xz & xu \\ yz & yu \end{bmatrix}.$$

Now the kernel of the integral operator in (15) can be written as

$$(a[1](t) \cdot e[1]) \,.\, \left(b[1](s) \cdot (e[1])^{\%T}\right) + (a[2](t) \cdot e[1]) \,.\, \left(b[2](s) \cdot (e[2])^{\%T}\right) +$$
$$+ (a[3](t) \cdot e[2]) \,.\, \left(b[3](s) \cdot (e[1])^{\%T}\right) + (a[4](t) \cdot e[2]) \,.\, \left(b[4](s) \cdot (e[2])^{\%T}\right),$$

and system (15) itself in the form

$$\begin{aligned} u(t) = {} & (a[1](t) \cdot e[1]) \,.\, \int_0^1 \left(b[1](s) \cdot (e[1])^{\%T} .u(s)\right) \mathrm{d}s + \\ & + (a[2](t) \cdot e[1]) \,.\, \int_0^1 \left(b[2](s) \cdot (e[2])^{\%T}\right) .u(s)\,\mathrm{d}s + \\ & + (a[3](t) \cdot e[2]) \,.\, \int_0^1 \left(b[3](s) \cdot (e[1])^{\%T}\right) .u(s)\,\mathrm{d}s + \\ & + (a[4](t) \cdot e[2]) \,.\, \int_0^1 \left(b[4](s) \cdot (e[2])^{\%T}\right) .u(s)\,\mathrm{d}s + h(t), \end{aligned} \tag{16}$$

where $u(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$, $h(t) = \begin{bmatrix} m(t) \\ n(t) \end{bmatrix}$. In this notation, the kernel of the integral operator is represented as the sum of products of matrices with separated variables:

$$A1(t) = a[1](t) \cdot e[1];\, B1(s) = b[1](s) \cdot (e[1])^{\%T};\, A2(t) = a[2](t) \cdot e[1];$$
$$B2(s) = b[2](s) \cdot (e[2])^{\%T};\, A3(t) = a[3](t) \cdot e[2];\, B3(s) = b[3](s) \cdot (e[1])^{\%T};$$
$$A4(t) = a[4](t) \cdot e[2];\, B4(s) = b[4](s) \cdot (e[2])^{\%T}.$$

Let us rewrite system (16) in the form

$$u(t) = A1(t) \,.\, \int_0^1 B1(s) .u(s)\,\mathrm{d}s + A2(t) \,.\, \int_0^1 B2(s) .u(s)\,\mathrm{d}s +$$
$$+ A3(t) \,.\, \int_0^1 B3(s) .u(s)\,\mathrm{d}s + A4(t) \,.\, \int_0^1 B4(s) .u(s)\,\mathrm{d}s + h(t).$$

Enter numbers

$$\begin{gathered}\int_0^1 B1(s).u(s)\,\mathrm{d}s = w1, \int_0^1 B2(s).u(s)\,\mathrm{d}s = w2,\\ \int_0^1 B3(s).u(s)\,\mathrm{d}s = w3, \int_0^1 B4(s).u(s)\,\mathrm{d}s = w4.\end{gathered} \tag{17}$$

Then system (17) takes the form

$$u(t) = A1(t)\cdot w1 + A2(t)\cdot w2 + A3(t)\cdot w3 + A4(t)\cdot w4 + h(t). \tag{18}$$

We multiply this equality successively by matrices $B1(t), B2(t), B3(t), B4(t)$ on the left and integrate the results over $t \in [0,1]$; we get

$$\begin{aligned}
w1 &= \left(\int_0^1 B1(t)\ .A1(t)\,\mathrm{d}t\right)\cdot w1 + \left(\int_0^1 B1(t)\ .A2(t)\,\mathrm{d}t\right)\cdot w2 +\\
&+\left(\int_0^1 B1(t)\ .A3(t)\,\mathrm{d}t\right)\cdot w3 + \left(\int_0^1 B1(t)\ .A4(t)\,\mathrm{d}t\right)\cdot w4 + \int_0^1 B1(t).h(t)\,\mathrm{d}t;\\
w2 &= \left(\int_0^1 B2(t)\ .A1(t)\,\mathrm{d}t\right)\cdot w1 + \left(\int_0^1 B2(t)\ .A2(t)\,\mathrm{d}t\right)\cdot w2 +\\
&\left(\int_0^1 B2(t)\ .A3(t)\,\mathrm{d}t\right)\cdot w3 + \left(\int_0^1 B2(t)\ .A4(t)\,\mathrm{d}t\right)\cdot w4 + \int_0^1 B2(t).h(t)\,\mathrm{d}t;\\
w3 &= \left(\int_0^1 B3(t)\ .A1(t)\,\mathrm{d}t\right)\cdot w1 + \left(\int_0^1 B3(t)\ .A2(t)\,\mathrm{d}t\right)\cdot w2 +\\
&+\left(\int_0^1 B3(t)\ .A3(t)\,\mathrm{d}t\right)\cdot w3 + \left(\int_0^1 B3(t)\ .A4(t)\,\mathrm{d}t\right)\cdot w4 + \int_0^1 B3(t).h(t)\,\mathrm{d}t;\\
w4 &= \left(\int_0^1 B4(t)\ .A1(t)\,\mathrm{d}t\right)\cdot w1 + \left(\int_0^1 B4(t)\ .A2(t)\,\mathrm{d}t\right)\cdot w2 +\\
&+\left(\int_0^1 B4(t)\ .A3(t)\,\mathrm{d}t\right)\cdot w3 + \left(\int_0^1 B4(t)\ .A4(t)\,\mathrm{d}t\right)\cdot w4 + \int_0^1 B4(t).h(t)\,\mathrm{d}t.
\end{aligned} \tag{19}$$

Since the matrices $A[i], B[j]$ are known and their product $B[j].A[i]$ is a scalar quantity, then (19) is a system of linear algebraic equations with respect to the unknowns $w1, w2, w3, w4$. Solving this system in Maple and substituting the found unknowns in (18), we find the solution of the original integral system (15).

Example 1. Solve a system of integral equations

$$\begin{aligned}
y(t) &= \int_0^1 6tsy(s)\ \mathrm{d}s + \int_0^1 3t^2sz(s)\ \mathrm{d}s + t^2 + 1,\\
z(t) &= \int_0^1 (3+t)(5s+3)\,y(s)\ \mathrm{d}s + \int_0^1 (8t+5)\,s^3z(s)\ \mathrm{d}s + 4t.
\end{aligned} \tag{20}$$

Solution. Enter the coefficients

$$\begin{gathered}a_1(t) := 6\,t; b_1(t) := t; a_2(t) := \ 3\,t^2; b_2(t) := t; a_3(t) := 3 + \ t;\\ b_3(t) := 5\,t + 3; a_4(t) := 8\,t + 5; b_4(t) := \ t^3; m(t) := t^2 + 1; n(t) := 4\,t;\end{gathered}$$

Enter vectors

$$e[1] := \begin{bmatrix}1\\0\end{bmatrix}; \quad e[2] := \begin{bmatrix}0\\1\end{bmatrix}; u(t) := \begin{bmatrix}y(t)\\z(t)\end{bmatrix}; h(t) := \begin{bmatrix}m(t)\\n(t)\end{bmatrix};$$

Enter matrices:

$A1(t) := a[1](t)\cdot e[1]; B1(s) := b[1](s)\cdot (e[1])^{\%T}; A2(t) := a[2](t)\cdot e[1];$
$B2(s) := b[2](s)\cdot (e[2])^{\%T}; A3(t) := a[3](t)\cdot e[2]; B3(s) := b[3](s)\cdot (e[1])^{\%T};$
$A4(t) := a[4](t)\cdot e[2]; B4(s) := b[4](s)\cdot (e[2])^{\%T};$

We compose and solve a system of equations for unknowns $w1, w2, w3, w4$:

$$w1 = \left(\int_0^1 B1(t).A1(t)\,\mathrm{d}t\right)\cdot w1 + \left(\int_0^1 B1(t).A2(t)\,\mathrm{d}t\right)\cdot w2 + \left(\int_0^1 B1(t).A3(t)\,\mathrm{d}t\right)\cdot w3 + \left(\int_0^1 B1(t).A4(t)\,\mathrm{d}t\right)\cdot w4 + \int_0^1 B1(t).h(t)\,\mathrm{d}t$$

$$w2 = \left(\int_0^1 B2(t).A1(t)\,\mathrm{d}t\right)\cdot w1 + \left(\int_0^1 B2(t).A2(t)\,\mathrm{d}t\right)\cdot w2 + \left(\int_0^1 B2(t).A3(t)\,\mathrm{d}t\right)\cdot w3 + \left(\int_0^1 B2(t).A4(t)\,\mathrm{d}t\right)\cdot w4 + \int_0^1 B2(t).h(t)\,\mathrm{d}t$$

$$w3 = \left(\int_0^1 B3(t).A1(t)\,\mathrm{d}t\right)\cdot w1 + \left(\int_0^1 B3(t).A2(t)\,\mathrm{d}t\right)\cdot w2 + \left(\int_0^1 B3(t).A3(t)\,\mathrm{d}t\right)\cdot w3 + \left(\int_0^1 B3(t).A4(t)\,\mathrm{d}t\right)\cdot w4 + \int_0^1 B3(t).h(t)\,\mathrm{d}t$$

$$w4 = \left(\int_0^1 B4(t).A1(t)\,\mathrm{d}t\right)\cdot w1 + \left(\int_0^1 B4(t).A2(t)\,\mathrm{d}t\right)\cdot w2 + \left(\int_0^1 B4(t).A3(t)\,\mathrm{d}t\right)\cdot w3 + \left(\int_0^1 B4(t).A4(t)\,\mathrm{d}t\right)\cdot w4 + \int_0^1 B4(t).h(t)\,\mathrm{d}t$$

Calculate the solution of the original integral system

$$u(t) = A1(t)\cdot w1 + A2(t)\cdot w2 + A3(t)\cdot w3 + A4(t)\cdot w4 + h(t);$$

Answer.

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{602}{381}t^2 - \frac{80}{127}t + 1 \\ -\frac{832}{381} + \frac{265}{381}t \end{bmatrix}.$$

Verification. Let us introduce the obtained solutions:

$$y(t) := -\frac{602}{381}t^2 - \frac{80}{127}t + 1;\ z(t) := -\frac{832}{381} + \frac{265}{381}t.$$

Let us calculate the difference between the left and right parts of the original system:

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} - \begin{bmatrix} \int_0^1 6tsy(s)\ \mathrm{d}s + \int_0^1 3t^2sz(s)\ \mathrm{d}s + t^2 + 1 \\ \int_0^1 (3+t)(5s+3)\,y(s)\ \mathrm{d}s + \int_0^1 (8t+5)\,s^3z(s)\ \mathrm{d}s + 4t \end{bmatrix}.$$

Got a vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thus, the solution to system (20) is the vector function

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{602}{381}t^2 - \frac{80}{127}t + 1 \\ -\frac{832}{381} + \frac{265}{381}t \end{bmatrix}.$$

Remark 1. We have considered the two-dimensional case of the integral system (20). It is clear that the described algorithm obviously extends to any integral systems of type (20) of order higher than the second.

3 Systems of integro-differential equations with a degenerate kernel and their reduction to integral systems

Systems of equations of the form

$$\frac{dy}{dt} = A(t)y + \lambda \int_0^T K(t,s)y(s)ds + h(t), y(0,\varepsilon) = y^0, t \in [0,T], \tag{21}$$

where $y = \{y_1(t),\ldots,y_n(t)\}$is an unknown function, $h(t) = \{h_1(t),\ldots,h_n(t)\}$ is the known function (inhomogeneity), $A(t)$, $K(t,s)$ are known matrices of size $n\times\ n$, are called *systems of integro-differential equations of the Fredholm type (or simply integro-differential systems).* They can be reduced to an integrated system. It is done like this.

Let us assume that $Y(t)$ is the fundamental matrix of solutions of the differential system $\frac{dz}{dt} = A(t)z$. Taking $H(t) \equiv \lambda \int_0^T K(t,s)y(s)ds + h(t)$ for the inhomogeneity of the differential system $dy/dt = A(t)y + H(t)$, we find its "solution"

$$\begin{gathered} y(t) = Y(t)y^0 + \lambda \int_0^t Y(t)Y^{-1}(\zeta)(\int_0^T K(\zeta,s)y(s)ds)d\zeta + \\ + \int_0^t Y(t)Y^{-1}(\zeta)h(\zeta)d\zeta. \end{gathered} \tag{22}$$

Denoting

$$h_0(t) \equiv Y(t)y^0 + \int_0^t Y(t)Y^{-1}(\zeta)h(\zeta)d\zeta \tag{23}$$

and changing the order of integration in the iterated integral (22), we have

$$y(t) = \lambda \int_0^T (\int_0^t Y(t)Y^{-1}(\zeta)K(\zeta,s)d\zeta)y(s)ds + h_0(t). \tag{24}$$

We have obtained an integral system (24) with a kernel

$$G(t,s) \equiv \int_0^t Y(t)Y^{-1}(\zeta)K(\zeta,s)d\zeta. \tag{25}$$

It is easy to show that the system (24) is equivalent to the system (21). The following result is obtained.

Lemma 1. If $Y(t)$ is a fundamental matrix of solutions of a homogeneous system $\dot{z} = A(t)z$ (it is assumed that it exists on a segment $[0,T]$), then the integro-differential system (21) is equivalent to the integral Fredholm type:

$$y(t) = \lambda \int_0^1 G(t,s)y(s)ds + h_0(t), \tag{26}$$

where $h_0(t) \equiv Y(t)y^0 + \int_0^t Y^{-1}(\zeta)h(\zeta)d\zeta$, and the kernel $G(t,s)$ has the form (25).

For equations (26) of the Fredholm type, statements about solvability look rather complicated.

Theorem 1. Let in the system (21) the matrices $A(t) \in \mathrm{C}([0,T],\mathbb{C}^{n\times n})$,$K(t,s) \in \mathrm{C}(0 \le s,t \le \le T,\mathbb{C}^{n\times n})$,$h(t) \in \mathrm{C}([0,T],\mathbb{C}^{n})$. Then the following statements are true:

a) if λ is not a characteristic value of the kernel (25), then the integro-differential system (21) is solvable for any right-hand side $h(t)$ and, moreover, uniquely; in this case, its solution is given by the formula

$$y(t) = h_0(t) + \lambda \int_0^T R_\lambda(t,s)h_0(s)ds,$$

where$R_\lambda(t,s)$ *is the resolvent of the kernel (25),* $h_0(t)$ *is the function (23);*

b) if λ is the characteristic value of the kernel (25) of rank r, then system (21) is solvable in the space $\mathrm{C}^1([0,T],\mathbb{C}^n)$ if and only if the inhomogeneity (23) is orthogonal to all solutions of the homogeneous adjoint system $z(t) = \bar{\lambda}\int_0^T \overline{G^T(s,t)}z(s)ds$, i.e.

$$\int_0^T (h_0(t), z^{(j)}(t))dt = 0,\, j = \overline{1,r},$$

where $z^{(1)}(t),\dots,z^{(r)}$ is the basic system of solutions of the homogeneous adjoint system. In this case, the solution of the integro-differential system (21) is given by the formula

$$y(t) = \sum_{j=1}^{r} \alpha_j y^{(j)}(t) + \tilde{y}(t),$$

where $y^{(1)}(t),\dots,y^{(r)}(t)$ is the basic system of solutions of the homogeneous system, $y(t) = \lambda\int_0^T G(t,s)y(s)ds$, $\tilde{y}(t)$ is a particular solution of the integral system (26), and $\alpha_1,\dots,\alpha_r$ are arbitrary constants.

Now let the kernel in the original equation (21) be degenerate, i.e.

$$K(t,s) = \sum_{j=1}^{m} A_j(t)B_j(s), \tag{27}$$

where all $A_j(t)$ are matrices of the size $n\times p$, and all $B_j(s)$ are matrices of the size $p\times n$, $j = \overline{1,m}$ (we assume that the columns of the matrix $S(t) = (A_1(t),\dots,A_m(t))$ are linearly independent on the segment $[0,T]$). Then the kernel of equation (26) will have the form

$$\begin{aligned} G(t,s) &= \int_0^t Y(t)Y^{-1}(\zeta)K(\zeta,s)d\zeta = \\ &= \textstyle\sum_{j=1}^{m} Y(t)\left(\int_0^t Y^{-1}(\zeta)A_j(\zeta)d\zeta\right)B_j(s) \equiv \sum_{j=1}^{m}\Phi_j(t)B_j(s), \end{aligned} \tag{28}$$

where denoted: $\Phi_j(t) \equiv Y(t)\int_0^t Y^{-1}(\zeta)A_j(\zeta)d\zeta, j = \overline{1,m}$. Hence, the degenerate kernel (27) of the original integro-differential system (21) generates the degenerate kernel (28) of the integral system (24), equivalent to it, therefore, to construct a solution to system (24), we can apply the procedure developed above. We will show how this is done using the Maple program in the following example.

Example 2. Let's try to get the solution of the system

$$\begin{aligned}
&\frac{d}{dt}\, y(t) = -y(t) + \int_0^1 a_1(t)\, b_1(s)\, y(s)\ ds + \int_0^1 a_2(t)\, b_2(s)\, z(s)\ ds + m(t),\\
&\frac{d}{dt}\, z(t) = -2\, z(t) + \int_0^1 a_3(t)\, b_3(s)\, y(s)\ ds + \int_0^1 a_4(t)\, b_4(s)\, z(s)\ ds + n(t),\\
&y(0) = a, z(0) = b,
\end{aligned} \tag{29}$$

where, for the sake of simplicity, the following data are taken:

$$\begin{aligned}
&a1(t) = t; a2(t) = t^2; a3(t) = 2\,t; a4(t) = t+1; b1(t) = 3\,t;\\
&b2(t) = 2\,t^2; b3(t) = t; b4(t) = t-1; m(t) = 2\,t; n(t) = t^2; a = 1;\ b = 3.
\end{aligned}$$

*Solution.**
restart:
with(linalg):
Enter the coefficients:

$$\begin{aligned}
&a1(t) := t; a2(t) := t^2; a3(t) := 2\,t; a4(t) := t+1; b1(t) := 3\,t;\\
&b2(t) := 2\,t^2; b3(t) := t; b4(t) := t-1; m(t) := 2\,t; n(t) := t^2; a := 1;\ b := 3;
\end{aligned}$$

Enter kernel:

$$\begin{aligned}
&(a1(t)\cdot e[1]) . \left(b1(s)\cdot (e[1])^{\%T}\right) + (a2(t)\cdot e[1]) . \left(b2(s)\cdot (e[2])^{\%T}\right)\\
&+ (a3(t)\cdot e[2]) . \left(b3(s)\cdot (e[1])^{\%T}\right) + (a4(t)\cdot e[2]) . \left(b4(s)\cdot (e[2])^{\%T}\right);
\end{aligned}$$

Enter vectors:

$$h(t) := \begin{bmatrix} m(t) \\ n(t) \end{bmatrix};$$
$$u(t) := \begin{bmatrix} y(t) \\ z(t) \end{bmatrix};$$

Then the integro-differential system (29) takes the form:

$$\begin{aligned}
&map(diff,\ u(t), t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} .u(t) + (a1(t)\cdot e[1]) . \int_0^1 \left(b1(s)\cdot (e[1])^{\%T} .u(s)\right) ds\\
&+ (a2(t)\cdot e[1]) . \int_0^1 \left(b2(s)\cdot (e[2])^{\%T}\right) .u(s)\, ds\\
&+ (a3(t)\cdot e[2]) . \int_0^1 \left(b3(s)\cdot (e[1])^{\%T}\right) .u(s)\, ds\\
&+ (a4(t)\cdot e[2]) . \int_0^1 \left(b4(s)\cdot (e[2])^{\%T}\right) .u(s)\, ds + h(t);
\end{aligned}$$

Enter matrices:

$$\begin{aligned}
&A1(t) := a1(t)\cdot e[1]; B1(s) := b1(s)\cdot (e[1])^{\%T}; A2(t) := a2(t)\cdot e[1];\\
&B2(s) := b2(s)\cdot (e[2])^{\%T}; A3(t) := a3(t)\cdot e[2]; B3(s) := b3(s)\cdot (e[1])^{\%T};\\
&A4(t) := a4(t)\cdot e[2]; B4(s) := b4(s)\cdot (e[2])^{\%T};
\end{aligned}$$

* Maple does not put punctuation marks.

Then the IDE system (29) can be rewritten as:

$$map\left(diff,\ u(t),t\right)=\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}.u(t)+A1(t).\int\limits_0^1 B1(s).u(s)\,\mathrm{d}s$$
$$+A2(t).\int\limits_0^1 B2(s).u(s)\,\mathrm{d}s+A3(t).\int\limits_0^1 B3(s).u(s)\,\mathrm{d}s$$
$$+A4(t).\int\limits_0^1 B4(s).u(s)\,\mathrm{d}s+h(t);$$

We find the fundamental decision matrix:

$$dsolve\left(\left\{\frac{\mathrm{d}}{\mathrm{d}\,t}\,y(t)=-y(t),\frac{\mathrm{d}}{\mathrm{d}\,t}\,z(t)=-2\,z(t)\right\}\right);$$

$$Y(t):=\begin{bmatrix}\mathrm{e}^{-t} & 0\\ 0 & \mathrm{e}^{-2\,t}\end{bmatrix};$$

Change the inhomogeneity:

$$h1(t):=Y(t).\begin{bmatrix}a\\ b\end{bmatrix}+map\left(int,Y(t).Y^{-1}(x).h(x),x=0..t\right);$$

Enter matrices:

$$F1(t):=map\left(int,Y(t).Y^{-1}(x).A1(x),x=0..t\right);$$
$$F2(t):=map\left(int,Y(t).Y^{-1}(x).A2(x),x=0..t\right);$$
$$F3(t):=map\left(int,Y(t).Y^{-1}(x).A3(x),x=0..t\right);$$
$$F4(t):=map\left(int,Y(t).Y^{-1}(x).A4(x),x=0..t\right);$$

Denote $wj=\int\limits_0^1 Bj(s).y(s)\ \mathrm{d}s, j=\overline{1,4}$:

Then the equivalent integral system can be written as:

$$u(t)=F1(t)\cdot w1+F2(t)\cdot w2+F3(t)\cdot w3+F4(t)\cdot w4+h1(t):$$

Multiply this equation on the left sequentially by the matrices $B1(t),B2(t),B3(t),B4(t)$ and integrate the results obtained over $t\in[0,1]$. We obtain the system of algebraic equations:

$$eq1:=w1=map\left(\begin{array}{l}int,w1\cdot B1(t).F1(t)\ +w2\cdot B1(t).F2(t)\\ +w3\cdot B1(t).F3(t)\ +w4\cdot B1(t).F4(t)\\ +B1(t).h1(t),t=0..1\end{array}\right);$$
$$eq2:=w2=map\left(\begin{array}{l}int,w1\cdot B2(t).F1(t)\ +w2\cdot B2(t).F2(t)\\ +w3\cdot B2(t).F3(t)\ +w4\cdot B2(t).F4(t)\\ +B2(t).h1(t),t=0..1\end{array}\right);$$
$$eq3:=w3=map\left(\begin{array}{l}int,w1\cdot B3(t).F1(t)\ +w2\cdot B3(t).F2(t)\\ +w3\cdot B3(t).F3(t)\ +w4\cdot B3(t).F4(t)\\ +B3(t).h1(t),t=0..1\end{array}\right);$$
$$eq4:w4=map\left(\begin{array}{l}int,w1\cdot B4(t).F1(t)\ +w2\cdot B4(t).F2(t)\\ +w3\cdot B4(t).F3(t)\ +w4\cdot B4(t).F4(t)\\ +B4(t).h1(t),t=0..1\end{array}\right);$$

Let's solve this system:

$$solve\left([eq1,eq2,eq3,eq4],\{w1,w2,w3,w4\}\right):$$

and activate the found solutions with the assignment operator (:=).

We write down the solution of the original integro-differential system (29):

$$F1(t)\cdot w1 + F2(t)\cdot w2 + F3(t)\cdot w3 + F4(t)\cdot w4 + h1(t);$$

$$\begin{aligned} y(t) := & -\frac{9}{20}\frac{\left(394416e^{-1}e^{-2}+10800e^{-1}-139299e^{-2}-10975\right)\left(e^t t-e^t+1\right)e^{-t}}{9864e^{-1}e^{-2}+8664e^{-1}-3561e^{-2}-1831} \\ & -\frac{1}{10}\frac{\left(739476e^{-1}e^{-2}-105660e^{-1}-154209e^{-2}+17035\right)\left(e^t t^2-2e^t t+2e^t-2\right)e^{-t}}{9864e^{-1}e^{-2}+8664e^{-1}-3561e^{-2}-1831}+ \\ & e^{-t}+\left(2e^t t-2e^t+2\right)e^{-t}; \end{aligned} \tag{30}$$

$$\begin{aligned} z(t) := & -\frac{3}{40}\frac{\left(394416e^{-1}e^{-2}+10800e^{-1}-139299e^{-2}-10975\right)\left(2te^{2t}-e^{2t}+1\right)e^{-2t}}{9864e^{-1}e^{-2}+8664e^{-1}-3561e^{-2}-1831} \\ & -\frac{1}{40}\frac{\left(98208e^{-1}e^{-2}+57696e^{-1}-26187e^{-2}-11719\right)\left(2te^{2t}+e^{2t}-1\right)e^{-2t}}{9864e^{-1}e^{-2}+8664e^{-1}-3561e^{-2}-1831} \\ & +3e^{-2t}+\left(\frac{1}{2}t^2e^{2t}-\frac{1}{2}te^{2t}+\frac{1}{4}e^{2t}-\frac{1}{4}\right)e^{-2\,t}; \end{aligned}$$

The verification is carried out by substituting the solution into the difference between the left and right parts of the system (29):

$$\begin{bmatrix} \frac{d}{dt}y(t) \\ \frac{d}{dt}z(t) \end{bmatrix} - \begin{bmatrix} -y(t)+2t \\ -2z(t)+t^2 \end{bmatrix} - \left(\begin{array}{l} \begin{bmatrix} t \\ 0 \end{bmatrix}\cdot\left(\int_0^1 3sy(s)\,ds\right) + \begin{bmatrix} t^2 \\ 0 \end{bmatrix}\cdot\left(\int_0^1 2s^2z(s)\,ds\right) \\ + \begin{bmatrix} 0 \\ 2t \end{bmatrix}\cdot\left(\int_0^1 sy(s)\,ds\right) + \begin{bmatrix} 0 \\ t+1 \end{bmatrix}\cdot\left(\int_0^1 (s-1)z(s)\,ds\right) \end{array} \right)$$

$\xrightarrow{\text{simplify symbolic}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Consequently, functions (30) satisfy system (29).

Remark 2. When entering data in a Maple file, take into account that exponents and signs of differentials are entered as operators.

In conclusion, we note that the developed procedure, with some modifications, will be used to study linear and nonlinear singularly perturbed systems of integral and integro-differential equations with rapidly oscillating coefficients and inhomogeneities [10–14].

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Өзгешеленген ядросы бар интегралдық теңдеулер жүйесі және оларды Maple бағдарламасымен шешу алгоритмі

Математикалық әдебиеттерде өзгешеленген ядросы бар скалярлық интегралдық теңдеулер жақсы сипатталған (төменде (1) қараңыз, мұнда барлық жазылған функциялар скаляр шамалар). Авторларға матрицалардың көбейтіндісі түріндегі ядролары бар (1) типті интегралдық теңдеулер жүйесі егжей-тегжейлі қарастырылатын жарияланымдар белгісіз. Әдетте мұндай жүйелерді шешу әдістемесі скаляр жағдайдан векторлық жағдайға оңай ауыстырылады деп айтылады (мысалы, А.Л. Калашниковтың "Методы приближенного решения интегральных уравнений второго рода" (Нижний Новгород: ННГУ, 2017) монографиясында өзгешеленген ядролы теңдеулер жүйесінің қысқаша сипаттамасы берілген, мұнда өзгешеленген ядроның рөлін матрицалық функциялар емес, скалярлық функциялардың көбейтіндісі атқарады делінген). Алайда, қарапайым мысалдар көрсеткендей, матрицалық функциялардың көбейтіндісінің қосындысы түріндегі ядролы интегралдық жүйелер жағдайына скалярлық жағдайдың идеяларын жалпылау біршама түсініксіз, дегенмен бұл жағдайда интегралдық жүйені алгебралық теңдеулер жүйесіне келтіру идеясы қолданылады. Сонымен қатар, сәйкес алгебралық жүйенің шешімділік шарттарына сүйене отырып, ортогоналдылық шарттары түріндегі интегралдық жүйенің шешімділік шарттарын алу процесі бұрын сипатталмаған. Қолданбалы есептердегі интегралдық теңдеулер теориясының кең қолданылуын ескере отырып, авторлар көпөлшемді жағдайда ядролары өзгешеленген интегралдық жүйелерді шешудің егжей-тегжейлі схемасын беруді және бұл схеманы Maple бағдарламасында енгізуді қажет деп санады. Maple бағдарламасында тек скалярлық интегралдық теңдеулер *intsolve* процедурасы арқылы шешілетінін ескеріңіз. Авторлар интегралдық теңдеулер жүйесін шешудің ұқсас процедурасын таппады, сондықтан олар өздерінің процедурасын жасады.

Кілт сөздер: интегралдық оператор, өзгешеленген ядро, Maple бағдарламасының процедурасы, скалярлық интегралдық теңдеу.

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Системы интегральных уравнений с вырожденным ядром и алгоритм их решения с помощью программы Maple

В математической литературе хорошо описано скалярное интегральное уравнение с вырожденным ядром (см. ниже (1), где все записанные функции являются скалярными величинами). Авторам неизвестны публикации, в которых подробно рассматривались бы системы интегральных уравнений типа (1) с ядрами в виде произведения матриц. Обычно говорят, что техника решения таких систем легко переводится со скалярного случая на векторный (например, в монографии А.Л. Калашникова «Методы приближенного решения интегральных уравнений второго рода» (Нижний Новгород: ННГУ, 2017). Дано краткое описание систем уравнений с вырожденными ядрами, где роль вырожденных ядер играют произведения скалярных, а не матричных функций). Однако, как показывают простейшие примеры, обобщение идей скалярного случая на случай целочисленных систем с ядрами в виде суммы произведений матриц-функций весьма неясно, хотя в этом случае используется идея сведения интеграла уравнения к алгебраической системе. В то же время процесс получения условий разрешимости интегральной системы в виде условий ортогональности на основе условий разрешимости соответствующей алгебраической системы, как нам кажется, ранее не описывался. Учитывая широкое применение теории интегральных уравнений в прикладных задачах, авторы сочли необходимым привести подробную схему решения интегральных систем с вырожденными ядрами в многомерном случае и реализовать эту схему в программе Maple. Обратите внимание, что в Maple с помощью процедуры *intsolve* решаются только скалярные интегральные уравнения. Авторы не нашли аналогичной методики решения систем интегральных уравнений, поэтому разработали собственную методику.

Ключевые слова: интегральный оператор, вырожденное ядро, программная процедура Maple, скалярное интегральное уравнение.

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DOI 10.31489/2022M4/76-85
UDC 51-72

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Analytical and numerical research based on one modified refined bending theory

In the article, an analytical and numerical study based on one modified refined bending theory is presented. By the finite difference method, a general numerical calculation algorithm is developed. The solution obtained by the proposed method is compared with the results of known solutions, namely, with the solution of the classical theory, the exact solution, the solution in trigonometric series, as well as with experimental data. Comparison of the results obtained by the method given in the article with the solutions determined by other methods shows sufficient accuracy, which indicates the reliability of the proposed method based on one option of the modified refined bending theory. Classical theory is not applicable to such problems under consideration.

Keywords: modified refined bending theory, finite difference method, lagrange variational principle, differential operator, discretization of a system of equations

Introduction

The rapid development of scientific and technological progress requires the creation and implementation of new progressive materials and structures with predetermined properties. These requirements are fully met by composite materials, in particular, multilayer composites, which have a wide range of performance properties that cannot be achieved using traditional materials.

The use of multilayer composite materials in modern apparatuses and devices required taking into account their structural features, the physical and mechanical properties of the materials used, the number, structure and arrangement of layers for the composite material in mathematical research, as well as the creation of new methods that refine existing theories for the mathematical calculation of the stress-strain state of such structures.

In multilayer composite structures, the layers are made of such a material and these layers are arranged so as to endow the structure with a number of predetermined positive properties. At the same time, the materials are selected in such a way that, in an optimal combination, they give a qualitatively new type of construction. Or, in other words, in multilayer composite structures, the layers are arranged so that, under operational conditions, the structure better corresponds to its functional purpose.

The technical, mathematical and mechanical properties of structures made of multilayer inhomogeneous materials differ significantly in the thickness of their packages. Therefore, the features study for the operation of structures made of multilayer inhomogeneous materials in the thickness of their package by use refined theories is important in the mathematical study and the design of new innovative lightweight structures made of multilayer materials. Bending theories clarifying mathematical and technical theory should take into account the most important operational characteristics of multilayer

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composites, such as strain in the orthogonal direction to the layers, interaction of layers, strength, high resistance to fracture, etc. Each construction of a new multilayer composite that provides an increase in technical characteristics, as a rule, requires the development of new calculation methods based primarily on analytical mathematical research, and later on the numerical implementation of this research and its practical application.

One of the elements for multilayer composite spatial systems is a rectangular plate, which has numerous independent applications. An example of a rectangular plate, clamped with one edge, is a vertical panel, and an example of a plate, elastically clamped with three edges, is the wall of a rectangular reservoir. It should be noted that multilayer plates are a very extensive type of plates and are more often used in many fields of science and technology. The plate can be applied as an independent structure or can be part of the used lamellar system. For example, in the construction plates have all kinds of applications in the form of floorings and wall panels, reinforced concrete slabs to cover industrial and residential buildings, slabs for the foundations of massive structures, etc. Therefore, knowledge of the theory for rectangular plates bending and of classical methods for calculating them is necessary for a modern scientist.

Many analytical and numerical calculation methods are used to study the problems of plate bending [1–12]. An exact solution in analytical form for such problems is possible only in some particular cases for the geometrical type of the plate, the load and the conditions for its fixation on the supports, therefore, for practical applications, numerical, but sufficiently accurate methods for solving the considered problem are of special importance.

When considering the plate bending problems, the finite difference method is the most interesting because of connection with their possible numerical implementation in software package.

1 Initial positions and hypotheses

We consider a rectangular plate made of a multilayer composite material. The sides of the plate are equal to a and b, the thickness of the plate equals h. The study of the deformation of the plate is carried out in a rectangular coordinate system x_1, x_2, $x_3 = z$. The number of layers is arbitrary. The layers of the plate are orthotropic. Orthotropic materials are more difficult to analyze than isotropic materials, because their properties depend on the direction, so we place the directions of the Ox_1 and Ox_2 axes on the axes of the orthotropy of the layers. There is a coordinate plane at an arbitrary height of the plate section. The axes Ox_1 and Ox_2 lie on this coordinate plane.

The total number of layers in the plate is denoted by n. We number the layers as usual, starting from the bottom edge of the plate. The number of an arbitrary layer of the plate is denoted by k. The layer number in the coordinate plane is denoted by m. The totality of all n layers of the plate is called a package of layers.

In the general case, let's assume that the layers of the package have different thicknesses and different stiffness, the mechanical properties of which do not change in their thickness [13]. We suppose that the number of layers and their placement in the package are arbitrary.

During the transition from layer to layer we assume that static conditions

$$\sigma_{i3}^{k} = \sigma_{i3}^{k-1}, \quad \sigma_{33}^{k} = \sigma_{33}^{k-1}$$

and kinematic conditions

$$u_i^k = u_i^{k-1} \quad (i = 1, 2, 3)$$

are fulfilled, where σ_{ij}^k $(i, j = 1, 2, 3)$ are stresses, u_i^k $(i = 1, 2, 3)$ are displacements of the k-th layer. This corresponds to the operation of their layers without slipping and tearing.

Let a normal load $q(x_1, x_2)$ act on the upper surface of the plate. The normal load $q(x_1, x_2)$ varies according to an arbitrary law. The positive direction of the normal load coincides with the direction of the normal axis $x_3 = z$.

On the plate surface, the boundary conditions take the form

$$\sigma_{33}^{n} = q(x_1, x_2), \quad \sigma_{i3}^{n} = 0, \quad \sigma_{j3}^{1} = 0, \quad i = 1,2, \quad j = 1,2,3.$$

The conditions of the deformation continuity for the coordinate surface have the form [13]

$$\chi_{ii,l} - \chi_{12,i} = 0,$$

$$\varepsilon_{11,12} - 2\varepsilon_{12,12} + \varepsilon_{22,11} = 0$$

where $\varepsilon_{i,j}$ is strains, $\chi_{ij,l}$ is the shear function of the coordinate surface.

As the main assumptions for constructing a new refined model of the stress-strain state for a layered plate of an asymmetric structure with orthotropic layers, we accept the following system of hypotheses

$$\sigma_{i3}^{k} = G_{i3}^{k}\psi_{i,3}^{k}(z)\chi_{,i}, \quad \sigma_{33}^{k} = -\sum_{i=1}^{2}\eta_{3i}^{k}(z)\chi_{,ii}, \quad u_3^n = W \tag{1}$$

where G_{i3}^{k} is the shear modulus of the material, ψ, η are distribution functions for the k-th layer of the package, W and χ are the sought deflection function and the sought shift function of the coordinate surface, depending on the coordinates x_1 and x_2. The distribution functions depend on the z coordinate.

Hypotheses (1) are derived from the hypotheses made by Prof. A.Sh. Bozhenov [1], with the exception of those components that are not of great importance in calculating the stress-strain state of the plate. Hypotheses (1) guarantee the joint operation of layers without separation from each other and displacement, as well as conditions on the plate surfaces and determine the nonlinear law of variation of transverse shear stresses and normal transverse stresses in the plate thickness. It is assumed that normal displacements are equal to deflections.

For the distribution function in expressions (1), we have the following formulas

$$\psi_{i,3}^{k}(z) = \frac{1}{G_{i3}^{k}}\left[\eta_{2i}^{k}(z) - \eta_{1i}^{k}(z)\delta_i^*\right], \quad \eta_{1i}^{k}(z) = \int_{b_{k-1}-\delta_1}^{z} A_i^k dz + \sum_{j=1}^{k-1}\int_{b_{j-1}-\delta_1}^{b_j-\delta_1} A_i^j dz,$$

$$\eta_{2i}^{k}(z) = \int_{b_{k-1}-\delta_1}^{z} B_i^k z dz + \sum_{j=1}^{k-1}\int_{b_{j-1}-\delta_1}^{b_j-\delta_1} B_i^j z dz, \quad \eta_{3i}^{k}(z) = \int_{b_{k-1}-\delta_1}^{z} G_{i3}^{k}\psi_{i,3}^{k}(z)dz + C_{3i}^{k}, \tag{2}$$

where δ_1 is the distance from the coordinate plane to the bottom edge of the plate, and the constants have the form

$$C_{3i}^{k} = \sum_{j=1}^{k-1}\int_{b_{j-1}-\delta_1}^{b_j-\delta_1} G_{i3}^{j}\psi_{i,3}^{j}(z)dz.$$

Here and in what follows, the notation introduced in [1] is adopted. For the components in formulas (2), we have the following expressions

$$A_i^k = 0.5\{B_{ii}^k(1+\nu_{ii}^k) + G_{12}^k\}, \quad B_i^k = 0.5B_{ii}^k(1+\nu_{ii}^k) + G_{12}^k, \quad B_{ii}^k = E_i^k\nu_0^k,$$

$$\delta_i^* = \frac{\eta_{2i}^k}{\eta_{1i}^k}, \quad \nu_0^k = \left(1 - \nu_{12}^k\nu_{21}^k\right)^{-1}, \quad B_{i3}^k = \left(\nu_{3i}^k + \nu_{li}^k\nu_{3l}^k\right)\nu_0^k,$$

where E_i^k is the modulus of elasticity and ν_{ij}^k is the Poisson modulus for the k-th layer of the plate [13].

2 Analytical research

Based on the hypotheses (1) we have adopted, we carry out an analytical study of the stress-strain state for the layer package. First, we present the relationships that we use to derive the calculation formulas for stresses and strains.

We have the relations of Hooke's law

$$\sigma_{ii}^{k} = B_{ii}^{k} e_{ii}^{k} + 2B_{12}^{k} e_{22}^{k} + B_{i3}^{l} e_{33}^{k}, \quad \sigma_{12}^{k} = 2G_{12}^{k} e_{12}^{k}, \quad \sigma_{i3}^{k} = 2G_{i3}^{k} e_{i3}^{k}. \tag{3}$$

Inverse expressions of Hooke's law have the form

$$e_{ii}^{k} = \frac{1}{E_i^k}\sigma_{ii}^{k} - \frac{\nu_{il}^{k}}{E_l^k}\sigma_{ll}^{k} - \frac{\nu_{i3}^{k}}{E_3^k}\sigma_{33}^{k},$$

$$e_{33}^{k} = \frac{1}{E_3^k}\sigma_{33}^{k} - \frac{\nu_{31}^{k}}{E_1^k}\sigma_{11}^{k} - \frac{\nu_{32}^{k}}{E_2^k}\sigma_{22}^{k},$$

$$2e_{i3}^{k} = \frac{1}{G_{i3}^k}\sigma_{i3}^{k}, \quad 2e_{12}^{k} = \frac{1}{G_{12}^k}\sigma_{12}^{k}.$$

The Cauchy relationships are the following formulas

$$e_{ii}^{k} = u_{i,i}^{k}, \quad 2e_{12}^{k} = u_{1,2}^{k} + u_{2,1}^{k}. \quad 2e_{i3}^{k} = u_{i,3}^{k} + u_{3,i}^{k}, \quad e_{33}^{k} = u_{3,3}^{k}. \tag{4}$$

We determine the transverse shear strain $e_{i3}^{k}(x_1, x_2, z)$ from Hooke's law (3) by substituting the hypothesis expression for transverse tangential stresses (1)

$$2e_{i3}^{k} = \psi_{i,3}^{k}\chi_{,i}. \tag{5}$$

We find normal transverse strains $e_{33}^{k}(x_1, x_2, z)$ from the last Cauchy relation taking into account (1)

$$e_{33}^{k} = 0. \tag{6}$$

Integrating the third Cauchy relation (4) with respect to z, as well as using the relations (1) and (5), we obtain formulas for calculating tangential displacements

$$u_i^k = u_i - zW_{,i} + \psi_{i,i}^{k}\chi_{,i}, \tag{7}$$

where u_i are tangential displacements and $W_{,i}$ are the sought deflection functions of the coordinate surface, depending on the coordinates x_1 and x_2. Normal displacements are considered equal to deflections.

From the conditions for the joint work of the layers of the package

$$u_i^k = u_i^{k-1} (i = 1, 2, 3)$$

and conditions on the layer located in the coordinate surface

$$u_i^m(x_1, x_2, 0) = u(x_1, x_2)$$

we find the distribution function ψ_i^k in the form of the following expression

$$\psi_i^k = \int_{b_{k-1}-\delta_1}^{z} \psi_{i,3}^{k} dz + \sum_{j=1}^{k-1} \int_{b_{j-1}-\delta_1}^{b_j-\delta_1} \psi_{i,3}^{j} dz + \int_{0}^{b_{m-1}-\delta_1} \psi_{i,3}^{m} dz - \sum_{j=1}^{m-1} \int_{b_{j-1}-\delta_1}^{b_j-\delta_1} \psi_{i,3}^{j} dz.$$

Tangential deformations are found from the first Cauchy relations (4), substituting expressions for tangential displacements (7) into them. As a result, tangential strains are expressed by the following formulas

$$\begin{aligned} e_{11}^k &= \varepsilon_{11} - zW_{,11} + \psi_1^k \chi_{,11}, \\ e_{22}^k &= \varepsilon_{22} - zW_{,22} + \psi_2^k \chi_{,22}, \\ e_{21}^k &= \varepsilon_{21} - zW_{,21} + 0.5\big(\psi_2^k + \psi_1^k\big)\chi_{,21}. \end{aligned} \tag{8}$$

Taking into account formulas (1) and expressions for tangential strains (8), the stresses of the generalized Hooke's law (3) are found by the formulas [13]

$$\sigma_{11}^k = B_{11}^k\big(\varepsilon_{11} - zW_{,11} + \psi_1^k\chi_{,11}\big) + B_{12}^k\big(\varepsilon_{22} - zW_{,22} + \psi_2^k\chi_{,22}\big) - B_{13}^k\sum_{i=1}^{2}\eta_{3i}^k\chi_{,ii},$$

$$\sigma_{22}^k = B_{22}^k\big(\varepsilon_{22} - zW_{,22} + \psi_2^k\chi_{,22}\big) + B_{12}^k\big(\varepsilon_{11} - zW_{,11} + \psi_1^k\chi_{,11}\big) - B_{13}^k\sum_{i=1}^{2}\eta_{3i}^k\chi_{,ii},$$

$$\sigma_{12}^k = 2G_{12}^k\big[\varepsilon_{22} + 0.5\big(\psi_1^k + \psi_2^k\big)\chi_{,12} - zW_{,12}\big].$$

Based on formulas for calculating displacements (1), (7) and strains (5), (6), (8) it is possible to determine the components of the stress-strain state of the plate at an arbitrary point of the k-th layer.

Using the Lagrange variational principle and the relationships derived taking into account hypotheses (1), we obtain a system of equations for bending plates made of multilayer composite material with orthotropic layers. We notice that the number and arrangement of layers is arbitrary. Then we introduce force functions into the system of equations and obtain this system of equations in a mixed form

$$\begin{aligned} \triangle_F^2\phi + \triangle_{1S}^2 W - \big(\triangle_{2S}^2 - \triangle_{13}^2\big)\chi &= 0, \\ \triangle_{1S}^2\phi + \big(\triangle_{3S}^2 - \triangle_D^2\big)W + \big(\triangle_p^2 - \triangle_{23}^2 - \triangle_{4S}^2\big)\chi &= -q, \\ \triangle_{2S}^2\phi + \big(\triangle_{5S}^2 - \triangle_p^2\big)W + \big(\triangle_{P1}^2 - \triangle_{33}^2 - \triangle_{P3}\big)\chi &= 0. \end{aligned} \tag{9}$$

This system describes the bending of a multilayer plate with an asymmetric thickness structure with orthotropic layers.

The system of resolving equations of a layered plate is presented in a transformed form in [1].

The general order of the system of equations (9) is equal to twelve. The system of equations (9) takes into account the transverse shear and the interaction of layers. The functions of the coordinate plane, namely the force function ϕ, the deflection function W and the shear function χ are the sought functions in the system of equations (9).

There are differential operators in the system of equations (9). $\triangle$ is a second order differential operator, and $\triangle^2$ is a fourth order differential operator. These differential operators have the following form

$$\begin{aligned} \triangle_f^2 &= A_1^*\big(...\big)_{,1111} + A_2^*\big(...\big)_{,1122} + A_3^*\big(...\big)_{,2222}, \\ \triangle_g &= B_1^*\big(...\big)_{,11} + B_2^*\big(...\big)_{,22}, \end{aligned} \tag{10}$$

where A_j^* $(j = 1, 2, 3)$ and B_i^* $(i = 1, 2)$ are coefficients in equations (10). These coefficients depend on the stiffness of the package layers.

For different values of f and g, the coefficients of the operators take different values, which are shown in Table 1 [14]. When solving the system of equations (9), one should take into account the boundary conditions for fixing the edges of the plate with respect to the force function, the deflection function, and the shear function [12].

3 Numerical calculation

Using the finite difference method, the system of equations (9) and the boundary conditions of the plate were discretized. [3, 15]. The exclusion of unknown functions of the system of equations (9) outside the grid area of the plate is made in a matrix form.

An algorithm for numerical calculating the bending of multilayer composite plates with orthotropic layers, where the number of layers, their structure and arrangement are arbitrary, was developed by the finite difference method. This algorithm is implemented by a software package on a PC. This software package consists of a head program and several subroutines when using the FortRUN programming language.

The flowchart of the head program is divided into several blocks. Each block is autonomous and designed to perform specific functions. For the convenience of performing calculations, all magnitudes with dimensions are determined in a dimensionless form [12].

Below we describe the functions for these blocks of the flowchart of the head program.

In the first block, all the initial data and parameters of the task are introduced. In the second block, the stiffness characteristics are set for a multilayer composite plate with orthotropic layers. In the third block, systems of equilibrium equations for the plate under consideration are compiled and then solved. In the fourth block, the stress-strain state of the multilayer plate is calculated.

Conclusion

Using hypotheses (1), Lagrange's variational principle, the system of equations of the twelfth order is obtained. This system of equations describes the bending for a multilayer plate of an asymmetric structure in thickness with orthotropic layers. Three functions of the coordinate surface are unknown: the function of forces, the function deflection and the function shear.

The boundary conditions consist of two groups of relations. The first group of boundary conditions is similar in form to the conditions of the classical theory of plate bending and describes the boundary conditions for the coordinate plane of the plate [12]. The second group of equations simulates the type of deformation of the end surface for the plate and assumes the presence of various types of diaphragms at the end of the multilayer plate. The combination of the conditions from the two groups makes it possible to obtain various design features on the contour of the plate, i.e. it allows you to vary the boundary conditions on the edges of the plate.

In Table 2 [14], the solutions calculated by the method described above are checked against the results of solutions determined by known methods, namely, with the solution of the classical theory, the exact solution, the solution in trigonometric series, and the error of the solutions is calculated. In Table 3 [14], a comparison of the obtained solution with experimental data for three-layer plates with different plate parameters is presented.

Comparison of the obtained solution by the finite difference method with solutions determined using known methods, as well as with experimental data, shows a sufficiently acceptable accuracy in solving such problems and indicates the reliability of the proposed relations. It is impossible to apply classical theory for the problems under consideration.

It should be noted that when calculating the multilayer plates with orthotropic layers by analytical methods in the most general formulation: with arbitrary boundary conditions (including elastic), different types of load, complex shapes and different sizes of plates, different thickness of layers and different elastic characteristics, etc., we have to face with great mathematical difficulties, and in most cases to obtain an analytical solution of the problem under consideration is not possible. Such problems can be solved by applying a very efficient finite difference method, which gives a sufficiently high accuracy of solutions.

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Бір модификацияланған нақтыланған иілу теориясы негізінде аналитикалық және сандық зерттеу

Мақалада бір модификацияланған нақтыланған иілу теориясы негізінде аналитикалық және сандық зерттеулер жүргізілген. Ақырлы айырмашылықтар әдісі негізінде сандық есептеудің жалпы алгоритмі жасалған. Ұсынылған әдістеме бойынша алынған шешім белгілі шешімдердің нәтижелерімен, атап айтқанда, классикалық теорияның шешімімен, дәл шешіммен, тригонометриялық қатарлардағы шешіммен, сонымен қатар эксперименттік мәліметтермен салыстырылады. Мақалада көрсетілген әдіспен алынған нәтижелерді басқа әдістермен анықталған шешімдермен салыстыру жеткілікті дәлдікті көрсетеді. Бұл иілудің модификацияланған нақтыланған теориясының бір нұсқасы негізінде ұсынылған әдістің сенімділігін дәлелдейді. Қарастырылып отырған есептер үшін классикалық теория қолданылмайды.

Кілт сөздер: модификацияланған нақтыланған иілу теориясы, ақырлы айырмашылықтар әдісі, Лагранж вариациялық принципі, дифференциалдық оператор, теңдеулер жүйесін дискретизациялау.

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Аналитическое и численное исследование на основе одной модифицированной уточненной теории изгиба

В статье проведено аналитическое и численное исследование на основе одной модифицированной уточненной теории изгиба. На основе метода конечных разностей разработан общий алгоритм численного расчета. Полученное по предложенной методике решение сопоставлено с результатами известных решений, а именно с решением классической теории, с точным решением, с решением в тригонометрических рядах, а также с экспериментальными данными. Сравнение результатов, полученных по данной в статье методике, с решениями, определенными другими методами, показывает достаточную точность, что свидетельствует о достоверности предлагаемой методики на основе одного варианта модифицированной уточненной теории изгиба. Классическая теория для рассматриваемых задач не применима.

Ключевые слова: модифицированная уточненная теория изгиба, метод конечных разностей, вариационный принцип Лагранжа, дифференциальный оператор, дискретизация системы уравнений.

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DOI 10.31489/2022M/86-97
UDC 517.9

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On the approximation of solutions of one singular differential equation on the axis

In this paper we study the problem of the best approximation by linear methods of solutions to one Triebel-type equation. This problem was solved by using estimates of the linear widths of the unit ball in corresponding spaces of differentiable functions. According to the definition, linear widths give the best estimates for the approximation of compact sets in a given normed space by linear methods which are implemented through finite-dimensional operators. The problem includes answers to the questions about the solvability of the studied equation, the construction of the corresponding weighted space of differentiable functions, the development of a method for estimating linear widths of compact sets in weighted polynomial Sobolev space. In this work, conditions are obtained under which the considered operator has a bounded inverse. The weighted Sobolev space corresponding to the posed problem is determined. Upper estimates are obtained for the counting function for a sequence of linear widths, which correspond to the posed problem. One example is constructed in which two-sided estimates of linear widths are given. The method for solving this problem can be applied to the numerical solution of non-standard ordinary differential equations on an infinite axis.

Keywords: differential equations, Triebel equations, approximation of sets by linear methods, widths of sets, weighted Sobolev spaces.

1 Introduction and Main results

In this paper, we consider the problem of the best linear approximations of solutions to the equation

$$Ty \equiv -\rho_0^{\mu}(x)y'' + q_1(x)y' + (q_0(x) + \rho_0^{\nu}(x))y_0 = f \tag{1}$$

with the right side in the Hilbert space $L_2(I)$, T is an operator satisfying conditions from the Triebel class $U^1_{\mu,\nu}(I,\rho_0)$ $(\nu > \mu + 2,\ \mu > 0)$, where $I = [0,\infty)$, on $+\infty$, i.e. [1]: $\rho_0 \geq 1$ and q_i $(i = 0, 1)$ are functions infinitely differentiable in I such that

i) $\lim\limits_{x\to\infty} \rho_0(x) = \infty$,

ii) $|\rho_0^{(k)}(x)| \leq O(\rho_0^{1+k}(x)),\ k = 0, 1, ...,$

iii) $q_0^{(k)}(x) = o(\rho_0^{\nu+k}(x)),\ q_1^{(k)}(x) = o(\rho_0^{(\nu+\mu)/2+k}(x))$ for $x \to \infty$ $(k = 0, 1, 2, ...)$.

To solve the problem, we used a modified method of localization of estimates for the widths of compact sets in weighted spaces of differentiable functions [1; 104], [2–7], as well as coercive estimates for differential operators [8, 9]. The method of local estimates developed in this paper on intervals of adjustable variable length can be used in the theory of numerical solutions of a certain class of singular differential equations on an infinite axis. All results presented in this paper are new.

We denote $V^2_{p,(\mu,\nu)}(I)$ the completion of the class $C_0^{\infty}(I)$ of functions infinitely differentiable and finite in I with respect to the norm

$$\left\| y; V^2_{p,(\mu,\nu)}(I) \right\| = \left[\sum_{k=0}^{2} \int_0^{\infty} \left| \rho_0^{l_k} y^{(k)} \right|^p dx \right]^{1/p},\ 1 \leq p < \infty,$$

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where $l_k = 2^{-1}((2-k)\nu + k\mu)$ for $k = 0,1,2$. Let $V = V^2_{2,(\mu,\nu)}(I)$.

If

$$\sup_{x\geq 0}\left|q_0(x)\rho^{-\nu}(x)\right| = \beta_0 < 1, \tag{2}$$

$$\sup_{x\geq 0}|q_1(x)|\rho_0^{(\nu+\mu)/2}(x) = \beta < \infty,$$

then for the minimal operator

$$T_0 y = Ty,\ y \in C_0^\infty(I),$$

the following inequality holds

$$\|T_0 y; L_2(I)\| \leq (2+\beta)\,\|y\|_V\,.$$

Therefore, the operator T_0 has a closed extension

$$\widetilde{T}y = Ty,\ y \in D(\widetilde{T}) = V^2_{2,(\mu,\nu)} \subset L_2(I).$$

We defined the operator $T \overset{det}{=} \widetilde{T}$ in (1), where the norm $\|T\| \leq 2+\beta$.

Let y_i be a sequence of functions $y_i \in C_0^\infty(I)$ fundamental in the norm $\|\cdot\|_V$. Then each of the sequences $\{a_r y_j^{(r)}\}$, $\{y_j^{(r)}\}$ $(a_r = \rho_0^{lr},\ r = 0,1,2)$ are fundamental in $L_2(I)$. Therefore, $y = \lim y_j$ in $L_2(I)$ has finite a.e. in I derivatives $y'(x)$, $y''(x)$, and $\|y\|_V < \infty$.

Theorem 1. Let condition (2) be satisfied and

$$\frac{1}{2(1-\beta_0)}\sup_{x\geq 0}\left[\left((\rho_0^\mu)'(x)\right)^2 + (q_1(x))^2\right]\rho_0^{-(\nu+\mu)}(x) = \beta_1 < 1, \tag{3}$$

$$C_{\mu,\nu} = \frac{1}{(1-\beta_0)(1-\beta_1)}.$$

Then the operator T in (1) has an inverse T^{-1}. Therefore

$$\left\|T^{-1}\right\| \leq C_{\mu,\nu}. \tag{4}$$

Let $F = \{y \in V : \|Ty; L_2(I)\| \leq 1\}$. From (4) it follows that

$$F \subset \{y \in V : \|y\|_V \leq c\},\ c = c_{\mu,\nu}.$$

Let $\mathbb{C}$ be a bounded set in a Banach space X, containing 0, $\mathfrak{U}_k(\mathbb{C},X)$ be the class of all continuous linear operators $U : X \to X$ of dimension $\leq k$ and such that $\mathbb{C} \subset D(U)$. The value

$$\lambda_k(\mathbb{C},X) = \inf_{U\in\mathfrak{U}_k(X)}\sup_{x\in\mathbb{C}}\|x - \Lambda x\|_X$$

is called the linear k-width of $\mathbb{C}$ in X [10; 16].

The widths $\lambda_k(\mathbb{C},X)$ are related to the problem of the best (linear) method for approximating the set $\mathbb{C}$ in X.

Let $\mathcal{N}(\lambda;\mathbb{C},X) = \sum\limits_{\lambda_k(\mathbb{C},X)>\lambda} 1$ (number of widths $\lambda_k(\mathbb{C},X) > \lambda$). In this paper, we obtain an estimate (from above) for the counting function $\mathcal{N}(\lambda;F,L_2(I))$.

Let Ω be a (Lebesgue) measurable set in $\mathbb{R}$. Here and below BX is the unit ball of space X, $L_p(\Omega)$ is the space of functions f in Ω with the seminorm $\|f;L_p(\Omega)\| = \left(\int_\Omega |f(x)|^p\,dx\right)^{1/p} < \infty$, $L_{p,loc}(I)$ is the space of all functions f in I such that $f \in L_p(G)$ for any compact $G \subset I$, $L^+_{loc}(I) = L^+_{1,loc}(I)$ is the class of non-negative locally summable (weight) functions f in I for which Lebesgue measure

$|\{x \geq t : f(x) > 0\}| > 0$ for all $t > 0$. Let $\rho, v \in L^+_{loc}(I)$, $\rho > 0$, $\rho^{-1} = 1/\rho \in L_{loc}(I)$, $\delta \in (0,1)$. Assume that

$$S_{(\delta)}(x,h;v) = \inf_{\{e\}_\delta} \int\limits_{\Delta\setminus e} v dt,\ \Delta = [x, x+h],$$

where infimum is taken over the set $\{e\}_\delta$ of all closed $e \subset \Delta$ with measure $|e| \leq \delta\,|\Delta| = \delta h$,

$$\mathcal{M}_{(\delta)}(x,h;\rho,v) = h \left(\int\limits_{\Delta} \rho^{-1}\right)^{1/2} \left(S_{(\delta)}(x,h;v)\right)^{1/2}.$$

Let $h(\cdot)$ be a finite positive function in I. The function $h(\cdot)$ is called the length function in I (with respect to the pair (ρ, v)), if

$$\mathcal{M}_{(\delta)}(x, h(x); \rho, v) \geq 1,\ (x \in I). \tag{5}$$

We set

$$h_{(\delta)}(x;\rho,v) = \sup\left\{h > 0 :\ \mathcal{M}_{(\delta)}(x,h;\rho,v) \leq 1\right\}. \tag{6}$$

Proposition 1. a) If

$$0 < h_{(\delta)}(x;\rho,v) < \infty, \tag{7}$$

then

$$M_{(\delta)}(x, h_{(\delta)}(x;\rho,v);\rho,v) = 1. \tag{8}$$

b) If $\rho = \rho_0^{2\mu}$, $v = (q_0 + \rho_0^{\nu})^2$, then

$$h_{(\delta)}(x) = h_{(\delta)}(x;\rho,v) < \infty\ (x \geq 0).$$

Remark 1. The equality (8) implies the realization of condition (5). Then every finite positive function $h_{(\delta)}(x;\rho,v)$ is the length function with respect to the pair (ρ, v).

Remark 2. If $h(\cdot)$ is a length function with respect to the pair (ρ, v), then $0 < h_{(\delta)}(x;\rho,v) \leq h(x)$, $(x \geq 0)$.

We introduce a maximal operator with respect to the interval basis associated with the length function $h(\cdot)$ in I. Let

$$\mathcal{B} = \bigcup_{y\geq 0} \{\Delta = [\alpha,\beta) : y \leq \alpha < \beta \leq y + h(y)\},\ \mathcal{B}_x = \{\Delta = \mathcal{B} : x \in \Delta\}.$$

We define a maximal operator with respect to basis $\mathcal{B}$ [11; 43].

$$M^* f(x, h(\cdot)) = \sup_{\Delta\in\mathcal{B}_x} \frac{1}{|\Delta|} \int\limits_{\Delta} |f(t)| dt,\ f \in L_{loc}(I).$$

Let

$$M^*_{(\delta)} f(x) = M^* f(x, h_{(\delta)}(\cdot)),$$

$$\mathcal{K}_{(\delta)}(x) = (h_{(\delta)}(x))^{3/2} \left[\int\limits_{\Delta} \rho_0^{-2\mu}(t) dt\right]^{1/2},\ \Delta = [x, x + h_{(\delta)}(x)].$$

Theorem 2. Let

$$\lim_{x\to\infty} K_{(\delta)}(x) = 0 \ (0 < \delta < 1/2).$$

There is $c(\delta) > 1$, such that

$$\mathcal{N}(\lambda; F, L_2(I)) \le (c^{-1}\lambda)^{-1/2} \int\limits_{G(c^{-1}\lambda)} \left(M^*_{(\delta)}\rho_0^{-2\mu}\right)^{1/4} dx,$$

where

$$G(\lambda) = \{x > 0 : h_{(\delta)}(x) \left(M^*_{(\delta)}\rho_0^{-2\mu}\right)^{1/4} > \lambda^{1/2}\},$$

$c = 4c(\delta)c(\mu,\nu)$.

Example 1. Consider the equation

$$Ty \equiv -(3+x)^{\mu}y'' + q_1(x)y' + (q_0(x) + (3+x)^{2\nu})y = f \tag{9}$$

under the following conditions: $q_i \in C^{\infty}(I)$ $(i = 0,1)$ satisfy conditions iii) with respect to $\rho_0(x) = 3+x$, and also

$$\sup_{x\ge 0} |q_0(x)|(3+x)^{-\nu} = \beta_0 \le 3/4,$$

$$\sup_{x\ge 0} |q_1(x)|(3+x)^{(\nu+\mu)/2} = \beta < 1/2,$$

$$\frac{1}{2} < \mu < 2 < \nu - \mu.$$

Then $\beta_1 < \frac{3}{5}$ and, by virtue of Theorem 1, for the operator T in (9) there exists T^{-1} with the norm

$$\left\|T^{-1}\right\| \le ((1-\beta_0)(1-\beta_1))^{-1} < 4\cdot\frac{5}{2} = 10.$$

Therefore, the solution set of the equation (9) with the right side $f \in L_2(I)$ is contained in the ball $10BV$.

Let $c_\mu = (3^{\mu}4)^{1/2}$, $u = (4c_\mu^2(10\lambda^{-1}))^{1/2}$. By virtue of Theorem 2 we have

$$\mathcal{N}(c\lambda; 10BV, L_2(I)) = \mathcal{N}(10^{-1}c\lambda; BV; L_2(I)) \le$$

$$\le c_\mu(10^{-1}\lambda)^{-1/2} \int\limits_{G(10^{-1}\lambda)} (\beta+x)^{-\mu/2}dx \le c_\mu(10^{-1}\lambda)^{-1/2}\int\limits_0^u x^{-\mu/2}dx = \tag{10}$$

$$= \frac{1}{2-\mu}\left(3^{\mu} + 6(10^{-1}\lambda)^{-1}\right)^{(\nu-\mu+2)/2\nu}.$$

Let $c\lambda = \lambda_n(10BV, L_2(I))$. By (10) for any solution y of the equation (9) with $f \in BL_2(I)$ we have

$$\inf_{U\in\mathfrak{U}_n(L_2(I))} \|y - Uy; L_2(I)\| \le \lambda_n\left(10BV; L_2(I)\right) \le \kappa n^{-2(1-\frac{2-\mu}{2-\mu+\nu})},$$

where $\kappa = 160c3^{\mu}(\frac{1}{2-\mu})^{27/(242-\mu)}$.

2 Proof of main results

Proof of Theorem 1. Let $a_0(x) = q_0(x) + \rho_0^\nu(x)$, $a_1(x) = q_1(x)$, $a_2(x) = \rho_0^\mu(x)$.

1. Let $y \in C_0^\infty(I)$. In this case

$$(Ty, y) = \int_0^\infty \left[\left(\sqrt{a_2(x)}y'\right)^2 + \left(a_2'(x) + a_1(x)\right) y'y + a_0(x)y^2 \right] dx =$$

$$= \int_0^\infty \left[\sqrt{a_2(x)}y' + \frac{a_2'(x) + a_1(x)}{2\sqrt{a_2(x)}} y \right]^2 dx + \int_0^\infty w(x)y^2 dx \geq \int_0^\infty w(x)y^2 dx,$$

where $w(x) = a_0(x) - (a_2'(x) + a_1(x))^2/4a_2(x)$. By conditions (2) and (3) we get

$$a_0(x) = \rho^\nu(x)(1 + q_0(x)\rho^{-\nu}(x)) \geq (1 - \beta_0)\rho_0^\nu,$$

$$\frac{(a_2'(x) + a_1(x))^2}{4a_0(x)a_2(x)} \leq \frac{1}{2(1 - \beta_0)} \left[\left((\rho_0^\mu)'(x) \right)^2 + (q_1(x))^2 \right] \rho^{-(\nu+\mu)} \leq \beta_1,$$

where

$$\inf_{x>0} w(x) \geq 1 - \beta_0,$$

$$\|Ty; L_2(I)\| \geq (1 - \beta_1) \|y; L_2(I)\|. \tag{11}$$

It follows from (11) that the operator

$$T_0 y = Ty, \; y \in C_0^\infty(I),$$

has a bounded inverse operator T_0^{-1} c $D(T_0^{-1}) \subset L_2(I)$. Wherein

$$\left\|T_0^{-1}\right\| \leq C_{\mu,\nu}. \tag{12}$$

The estimate (4) follows from (12).

2. Let $y \in V^2_{2,(\mu,\nu)}$, $\{y_j\}$ be a sequence from $C_0^\infty(I)$ converging to y in $V^2_{2,(\mu,\nu)}$. Since

$$\|Ty - Ty_j; L_2(I)\| \leq 2(2 + \beta) \|y - y_j\|_V \;\; (j \geq 1).$$

By (11) we get

$$\|Ty; L_2(I)\| = \lim_{j\to\infty} \|Ty_j; L_2(I)\| \geq C_{\mu,\nu}^{-1} \|y; L_2(I)\|.$$

Therefore, there exists an inverse operator T^{-1} and the estimate (4) takes place.

Proof of Proposition 1. Let the function $h_{(\delta)}(x; \rho, v)$ in (6) satisfies the condition (7).

a) There is a sequence $h_j > 0$ ($j \in \mathbb{N}$) converging to $h = h_\delta(x)$, such that $\mathcal{M}_{(\delta)}(x, h_j; \rho, v) \leq 1$. Let $\Delta_j = [x, x + h_j]$. Passing to the limit (for $j \to \infty$) in the estimate

$$h_j^2 \int_{\Delta_j} \rho^{-1} d\xi S_{(\delta)}(x, h; \rho, v) \leq \mathcal{M}^2_{(\delta)}(x, h; \rho, v) + h_j^2 \int_{\Delta_j} \rho^{-1} d\xi \int_{x+h_j}^{x+h} v d\xi,$$

leads to inequality

$$M_{(\delta)}(x, h_{(\delta)}(x; \rho, v); \rho, v) \leq 1.$$

On the other hand, there is a sequence $h'_j > h$, $M_{(\delta)}(x, h''_j; \rho, v) > 1$. Let $\Delta'_j = [x, x+h]$ with measure $|e| \le \delta h$

$$(h'_j)^2 \int\limits_{\Delta'_j} \rho^{-1} d\xi \int\limits_{\Delta'_j \setminus e} v d\xi > 1. \tag{13}$$

Passing to the limit in (13) (for $j \to \infty$) leads to the estimates

$$h^2 \int\limits_{(\Delta)} \rho^{-1} d\xi \int\limits_{\Delta \setminus e} v d\xi \ge 1,$$

$$M_{(\delta)}(x, h_{(\delta)}(x; \rho, v)) \ge 1.$$

b) The assertion follows from the estimates

$$(1-\delta)h < S_{(\delta)}(x, h; \rho_0^{2\nu}) \le \int\limits_x^{x+h} \rho_0^{2\nu} d\xi.$$

Let

$$K_{(\delta)}(x, h; \rho) = h^{3/2} \left(\int\limits_x^{x+h} \rho^{-1} d\xi \right)^{1/2}.$$

Lemma 1. Let $M_{(\delta)}(x, h; \rho, v) \ge 1$. There is $c(\delta) > 1$, such that for all $y \in C_0^\infty(I)$

$$\left(\int\limits_x^{x+h} |y|^2 d\xi \right)^{1/2} \le c(\delta) K_{(\delta)}(x, h; \rho, v) \left[\int\limits_x^{x+h} (\rho(\xi) \left| y'' \right|^2 + v(\xi) \left| y \right|^2) d\xi \right]^{1/2}. \tag{14}$$

The proof of Lemma 1 is essentially a repetition of the main lemma in [2].

Let on intervals $\Delta \subset I$ $(1 \le \rho < \infty)$ the following equality holds

$$\|y; W(\Delta)\| = \left\| \rho^\mu y''; L_2(\Delta) \right\| + \|\rho^\nu y; L_2(\Delta)\|.$$

Let $W(\Delta)$ be the space $C^\infty(\Delta)$ with the norm $W(\Delta)$. $W = W^2_{2,(\mu,\nu)}$ denote the completion of the class $C_0^\infty(I)$ in the norm $\|y; W(I)\|$. It is easy to see that

$$K_{(\delta)}(x) = K_{(\delta)}(x, h_{(\delta)}(x); \rho^{2\mu}) \le (1-\delta)^{-1/2}. \tag{15}$$

Indeed, taking $h = h_{(\delta)}(x)$, $\Delta = [x, x+h]$, from (8) we derive

$$h \left(\int\limits_\Delta \rho^{2\mu} d\xi \right)^{1/2} = \left(\inf_{\{e\}_\delta} \int\limits_{\Delta \setminus e} \rho^{2\nu} \right)^{-1/2} \le [(1-\delta)h]^{-1/2},$$

which inplies (15).

Lemma 2. The following estimate is true

$$\|y; L_2(I)\| \le c(\delta) \left\| y; W^2_{2,(\mu,\nu)}(I) \right\|, \; y \in C_0^\infty(I). \tag{16}$$

Proof. Let supp$y \subset [\xi_0, \xi_1]$, $0 \le \xi_0 < \xi_1 < \infty$. Let us show that

$$\inf_{\xi_0 \le x \le \xi_1} h_\delta(x) \ge \gamma > 0. \tag{17}$$

If $h_\delta(x) < 1$ ($x \in [\xi_0, \xi_1]$), then by (8)

$$h_\delta(x) \ge \left[\int\limits_{\xi_0}^{\xi_1+1} \rho^{2\nu} d\xi \right]^{-1/2} = b > 0.$$

We take $\gamma = \min\{1, b\}$. From (17) it follows that $[\xi_0, \xi_1] \subset \bigcup_{j=1}^{\mathbb{N}} \Delta(x_j)$ $(N < \infty)$, where $x_1 = \xi_0$, $x_{j+1} = x_j + h_{(\delta)}(x_j)$, $\Delta(x_j) = [x_j, x_{j+1})$. Using the estimates (14) and (15), we derive the inequality (16) namely:

$$\|y; L_2(I)\| \le \left(\sum_{j=1}^{N} \int\limits_{\Delta(x_j)} |y|^2 \, d\xi \right)^{1/2} \le c(\delta) \left\| y; W^2_{2,(\mu,\nu)}(I) \right\|.$$

Lemma 3. The following statements are true: a)

$$0 < \widetilde{h}(x, \lambda) = \sup\{h > 0; K_{(\delta)}(x, h; \rho_0^{2\mu}) \le \lambda\} < \infty \ (x > 0),$$

$$\widetilde{h}(x, \lambda) < h_{(\delta)}(x), \text{ if } K_{(\delta)}(x) > \lambda, \tag{18}$$

$$K_{(\delta)}(x, \widetilde{h}(x, \lambda); \rho^{2\mu}) = \lambda, \tag{19}$$

b) on each $\widetilde{\Delta} = \widetilde{\Delta}(x; \lambda) = [x, x + \widetilde{h}(x, \lambda)]$ the counting function

$$\mathcal{N}(\lambda; BW(\widetilde{\Delta}), L_2(\widetilde{\Delta})) \le 1. \tag{20}$$

Proof. a) The estimates $0 < \widetilde{h}(x, \lambda) < \infty$ follow from the limit equalities

$$\lim_{h \to 0+} K_{(\delta)}(x, h; \rho_0) = 0, \ \lim_{h \to \infty} K_{(\delta)}(x, h; \rho_0) = \infty.$$

The statement (18) is trivial. Equality (19) is proved as equality (8).

b) Let $U_x y(t) = y(x) + y'(x)(t - x)$, $y \in W(\widetilde{\Delta})$. The operator $U_x \in \mathfrak{U}_2(BW(\widetilde{\Delta}), L_2(\widetilde{\Delta}))$, because $\dim U_x \le 2$ and the norm

$$\left\| U_x y; L_2(\widetilde{\Delta}) \right\| \le (|\widetilde{\Delta}|^{1/2} + |\widetilde{\Delta}|^{3/2})(|y(x)| + |y'(x)|) = b_x < \infty.$$

Use the Taylor formula with integral remainder

$$\left\| y - U_x y; L_2(\widetilde{\Delta}) \right\| = \left[\int\limits_{\widetilde{\Delta}} \left| \int\limits_x^t (t - \xi) y''(\xi) d\xi \right|^2 dt \right]^{1/2} \le \widetilde{h}(x, \lambda) \left[\int\limits_{\widetilde{\Delta}} \rho_0^{-2\mu} d\xi \right]^{1/2} \left[\int\limits_{\widetilde{\Delta}} \left| \rho_0^\mu y'' \right|^2 d\xi \right]^{1/2} \le \lambda.$$

Therefore, estimate (20) holds.

Lemma 4. Let $K_{(\delta)}(x) \to 0$ for $x \to \infty$. Let

$$\Delta_j = \begin{cases} [x_j, x_j + h_{(\delta)}(x_j)), & \text{if } K_{(\delta)}(x_j) \le \lambda, \\ [x_j, x_j + \widetilde{h}(x_j, \lambda)), & \text{if } K_{(\delta)}(x_j) > \lambda, \end{cases} \ (j = 1, 2, ...), \ x_1 = 0.$$

There are estimates

$$\mathcal{N}(2c\lambda; BW(I)) \le \sum_{j: K_{(\delta)}(x_j) > \lambda} \mathcal{N}(\lambda; BW(\Delta_j), L_2(\Delta_j)), \tag{21}$$

where $c = c(\delta)$ is the constant from Lemma 1.

Proof. Let $K_{(\delta)}(x_j) \le \lambda$. By (14)

$$n_j = \mathcal{N}(c\lambda; BW(\Delta_j), L_2(\Delta_j)) = 0.$$

Let $\Lambda = \{j \in \mathbb{N} : K_{(\delta)}(x_j) > \lambda\}$. Since $K_{(\delta)}(x) \to 0$ for $x \to \infty$, then $\Lambda \subset \{1, 2, ..., m\}$, where $m \in \mathbb{N}$ suffices big. If $n_j > 0$ $(j \in \Lambda)$, then for all $n \ge n_j$

$$\inf_{U \in \mathfrak{U}_{n_j}(BW(\Delta_j), L_2(\Delta_j))} \sup_{y \in BW(\Delta_j)} \|y - Uy; L_2(\Delta_j)\| \le c\lambda.$$

Therefore, for an arbitrarily small $\eta > 0$ there is an operator $U_j \in \mathfrak{U}_{n_j}(BW(\Delta_j), L_2(\Delta_j))$, for which

$$\sup_{y \in BW(\Delta_j)} \|y - U_j y; L_2(\Delta_j)\| \le (1 + \eta)c\lambda. \tag{22}$$

Let χ_j be the characteristic function of the interval $[x_j, x_j + h(x_j, \lambda))$, $\Lambda_+ = \{j \in \Lambda : n_j > 0\}$. Operator

$$Uy = \sum_{j \in \Lambda_+} \chi_j U_j(\chi, y), \ y \in L_2(I_\epsilon),$$

has finite dimension

$$\dim U \le \sum_{j \in \Lambda_+} n_j.$$

Moreover, for any $y \in BW$ it follows from Lemma 1 and (22) that

$$\int_0^\infty |y - Uy|^2 \, dx = \sum_{j \in \Lambda_+} \int_{\Delta_j} |\chi_j y - U_j(\chi, y)|^2 \, dx + \sum_{j \notin \Lambda_+} \int_{\Delta_j} |y|^2 dx \le$$

$$\le \sum_{j \in \Lambda_+} ((1 + \eta)c\lambda)^2 \|y; W(\Delta_j)\|^2 + \sum_{j \notin \Lambda_+} (c\lambda)^2 \|y; W(\Delta_j)\|^2 \le ((2 + \eta)c\lambda)^2 \|y\|_W^2 \le ((2 + \eta)\lambda)^2. \tag{23}$$

The passage to the limit in (23) leads to the following estimates:

$$\|y - Uy; L_2(I)\| \le 2c\lambda, \ y \in BW,$$

$$\lambda_n(BW, L_2(I)) \le 2c\lambda, \ \text{if} \ n \ge \sum_{j \in \Lambda_+} n_j,$$

$$\mathcal{N}(2c\lambda; BW, L_2(I)) \le \sum_{j \in \Lambda_+} n_j = \sum_{j: K_0(x_j) > \lambda} \mathcal{N}(c\lambda; BW(\Delta_j), L_2(\Delta)).$$

Proof of Theorem 2. It follows from (19), (21)

$$N(2c\lambda; BW, L_2(I)) \le \lambda^{-1/2} \sum_{j\in\Lambda} \left[K(x_j, h_j; \rho_0^{2\mu})\right]^{1/2}, \tag{24}$$

where $\Lambda = \{j \in \mathbb{N} : K_{(\delta/2)}(x_j) > \lambda\}$, $\widetilde{h}_j = \widetilde{h}(x_j, \lambda)$ and $\Delta_j = [x_j; x_j + h_j)$ $(j \in \Lambda)$ do not intersect. Since $\Delta_j' = [x_j; x_j + h_j/2) \in B_{t,(\delta/2)}$ for all $t \in \Delta_j'$, then

$$h_j^{-1} \int\limits_{\Delta_j} \rho_0^{-2\mu} d\xi \le M^*_{(\delta/2)} \rho_0^{-2\mu}(t),\ t \in \Delta_j'. \tag{25}$$

Therefore

$$(K(x_j, h_j))^{1/2} = h_j \left(\frac{1}{|\Delta_j|} \int\limits_{\Delta_j} \rho_0^{-2\mu} d\xi \right)^{1/4} \le 2 \int\limits_{\Delta_j'} \left(M^*_{(\delta/2)} \rho_0^{-2\mu} \right)^{1/4} d\xi,$$

and by (24) and embedding $BV \subset BW$ we have

$$\mathcal{N}(2c\lambda; BV, L_2(I)) \le 2\lambda^{-1/2} \sum_{j\in\Lambda} \int\limits_{\Delta_j'} \left(M^*_{(\delta/2)} \rho_0^{-2\mu} \right)^{1/4} d\xi. \tag{26}$$

Let $\Delta = [x, x+h)$ $(h > 0)$, $E_\delta(\Delta) = \{e : e = \overline{e} \subset [x, x+h]$ and $|e| \le \delta h\}$. With $0 < \delta < 1/2$

$$E_\delta(\Delta) \subset E_{2\delta}(\Delta), \tag{27}$$

and for $t \in [x, x+h/2]$, $\Delta_t = [t, t+h/2)$

$$\{e_t = e \cap [t, t+h/2],\ e \in E_\delta(\Delta)\} \subset E_{2\delta}(\Delta_t). \tag{28}$$

(27) and (28) allow us to show (using simple reasoning) that

$$h_{(\delta)}(x) \ge h_{(\delta/2)}(x), \tag{29}$$

$$M^*_{(\delta)} f(x) \ge M^*_{(\delta/2)} f(x), \tag{30}$$

$$h_{(\delta)}(t) \ge h_{(\delta/2)}(x)/2,\ \text{if } t \in [x, x+h/2]. \tag{31}$$

Now from (25), (29)–(31) we deduce that

$$\Delta_j' \subset G(\lambda/2), \tag{32}$$

namely that for all $t \in \Delta_j'$ we have

$$(h_\delta(t))^2 \left(M^*_{(\delta)} \rho_0^{-2\mu}(t) \right)^{1/2} \ge \frac{1}{4} h_j^2 \left(h_j^{-1} \int\limits_{\Delta_j} \rho_0^{-2\mu} d\xi \right)^{1/2} = \frac{1}{4}\lambda.$$

Since $F \subset aBV \subset aBW$, $a = c_{\mu,\nu}$, then by (26) and (32)

$$\mathcal{N}(\lambda; F, L_2(I)) \le 2(\frac{ac}{\lambda})^{1/2} \int\limits_{G(\lambda/ac)} \left(M^*_{(\delta)} \rho_0^{-2\mu} \right)^{1/4} d\xi.$$

Acknowledgments

This study was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08856104).

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Бір сингулярлы дифференциалдық теңдеудің шешімдерінің аппроксимациясы туралы

Мақалада Трибел типті дифференциалдық теңдеудің шешімдерін сызықтық әдістермен ең жақсы жуықтау мәселесі зерттелді. Бұл есептер дифференциалданатын функциялардың сәйкес кеңістіктеріндегі бірлік шарының көлденең сызығын бағалау арқылы шешілді. Анықтамаға сәйкес, көлденең сызық берілген нормаланған кеңістіктегі компакт жиындарды сызықтық ақырлыөлшемді операторлар арқылы жүзеге асырылатын сызықтық әдістермен жуықтаудың ең жақсы бағалауын береді. Тапсырмада зерттелетін теңдеудің шешілетіндігі туралы, сәйкес дифференциалданатын функциялардың салмақты кеңістігін құру, Соболев салмақты полиномиалды кеңістігіндегі компакт жиындардың көлденең сызығын бағалау үшін әдістемесін құру туралы мәселелер қамтылды. Бұл жұмыста қарастырылған оператордың шектелген кері операторы болудың шарттары алынды. Қойылған мәселеге сәйкес Соболев салмақты кеңістігі анықталды, көлденең сызық тізбегі үшін санау функциясының жоғарғы бағалаулары алынды. Сонымен қатар, көлденең сызықтың екі жақты бағалаулары берілген мысал құрастырылды. Бұл есептің шешу әдісін шексіз осьте стандартты емес кәдімгі дифференциалдық теңдеулерді сандық түрде шешу үшін қолдануға болады.

Кілт сөздер: дифференциалдық теңдеулер, Трибел теңдеулері, жиындарды сызықтық әдістермен жуықтау, көлденең жиындар, Соболевтің салмақты кеңістіктері.

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Об аппроксимации решений одного сингулярного дифференциального уравнения на оси

В статье исследована задача о наилучшем приближении линейными методами решений одного уравнения типа Трибеля. Эта задача решалась с помощью оценок линейных поперечников единичного шара в соответствующих пространствах дифференцируемых функций. Согласно определению, линейные поперечники дают наилучшие оценки аппроксимации компактов в заданном нормированном пространстве линейными методами, реализуемыми через конечномерные операторы. Задача включает ответы на вопросы о разрешимости изучаемого уравнения, построение соответствующего весового пространства дифференцируемых функций, разработку метода для оценки линейных поперечников компактов в весовом полиномиальном пространстве Соболева. В работе получены условия, при которых рассматриваемый оператор становится ограниченно обратным. Определено весовое пространство Соболева, соответствующее поставленной задаче. Получены верхние оценки считающей функции для последовательности линейных поперечников, соответствующих поставленной проблеме. Построен один пример, в котором даны двусторонние оценки линейных поперечников. Метод решения этой задачи может быть применен к численному решению нестандартных обыкновенных дифференциальных уравнений на бесконечной оси.

Ключевые слова: дифференциальные уравнения, уравнения Трибеля, аппроксимация множеств линейными методами, поперечники множеств, весовые пространства Соболева.

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DOI 10.31489/2022M4/98-106

UDC 517.95

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On the non-uniqueness of the solution to a boundary value problem of heat conduction with a load in the form of a fractional derivative

The paper deals with the second boundary value problem for the loaded heat equation in the first quadrant. The loaded term contains a fractional derivative in the Caputo sense of an order α, $2 < \alpha < 3$. The boundary value problem is reduced to an integro-differential equation with a difference kernel by inverting the differential part. It is proved that a homogeneous integro-differential equation has at least one non-zero solution. It is shown that the solution of the homogeneous boundary value problem corresponding to the original boundary value problem is not unique, and the load acts as a strong perturbation of the boundary value problem.

Keywords: second boundary value problem, loaded equation, Caputo fractional derivative, non-unique solvability, strong perturbation.

Introduction

Loaded differential equations today have a wide practical application in many areas of natural science. Moreover, loaded equations are a special class of equations that require separate consideration. In addition, loaded equations can act as one of the ways to introduce generalized solutions of wide classes of partial differential equations and as an effective method for finding approximate solutions to boundary value problems for differential equations. A significant contribution to the development of the theory of loaded equations was made by the work of A.M. Nakhushev [1] (and his other works), where definitions of loaded differential, loaded integro-differential, loaded functional equations and their numerous applications are given. In papers [2–5], the theory of loaded equations was further developed. [3] considers boundary value problems for a loaded differential operator, which are interpreted as perturbations of the corresponding differential operators. It is shown that the loaded part is a weak or strong perturbation, depending on the derivative order in the loaded term, as well as on the manifold on which the trace of the BVP solution is given.

There are many books devoted to fractional analysis today [6–21]. In recent years, an intensive study of loaded differential equations has been carried out, associated with various applied problems of mechanics, biology, ecology and chemistry, modeled using loaded equations. To date, many books have been devoted to fractional analysis (various applications in physics, mechanics, and simulation) [7], [14–20]. Among the variety of works, the monograph [6], covering a huge range of ideas. Monograph presents classical and modern results in the theory of fractional analysis, and gives their applications to integral and differential equations and function theory.

From a mathematical point of view, it is interesting to study the boundary value problems for the heat equation with a fractional load, when the loaded term is considered in the form of a fractional derivative or a fractional integral. In [21, 22] the load moves with a constant velocity, namely, it moves along the line $x = t$. The loaded term contains a fractional derivative in the Riemann-Liouville sense. The boundary value problem was reduced to the Volterra integral equation with a kernel containing a

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generalized hypergeometric series. The integral equation has a nonempty spectrum for certain values of the fractional derivative order and for the spectral parameter.

We also note that the boundary value problems of heat conduction and the Volterra integral equations arising in their study with singularities in the kernel, similar to the singularities in this paper, were considered in [23, 24].

In [25–28] fractionally loaded boundary value problems of heat conduction are investigated, the loaded term is represented in the form of the fractional derivative. The derivative order in the loaded term is less than the order of the differential part. In [25, 26] the loaded term is represented in the form of the Caputo fractional derivative with respect to the spatial variable. In [25], it is proved that there is continuity on the right in the order of the fractional derivative. There is no continuity on the left. In [26] there is continuity in the order of the derivative in the loaded term of the problem. In [27, 28], the loaded term has the form of a fractional Riemann-Liouville derivative with respect to the time variable. The kernel of the resulting integral equation contains a special function, for example, a generalized hypergeometric function in [25] or the Wright function in [27]. Conditions for the unique solvability of the integral equation are established by estimating the integral kernel. It is shown that the existence and uniqueness of solutions to the integral equation depends on the order of the fractional derivative in the loaded term.

In [29] the first boundary value problem for essentially loaded equation of heat conduction is considered. It is shown that if the point of load is fixed, then the stated boundary problem is uniquely solvable.

In this paper, the second boundary value problem for the loaded heat equation is considered in the domain $Q = \{(x,t) \mid x > 0, \quad t > 0\}$. The load is presented as a Caputo fractional derivative. The fractional derivative is greater than the order of the differential part of the BVP. The boundary value problem is reduced to an integro-differential equation by representing the problem solution in terms of the Green's function. Solvability of the integro-differential equation depends on the fractional derivative order in the loaded term of the BVP. The integro-differential equation has an eigenfunction. The solution of the stated boundary problem is determined by the solution of the obtained integro-differential equation in explicit form. Since the uniqueness of the BVP solution is violated, in this case the load can be interpreted as a strong perturbation the BVP.

The article is structured as follows. Section 1 includes some necessary concepts, definitions, auxiliary assertions, and preliminary assumptions about the classes of the BVP solution and the data included in the problem under study. In Section 2, we set the BVP that we are going to solve. In Section 3, the problem is reduced to an integro-differential equation with a difference kernel. In Section 4 we solve the resulting integro-differential equation by Laplace integral transform method. We write out the solution of the resulting equation in explicit form and formulate the corresponding results on the non-uniqueness of the solution to the BVP and the solution to the associated integro-differential equation.

1 Preliminaries

We first give some definitions and useful information.

Definition 1 ([6]). Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville derivative of the order β is defined as follows

$$ {}_rD^{\beta}_{a,t}f(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}}d\tau, \;\; \beta, a \in R, \;\; n-1 < \beta < n. \qquad (1)$$

Definition 2. Let $f(t) \in \mathrm{AC}^n[a,b]$ (i.e. $f^{(n-1)}(t)$ is an absolutely continuous function). Then, the Caputo derivative of the order β is defined as follows

$$ {}_cD^{\beta}_{a,\,t}\, f(t) = \frac{1}{\Gamma(n-\beta)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}}d\tau; \;\; \beta, \;\; a \in \Re, \;\; n-1 < \beta < n, \qquad (2)$$

From formula (1) it follows that

$$ {}_rD^0_{a,t}f(t) = f(t), \quad {}_rD^n_{a,t}f(t) = f^{(n)}(t), \quad n \in N. $$

We study a BVP for the loaded heat equation, when the loaded term is represented in the form of a fractional derivative. To study the formulated boundary problem, we need a formula for inverting the differential part of the equation.

It's known [30; 57] that in the domain $Q = \{(x,t) \mid x > 0, \quad t > 0\}$ the following boundary value problem of heat conduction

$$ u_t = a^2 u_{xx} + F(x,t), $$

$$ u\,|_{t=0} = f(x), \quad u_x\,|_{x=0} = g(x), $$

has the solution $u(x,t)$ described by the formula

$$ u(x,t) = \int_0^\infty G(x,\xi,t)\, f(\xi)\, d\xi - a\int_0^t G(x,\,0,\,t-\tau)\, g(\tau)\, d\tau + $$

$$ + \int_0^t \int_0^\infty G(x,\xi,t-\tau)\, F(\xi,\tau)\, d\xi d\tau, \tag{3} $$

where

$$ G(x,\,\xi,\,t) = \frac{1}{2\sqrt{\pi\, a\, t}} \left\{ \exp\left(-\frac{(x-\xi)^2}{4\,a\,t}\right) + \exp\left(-\frac{(x+\xi)^2}{4\,a\,t}\right) \right\}. $$

The following equality holds true for the Green function $G(x,\xi,t)$

$$ \int_0^\infty G(x,\xi,t)\, d\xi = 1. \tag{4} $$

It follows from the definitions that for the existence of a derivative of $f(t)$ in the sense of Riemann-Liouville (1) it is sufficient that $f(t)$ belongs to the class of summable functions, for the existence of a derivative in the sense of Caputo (2) it is sufficient that the $n-1$st derivative of the function $f(t)$ be an absolutely continuous function, where n-1 is the integer part of the derivative order, i.e. $f(t) \in \mathrm{AC}^n[a,b]$ and there is the next relation formula for these derivatives

$$ {}_cD^\beta_{a,t}\, f(t) = {}_rD^\beta_{a,t} \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k \right]. $$

So we assume that the solution $u(x,t)$ belongs to the class

$$ u(x,t) \in \mathrm{AC}^3\,(t \in [0,T]), \tag{5} $$

The right side of the BVP equation vanishes at $t < 0$ and belongs to the class

$$ f(x,t) \in L_\infty(A) \cap C(B), \tag{6} $$

where $A = \{(x,t)\,|\,x > 0,\ t \in [0,T]\}$, $B = \{(x,t)\,|\,x > 0,\ t \geq 0\}$, $T - const > 0$, also we assume

$$ f_1(x,t) = \int_0^t \int_0^\infty G(x,\xi,t-\tau)\, f(\xi,\tau)\, d\xi d\tau \in \mathrm{AC}^3\,(t \in [0,T]). \tag{7} $$

The classes in which the problem is studied are determined from the natural requirement for the existence and convergence of improper integrals that arise in the study.

2 Statement of the fractionally loaded BVP of heat conduction

In a domain $Q = \{(x,t) : x > 0, t > 0\}$ we consider a BVP

$$u_t - u_{xx} + \lambda \left\{ {}_cD^{\alpha}_{0t} u(x,t) \right\} \Big|_{x=\gamma(t)} = f(x,t), \tag{8}$$

$$u\,(x,0) = 0, \quad u_x\,(0,t) = 0, \tag{9}$$

where λ is a complex parameter,

$${}_cD^{\alpha}_{0t} u(x,t) = \frac{1}{\Gamma(3-\beta)} \int_0^t \frac{u_{\tau^3(x,\tau)}}{(t-\tau)^{\alpha-2}} d\tau$$

is Caputo derivative (2) of an order α, $2 < \alpha < 3$, $\gamma(t)$ is a continuous increasing function, $\gamma(0) = 0$ or $\gamma(t)$ is a positive *const*.

The solution of the problem and the right side of the equation belong to the classes (5) and (6), respectively.

3 Reducing the problem to a Volterra integro-differential equation of the second kind

Lemma 1. Boundary value problem (8)–(9) is reduced to a Volterra integro-differential equation of the second kind.

Proof. We invert the differential part of problem (8)–(9) by formula (3):

$$u(x,t) = -\lambda \int_0^t \int_0^{\infty} \left\{ {}_cD^{\alpha}_{0t} u(x,t) \right\} \Big|_{x=\gamma(t)} G(x,\xi,t-\tau) d\xi d\tau +$$
$$+ \int_0^t \int_0^{\infty} G\,(x,\xi,t-\tau)\, f\,(\xi,\tau)\, d\xi\, d\tau.$$

Taking into account relation (4) and introducing the notation

$$f_1\,(x,t) = \int_0^t \int_0^{\infty} G\,(x,\xi,t-\tau)\, f\,(\xi,\tau)\, d\xi\, d\tau,$$

we get the following representation of the solution to problem (8)–(9):

$$u(x,t) = -\lambda \int_0^t \mu(\tau) d\tau + f_1(x,t), \tag{10}$$

where

$$\mu(t) = \left\{ {}_cD^{\alpha}_{0t} u(x,t) \right\} \Big|_{x=\gamma(t)}. \tag{11}$$

From representation (10) we take the derivative of the order $2 < \alpha < 3$ with respect to the variables t on both sides and put $x = \gamma(t)$. On the left side, we get the function $\mu\,(t)$. We also introduce the notation according to formula (11).

Then BVP (8)–(9) is reduced to the integro-differential equation:

$$\mu(t) + \lambda \int_0^t K_{\alpha}\,(t,\tau)\, \mu''(\tau) d\tau = f_2(t), \tag{12}$$

with conditions $\mu(0) = \mu'(0) = 0$, where

$$K_{\alpha}\,(t,\tau) = \frac{1}{(3-\alpha)(t-\tau)^{\alpha-2}} \tag{13}$$

and

$$f_2(t) = \left\{ {}_cD^{\alpha}_{0t} f_1(x,t) \right\} \Big|_{x=\gamma(t)}. \tag{14}$$

Lemma 1 on reducing the BVP to an integro-differential equation is proved.

4 Study of the integro-differential equation. Main result

Lemma 2. The homogeneous integro-differential equation (12) has a non-trivial solution.

Proof. We denote the Laplace transforms of $\mu(t)$ and $f_2(t)$ as

$$\overline{\mu(p)} = L[\mu(t)] = \int_0^\infty e^{-pt}\mu(t)dt, \quad \overline{f_2(p)} = L[f_2(t)].$$

Since

$$L\left[\frac{1}{t^{\alpha-2}}\right] = \frac{\Gamma(3-\alpha)}{p^{3-\alpha}},$$

then applying Laplace transform to equation (12) with the condition $\mu(0) = \mu'(0) = 0$, we get

$$\overline{\mu(p)} = \frac{\overline{f_2(p)}}{1+\lambda p^{\alpha-1}}. \tag{15}$$

Consider equation (15) for $\overline{f_2(p)} \equiv 0$.

$$\overline{\mu(p)}(1+\lambda p^{\alpha-1}) = 0. \tag{16}$$

Let's solve the equation:

$$1+\lambda p^{\alpha-1} = 0. \tag{17}$$

For $\lambda \in \mathbb{C}$ and $2 < \alpha < 3$. Then $\alpha - 1$ is a real number. Let's consider cases.

I. $\alpha \in Q$. In case for $\alpha \in Q$, there can be finite number $p_1, p_2, ..., p_n$ are solutions to equation (17). Then nonzero solutions to (16) are

$$\overline{\mu_k(p)} = C_k\delta(p-p_k),$$

here $\delta(x)$ is the delta function, $C_k = const$; p_k are solutions of equation (17), $k = 1, ..., n$, n is a denominator of the rational number $\alpha - 1$. Here and below, the numbers are in the left half-plane of the complex plane, i.e. $Re,\ p_k < 0$.

I. $\alpha \in Q$. Applying the inverse Laplace transform to the last equation, we get

$$\mu_k(t) = C_k\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\delta(p-p_k)e^{pt}dp = C_k e^{p_k t}.$$

Integral is taken along the line $Re\, p = \sigma$ and is considered in the form of the main value. That's why, if $p = p_k$ are the solutions of equation (17), then eigenfunctions of equation (12) have the following form

$$\mu_k(t) = C_k e^{p_k t}. \tag{18}$$

Remark 1. The power of the complex number z with rational power $z^{\frac{m}{n}}$ is defined as:

$$z^{\frac{m}{n}} = (\sqrt[n]{z})^m.$$

II. $\alpha \in R$ *but* $\alpha \notin Q$. Then $\alpha - 1$ is an irrational number

Remark 2. Power of the complex number z with real irrational index of power $0 < s < \alpha - 1$ is defined as the limit

$$z^s = \lim_{n\to\infty} z^{\frac{\alpha_n}{\beta_n}};$$
$$\frac{\alpha_n}{\beta_n} \to s,$$

here α_n and β_n are sequences of natural numbers.

Based on Remark 2 we can claim that equation (17) has at least one solution p_0 for $\lambda \in C$ and $2 < \alpha < 3$.

Then equation (12) has at least one eigenfunction (18). The number of eigenfunctions depends on the values of parameters α and λ.

Now let's find a solution of nonhomogeneous equation (12) ($f_2(t) \neq 0$).

Equation (15) can be rewritten:

$$\overline{\mu(p)} = \overline{f_2(p)} - \lambda \frac{p^{\alpha-1}}{1+\lambda p^{\alpha-1}} \overline{f_2(p)}. \tag{19}$$

Now we apply the inverse Laplace transform to equation (19)

$$L^{-1}\left[\frac{p^{\alpha-1}}{1+\lambda p^{\alpha-1}}\right] = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{A(p)}{1+\lambda p^{\alpha-1}} dp = R_\lambda(t,\alpha), \tag{20}$$

here $A(p) = p^{\alpha-1}e^{pt}$.

The integral in (20) is considered as the main value and the integration is taken along the contour which doesn't include p_k on the left side. Then the solution of equation (12) can be written as: the solution of equation (12)

$$\mu(t) = f_2(t) + \sum_k C_k e^{p_k t} - \lambda \int_0^t R_\lambda(t-\tau;\alpha) f_2(\tau) d\tau, \tag{21}$$

here p_k are the roots of equation (17), C_k are arbitrary constants and $R_\lambda(t;\alpha)$ is defined as in (20). The zeros of the denominator of the integral function in (20) are the numbers p_k so that $A(p_k)) \neq 0$. Therefore

$$R_\lambda(t,\alpha) = \sum_k res_{p=p_k} \frac{A(p)}{1+\lambda p^{\alpha-1}} = \sum_k \frac{A(p_k)}{\lambda(\alpha-1)p_k^{\alpha-2}} = \sum_k \frac{p_k e^{p_k t}}{\lambda(\alpha-1)}.$$

Then (21) can be rewritten as

$$\mu(t) = f_2(t) + \sum_k C_k e^{p_k t} - \sum_k \frac{p_k}{\alpha-1} \int_0^t e^{p_k(t-\tau)} f_2(\tau) d\tau. \tag{22}$$

Thus, the following theorem has been proved.

Theorem. Integro-differential equation (12) with kernel and right side defined by the formulas (13) ($2 < \alpha < 3$) and (14), respectively, has a solution defined by the formula (22), moreover, the corresponding homogeneous equation (12) (when $f_2(t) = 0$) has a nonzero solution

$$\mu(t) = \sum_k e^{p_k t},$$

where here p_k are the roots of equation (17) and $Re\, p_k < 0$.

Conclusion

So function (22) is the solution of equation (12). Then the solution of BVP (8)–(9) has the form of (10)

$$u(x,t) = -\lambda \int_0^t \mu(\tau) d\tau + f_1(x,t),$$

where the function $f_1(x,t)$ is defined by the formula (7).

In such a way it can be claimed that term with a load in equation for BVP (8)–(9) is considered a strong perturbation, since according to (22) and (10) the homogeneous BVP (8)–(9) (when $f(x,t)=0$) has non-zero solutions in the form of:

$$u(x,t)=\sum_k \frac{\lambda}{p_k}(e^{p_k t}-1),$$

here p_k are solutions of equation (17) and $Re\, p_k<0$.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP09259780, 2021–2023.)

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Бөлшек туындысы түріндегі жүктемемен жылу өткізгіштіктің шекаралық есебін шешудің бірегей еместігі туралы

Бірінші квадрантта бөлшектік-жүктелген жылуөткізгіштік теңдеуі үшін екінші шеттік есеп қарастырылған. Жүктеме қосылғышы $2 < \alpha < 3$ ретті Капуто бөлшек туындысы ретінде берілген. Шеттік есеп дифференциалдық бөлігін ауыстыру арқылы айырма өзекті интегро-дифференциалдық теңдеуге келтіріледі. Біртекті интегро-дифференциалдық теңдеудің кем дегенде бір нөлдік емес шешімі бар екені дәлелденді. Біртекті шекаралық есептің шешімі бірегей емес, ал жүктеме шекаралық есептің қатты ауытқуы болып табылатыны көрсетілген.

Кілт сөздер: екінші шеттік есеп, жүктелген теңдеу, Капуто бөлшектік туындысы, көп мағыналы шешілім, қатты ауытқу.

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О неединственности решения краевой задачи теплопроводности с нагрузкой в виде дробной производной

В статье рассмотрена вторая краевая задача для нагруженного уравнения теплопроводности в первом квадранте. Нагруженное слагаемое содержит дробную производную в смысле Капуто порядка $2 < \alpha < 3$. Краевая задача сводится к интегро-дифференциальному уравнению с разностным ядром изменения дифференциальной части. Доказано, что однородное интегро-дифференциальное уравнение имеет хотя бы одно ненулевое решение. Показано, что решение однородной краевой задачи, соответствующей исходной краевой задаче, неединственно, а нагрузка выступает как сильное возмущение краевой задачи.

Ключевые слова: вторая краевая задача, нагруженное уравнение, дробная производная Капуто, неоднозначная разрешимость, сильное возмущение.

DOI 10.31489/2022M4/107-116
UDC 519.624

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On bounded solutions of linear systems of differential equations with unbounded coefficients

This paper deals with a problem of finding a bounded solution of a system of nonhomogeneous linear differential equations with an unbounded matrix of coefficients on a finite interval. The right-hand side of the equation belongs to a space of continuous functions bounded with some weight; the weight function is chosen taking into account the behavior of the coefficient matrix. The problem is studied using a modified version of the parameterization method with non-uniform partitioning. Necessary and sufficient conditions of well-posedness of the problem are obtained in terms of a bilaterally infinite matrix of special structure.

Keywords: ordinary differential equation, singular boundary-value problem, well-posedness, parameterization method, bounded solution, linear system, unbounded coefficients.

In various branches of applied mathematics there arise problems leading to systems of ordinary differential equations involving singularities or defined on an infinite interval. Numerous works [1–12] have been studied the existence of bounded solutions of such problems. In [6], the boundedness condition for a solution at a singular point is replaced by an equivalent relation in a neighborhood of this point, namely, the equation of a stable initial manifold generated in the neighborhood of the singular point by the total set of bounded solutions of the system. In [8], the existence and approximation of a bounded (on the whole axis) solution of a linear ordinary differential equation are investigated by using the parameterization method. In this paper, we apply the parameterization method with non-uniform partition of the interval (0, T) to the linear differential equation

$$\frac{dx}{dt} = A(t)x + f(t), \quad x \in \mathbb{R}^n, \quad t \in (0,T), \tag{1}$$

where $A(t)$ and $f(t)$ are continuous on $(0,T)$, $\|A(t)\| = \max\limits_i \sum\limits_{j=1}^{n} |a_{i,j}(t)| = \alpha(t)$. We assume that the function $\alpha(t)$ is continuous on $(0,T)$ and satisfies the following conditions:

$$\int\limits_0^{T/2} \alpha(t)dt = \infty, \quad \lim_{t\to 0+0} \alpha(t) = \infty, \quad \int\limits_{T/2}^{T} \alpha(t)dt = \infty, \quad \lim_{t\to T-0} \alpha(t) = \infty.$$

We introduce the following spaces:
$\widetilde{C}((0,T),\mathbb{R}^n)$ is the space of functions $x : (0,T) \to \mathbb{R}^n$ that are continuous and bounded on $(0,T)$, equipped with the norm

$$\|x\|_1 = \sup_{t\in(0,T)} \|x(t)\|;$$

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$\widetilde{C}_{1/\alpha}((0,T),\mathbb{R}^n)$ is the space of functions $f:(0,T)\to\mathbb{R}^n$ that are continuous and bounded on $(0,T)$ with the weight $1/\alpha(t)$, equipped with the norm

$$\|f\|_\alpha = \sup_{t\in(0,T)} \|f(t)\alpha(t)\|;$$

m_n is the space bounded bilaterally infinite sequences $\lambda_s \in \mathbb{R}^n$ with the norm

$$\|\lambda\|_2 = \|(\ldots,\lambda_r,\lambda_{r+1},\ldots)'\|_2 = \sup_r \|\lambda_r\|, \quad r\in\mathbb{Z};$$

$L(m_n)$ is the space of all bounded linear operators mapping m_n into itself, equipped with the induced norm.

Let us choose a number $\theta > 0$ and make the partition $(0,T) = \bigcup\limits_{r=-\infty}^{\infty} [t_{r-1},t_r)$ by the points t_r, $r\in\mathbb{Z}$, defined as follows: $t_0 = T/2$, $\int\limits_{t_{r-1}}^{t_r} \alpha(t)dt = \theta$.

Let $\overline{h}(\theta)$ be the bilaterally infinite sequence of the partition step-sizes $h_r = t_r - t_{r-1}$, $r\in\mathbb{R}$. We denote by $x_r(t)$ the restriction of a function $x(t)\in\widetilde{C}((0,T),\mathbb{R}^n)$ to the r-th subinterval and introduce one more space $m_n(\overline{h})$ of bounded bilaterally infinite sequences of functions $x_r(t)$, $r\in\mathbb{Z}$, that are continuous and bounded on $[t_{r-1},t_r)$, equipped with the norm

$$\|x[t]\|_3 = \|(\ldots,x_r(t),x_{r+1}(t),\ldots)'\|_3 = \sup_r \sup_{t\in[t_{r-1},t_r)} \|x_r(t)\|.$$

Definition 1. We call Problem 1_α the problem of finding a bounded on $(0,T)$ solution of Eq. (1) with $f(t)\in\widetilde{C}_{1/\alpha}((0,T),\mathbb{R}^n)$.

The existence of a solution $x(t)\in\widetilde{C}((0,T),\mathbb{R}^n)$ of Problem 1_α is equivalent to the existence of a solution $x[t]\in m_n(\overline{h})$ of the multipont problem for the equations

$$\frac{dx_r}{dt} = A(t)x_r + f(t), \quad t\in[t_{r-1},t_r), \tag{2}$$

subject to the gluing conditions for $x(t)$ at the interior partition points:

$$\lim_{t\to t_r-0} x_r(t) = x_{r+1}(t_r), \quad r\in\mathbb{Z}. \tag{3}$$

Note that the derivative $\frac{dx_r}{dt}\big|_{t=t_{r-1}}$ in Eq. (2) is understood as the right-sided limit $\lim\limits_{t\to t_{r-1}+0} \frac{dx_r}{dt}$.

Indeed, let $\widehat{x}(t)$ be a solution of Problem 1_α. Let us show that the system of its restrictions to the partition subintervals, $\widehat{x}[t] = (\ldots,\widehat{x}_r(t),\widehat{x}_{r+1}(t),\ldots)'$, belongs to $m_n(\overline{h})$ and satisfies Eq. (2) and conditions (3).

Since $\widehat{x}(t)$ is a solution of Eq. (1), it is continuously differentiable on $(0,T)$. Hence $\widehat{x}_r(t)$ and $\frac{d\widehat{x}_r}{dt}$, $r\in\mathbb{Z}$, are continuous on $[t_{r-1},t_r)$. The boundedness of the function $\widehat{x}(t)$ on $(0,T)$ implies that the functions $\widehat{x}_r(t)$, $r\in\mathbb{Z}$, are bounded on $[t_{r-1},t_r)$, and $\widehat{x}[t]\in m_n(\overline{h})$.

The function system $\widehat{x}[t]$ satisfies Eq. (2) for all $t\in[t_{r-1},t_r)$, $r\in\mathbb{Z}$:

$$\frac{d\widehat{x}_r(t)}{dt} = \frac{d\widehat{x}(t)}{dt} = A(t)\widehat{x}(t) + f(t) = A(t)\widehat{x}_r(t) + f(t).$$

The continuity of $\widehat{x}(t)$ on $(0,T)$ implies the existence of the left-sided limits

$$\lim_{t\to t_r-0} \widehat{x}_r(t) = \lim_{t\to t_r-0} \widehat{x}(t) = \widehat{x}(t_r), \quad r\in\mathbb{Z},$$

that is, conditions (3) are satisfied:

$$\lim_{t\to t_r-0}\widehat{x}_r(t)=\widehat{x}(t_r)=\widehat{x}_{r+1}(t_r).$$

Let us now show that if $\widetilde{x}[t]=(\ldots,\widetilde{x}_r(t),\widetilde{x}_{r+1}(t),\ldots)'\in m_n(\overline{h})$ is a solution of problem (2),(3), then the function $\widetilde{x}(t)$, defined as $\widetilde{x}(t)=\widetilde{x}_r(t)$, $t\in[t_{r-1},t_r)$, $r\in\mathbb{Z}$, is a solution of Problem 1_α.

It follows from (3) that $\widetilde{x}(t)$ is continuous on $(0,T)$. Since the functions $\widetilde{x}_r(t)$, $r\in\mathbb{Z}$, satisfy Eq. (2) for all $t\in[t_{r-1},t_r)$, the function $\widetilde{x}(t)$ is continuously differentiable for all $x\in(0,T)$ except the points $t=t_r$, $r\in\mathbb{Z}$, and

$$\frac{d\widetilde{x}(t)}{dt}=\frac{d\widetilde{x}_r(t)}{dt}=A(t)\widetilde{x}_r(t)+f(t)=A(t)\widetilde{x}(t)+f(t),$$
$$t\in(0,T)\setminus\{t=t_r,r\in\mathbb{Z}\}.$$

The function $\widetilde{x}(t)$ has the right-hand derivative at the points $t=t_r$, $r\in\mathbb{Z}$. Let t_k be one of these points, and let us consider Eq. (1) on the intervals $[t_{k-1},t_k)$ and $[t_k,t_{k+1})$:

$$\frac{d\widetilde{x}(t)}{dt}=A(t)\widetilde{x}(t)+f(t),\quad [t_{k-1},t_k), \tag{4}$$

$$\frac{d\widetilde{x}(t)}{dt}=A(t)\widetilde{x}(t)+f(t),\quad [t_k,t_{k+1}). \tag{5}$$

From (4) and the continuity of $A(t)$, $f(t)$, and $\widetilde{x}(t)$ on $(0,T)$, we have

$$\lim_{t\to t_k-0}\frac{d\widetilde{x}}{dt}=A(t_k)\widetilde{x}(t_k)+f(t_k),$$

i.e., at $t=t_k$ there exists the left-hand derivative of $\widetilde{x}(t)$:

$$\dot{\widetilde{x}}(t_k-0)=A(t_k)\widetilde{x}(t_k)+f(t_k).$$

Taking into account (5) and the existence of $\dot{\widetilde{x}}(t_k+0)=A(t_k)\widetilde{x}(t_k)+f(t_k)$, we obtain that the continuous derivative of $\widetilde{x}$ exists at $t=t_k$, and Eq. (1) holds at this point.

Thus, the function $\widetilde{x}(t)$ is continuously differentiable on $(0,T)$ and satisfies Eq. (1) for all $x\in(0,T)$. It follows from $\widetilde{x}[t]\in m_n(\overline{h})$ that $\widetilde{x}(t)$ is a bounded solution of Eq. (1).

Let λ_r denote the values of $x_r(t)$ at $t=t_{r-1}$, $r\in\mathbb{Z}$. Setting $u_r(t)=x_r(t)-\lambda_r$ on each partition subinterval $[t_{r-1},t_r)$, we obtain the following boundary value problem with parameter:

$$\frac{du_r}{dt}=A(t)[u_r+\lambda_r]+f(t),\quad t\in[t_{r-1},t_r),\quad u_r(t_{r-1})=0, \tag{6}$$

$$\lim_{t\to t_r-0}u_r(t)+\lambda_r=\lambda_{r+1},\quad r\in\mathbb{Z} \tag{7}$$

$$(\lambda,u[t])\in m_n\times m_n(\overline{h}). \tag{8}$$

If a pair $(\lambda^*,u^*[t])\in m_n\times m_n(\overline{h})$ is a solution of problem (6)-(8), then the function $x^*(t)$, obtained by gluing the function systems $(\lambda_r^*+u_r^*[t])$, $r\in\mathbb{Z}$, belongs to the space $\widetilde{C}((0,T),\mathbb{R}^n)$ and satisfies Eq. (1) for all $t\in(0,T)$. Conversely, if $x(t)$ is a solution of Problem 1_α, then the pair $(\lambda,u[t])$ (with $\lambda=(\ldots,x_r(t_{r-1}),x_{r+1}(t_r),\ldots)$ and $u[t]=(\ldots,x_r(t)-x_r(t_{r-1}),x_{r+1}(t)-x_{r+1}(t_r),\ldots)$, where $x_r(t)$ are the restrictions of $x(t)$ to the $r-$th subintervals, $r\in\mathbb{Z}$) belongs to $m_n\times m_n(\overline{h})$ and satisfies Eq. (6) and conditions (7).

Since (6) is an initial-value problem with parameter, we obtain the integral representation of $u_r(t)$ for fixed parameter values λ_r:

$$u_r(t) = \int\limits_{t_{r-1}}^{t} A(\tau)[u_r(\tau) + \lambda_r]d\tau + \int\limits_{t_{r-1}}^{t} f(\tau)d\tau, \quad r \in \mathbb{Z}. \tag{9}$$

Replacing $u_r(\tau)$ with the right-hand side of (9) and repeating this procedure ν times ($\nu = 1, 2, \ldots$), we obtain

$$u_r(t) = D_{\nu,r}(t)\lambda_r + F_{\nu,r}(t) + G_{\nu,r}(u,t), \quad t \in [t_{r-1}, t_r), \tag{10}$$

where

$$D_{\nu,r}(t) = \sum_{j=0}^{\nu-1} \int\limits_{t_{r-1}}^{t} A(\tau_1) \ldots \int\limits_{t_{r-1}}^{\tau_j} A(\tau_{j+1})d\tau_{j+1} \ldots d\tau_1,$$

$$F_{\nu,r}(t) = \int\limits_{t_{r-1}}^{t} f(\tau_1)d\tau_1 + \sum_{j=1}^{\nu-1} \int\limits_{t_{r-1}}^{t} A(\tau_1) \ldots \int\limits_{t_{r-1}}^{\tau_{j-1}} A(\tau_j) \int\limits_{t_{r-1}}^{\tau_j} f(\tau_{j+1})d\tau_{j+1}d\tau_j \ldots d\tau_1,$$

$$G_{\nu,r}(u,t) = \int\limits_{t_{r-1}}^{t} A(\tau_1) \ldots \int\limits_{t_{r-1}}^{\tau_j} A(\tau_{j+1})u_r(\tau_{j+1})d\tau_{j+1} \ldots d\tau_1, \quad \tau_0 = t, \quad r \in \mathbb{Z}.$$

Now, substituting the values $\lim\limits_{t \to t_r - 0} u_r(t)$, $r \in \mathbb{Z}$, determined from (9), into equations (10), we obtain the bilaterally infinite system of algebraic equations in parameters λ_r:

$$[I + D_{\nu,r}(h_r)]\lambda_r - \lambda_{r+1} = -F_{\nu,r}(h_r) - G_{\nu,r}(u, h_r), \quad r \in \mathbb{Z}. \tag{11}$$

Here I is the identity matrix of order n.

Let us denote by $Q_{\nu,\overline{h}(\theta)}$ the bilaterally infinite block-banded matrix corresponding to the left-hand side of system (11). The only non-zero terms in each block row of $Q_{\nu,\overline{h}(\theta)}$ are $I + D_{\nu,r}(h_r)$ and $-I$. Hence, for any sequence $\overline{h}(\theta)$, the matrix $Q_{\nu,\overline{h}(\theta)}$ maps the space m_n into itself, and the following estimate holds:

$$\|Q_{\nu,\overline{h}(\theta)}\|_{L(m_n)} \leq 2 + \sum_{j=1}^{\nu} \frac{\theta^j}{j!}.$$

The matrix form of system (11) is

$$Q_{\nu,\overline{h}(\theta)}\lambda = -F_\nu(\overline{h}) - G_\nu(u, \overline{h}), \quad \lambda \in m_n,$$

where

$$F_\nu(\overline{h}) = (\ldots, F_{\nu,r}(h_r), F_{\nu,r+1}(h_{r+1}), \ldots)' \in m_n,$$

$$G_\nu(u, \overline{h}) = (\ldots, G_{\nu,r}(u, h_r), G_{\nu,r+1}(u, h_{r+1}), \ldots)' \in m_n$$

for all $u[t] \in m_n(\overline{h})$ and $\overline{h}(\theta)$.

Definition 2. Problem 1_α is well-posed if it has a unique solution $x(t) \in \widetilde{C}((0,T), \mathbb{R}^n)$ for any $f(t) \in \widetilde{C}_{1/\alpha}((0,T), \mathbb{R}^n)$, and $\|x\|_1 \leq K\|f\|_\alpha$, where K is a constant independent of $f(t)$.

Theorem 1. Let $Q_{\nu,\overline{h}(\theta)}$ have an inverse for some $\overline{h}(\theta)$ and ν ($\nu = 1, 2, \ldots$), and let

$$\|Q^{-1}_{\nu,\overline{h}(\theta)}\|_{L(m_n)} \leq \gamma_\nu(\overline{h}), \tag{12}$$

$$q_\nu(\overline{h}) = \gamma_\nu(\overline{h}) \left(e^\theta - 1 - \theta - \ldots - \frac{\theta^\nu}{\nu!} \right) < 1. \tag{13}$$

Then Problem 1_α is well-posed and its solution satisfies the estimate

$$\|x^*\|_1 \leq e^\theta \left[\frac{\gamma_\nu(\overline{h}}{1 - q_\nu(\overline{h})} \frac{\theta^\nu}{\nu!} (\gamma_\nu(\overline{h})(e^\theta - 1)^2 + \theta e^\theta) + \gamma_\nu(\overline{h})(e^\theta - 1) + \theta \right] \|f\|_\alpha.$$

The proof of Theorem 1 follows the scheme of Theorem 1 in [7].

Let $x^*(t)$ be the solution of Problem 1_α. Then the pair $(\lambda^*, u^*[t])$ with components $\lambda^*_r = x^*_r(t_{r-1})$ and $u^*_r(t) = x^*(t) - x^*(t_{r-1})$, $t \in [t_{r-1}, t_r)$, $r \in \mathbb{Z}$, is the solution of problem (6)–(8). Moreover, there exist numbers δ_1 and δ_2 such that $\|\lambda^*\| \leq \delta_1$ and $\|u^*_r(t)\| \leq \delta_2$, $t \in [t_{r-1}, t_r)$, $r \in \mathbb{Z}$, and for any $\nu \in \mathbb{N}$ the following identities hold:

$$u^*_r(t) = D_{\nu,r}(t)\lambda^*_r + F_{\nu,r}(t) + G_{\nu,r}(u^*, t), \quad t \in [t_{r-1}, t_r), \quad r \in \mathbb{Z}, \tag{14}$$

$$Q_{\nu,\overline{h}(\theta)}\lambda^* = -F_\nu(\overline{h}) - G_\nu(u^*, \overline{h}). \tag{15}$$

It can be easily shown that $\|G_\nu(u^*, \overline{h})\|_2 \leq \frac{\theta^\nu}{\nu!}\|u^*[t]\|_3 \leq \frac{\theta^\nu}{\nu!}\delta_2$, and $D_{\nu,r}(t)$ and $F_{\nu,r}(t)$ converge uniformly to

$$D_{*,r}(t) = \sum_{j=0}^{\infty} \int\limits_{t_{r-1}}^{t} A(\tau_1) \ldots \int\limits_{t_{r-1}}^{\tau_j} A(\tau_{j+1}) d\tau_{j+1} \ldots d\tau_1,$$

and

$$F_{*,r}(t) = \int\limits_{t_{r-1}}^{t} f(\tau_1)d\tau_1 + \sum_{j=1}^{\infty} \int\limits_{t_{r-1}}^{t} A(\tau_1) \ldots \int\limits_{t_{r-1}}^{\tau_{j-1}} A(\tau_j) \int\limits_{t_{r-1}}^{\tau_j} f(\tau_{j+1}) d\tau_{j+1} d\tau_j \ldots d\tau_1,$$

respectively. Then, letting $\nu \to \infty$ in (14), (15), and dividing both sides of (15) by $\theta > 0$, we obtain

$$u^*_r(t) = D_{*,r}(t)\lambda^*_r + F_{*,r}(t), \quad t \in [t_{r-1}, t_r), \quad r \in \mathbb{Z}, \tag{16}$$

$$\frac{1}{\theta} Q_{*,\overline{h}(\theta)}\lambda^* = -F_*(A, f, \overline{h}(\theta)), \quad \lambda^* \in m_n. \tag{17}$$

Here $F_*(A, f, \overline{h}(\theta)) = \lim\limits_{\nu\to\infty} \frac{1}{\theta} F_\nu(\overline{h})$.

Thus, if $(\lambda^*, u^*[t])$ is a solution of problem (6)–(8), then the parameter $\lambda^* = (\ldots, \lambda^*_r, \lambda^*_{r+1}, \ldots)' \in m_n$ satisfies Eq.(17), and the solutions $u^*_r(t)$ of the Cauchy problems (6), corresponding to λ^*_r, $r \in \mathbb{Z}$, are of the form (16).

We now assume that $\widehat{\lambda} = (\ldots, \widehat{\lambda}_r, \widehat{\lambda}_{r+1}, \ldots)' \in m_n$ is a solution of the system

$$\frac{1}{\theta}[I + D_{*,r}(t_r)]\lambda_r - \frac{1}{\theta}\lambda_{r+1} = -\frac{1}{\theta} F_{*,r}(t_),$$

or

$$\frac{1}{\theta} Q_{*,\overline{h}(\theta)}\widehat{\lambda} = -F_*(A, f, \overline{h}(\theta)), \tag{18}$$

and $\widehat{u}[t] = (\ldots, \widehat{u}_r(t), \widehat{u}_{r+1}(t), \ldots)'$ is the system of solutions of the Cauchy problem (6) on $[t_{r-1}, t_r)$ with $\lambda_r = \widehat{\lambda}_r$, $r \in \mathbb{Z}$. Let us show that the pair $(\widehat{\lambda}, \widehat{u}[t])$ is the solution of problem (6)–(8). Since $\widehat{u}_r(t)$

is the solution of the Cauchy problem (6) with $\lambda_r = \widehat{\lambda}_r$, it follows from (16) and the unique solvability of the Cauchy problem (6) for fixed parameter values λ_r that

$$\widehat{u}_r(t) = D_{*,r}(t)\widehat{\lambda}_r + F_{*,r}(t), \quad t \in [t_{r-1}, t_r), \quad r \in \mathbb{Z}. \tag{19}$$

In view of (18), we have

$$\widehat{\lambda}_r + [D_{*,r}(t_r)\widehat{\lambda}_r + F_{*,r}(t_r)] = \widehat{\lambda}_{r+1}, \quad r \in \mathbb{Z}. \tag{20}$$

Then, by (19) the expressions in square brackets in (20) are equal to $\lim\limits_{t \to t_r - 0} \widehat{u}_r(t)$, $r \in \mathbb{Z}$, and the pair $(\widehat{\lambda}, \widehat{u}[t])$ satisfies (7) as well.

Theorem 2. Problem 1_α is well-posed iff, given an arbitrary $\nu \in \mathbb{N}$, there is a $\theta(\nu) > 0$ such that the matrix $Q_{\nu,\overline{h}(\theta)}$ has an inverse for all $\overline{h}(\theta) = (\ldots, h_r(\theta), h_{r+1}(\theta), \ldots)$ and the inequalities (12) and (13) hold.

Proof. The *sufficiency* of the conditions of Theorem 2 for the well-posedness of Problem 1_α follows from Theorem 1.

Necessity. Let us consider the equation

$$\frac{1}{\theta} Q_{*,\overline{h}(\theta)} \lambda = b, \quad \lambda, b \in m_n.$$

Obviously, the kernel of the matrix $\frac{1}{\theta} Q_{*,\overline{h}(\theta)}$ consists only of the zero vector of the space m_n. Suppose, contrary to this claim, that there is a $\overline{\lambda} \in m_n$ such that $\frac{1}{\theta} Q_{*,\overline{h}(\theta)} \overline{\lambda} = 0$, $\|\overline{\lambda}\| \neq 0$. Hence, as shown above, the pair $(\overline{\lambda}, \overline{u}[t])$, with $\overline{u}[t] = (\ldots, \overline{u}_r(t), \overline{u}_{r+1}(t), \ldots)'$ being the system of solutions of the Cauchy problems (6) with $\lambda_r = \overline{\lambda}_r$ on $[t_{r-1}, t_r)$, is the solution of problem (6)–(8) with $f(t) = 0$. The function $\overline{x}(t)$, obtained by gluing the function systems $(\overline{\lambda}_r + \overline{u}_r(t))$, $r \in \mathbb{Z}$, belongs to $\widetilde{C}((0,T), \mathbb{R}^n)$ and satisfies the equation $\frac{dx}{dt} = A(t)x$. But $\sup\limits_{t \in (0,T)} \|\overline{x}(t)\| \neq 0$, which contradicts the well-posedness of Problem 1_α. Thus, the matrix $Q_{*,\overline{h}(\theta)}$ has an inverse.

Let us fix $\varepsilon > 0$ and choose $\theta_0(\varepsilon) > 0$ satisfying the inequality

$$\frac{1}{\theta}(e^\theta - 1 - \theta) \leq \frac{\varepsilon/2}{2(1+\varepsilon/4)(1+\varepsilon/2)}. \tag{21}$$

Then, by Lemma in [12], for arbitrary $b_r \in \mathbb{R}^n$, $r \in \mathbb{Z}$, the functions $f_{b_r} \in C([t_{r-1}, t_r], \mathbb{R}^n)$ can be constructed such that

$$F_*(A, f_{b_r}) = b_r, \quad \max_{t \in [t_{r-1}, t_r]} \|f_{b_r}(t)/\alpha(t)\| \leq (1+\varepsilon/2)\|b_r\|.$$

Hence, the function $f_b(t)$ defined as $f_b(t) = f_{b_r}(t)$, $t \in [t_{r-1}, t_r]$, satisfies the relations

$$f_b(t) \in \widetilde{C}((0,T), \mathbb{R}^n), \quad \|f_b\|_\alpha \leq (1+\varepsilon/2)\|b\|_2, \quad F_*(A, f_b, \overline{h}(\theta)) = b.$$

The well-posedness of Problem 1_α implies that Eq.(17) has a unique solution $\lambda_b \in m_n$ for any $f_b(t) \in \widetilde{C}_{1/\alpha}((0,T), \mathbb{R}^n)$, and

$$\|\lambda_b\|_2 = \sup_{r \in \mathbb{Z}} \|\lambda_{b_r}\| = \sup_{r \in \mathbb{Z}} \|x_b(t_{r-1})\| \leq \sup_{t \in (0,T)} \|x_b(t)\| \leq K\|f_b\|_\alpha \leq K(1+\varepsilon/2)\|b\|_2.$$

Taking into account that $\|\lambda_b\|_2 = \|[\frac{1}{\theta} Q_{*,\overline{h}(\theta)}]^{-1} b\|_2$, the latter estimate yields

$$\frac{\|[\frac{1}{\theta} Q_{*,\overline{h}(\theta)}]^{-1} b\|_2}{\|b\|_2} \leq \left(1 + \frac{\varepsilon}{2}\right) K, \quad \forall b \in m_n.$$

This gives

$$\|[\frac{1}{\theta}Q_{*,\overline{h}(\theta)}]^{-1}\|_{L(m_n)} \leq \left(1+\frac{\varepsilon}{2}\right)K, \quad \forall\theta \in (0,\theta_0].$$

Hence, choosing $\theta \in (0,\theta_0]$ such that

$$\frac{(1+\varepsilon/2)K}{\theta}\left(e^{\theta}-1-\theta-\ldots-\frac{\theta^{\nu}}{\nu!}\right) < \frac{\varepsilon}{2(1+\varepsilon)}$$

and taking into account

$$\left\|\frac{1}{\theta}Q_{*,\overline{h}(\theta)}-\frac{1}{\theta}Q_{\nu,\overline{h}(\theta)}\right\|_{L(m_n)} \leq \frac{1}{\theta}\left(e^{\theta}-1-\theta-\ldots-\frac{\theta^{\nu}}{\nu!}\right),$$

by the theorem on small perturbations of boundedly invertible operators, we obtain that the matrix $Q_{\nu,\overline{h}(\theta)}$ has a bounded inverse satisfying the estimate

$$\|\left[\frac{1}{\theta}Q_{\nu,\overline{h}(\theta)}\right]^{-1}\|_{L(m_n)} \leq (1+\varepsilon)K.$$

Finally, (17) yields

$$q_{\nu}(\overline{h}(\theta)) = (1+\varepsilon)\frac{K}{\theta}\left(e^{\theta}-1-\theta-\ldots-\frac{\theta^{\nu}}{\nu!}\right) < \frac{\varepsilon}{2+\varepsilon} < 1,$$

which completes the proof.

Theorem 3. Problem 1_α is well-posed iff, given an arbitrary $\nu \in \mathbb{N}$, there is a $\theta_0(\nu)$ such that the matrix $Q_{\nu,\overline{h}(\theta)}$ has an inverse for all sequences $\overline{h}(\theta)$, $\theta \in (0,\theta_0]$, and

$$\|\left[Q_{\nu,\overline{h}(\theta)}\right]^{-1}\|_{L(m_n)} \leq \frac{\gamma}{\theta}, \tag{22}$$

where γ is a constant independent of $\overline{h}(\theta)$.

Moreover, if the well-posedness constant K is known, then for any $\varepsilon > 0$ there exists $\overline{\theta}(\varepsilon,\nu) > 0$ such that estimate (22) holds with constant $\gamma = (1+\varepsilon)K$ for all $\theta \in (0,\overline{\theta}(\varepsilon,\nu)]$. Conversely, if estimate (22) holds, then $K = \gamma$.

Proof. Necessity. Let Problem 1_α be well-posed with constant K. Given $\varepsilon > 0$, we choose $\overline{\theta}(\varepsilon,\nu) \in (0,\theta_0(\varepsilon)]$, where $\theta_0(\varepsilon)$ satisfies condition (21). Then, as it was shown in Theorem 2, the matrix $Q_{\nu,\overline{h}(\theta)}$ is invertible for all $\theta \in (0,\overline{\theta}(\varepsilon,\nu)]$ and $\|\left[Q_{\nu,\overline{h}(\theta)}\right]^{-1}\|_{L(m_n)} \leq \frac{(1+\varepsilon)K}{\theta}$, i.e. $\gamma = (1+\varepsilon)K$.

Sufficiency. Let estimate (22) hold. let us choose θ so that $q_\nu(\overline{h}(\theta)) < 1$. Then, by Theorem 1, Problem 1_α is well-posed and

$$\|x^*\|_1 \leq e^{\theta}\left[\frac{\gamma}{\theta}\cdot\frac{1}{1-q_\nu(\overline{h}(\theta))}\cdot\frac{\theta^{\nu}}{\nu!}\left(\frac{\gamma}{\theta}(e^{\theta}-1)^2+\theta e^{\theta}\right)+\frac{\gamma}{\theta}(e^{\theta}-1)+\theta\right]\|f\|_{\alpha}.$$

Letting $\theta \to 0$, we obtain

$$\|x^*\|_1 \leq \gamma\|f\|_{\alpha},$$

i.e. $K = \gamma$, which completes the proof.

Acknowledgments

This research is funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP09258829).

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Коэффициенттері шектелмеген дифференциалдық теңдеулер сызықты жүйелерінің шектелген шешімдері туралы

Мақалада шектелмеген коэффициенттер матрицасы бар біртекті емес сызықты дифференциалдық теңдеулер жүйесі үшін ақырлы интервалда шектелген шешімін табу есебі қарастырылған. Теңдеудің оң жағы үзіліссіз және қандай да бір салмақпен шектелген функциялар кеңістігіне жатады; салмақтық функция коэффициенттер матрицасының әрекетін ескере отырып таңдалды. Қарастырылып отырған есепті зерттеу үшін біркелкі емес бөліммен параметрлеу әдісінің модификациясы қолданылды. Арнайы құрылымды екі жақты шексіз матрицасы тұрғысынан зерттелген есептің дұрыс шешілімділігіне қажетті және жеткілікті шарттар алынған.

Кілт сөздер: жәй дифференциалдық теңдеулер, сингулярлы шеттік есеп, корректі шешілімділік, параметрлеу әдісі, шектелген шешім, сызықты жүйе, шектелмеген коэффициенттер.

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Об ограниченных решениях линейных систем дифференциальных уравнений с неограниченными коэффициентами

В статье рассмотрена задача нахождения ограниченного на конечном интервале решения системы неоднородных линейных дифференциальных уравнений с неограниченной матрицей коэффициентов. Правая часть уравнения принадлежит пространству непрерывных и ограниченных с некоторым весом функций; весовая функция выбирается с учетом поведения матрицы коэффициентов. Для исследования рассматриваемой задачи применена модификация метода параметризации с неравномерным разбиением. Получены необходимые и достаточные условия корректной разрешимости рассматриваемой задачи в терминах двусторонне-бесконечной матрицы специальной структуры.

Ключевые слова: обыкновенные дифференциальные уравнения, сингулярная краевая задача, корректная разрешимость, метод параметризации, ограниченное решение, линейная система, неограниченные коэффициенты.

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DOI 10.31489/2022M4/117-124

UDC 510.67

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Existentially prime Jonsson quasivarieties and their Jonsson spectra

This article is devoted to the study of Jonsson quasivarieties in a signature enriched with new predicate and constant symbols. New concepts of semantic Jonsson quasivariety and fragment-conservativeness of the center of the Jonsson theory are introduced. The cosemanticness classes of the Jonsson spectrum constructed for a semantic Jonsson quasvariety are considered. In this case, the Kaiser hull of the semantic Jonsson quasivariety is assumed to be existentially prime. By constructing a central type for classes of theories from the Jonsson spectrum, the following results are formulated and proved. In the first main result, the necessary and sufficient condition is given for the center of the cosemanticness class of an existentially prime semantic Jonsson quasivariety to be λ-stable. The second result is the criterion for the center of the class of theories to be ω-categorical in the enriched language. The obtained theorems can be useful in continuing studies of various Jonsson algebras, in particular, Jonsson quasivarieties.

Keywords: Jonsson theory, perfect Jonsson theory, variety, quasivariety, semantic Jonsson quasivariety, Jonsson spectrum, existentially prime theory, central type, orbital type, central-orbital type, fragments of Jonsson sets.

Introduction

It is well-known fact that the greatest part of considered objects in Model Theory is connected with the study of incomplete theories. Many classical algebras, such as groups, fields, R-modules and many others, are axiomatized by incomplete theories. Nevertheless, this class of theories is too vast and, consequently, complicated for considering in detail. This is the reason why we need to introduce some conditions that clarify the subject of our research and allow studying various algebras, as well as their syntactic and semantic properties.

Thus, a subclass of incomplete theories where we do our research is Jonsson theories. One can find basic material in [1, 2] and more specific information on the connection between Jonsson theories, for example, in [3–5]. In this paper we mainly deal with semantic Jonsson quasivarieties and central-orbital type that play a significant role in the apparatus of Jonsson theories. In Section 1, necessary information on Jonsson theories is given. Section 2 is devoted to considering some specific properties of the Jonsson spectra of semantic Jonsson quasivarieties in the case of existential primeness. The main results are connected with constructing of the central type and stability and categoricity of cosemanticness classes.

All definitions that are not given in this article can be found in [2].

1 *Preliminary information*

We start with the main definitions and facts concerning the subject of the study. Recall the definitions of Jonsson theory and related concepts.

We are working within the framework of the following definition of Jonsson theory published in the Russian edition of [1].

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Definition 1. [1; 80] A theory T is called Jonsson if the following conditions hold for T:
1. T has at least one infinite model;
2. T is an inductive theory;
3. T has the amalgam property (AP);
4. T has the joint embedding property (JEP).

Classical examples of Jonsson theories include:
1) group theory;
2) the theory of abelian groups;
3) the theory of Boolean algebras;
4) the theory of linear orders;
5) field theory of characteristic p, where p is zero or a prime number;
6) the theory of ordered fields;
7) the theory of modules.

The following notions and facts form a necessary apparatus for studying Jonsson theories.

Definition 2. [2; 155] Let T be a Jonsson theory. A model C_T of power $2^{|T|}$ is called to be a semantic model of the theory T if C_T is a $|T|^+$-homogeneous $|T|^+$-universal model of the theory T.

Theorem 1. [2; 155] T is Jonsson iff it has a semantic model C_T.

The following definition was introduced by T.G. Mustafin.

Definition 3. [2; 155] A Jonsson theory T is called perfect if its semantic model C_T is saturated.

Definition 4. [2; 161] The elementary theory of a semantic model of the Jonsson theory T is called the center of this theory. The center is denoted by T^*, i.e. $Th(C) = T^*$.

The following theorem represents one of the most considerable facts describing perfect Jonsson theories.

Theorem 2. [2; 162] Let T be a perfect Jonsson theory. Then the following statements are equivalent:
1) T^* is the model companion of T;
2) $ModT^* = E_T$;
3) $T^* = T^f$, where T^f is a forcing companion of the theory T.

Some classical examples of perfect Jonsson theories one can find in [4], while non-perfect Jonsson theories are considered in [5].

Theorem 3. [5] Let T be a Jonsson theory. Then for any model $A \in E_T$ the theory $T^0(A)$ is Jonsson, where $T^0(A) = Th_{\forall\exists}(A)$.

We can see that in the case of perfectness of T its center T^* is also a perfect Jonsson theory.

The following definition will help us to specify the class of Jonsson theories which we will deal with in this paper.

Definition 5. [6; 120] A Jonsson theory is said to be hereditary if, in any of its permissible enrichment, it preserves the Jonssonness.

Unfortunately, there is no complete description of this notion. However, one can find some useful information in [7] on hereditary Jonsson theories.

One more specific notion that is widely used in the study of Jonsson theories is a Jonsson set. Recall its definition.

Definition 6. [8] Let T be a Jonsson theory and C be its semantic model. A Σ-definable subset of C is called a Jonsson set for the theory T, if $dcl(X) = M$, $M \in E_T$. A theory $Th_{\forall\exists}(M)$ is called a fragment of the Jonsson set X.

Some research methods, where this notion is used, are revealed in [9].

The following class of theories was specified by Yeshkeyev A.R.

Definition 7. [10] A theory T is called existentially prime, if $AP_T \cap E_T \neq \emptyset$, where AP_T is the class of algebraically prime models of T.

Let T be a Jonsson theory and $S^J(Y)$ be a set of all existentially complete n-types over Y that are consistent with T, for any finite n.

Definition 8. [11] A Jonsson theory T is $J-\lambda$-stable if for any T-existentially closed model A and for any subset Y of A from the inequality $|Y| \leq \lambda$ it follows that $|S^J(Y)| \leq \lambda$.

In [11], the authors proved the following result that shows the connection between Jonsson stability and stability in the classical sense.

Theorem 4. [11] Let T be a perfect Jonsson theory and let T be complete for existential sentences. Let $\lambda \geq \omega$. Then the following statements are equivalent:

1) T is $J-\lambda$-stable;

2) T^* is λ-stable, where $T^* = Th(C)$, C is a semantic model of T.

Let L be a first-order language of a signature σ and let K be a class of L-structures. Then we can consider a Jonsson spectrum for K, which can be defined as follows.

Definition 9. [5] A set $JSp(K)$ of Jonsson theories of L, where

$$JSp(K) = \{T \,|T \text{ is a Jonsson theory and } \; K \subseteq Mod(T)\},$$

is called a Jonsson spectrum of K.

Jonsson spectra are well-described in [12]

Let T_1 and T_2 be Jonsson theories, T_1^* and T_2^* be their centres, respectively.

Definition 10. [2; 40] T_1 and T_2 are said to be cosemantic Jonsson theories (denoted by $T_1 \bowtie T_2$), if $T_1^* = T_2^*$.

It is easy to see that the relation of cosemanticness between two Jonsson theories is an equivalence relation:

1) this relation is reflexive since for every Jonsson theory T the equation $T^* = T^*$ holds,

2) it is symmetric as soon as, for any Jonsson theories T_1 and T_2, if $T_1^* = T_2^*$ then $T_2^* = T_1^*$,

3) finally, " $\bowtie$ " is transitive, that follows from the fact that, for any Jonsson theories T_1, T_2 and T_3, if $T_1^* = T_2^*$, $T_2^* = T_3^*$ then $T_1^* = T_3^*$.

This means that, when introducing the relation of cosemanticness on the Jonsson spectrum $JSp(K)$, we get a partition of $JSp(K)$ into cosemanticness classes. The obtained factor-set is denoted by $JSp(K)_{/\bowtie}$.

Now let us consider the notion of a semantic Jonsson quasivariety. One should note that this concept differs significantly from the concept of a Jonsson quasivariety introduced in [13].

Let K be a quasivariety in the usual sense as in [14; 269]. We construct a set $\forall\exists(K)$, where $\forall\exists(K)$ is a set of Jonsson theories and obtained as follows:

$$\forall\exists(K) = \{Th(K) \,\cup\, \varphi \,|\, \varphi \text{ is an } \forall\exists\text{-sentence and } \varphi \cup Th(K) \text{ is consistent}\}. \tag{1}$$

In other words, the set $\forall\exists(K) = \{T_1, T_2, ...\}$ is a list of all Jonsson theories that satisfy Condition 1. Then C_i is a semantic model of T_i from this list. Let us consider the following set:

$$JK = \{C_i \,|\, C_i \text{ is a semantic model of } T_i,\, T_i \in \forall\exists(K)\}.$$

Definition 11. The set JK is a semantic Jonsson quasivariety, if the theory $T^0(JK) = Th_{\forall\exists}(JK)$ is Jonsson.

The theory $T^0(JK)$ is called a Kaiser hull of the class JK.

Definition 12. A set of theories $JSp(JK)$, where

$$JSp(JK) = \{T^0(JN) \mid N \text{ is a subquasivariety of } K\},$$

is said to be a Jonsson spectrum of a semantic Jonsson quasivariety JK.

2 Central types for cosemanticness classes of $JSp(JK)$

In this section, we consider the Jonsson spectrum of a semantic Jonsson quasivariety from a position of central-orbital types and existential primeness. The main definitions and facts related to central type can be found in the papers of the first author Yeshkeyev A.R., for example [15–17]. A special role is played by the work [18] where the author defined the notion of central-orbital type for the Jonsson case by analogy with [19]. Here we apply the results of [18] to semantic Jonsson quasivarieties.

Currently, the class of Jonsson quasivarieties is not studied well enough. Generally speaking, in contrast to the case of complete theories, the apparatus for studying incomplete theories (including Jonsson ones) is not developed at a sufficient level. This is why we have to restrict this research by introducing some specific conditions.

First of all, we have to refine that all the Jonsson theories in this section are hereditary (Definition 5) by our assumption.

Another necessary restriction is formulated by the following definition.

Definition 13. The center T^* of a Jonsson theory T is said to be fragment-conservative if the semantic model of any fragment of T^* is an existentially closed submodel of the semantic model C of T.

Further in this paper, we work with Jonsson theories whose centers are fragment-conservative.

We work in a first-order language L of a signature σ. Let JK be an existentially prime semantic Jonsson quasivariety, which means that the theory $T^0(JK)$ from Definition 11 is existentially prime as it is described in Definition 7. Let $JSp(JK)$ be a Jonsson spectrum of JK. We introduce the relation of cosemanticness on $JSp(JK)$. In this manner, we have a factor-set $JSp(JK)_{/\bowtie}$ consisting of all Jonsson theories that satisfy Definition 12. Let us consider some class $[T]_{/\bowtie} \in JSp(JK)_{/\bowtie}$. Let C be a semantic model of each theory $T \in [T]_{/\bowtie}$, $X \subseteq C$ be a Jonsson set.

To consider the properties of JK and $JSp(JK)$ through constructing the central type for the cosemanticness classes, firstly we need to enrich the signature σ by new constant c and predicate P symbols as follows.

Let $\sigma_\Gamma(X) = \sigma \cup \{c_a, a \in X\} \cup \Gamma$, $\Gamma = \{P\} \cup \{c\}$. We consider a class of theories $[T_X^C]$ in the new enriched signature $\sigma_\Gamma(X)$ for each cosemanticness class $[T]_{/\bowtie}$, where $T_X^C \in [T_X^C]$ is constructed as follows:

$$T_X^C = T \cup Th_{\forall\exists}(C, a)_{a \in X} \cup \{P(c_a), a \in X\} \cup \{P(c)\} \cup \{P, \subseteq\}.$$

Here P is a new 1-ary predicate symbol interpretations of which are an existentially closed submodel M of the semantic model C, i.e. $P(C) = M, M \in E_T$, $T \in [T]_{/\bowtie}$.

As soon as any theory $T \in [T]_{/\bowtie}$ is hereditary by our assumption and the introduced enrichments are permissible, every theory T_X^C in the class $[T_X^C]$ is also Jonsson. Therefore, there is a semantic model C' for $[T_X^C]$. It is easy to see, that the semantic models of the theories from $[T_X^C]$ coincide, so we denote it by C'. Let $T' = Th(C')$ be a center for the class $[T_X^C]$. Now we will consider the theory T' in a restricted signature $\sigma_\Gamma(X) \backslash \{c\}$ so that T' becomes a complete type of c.

Definition 14. A complete type described above is called a central type for the Jonsson theory T with respect to the Jonsson set X (denoted by p_X^C).

In case when a central type coincides with an orbital type of a Jonsson theory the obtained type is called a central-orbital type. Some properties related to central-orbital types of a Jonsson theory are considered in [18]. Since a central-orbital type is central and orbital at the same time, all statements that are connected with central types and mentioned in this section can be considered in terms of central-orbital types.

Here we work with the cosemanticness classes of theories, not single theories. Taking into consideration this fact and the results from [18], we can get the following theorems.

Theorem 5. Let JK be an existentially prime semantic Jonsson quasivariety, $JSp(JK)$ be its Jonsson spectrum, $[T]_{/\bowtie} \in JSp(JK)_{/\bowtie}$, $T_i \in [T]_{/\bowtie} (i \in I)$, T^* is a center for the class $[T]_{/\bowtie}$. Let X_i be Jonsson sets for the theories T_i respectively, $dcl(X_i) = M_i$, $M_i \in E_{T_i}$. $[T_X^C]$ is the class of the theories in the enriched signature as it is described above. If $\lambda \geq \omega$, then T^* is $J - \lambda$-stable if and only if S is λ-stable for any theory $S \in [T_X^C]$.

Proof. The proof follows from Theorem 4 from Section 1 and Theorem 1 from [18], applying this result to each arbitrary theory from $[T_X^C]$.

The following result demonstrates the connection between the categoricity of a center for the cosemanticness class of Jonsson fragments and the categoricity of the corresponding theories in the enriched signature. Here we need to introduce the following notation. Let a cosemanticness class $[T]_{/\bowtie}$ consist of theories $T_i, i \in I$. As soon as all theories are inductive, for any T_i there exists a non-empty class of existentially closed models E_{T_i}. For each i, we consider a Jonsson set X_i such that a model $M_i \in E_{T_i}$ is a definable closure of X_i, i.e. $dcl(X_i) = M_i$. After this, we can construct a theory $Th_{\forall\exists}(M_i)$, which is called a Jonsson fragment of the Jonsson set X_i, for each theory T_i. Thus we get a class of all Jonsson fragments for the corresponding cosemanticness class $[T]_{/\bowtie}$. We denote the obtained class by $[T_X]$. Note that every theory in this class is Jonsson, which means that it has a semantic model. It is easy to see that the semantic models for each $T_X \in [T_X]$ coincide, so let us denote the center for this class by T_X^*.

The following lemma is true for $[T_X]$ because of heredity.

Lemma 1. Every theory $T_X \in [T_X]$ is Jonsson in the new signature $\sigma_\Gamma(X)$.

Theorem 6. Let K be an existentially prime semantic Jonsson quasivariety, $JSp(JK)$ be its Jonsson spectrum, $[T]_{/\bowtie} \in JSp(JK)_{/\bowtie}$, $T_i \in [T]_{/\bowtie} (i \in I)$, T^* is a center for the class $[T]_{/\bowtie}$, and let $[T_X]$ be as it is describe above. Then T_X^* is ω-categorical if and only if each $S \in [T_X^C]$ is ω-categorical.

Proof. The proof can be obtained by applying Theorem 2 of [18] to arbitrary theories of the mentioned classes.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP09260237).

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Экзистенциалды жай йонсондық квазикөптүрліліктер және олардың йонсондық спектрлері

Мақала жаңа предикаттық және тұрақты символдармен байытылған сигнатурадағы йонсондық квазикөптүрліліктерді зерттеуге арналған. Семантикалық йонсондық квазикөптүрліліктер мен йонсондық теорияның фрагмент-консервативтілігі туралы жаңа түсініктер енгізілді. Йонсондық квазикөптүрліліктер үшін құрылған йонсондық спектрдің косемантты кластары қарастырылған. Бұл жағдайда йонсондық квазикөптүрліліктің Кайзер қабықшасы экзистенциалды жай деп ұйғарылады. Йонсондық спектрдің теория кластары үшін центрлік типті құру арқылы келесі нәтижелер тұжырымдалды және дәлелденді. Бірінші негізгі нәтижеде экзистенциалды жай йонсондық квазикөптүрліліктің косемантты класының центрі Λ-тұрақты болуы үшін қажетті және жеткілікті шарттар келтірілген. Екінші нәтиже, байытылған тілдің теориялар класы центрінің ω-категориялығының критерийі болып табылады. Алынған теоремалар әртүрлі йонсондық алгебраларды, атап айтқанда, йонсондық квазикөптүрліліктерді зерттеуді жалғастыру үшін пайдалы болуы мүмкін.

Кілт сөздер: йонсондық теория, кемел йонсондық теория, көптүрлілік, квазикөптүрлілік, семантикалық йонсондық квазикөптүрлілік, йонсондық спектр, экзистенциалды жай спектр, центральді тип, орбитальді тип, центральді-орбитальді тип, йонсондық жиынның фрагменттері.

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Экзистенциально простые йонсоновские квазимногообразия и их йонсоновские спектры

Статья посвящена изучению йонсоновских квазимногообразий в сигнатуре, обогащенной новым предикатным и константным символами. Введены новые понятия семантического йонсоновского квазимногообразия и фрагмент-консервативности центра йонсоновской теории. Рассмотрены классы косемантичности йонсоновского спектра, построенного для йонсоновского квазимногообразия. При этом оболочка Кайзера йонсоновского квазимногообразия предполагается экзистенциально простой. С помощью построения центрального типа для классов теорий из йонсоновского спектра формулируются и доказываются следующие результаты. В первом основном результате приведено необходимое и достаточное условие для того, чтобы центр класса косемантичности экзистенциально простого йонсоновского квазимногообразия являлся λ-стабильным. Второй результат является критерием ω-категоричности центра класса теорий обогащенного языка. Полученные теоремы могут быть полезны для продолжения исследований различных йонсоновских алгебр, в частности, йонсоновских квазимногообразий.

Ключевые слова: йонсоновская теория, совершенная йонсоновская теория, многообразие, квазимногообразие, семантическое йонсоновское квазимногообразие, йонсоновский спектр, экзистенциально простая теория, центральный тип, орбитальный тип, центрально-орбитальный тип, фрагменты йонсоновского множества.

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Index of articles published in the «Bulletin of the Karaganda University» in 2022 y. «Mathematics» Series

MATHEMATICS

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