ISSN 2518-7929 (Print)
ISSN 2663-5011 (Online)

BULLETIN
OF THE KARAGANDA UNIVERSITY

MATHEMATICS
Series

№ 3(107)/2022

ISSN 2518-7929 (Print)
ISSN 2663–5011(Online)
Индексі 74618
Индекс 74618

ҚАРАҒАНДЫ УНИВЕРСИТЕТІНІҢ **ХАБАРШЫСЫ**

ВЕСТНИК КАРАГАНДИНСКОГО УНИВЕРСИТЕТА

BULLETIN OF THE KARAGANDA UNIVERSITY

МАТЕМАТИКА сериясы

Серия МАТЕМАТИКА

MATHEMATICS Series

№ 3(107)/2022

Шілде–тамыз–қыркүйек
30 қыркүйек 2022 ж.

Июль–август–сентябрь
30 сентября 2022 г.

July-August-September
September, 30th, 2022

1996 жылдан бастап шығады
Издается с 1996 года
Founded in 1996

Жылына 4 рет шығады
Выходит 4 раза в год
Published 4 times a year

Қарағанды, 2022
Караганда, 2022
Karaganda 2022

Postal address: 28, University Str., Karaganda, 100024, Kazakhstan
Tel.: (7212) 77-04-38 (add. 1026); fax: (7212) 35-63-98.
E-mail: vestnikku@gmail.com. Web-site: mathematics-vestnik.ksu.kz

Editors
Zh.T. Nurmukhanova, S.S. Balkeyeva, Z.E. Ramazanova
Computer layout
G.K. Kalel

Bulletin of the Karaganda University. Mathematics series.
ISSN 2518-7929 (Print). ISSN 2663–5011 (Online).

Proprietary: NLC «Karagandy University of the name of academician E.A. Buketov».

Registered by the Ministry of Information and Social Development of the Republic of Kazakhstan. Rediscount certificate No. KZ43VPY00027385 dated 30.09.2020.

Signed in print 29.09.2022. Format 60×84 1/8. Offset paper. Volume 21,87 p.sh. Circulation 200 copies. Price upon request. Order № 98.

Printed in the Publishing house of NLC «Karagandy University of the name of acad. E.A. Buketov».

28, University Str., Karaganda, 100024, Kazakhstan. Tel.: (7212) 35-63-16. E-mail: izd_kargu@mail.ru

CONTENT

MATHEMATICS

DOI 10.31489/2022M3/4-16

UDC 517.949.8, 533.7

A.Sh. Akysh (Akishev)

Almaty, Kazakhstan
(E-mail: akysh41@mail.ru)

Global solvability of a nonlinear Boltzmann equation

In this paper, based on the splitting method scheme, the existence and uniqueness theorem on the whole time interval $t \in [0, T), T \leq \infty$ for the full nonlinear Boltzmann equation in the nonequilibrium case is proved where the intermolecular interactions are hard-sphere molecule and central forces. Considering the existence of a bounded solution in the space **C**, the strict positivity of the solution to the full nonlinear Boltzmann equation is proved when the initial function is positive. On the basis of this some mathematical justification of the $H-$theorem of Boltzmann is shown.

Keywords: full nonlinear Boltzmann equation, splitting method, existence and uniqueness theorem on the whole time for the nonlinear Boltzmann equation, positivity of the solution to the nonlinear Boltzmann equation, Boltzmann's $H-$theorem.

Introduction

The Boltzmann equation [1] is a complex nonlinear integro-differential equation and refers to difficult-to-study mathematical objects. Proof of the existence and uniqueness theorem for a solution of the Cauchy problem for a spatially homogeneous Boltzmann equation begins with the work of T. Carleman [2].

H. Grad [3] proved the first existence theorem in the "small" for spatially nonhomogeneous Boltzmann equation in the case of Maxwellian molecules when the initial function tends to Maxwellian distribution function in a special norm.

The world's leading experts on kinetic equations provided a review monograph [4], on the current state of mathematical theory of the Boltzmann equation starting with its derivation, theorem existence and uniqueness and methods of solution. They wrote: «... For over 110 years this equation attracts the attention of researchers, but only in recent years it has proved *global solvability* spatially − nonhomogeneous problem in the case of a small deviation of the gas state from equilibrium positions - more general results are not obtained to this day ...» [4].

T. Carleman in [2] pointed out that solving the full Boltzmann equation for practical problems can be only done through approximate mathematical methods. In this connection, we have chosen the splitting method to solve the full nonlinear Boltzmann equation. Splitting methods for solving a class of various applied problems were developed by G. I. Marchuk [5].

In Kazakhstan, the study of the nonlinear equation and its corresponding discrete models began in S.K. Godunov and U.M. Sultangazin works [6].

In this connection, to solve the full nonlinear Boltzmann equation in the class of positive initial functions, the splitting method was applied [7], [8]. First, based on this method boundedness of positive solutions in the space continuous functions was got. With the help of the boundedness of the solution and of the established a priori estimates, the convergence of the scheme splitting method and uniqueness of the limiting element were proved. The found limiting element satisfies the equivalent integral Boltzmann equation. Thus, a weak solvability of the nonlinear Boltzmann equation as a whole in time.

From modern bibliographic sources it follows that there are no the existence and uniqueness theorems as a whole in time for the nonlinear Boltzmann equation in a nonequilibrium case when intermolecular interactions are hard-sphere molecules or central forces.

1 *Statement of the problem for a nonlinear Boltzmann equation*

Cauchy problem for the full nonlinear Boltzmann equation for molecules − hard spheres of radius χ in the domain

$$Q = \Big(t \in [0,T),\ T \le \infty;\ \mathbf{x} = (x_1,x_2,x_3) \in G \equiv \{\, 0 \le x_\alpha \le 1,\ \alpha = \overline{1,3}\, \};$$
$$\mathbf{v} = (\xi_1,\xi_2,\xi_3) \in V_3 \equiv \{ -\infty \le \xi_\alpha \le \infty,\ \alpha = \overline{1,3}\, \} \Big)$$

with respect to the distribution function $f = f(t,\mathbf{x},\mathbf{v})$ is written as [1], [2]:

$$\frac{\partial f}{\partial t} + (\mathbf{v},\nabla) f = \mathbf{J}(f) - f\mathbf{S}(f) \equiv \mathbf{B}(f,f), \tag{1}$$

with an initial

$$f(t,\mathbf{x},\mathbf{v}) \mid_{t=0} = \varphi(\mathbf{x},\mathbf{v}) \tag{2}$$

and periodic boundary conditions*

$$f(t,\mathbf{x},\mathbf{v})\big|_{\Gamma_{0x_\alpha}} = f(t,\mathbf{x},\mathbf{v})\big|_{\Gamma_{1x_\alpha}},\ \ \alpha = \overline{1,3}, \tag{3}$$

where

$$\mathbf{S}(f) = \iiint\limits_{-\infty}^{\infty} \int\limits_0^{\pi/2} \int\limits_0^{2\pi} f(t,\mathbf{x},\mathbf{v}_1) K(\theta,\mathbf{w}) d\varepsilon d\theta d\mathbf{v}_1 \equiv \int\limits_{V_3\times\Sigma} f(t,\mathbf{x},\mathbf{v}_1) K(\theta,\mathbf{w}) d\sigma d\mathbf{v}_1;$$

$$\mathbf{J}(f) = \int\limits_{V_3\times\Sigma} f(t,\mathbf{x},\mathbf{v}') f(t,\mathbf{x},\mathbf{v}_1') K(\theta,\mathbf{w}) d\sigma d\mathbf{v}_1,\ \ K(\theta,\mathbf{w}) = 0.25\chi^2 \mid \mathbf{w} \mid \sin(2\theta),$$

$\mathbf{v}$, $\mathbf{v}_1$ are the velocity vectors of two colliding molecules, $\mathbf{w} = \mathbf{v} - \mathbf{v}_1$ is relative velocity vector; velocity of molecules after collisions $\mathbf{v}', \mathbf{v}_1'$, are related to $\mathbf{v}, \mathbf{v}_1$ by the dynamic relation: $\mathbf{v}' = \mathbf{v} + \mathbf{g}(\mathbf{g},\mathbf{w})$, $\mathbf{v}_1' = \mathbf{v}_1 - \mathbf{g}(\mathbf{g},\mathbf{w})$; $\mathbf{g}$ *is* unit vector in the direction of scattering of molecules:
$\mathbf{g} = \big(\sin\theta\cos\varepsilon,\ \sin\theta\sin\varepsilon,\ \cos\theta\big)$; $(\theta,\varepsilon) \in \mathbf{\Sigma} \equiv \big\{0 \le \theta \le \pi;\ 0 \le \varepsilon \le 2\pi\big\}$; $\Gamma_{\rho x_\alpha}-$ edge cube G perpendicular to the axis x_α, passing through $x_\alpha = \rho$, ρ takes the value either 0 or 1.

The initial function $\varphi(\mathbf{x},\mathbf{v})$ satisfies condition (3) and it is such that

$$\begin{cases} 0 < \varphi(\mathbf{x},\mathbf{v}) \in C(G\times V_3) \wedge \Big(\|\varphi(\mathbf{v})\|_{L_\infty(G)} \le \frac{const}{(1+|\mathbf{v}|^2)^{\frac{\gamma}{2}}},\ \gamma > 6 \Big); \\ \mathbf{J}(\varphi) \le \int\limits_{V_3\times\Sigma} \|\varphi(\mathbf{v}')\|_{L_\infty(G)} \cdot \|\varphi(\mathbf{v}_1')_{L_\infty(G)}\| K(\theta,\mathbf{w}) d\sigma d\mathbf{v}_1 = q_1(\mathbf{v}) < \infty; \\ \mathbf{S}(\varphi) \le \int\limits_{V_3\times\Sigma} \|\varphi(\mathbf{v}_1)\|_{L_\infty(G)} K(\theta,\mathbf{w}) d\sigma d\mathbf{v}_1 = q_2(\mathbf{v}) < \infty, \end{cases} \tag{4}$$

where $\|\varphi(\mathbf{v})\|_{L_\infty(G)} = \sup\limits_{\mathbf{x}\in G} \mid \varphi(\mathbf{x},\mathbf{v}) \mid$ at every $\mathbf{v} \in V_3$, $\int\limits_{V_3} q_k(\mathbf{v}) d\mathbf{v} = const$, $k = 1,2$. Following [2], requirements (4) for the initial function were taken into account, that improper integrals were convergent in the velocity space.

Lemma 1. For periodic functions, the following integration-by-parts formula over the cube G_3 is valid

$$\int\limits_{G_3} V \Delta U\, \mathbf{dx} = - \int\limits_{G_3} \nabla V\, \nabla U\, \mathbf{dx}. \tag{5}$$

*or mirror reflections of molecules from the boundary of the G domain.

Proof. Let us write down the integration-by-parts formula

$$\int\limits_{\Omega} V\Delta U\,\mathbf{dx} = -\int\limits_{\Omega} \nabla V\,\nabla U\,\mathbf{dx} + \int\limits_{\partial\Omega} V\frac{\partial U}{\partial \mathbf{n}}\,\mathbf{dx}. \tag{6}$$

From the properties of the cube surface, it follows

$$\int\limits_{\partial G_3} V\frac{\partial U}{\partial \mathbf{n}}\,\mathbf{dx} = \sum_{\kappa=1}^{3}\Bigg(\int\limits_{\Gamma_{0,x_\kappa}} V\frac{\partial U}{\partial x_\kappa} n_\kappa\,dx_\beta dx_\gamma + \\ + \int\limits_{\Gamma_{1,x_\kappa}} V\frac{\partial U}{\partial x_\kappa} n_\kappa\,dx_\beta dx_\gamma\Bigg),\ (\beta,\gamma)\in\{1,2,3\}\wedge(\beta,\gamma)\neq\kappa, \tag{7}$$

where $\mathbf{n}$ is the outward normal vector to the cube surface, then considering the value of the normal component in formula (7), $n_\kappa = \begin{cases} -1, & \rho = 0, \\ +1, & \rho = 1, \end{cases}$ we have

$$\int\limits_{\partial G_3} V\frac{\partial U}{\partial \mathbf{n}}\,\mathbf{dx} = \sum_{\kappa=1}^{3}\Bigg(\int\limits_{\Gamma_{0,x_\kappa}} V\frac{\partial U}{\partial x_\kappa} n_\kappa\,dx_\beta dx_\gamma + \int\limits_{\Gamma_{1,x_\kappa}} V\frac{\partial U}{\partial x_\kappa} n_\kappa\,dx_\beta dx_\gamma\Bigg) = 0.$$

Taking into account this relation, from (6) we get (5).

2 Existence and uniqueness theorems

To solve problem (1)−(3) we use the of splitting method [5]. On $[0,T)$ we introduce the time grid $\omega^\tau = \{t_n = n\tau \le \infty,\ \tau > 0,\ n = 0,1,\cdots;\}$, and*

$$\tau < 1/\int\limits_{V_3}(q_1+q_2)d\mathbf{v}. \tag{8}$$

Suppose an approximation is known $f^n(\mathbf{x},\mathbf{v})$, at time $n\tau$. Then the schemes of the splitting method corresponding to the problem (1)–(3) are written as follows:

$$\frac{f^{n+1/2}-f^n}{\tau} = \mathbf{B}(f^n,f^n), \tag{9}$$

$$\frac{f^{n+1}-f^{n+1/2}}{\tau} + (\mathbf{v},\nabla)f^{n+1} = 0, \tag{10}$$

with initial and periodic boundary conditions

$$f^0(\mathbf{x},\mathbf{v}) = \varphi(\mathbf{x},\mathbf{v}), \quad f^{n+1}\big|_{\Gamma_{0x_\alpha}} = f^{n+1}\big|_{\Gamma_{1x_\alpha}}. \tag{11}$$

Let the known approximation $f^n(\mathbf{x},\mathbf{v})$ has all the properties (4) of the initial function.

Introduce a shift operator $\mathbf{T}^{-1/2}$ such that $\mathbf{T}^{-1/2}f^n = f^{n-1/2}$, that is, the operator $\mathbf{T}^{-1/2}-$ acting on the function f^n returns its value obtained by the previous fractional step of the splitting method. Acting this operator on scheme (10), we find the difference-differential analog of the continuity equation

*Condition (8) on the step τ is necessary for the solution positivity of the splitting method schemes.

(or mass conservation equation) at each $\mathbf{v} \in V_3$ corresponding to the first fractional step of the splitting method, that is

$$\frac{f^{n+1/2} - f^n}{\tau} + (\mathbf{v}, \nabla) f^{n+1/2} = 0, \quad f^{n+1/2}\big|_{\Gamma_{0x_\alpha}} = f^{n+1/2}\big|_{\Gamma_{1x_\alpha}}. \tag{12}$$

The boundary condition was obtained from (9), since the function f^n has this property.

It is easy to see that there is the maximum principle on spatial variable $\mathbf{x} \in G$ for problems (10)-(11) and (12).

Let us first consider problem (12) in the form

$$f^{n+1/2} + \tau(\mathbf{v}, \nabla) f^{n+1/2} = f^n, \ f^{n+1/2}\big|_{\Gamma_{0x_\alpha}} = f^{n+1/2}\big|_{\Gamma_{1x_\alpha}}.$$

Applying the maximum principle to this problem, we find an estimate for the solution $f^{n+1/2}(\mathbf{x}, \mathbf{v})$ in the space $C(G)$

$$\sup_{\mathbf{x} \in G} |f^{n+1/2}(\mathbf{v})| \le \sup_{\mathbf{x} \in G} |f^n(\mathbf{v})|, \ \forall \mathbf{v} \in V_3.$$

Then in the same way from problem (10), (11) we obtain an estimate

$$\sup_{\mathbf{x} \in G} |f^{n+1}(\mathbf{v})| \le \sup_{\mathbf{x} \in G} |f^{n+1/2}(\mathbf{v})|, \ \forall \mathbf{v} \in V_3.$$

Combining the found estimates, we have

$$\sup_{\mathbf{x} \in G} |f^{n+1}(\mathbf{v})| \le \sup_{\mathbf{x} \in G} |f^{n+1/2}(\mathbf{v})| \le \sup_{\mathbf{x} \in G} |f^n(\mathbf{v})|, \ \forall \mathbf{v} \in V_3.$$

From here, summing over n, we find the main estimate

$$\sup_{\mathbf{x} \in G} |f^{n+1}(\mathbf{v})| \le \sup_{\mathbf{x} \in G} |f^{n+1/2}(\mathbf{v})| \le \|\varphi(\mathbf{v})\|_{L_\infty(G)} = q_0(\mathbf{v}), \ \forall \mathbf{v} \in V_3 \tag{13}$$

that allows us to obtain estimates for the nonlinear terms of the equation (9).

Consider first

$$\mathbf{J}(f^n) = \int\limits_{V_3 \times \Sigma} f^n(\mathbf{x}, \mathbf{v}') \cdot f^n(\mathbf{x}, \mathbf{v}_1') K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1.$$

From here

$$\begin{aligned}
1.\ |\mathbf{J}(f^n)| \le \int\limits_{V_3 \times \Sigma} |f^n(\mathbf{x}, \mathbf{v}')| \cdot |f^n(\mathbf{x}, \mathbf{v}_1')| K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 \le \\
\le \int\limits_{V_3 \times \Sigma} \sup_{\mathbf{x} \in G} |f^n(\mathbf{x}, \mathbf{v}')| \cdot \sup_{\mathbf{x} \in G} |f^n(\mathbf{x}, \mathbf{v}_1')| K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 = \\
= \int\limits_{V_3 \times \Sigma} \|f^n(\mathbf{v}')\|_{L_\infty(G)} \cdot \|f^n(\mathbf{v}_1')\|_{L_\infty(G)} K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 \le \\
\le \int\limits_{V_3 \times \Sigma} \|\varphi(\mathbf{v}')\|_{L_\infty(G)} \cdot \|\varphi(\mathbf{v}_1')\|_{L_\infty(G)} K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 = q_1(\mathbf{v}) < \infty.
\end{aligned} \tag{14}$$

$$2.\ f^n(\mathbf{x},\mathbf{v}) \mid \mathbf{S}(f^n(\mathbf{x},\mathbf{v}_1)) \mid\leq \sup_{\mathbf{x}\in G}|f^n(\mathbf{v})| \int\limits_{V_3\times\Sigma} \mid f^n(\mathbf{x},\mathbf{v}_1)\mid K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 \leq$$

$$\leq q_0(\mathbf{v}) \int\limits_{V_3\times\Sigma} \|f^n(\mathbf{v}_1\|_{L_\infty(G)} K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 \leq$$

$$\leq q_0(\mathbf{v}) \int\limits_{V_3\times\Sigma} \|\varphi(\mathbf{v}_1)\|_{L_\infty(G)} K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 = q_2(\mathbf{v}) < \infty. \quad (15)$$

It is now easy to obtain an estimate for the difference derivative $f_{\bar{t}}^{n+1/2}$ using (14), (15), based on the equation (9):

$$\sup_{\mathbf{x}\in G} \mid (f^{n+1/2} - f^n)/\tau \mid\leq\mid \mathbf{B}(f^n,f^n) \mid\leq\mid \mathbf{J}(f^n)| + |f^n(\mathbf{v})| \mid \mathbf{S}(f^n) \mid\leq q_1(\mathbf{v}) + q_2(\mathbf{v}) = q_3(\mathbf{v}). \quad (16)$$

Adding equations (9), (10), on the integer step we obtain the difference-differential Boltzmann equation

$$(f^{n+1} - f^n)/\tau + (\mathbf{v},\nabla)f^{n+1} = \mathbf{B}(f^n,f^n). \quad (17)$$

with initial and boundary conditions

$$f^0(\mathbf{x},\mathbf{v}) = \varphi(\mathbf{x},\mathbf{v}), \quad f^{n+1}\big|_{\Gamma_{0x_\alpha}} = f^{n+1}\big|_{\Gamma_{1x_\alpha}}. \quad (18)$$

From here

$$\mid (f^{n+1} - f^n)/\tau \mid\leq\mid \mathbf{B}(f^n,f^n) \mid + \mid (\mathbf{v},\nabla)f^{n+1} \mid .$$

When the function $f^{n+1}(\mathbf{x},\mathbf{v})$ reaches its maximum value at extremum points reaches $\mathbf{x}^e$ in G for every $\mathbf{v}$ in V_3 by virtue of the maximum principle, we have

$$\mid (f^{n+1} - f^n)/\tau \mid (\mathbf{x}) \leq\mid (f^{n+1} - f^n)/\tau \mid (\mathbf{x}^e) \leq\mid \mathbf{B}(f^n,f^n) \mid (\mathbf{x}^e).$$

From here

$$\sup_{\mathbf{x}\in G} \left|(f^{n+1} - f^n)/\tau\right| \leq \left|\mathbf{B}(f^n,f^n)\right| = q_3(\mathbf{v}). \quad (19)$$

Now from (17) we find

$$\sup_{\mathbf{x}\in G} \left|(\mathbf{v},\nabla)f^{n+1}\right| \leq 2q_3(\mathbf{v}). \quad (20)$$

Remark 1. The functions $q_k(\mathbf{v}) \in C(V_3)$. $k = \overline{0,3}$, i.e., they are positive continuous summable functions and continuous depending on the integral of norm for the initial function $\varphi(\mathbf{v})$ over the domain V_3.

Proposition 1. Each problem (12) and (10)–(11) has a unique positive continuous solution that is bounded in Q, and it is periodic function over x_α, i.e., it possesses properties (4) of the initial function, since the approximation $f^n(\mathbf{x},\mathbf{v})$ is such. The periodicity is shown in the same ways as in [7], and the rest of the properties have already been proven.

3 Compact solutions and existence

We denote the set of found approximate solutions to problems (9), (10)–(11) by $\{f^\tau\}$, and the the set of interpolated values on the interval $[0,T)$ by $\tilde{f}^\tau$.

In the velocity space V_3, we introduce a ball V_{R^τ} with the center at the origin of coordinates and with the radius $R^\tau = O(1/\tau^k) < \infty$, where $1 \ll k = const \wedge k \in N$ resulting in a finite bounded domain $Q_{R^\tau} = [0,T) \times G \times V_{R^\tau} \subset Q$.

Since all the estimates are established in the domain Q then they are valid in Q_{R^τ}. The validity of the estimates are not violated when the radius of the ball R^τ increases arbitrarily large as τ tends to zero.

Moreover, from estimates of (13), (14), (15), (16) and (19), (20) it follows the uniform boundedness of the norms for the interpolated functions

$$\tilde{f}^\tau, \quad \mathbf{J}(\tilde{f}^\tau), \quad \mathbf{S}(\tilde{f}^\tau), \quad \tilde{f}^\tau_{\bar{t}}, \ (\mathbf{v}, \nabla)\tilde{f}^\tau$$

in $C(Q)$ at $\tau \to 0$. From here it follows the equicontinuity of $\{\tilde{f}^\tau\}$ in $C(Q)$. Hence, the set $\tilde{f}^\tau$ is compact in the space $C(Q)$. A convergent subsequence can be distinguished from this set. It converges in $C(Q)$ to some element $f(t, \mathbf{x}, \mathbf{v}) \in C(Q)$. Due to compactness, the following limit transitions take place at $\tau \to 0$:

$$\tilde{f}^\tau \to f, \ \tilde{f}^\tau_{\bar{t}} \to f_t, \ \mathbf{J}(\tilde{f}^\tau) \to \mathbf{J}(f), \ \tilde{f}^\tau \mathbf{S}(\tilde{f}^\tau) \to f\mathbf{S}(f),$$
$$\tilde{f}^\tau(t, \mathbf{x}, \mathbf{v}) \mid_{t=0} \to f(t, \mathbf{x}, \mathbf{v}) \mid_{t=0} = \varphi(\mathbf{x}, \mathbf{v}), \ (\mathbf{v}, \nabla)\tilde{f}^\tau \to (\mathbf{v}, \nabla)f,$$
$$\tilde{f}^\tau(t, \mathbf{x}, \mathbf{v})\big|_{\Gamma_{0x_\alpha}} = \tilde{f}^\tau(t, \mathbf{x}, \mathbf{v})\big|_{\Gamma_{1\mathbf{x}_\alpha}} \to f(t, \mathbf{x}, \mathbf{v})\big|_{\Gamma_{0x_\alpha}} = f(t, \mathbf{x}, \mathbf{v})\big|_{\Gamma_{1x_\alpha}}, \alpha = \overline{1,3},$$
$$Q_{R^\tau} \to Q.$$

Thus, going to the limit in the difference-differential problem (17)–(18) we make sure that the limit element $f(t, \mathbf{x}, \mathbf{v})$ uniformly satisfies problem (1)–(3) for the nonlinear Boltzmann equation.

4 Uniqueness

Let there be two solutions $f(t, \mathbf{x}, \mathbf{v})$ and $F(t, \mathbf{x}, \mathbf{v})$ of problem (1)$-$(3). Let us write down the equations for their difference $U = f - F$:

$$\frac{\partial U}{\partial t} + (\mathbf{v}, \nabla)U = \mathbf{B}(f, f) - \mathbf{B}(F, F), \tag{21}$$

in the domain $Q = [0, T) \times G \times V_3$ with zero initial $U \mid_{t=0} = 0$ and periodic boundary condition

$$U(t, \mathbf{x}, \mathbf{v})\Big|_{\Gamma_{0x_\alpha}} = U(t, \mathbf{x}, \mathbf{v})\Big|_{\Gamma_{1x_\alpha}}, \ \alpha = \overline{1,3}. \tag{22}$$

Note that all improper integrals in the calculations make sense, i.e. they are converging integrals. Multiply equation (21) by $2U$ and integrate by domain V_3:

$$\frac{\partial}{\partial t}\int\limits_{V_3} U^2 d\mathbf{v} + \int\limits_{V_3} (\mathbf{v}, \nabla)U^2 d\mathbf{v} = 2\int\limits_{V_3} U\Big(\mathbf{B}(f, f) - \mathbf{B}(F, F)\Big)d\mathbf{v}. \tag{23}$$

Remark 2. In ([2], p. 13), there are formulas (8), (9) of the involutive transformation. For transformation (8), properties are briefly written as

$$U' = \mathbf{P}(U),$$

a) $\mathbf{P}$ is an involutive transformation, i.e. $\mathbf{P}\big(\mathbf{P}(U) = U\big)$,
b) Transformation $\mathbf{P}$ preserves the volume element $d\sigma d\mathbf{v_1} d\mathbf{v}$.

Definition 1. We call two single-valued functions sign equivalent, i.e., $U \sim W$, in the domain Q for $\forall t \in [0, T)$ such that

$$U(t, \mathbf{x}, \mathbf{v}), W(t, \mathbf{x}, \mathbf{v}) \in C(G \times V_3) \cap L_1(V_3), \forall t \in [0, T),$$

and properties
a) $s$ignU = signW in Q,
b) $U(M^j) = W(M^j) = 0$, where $M^j, j = 0, 1, \cdots$, are zeros of these functions in Q.

Lemma 2. There is the inequality

$$\int\limits_{V_3} U\Big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\Big)d\mathbf{v} \le 0. \tag{24}$$

Proof. Consider the expression system

$$\begin{cases} \int\limits_{V_3} \Phi\mathbf{B}(f,f)d\mathbf{v} = \int\limits_{V_3^2\times\Sigma} \Phi[f,f]d\sigma d\mathbf{v}_1 d\mathbf{v}, \\ \\ \int\limits_{V_3} \Phi\mathbf{B}(F,F)d\mathbf{v} = \int\limits_{V_3^2\times\Sigma} \Phi[F,F]d\sigma d\mathbf{v}_1 d\mathbf{v}, \end{cases} \tag{25}$$

where

$$[f,f] = \big(f'f_1' - ff_1\big)K(\theta,\mathbf{w}), \tag{26}$$

$V_3^2 = V_3\times V_3$, $\Phi = \Phi(t,\mathbf{x},\mathbf{v})$ is an arbitrary continuous in Q and summable in V_3 function.

From the first expression of system (25), subtracting the second expression, respectively, we get

$$\int\limits_{V_3} \Phi\big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\big)d\mathbf{v} = \int\limits_{V_3^2\times\Sigma} \Phi\big([f,f]-[F,F]\big)d\sigma d\mathbf{v}_1 d\mathbf{v}.$$

Here we use the well-known involutive transformation $\mathbf{P}$ (see Remark 2).

Applying $\mathbf{P}$ to the integrand on the right parts, we have

$$\int\limits_{V_3} \Phi\big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\big)d\mathbf{v} = -\int\limits_{V_3^2\times\Sigma} \Phi'\big([f,f]-[F,F]\big)d\sigma d\mathbf{v}_1 d\mathbf{v}.$$

Adding the latter with the previous expression, we find

$$\int\limits_{V_3} \Phi\big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\big)d\mathbf{v} = \frac{1}{2}\int\limits_{V_3^2\times\Sigma} (\Phi-\Phi')\big([f,f]-[F,F]\big)d\sigma d\mathbf{v}_1 d\mathbf{v}.$$

In this formula, we make the change of variables $\mathbf{v}_1 \rightleftarrows \mathbf{v}$ and find

$$\int\limits_{V_3} \Phi\big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\big)d\mathbf{v} = \frac{1}{4}\int\limits_{V_3^2\times\Sigma} (\Phi+\Phi_1-\Phi'-\Phi_1')\big([f,f]-[F,F]\big)d\sigma d\mathbf{v}_1 d\mathbf{v}.$$

Hence the square brackets on the right side, replacing the expressions according to the formula (26), we find

$$\int\limits_{V_3} \Phi\big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\big)d\mathbf{v} = \frac{1}{4}\int\limits_{V_3^2\times\Sigma} (\Phi+\Phi_1-\Phi'-\Phi_1')\times$$
$$\times\Big(\big(f'f_1' + FF_1\big) - \big(ff_1 + F'F_1'\big)\Big)d\sigma d\mathbf{v}_1 d\mathbf{v}. \tag{27}$$

If we put $\Phi = \ln(f/F)$, then from (27) we arrive at formula

$$\int\limits_{V_3} \ln(f/F)\big(\mathbf{B}(f,f)-\mathbf{B}(F,F)\big)d\mathbf{v} =$$
$$= \frac{1}{4}\int\limits_{V_3^2\times\Sigma} \ln\Big(\frac{ff_1F'F_1'}{f'f_1'FF_1}\Big)\Big(\big(f'f_1' + FF_1\big) - \big(ff_1 + F'F_1'\big)\Big)d\sigma d\mathbf{v}_1 d\mathbf{v}. \tag{28}$$

We must define the sign definiteness of the complex integral (28). In this case, it is difficult to check the sign of the second the integrand in the domain Q. Since we are interested in only integral is definite in sign, then using Definition 1 we write the sign equivalence functions for the terms of the second integrand

$$\big(f'f_1' - ff_1\big) \sim ln\frac{f'f_1'}{ff_1},\ \big(FF_1 - F'F_1'\big) \sim ln\frac{FF_1}{F'F_1'}.$$

Using these relations, we rewrite the integral (28) sign equivalent form

$$\int\limits_{V_3} \ln(f/F)\big(\mathbf{B}(f,f) - \mathbf{B}(F,F)\big)d\mathbf{v} \sim \frac{1}{4}\int\limits_{V_3^2\times\Sigma} \ln\Big(\frac{ff_1F'F_1'}{f'f_1'FF_1}\Big)\Big(ln\frac{f'f_1'}{ff_1}+$$

$$+ln\frac{FF_1}{F'F_1'}\Big)K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 d\mathbf{v} = \frac{1}{4}\int\limits_{V_3^2\times\Sigma} \ln\Big(\frac{ff_1F'F_1'}{f'f_1'FF_1}\Big)ln\Big(\frac{f'f_1'FF_1}{ff_1F'F_1'}\Big)K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 d\mathbf{v} =$$

$$= -\frac{1}{4}\int\limits_{V_3^2\times\Sigma} \ln^2\Big(\frac{ff_1F'F_1'}{f'f_1'FF_1}\Big)K(\theta,\mathbf{w})d\sigma d\mathbf{v}_1 d\mathbf{v} \le 0.\ \forall t\in[0,T). \quad (29)$$

According to Definition 1, the functions $U = f - F$ and $\Phi = \ln(f/F)$ are also sign equivalent that is, $U \sim \Phi$, since $signU = sign\Phi$ in Q, Thus, (29) implies inequality (24), since $U = f - F \wedge \Phi = \ln(f/F)$, then we will see that $signU = sign\Phi$ in Q, As a result, we arrive at the inequality (24).

$$\int\limits_{V_3} \ln(f/F)\big(\mathbf{B}(f,f) - \mathbf{B}(F,F)\big)d\mathbf{v} \le 0, \Longrightarrow \int\limits_{V_3} U\big(\mathbf{B}(f,f) - \mathbf{B}(F,F)\big)d\mathbf{v} \le 0.$$

Lemma 2 is proved.

Now for functional equation (23), integrating over the domain G taking into account the boundary condition (22) and Lemmas 1 and 2, we obtain the main the inequality for the uniqueness of the solution

$$\frac{d}{dt}\int\limits_{G\times V_3} U^2(t,\mathbf{x},\mathbf{v})d\mathbf{v}d\mathbf{x} \le 2\int\limits_{G\times V_3} U\Big(\mathbf{B}(f,f) - \mathbf{B}(F,F)\Big)d\mathbf{v}d\mathbf{x} \le 0.$$

The latter we will rewrite

$$\frac{d}{dt}\int\limits_{G\times V_3} U^2(t,\mathbf{x},\mathbf{v})d\mathbf{v}d\mathbf{x} - 2\int\limits_{G\times V_3} U\Big(\mathbf{B}(f,f) - \mathbf{B}(F,F)\Big)d\mathbf{v}d\mathbf{x} \le 0,$$

from the left side, discarding the non-negative bounded integral justified in estimates (13)–(16) and, integrating over t, we obtain

$$\int\limits_{G\times V_3} U^2(t,\mathbf{x},\mathbf{v})d\mathbf{v}d\mathbf{x} \le \int\limits_{G\times V_3} U^2(0,\mathbf{x},\mathbf{v})d\mathbf{v}d\mathbf{x},\ \forall t\in[0,T).$$

From here $\int\limits_{G\times V_3} U^2(t,\mathbf{x},\mathbf{v})d\mathbf{v}d\mathbf{x} \le 0,\ \Longrightarrow U(t,\mathbf{v},\mathbf{x}) \equiv 0.\ \forall(t,\mathbf{x},\mathbf{v})\in Q.$

As a result, we show the existence and uniqueness of the positive solution to the full nonlinear Boltzmann equation from the space

$$f(t,\mathbf{x},\mathbf{v}) \in C^1\big(0,T; C(G\times V_3)\cap L_1(V_3)\big) \wedge \big((\mathbf{v},\nabla f); \mathbf{B}(f,f)\big) \in C(Q)\cap L_1(V_3), \quad (30)$$

it consists of the union of some functional spaces, as the space of continuously differentiable functions $f(t,\mathbf{x},\mathbf{v})$ by $t \in [0,T)$ and at each t continuous in $(\mathbf{x},\mathbf{v})$ in the domain $G \times V_3$ and summable over $\mathbf{v}$ in V_3, and the functions $(\mathbf{v},\nabla)f$; $\mathbf{B}(f,f)$ at each t continuous over all variables in Q and summable over $\mathbf{v}$ in V_3.

Definition 2. The solution $f(t,\mathbf{x},\mathbf{v})$ with properties (30) uniformly satisfying the Boltzmann equation (1) with initial boundary conditions (2), (3) in the domain Q will be called strong.

As a result, it was proved next main theorems

Theorem 1. If the initial function satisfies conditions (3), (4), then there is a unique strong positive solution to (1)–(3) for the whole time interval $t \in [0,T), T \leq \infty$ satisfying uniformly the Boltzmann equation (1) everywhere in Q.

When intermolecular interactions are determined by central forces, then $K(\theta,\mathbf{w})$ is determined by the formula (see [2], p. 15)

$$K_c(\theta,\mathbf{w}) =\mid \mathbf{w} \mid \rho; \quad \mathbf{w} = \mathbf{v} - \mathbf{v}_1; \quad \mathbf{\Sigma} \equiv \big\{0 \leq \rho \leq \rho_0; \ 0 \leq \theta \leq 2\pi\big\}, \quad d\sigma = d\rho d\theta,$$

where ρ is the target distance of the colliding molecules, ρ_0 is the radius of action of the molecule. Initial function $\varphi(\mathbf{x},\mathbf{v})$ satisfies condition (3) and such that

$$\begin{cases} 0 < \varphi(\mathbf{x},\mathbf{v}) \in C(G \times V_3) \wedge \Big(\|\varphi(\mathbf{v})\|_{C(G)} \leq \frac{const}{(1+|\mathbf{v}|^2)^{\frac{\gamma}{2}}}, \ \gamma > 6\Big); \\ \mathbf{J}(\varphi) \leq \int\limits_{V_3 \times \Sigma} \|\varphi(\mathbf{v}')\| \cdot \|\varphi(\mathbf{v}_1')\| K_c(\theta,\mathbf{w}) d\sigma d\mathbf{v}_1 = h_1(\mathbf{v}) < \infty; \\ \mathbf{S}(\varphi) \leq \int\limits_{V_3 \times \Sigma} \|\varphi(\mathbf{v}_1)\| K_c(\theta,\mathbf{w}) d\sigma d\mathbf{v}_1 = h_2(\mathbf{v}) < \infty; \end{cases} \tag{31}$$

where $\int\limits_{V_3} h_k(\mathbf{v}) d\mathbf{v} = const, \ k = 1,2.$

The existence and uniqueness theorem of the Cauchy problem for the Boltzmann equation with intermolecular interaction $K_c(\theta,\mathbf{w})$ is also proved as Theorem 1, by a literal repetition, the formulation will be:

Theorem 2. If the initial function satisfies conditions (3) (31), then there exists a unique strong positive solution of problem (1)-(3) on the whole time interval $t \in [0,T), T \leq \infty$ satisfying uniformly the Boltzmann equation (1) everywhere in Q.

Corollary 1. The existence and uniqueness theorems 1 for the nonlinear Boltzmann equation (1) are trivial for the Boltzmann equation in the case of Maxwellian molecules with corresponding relaxations of the requirement from the initial function.

5 *Positivity of the solution to the Boltzmann equation*

Lemma 3. Since there exists a bounded solution of the Boltzmann equation (1) with positive initial condition (2), then the value $\mathbf{B}(f,f)$ of the collisions integral makes sense and the solution $f(t,\mathbf{x},\mathbf{v}) \in Q$ is positive.

Proof. The Boltzmann equation (1) is written along the trajectory

$$\frac{d}{d\tau} f(t,\mathbf{x} - \mathbf{v}(t-\tau),\mathbf{v}) = \frac{\partial f}{\partial t} + (\mathbf{v},\nabla) f(t,\mathbf{x} - \mathbf{v}(t-\tau),\mathbf{v}) = \mathbf{B}(f,f)(t,\mathbf{x} - \mathbf{v}(t-\tau),\mathbf{v}). \tag{32}$$

We put $f = U^{-1}$, since f exists and it is a bounded solution of the Boltzmann equation, then (32) can be rewritten as

$$\frac{d}{d\tau} U(t,\mathbf{x} - \mathbf{v}(t-\tau),\mathbf{v}) = -U\mathbf{B}(U,U)(t,\mathbf{x} - \mathbf{v}(t-\tau),\mathbf{v}).$$

From here, integrating we find

$$U(t, \mathbf{x}, \mathbf{v}) = \varphi(\mathbf{x}, \mathbf{v}) \exp(-\mathbf{B}(U, U)t) > 0, \forall (\mathbf{x}, \mathbf{v}) \in G \times V_3,$$

it was required to prove.

6 H−Boltzmann theorem

Let us multiply the Boltzmann equation (1) by the function $1 + \ln f(t, \mathbf{x}, \mathbf{v})$. Then integrate over the domain $G \times V_3$ and, considering mass conservation [2], we find

$$\frac{d}{dt} \int\limits_{G\times V_3} f \ln f d\mathbf{v} d\mathbf{x} + \int\limits_{G\times V_3} (\mathbf{v}, \nabla) f \ln f d\mathbf{v} d\mathbf{x} =$$

$$= \int\limits_{G\times V_3} \int\limits_{V_3\times\Sigma} \ln f \big(f' f_1' - f f_1\big) K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 d\mathbf{v} d\mathbf{x}. \quad (33)$$

Hence, the second summand of the left part, integrating over the parts, taking into account the boundary condition (3) and using the lemma 1, we have:

$$\int\limits_{G\times V_3} (\mathbf{v}, \nabla) f \ln f d\mathbf{v} d\mathbf{x} = 0. \quad (34)$$

Using the involutive transformation $\mathbf{P}$ (see note 2), the integral in the right-hand side (33) can be written as

$$\int\limits_{G\times V_3} \int\limits_{V_3\times\Sigma} \ln f \big(f' f_1' - f f_1\big) K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 d\mathbf{v} d\mathbf{x} =$$

$$-\frac{1}{4} \int\limits_{G\times V_3^2\times\Sigma} \Big(\ln f' + \ln f_1' - \ln f - \ln f_1\Big)\big(f' f_1' - f f_1\big) K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 d\mathbf{v} d\mathbf{x} =$$

$$= -\frac{1}{4} \int\limits_{G\times V_3^2\times\Sigma} \ln \frac{f' f_1'}{f f_1} \big(f' f_1' - f f_1\big) K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 d\mathbf{v} d\mathbf{x}. \quad (35)$$

Wherefore, using the signequivalence of the function $ln\frac{f' f_1'}{f f_1} \sim \big(f' f_1' - f f_1\big)$ and denoting

$$H(t) = \int\limits_{G\times V_3} f(t, \mathbf{x}, \mathbf{v}) \ln f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} d\mathbf{x},$$

considering (34), (35) from (33), we find

First case:

$$\frac{d}{dt} H \sim -\frac{1}{4} \int\limits_{G\times V_3} \int\limits_{V_3\times\Sigma} ln^2 \frac{f' f_1'}{f f_1} K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 d\mathbf{v} d\mathbf{x} \le 0.$$

Second case:

$$\frac{d}{dt} H \sim -\frac{1}{4} \int\limits_{G\times V_3} \int\limits_{V_3\times\Sigma} \big(f' f_1' - f f_1\big)^2 K(\theta, \mathbf{w}) d\sigma d\mathbf{v}_1 d\mathbf{v} d\mathbf{x} \le 0.$$

From these cases it follows that

$$\frac{d}{dt} H \le 0. \quad (36)$$

Lemma 4. If the positive initial function $\varphi(\mathbf{x}, \mathbf{v})$ is an additionally function to the requirements (4) and satisfies the condition

$$\int_{G\times V_3} \varphi(\mathbf{x}, \mathbf{v})|\ln \varphi(\mathbf{x}, \mathbf{v})|d\mathbf{v}d\mathbf{x} < \infty,$$

then the following inequality holds

$$|\int_{G\times V_3} f(t, \mathbf{x}, \mathbf{v}) \ln f(t, \mathbf{x}, \mathbf{v})d\mathbf{v}d\mathbf{x}| < \int_{G\times V_3} \varphi(\mathbf{x}, \mathbf{v})|\ln \varphi(\mathbf{x}, \mathbf{v})|d\mathbf{v}d\mathbf{x} < \infty, \forall t. \tag{37}$$

Integrating inequality (36) over t in the range from 0 to t, we obtain (37).

Above we proved the strict positivity of the solution $f(t, \mathbf{x}, \mathbf{v})$ to problem (1)-(3), when the initial function $\varphi(\mathbf{x}, \mathbf{v})$ is positive, thus the logarithm function of the distribution is lawful and, moreover, it follows from (30), (37) that $\exists \ln f(t, \mathbf{x}, \mathbf{v})$ for all $(t, \mathbf{x}, \mathbf{v}) \in Q$.

It follows from (36) that the $H(t)$ function never increases in time and is constant if and only if the distribution function is locally-Maxwellian. Indeed, the equality in (36) is achieved, if and only if: in the first case $\ln^2 \frac{f'f'_1}{ff_1} = 0, \Longrightarrow \ \ln \frac{f'f'_1}{ff_1} = 0$, from here

$$\ln f(\mathbf{v}') + \ln f(\mathbf{v}'_1) - \ln f(\mathbf{v}) - \ln f(\mathbf{v}_1) = 0, \tag{38}$$

and in the second case $f'f'_1 = ff_1$, By logarithmizing both parts of the latter, we have the ratio

$$\ln f(\mathbf{v}) + \ln f(\mathbf{v}_1) - \ln f(\mathbf{v}') - \ln f(\mathbf{v}'_1) = 0. \tag{39}$$

Eventually, we see that equations (38), (39) coincide and it follows from them that $\ln f(t, \mathbf{x}, \mathbf{v})$ is a summator invariant, i.e.,

$$\ln f(t, \mathbf{x}, \mathbf{v}) = a + \mathbf{b}\mathbf{v} + c|\mathbf{v}|^2, \ \forall(t, \mathbf{x}) \in [0, T) \times G,$$

where a, c are scalar function and $\mathbf{b}$ is a vector constant. Hence, following [2], we obtain

$$f \equiv f_0 = C \exp(\big[-\alpha(|v| - |v_0|)^2\big]),$$

where f_0 is the local-Maxwell distribution.

$$C > 0 \quad \text{and} \quad \alpha > 0, \text{and} \alpha = 1/(2kT),$$

k is the Boltzmann constant, T is the temperature, v_0 is the average velocity, $C = \rho(2\pi kT)^{-\frac{3}{2}}$, ρ is the density. ρ, T, v_0 can depend on $(t, \mathbf{x})$.

References

1 Больцман Л.Э. Лекции по теории газов / Л.Э. Больцман. — М.: Гостехиздат, 1953. — 554 с.

2 Карлеман Т. Математические задачи кинетической теории газов / Т. Карлеман. — М.: Изд-во ИЛ, 1960. — 150 с.

3 Град Г. Кинетическая теория газов / Г. Град // Сб. ст.: Термодинамика газов. — М.: Машиностроение, 1970. — С. 5–109.

4 Либовиц Дж.Л. Неравновесные явления. Уравнение Больцмана / Дж.Л. Либовиц, Е.У. Монтролл (ред.). — М.: Мир, 1986. — 272 с.

5 Марчук Г.И. Методы расщепления / Г.И. Марчук. — М.: Наука, 1988. — 263 с.

6 Годунов С.К. О дискретных моделях кинетического уравнения Больцмана / С.К. Годунов, У.М. Султангазин // УМН. — 1971. — Т. 26. — Вып. 3(159). — С. 3–51.

7 Akysh A.Sh. Convergence of a splitting method for the nonlinear Boltzmann equation / A.Sh. Akysh // Numerical Analysis and Applications. — 2013. — 6. — No. 2. — P. 111–118.

8 Акыш А.Ш. О разрешимости нелинейного уравнения Больцмана / А.Ш. Акыш // В книге: Неклассические уравнения математической физики. — Новосибирск: Ин-т математики им. С.Л. Соболева СО РАН, 2007. — С. 15–23.

Ә.Ш. Ақыш (Ақышев)

Алматы, Қазақстан

Бейсызықты Больцман теңдеуінің барлық уақытта шешілетіндігі

Жұмыста ыдырату әдісінің негізінде толық бейсызықты Болцман теңдеуінің барлық уақыт аралығында $t \in [0, T),\ T \leq \infty$ молекулалардың тепетеңдіксіз күйі ортасында және олардың әсерлесуі қатты сфералы молекулалар болса немесе қақтығысуы орталық күш арқылы орындалса жалқы шешуінің болатындығы теоремасы дәлелденген. Үзіліссіз функциялар кеңістігінде тұйық шешуі болғандықтан бейсызықты Болцман теңдеуінің бастапқы шарты оң болғанда, шешудің әрқашанда оң болатыны дәлелденді. Соңғының негізінде Болцман $H-$теоремасының кейбір математикалық негіздеуі көрсетілген.

Кілт сөздер: толық бейсызықты Болцман теңдеуі, ыдырату әдісі, барлық уақыт аралығында бейсызықты Болцман теңдеуінің жалқы шешуінің болатындығы теоремасы, бейсызықты Болцман теңдеуінің оң шешуі, Болцман $H-$теоремасының математикалық негіздемесі.

А.Ш. Акыш (Акишев)

Алматы, Казахстан

Глобальная разрешимость нелинейного уравнения Больцмана

В статье с помощью схемы метода расщепления доказана теорема существования и единственности на всем промежутке времени $t \in [0, T),\ T \leq \infty$, для полного нелинейного уравнения Больцмана в неравновесном случае, когда межмолекулярные взаимодействия являются молекул-твердыми сферами и центральными силами. На основе существования ограниченного решения в пространстве $\mathbf{C}$ подтверждена строгая положительность решений полного нелинейного уравнения Больцмана, когда начальная функция положительна. На основании этого показано некоторое математическое обоснование $H-$теоремы Больцмана.

Ключевые слова: полное нелинейное уравнение Больцмана, метод расщепления, теорема существования и единственности на всем промежутке времени для нелинейного уравнения Больцмана, положительность решений нелинейного уравнения Больцмана, H-теорема Больцмана.

References

1 Boltzmann, L. (1953). *Lektsii po teorii gazov [Lectures on gas theory].* Moscow: Gostekhizdat [in Russian].

2 Carleman, T. (1960). *Matematicheskie zadachi kineticheskoi teorii gazov [Mathematical problems of kinetic theory of gases].* Moscow: Izdatelstvo inostrannoi literatury [in Russian].

3 Grad, H. (1970). *Kineticheskaia teoriia gazov [Kinetic theory of gases].* V sbornike statei: Termodinamika gazov — In collection articles: Thermodynamics of gases. Moscow: Mashinostroenie, 5–109 [in Russian].

4 Liebovitz, J.L., & Montroll, E.U. (1986). *Neravnovesnye yavleniia. Uravnenie Boltsmana [Nonequilibrium phenomena. Boltzmann equation].* Moscow: Mir [in Russian].

5 Marchuk, G.I. (1988). *Metody rasshchepleniia [Splitting methods].* Moscow: Nauka [in Russian].

6 Godunov, S.K., & Sultangazin, U.M. (1971). O diskretnykh modeliakh kineticheskogo uravneniia Boltsmana [About discrete models of the kinetic Boltzmann equation]. *Uspekhi matematicheskikh nauk — Russian Mathematical Surveys, 26,* 3(159), 3–51 [in Russian].

7 Akysh, A.Sh. (2013). Convergence of a splitting method for the nonlinear Boltzmann equation. *Numerical Analysis and Applications, 6*(2), 111–118.

8 Akysh, A.Sh. (2007). O razreshimosti nelineinogo uravneniia Boltsmana [On the solvability of a nonlinear Boltzmann equation]. *V knige: Neklassicheskie uravneniia matematicheskoi fiziki — In the book: Nonclassical equations of mathematical Physics.* Novosibirsk: Institut matematiki imeni S.L. Soboleva SO RAN [in Russian].

DOI 10.31489/2022M3/17-24

UDC 517.95

V. Ala*

Mersin University, Science and Letters Faculty, Mathematics Department, Mersin, Turkey

New exact solutions of space-time fractional Schrödinger-Hirota equation

In this study, improved Bernoulli sub-equation function method (IBSEFM) is presented to construct the exact solutions of the nonlinear conformable fractional Schrödinger-Hirota equation (FSHE). By using the traveling wave transformation FSHE turns into the ordinary differential equation (ODE) and by the aid of symbolic calculation software, new exact solutions are obtained. 2D, 3D figures and contour surfaces acquired from the values of the solutions are plotted. The results show that the proposed method is powerful, effective and straightforward for formulating new solutions to various types of nonlinear fractional partial differential equations in applied sciences.

Keywords: conformable fractional derivative, Schrödinger-Hirota equation, improved Bernoulli sub-equation function method (IBSEFM).

1 Introduction

Fractional differential equations are the generalization of classical differential equations with integer order. In recent years, fractional differential equations become the field of scientists to investigate the expediency of non-integer order derivatives in different areas of physics and mathematics. These equations have become a useful tool for describing numerous nonlinear phenomena of physics such as heat conduction systems, nonlinear chaotic systems, viscoelasticity, plasma waves, acustic gravity waves, diffusion processes [1–3]. Many numerical and analytical methods have been developed and successfully employed to solve these equations such as modified Kudryashov method [4], homotopy perturbation method [5], new extended direct algebraic method [6], fractional Riccati expansion method [7], modified extended tanh method [8].

During the last few years, a straightforward definition of conformable derivative has been given [9]. The conformable derivative operator which is compatible to many real-world problems provides some properties of classical calculus: derivative of the quotient of two functions, the chain rule, the product of two functions [10]. In addition, many techniques have been applied to find exact solutions for conformable nonlinear partial differential equations [11–16].

In this study, FSHE is considered as follows:

$$iq_t^{(\mu)} + \frac{1}{2}q_{xx} + |q|^2 q + i\lambda q_{xxx} = 0,\ t \geq 0,\ 0 < \mu \leq 1, i = \sqrt{-1}, \tag{1}$$

where λ is a nonlinear dispersion term, q is the function of the independent variables of x and t. The operator $q_t^{(\mu)}$ represents a conformal derivative operator defined only for a positive domain of t [10]. Before beginning the solution procedure, let us give some properties of the conformable derivative:

The conformable derivative of order α with respect to the independent variable t is defined as [9]:

$$D_t^{\alpha}(y(t)) = \lim_{\tau \to 0} \frac{y(t + \tau t^{1-\alpha}) - y(t)}{\tau},\ t > 0,\ \alpha \in (0, 1],$$

for a function $y = y(t) : [0, \infty) \to \mathbb{R}$.

*Corresponding author.

E-mail: volkanala@mersin.edu.tr

Theorem 1. Assume that the order of the derivative $\alpha \in (0,1]$ and suppose that $u = u(t)$ and $v = v(t)$ are $\alpha-$ differentiable for all positive t. Then,

i. $D_t^a(c_1u + c_2v) = c_1D_t^a(u) + c_2D_t^a(v)$, for $\forall\ c_1, c_2 \in \mathbb{R}$.
ii. $D_t^a(t^k) = kt^{k-a}, \forall\ k \in \mathbb{R}$.
iii. $D_t^a(\lambda) = 0$, for all constant function $u(t) = \lambda$.
iv. $D_t^a\,(uv) = uD_t^a(v) + vD_t^a(u)$.
v. $D_t^a\left(\frac{u}{v}\right) = \frac{vD_t^a(u) - uD_t^a(v)}{v^2}$.
vi. $D_t^a\,(u)\,(t) = t^{1-\alpha}\frac{du}{dt}$.

The conformable differential operator satisfies some critical fundamental properties like the chain rule, Taylor series expansion, and Laplace transform.

Theorem 2. Let $u = u(t)$ be an $\alpha-$ conformable differentiable function and assume that v is a differentiable function. Then,

$$D_t^a(u \circ v)(t) = t^{1-\alpha}v'(t)u'(v(t)).$$

The proofs of these properties are given in [17] and [9] respectively.

The rest of the paper is organized as follows: in the second section, descripton of the IBSEFM is given; in the third section, the application of IBSEFM is mentioned; in the last section, this study provides conclusions.

2 Description of the IBSEFM

In this section, we give the fundamental properties of the IBSEFM. This method is direct, significant, advanced algebraic method for establishing reliable exact solutions for both nonlinear and nonlinear fractional partial differential equations [11, 12, 18–21]. We present five main steps of the IBSEFM as follows:

Step 1: Let us take account of the following conformable partial differential equation of the style

$$P(v, D_t^{(\alpha)}v, D_x^{(\alpha)}v, D_{xt}^{(2\alpha)}v, ...) = 0, \tag{2}$$

where $D_t^{(\alpha)}$ is the conformable fractional derivate operator, $v(x,t)$ is an unknown function, P is a polynomial and its partial derivatives contain fractional derivatives. The aim is to convert conformable nonlinear partial differential equation with a suitable fractional transformation into the ordinary differantial equation. The wave transformation as

$$v(x,t) = V\left(\xi\right), \quad \xi = \xi\left(x, t^{\alpha}\right). \tag{3}$$

Using (3) and the properties of conformable fractional derivate, it enables us to convert (2) into an ODE in the form

$$N(V, V', V'', ...) = 0. \tag{4}$$

If we integrate (4) term to term, we obtain integration constants which can be determined later.

Step 2: Hypothesize the solution of (4) can be presented as follows:

$$V(\xi) = \frac{\sum\limits_{i=0}^{n} a_iF^i(\xi)}{\sum\limits_{j=0}^{m} b_jF^j(\xi)} = \frac{a_0 + a_1F(\xi) + a_2F^2(\xi) + ...a_nF^n(\xi)}{b_0 + b_1F(\xi) + b_2F^2(\xi) + ...b_mF^m(\xi)}, \tag{5}$$

where $a_0, a_1, ..., a_n$ and $b_0, b_1, ..., b_m$ are chosen arbitrary constants of the balance principle and the form of Bernoulli differential equation is as follows:

$$F'(\xi) = \sigma F(\xi) + dF^M(\xi),\ d \neq 0, \sigma \neq 0,\ M \in \mathbb{R}/\left\{0, 1, 2\right\}, \tag{6}$$

where $F(\xi)$ is a polynomial.

Step 3: The positive integers m, n, M are found by balance principle that is both nonlinear term and the highest order derivative term of (4).

Substituting (5) and (6) into (2) it gives us an equation of polynomial $\Theta(F)$ of F as follows;

$$\Theta(F(\xi)) = \rho_s F(\xi)^s + ... + \rho_1 F(\xi) + \rho_0 = 0,$$

where $\rho_i, i = 0, ..., s$ are to be determined later.

Step 4: Equating all the coefficients of $\Theta(F(\xi))$ which yields us an algebraic equation system;

$$\rho_i = 0, i = 0, .., s.$$

Step 5: When we solve (4), we get the following two cases with respect to σ and d,

$$F(\xi) = \left[\frac{-de^{\sigma(\epsilon-1)} + \epsilon\sigma}{\sigma e^{\sigma(\epsilon-1)\xi}}\right]^{\frac{1}{1-\epsilon}},\quad d \neq \sigma, \tag{7}$$

$$F(\xi) = \left[\frac{(\epsilon-1) + (\epsilon+1)\tanh\left(\sigma(1-\epsilon)\frac{\xi}{2}\right)}{1 - \tanh\left(\sigma(1-\epsilon)\frac{\xi}{2}\right)}\right], d = \sigma,\ \epsilon \in \mathbb{R}.$$

Using a complete discrimination system of $F(\xi)$, we obtain the analytical solutions of (4) via Wolfram Mathematica and categorize the exact solutions of (4). To achieve better results, we can plot two and three-dimensional figures of analytical solutions by considering proper values of parameters.

3 Application of the IBSEFM

In this section, we will applicate the IBSEFM to obtain the exact solutions to space-time fractional Schrödinger-Hirota equation. Let us consider the following wave transform:

$$q(x,t) = U(\xi)\exp\left(i\left(\omega x + \eta\frac{t^\mu}{\mu}\right)\right),\quad \xi = x - 2\omega\frac{t^\mu}{\mu}, \tag{8}$$

where the coefficient η and ω are constants that represent soliton frequency and soliton wave number respectively. Introducing (8) we get

$$q_t^{(\mu)} = \left(-2\omega U' + i\eta U\right)\exp\left(i\left(\omega x + \eta\frac{t^\mu}{\mu}\right)\right), \tag{9}$$

$$q_{xx} = \left(U'' + 2i\omega U' - \omega^2 U\right)\exp\left(i\left(\omega x + \eta\frac{t^\mu}{\mu}\right)\right), \tag{10}$$

$$q_{xxx} = \left(U''' + 3i\omega U'' - 3\omega^2 U' - i\omega^3 U\right)\exp\left(i\left(\omega x + \eta\frac{t^\mu}{\mu}\right)\right). \tag{11}$$

Substituting (9)–(11) into (1) and detaching the real and imaginary parts yield $\omega = -\frac{1}{3\lambda}$ and $U(\xi)$ satisfy the following ordinary differential equation:

$$-\left(\frac{5}{54\lambda^2}+\eta\right)U+\frac{3}{2}U''+U^3=0, \tag{12}$$

where $U''=\frac{d^2U}{d\xi^2}$. When we reconsider (12) for balance principle, considering among U'' and U^3, the relationship as follow:

$$M=n-m+1. \tag{13}$$

(13) shows us the different cases of the solutions of (12) and some analytical solutions can be constructed. According to the balance, we consider $M=3, m=1, n=3$ for (12) and (13), the following equations hold:

$$U(\xi)=\frac{a_0+a_1F(\xi)+a_2F^2(\xi)+a_3F^3(\xi)}{b_0+b_1F(\xi)}\equiv\frac{\Upsilon(\xi)}{\Psi(\xi)}, \tag{14}$$

$$U'(\xi)\equiv\frac{\Upsilon'(\xi)\Psi(\xi)-\Upsilon(\xi)\Psi'(\xi)}{\Psi^2(\xi)}, \tag{15}$$

and

$$U''(\xi)\equiv\frac{\Upsilon'(\xi)\Psi(\xi)-\Upsilon(\xi)\Psi'(\xi)}{\Psi^2(\xi)}-\frac{\left[\Upsilon(\xi)\Psi'(\xi)\right]'\Psi^2(\xi)-2\Upsilon(\xi)[\Psi'(\xi)]^2\Psi(\xi)}{\Psi^4(\xi)}, \tag{16}$$

where $F'=\sigma F+dF^3$, $a_3\neq 0, b_1\neq 0, \sigma\neq 0, d\neq 0$. Using (14)-(16) in (13), obtained from coefficients of polynomial of F we get:

$F^0: a_0^3-\eta a_0b_0^2-\frac{5a_0b_0^2}{54\lambda^2}=0,$

$F: 3a_0^2a_1-\eta a_1b_0^2-\frac{5a_1b_0^2}{54\lambda^2}+\frac{3}{2}\sigma^2a_1b_0^2-2\eta a_0b_0b_1-\frac{5a_0b_0b_1}{27\lambda^2}-\frac{3}{2}\sigma^2a_0b_0b_1=0,$

$F^2: 3a_0a_1^2+3a_0^2a_2-\eta a_2b_0^2-\frac{5a_2b_0^2}{54\lambda^2}+6\sigma^2a_2b_0^2-2\eta a_1b_0b_1-\frac{5a_1b_0b_1}{27\lambda^2}-\frac{3}{2}\sigma^2a_1b_0b_1-\eta a_0b_1^2-\frac{5a_0b_1^2}{54\lambda^2}+\frac{3}{2}\sigma^2a_0b_1^2=0,$

$F^3: a_1^3+6a_0a_1a_2+3a_0^2a_3+6d\sigma a_1b_0^2-\eta a_3b_0^2-\frac{5a_3b_0^2}{54\lambda^2}+\frac{27}{2}\sigma^2a_3b_0^2-6d\sigma a_0b_0b_1-2\eta a_2b_0b_1-\frac{5a_2b_0b_1}{27\lambda^2}+\frac{9}{2}\sigma^2a_2b_0b_1-\eta a_1b_1^2-\frac{5a_1b_1^2}{54\lambda^2}=0,$

$F^4: 3a_1^2a_2+3a_0a_2^2+6a_0a_1a_3+18d\sigma a_2b_0^2-2\eta a_3b_0b_1-\frac{5a_3b_0b_1}{27\lambda^2}+\frac{33}{2}\sigma^2a_3b_0b_1-\eta a_2b_1^2-\frac{5a_2b_1^2}{54\lambda^2}+\frac{3}{2}\sigma^2a_2b_1^2=0,$

$F^5: 3a_1a_2^2+3a_1^2a_3+6a_0a_2a_3+\frac{9}{2}d^2a_1b_0^2+36d\sigma a_3b_0^2-\frac{9}{2}d^2a_0b_0b_1+18d\sigma a_2b_0b_1-\eta a_3b_1^2-\frac{5a_3b_1^2}{54\lambda^2}+6\sigma^2a_3b_1^2=0,$

$F^6: a_2^3+6a_1a_2a_3+3a_0a_3^2+12d^2a_2b_0^2+\frac{3}{2}d^2a_1b_0b_1+48d\sigma a_3b_0b_1-\frac{3}{2}d^2a_0b_1^2+6d\sigma a_2b_1^2=0,$

$F^7: 3a_2a_3^2+\frac{63}{2}d^2a_3b_0b_1+\frac{9}{2}d^2a_2b_1^2=0,$

$F^8: a_3^3+12d^2a_3b_1^2=0,$

$F^9: 3a_2^2a_3+3a_1a_3^2+\frac{45}{2}d^2a_3b_0^2+\frac{27}{2}d^2a_2b_0b_1+18d\sigma a_3b_1^2=0.$

When we solve above the system of the equations of F using Wolfram Mathematica, the coefficients are obtained as:

Case 1. For $\sigma\neq d$,

$$a_0=\frac{i\sqrt{\frac{-5}{6}-9\eta\lambda^2}\,b_0}{3\lambda};\ a_1=\frac{i\sqrt{\frac{-5}{6}-9\eta\lambda^2}\,b_1}{3\lambda}; a_2=2i\sqrt{3}db_0;\ a_3=2i\sqrt{3}db_1;\ \sigma=-\frac{i\sqrt{\frac{-5}{2}-27\eta\lambda^2}}{9\lambda}.$$

Substituting these coefficients along with (7) in (14), we obtain the following solution to (1) as follows:

$$q_1(x,t)=\left(\frac{i\sqrt{\frac{-5}{6}-9\eta\lambda^2}}{3\lambda}-\frac{2i\sqrt{3}d}{e^{\frac{2\sqrt{\frac{-5}{2}-27\eta\lambda^2}\left(x-2\omega\frac{t^\mu}{\mu}\right)}{9\lambda}}\epsilon+\frac{9d\lambda}{\sqrt{\frac{-5}{2}-27\eta\lambda^2}}}\right)e^{i\left(\omega x+\eta\frac{t^\mu}{\mu}\right)}.$$

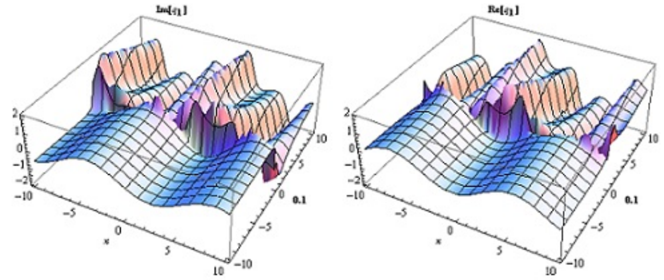

Figure 1. 3D- plots of $q_1(x,t)$ for the values $d=0.4$; $\omega=0.5$; $\mu=0.3$; $\epsilon=0.2$; $\lambda=0.3$; $\eta=0.1$; $\lambda=0.3$; $t=0.1$; $-10<x<10$, $-10<t<10$.

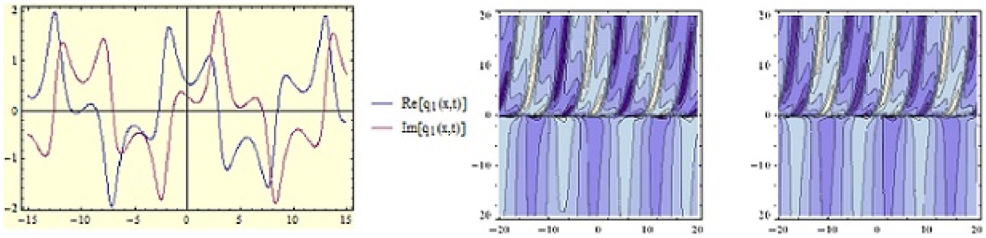

Figure 2. 2D- plots and contour surfaces of $q_1(x,t)$.

Case 2. For $\sigma\neq d$, $a_0=i\sqrt{3}\;\sigma b_0$; $a_1=i\sqrt{3}\;\sigma b_1$; $a_2=2i\sqrt{3}\;db_0$; $a_3=2i\sqrt{3}\;db_1$; $\lambda=-\frac{i\sqrt{\frac{5}{6}}}{3\sqrt{-\eta-3\sigma^2}}$.

Substituting these coefficients along with (7) in (14), we obtain the following solution to (1) as follows:

$$q_2(x,t)=i\sqrt{3}\left(\frac{2d}{e^{-2x\sigma+\frac{4t^\mu\sigma\omega}{\mu}}\epsilon-\frac{d}{\sigma}}+\sigma\right)e^{i\left(\omega x+\eta\frac{t^\mu}{\mu}\right)}.$$

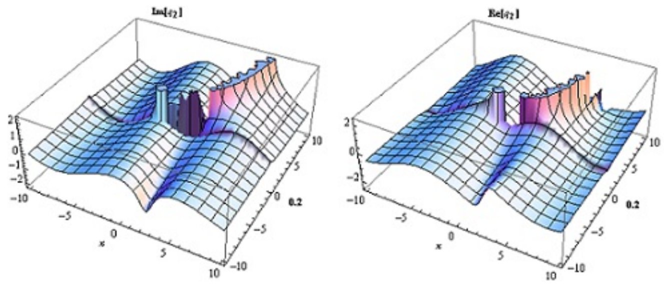

Figure 3. 3D- plots of $q_2(x,t)$ for the values $d=0.3$; $\sigma=0.5$; $\mu=0.3$; $\epsilon=0.2$; $\omega=0.5$; $\eta=0.1$; $t=0.2$;$-10<x<10$, $-10<t<10$.

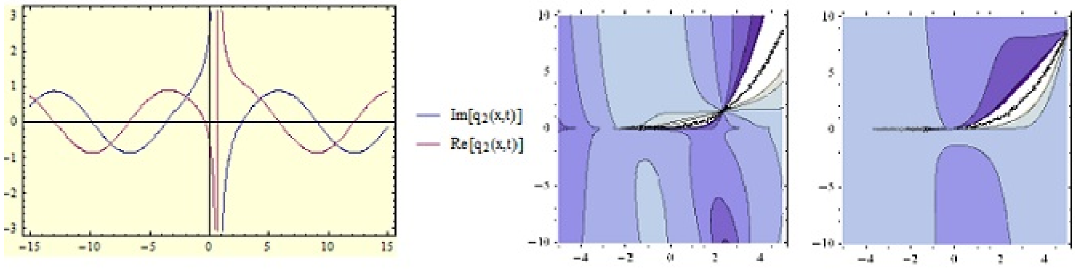

Figure 4. 2D- plots and contour surfaces of $q_2(x,t)$.

4 Conclusions

In this paper, the IBSEFM is applied for FSHE. Using a wave transformation, FSHE has been converted into the ODE which can be solved according to the IBSEFM. By means of this method, exact solutions are obtained. The contourplot surfaces, 3D and 2D figures (Figures 1–4) of all solutions obtained by IBSEFM under the suitable values of parameters are plotted by showing the main characteristic physical properties of the solutions with the help of symbolic software. According to the results, the formats of traveling wave solutions in two and three-dimensional surfaces are similar to the physical meaning of results.

The solutions are solitary wave solutions. It is also clear that the more steps are developed and the better approximations are obtained. The conclusions show that the IBSEFM is simple, effective, and powerful. Thus, in mathematical physics, it is applicable to solve other conformable partial differential equations. In summary, the improved Bernoulli Sub-equation funtion method is influential and suitable for solving other types of nonlinear differential equations in which the balance principle is satisfied.

References

1 Baleanu, D., Machado, T., Jose, A., & et al. (2012). New properties of conformable derivative. *Fractional Dynamics and Control*, 49–57.

2 Miller, K.S., & Ross, B. (1993). A new definition of fractional derivative. *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley, New York.

3 Podlubny, I. (1999). *Fractional Differential Equations.* California: Academic Press.

4 Rezazadeh, H., Osman, M.S., Eslami, M., & et al. (2018). Mitigating Internet bottleneck with fractional temporal evolution of optical solitons having quadratic-cubic nonlinearity. *Optik, 164*, 84–92. https://doi.org/10.1016/j.advwatres.2009.09.003

5 Yıldırım, A., & Koçak, H. (2009). Homotopy perturbation method for solving the space-time fractional advection-dispersion equation. *Advances in Water Resources, 32*(12), 1711–1716.

6 Rezazadeh, H., Tariq, H.,Eslami, M., & et al. (2018). New exact solutions of nonlinear conformable time-fractional Phi-4 equation. *Chinese Journal of Physics, 56*(6), 2805–2816.

7 Abdel-Salam, A., Emad, B.M., & Eltyabev, A.Y. (2013). Solution of nonlinear space-time fractional differential equations using the fractional Riccati expansion method. *Math. Probl. Eng., 2013*, 1–6. https://doi.org/10.1155/2013/846283

8 Shallal, M.A., Jabbar, H.N., & Ali, K.K. (2018). Analytic solution for the space-time fractional Klein-Gordon and coupled conformable Boussinesq equations. *Results in Physics, 8*, 372–378. https://doi.org/10.1016/j.rinp.2017.12.051

9 Abdeljawad, T. (2015). On Conformable Fractional Calculus. *J. Comput. Appl. Math., 279*, 57–66. https://doi.org/ 10.1016/j.cam.2014.10.016

10 Khalil, R., Horani, M., Yousef, A., & et al. (2014). A new definition of fractional derivative. *J. Comput. Appl. Math., 264* 65–70. https://doi.org/10.1016/j.cam.2014.01.002

11 Ala, V., Demirbilek, U., & Mamedov, Kh.R. (2020). An application of improved Bernoulli sub-equation function method to the nonlinear conformable time-fractional SRLW equation. *AIMS Mathematics, 5*(4), 3751–3761. https://doi.org/10.3934/math.2020243

12 Başkonuş, H.M., & Bulut, H. (2016). Exponential prototype structures for $(2+1)$ dimensional Boiti-Leon-Penpinelli systems in mathematical physics. *Waves Random Complex Media, 26*(2), 189–196. https://doi.org/10.1080/17455030.2015.1132860

13 Durur, H., Taşbozan, O., & Kurt A. (2020). New Analytical Solutions of Conformable Time Fractional Bad and Good Modified Boussinesq Equations. *Applied Mathematics and Nonlinear Sciences, 5*(1), 447–454. https://doi.org/10.2478/amns.2020.1.00042

14 Kurt, A. (2020). New analytical and numerical results for fractional Bogoyavlensky-Konopelchenko equation arising in fluid dynamics. *Applied Mathematics-A Journal of Chinese Uni., 35*(1), 101–112. https://doi.org/10.1007/s11766-020-3808-9

15 Şenol, M. (2020). New Analytical Solutions of Fractional Symmetric Regularized-Long-Wave Equation. *Revista Mexicana de Fisica, 66*(3). https://doi.org/10.31349/RevMexFis.66.297

16 Yel, G., Sulaiman, T.A., & Başkonuş, H.M. (2020). On the complex solutions to $(3+1)$-dimensional conformable fractional modified KdV-Zakharov-Kuznetsov equation. *Modern Physics Letters B, 34*(5). https://doi.org/10.1142/S0217984920500694

17 Atangana, A., Baleanu, D., & Alsaedi, A. (2015). New properties of conformable derivative. *Open Math., 13*, 889–898. https://doi.org/10.1515/math-2015-0081

18 Başkonuş, H.M., & Bulut, H. (2016). On the complex structures of Kundu-Eckhaus equation via improved Bernoulli sub-equation function method. *Waves Random Complex Media, 25*(4), 720–728. https://doi.org/10.1080/17455030.2015.1080392

19 Başkonuş, H.M., Koç D., & Bulut, H. (2016). New travelling wave prototypes to the nonlinear Zakharov-Kuznetsov equation with power law nonlinearity. *Nonlin. Sci. Lett. A, 7*, 67–76.

20 Demirbilek, U., Ala V., & Mamedov, Kh.R. (2021). An application of improved Bernoulli sub-equation function method to the nonlinear conformable time-fractional equation. *Tbilisi Mathematical Journal, 14*(3), 67–76. https://doi.org/10.32513/tmj/19322008142

21 Duşunceli, F. (2019). New Exponential and Complex Traveling Wave Solutions to the Konopelchenko-Dubrovsky model. *Adv. Math. Phys., 2019.* https://doi.org/10.1016/j.cam.2014.01.002

В. Ала

Мерсин университеті, ғылым және әдебиет факультеті, математика факультеті, Мерсин, Түркия

Кеңістік пен уақыттан тәуелді бөлшек Шредингер-Хирота теңдеуінің жаңа нақты шешімдері

Мақалада сызықты емес бөлшек тәріздес Шредингер-Хирота теңдеуінің (FSHE) дәл шешімдерін құру үшін жақсартылған Бернулли қосалқы теңдеуі функциясының әдісі (IBSEFM) ұсынылған. Жылжымалы толқын түрлендіруінің көмегімен FSHE кәдімгі дифференциалдық теңдеуге (ODE) түрлендіріледі және символдық есептеуіш бағдарламалық қамтамасыз етудің көмегімен жаңа нақты шешімдер алынады. 2D, 3D фигуралары мен шешімдердің мәндерінен алынған контур беттер салынған.

Нәтижелер көрсеткендей, ұсынылып отырған әдіс қолданбалы ғылымдардағы әртүрлі типті сызықты емес бөлшек дербес туындылы дифференциалдық теңдеулердің жаңа шешімдерін жасау үшін қуатты, тиімді және қарапайым әдіс.

Кілт сөздер: бөлшек тәріздес туынды, Шредингер-Хирота теңдеуі, жақсартылған Бернулли қосалқы теңдеуінің функциясы әдісі (IBSEFM).

В. Ала

Университет Мерсин, Мерсин, Турция

Новые точные решения пространственно-временного дробного уравнения Шрёдингера-Хироты

В статье представлен усовершенствованный метод функций подуравнений Бернулли для построения точных решений нелинейного дробно-подобного уравнения Шрёдингера-Хироты (FSHE). С помощью преобразования бегущей волны FSHE превращается в обыкновенное дифференциальное уравнение, а с использованием программного обеспечения для символьных вычислений получаются новые точные решения. Строятся 2D, 3D фигуры и контурные поверхности, полученные из значений решений. Результаты показывают, что предложенный метод является мощным, эффективным и простым способом для разработки новых решений различных типов нелинейных дробных дифференциальных уравнений в частных производных в прикладных науках.

Ключевые слова: дробно-подобная производная, уравнение Шрёдингера-Хироты, усовершенствованный метод функций подуравнений Бернулли.

DOI 10.31489/2022M3/25-34
UDC 519.63

A. Ashyralyev[1,2,3,*], B. Haso[4,5]

[1] *Bahcesehir University, Istanbul, Turkey;*
[2] *Peoples Friendship University of Russia, Moscow, Russian Federation;*
[3] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[4] *Near East University, Nicosia, Turkey;*
[5] *Department of Mathematics, High School, Duhok, Iraq*
(E-mail: aallaberen@gmail.com, bashar.chato1988@gmail.com)

Stability of the time-dependent identification problem for delay hyperbolic equations

Time-dependent and space-dependent source identification problems for partial differential and difference equations take an important place in applied sciences and engineering, and have been studied by several authors. Moreover, the delay appears in complicated systems with logical and computing devices, where certain time for information processing is needed. In the present paper, the time-dependent identification problem for delay hyperbolic equation is investigated. The theorems on the stability estimates for the solution of the time-dependent identification problem for the one dimensional delay hyperbolic differential equation are established. The proofs of these theorems are based on the Dalambert's formula for the hyperbolic differential equation and integral inequality.

Keywords: hyperbolic equation, time delay, Hilbert space, source identification, stability.

Introduction

There is always a major interest for the theory of source identification problems for partial differential equations since they have widespread applications in modern physics and technology. Subsequently, the stability of various source identification problems for partial differential and difference equations have been studied extensively by many researchers (see, e.g., [1–25] and the references given therein). In many fields of the contemporary science and technology, systems with delaying terms appear. The dynamical processes are described by systems of delay ordinary and partial differential and difference equations. The stability of the delay differential and difference equations have also been studied in many papers (see, e.g., [26–35] and the references given therein). In the present paper, the time-dependent identification problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = b\frac{\partial^2 u(t-\omega,x)}{\partial x^2} + p(t)q(x) + f(t,x), \\ 0 < t < \infty, x \in (-\infty,\infty), \\ u(t,x) = g(t,x), -\omega \le t \le 0, x \in (-\infty,\infty), \\ \int\limits_{-\infty}^{\infty} \alpha(x)u(t,x)dx = \zeta(t), t \ge 0 \end{cases} \tag{1}$$

for one-sdimensional delay hyperbolic equation is considered. Here $u(t,x)$ and $p(t)$ are unknown functions. Under compatibility conditions, problem (1) has a unique solution $(u(t,x), p(t))$ for the smooth functions $f(t,x)((t,x) \in (0,\infty) \times (-\infty,\infty))$, $g(t,x)((t,x) \in [-\omega,0] \times (-\infty,\infty))$, $\zeta(t)(t \ge 0)$, $q(x)$, and $\alpha(x)$,
$x \in (-\infty,\infty)$. Here b is a constant.

[*] Corresponding author.
E-mail: aallaberen@gmail.com

The theorems on stability

We have the following theorems on the stability of problem (1).

Theorem 1. Assume that $\int\limits_{-\infty}^{\infty} \alpha(x)q(x)dx \neq 0$ and $\int\limits_{-\infty}^{\infty} |\alpha(x)|\, dx \leq \alpha < \infty$. Then for the solution of problem (1) the following stability estimates holds:

$$\max_{0\leq t\leq\omega} |p(t)|\,,\ \max_{0\leq t\leq\omega} \|u_{tt}\|_{C(-\infty,\infty)}\,,\ \max_{0\leq t\leq\omega} \|u_t\|_{C^{(1)}(-\infty,\infty)}\,,\ \max_{0\leq t\leq\omega} \|u\|_{C^{(2)}(-\infty,\infty)}$$

$$\leq M\,(q,\alpha) \left[a_0 + \max_{0\leq t\leq\omega} \left\|f'(t)\right\|_{C(-\infty,\infty)} + \|f\,(0)\|_{C(-\infty,\infty)} + \max_{0\leq t\leq\omega} \left|\zeta''\right|\right],$$

$$a_0 = \max\left\{ \max_{-\omega\leq t\leq 0} \|g_{tt}(t)\|_{C(-\infty,\infty)}\,,\ \max_{-\omega\leq t\leq 0} \|g_t(t)\|_{C^{(1)}(-\infty,\infty)}\,,\right.$$

$$\left.\max_{-\omega\leq t\leq 0} \|g(t)\|_{C^{(2)}(-\infty,\infty)}\right\},$$

and

$$\max_{n\omega\leq t\leq(n+1)\omega} |p(t)|\,,\quad \max_{n\omega\leq t\leq(n+1)\omega} \|u_{tt}\|_{C(-\infty,\infty)}\,,\ \max_{n\omega\leq t\leq(n+1)\omega} \|u_t\|_{C^{(1)}(-\infty,\infty)}\,,$$

$$\max_{n\omega\leq t\leq(n+1)\omega} \|u\|_{C^{(2)}(-\infty,\infty)} \leq M\,(q,\alpha) \left[a_n + \max_{(n-1)\omega\leq t\leq n\omega} |p(t)|\right.$$

$$\left. + \max_{n\omega\leq t\leq(n+1)\omega} \left\|f'(t)\right\|_{C(-\infty,\infty)} + \|f\,(n\omega)\|_{C(-\infty,\infty)} + \max_{n\omega\leq t\leq(n+1)\omega} \left|\zeta''\right|\right],$$

$$a_n = \max\left\{ \max_{(n-1)\omega\leq t\leq n\omega} \|u_{tt}(t)\|_{C(-\infty,\infty)}\,,\ \max_{(n-1)\omega\leq t\leq n\omega} \|u_t(t)\|_{C^{(1)}(-\infty,\infty)}\,,\right.$$

$$\left.\max_{(n-1)\omega\leq t\leq n\omega} \|u(t)\|_{C^{(2)}(-\infty,\infty)}\right\}, n = 1, 2, \cdots .$$

Here $C(-\infty,\infty)$ refers to the vector space of continuous functions $w(x)$ from the entire real line to $R = (-\infty,\infty)$ with norm

$$\|w\|_{C(-\infty,\infty)} = \sup_{x\in(-\infty,\infty)} |w\,(x)|\,.$$

Proof. We will seek $u(t,x)$, using the substitution

$$u(t,\ x) = w(t,\ x) + \eta(t)q(x), \tag{2}$$

where $\eta(t)$ is the function defined by the formula

$$\eta(t) = \int_{(n-1)\omega}^{t} (t-s)p(s)ds,\ \ \eta((n-1)\,\omega) = \eta'((n-1)\,\omega) = 0, n = 1, 2, \cdots .$$

It is easy to see that $w(t,\,x)$ is the solution of the problems

$$\begin{cases} \frac{\partial^2 w(t,x)}{\partial t^2} - \frac{\partial^2 w(t,x)}{\partial x^2} = \eta(t)q''(x) + bg_{xx}(t-\omega,x) + f(t,x), \\ 0 < t < \omega, x \in (-\infty,\infty)\,, \\ w(0,\ x) = g(0,x),\ \ w_t(0,\ x) = g_t(0,x),\ \ x \in (-\infty,\infty)\,, \end{cases} \tag{3}$$

and

$$\begin{cases} \frac{\partial^2 w(t,x)}{\partial t^2} - \frac{\partial^2 w(t,x)}{\partial x^2} = b\frac{\partial^2 w(t-\omega,x)}{\partial x^2} \\ + (\eta(t) + b\eta(t-\omega))\, q''(x) + f(t,x), \\ (n-1)\,\omega < t < n\omega, x \in (-\infty,\infty)\,, \quad n = 2,3,\cdots, \\ w((n-1)\,\omega+,\ x) = w((n-1)\,\omega-,\ x), \\ w_t((n-1)\,\omega+,\ x) = w_t((n-1)\,\omega-,\ x), \\ x \in (-\infty,\infty)\,, n = 2,3,\cdots. \end{cases} \tag{4}$$

Now we will take an estimate for $|p(t)|$. Applying the integral overdetermined condition

$$\int_{-\infty}^{\infty} \alpha(x)u(t,x)dx = \zeta(t)$$

and substitution (2), we get

$$\eta(t) = \frac{\zeta(t) - \int\limits_{-\infty}^{\infty} \alpha(x)w(t,x)dx}{\int\limits_{-\infty}^{\infty} \alpha(x)q(x)dx}.$$

From that and $p(t) = \eta''(t)$, it follows that

$$p(t) = \frac{\zeta''(t) - \int\limits_{-\infty}^{\infty} \alpha(x)\frac{\partial^2}{\partial t^2}w(t,x)dx}{\int\limits_{-\infty}^{\infty} \alpha(x)q(x)dx}.$$

Then, using the triangle inequality, we obtain

$$|p(t)| \le \frac{|\zeta''(t)| + \int\limits_{-\infty}^{\infty} \left|\alpha(x)\frac{\partial^2}{\partial t^2}w(t,x)\right| dx}{\left|\int\limits_{-\infty}^{\infty} \alpha(x)q(x)dx\right|} \tag{5}$$

$$\le k(q,\alpha)\left[\left|\zeta''(t)\right| + \left\|\frac{\partial^2}{\partial t^2}w(t,.)\right\|_{C(-\infty,\infty)}\right]$$

for all $t \in (0,\infty)$. Now, using substitution (2), we get

$$\frac{\partial^2 u\,(t,x)}{\partial t^2} = \frac{\partial^2 w\,(t,x)}{\partial t^2} + p(t)q(x).$$

Applying the triangle inequality, we obtain

$$\left\|\frac{\partial^2 u\,(t,\cdot)}{\partial t^2}\right\|_{C(-\infty,\infty)} \le \left\|\frac{\partial^2 w\,(t,\cdot)}{\partial t^2}\right\|_{C(-\infty,\infty)} + |p(t)|\,\|q\|_{C(-\infty,\infty)}$$

for all $t \in (0,\infty)$. Therefore, the proof of Theorem 1 is based on the following theorem.

Theorem 2. Under assumptions of Theorem 1, for the solution of problems (3) and (4) the following stability estimates holds:

$$\max_{0\le t\le\omega}\|w_{tt}\|_{C(-\infty,\infty)},\ \max_{0\le t\le\omega}\|w_t\|_{C^{(1)}(-\infty,\infty)},\ \max_{0\le t\le\omega}\|w\|_{C^{(2)}(-\infty,\infty)} \tag{6}$$

$$\le M(q,\alpha)\left[a_0+\max_{0\le t\le\omega}\left\|f'(t)\right\|_{C(-\infty,\infty)}+\|f(0)\|_{C(-\infty,\infty)}+\max_{0\le t\le\omega}\left|\zeta''\right|\right],$$

$$a_0=\max\left\{\max_{-\omega\le t\le 0}\|g_{tt}(t)\|_{C(-\infty,\infty)},\ \max_{-\omega\le t\le 0}\|g_t(t)\|_{C^{(1)}(-\infty,\infty)},\ \max_{-\omega\le t\le 0}\|g(t)\|_{C^{(2)}(-\infty,\infty)}\right\},$$

$$\max_{n\omega\le t\le(n+1)\omega}\|w_{tt}\|_{C(-\infty,\infty)},\ \max_{n\omega\le t\le(n+1)\omega}\|w_t\|_{C^{(1)}(-\infty,\infty)},\ \max_{n\omega\le t\le(n+1)\omega}\|w\|_{C^{(2)}(-\infty,\infty)} \tag{7}$$

$$\le M(q,\alpha)\left[a_n+\max_{n\omega\le t\le(n+1)\omega}\left\|f'(t)\right\|_{C(-\infty,\infty)}+\|f(n\omega)\|_{C(-\infty,\infty)}+\max_{n\omega\le t\le(n+1)\omega}\left|\zeta''\right|\right],$$

$$a_n=\max\left\{\max_{(n-1)\omega\le t\le n\omega}\|w_{tt}(t)\|_{C(-\infty,\infty)},\max_{(n-1)\omega\le t\le n\omega}\|w_t(t)\|_{C^{(1)}(-\infty,\infty)},\right.$$

$$\left.\max_{(n-1)\omega\le t\le n\omega}\|w(t)\|_{C^{(2)}(-\infty,\infty)}\right\},n=1,2,\cdots.$$

Proof. First, we will prove that

$$\max_{0\le t\le\omega}\|w_{tt}\|_{C(-\infty,\infty)}\le M(q,\alpha)\left[a_0+\max_{0\le t\le\omega}\left\|f'(t)\right\|_{C(-\infty,\infty)}+\|f(0)\|_{C(-\infty,\infty)}+\max_{0\le t\le\omega}\left|\zeta''\right|\right]. \tag{8}$$

Applying the Dalambert's formula, we get the following formula

$$w(t,x)=\frac{g(0,x+t)+g(0,x-t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}g_t(0,\xi)d\xi$$

$$+\int_0^t\frac{1}{2}\int_{x-(t-\tau)}^{x+(t-\tau)}\left[\eta(\tau)q''(\xi)+bg_{\xi\xi}(\tau-\omega,\xi)+f(\tau,\xi)\right]d\xi d\tau$$

for any $t\in[0,\omega], x\in(-\infty,\infty)$. From that it follows that

$$w(t,x)=\frac{g(0,x+t)+g(0,x-t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}g_t(0,\xi)d\xi$$

$$+\int_0^t\frac{\eta(\tau)}{2}\left[q_{x+(t-\tau)}(x+(t-\tau))-q_{x-(t-\tau)}(x-(t-\tau))\right]d\tau$$

$$+\int_0^t\frac{b}{2}\left[g_{x+(t-\tau)}(\tau-\omega,x+(t-\tau))-g_{x-(t-\tau)}(\tau-\omega,x-(t-\tau))\right]d\tau$$

$$+\int_0^t \frac{1}{2}\int_{x-(t-\tau)}^{x+(t-\tau)} f(\tau,\xi)d\xi d\tau.$$

Taking the derivatives, we get

$$w_t\left(t,x\right)=\frac{g_t(0,x+t)+g_t(0,x-t)}{2}+\frac{1}{2}\left[g_t(0,x+t)-g_t(0,x-t)\right]$$

$$+\int_0^t \frac{\eta(\tau)}{2}\left[q_{x+(t-\tau),t}(x+(t-\tau))-q_{x-(t-\tau),t}(x-(t-\tau))\right]d\tau$$

$$+\int_0^t \frac{b}{2}\left[g_{x+(t-\tau),t}(\tau-\omega,x+(t-\tau))-g_{x-(t-\tau),t}(\tau-\omega,x-(t-\tau))\right]d\tau$$

$$+\int_0^t \frac{1}{2}\left[f(\tau,x+(t-\tau))-f(\tau,x-(t-\tau))\right]d\tau,$$

$$w_{tt}\left(t,x\right)=\frac{g_{tt}(0,x+t)+g_{tt}(0,x-t)}{2}+\frac{1}{2}\left[g_{tt}(0,x+t)-g_{tt}(0,x-t)\right]$$

$$+\int_0^t \frac{\eta(\tau)}{2}\left[q_{x+(t-\tau),tt}(x+(t-\tau))-q_{x-(t-\tau),tt}(x-(t-\tau))\right]d\tau$$

$$+\int_0^t \frac{b}{2}\left[g_{tt}(-\omega,x+t)-g_{tt}(-\omega,x-t)\right]d\tau$$

$$+\int_0^t \frac{1}{2}\left[f_t(\tau,x+(t-\tau))-f_t(\tau,x-(t-\tau))\right]d\tau.$$

Applying this formula and the triangle inequality and estimate (5), we get

$$\begin{aligned}\|w_{tt}(t,\cdot)\| \quad &\leq \quad M\left(q,\alpha\right)\left[a_0+\max_{0\leq t\leq\omega}\left\|f'(t)\right\|_{C(-\infty,\infty)}+\left\|f\left(0\right)\right\|_{C(-\infty,\infty)}+\left|\zeta''(t)\right|\right]\\ &+M\left(q\right)\int_0^t \|w_{\tau\tau}(\tau,\cdot)\|\,d\tau\end{aligned}$$

for any $t\in[0,\omega]$. By the integral inequality, we get the estimate (8). Applying equation (3) and triangle inequality and estimate (8), we get estimate (6).

Second, we will prove that

$$\max_{n\omega\leq t\leq(n+1)\omega}\left\|\frac{\partial^2 w\left(t,\cdot\right)}{\partial t^2}\right\|_{C(-\infty,\infty)}\leq M\left(q,\alpha\right)\left[a_n\right.$$

$$\left.+\max_{n\omega\leq t\leq(n+1)\omega}\left\|f'(t)\right\|_{C(-\infty,\infty)}+\left\|f\left(n\omega\right)\right\|_{C(-\infty,\infty)}+\max_{n\omega\leq t\leq(n+1)\omega}\left|\zeta''\right|\right], n=1,2,\cdots$$

Applying the Dalambert's formula, we get the following formula

$$w\left(t,x\right)=\frac{w(n\omega,x+t)+w(n\omega,x-t)}{2}+\frac{1}{2}\int\limits_{x-t}^{x+t}w_t(n\omega,\xi)d\xi$$

$$+\int\limits_{n\omega}^{t}\frac{1}{2}\int\limits_{x-(t-\tau)}^{x+(t-\tau)}\left[\left(\eta(\tau)+b\eta(\tau-\omega)\right)q''(\xi)+bw_{\xi\xi}(\tau-\omega,\xi)+f(\tau,\xi)\right]d\xi d\tau.$$

for any $t\in\left[n\omega,\left(n+1\right)\omega\right], x\in\left(-\infty,\infty\right)$. From that it follows that

$$w\left(t,x\right)=\frac{w(n\omega,x+t)+w(n\omega,x-t)}{2}+\frac{1}{2}\int\limits_{x-t}^{x+t}w_t(n\omega,\xi)d\xi$$

$$+\int\limits_{n\omega}^{t}\frac{\left(\eta(\tau)+b\eta(\tau-\omega)\right)}{2}\left[q_{x+(t-\tau)}(x+(t-\tau))-q_{x-(t-\tau)}(x-(t-\tau))\right]d\tau$$

$$+\int\limits_{n\omega}^{t}\frac{b}{2}\left[w_{x+(t-\tau)}(\tau-\omega,x+(t-\tau))-w_{x-(t-\tau)}(\tau-\omega,x-(t-\tau))\right]d\tau$$

$$+\int\limits_{n\omega}^{t}\frac{1}{2}\int\limits_{x-(t-\tau)}^{x+(t-\tau)}f(\tau,\xi)d\xi d\tau.$$

Taking the derivatives, we get

$$w_t\left(t,x\right)=\frac{w_t(n\omega,x+t)+w_t(n\omega,x-t)}{2}$$

$$+\frac{1}{2}\left[w_t(n\omega,x+t)-w_t(n\omega,x-t)\right]$$

$$+\int\limits_{n\omega}^{t}\frac{\left(\eta(\tau)+b\eta(\tau-\omega)\right)}{2}\left[q_{x+(t-\tau),t}(x+(t-\tau))-q_{x-(t-\tau),t}(x-(t-\tau))\right]d\tau$$

$$+\int\limits_{n\omega}^{t}\frac{b}{2}\left[w_{x+(t-\tau),t}(\tau-\omega,x+(t-\tau))-w_{x-(t-\tau),t}(\tau-\omega,x-(t-\tau))\right]d\tau$$

$$+\int\limits_{n\omega}^{t}\frac{1}{2}\left[f(\tau,x+(t-\tau))-f(\tau,x-(t-\tau))\right]d\tau,$$

$$w_{tt}\left(t,x\right)=\frac{w_{tt}(n\omega,x+t)+w_{tt}(n\omega,x-t)}{2}$$

$$+\frac{1}{2}\left[w_{tt}(n\omega,x+t)-w_{tt}(n\omega,x-t)\right]$$

$$+\int\limits_{n\omega}^{t}\frac{(\eta(\tau)+b\eta(\tau-\omega))}{2}\left[q_{x+(t-\tau),tt}(x+(t-\tau))-q_{x-(t-\tau),tt}(x-(t-\tau))\right]d\tau$$

$$+\int\limits_{n\omega}^{t}\frac{b}{2}\left[w_{tt}(-\omega,x+t)-w_{tt}(-\omega,x-t)\right]d\tau$$

$$+\int\limits_{n\omega}^{t}\frac{1}{2}\left[f_t(\tau,x+(t-\tau))-f_t(\tau,x-(t-\tau))\right]d\tau.$$

Applying this formula and the triangle inequality and estimate (5), we get

$$\|w_{tt}(t,\cdot)\|\leq M(q,\alpha)\,[a_n$$

$$\left.+\max_{n\omega\leq t\leq(n+1)\omega}\left\|f'(t)\right\|_{C(-\infty,\infty)}+\|f(n\omega)\|_{C(-\infty,\infty)}+\max_{n\omega\leq t\leq(n+1)\omega}\left|\zeta''\right|\right]$$

$$+M(q)\int\limits_{n\omega}^{t}\|w_{\tau\tau}(\tau,\cdot)\|\,d\tau$$

for any $t\in[n\omega,(n+1)\,\omega]$. By the integral inequality, we get the estimate (6). Applying equation (4) and triangle inequality and estimate (6), we get estimate (7). This completes the proof of Theorem 2.

Moreover, we have that

Theorem 3. Assume that $\int\limits_{-\infty}^{\infty}\alpha(x)q(x)dx\neq 0$ and $\int\limits_{-\infty}^{\infty}|\alpha(x)|^q\,dx\leq\alpha<\infty, 1\leq q<\infty, \frac{1}{q}+\frac{1}{p}=1.$

Then for the solution of problem (1) the following stability estimates holds:

$$\max_{0\leq t\leq\omega}|p(t)|,\ \max_{0\leq t\leq\omega}\|u_{tt}\|_{L_p(-\infty,\infty)},\ \max_{0\leq t\leq\omega}\|u_t\|_{W_p^1(-\infty,\infty)},\ \max_{0\leq t\leq\omega}\|u\|_{W_p^2(-\infty,\infty)}$$

$$\leq M(q,\alpha)\left[a_0+\max_{0\leq t\leq\omega}\left\|f'(t)\right\|_{L_p(-\infty,\infty)}+\|f(0)\|_{L_p(-\infty,\infty)}+\max_{0\leq t\leq\omega}\left|\zeta''\right|\right],$$

$$a_0=\max\left\{\max_{-\omega\leq t\leq 0}\|g_{tt}(t)\|_{L_p(-\infty,\infty)},\max_{-\omega\leq t\leq 0}\|g_t(t)\|_{W_p^1(-\infty,\infty)},\right.$$

$$\left.\max_{-\omega\leq t\leq 0}\|g(t)\|_{W_p^2(-\infty,\infty)}\right\},$$

$$\max_{n\omega\leq t\leq(n+1)\omega}|p(t)|,\quad\max_{n\omega\leq t\leq(n+1)\omega}\|u_{tt}\|_{L_p(-\infty,\infty)},\ \max_{n\omega\leq t\leq(n+1)\omega}\|u_t\|_{W_p^1(-\infty,\infty)},$$

$$\max_{n\omega\leq t\leq(n+1)\omega}\|u\|_{W_p^2(-\infty,\infty)}\leq M(q,\alpha)\left[a_n+\max_{(n-1)\omega\leq t\leq n\omega}|p(t)|\right.$$

$$\left.+\max_{n\omega\leq t\leq(n+1)\omega}\left\|f'(t)\right\|_{L_p(-\infty,\infty)}+\|f(n\omega)\|_{L_p(-\infty,\infty)}+\max_{n\omega\leq t\leq(n+1)\omega}\left|\zeta''\right|\right],$$

$$a_n=\max\left\{\max_{(n-1)\omega\leq t\leq n\omega}\|u_{tt}(t)\|_{L_p(-\infty,\infty)},\max_{(n-1)\omega\leq t\leq n\omega}\|u_t(t)\|_{W_p^1(-\infty,\infty)},\right.$$

$$\left.\max_{(n-1)\omega\leq t\leq n\omega}\|u(t)\|_{W_p^2(-\infty,\infty)}\right\}, n=1,2,\cdots.$$

Here $L_p(-\infty,\infty)$ refers to the vector space of functions $w(x)$ from the entire real line to $R=(-\infty,\infty)$ satisfy the condition

$$\int_{-\infty}^{\infty} |w(x)|^p \, dx < \infty.$$

Conclusion

This paper is devoted to the time-dependent identification problems for delay hyperbolic partial differential equations with unknown parameter p(t). The theorems on stability estimates for the solution of this problem are established.

Acknowledgements

The publication was prepared with the support of the "RUDN University Program 5-100" and published under target program BR05236656 of the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan.

References

1 Blasio, G.Di., & Lorenzi, A. (2007). Identification problems for parabolic delay differential equations with measurement on the boundary. *Journal of Inverse and Ill-Posed Problems, 15*(7), 709–734.

2 Orazov, I., & Sadybekov, M.A. (2012). On a class of problems of determining the temperature and density of heat source given initial and final temperature. *Siberian Mathematical Journal, 53*, 146–151.

3 Sadybekov, M.A., Dildabek, G., & Ivanova, M.B. (2018a). On an inverse problem of reconstructing a heat conduction process from nonlocal data. *Advances in Mathematical Physics*, 8301656.

4 Sadybekov, M.A., Oralsyn, G., & Ismailov, M. (2018b). Determination of a time-dependent heat source under not strengthened regular boundary and integral overdetermination conditions. *Filomat, 32*(3), 809–814.

5 Saitoh, S., Tuan, V.K., & Yamamoto, M. (2002). Reverse convolution inequalities and applications to inverse heat source problems. *J. of Inequalities in pure and Applied Mathematics, 5*, 80–91.

6 Sakamoto, K., & Yamamoto, M. (2011). Initial-boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math.Anal.Appl., 382*, 426–447.

7 Kabanikhin, S.I. (2014). Method for solving dynamic inverse problems for hyperbolic equations. *J. Inverse Problems, 12*, 493–517.

8 Samarskii, A.A., & Vabishchevich, P.N. (2007). *Numerical Methods for Solving Inverse Problems of Mathematical Physics.* Inverse and Ill-Posed, Problems Series, Walter de Gruyter, Berlin-New York.

9 Ashyralyev, A., & Agirseven, D. (2014a). On source identification problem for a delay parabolic equation. *Nonlinear Analysis: Modelling and Control, 19*(3), 335–349.

10 Ashyralyev, A., & Ashyralyyev, C. (2014b). On the problem of determining the parameter of an elliptic equation in a Banach space. *Nonlinear Analysis Modelling and Control, 3*, 350–366.

11 Ashyralyev, A., Agirseven, D., & Agarwal, R.P. (2020a). Stability estimates for delay parabolic differential and difference equations. *Appl. Comput. Math., 19*, 175–204.

12 Ashyralyev, A., & Al-Hammouri, A. (2020b). Stability of the space identification problem for the elliptic-telegraph differential equation. *Mathematical Methods in the Applied Sciences, 44*(1), 945–959.

13 Ashyralyev, A., & Emharab, F. (2019). Source identification problems for hyperbolic differential and difference equations. *Journal of Inverse and Ill-posed Problems, 27*(3), 301–315.

14 Emharab, F. (2019). *Source Identification Problems for Hyperbolic Differential and Difference Equations*. PhD Thesis, Near East University, Nicosia, 135 p.

15 Ashyralyev, A., Al-Hammouri, A., & Ashyralyyev, C. (2021). On the absolute stable difference scheme for the space-wise dependent source identification problem for elliptic-telegraph equation. *Numerical Methods for Partial Differential Equations, 37*(2), 962–986.

16 Al-Hammauri, A.M.S. (2020). *The Source Identification Problem for Elliptic-Telegrah Equations*. PhD Thesis, Near East University, Nicosia.

17 Ashyralyev, A., & Urun, M. (2021). Time-dependent source identification Schrodinger type problem. *International Journal of Applied Mathematics, 34*(2), 297–310.

18 Erdogan, A.S. (2010). *Numerical Solution of Parabolic Inverse Problem with an Unknown Source Function*. PhD Thesis, Yəldəz Technical Universty, Istanbul.

19 Ashyralyyev, C. (2017). Stability estimates for solution of Neumann-type overdetermined elliptic problem. *Numerical Functional Analysis and Optimization, 38*(10), 1226–1243.

20 Ashyraliyev, M., Ashyralyyeva, M.A., & Ashyralyev, A. (2020). A note on the hyperbolic-parabolic identification problem with involution and Dirichlet boundary condition. *Bulletin of the Karaganda University-Mathematics, 99*(3), 120–129.

21 Ashyraliyev, M. (2021). On hyperbolic-parabolic problems with involution and Neumann boundary condition. *International Journal of Applied Mathematics, 34*(2), 363–376.

22 Ashyralyev, A., Ashyraliyev, M., & Ashyralyyeva, M.A. (2020). A note on the hyperbolic-parabolic identification problem with involution and Dirichlet boundary condition. *Computational Mathematics and Mathematical Physics, 60*(8), 1294–1305.

23 Ashyralyev, A., & Erdogan, A.S. (2014). Well-posedness of the right-hand side identification problem for a parabolic equation. *Ukrainian Mathematical Journal, 2*, 165–177.

24 Ashyralyev, A., & Urun, M. (2021). On the Crank-Nicholson difference scheme for the time-dependent source identification problem. *Bulletin of the Karaganda University-Mathematics, 99*(2), 35–40.

25 Ashurov, R.R., & Shakarova, M.D. (2022). Time-dependent source identification problem for fractional Shrцdinger type equations. *Labachevskii Journal of Mathematics, 43*, 1053–1064.

26 Al-Mutib, A.N. (1984). Stability properties of numerical methods for solving delay differential equations. *J. Comput. and Appl. Math., 10*(1), 71–79.

27 Ashyralyev, A., & Akca, H. (2001). Stability estimates of difference schemes for neutral delay differential equations. *Nonlinear Analysis: Theory, Methods and Applications, 44*(4), 443–452.

28 Ashyralyev, A., & Sobolevskii, P.E. (2001). On the stability of the delay differential and difference equations. *Abstract and Applied Analysis, 6*(5), 267–297.

29 Torelli, L. (1989). Stability of numerical methods for delay differential equations, *J. Comput. and Appl. Math., 25*, 15–26.

30 Musaev, H. (2021). The Cauchy problem for degenerate parabolic convolution equation. *TWMS J. Pure Appl. Math., 12*, 278–288.

31 Bellen, A., Jackiewicz, Z., & Zennaro, M. (1988). Stability analysis of one-step methods for neutral delay-differential equations. *Numer. Math., 52*(6), 605–619.

32 Yenicerioglu, A.F., & Yalcinbas, S. (2004). On the stability of the second-order delay differential equations with variable coefficients. *Applied Mathematics and Computation, 152*(3), 667–673.

33 Yenicerioglu, A.F. (2008). Stability properties of second order delay integro-differential equations. *Computers and Mathematics with Applications, 56*(12), 309–311.

34 Ashyralyev, A., & Agirseven, D. (2020). On the stable difference scheme for the Schrodinger equation with time delay. *Computational Method in Applied Mathematics, 20*(1), 27–38.

35 Agirseven, D. (2018). On the stability of the Schrodinger equation with time delay. *Filomat, 32*(3), 759–766.

А. Ашыралыев[1,2,3], Б. Хасо[4,5]

[1] *Бахчешехир университеті, Стамбул, Түркия;*
[2] *Ресей халықтар достығы университеті, Мәскеу, Ресей Федерациясы;*
[3] *Математика және математикалық модельдеу институты, Алматы, Қазақстан;*
[4] *Таяу Шығыс университеті, Никосия, Түркия;*
[5] *Математика факультеті, орта мектеп, Духок, Ирак*

Гиперболалық кешігу теңдеулері үшін стационарлы емес сәйкестендіру есебінің тұрақтылығы

Дербес туындылы дифференциалдық және айырымдық теңдеулер үшін уақытқа және кеңістікке тәуелді көзді анықтау есептері қолданбалы ғылымдар мен техникада маңызды орын алады және бірнеше авторлармен зерттелген. Сонымен қатар, кешігу логикалық және есептеуіш құрылғылары бар күрделі жүйелерде туындайды, мұнда ақпаратты өңдеу үшін белгілі бір уақыт қажет. Мақалада кешігуі бар гиперболалық теңдеу үшін стационарлы емес сәйкестендіру есебі зерттелген. Кешігуі бар бірөлшемді гиперболалық дифференциалдық теңдеу үшін стационарлы емес сәйкестендіру есебін шешу үшін орнықтылықты бағалау туралы теоремалар анықталған. Бұл теоремаларды дәлелдеу гиперболалық дифференциалдық теңдеу мен интегралдық теңсіздік үшін Даламбер формуласына негізделген.

Кілт сөздер: гиперболалық теңдеу, кешігу, Гильберт кеңістігі, көзді анықтау, тұрақтылық.

А. Ашыралыев[1,2,3], Б. Хасо[4,5]

[1] *Университет Бахчешехир, Стамбул, Турция;*
[2] *Университет дружбы народов России, Москва, Россия;*
[3] *Институт математики и математического моделирования, Алматы, Казахстан;*
[4] *Ближневосточный университет, Никосия, Турция;*
[5] *Средняя школа, Духок, Ирак*

Устойчивость нестационарной задачи идентификации для гиперболических уравнений с запаздыванием

Зависящие от времени и пространства задачи идентификации источника для дифференциальных и разностных уравнений в частных производных занимают важное место в прикладных науках и технике и изучались несколькими авторами. Кроме того, задержка возникает в сложных системах с логическими и вычислительными устройствами, где требуется определенное время для обработки информации. В настоящей работе исследована нестационарная задача идентификации для гиперболического уравнения с запаздыванием. Установлены теоремы об оценках устойчивости решения нестационарной задачи идентификации для одномерного гиперболического дифференциального уравнения с запаздыванием. Доказательства этих теорем основаны на формуле Даламбера для гиперболического дифференциального уравнения и интегрального неравенства.

Ключевые слова: гиперболическое уравнение, запаздывание, гильбертово пространство, идентификация источника, устойчивость.

DOI 10.31489/2022M3/35-50
UDC 517.946

K.A. Bekmaganbetov[1,2], G.A. Chechkin[2,3,4], A.M. Toleubay[2,5,*]

[1] *M.V. Lomonosov Moscow State University, Kazakhstan Branch, Nur-Sultan, Kazakhstan;*
[2] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[3] *M.V. Lomonosov Moscow State University, Moscow, Russia;*
[4] *Institute of Mathematics with Computing Center Subdivision of the Ufa Federal Research Center of Russian Academy of Science, Ufa, Russia;*
[5] *L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan*
(E-mail: bekmaganbetov-ka@yandex.kz, chechkin@mech.math.msu.su, altyn.15.94@mail.ru)

Attractors of 2D Navier–Stokes system of equations in a locally periodic porous medium

This article deals with two-dimensional Navier–Stokes system of equations with rapidly oscillating terms in the equations and boundary conditions. Studying the problem in a perforated domain, the authors set homogeneous Dirichlet condition on the outer boundary and the Fourier (Robin) condition on the boundary of the cavities. Under such assumptions it is proved that the trajectory attractors of this system converge in some weak topology to trajectory attractors of the homogenized Navier–Stokes system of equations with an additional potential and nontrivial right hand side in the domain without pores. For this aim, the approaches from the works of A.V. Babin, V.V. Chepyzhov, J.-L. Lions, R. Temam, M.I. Vishik concerning trajectory attractors of evolution equations and homogenization methods appeared at the end of the XX-th century are used. First, we apply the asymptotic methods for formal construction of asymptotics, then, we verify the leading terms of asymptotic series by means of the methods of functional analysis and integral estimates. Defining the appropriate axillary functional spaces with weak topology, we derive the limit (homogenized) system of equations and prove the existence of trajectory attractors for this system. Lastly, we formulate the main theorem and prove it through axillary lemmas.

Keywords: attractors, homogenization, system of Navier–Stokes equations, weak convergence, perforated domains, rapidly oscillating terms, porous medium.

Introduction

In this paper, we study the asymptotic behavior of attractors to initial-boundary-value problems for two-dimensional Navier-Stokes systems of equations in perforated domains as the small parameter ε, characterizing the microinhomogeneous structure of the domain, tends to zero.

One can find some results for homogenization problems in perforated domains and a detailed bibliography in monographs [1–3]. This paper presents the case of the appearance of a potential in the limit (homogenized) equation (cf. similar problem in [4–10]).

We study a weak convergence and limit behavior of attractors to the given system of equations as the small parameter converges to zero. There are recent works (cf. [11–13]) on homogenization of attractors used for this study. Overall results on the theory of attractors and the homogenization of attractors cf., for example, in monographs [14–16], and also see the bibliography in these monographs.

We prove that the trajectory attractors $\mathfrak{A}_\varepsilon$ of the two-dimensional Navier–Stokes system of equations in (cf. also [17–19]) a perforated domain weakly converge as $\varepsilon \to 0$ to the trajectory attractor $\overline{\mathfrak{A}}$ to the homogenized system of equations in the corresponding function space. The small parameter ε characterizes the cavity diameter, as well as the distance between cavities in the perforated medium.

In Section 1, we define main notions and formulate theorems on trajectory attractors of autonomous evolution equations. In Section 2, we describe the geometric structure of a perforated domain, formulate

*Corresponding author.
E-mail: altyn.15.94@mail.ru

the problem under consideration, and introduce some function spaces. Section 3 is devoted to homogenization of attractors to the autonomous two-dimensional system of Navier–Stokes equations with rapidly oscillating terms in a perforated domain.

1 Trajectory Attractors of Evolution Equations

We describe a general scheme of constructing trajectory attractors of autonomous evolution equations. This scheme is used in Section 2 to study trajectory attractors of a two-dimensional system of Navier–Stokes equations in a perforated domain with rapidly oscillating terms in equations and boundary conditions and the corresponding homogenized equation.

We consider the abstract autonomous evolution equations

$$\frac{\partial u}{\partial t} = A(u), \quad t \geq 0, \tag{1}$$

where $A(\cdot) : E_1 \to E_0$ is a given nonlinear operator, E_1 and E_0 are Banach spaces such that $E_1 \subseteq E_0$. For example, $A(u) = \nu \Delta u - (u, \nabla u) + g(\cdot)$ (cf. Section 2).

We study a solution $u(s)$ to equation (1) globally, as a function of variable $s \in \mathbb{R}_+$. Here, $s \equiv t$ denotes the time-variable. The set of solutions to equation (1) is called the *trajectory space* of equation (1) and is denoted by $\mathcal{K}^+$. We describe the trajectory space $\mathcal{K}^+$ in detail.

First of all, we consider the solution $u(s)$ to equation (1), defined on a fixed time-segment $[t_1, t_2]$ in $\mathbb{R}$. We study solutions to equation (1) in the Banach space $\mathcal{F}_{t_1,t_2}$, which depends on t_1 and t_2. The space $\mathcal{F}_{t_1,t_2}$ consists of functions, $f(s), s \in [t_1, t_2]$, such that $f(s) \in E$ for almost all $s \in [t_1, t_2]$, where E is a Banach space. It is assumed that $E_1 \subseteq E \subseteq E_0$.

For example, for $\mathcal{F}_{t_1,t_2}$ we can take the space $C([t_1, t_2]; E)$ the space $L_p(t_1, t_2; E)$, or $p \in [1, \infty]$, or the intersection of such spaces (cf. section 2). We assume that $\Pi_{t_1,t_2} \mathcal{F}_{\tau_1,\tau_2} \subseteq \mathcal{F}_{t_1,t_2}$ and

$$\|\Pi_{t_1,t_2} f\|_{\mathcal{F}_{t_1,t_2}} \leq C(t_1, t_2, \tau_1, \tau_2) \|f\|_{\mathcal{F}_{\tau_1,\tau_2}}, \quad \forall f \in \mathcal{F}_{\tau_1,\tau_2}, \tag{2}$$

where $[t_1, t_2] \subseteq [\tau_1, \tau_2]$ and Ψ_{t_1,t_2} is the restriction operator on $[t_1, t_2]$. Constant $C(t_1, t_2, \tau_1, \tau_2)$ is independent of f. Usually, one consider the homogeneous case of the space where $C(t_1, t_2, \tau_1, \tau_2) = C(t_2 - t_1, \tau_2 - \tau_1)$.

Let $S(h)$ for $h \in \mathbb{R}$ denote the translation operator

$$S(h) f(s) = f(h + s).$$

It is obvious that if the variable s of $f(\cdot)$ belongs to $[t_1, t_2]$, then the variable s of $S(h) f(\cdot)$ belongs to $[t_1 - h, t_2 - h]$ for $h \in \mathbb{R}$. We assume that the mapping $S(h)$ is an isomorphism from F_{t_1,t_2} to F_{t_1-h,t_2-h} and

$$\|S(h) f\|_{\mathcal{F}_{t_1-h,t_2-h}} = \|f\|_{\mathcal{F}t_1,t_2}, \quad \forall f \in \mathcal{F}_{t_1,t_2}. \tag{3}$$

This assumption is natural, for example, for the homogeneous space.

We assume that if $f(s) \in \mathcal{F}_{t_1,t_2}$, then $A(f(s)) \in \mathcal{D}_{t_1,t_2}$, where $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$. The derivative $\frac{\partial f(t)}{\partial t}$ is a generalized function taking the values in E_0, $\frac{\partial f}{\partial t} \in D'((t_1, t_2); E_0)$ We assume that $\mathcal{D}_{t_1,t_2} \subseteq D'((t_1, t_2); E_0)$ for all $(t_1, t_2) \subset \mathbb{R}$. A function $u(s) \in \mathcal{F}_{t_1,t_2}$ is called a *solution* to equation (1) in the space $\mathcal{F}_{t_1,t_2}$ (on the interval (t_1, t_2)) if $\frac{\partial u}{\partial t}(s) = A(u(s))$ if in the sense of distributions in $D'((t_1, t_2); E_0)$.

We also introduce the space

$$\mathcal{F}_+^{loc} = \{f(s),\, s \in \mathbb{R}_+ \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2}, \quad \forall\, [t_1, t_2] \subset \mathbb{R}_+\}. \tag{4}$$

For example, $\mathcal{F}_{t_1,t_2} = C([t_1, t_2]; E)$ implies $\mathcal{F}_+^{loc} = C(\mathbb{R}_+; E)$, and $\mathcal{F}_{t_1,t_2} = L_p(t_1, t_2; E)$, implies $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+; E)$.

A function $u(s) \in \mathcal{F}_+^{loc}$ is a solution to equation (1) in $\mathcal{F}_+^{loc}$, if $\Pi_{t_1,t_2}u(s) \in \mathcal{F}_{t_1,t_2}$, and the function $\Pi_{t_1,t_2}u(s)$ is a solution to equation (1) for any time-segment $[t_1, t_2] \subset \mathbb{R}_+$.

Let $\mathcal{K}^+$ be a set of solutions to equation (1), on the space $\mathcal{F}_+^{loc}$, but does not necessarily coincides with the set of *all* solutions to equation (1) in $\mathcal{F}_+^{loc}$. Elements of $\mathcal{K}^+$ are called *trajectories*, and $\mathcal{K}^+$ is said to be the *the trajectory space* of equation (1).

We assume that the trajectory space $\mathcal{K}^+$ is *translation invariant* in the following sense: if $u(s) \in \mathcal{K}^+$, then $u(h+s) \in \mathcal{K}^+$ for any $h \geq 0$. This condition is natural for solutions to autonomous equations in homogeneous spaces. We consider the translation operators $S(h)$ in $\mathcal{F}_+^{loc}$:

$$S(h)f(s) = f(s+h), \quad h \geq 0.$$

It is clear that $\{S(h), h \geq 0\}$ is a semigroup in $\mathcal{F}_+^{loc}$: $S(h_1)S(h_2) = S(h_1+h_2)$ for $h_1, h_2 \geq 0$ and $S(0) = I$ is the identity mapping. We replace the variable h with the time-variable t. The semigroup $\{S(t), t \geq 0\}$ is called the *translation semigroup*. By assumption, the translation semigroup maps the trajectory space $\mathcal{K}^+$ onto itself:

$$S(t)\mathcal{K}^+ \subseteq \mathcal{K}^+ \quad \forall\, t \geq 0. \tag{5}$$

In what follows, we study the attraction property of the translation semigroup $\{S(t)\}$, acting on the trajectory space $\mathcal{K}^+ \subset \mathcal{F}_+^{loc}$. We introduce a topology in $\mathcal{F}_+^{loc}$.

Let $\rho_{t_1,t_2}(\cdot,\cdot)$ be a group defined on the space $\mathcal{F}_{t_1,t_2}$ for all segments $[t_1,t_2] \subset \mathbb{R}$. As in (2) and (3) we assume that

$$\rho_{t_1,t_2}(\Pi_{t_1,t_2}f, \Pi_{t_1,t_2}g) \leq D(t_1,t_2,\tau_1,\tau_2)\rho_{\tau_1,\tau_2}(f,g), \quad \forall f,g \in \mathcal{F}_{\tau_1,\tau_2},\ [t_1,t_2] \subseteq [\tau_1,\tau_2],$$
$$\rho_{t_1-h,t_2-h}(S(h)f, S(h)g) = \rho_{t_1,t_2}(f,g), \quad \forall f,g \in \mathcal{F}_{t_1,t_2},\ [t_1,t_2] \subset \mathbb{R},\ h \in \mathbb{R}.$$

(For a homogeneous space $D(t_1,t_2,\tau_1,\tau_2) = D(t_2-t_2, \tau_2-\tau_1)$.)

We denote by Θ_{t_1,t_2} the corresponding metric space on $\mathcal{F}_{t_1,t_2}$. For example, ρ_{t_1,t_2} can be the metric generated by the norm $\|\cdot\|_{\mathcal{F}_{t_1,t_2}}$ in the Banach space $\mathcal{F}_{t_1,t_2}$. In applications, it can happen that the metric ρ_{t_1,t_2} generates a weaker topology in Θ_{t_1,t_2} than the strong convergence topology in the Banach space $\mathcal{F}_{t_1,t_2}$.

We denote by Θ_+^{loc} the space $\mathcal{F}_+^{loc}$, equipped with the local convergence topology in Θ_{t_1,t_2} for any $[t_1,t_2] \subset \mathbb{R}_+$. More exactly, by definition, a sequence of functions $\{f_k(s)\} \subset \mathcal{F}_+^{loc}$ converges to a function $f(s) \in \mathcal{F}_+^{loc}$ in $k \to \infty$ as Θ_+^{loc}, if $\rho_{t_1,t_2}(\Pi_{t_1,t_2}f_k, \Pi_{t_1,t_2}f) \to 0$ as $k \to \infty$ for any $[t_1,t_2] \subset \mathbb{R}_+$. It is easy to prove that the topology in Θ_+^{loc} in metrizable by using the Frechet metric

$$\rho_+(f_1,f_2) := \sum_{m\in\mathbb{N}} 2^{-m} \frac{\rho_{0,m}(f_1,f_2)}{1+\rho_{0,m}(f_1,f_2)}. \tag{6}$$

If all metric spaces Θ_{t_1,t_2} are complete, then the metric space Θ_+^{loc} is also complete.

We note that the translation semigroup $\{S(t)\}$ is continuous in the topology of the space Θ_+^{loc}. This fact directly follows from the definition of the topological space Θ_+^{loc}.

We define the Banach space

$$\mathcal{F}_+^b := \{f(s) \in \mathcal{F}_+^{loc} \mid \|f\|_{\mathcal{F}_+^b} < +\infty\}, \tag{7}$$

equipped with the norm

$$\|f\|_{\mathcal{F}_+^b} := \sup_{h\geq 0} \|\Pi_{0,1}f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{8}$$

For example, if $\mathcal{F}_+^{loc} = C(\mathbb{R}_+;E)$, then $\mathcal{F}_+^b = C^b(\mathbb{R}_+;E)$ is equipped with the norm $\|f\|_{\mathcal{F}_+^b} = \sup_{h\geq 0}\|f(h)\|_E$, and if $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+;E)$, then $\mathcal{F}_+^b = L_p^b(\mathbb{R}_+;E)$ is equipped with the norm $\|f\|_{\mathcal{F}_+^b} = \left(\sup_{h\geq 0} \int\limits_h^{h+1} \|f(s)\|_E^p ds\right)^{1/p}$.

We note that $\mathcal{F}_+^b \subseteq \Theta_+^{loc}$. The Banach space $\mathcal{F}_+^b$ is necessary to introduce bounded sets in the trajectory space $\mathcal{K}^+$. To construct a trajectory attractor in $\mathcal{K}^+$, we use the weaker local convergence topology in Θ_+^{loc}, instead of the uniform convergences in the topology of the space $\mathcal{F}_+^b$.

We assume that $\mathcal{K}^+ \subseteq \mathcal{F}_+^b$, i.e., any trajectory $u(s) \in \mathcal{K}^+$ of equation (1) has finite norm (8). We define an attracting set and a trajectory attractor of the translation semigroup $\{S(t)\}$, acting on $\mathcal{K}^+$.

Definition 1.1. A set $\mathcal{P} \subseteq \Theta_+^{loc}$ is called an *attracting set* set of the translation semigroup $\{S(t)\}$, acting on $\mathcal{K}^+$, in the topology of Θ_+^{loc}, if for any bounded set $\mathcal{F}_+^b$ in $\mathcal{B} \subseteq \mathcal{K}^+$ the set $\mathcal{P}$ attracts $S(t)\mathcal{B}$ as $t \to +\infty$ in the topology of Θ_+^{loc},i.e., for any ε-neighborhood $O_\varepsilon(\mathcal{P})$ in Θ_+^{loc} there exists $t_1 \geq 0$ such that $S(t)\mathcal{B} \subseteq O_\varepsilon(\mathcal{P})$ for any $t \geq t_1$. The attraction property of $\mathcal{P}$ can be formulated in the equivalent form: for any bounded set $\mathcal{B} \subseteq \mathcal{K}^+$ in $\mathcal{F}_+^b$ and any $M > 0$

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathcal{P}) \to 0 \quad (t \to +\infty),$$

where

$$\mathrm{dist}_{\mathcal{M}}(X, Y) := \sup_{x\in X} \mathrm{dist}_{\mathcal{M}}(x, Y) = \sup_{x\in X} \inf_{y\in Y} \rho_{\mathcal{M}}(x, y)$$

where denotes the Hausdorff semi-distance between sets X and Y in the metric space $\mathcal{M}$.

Definition 1.2.([15]) A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called a *trajectory attractor* of the translation semigroup $\{S(t)\}$ on $\mathcal{K}^+$ in the topology of Θ_+^{loc} if the following conditions are satisfied: **(i)** $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}, **(ii)** $\mathfrak{A}$ is strictly invariant under the translation semigroup: $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$, and **(iii)** $\mathfrak{A}$ is an attracting set of the translation semigroup in the topology of $\{S(t)\}$ for $\mathcal{K}^+$ in the topology of Θ_+^{loc}, i.e., for any $M > 0$

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathfrak{A}) \to 0 \quad (t \to +\infty).$$

Remark 1.1. Using the terminology of [14], we can say that a trajectory attractor $\mathfrak{A}$ is *global* $(\mathcal{F}_+^b, \Theta_+^{loc})$-*attractor* of the translation semigroup $\{S(t)\}$, acting on $\mathcal{K}^+$, i.e., $\mathfrak{A}$ attracts $S(t)\,\mathcal{B}$ as $t \to +\infty$ in the topology of Θ_+^{loc}, where $\mathcal{B}$ is any bounded (in $\mathcal{F}_+^b$) set in $\mathcal{K}^+$:

$$\mathrm{dist}_{\Theta_+^{loc}}(S(t)\,\mathcal{B}, \mathfrak{A}) \to 0 \quad (t \to +\infty).$$

We formulate the main result concerning the existence and structure of a trajectory attractor of equation (1).

Theorem 1.1.([14, 15, 20]) Let the trajectory space $\mathcal{K}^+$, corresponding to equation (1), be closed in $\mathcal{F}_+^b$ and satisfy the condition (5). Let the translation semigroup $\{S(t)\}$ have an attracting set $\mathcal{P} \subseteq \mathcal{K}^+$, that is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Then the translation semigroup $\{S(t), t \geq 0\}$, acting on $\mathcal{K}^+$, has a trajectory attractor $\mathfrak{A} \subseteq \mathcal{P}$. The set $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. We describe the structure of trajectory attractors $\mathfrak{A}$ of equation (1) in terms of complete trajectories of this equation. We consider equation (1) on the whole time-axis

$$\frac{\partial u}{\partial t} = A(u),\ t \in \mathbb{R}. \tag{9}$$

Now, we extend the notion of the trajectory space $\mathcal{K}^+$ of equation (9) introduced on $\mathbb{R}_+$. To the case of the whole axis $\mathbb{R}$. If a function $f(s),\ s \in \mathbb{R}$, is given on the whole time-axis, then the translations $S(h)f(s) = f(s+h)$ are also defined for negative h. A function $u(s), s \in \mathbb{R}$ is called a *complete trajectory* of equation (9), if $\Pi_+ u(s+h) \in \mathcal{K}^+$ for any $h \in \mathbb{R}$. Here, $\Pi_+ = \Pi_{0,\infty}$ denotes the operator of restriction onto the half-axis $\mathbb{R}_+$.

We introduced the spaces $\mathcal{F}_+^{loc}, \mathcal{F}_+^b$ and Θ_+^{loc}. Now, we can introduce the space $\mathcal{F}^{loc}, \mathcal{F}^b$ and Θ^{loc} as follows:

$$\mathcal{F}^{loc} := \{f(s), s \in \mathbb{R} \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2}\ \forall\ [t_1, t_2] \subseteq \mathbb{R}\};$$
$$\mathcal{F}^b := \{f(s) \in \mathcal{F}^{loc} \mid \|f\|_{\mathcal{F}^b} < +\infty\},$$

where

$$\|f\|_{\mathcal{F}^b} := \sup_{h\in\mathbb{R}} \|\Pi_{0,1} f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{10}$$

The topological space Θ^{loc} coincides (as a set) with $\mathcal{F}^{loc}$, and by definition $f_k(s) \to f(s)\ as(k\to\infty)$ in Θ^{loc}, if $\Pi_{t_1,t_2} f_k(s) \to \Pi_{t_1,t_2} f(s)\ as(k\to\infty)$ in Θ_{t_1,t_2} for any $[t_1,t_2]\subseteq\mathbb{R}$. It is clear that Θ^{loc} is a metric space, as well as Θ^{loc}_+.

Definition 1.3. The *kernel* $\mathcal{K}$ in the space $\mathcal{F}^b$ of equation (9) is the union of all complete trajectories $u(s), s\in\mathbb{R}$, of equation (9), that are bounded in $\mathcal{F}^b$ in the norm (10):

$$\|\Pi_{0,1} u(h+s)\|_{\mathcal{F}_{0,1}} \le C_u, \quad \forall h\in\mathbb{R}.$$

Theorem 1.2. Let the assumptions of Theorem 1.1 hold.Then

$$\mathfrak{A} = \Pi_+\mathcal{K}.$$

The set $\mathcal{K}$ is compact in Θ^{loc} and bounded in $\mathcal{F}^b$.

The full proof is given in [15, 20]. To prove that some ball in $\mathcal{F}^b_+$ is compact in Θ^{loc}_+ we use the following lemma. Let E_0 and E_1 be the Banach spaces such that $E_1\subset E_0$. We consider the Banach spaces

$$W_{p_1,p_0}(0,M;E_1,E_0) = \left\{\psi(s), s\in 0,M \mid \psi(\cdot)\in L_{p_1}(0,M;E_1),\ \psi'(\cdot)\in L_{p_0}(0,M;E_0)\right\},$$
$$W_{\infty,p_0}(0,M;E_1,E_0) = \left\{\psi(s), s\in 0,M \mid \psi(\cdot)\in L_{\infty}(0,M;E_1),\ \psi'(\cdot)\in L_{p_0}(0,M;E_0)\right\},$$

(where $p_1\ge 1$ e $p_0>1$) with the norms

$$\|\psi\|_{W_{p_1,p_0}} := \left(\int_0^M \|\psi(s)\|^{p_1}_{E_1} ds\right)^{1/p_1} + \left(\int_0^M \|\psi'(s)\|^{p_0}_{E_0} ds\right)^{1/p_0},$$

$$\|\psi\|_{W_{\infty,p_0}} := \operatorname{ess\,sup}\{\|\psi(s)\|_{E_1} \mid s\in[0,M]\} + \left(\int_0^M \|\psi'(s)\|^{p_0}_{E_0} ds\right)^{1/p_0}.$$

Lemma 1.1.(Aubin-Lions-Simon, [21]) Let $E_1 \Subset E \subset E_0$. Then the following embeddings are compact:

$$W_{p_1,p_0}(0,T;E_1,E_0) \Subset L_{p_1}(0,T;E), \qquad W_{\infty,p_0}(0,T;E_1,E_0) \Subset C([0,T];E).$$

In the next section, two-dimensional systems of Navier-Stokes equations and their trajectory attractors depending on a small parameter $\varepsilon>0$ will be studied.

Definition 1.4. We say that trajectory attractors $\mathfrak{A}_\varepsilon$ *converge* to a trajectory attractor $\overline{\mathfrak{A}}$ as $\varepsilon\to 0$ in the topological space Θ^{loc}_+, if for any neighborhood $\mathcal{O}(\mathfrak{A})$ in Θ^{loc}_+ there is $\varepsilon_1\ge 0$ such that $\mathfrak{A}_\varepsilon\subseteq\mathcal{O}(\overline{\mathfrak{A}})$ for any $\varepsilon<\varepsilon_1$, i.e., for any $M>0$

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0\ (\varepsilon\to 0).$$

2 Notation and Setting of the Problem

First, we define a perforated domain. Let Ω be a smooth bounded domain $\mathbb{R}^2$. Denote

$$\Upsilon_\varepsilon = \left\{j\in\mathbb{Z}^2 : \operatorname{dist}(\varepsilon j, \partial\Omega) \ge \sqrt{2}\varepsilon\right\}, \quad \square \equiv \left\{\xi : -\frac{1}{2} < \xi_k < \frac{1}{2}, k=1,2\right\}.$$

Given an 1-periodic in ξ smooth function $F(x,\xi)$ such that $F(x,\xi)|_{\xi\in\partial\Box} \geq \text{const} > 0$, $F(x,0) = -1$, $\nabla_\xi F \neq 0$ as $\xi \in \Box \setminus \{0\}$, we set

$$G_j^\varepsilon = \left\{x \in \varepsilon\,(\Box + j) \mid F\left(x, \frac{x}{\varepsilon}\right) \leq 0\right\}, \qquad G_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon} G_j^\varepsilon$$

and introduce the perforated domain as follows:

$$\Omega^\varepsilon = \Omega \setminus G_\varepsilon.$$

Denote by $G(x)$ the domain $G(x,\xi)$ in a stretched space ξ. Afterwards, we often interprete 1-periodic in ξ functions as functions defined on 2-dimensional torus $\mathbb{T}^2 \equiv \{\xi \,:\, \xi \in \mathbb{R}^2/\mathbb{Z}^2\}$. According to the above construction, the boundary $\partial\Omega_\varepsilon$ consists of $\partial\Omega$ and the boundary of the cavities $\partial G_\varepsilon \subset \Omega$.

We introduce the function spaces:

$\mathbf{H} := [L_2(\Omega)]^2$, $\mathbf{H}_\varepsilon := [L_2(\Omega_\varepsilon)]^2$, $\mathbf{V} := [H_0^1(\Omega)]^2$, $\mathbf{V}_\varepsilon := [H^1(\Omega_\varepsilon;\partial\Omega)]^2$ is the set of vector-valued functions in $[H^1(\Omega_\varepsilon)]^2$ with zero trace on $\partial\Omega$. The norms in these spaces are defined by

$$\|v\|^2 := \int_\Omega \sum_{i=1}^2 |v^i(x)|^2 dx, \; \|v\|_\varepsilon^2 := \int_{\Omega_\varepsilon} \sum_{i=1}^2 |v^i(x)|^2 dx,$$
$$\|v\|_1^2 := \int_\Omega \sum_{i=1}^2 |\nabla v^i(x)|^2 dx, \; \|v\|_{1\varepsilon}^2 := \int_{\Omega_\varepsilon} \sum_{i=1}^2 |\nabla v^i(x)|^2 dx.$$

We study the asymptotic behavior of trajectory attractors of the following initial-boundary-value problem for the autonomous two-dimensional system of Navier–Stokes equations:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \nu\Delta u_\varepsilon + (u_\varepsilon,\nabla)u_\varepsilon = g\big(x, \dfrac{x}{\varepsilon}\big), & x \in \Omega_\varepsilon, \\ (\nabla, u_\varepsilon) = 0, & x \in \Omega_\varepsilon, \\ \nu\dfrac{\partial u_\varepsilon}{\partial n} + B\big(x, \dfrac{x}{\varepsilon}\big)u_\varepsilon = h\big(x, \dfrac{x}{\varepsilon}\big), & x \in \partial G_\varepsilon, t \in (0,+\infty), \\ u_\varepsilon = 0, & x \in \partial\Omega \\ u_\varepsilon = U(x), & x \in \Omega_\varepsilon, t = 0. \end{cases} \tag{11}$$

Here $u_\varepsilon = u_\varepsilon(x,t) = (u_\varepsilon^1, u_\varepsilon^2)$, $g_\varepsilon(x) = g\left(x, \frac{x}{\varepsilon}\right) = (g^1, g^2) \in \mathbf{H}$, $h_\varepsilon(x) = h\left(x, \frac{x}{\varepsilon}\right) = (h^1, h^2) \in \mathbf{H}$, n is the outward normal vector to the boundary, and $\nu > 0$.

Further,

$$B(x,\xi) = \begin{pmatrix} b^1(x,\xi) & 0 \\ 0 & b^2(x,\xi) \end{pmatrix},$$

functions $b^k(x,\xi) \in C(\Omega\times\mathbb{R}^2)$ such that $b^k(x,\xi)$ is 1-periodic by variable ξ functions on $\Omega\times\mathbb{R}^2$ and satisfy the condition

$$\int\limits_{\partial G(x)} b^k(x,\xi)\, d\sigma = 0, \qquad k = 1, 2,$$

here, $d\sigma$ is the length element of the curve $\partial G(x)$.

Similarly, vector–function components $h(x,\xi)$ satisfy the conditions: $h^k(x,\xi) \in C(\Omega\times\mathbb{R}^2)$, $h^k(x,\xi)$ is 1-periodic by variable ξ functions on $\Omega\times\mathbb{R}^2$ and

$$\int\limits_{\partial G(x)} h^k(x,\xi)\, d\sigma = 0, \qquad k = 1, 2.$$

For $U \in \mathbf{H}$, there exists a weak solution $u(s)$ to the problem (11) in the space $\mathbf{L}_{2,w}^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_{\infty,*w}^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon)$, such that $u(0) = U$. Moreover $\dfrac{\partial u_\varepsilon}{\partial t} \in \mathbf{L}_{2,w}^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon)$. We consider weak solutions to the problem, i.e., [20], [22].

This satisfies the problem (11) in the sense of distributions, i.e.,

$$u_\varepsilon(x,s) \in \mathbf{L}_{2,w}^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_{\infty,*w}^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \left\{ v : \frac{\partial u_\varepsilon}{\partial t} \in \mathbf{L}_{2,w}^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon) \right\}$$

that satisfy the problem (11) in the sense of distributions, i.e.,

$$\int\limits_{Q_\varepsilon} \frac{\partial u_\varepsilon}{\partial t} \cdot \psi \, dxdt + \nu \int\limits_{Q_\varepsilon} \nabla u_\varepsilon \cdot \nabla \psi \, dxdt + \int\limits_{Q_\varepsilon} (u_\varepsilon, \nabla) u_\varepsilon \cdot \psi \, dxdt +$$

$$+ \sum_{j \in \Upsilon_\varepsilon} \int\limits_0^{+\infty} \int\limits_{\partial G_\varepsilon^j} B\big(x, \frac{x}{\varepsilon}\big) u_\varepsilon \cdot \psi \, d\sigma dt = \int\limits_{Q_\varepsilon} g_\varepsilon(x) \cdot \psi \, dxdt + \sum_{j \in \Upsilon_\varepsilon} \int\limits_0^{+\infty} \int\limits_{\partial G_\varepsilon^j} h_\varepsilon(x) \cdot \psi \, d\sigma dt$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+;\mathbf{H}_\varepsilon)$. Here $y_1 \cdot y_2$ denotes the inner product vectors $y_1, y_2 \in \mathbb{R}^2$.

To describle the trajectory space $\mathcal{K}_\varepsilon^+$ of the problem (11), we follow the general scheme of and on every segment, introduce the Banach space $[t_1, t_2] \in \mathbb{R}$

$$\mathcal{F}_{t_1,t_2} := \mathbf{L}_{2,w}^{loc}(t_1,t_2;\mathbf{V}_\varepsilon) \cap \mathbf{L}_{\infty,*w}^{loc}(t_1,t_2;\mathbf{H}_\varepsilon) \cap \left\{ v : \frac{\partial u_\varepsilon}{\partial t} \in \mathbf{L}_{2,w}^{loc}(t_1,t_2;\mathbf{H}_\varepsilon) \right\}$$

equipped with the norm

$$\|v\|_{\mathcal{F}_{t_1,t_2}} := \|v\|_{\mathbf{L}_2(t_1,t_2;\mathbf{V})} + \|v\|_{\mathbf{L}_\infty(t_1,t_2;\mathbf{H})} + \left\| \frac{\partial v}{\partial t} \right\|_{\mathbf{L}_2(t_1,t_2;\mathbf{H})}. \tag{12}$$

It is obvious that the condition (2) holds for the norm (12) and the translation semigroup $\{S(h)\}$ satisfies (3).

Setting $\mathcal{D}_{t_1,t_2} = \mathbf{L_2}(t_1,t_2;\mathbf{V})$ we find that $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$, if $u(s) \in \mathcal{F}_{t_1,t_2}$, then $A(u(s)) \in \mathcal{D}_{t_1,t_2}$. Further, we can consider a weak solution to the problem (11) as a solution to the system of equations in accordance with the general scheme .

Introducing the space (4), we find

$$\mathcal{F}_+^{loc} = \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}) \cap \mathbf{L}_\infty^{loc}(\mathbb{R}_+;\mathbf{H}) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{H}) \right\},$$

$$\mathcal{F}_{\varepsilon,+}^{loc} = \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon) \right\}.$$

We denote by $\mathcal{K}_\varepsilon^+$ a set of all weak solutions to the problem (11). We recall that for any function $U \in \mathbf{H}$ there exists at least one trajectory $u(\cdot) \in \mathcal{K}_\varepsilon^+$ such that $u(0) = U(x)$. Consequently, the trajectory space $\mathcal{K}_\varepsilon^+$ of the problem (11) is not empty.

It is clear that $\mathcal{K}_\varepsilon^+ \subset \mathcal{F}_+^{loc}$ and the trajectory space $\mathcal{K}_\varepsilon^+$ is translation invariant, i.e., if $u(s) \in \mathcal{K}_\varepsilon^+$, then and $u(h+s) \in \mathcal{K}_\varepsilon^+$ for any $h \geq 0$. Consequently,

$$S(h)\mathcal{K}_\varepsilon^+ \subseteq \mathcal{K}_\varepsilon^+, \quad \forall h \geq 0.$$

Further, using the $\mathbf{L}_2(t_1,t_2;\mathbf{V})$–norms, we introduce the metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ in the spaces $\mathcal{F}_{t_1,t_2}$ as follows:

$$\rho_{0,M}(u,v) = \left(\int_0^M \|u(s) - v(s)\|^2 ds \right)^{1/2}, \quad \forall\, u(\cdot), v(\cdot) \in \mathcal{F}_{0,M}.$$

These metrics generate the topology of Θ_+^{loc} in the space $\mathcal{F}_+^{loc}$ (respectively $\Theta_{\varepsilon,+}^{loc}$ in $\mathcal{F}_{\varepsilon,+}^{loc}$). We recall that a sequence $\{v_k\} \subset \mathcal{F}_+^{loc}$ converges to a function $v \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc}, if $\|v_k(\cdot) - v(\cdot)\|_{\mathbf{L}_2(0,M;\mathbf{H})} \to 0$ $(k \to \infty)$ for any $M > 0$. The topology of Θ_+^{loc} is metrizable (nf. (6)) and the corresponding metric space is complete. We consider the topology in the trajectory space $\mathcal{K}_\varepsilon^+$ of the problem (11). The translation semigroup $\{S(t)\}$, acting on $\mathcal{K}_\varepsilon^+$ is continuous in the topology of the space Θ_+^{loc}.

Following the general scheme of 1, we consider the bounded set in $\mathcal{K}_\varepsilon^+$ by using the Banach space $\mathcal{F}_+^b$ (cf. (7)). It is clear that

$$\mathcal{F}_+^b = \mathbf{L}_2^b(\mathbb{R}_+;\mathbf{V}) \cap \mathbf{L}_\infty(\mathbb{R}_+;\mathbf{H}) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L}_2^b(\mathbb{R}_+;\mathbf{H}) \right\}$$

and $\mathcal{F}_+^b$ is a subspace of the space $\mathcal{F}_+^{loc}$.

We consider the translation semigroup $\{S(t)\}$ on $\mathcal{K}_\varepsilon^+$, $S(t) : \mathcal{K}_\varepsilon^+ \to \mathcal{K}_\varepsilon^+$, $t \geq 0$.

Let $\mathcal{K}_\varepsilon$ denote the kernel of the problem (11), consisting of all weak solutions $u(s), s \in \mathbb{R}$ bounded in the space

$$\mathcal{F}^b = \mathbf{L}_2^b(\mathbb{R};\mathbf{V}) \cap \mathbf{L}_\infty(\mathbb{R};\mathbf{H}) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L}_2^b(\mathbb{R};\mathbf{H}) \right\}$$

Proposition 2.1. The problem (11) has trajectory attractors $\mathfrak{A}_\varepsilon$ in the topological space Θ_+^{loc}. The set $\mathfrak{A}_\varepsilon$ is uniformly (with respect to $\varepsilon \in (0,1)$) bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Furthermore,

$$\mathfrak{A}_\varepsilon = \Pi_+ \mathcal{K}_\varepsilon,$$

the kernel $\mathcal{K}_\varepsilon$ is nonempty and uniformly (with respect to $\varepsilon \in (0,1)$) bounded in $\mathcal{F}^b$. We recall that the spaces $\mathcal{F}_+^b$ and Θ_+^{loc} depend on ε.

The proof of Proposition 2.1 is similar to the proof in [15] given in a particular case.

3 Homogenization of attractors of initial boundary value problem for the Navier-Stokes system of equations in a perfected domain

3.1 The main assertion

In this subsection, we study the limit behavior of attractors $\mathfrak{A}_\varepsilon$ of the Navier-Stokes equations (11) as $\varepsilon \to 0+$ as and their convergence to a trajectory attractor of the corresponding homogenized equation.

The homogenized (limit) problem has the form:

$$\begin{cases} \dfrac{\partial u_0}{\partial t} - \nu \displaystyle\sum_{i,l=1}^{2} \frac{\partial}{\partial x_i}\left(\widehat{a}_{il}(x)\frac{\partial u_0}{\partial x_l}\right) + (u_0,\nabla)u_0 + V(x)u_0 = \bar{g}(x) + H(x), & x \in \Omega, \\ (\nabla, u_0) = 0, & x \in \Omega, \\ u_0 = 0, & x \in \partial\Omega \\ u_0 = U(x), & x \in \Omega, t = 0, \end{cases} \tag{13}$$

where

$$\widehat{a}_{il}(x) = \int\limits_{Y\setminus G(x)} \left(\frac{\partial N_l(x,\xi)}{\partial \xi_i} + \delta_{il}\right) d\xi, \qquad \bar{g}(x) = \int\limits_{Y\setminus G(x)} g(x,\xi)\, d\xi,$$

$$m_k(x) = -\int\limits_{\partial G(x)} b^k(x,\xi) M^k(x,\xi)\, d\sigma, \qquad V(x) = \begin{pmatrix} m_1(x) & 0 \\ 0 & m_2(x) \end{pmatrix},$$

$$H_k(x) = -\int\limits_{\partial G(x)} h^k(x,\xi) M^k(x,\xi)\, d\sigma, \qquad H(x) = \begin{pmatrix} H_1(x) \\ H_2(x) \end{pmatrix}.$$

Here $M^k(\xi)$ and $N_l(\xi)$ are 1-periodic functions of ξ satisfying the problems

$$\Delta M^k = 0 \text{ in } Y \setminus G(x), \qquad \frac{\partial M^k}{\partial n} = -b^k(x,\xi) \text{ on } \partial G(x),$$

$$\Delta N_l = 0 \text{ in } Y \setminus G(x), \qquad \frac{\partial N_l}{\partial n} = -n_l \text{ on } \partial G(x)$$

and having zero mean over the periodicity cell.

We consider the weak solution to the problem (13), i.e., a function

$$u_0(x,s) \in \mathbf{L}_{2,w}^{loc}(\mathbb{R}_+;\mathbf{V}) \cap \mathbf{L}_{\infty,*w}^{loc}(\mathbb{R}_+;\mathbf{H}) \cap \left\{ v : \frac{\partial u_\varepsilon}{\partial t} \in \mathbf{L}_{2,w}^{loc}(\mathbb{R}_+;\mathbf{H}) \right\}$$

satisfying the problem (11) in the sense of distributions,i.e.,

$$\int\limits_Q \frac{\partial u_0}{\partial t} \cdot \psi\; dxdt + \nu \int\limits_Q \sum_{i,l=1}^{2} \widehat{a}_{il}(x) \frac{\partial u_0}{\partial x_i} \cdot \frac{\partial \psi}{\partial x_l}\; dxdt + \int\limits_Q (u_0,\nabla) u_0 \cdot \psi\; dxdt +$$

$$+ \int\limits_Q V u_0 \cdot \psi\; dxdt = \int\limits_Q \bar{g}(x) \cdot \psi\; dxdt + \int\limits_Q H \cdot \psi\; dxdt$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+;\mathbf{H})$.

Remark 3.1. Denote by $m_k = \sup\limits_{\Omega} m_k(x)$.The coercivity of the limit operator (13), is a delicate problem since the constants m_k are always positive. In particular, the well-posedness of the problem (13), connected with the coercivity of the operator is guaranteed by the inequalities

$$\lambda_0 > \max\{m_1, m_2\}, \tag{14}$$

where λ_0 is the first eigenvalue of the operator $\nu \sum\limits_{i,l=1}^{2} \frac{\partial}{\partial x_i}\left(\widehat{a}_{il}(x)\frac{\partial}{\partial x_l}\right)$ in the space $H^1(\Omega)$. The proof of this assertion can be found in [8].

Under the condition (14) (cf. remark 3.1) the problem (13) has a trajectory attractor $\overline{\mathfrak{A}}$ in the trajectory space $\overline{\mathcal{K}}^+$, of the problem (13); moreover,

$$\overline{\mathfrak{A}} = \Pi_+ \overline{\mathcal{K}}$$

whera $\overline{\mathcal{K}}$ is the kernel of the problem (13) in $\mathcal{F}^b$.

We formulate the main theorem on homogenization of attractors of the system of Navier–Stokes equations.

Theorem 3.1. Let $\lambda_0 > \max\{m_1, m_2\}$, then is topological space Θ_+^{loc} correctly limited relation

$$\mathfrak{A}_\varepsilon \to \overline{\mathfrak{A}} \quad \text{if} \quad \varepsilon \to 0+. \tag{15}$$

Moreover,

$$\mathcal{K}_\varepsilon \to \overline{\mathcal{K}} \;\text{ if }\; \varepsilon \to 0+ \;\text{ in } \Theta^{loc}. \tag{16}$$

Remark 3.2. We recall that the spaces in theorem 3.1 depend on ε. We assume that all functions under consideration can be extended over the holes with preserving the norms.

3.2 Auxiliaries

We use some results of [8] below.

We consider the auxiliary problem

$$\begin{cases} -\nu\Delta_{xx}u_\varepsilon^k = g^k\big(x,\frac{x}{\varepsilon}\big), & x\in\Omega_\varepsilon,\\ \nu\frac{\partial u_\varepsilon^k}{\partial n} + b^k\big(x,\frac{x}{\varepsilon}\big)u_\varepsilon^k = h^k\big(x,\frac{x}{\varepsilon}\big), & x\in\partial G_\varepsilon, \qquad k=1,2.\\ u_\varepsilon^k = 0, & x\in\partial\Omega. \end{cases} \tag{17}$$

We also require that

$$\int_{\partial G(x)} b^k(x,\xi)d\sigma = 0, \qquad \int_{\partial G(x)} h^k(x,\xi)d\sigma = 0. \tag{18}$$

We look for a solution in the form of a series

$$u_\varepsilon^k = u_0^k(x) + \varepsilon u_1^k(x,\xi) + \varepsilon^2 u_2^k(x,\xi) + \ldots, \qquad \xi = \frac{x}{\varepsilon}. \tag{19}$$

Substituting the series (19) into (17) and collecting terms with ε , of the same order in the equation and boundary conditions, we find a recurrent sequence of problems such that the first one has the form

$$\begin{cases} -\nu\Delta_{\xi\xi}u_1^k + \frac{\partial^2 u_0^k}{\partial\xi_1\partial x_1} + \frac{\partial^2 u_0^k}{\partial\xi_2\partial x_2} = 0, & x\in Y\setminus G(x),\\ \nu\frac{\partial u_1^k}{\partial n_\xi} + \nu\frac{\partial u_0^k}{\partial n_x} + b^k\big(x,\xi\big)u_0^k = h^k(x,\xi), & x\in\partial G(x). \end{cases} \tag{20}$$

The integral identity for the problem (20) is a follows:

$$\begin{gathered} \iint_{Y\setminus G(x)} \left(\frac{\partial u_1^k}{\partial\xi_1}\frac{\partial v}{\partial\xi_1} + \frac{\partial u_1^k}{\partial\xi_2}\frac{\partial v}{\partial\xi_2}\right)d\xi_1 d\xi_2 + \iint_{Y\setminus G(x)} \left(\frac{\partial u_0^k}{\partial x_1}\frac{\partial v}{\partial\xi_1} + \frac{\partial u_0^k}{\partial x_2}\frac{\partial v}{\partial\xi_2}\right)d\xi_1 d\xi_2 + \\ + \int_{\partial G(x)} b^k\big(x,\xi\big)u_0^k v d\sigma = \int_{\partial G(x)} h^k\big(x,\xi\big) v d\sigma, \end{gathered} \tag{21}$$

where $v\in H^1_{per}(Y\setminus G(x))$.

From the form of the integral identity we can propose that the functions $u_1^k(x,\xi)$ have the following structure:

$$u_1^k(x,\xi) = L^k(\xi) + M^k(\xi)u_0^k(x) + N_1(\xi)\frac{\partial u_0^k}{\partial x_1} + N_2(\xi)\frac{\partial u_0^k}{\partial x_2}.$$

Substituting the last expression into (21) and collecting the corresponding terms, we obtain the following problem for the functions $N_l(\xi)$ and $M^k(\xi)$:

$$\iint_{Y\setminus G(x)} \left(\frac{\partial N_l}{\partial\xi_1}\frac{\partial v}{\partial\xi_1} + \frac{\partial N_l}{\partial\xi_2}\frac{\partial v}{\partial\xi_2}\right)d\xi_1 d\xi_2 + \iint_{Y\setminus G(x)} \frac{\partial v}{\partial\xi_1}d\xi_1 d\xi_2 = 0, \tag{22}$$

or, in the classical form:

$$\begin{cases} \Delta_{\xi\xi}(N_l+\xi_l) = 0, & x\in Y\setminus G(x),\\ \frac{\partial N_l}{\partial n_\xi} = n_l, & x\in\partial G(x); \end{cases}$$

$$\iint_{Y\setminus G(x)} \left(\frac{\partial M^k}{\partial\xi_1}\frac{\partial v}{\partial\xi_1} + \frac{\partial M^k}{\partial\xi_2}\frac{\partial v}{\partial\xi_2}\right)d\xi_1 d\xi_2 + \int_{\partial G(x)} b^k\big(x,\xi\big)v d\sigma = 0 \tag{23}$$

or

$$\begin{cases} \Delta_{\xi\xi}M^k = 0, & x\in Y\setminus G(x),\\ \frac{\partial M^k}{\partial n_\xi} + b^k\big(x,\xi\big) = 0, & x\in\partial G(x); \end{cases}$$

$$\iint_{Y\setminus G(x)} \left(\frac{\partial L^k}{\partial \xi_1} \frac{\partial v}{\partial \xi_1} + \frac{\partial L^k}{\partial \xi_2} \frac{\partial v}{\partial \xi_2} \right) d\xi_1 d\xi_2 = \int_{\partial G(x)} h^k\big(x,\xi\big) v d\sigma \tag{24}$$

or

$$\begin{cases} \Delta_{\xi\xi} L^k = 0, & x \in Y \setminus G(x), \\ \frac{\partial L^k}{\partial n_\xi} = h^k\big(x,\xi\big), & x \in \partial G(x). \end{cases}$$

The compatibility condition in the problem (22) can be easily verified by integrating by parts and using (18) in the problems (23) and (24). We note that the functions $L^k(\xi)$, $M^k(\xi)$, and $N_l(\xi)$ are defined up to an additive constant and the natural normalization conditions are the following:

$$\iint_{Y\setminus G(x)} L^k(\xi) d\xi = \iint_{Y\setminus G(x)} M^k(\xi) d\xi = \iint_{Y\setminus G(x)} N_l(\xi) d\xi = 0.$$

In what follows, we assume that these conditions are satisfied.

The next power of εyields the problem for $u_2^k(x,\xi)$:

$$\begin{cases} \Delta_{\xi\xi} u_2^k + 2\left(\frac{\partial^2 u_1^k}{\partial \xi_1 \partial x_1} + \frac{\partial^2 u_1^k}{\partial \xi_2 \partial x_2} \right) + \Delta_{xx} u_0^k = -g^k, & x \in Y \setminus G(x), \\ \frac{\partial u_2^k}{\partial n_\xi} + \frac{\partial u_1^k}{\partial n_x} + b^k\big(x, \frac{x}{\varepsilon}\big) u_1^k + h^k\big(x,\frac{x}{\varepsilon}\big) u_0^k = 0, & x \in \partial G(x). \end{cases} \tag{25}$$

The following statement is true.

Lemma 3.1. The functions $M^k(\xi)$ and $N_l(\xi)$ are connected by the integral identity

$$\frac{\partial u_0^k(x)}{\partial x_l} \left(\iint_{Y\setminus G(x)} \frac{\partial M^k}{\partial \xi_l} d\xi_1 d\xi_2 - \int_{\partial G(x)} b^k N_l d\sigma \right) = 0.$$

We also need the integral identity corresponding to the problem (25)

$$\iint_{Y\setminus G(x)} \left(\frac{\partial u_2^k}{\partial \xi_1} \frac{\partial v}{\partial \xi_1} + \frac{\partial u_2^k}{\partial \xi_2} \frac{\partial v}{\partial \xi_2} \right) d\xi_1 d\xi_2 + \iint_{Y\setminus G_(x)} \left(\frac{\partial u_1^k}{\partial x_1} \frac{\partial v}{\partial \xi_1} + \frac{\partial u_1^k}{\partial x_2} \frac{\partial v}{\partial \xi_2} \right) d\xi_1 d\xi_2 +$$
$$+ \int_{\partial G(x)} b^k\big(\xi\big) u_1^k v d\sigma + u_0^k(x) \int_{\partial G(x)} h^k\big(x,\xi\big) v d\sigma - \iint_{Y\setminus G(x)} \frac{\partial M^k}{\partial \xi_1} v d\xi_1 d\xi_2 \cdot \frac{\partial u_0^k}{\partial x_1} -$$
$$- \iint_{Y\setminus G(x)} \frac{\partial M^k}{\partial \xi_2} v d\xi_1 d\xi_2 \cdot \frac{\partial u_0^k}{\partial x_2} - \iint_{Y\setminus G(x)} \left(\frac{\partial N_1}{\partial \xi_1} + 1 \right) v d\xi_1 d\xi_2 \cdot \frac{\partial^2 u_0^k}{\partial x_1^2} -$$
$$- \iint_{Y\setminus G(x)} \left(\frac{\partial N_1}{\partial \xi_2} + \frac{\partial N_2}{\partial \xi_1} \right) v d\xi_1 d\xi_2 \cdot \frac{\partial^2 u_0^k}{\partial x_1 \partial x_2} - \iint_{Y\setminus G(x)} \left(\frac{\partial N_2}{\partial \xi_2} + 1 \right) v d\xi_1 d\xi_2 \cdot \frac{\partial^2 u_0^k}{\partial x_2^2} + \bar{g}^k = 0,$$

where $\bar{g}^k(x) = \iint_{Y\setminus G(x)} g^k(x,\xi) d\xi_1 d\xi_2$.

The solvability condition for the problem (25) leads to the equations for $u_0^k(x)$, which is the required formal homogenized equations. Applying Lemma 3.1, and considering the connection between $b(x,\xi)$ and $h(x,\xi)$ we can write it in the form

$$\nu \sum_{i,l=1}^{2} \frac{\partial}{\partial x_i} \left(\widehat{a}_{il}(x) \frac{\partial u_0^k}{\partial x_l} \right) - \int\limits_{\partial G(x)} b^k(x,\xi) M^k(\xi) d\sigma\, u_0^k(x) = \bar{g}^k(x) + \int\limits_{\partial G(x)} h^k(x,\xi) M^k(\xi) d\sigma,$$

where

$$\widehat{a}_{il}(x) = \iint_{Y\setminus G(x)} \left(\frac{\partial N_l}{\partial \xi_i} + \delta_{il} \right) d\xi_1 d\xi_2, \quad \delta_{il} \text{ is the Kroneker symbol.}$$

Thus, the homogenized problem can be written as

$$\begin{cases} \nu \sum_{i,l=1}^{2} \frac{\partial}{\partial x_i}\Big(\widehat{a}_{il}(x)\frac{\partial u_0^k}{\partial x_l}\Big) - m^k(x)u_0^k(x) = \bar{g}^k(x) + H^k(x), & x \in \Omega, \\ u_0^k(x) = 0, & x \in \partial\Omega, \end{cases} \tag{26}$$

where as $k = 1, 2$ we have $m^k(x) = \int_{\partial G(x)} b^k(x,\xi)M^k(x,\xi)d\sigma$,

$$H^k(x) = \int_{\partial G(x)} h^k(x,\xi)M^k(x,\xi)d\sigma = -\int_{\partial G(x)} b^k(x,\xi)L^k(x,\xi)d\sigma.$$

The following lemma is true (cf. [8]).

Lemma 3.2. If u_ε is a solution to the problem (17), and u_0 is a solution to the problem (26), then there is convergence

$$\nu \int\limits_{Q_\varepsilon} \nabla u_\varepsilon \cdot \nabla\psi \; dxdt + \sum_{j\in\Upsilon_\varepsilon} \int\limits_0^{+\infty} \int\limits_{\partial G_\varepsilon^j} B\big(x,\frac{x}{\varepsilon}\big)u_\varepsilon \cdot \psi \; d\sigma dt - \sum_{j\in\Upsilon_\varepsilon} \int\limits_0^{+\infty} \int\limits_{\partial G_\varepsilon^j} h\big(x,\frac{x}{\varepsilon}\big) \cdot \psi \; d\sigma dt -$$

$$-\int\limits_{Q_\varepsilon} g\big(x,\frac{x}{\varepsilon}\big)u_\varepsilon \cdot \psi \; d\sigma dt \longrightarrow \nu \int\limits_Q \widehat{a}\nabla u_0 \cdot \nabla\psi \; dxdt + \int\limits_Q Vu_0 \cdot \psi \; dxdt - \int\limits_Q H \cdot \psi \; dxdt - \int\limits_Q \bar{g} \cdot \psi \; dxdt$$

as $\varepsilon \to 0$.

Following [23] and taking into account Remark 3.2 , we show that

$$(u_\varepsilon, \nabla)u_\varepsilon \longrightarrow (u, \nabla)u \qquad \text{strongly in} \qquad L_2(Q). \tag{27}$$

For this purpose we use the estimate

$$\|(u_\varepsilon,\nabla)u_\varepsilon - (u,\nabla)u\|_{L_2(Q)} \le \|(u_\varepsilon - u,\nabla)u_\varepsilon\|_{L_2(Q)} + \|(u,\nabla)(u_\varepsilon - u)\|_{L_2(Q)} \le$$

$$\le C\Big(\int\limits_Q |u_\varepsilon - u|^2|\nabla u_\varepsilon|^2 dxds\Big)^{\frac{1}{2}} + C\Big(\int\limits_Q |u|^2|\nabla(u_\varepsilon - u)|^2 dxds\Big)^{\frac{1}{2}} \le$$

$$\le C_1\Big(\int\limits_Q |\nabla u_\varepsilon|^3 dxds\Big)^{\frac{1}{3}}\Big(\int\limits_Q |u_\varepsilon - u|^6 dxds\Big)^{\frac{1}{6}} + C_1\Big(\int\limits_Q |u|^6 dxds\Big)^{\frac{1}{6}}\Big(\int\limits_Q |\nabla(u_\varepsilon - u)|^3 dxds\Big)^{\frac{1}{3}}.$$

As proved in [15] the trajectory attractors $\mathfrak{A}_\varepsilon$ and $\overline{\mathfrak{A}}$ of equations (11) and (13) exist in the following space with a stronger topology: $H_w^{(2,2,1)}(Q)$, where

$$H_w^{(2,2,1)}(Q) = L_{2,w}\left(\mathbb{R}_+; \left[W_2^2(\Omega)\right]^2\right) \cap \left\{v : \frac{\partial v}{\partial t} \in L_{2,w}(\mathbb{R}_+; \mathbf{H})\right\}.$$

We set

$$H_{3,w}^{(1,1,0)}(Q) = L_{3,w}\left(\mathbb{R}_+; \left[W_3^1(\Omega)\right]^2\right).$$

Since $H^{(2,2,1)}(Q) \Subset H_3^{(1,1,0)}(Q)$ and $H^{(2,2,1)}(Q) \Subset L_6(Q)$, we find

$$\int\limits_Q |u_\varepsilon - u|^6 dxds \to 0, \qquad \int\limits_Q |\nabla(u_\varepsilon - u)|^3 dxds \to 0$$

as $\varepsilon \to 0$. Here, we used the uniform boundedness of the integral $\int\limits_Q |\nabla u_\varepsilon|^3 dxds \le M$. Thus, we have proved the convergence (27).

3.3 Proof of Theorem 3.1

Proof. It is clear that (16) implies (15). Therefore, it suffices to prove (16), i.e., for any neighborhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc} there is $\varepsilon_1 = \varepsilon_1(\mathcal{O}) > 0$ such that

$$\mathcal{K}_\varepsilon \subset \mathcal{O}(\overline{\mathcal{K}}) \text{ for all } \varepsilon < \varepsilon_1. \tag{28}$$

If (28) fails, then there exists a neighborhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\varepsilon_k \to 0+$ $(k \to \infty)$ and a sequence $u_{\varepsilon_k}(\cdot) = u_{\varepsilon_k}(s) \in \mathcal{K}_{\varepsilon_k}$ such that

$$u_{\varepsilon_k} \notin \mathcal{O}'(\overline{\mathcal{K}}) \ \text{ for all } k \in \mathbb{N}. \tag{29}$$

The sequence $\left\{ g\left(x, \frac{x}{\varepsilon_n}\right)\right\}$ is bounded in $\mathbf{H}$. Consequently, using the integral identity and the Cauchy-Bunyakovsky inequality, we conclude that the sequence of solutions $\{u_{\varepsilon_n}\}$ is bounded in $\mathcal{F}^b$. Passing to a subsequence, we can assume that

$$u_{\varepsilon_n} \to u_0 \ (n \to \infty) \ \text{ in } \Theta^{loc}.$$

We assert that $u_0 \in \overline{\mathcal{K}}$. The functions $u_{\varepsilon_n}(x,s)$ satisfy the equation

$$\frac{\partial u_{\varepsilon_n}}{\partial t} - \nu \Delta u_{\varepsilon_n} + (u_{\varepsilon_n}, \nabla) u_{\varepsilon_n} = g\big(x, \frac{x}{\varepsilon_n}\big), \quad t \in \mathbb{R}, \tag{30}$$

the condition

$$\nu \frac{\partial u_{\varepsilon_n}}{\partial n} + B\big(x, \frac{x}{\varepsilon_n}\big) u_{\varepsilon_n} = h\big(x, \frac{x}{\varepsilon_n}\big), \quad x \in \partial G_{\varepsilon_n},$$

and the energy identity

$$-\frac{1}{2}\int_{-M}^{M} \|u_{\varepsilon_n}(s)\|^2_{\mathbf{H}} \psi'(s) ds + \nu \int_{-M}^{M} \|u_{\varepsilon_n}(s)\|^2_{\mathbf{V}} \psi(s) ds + \sum_{j\in\Upsilon_\varepsilon} \int_{-M}^{M} \int\limits_{\partial G_\varepsilon^j} B(x,\xi) u^1_{\varepsilon_n}(x,s)\psi(s) d\sigma ds -$$

$$- \sum_{j\in\Upsilon_\varepsilon} \int_{-M}^{M} \int\limits_{\partial G_\varepsilon^j} h(x,\xi)\psi(s) d\sigma ds = \int_{-M}^{M} \left(g(x,\xi), u_\varepsilon(x,s)\right)_{\mathbf{H}} \psi(s) ds \tag{31}$$

for any $M > 0$ and any function $\psi \in C_0^\infty(]-M,M[)$, $\psi \ge 0$. Furthermore, $u_{\varepsilon_n}(s) \rightharpoonup u_0(s)$ $(n \to \infty)$ weakly in $\mathbf{L}_2(-M,M;\mathbf{V})$, $*$-weakly in $\mathbf{L}_\infty(-M,M;\mathbf{H})$ and $\dfrac{\partial u_{\varepsilon_n}(s)}{\partial t} \rightharpoonup \dfrac{\partial u_0(s)}{\partial t}$ $(n \to \infty)$ weakly in $\mathbf{L}_2(-M,M;\mathbf{H})$. By the known compactness theorem [22] we can assume that $u_{\varepsilon_n}(s) \to u_0(s)$ $(n \to \infty)$ strongly in $\mathbf{L}_2(-M,M;\mathbf{H})$ and $u_{\varepsilon_n}(x,s) \to u_0(x,s)$ $(n \to \infty)$ for almost all $(x,s) \in D \times (-M,M)$. In particular, $u_{\varepsilon_n}(s) \to u_0(s)$ $(n \to \infty)$ strongly in $\Theta^{loc}_+ = \mathbf{L}_2^{loc}(\mathbb{R};\mathbf{H})$.

Now, taking into account Lemma 3.2 and the convergence (27), we pass to the limit in (30) and (31) as $\varepsilon \to 0$, based on a standard argument in [22] (see the detailed proof in [15,17,20]). Consequently $u_0 \in \overline{\mathcal{K}}$, i.e., u_0 is a solution to the problem (13), satisfying the corresponding identity (31) with the exterior force $\bar{g}(x)$. At the same time, we have established that $u_{\varepsilon_n}(s) \to u_0(s)$ $(n \to \infty)$ in Θ^{loc}_+ and, consequently, $u_{\varepsilon_n}(s) \in \mathcal{O}'(u_0(s)) \subset \mathcal{O}'\left(\overline{\mathcal{K}}\right)$ for $\varepsilon_n \ll 1$. Thus, we arrive at a contradiction with (29). The theorem is proved.

Acknowledgments

The work of K.A. Bekmaganbetov and A.M. Toleubay in Sections 1, 2 and Subsection 3.2 is supported by the Committee of Science of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant AP14869553). The work of the G.A. Chechkin in Subsections 3.1 and 3.3 is partially supported by the Russian Science Foundation (project 20-11-20272).

References

1 Jikov, V.V., Kozlov, S.M., & Oleinik, O.A. (1994). *Homogenization of Differential Operators and Integral Functionals.* Berlin: Springer–Verlag.

2 Oleinik, O.A., Shamaev, A.S., & Yosifian, G.A. (1992). *Mathematical Problems in Elasticity and Homogenization.* Amsterdam: North–Holland.

3 Chechkin, G.A., Piatnitski, A.L., & Shamaev, A.S. (2007). *Homogenization: Methods and Applications.* Providence (RI): Am. Math. Soc.

4 Marchenko, V.A., & Khruslov, E.Ya. (2006). *Homogenization of partial differential equations.* Boston (MA): Birkhäuser.

5 Cioranescu, D., & Murat, F. (1982). Un terme étrange venu d'ailleurs I & II. *In Nonlinear Partial Differential equations and their Applications.* Collège de France Séminaire, Volume II & III, ed. H.Berzis, J.L.Lions. Research Notes in Mathematics, 60 & 70, London: Pitman, 98–138 & 154–178.

6 Conca, C., & Donato, P. (1988). Non-homogeneous Neumann problems in domains with small holes. *Modélisation Mathématique et Analyse Numérique (M^2AN), 22*(4), 561–607.

7 Cioranescu, D., & Donato, P. (1997). On a Robin Problem in Perforated Domains. *In: Homogenization and Applications to Material Sciences.* Edited by D.Cioranescu, A. Damlamian, and P. Donato. GAKUTO International Series. Mathematical Sciences and Applications. Tokyo: Gakkōtosho, 9, 123–136.

8 Belyaev, A.G., Piatnitski, A.L., & Chechkin, G.A. (1998). Asymptotic Behavior of Solution for Boundary-Value Problem in a Perforated Domain with Oscillating Boundary. *Siberian Math. Jour., 39*(4), 730–754. DOI: 10.1007/BF02673049.

9 Belyaev, A.G., Piatnitski, A.L., & Chechkin, G.A. (2001). Averaging in a Perforated Domain with an Oscillating Third Boundary Condition. *Sb. Math., 192*(7), 933–949. DOI: 10.4213/sm576.

10 Chechkin, G.A., & Piatnitski, A.L. (1999). Homogenization of Boundary–Value Problem in a Locally Periodic Perforated Domain. *Applicable Analysis, 71*(1-4), 215–235.

11 Bekmaganbetov, K.A., Chechkin, G.A., & Chepyzhov, V.V. (2020a). Attractors and a "strange term" in homogenized equation. *C R Mécanique, 348*(5), 351–359. DOI: 10.5802/crmeca.1.

12 Bekmaganbetov, K.A., Chechkin, G.A., & Chepyzhov, V.V. (2020b). Strong Convergence of Trajectory Attractors for Reaction–Diffusion Systems with Random Rapidly Oscillating Terms. *Communications on Pure and Applied Analysis, 19*(5), 2419–2443. DOI: 10.3934/cpaa.2020106.

13 Bekmaganbetov, K.A., Chechkin, G.A., & Chepyzhov, V.V. (2020c). "Strange Term" in Homogenization of Attractors of Reaction–Diffusion equation in Perforated Domain. *Chaos, Solitons & Fractals, 140,* Art. No. 110208. DOI: 10.1016/j.chaos.2020.110208.

14 Babin, A.V., & Vishik, M.I. (1992). *Attractors of evolution equations.* Amsterdam: North–Holland.

15 Chepyzhov, V.V., & Vishik, M.I. (2002). *Attractors for equations of mathematical physics.* Providence (RI): Amer. Math. Soc.

16 Temam, R. (1988). Infinite-dimensional dynamical systems in mechanics and physics. *Applied Mathematics Series, 68.* New York (NY): Springer-Verlag. DOI: 10.1007/978-1-4684-0313-8.

17 Vishik, M.I., & Chepyzhov, V.V. (2003). Approximation of trajectories lying on a global attractor of a hyperbolic equation with an exterior force that oscillates rapidly over time. *Sb. Math., 194,* 1273–1300. DOI: 10.4213/sm765.

18 Chepyzhov, V.V., & Vishik, M.I. (2002). Non-autonomous 2D Navier-Stokes system with a simple global attractor and some averaging problems. *ESAIM Control Optim. Calc. Var., 8,* 467–487. DOI: 10.1051/cocv:200205.

19 Chepyzhov, V.V., & Vishik, M.I. (2007). Non-autonomous 2D Navier-Stokes system with singularly oscillating external force and its global attractor. *J. Dynam. Diff. Eq., 19*(3), 655–684. DOI: 10.1007/s10884-007-9077-y.

20 Chepyzhov, V.V., & Vishik, M.I. (1997). Evolution equations and their trajectory attractors. *J. Math. Pures Appl., 76*(10), 913–964.

21 Boyer, F., & Fabrie, P. (2013). Mathematical Tools for the Study of the Incompressible Navier-Stokes equations and Related Models. *Applied Mathematical Sciences, 183.* New York (NY): Springer. DOI: 10.1007/978-1-4614-5975-0.

22 Lions, J.-L. (1969). *Quelques méthodes de résolutions des problèmes aux limites non linéaires.* Paris: Dunod, Gauthier-Villars.

23 Chepyzhov, V.V., & Vishik, M.I. (1996). Trajectory attractors for reaction-diffusion systems. *Top. Meth. Nonlin. Anal. J. Julius Schauder Center, 7*(1), 49–76. DOI: 10.12775/TMNA.1996.002.

Қ.А. Бекмаганбетов[1,2], Г.А. Чечкин[2,3,4], А.М. Төлеубай[2,5]

[1] *М.В.Ломоносов атындағы Мәскеу мемлекеттік университеті, Қазақстан филиалы, Нұр-Сұлтан, Қазақстан;*
[2] *Математика және математикалық модельдеу институты, Алматы, Қазақстан;*
[3] *М.В. Ломоносов атындағы Мәскеу мемлекеттік университеті, Мәскеу, Ресей;*
[4] *Компьютерлік орталығы бар математика институты – РҒА Уфа федеральды зерттеу орталығының бөлімшесі, Уфа, Ресей;*
[5] *Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Нұр-Сұлтан,Қазақстан*

Локальды периодты кеуекті ортадағы 2D Навье–Стокс теңдеулер жүйесінің аттракторлары

Мақалада теңдеулерде және шекаралық шарттарда тез тербелмелі мүшелері бар екі өлшемді Навье–Стокс теңдеулер жүйесінің қарастырылды. Тесік облыстағы есепті зерттей отырып, сыртқы шекарадағы Дирихленің біртекті шартын және қуыстардың шекарасындағы Фурье (Робен) шартын анықтаймыз. Осындай болжамдармен осы жүйенің траекториялық аттракторы кейбір әлсіз топологияларда қосымша потенциалымен және тривиалды емес оң жақ бөлігі бар тесігі жоқ облыстағы орташаланған Навье–Стокс теңдеулер жүйесінің траекториялық аттракторына жинақталатыны дәлелденген. Ол үшін А.В. Бабиннің, В.В. Чепыжовтың, Ж.–Л. Лионстың, Р. Темам және М.И. Вишиктің эволюциялық теңдеулердің траекториялық аттракторлары туралы мақалалары мен монографияларының әдістемесі қолданылған. Сондай-ақ, XX ғасырдың соңында пайда болған орташалау әдістері пайдаланылған. Алдымен асимптотикалық әдістерді асимптотиканы формальды құру үшін қолданып содан кейін асимптотикалық қатарлардың негізгі мүшелерін функционалды талдау және интегралды бағалау әдістерін қолдана отырып таңдалған. Сәйкесінше, көмекші әлсіз топологиялы функционалды кеңістікті анықтау арқылы теңдеулердің шекті (орташаланған) жүйесін алынған және осы жүйе үшін траекториялық аттракторлардың бар екені дәлелденген. Содан кейін негізгі теорема тұжырымдалып, ол көмекші леммалардың көмегімен нақтыланған.

Кілт сөздер: аттракторлар, орташалау, Навье–Стокс теңдеулер жүйесі, әлсіз жинақтылық, тесік облыс, тез тербелмелі мүшелер, кеуекті орта.

К.А. Бекмаганбетов[1,2], Г.А. Чечкин[2,3,4], А.М. Толеубай[2,5]

[1]*Московский государственный университет имени М.В. Ломоносова, Казахстанский филиал, Нур-Султан, Казахстан;*
[2]*Институт математики и математического моделирования КН МОН РК, Алматы, Казахстан;*
[3]*Московский государственный университет имени М.В. Ломоносова, Москва, Россия;*
[4]*Институт математики с компьютерным центром — подразделение Уфимского федерального исследовательского центра РАН, Уфа, Россия;*
[5]*Евразийский национальный университет имени Л.Н. Гумилева, Нур-Султан, Казахстан*

Аттракторы 2D системы уравнений Навье-Стокса в локально периодической пористой среде

Рассмотрена двумерная система уравнений Навье–Стокса с быстро осциллирующими членами в уравнениях и граничных условиях. Исследуя задачу в перфорированной области, мы задаем однородное условие Дирихле на внешней границе и условие Фурье (Робена) на границе полостей. При таких предположениях доказываем, что траекторные аттракторы этой системы сходятся в некоторой слабой топологии к траекторным аттракторам усредненной системы уравнений Навье–Стокса с дополнительным потенциалом и нетривиальной правой частью в области без пор. Для этого мы используем подход из статей и монографий А.В. Бабина, В.В. Чепыжова, Ж.-Л. Лионса, Р. Темама и М.И. Вишика о траекторных аттракторах эволюционных уравнений. Кроме того, применяем методы усреднения, появившиеся в конце XX века. Сначала используем асимптотические методы для формального построения асимптотик, далее мы выверяем главные члены асимптотических рядов с помощью методов функционального анализа и интегральных оценок. Определяя соответствующие вспомогательные функциональные пространства со слабой топологией, мы выводим предельную (усредненную) систему уравнений и доказываем существование траекторных аттракторов для этой системы. Затем формулируем основную теорему и доказываем ее с помощью вспомогательных лемм.

Ключевые слова: аттракторы, усреднение, система уравнений Навье-Стокса, слабая сходимость, перфорированная область, быстро осциллирующие члены, пористая среда.

DOI 10.31489/2022M3/51-58
UDC 517.9

S.Zh. Igisinov*, L.D. Zhumaliyeva, A.O. Suleimbekova, Ye.N. Bayandiyev

M.Kh. Dulaty Taraz Regional University, Taraz, Kazakhstan
(E-mail: igisinovsabit@mail.ru, lzdmath@mail.ru, suleimbekovaa@mail.ru, erikn87@mail.ru)

Estimates of singular numbers (s-numbers) for a class of degenerate elliptic operators

In this paper we study a class of degenerate elliptic equations with an arbitrary power degeneracy on the line. Based on the research carried out in the course of the work, the authors propose methods to overcome various difficulties associated with the behavior of functions from the definition domain for a differential operator with piecewise continuous coefficients in a bounded domain, which affect the spectral characteristics of boundary value problems for degenerate elliptic equations. It is shown the conditions imposed on the coefficients at the lowest terms of the equation, which ensure the existence and uniqueness of the solution. The existence, uniqueness, and smoothness of a solution are proved, and estimates are found for singular numbers (s-numbers) and eigenvalues of the semiperiodic Dirichlet problem for a class of degenerate elliptic equations with arbitrary power degeneration.

Keywords: elliptic operator, boundary value problem, singular numbers, power degeneracy, solution, uniqueness.

1 Introduction. Main results

Let $\Omega = \{(x, y) : -\pi < x < \pi, 0 < y < 1\}$. Consider the following problem

$$Lu = -k(y)u_{xx} - u_{yy} + a(y)u_x + c(y)u = f(x, y) \in L_2(\Omega), \tag{1}$$

$$u(-\pi, y) = u(\pi, y), u_x(-\pi, y) = u_x(\pi, y), \tag{2}$$

$$u(x, 0) = u(x, 1) = 0, \tag{3}$$

where $a(y), c(y)$ are piecewise continuous functions in $[0, 1]$, $k(y) > 0$ as $y \in (0, 1]$ and $k(0) = 0$. Let $C^{\infty}_{0,\pi}(\bar{\Omega})$ be a class of infinitely differentiable finite functions in $\bar{\Omega}$ and satisfying the conditions (2)–(3).

We also denote closure of the operator (1) by the norm of $L_2(\Omega)$ as L.

In the study of the smoothness and approximation properties of solutions to boundary value problems for some nonlinear equations we encounter questions of the spectral properties of linear degenerate elliptic equations. In contrast to elliptic operators, spectral questions for degenerate elliptic operators are poorly understood. Known results on this topic or those close to it in content are contained in the works of M. Smirnov [1], M. Keldysh [2], T. Kalmenov, M. Otelbaev [3], O. Oleinik [4], M. Vishik, V. Grushin [5, 6], and others.

However, in the general case, such traditional questions as asymptotic behavior and estimates of eigenvalues in general are far from complete.

The results of this work are close to those of M.B. Muratbekov [7–10], where differential operators of mixed and hyperbolic types were investigated. In contrast to the above works, here we investigate previously unconsidered degenerate elliptic equations with an arbitrary power-law degeneracy on the degeneracy line.

Definition 1. The function $u \in L_2(\Omega)$ is called a solution to (1)–(3) if there exists a sequence $\{u_k(x, y)\}_{k=1}^{\infty} \subset C^{\infty}_{0,\pi}(\overline{\Omega})$ such that

*Corresponding author.
E-mail: igisinovsabit@mail.ru

$$\|u_k - u\|_2 \to 0, \ \ \|Lu_k - f\|_2 \to 0 \text{ as } k \to \infty.$$

By $C([0,1], L_2(-\pi,\pi))$ we denote the space obtained by completing the set of continuous functions on the interval $[0,1]$ with values in $L_2(-\pi,\pi)$ relative to the norm

$$\|u\|_{C([0,1],L_2)} = \sup_{y\in[0,1]} \left(\int_{-\pi}^{\pi} |u(x,y)|^2 dx \right)^{\frac{1}{2}}$$

and $W_2^1(\Omega)$ is the Sobolev space with norm

$$\|u\|_{2,1,\Omega} = \left[\|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2 \right]^{\frac{1}{2}},$$

where $\|\cdot\|$ is a norm of $L_2(\Omega)$.

Definition 2. [11] Let A be a completely continuous linear operator and $|A| = \sqrt{A^*A}$. Eigenvalues of $|A|$ are called s-numbers of A.

Nonzero s-numbers of L^{-1} will be numbered in descending order, taking into account their multiplicity, so that

$$s_k(L^{-1}) = \lambda_k(|L^{-1}|), k = 1,2,3, ...$$

Theorem 1. Let $a(y), c(y)$ are piecewise continuous functions in $[0,1]$ and satisfying the conditions

$$i) a(y) \geq \delta_0 > 0, c(y) \geq \delta > 0.$$

Then there exists a unique solution $u(x,y)$ to (1)–(3) such that

$$\|u\|_{C(0,1),L_2} + \|u\|_{2,1,\Omega} \leq c_0 \|f\|_2$$

for all $f \in L_2(\Omega)$, where c_0 is a constant.

Theorem 2. Let the condition i) be fulfilled. Then the estimate

$$c_1 \frac{1}{k} \leq s_k \leq c_2 \frac{1}{k^{\frac{1}{2}}}, k = 1,2,3, ...$$

holds, where c_1, c_2 are constants, $0 < c_1 \leq c_2$, s_k is singular numbers (s-numbers) of L^{-1}.

2 Auxiliary lemmas

Lemma 1. The estimate

$$\|Lu\|_2 \geq c\|u\|_2 \tag{4}$$

holds for all $u \in D(L)$, where c_0 is a constant.

Proof. Let $C_{0,\pi}^{\infty}(\Omega)$. Integrate by parts and taking into account the boundary conditions we have

$$< Lu, u > \geq \int_{\Omega} (u_y^2 + c(y)u^2) dxdy + \int_{\Omega} k(y) u_x^2 dxdy$$

and

$$< Lu, u_x > = \int_{\Omega} a(y) u_x^2 dxdy.$$

From these relations we obtain (4) using the Cauchy inequality with "ϵ"and taking into account the condition i). Lemma 1 is proved.

We denote by l_n the closure of the operator

$$l_n u(y) = -u'' + (n^2 k(y) + ina(y) + c(y))u, n = 0, \pm 1, \pm 2, ...$$

defined on $C_0^\infty[0,1]$, where $C_0^\infty[0,1]$ is a set of infinitely differentiable functions satisfying the conditions (3).

Lemma 2. The estimates

$$\|l_n u\|_{L_2(0,1)} \geq c_1(\|u'\|_{L_2(0,1)} + \|u\|_{L_2(0,1)}), \tag{5}$$

$$\|l_n u\|_{L_2(0,1)} \geq c_2 \|u\|_{C[0,1]}, \tag{6}$$

hold for all $u(y) \in D(l_n)$, where c_1, c_2 are constants.

Proof. Let us compose quadratic form $(l_n u, u)$, $u \in C_0^\infty[0,1]$. Integrating by parts we obtain

$$|(l_n u, u)| = \left| \int_0^1 (l_n u)\bar{u} dy \right| = \left| \int_0^1 (u'^2 + (n^2 k(y) + ina(y) + c(y))|u|^2) dy \right|.$$

Hence, using the inequality $|\alpha + i\beta| \geq max(|\alpha|, |\beta|)$ $(\alpha, \beta \in R)$, the inequality Schwartz and the Cauchy inequality with "$\epsilon > 0$"we obtain

$$\|l_n u\|^2_{L_2(0,1)} \geq n^2 \delta^2 \|u\|^2_{L_2(0,1)},$$

$$c\|l_n u\|^2_{L_2(0,1)} \geq c_3 \int_0^1 (|u'|^2 + c(y)|u|^2) dy + \int_0^1 n^2 k(y)|u|^2) dy. \tag{7}$$

From (7) taking into account $k(y) \geq 0$ we have

$$\|l_n u\|^2_{L_2(0,1)} \geq c_4(\|u\|^2_{L_2(0,1)} + \|u'\|^2_{L_2(0,1)}) \geq c_1 \|u\|^2_{W_2^1(0,1)}.$$

Since the embedding operator of the Sobolev space $W_2^1(0,1)$ to $[0,1]$ is bounded it follows that

$$\|l_n u\|_{L_2(0,1)} \geq c_2(\|u\|_{C[0,1]}$$

which is true for all $u \in D(L)$. Lemma 2 is proved.

Lemma 3. The operator l_n is continuously invertible.

Proof. Taking into account (5) it is enough if we show the density of $D(l_n)$ in $L_2(\Omega)$. Assume the contrary. Let the set $D(l_n)$ is not density in $L_2(0,1)$. Then there exists nonzero element $w \in L_2(0,1)$ such that $(l_n u, w) = 0$ for $u \in D(l_n)$. Hence since the set $D(l_n)$ is not density in $L_2(0,1)$ we obtain that w is a solution to $l_n^* w = -w'' + (n^2 k(y) + c(y))w = 0$. From this equality it follows that $w'' \in L_2(0,1)$ by virtue of the continuous coefficients on [0,1]. Now we show that $w(y)$ satisfies the condition $w(0) = w(1) = 0$. Integrating by parts we obtain

$$0 = (u, l_n^* w) = (l_n u, w) - u'(1)\bar{w}(1) + u'(0)\bar{w}(0)$$

for all $u \in D(l_n)$. Last equality holds if $w(0) = w(1) = 0$. Therefore $w \in D(l_n)$. Then, we obtain

$$\|l_n w\|_{L_2(0,1)} \geq c\|w\|_{L_2(0,1)}$$

same as (5). It is shown that $w = 0$. The resulting contradiction proves the lemma 3.

Lemma 4. The following estimate holds for l_n^{-1}

$$\|l_n\|_{L_2(0,1) \to L_2(0,1)} \geq \frac{1}{n\delta_0}, n = \pm 1, \pm 2, ...$$

Proof. Taking into account the condition i) we have for any function $u \in C_0^\infty[0,1]$

$$|(l_n u, u)| \geq \left| \int_0^1 ina(y)|u|^2 dy \right| \geq |n|\delta_0 \|u\|^2_{L_2(0,1)}.$$

Hence, using the Cauchy inequality we obtain

$$\|l_n u\|_{L_2(0,1)} \geq |n|\delta_0 \|u\|_{L_2(0,1)}.$$

From the last estimates it follows Lemma 4.

3 Proofs of the main theorems

Proof of Theorem 1. The existence and continuity of l_n^{-1} follows from Lemma 3. Let $u_n(y) = (l_n^{-1} f_n)(y)$. By direct verification, we make sure that the function

$$u_k(x,y) = \sum_{n=-k}^{k} u_n(y) e^{inx} = \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx}$$

is a solution to (1) with the right side

$$f_k(x,y) = \sum_{n=-k}^{k} f_n(y) e^{inx}$$

which satisfies the condition (2)–(3). Moreover the following equality

$$\|u_k(x,y)\|^2_{L_2(-\pi,\pi)} = 2\pi \sum_{n=-k}^{k} |u_n(y)|^2$$

holds. Then from the estimate (5) it follows that

$$\sup_{y\in[0,1]} \|u_k(x,y)\|^2_{L_2(-\pi,\pi)} = 2\pi \sum_{n=-k}^{k} \sup_{y\in[0,1]} |u_n(y)|^2 \leq$$

$$\leq c_1 2\pi \sum_{n=-k}^{k} \|l_n u\|^2_{L_2(0,1)} \leq c_2 2\pi \sum_{n=-k}^{k} \|f_n(y)\|^2_{L_2(0,1)} = c\|f_k(x,y)\|_2^2. \quad (8)$$

From Lemma 4 we have

$$\|\frac{\partial u_k(x,y)}{\partial x}\|_2^2 = \|\frac{\partial}{\partial x} \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx})\|_2^2 = \|in \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx}\|_2^2 \leq$$

$$\leq \sum_{n=-k}^{k} |n|^2 \|l_n^{-1}\|^2_{L_2(0,1)\to L_2(0,1)} \|f_n\|^2_{L_2(0,1)} \leq \frac{1}{\delta_0^2} \sum_{n=-k}^{k} \|f_n\|^2_{L_2(0,1)} = \frac{1}{\delta_0^2} \|f_k(x,y)\|_2^2. \quad (9)$$

Similarly, using estimates (5), (6) we obtain

$$\|\frac{\partial u_k(x,y)}{\partial y}\|_2^2 + \|u_k\|_2^2 = \|\frac{\partial}{\partial y} \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx})\|_2^2 + \| \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx})\|_2^2 \leq$$

$$\leq \sum_{n=-k}^{k} \|f_n\|_2^2 + \sum_{n=-k}^{k} \|f_n\|_2^2 \leq= 2\|f_k(x,y)\|_2^2. \qquad (10)$$

It is known that a set of functions

$$f_k(x,y) = \sum_{n=-k}^{k} f_n(y)e^{inx} \quad (k=1,2,...)$$

is dense in $L_2(\Omega)$. Therefore, we can assume that $\|f_k(x,y) - f(x,y)\|_2 \to 0$ as $k \to \infty$. Then the sequence $\{f_k\}_{k=1}^{\infty}$ is fundamental, and by virtue of estimates (8)–(10)

$$\|u_k(x,y) - u_m(x,y)\|_{C([0,1],L_2)} + \|u_k(x,y) - u_m(x,y)\|_{2,1,\Omega} \leq$$

$$\leq c_6\|f_k(x,y) - f_m(x,y)\|_2 \to 0$$

as $k, m \to \infty$. Hence, since the spaces $C([0,1], L_2(-\pi,\pi))$ and $W_2^1(\Omega)$ are complete, it follows that the sequence $\{u_n(x,y)\}_{k=-\infty}^{\infty}$ has limit $u(x,y)$, for which, by virtue of (8)–(10), the estimate

$$\|u\|_{C([0,1],L_2)} + \|u\|_{2,1,\Omega} \leq c\|f\|_2$$

holds. Theorem 1 is proved.

Let us introduce the sets

$$M = \{u \in L_2(\Omega) : \|Lu\|_2 + \|u\|_2 \leq 1\},$$

$$\widetilde{M}_{c_1} = \{u \in C([0,1], L_2(-\pi,\pi)) : (\|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2)^{\frac{1}{2}} \leq c_1\},$$

$$\dot{M}_{c_1^{-1}} = \{u \in L_2(\Omega); (\|u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + \|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2)^{\frac{1}{2}} \leq c_1^{-1}\},$$

where $c_1 > 0$ и $c_1^{-1} > 1$.

The following lemma holds

Lemma 5. Let condition i) be satisfied. Then for some constant $c_1 > 1$ the inclusions

$$\dot{M}_{c_1^{-1}} \subseteq M \subseteq \widetilde{M}_{c_1}$$

hold.

Proof. Let $u(x,y) \in \dot{M}_{c_1^{-1}}$. Then, taking into account condition i), we obtain

$$\|Lu\|_2^2 + \|u\|_2^2 \leq c_2(\|u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + \|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2)^{\frac{1}{2}} \leq c_1^{-1}c_2,$$

where $c_2 = \max_{y\in[0,1]} \{|k(y)|, |a(y)|, |c(y)|\}$.

Hence, assuming $c_2 = c_1$, we have $\dot{M}_{c_1^{-1}} \subseteq M$.

Let $u \in M$. Then it follows from Theorem 1 that

$$(\|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2)^{\frac{1}{2}} \leq C(\|Lu\|_2^2 + \|u\|_2^2) \leq C,$$

i.e. $M \subseteq \widetilde{M}_C$. By choosing a constant c_1 such that $c_1 \geq c$ we obtain the assertion of the lemma. Lemma 5 is proved.

Lemma 6. Let condition i) be satisfied. Then the estimates

$$c^{-1}\dot{d}_k \leq d_k \leq c\widetilde{d}_k, \quad k = 1, 2, ...$$

hold, where $c > 0$ is an any constant, $\widetilde{d}_k, d_k, \dot{d}_k$ are the k-widths of the sets respectively $\widetilde{M}_c, M, \dot{M}_{c^{-1}}$.

The proof of this lemma it follows from Lemma 5 and the properties of the widths.

Let us introduce the functions

$$N(\lambda) = \sum_{d_k > \lambda} 1, \widetilde{N}(\lambda) = \sum_{\widetilde{d}_k > \lambda} 1, \dot{N}(\lambda) = \sum_{\dot{d}_k > \lambda} 1,$$

equal respectively to the number of widths $d_k(M)$, $\widetilde{d}_k$ and $\dot{d}_k$ are greater than $\lambda > 0$. Estimates (8) easily imply the inequalities

$$\dot{N}(c\lambda) \leq N(\lambda) \leq \widetilde{N}(c^{-1}\lambda)$$

Proof of Theorem 2. It is known that the estimates

$$c_0^{-1}\lambda^{-2} \leq \widetilde{N}(\lambda) \leq c_0\lambda^{-2}, \tag{11}$$

$$c_0^{-1}\lambda^{-1} \leq N(\lambda) \leq c_0\lambda^{-1}. \tag{12}$$

hold for the functions $\widetilde{N}(\lambda)$ and $N(\lambda)$. Let $\lambda = \widetilde{d}_k$. Then $\widetilde{N}(\widetilde{d}_k) = k$. Therefore from (11) and (12) it follows

$$C_0^{-1}\frac{1}{\sqrt{k}} \leq \widetilde{d}_k \leq C_0\frac{1}{\sqrt{k}}, \quad C_0^{-1}\frac{1}{k} \leq \dot{d}_k \leq C_0\frac{1}{k}.$$

respectively. Hence, taking into account estimates (7) and the equality $s_{k+1}(L^{-1}) = d_k$ we obtain

$$C_1\frac{1}{k} \leq s_k \leq C_2\frac{1}{k^{\frac{1}{2}}, \quad k = 1, 2, 3, ...}$$

Theorem 2 is proved.

Acknowledgments

This paper was supported by the grant AP08855802 of the Ministry of Education and Science of the Republic of Kazakhstan.

References

1 Смирнов М.М. Вырождающиеся эллиптические и гиперболические уравнения / М.М. Смирнов. — М.: Наука, 1966. — 292 с.

2 Келдыш М.В. О некоторых случаях вырождения уравнений эллиптического типа на границе области / М.В. Келдыш // ДАН СССР. — 1951. — Т. 77. — № 2. — С. 181–183.

3 Кальменов Т.Ш. О гладкости решений одного класса вырождающихся эллиптических уравнений / Т.Ш. Кальменов, М. Отелбаев // Дифференц. уравнения. — 1977. — Т. 13. — №7. — С. 1244–1255.

4 Олейник О.А. О гладкости решений вырождающихся эллиптических и параболических уравнений / О.А. Олейник // ДАН СССР. — 1965. — Т. 163. — №3. — С. 557–580.

5 Вишик М.И. Об одном классе вырождающихся эллиптических уравнений высших порядков /М.И. Вишик, В.В. Грушин // Мат. сб. — 1969. — Т. 79. — №1. — С. 3–36.

6 Вишик М.И. Уравнения 2-го порядка с неотрицательной характеристической формой / М.И. Вишик, В.В. Грушин // Мат. сб. — 1969. — Т. 80. — №4. — С. 445–491.

7 Muratbekov M.B. Estimates of spectrum for a class of mixed type operators / M.B. Muratbekov, M.M. Muratbekov // Differential equations. — 2007. — 43. — P. 143–146.

8 Muratbekov M.B. On existence of the resolvent and discreteness of the spectrum of a class of differential operators of hyperbolic type / M.B. Muratbekov, M.M. Muratbekov, A.M. Abylayeva // Electronic Journal of theory of differential equations. — 2013. — 63. — P. 1–10.

9 Muratbekov M. On discreteness of the spectrum of a class of mixed type singular differential operators / M.B. Muratbekov, T.Sh. Kalmenov, M.M Muratbekov // Complex Variables and Elliptic Equations. — 2015. — 60. — No.12. — P. 1752–1763.

10 Muratbekov M. On the existence of a resolvent and separability for a class of singular hyperbolic type differential operators on an unbounded domain / M. Muratbekov, M. Otelbaev // Eurasian mathematical Journal. — 2016. — 7. — No.1. — P. 50–67.

11 Gohberg I.C. Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space (Translations of mathematical monographs) / I.C. Gohberg, M.G. Krein. — American Mathematical society. — 1969. — Vol.18. — 378 p.

С.Ж. Игисинов, Л.Д. Жумалиева, А.О. Сулеймбекова, Е.Н. Баяндиев

М.Х. Дулати атындағы Тараз өңірлік университеті, Тараз, Қазақстан

Өзгешеленетін эллиптикалық операторлардың бір класы үшін сингулярлық сандарды (s-сандарды) бағалау

Мақалада түзудегі туынды дәрежеде өзгешеленуі бар өзгешеленетін эллиптикалық теңдеулердің бір класы зерттелген. Жұмыс барысында жүргізілген зерттеуге сүйене отырып, авторлар бөлікті-үзіліссіз коэффициенттері бар дифференциалды оператордың анықталу облысындағы функциялардың өзгеруіне байланысты туындайдын шенелген облыстағы шекаралық есептердің спектральдік сипаттамаларына әсер ететін әртүрлі қиындықтарды жеңудің әдістерін ұсынған осы жұмыста шешімнің болуы мен жалғыздығын қамтамасыз ететін теңдеудің кіші мүшелері коэффициенттері үшін қойылған шарттар көрсетілген. Шешімнің бар болуы, жалғыздығы мен тегістігі дәлелденді, сонымен қатар еркін дәрежелі өзгешеленетін эллиптикалық теңдеулердің бір класына қойылған жартылай периодты Дирихле есебінің сингулярлық сандары (s-сандардың) мен меншікті сандарының бағасы алынды.

Кілт сөздер: эллиптикалық оператор, шекаралық есеп, сингулярлық сандар, дәрежелік өзгешелену, шешім, жалғыздық.

С.Ж. Игисинов, Л.Д. Жумалиева, А.О. Сулеймбекова, Е.Н. Баяндиев

Таразский региональный университет имени М.Х. Дулати, Тараз, Казахстан

Оценки сингулярных чисел (s-чисел) для одного класса вырождающихся эллиптических операторов

В статье изучен один класс вырождающихся эллиптических уравнений с произвольным степенным вырождением на прямой. На основе исследований, проведенных в ходе работы, авторами предложены методы, позволяющие преодолеть различные трудности, связанные с поведением функций из области определения дифференциального оператора с кусочно-непрерывными коэффициентами в ограниченной области, которые влияют на спектральные характеристики краевых задач для вырождающихся эллиптических уравнений. Показаны условия, наложенные на коэффициенты при младших членах уравнения, обеспечивающие существование и единственность решения. Доказаны существование, единственность и гладкость решения, а также найдены оценки сингулярных чисел (s-чисел) и собственных чисел полупериодической задачи Дирихле для одного класса вырождающихся эллиптических уравнений с произвольным степенным вырождением.

Ключевые слова: эллиптический оператор, краевая задача, сингулярные числа, степенное вырождение, решение, единственность.

References

1 Smirnov, M.M. (1966). *Vyrozhdaiushchiesia ellipticheskie i giperbolicheskie uravneniia [Degenerate elliptic and hyperbolic equations]*. Moscow: Nauka [in Russian].

2 Keldysh, M.V. (1951). *O nekotorykh sluchaiakh vyrozhdeniia uravnenii ellipticheskogo tipa na granitse oblasti [On some cases of degeneration of equations of elliptic type on the boundary of the domain]. Doklady Akademii nauk SSSR — Proceedings of the Academy of Sciences USSR, 77*(2), 181–183 [in Russian].

3 Kalmenov, T.Sh., & Otelbaev, M. (1977). *O gladkosti reshenii odnogo klassa vyrozhdaiushchikhsia ellipticheskikh uravnenii [On the smoothness of solutions of a class of degenerate elliptic equations]. Differentsialnye uravneniia — Differential Equations, 13*(7), 1244–1255 [in Russian].

4 Oleinik, O.A. (1965). *O gladkosti reshenii vyrozhdaiushchikhsia ellipticheskikh i parabolicheskikh uravnenii [On the smoothness of solutions of degenerating elliptic and parabolic equations]. Doklady Akademii nauk SSSR — Proceedings of the Academy of Sciences USSR, 163*(3), 557–580 [in Russian].

5 Vishik, M.I., & Grushin, V.V. (1969). *Ob odnom klasse vyrozhdaiushchikhsia ellipticheskikh uravnenii vysshikh poriadkov [On a class of degenerate elliptic equations of higher orders]. Matematicheskii sbornik — Mathematical collection, 79*(1), 3–36 [in Russian].

6 Vishik, M.I., & Grushin, V.V. (1969). *Uravneniia 2-go poriadka s neotritsatelnoi kharakteristicheskoi formoi [Second-order equations with nonnegative characteristic form]. Matematicheskii sbornik — Mathematical collection, 80*(4), 445–491 [in Russian].

7 Muratbekov, M.B., & Muratbekov, M.M. (2007). Estimates of spectrum for a class of mixed type operators. *Differential equations, 43*, 143–146.

8 Muratbekov, M.B., Muratbekov, M.M., & Abylayeva, A.M. (2013). On existence of the resolvent and discreteness of the spectrum of a class of differential operators of hyperbolic type. *Electronic Journal of theory of differential equations, 63*, 1–10.

9 Muratbekov, M.B., Kalmenov, T.Sh., & Muratbekov, M.M. (2015). On discreteness of the spectrum of a class of mixed type singular differential operators. *Complex Variables and Elliptic Equations, 60*(12), 1752–1763.

10 Muratbekov, M., & Otelbaev, M. (2016). On the existence of a resolvent and separability for a class of singular hyperbolic type differential operators on an unbounded domain. *Eurasian mathematical journal, 7*(1), 50–67.

11 Gohberg, I.C., & Krein, M.G. (1969). *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space (Translations of mathematical monographs)*. American Mathematical society.

DOI 10.31489/2022M3/59-69
UDC 517.95

U.K. Koilyshov[1,2,*], K.A. Beisenbaeva[3], S.D. Zhapparova[1]

[1] *Al-Farabi Kazakh National University, Almaty, Kazakhstan;*
[2] *Institute of Mathematics and Mathematical Modelling, Almaty, Kazakhstan;*
[3] *Academy of Logistics and Transport, Almaty, Kazakhstan*
(E-mail: koylyshov@mail.ru, beysenbaeva56@mail.ru, zhapparova.saule@gmail.com)

A priori estimate of the solution of the Cauchy problem in the Sobolev classes for discontinuous coefficients of degenerate heat equations

Partial differential equations of the parabolic type with discontinuous coefficients and the heat equation degenerating in time, each separately, have been well studied by many authors. Conjugation problems for time-degenerate equations of the parabolic type with discontinuous coefficients are practically not studied. In this work, in an n-dimensional space, a conjugation problem is considered for a heat equation with discontinuous coefficients which degenerates at the initial moment of time. A fundamental solution to the set problem has been constructed and estimates of its derivatives have been found. With the help of these estimates, in the Sobolev classes, the estimate of the solution to the set problem was obtained.

Keywords: conjugation problem, heat equation, degenerating equations, discontinuous coefficients.

Partial differential equations of the parabolic type with discontinuous coefficients are studied in the works [1–8]. Time-degenerate equations of heat conduction are studied in the works [9, 10]. The conjugation problems for the periodic equations of the parabolic type with discontinuous coefficients are slightly studied. We consider the Cauchy problem for a degenerating equation with discontinuous coefficients: find functions $u_1(x,t), u_2(x,t)$ that satisfy the equations

$$t^p \frac{\partial u_1}{\partial t} = a_1^2 \Delta u_1 + f_1(x,t), \quad (x,t) \in D_{n+1}^- = \{(x,t), x' \in R^{n-1}, x_n < 0, t > 0\}, \tag{1}$$

$$t^p \frac{\partial u_2}{\partial t} = a_2^2 \Delta u_2 + f_2(x,t), \quad (x,t) \in D_{n+1}^+ = \{(x,t), x' \in R^{n-1}, x_n > 0, t > 0\}, \tag{2}$$

with initial conditions

$$u_1(x,0) = \varphi_1(x), \quad u_2(x,0) = \varphi_2(x), \tag{3}$$

and with conjugation conditions

$$u_1\Big|_{x_n=-0} = u_2\Big|_{x_n=+0}, \tag{4}$$

$$k_1 \frac{\partial u_1}{\partial x_n}\Big|_{x_n=-0} = k_2 \frac{\partial u_2}{\partial x_n}\Big|_{x_n=+0}, \tag{5}$$

where $x' = (x_1, x_2, ..., x_{n-1})$,
$k_i > 0$, $p < 1$, $(i = 1, 2)$.

The feature of the problem is that equations (1) and (2) with discontinuous coefficients degenerate at the initial moment $t = 0$.

[*]Corresponding author.
E-mail: koylyshov@mail.ru

Method of solving.

To solve problems (1)–(5) let us consider an auxiliary problem A: in the domain $D_{n+1}(x \in R^n$, $t > 0)$, find functions $u_1(x,t), u_2(x,t)$ that satisfy the equations

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + f_1(x,t), \quad (x,t) \in D^-_{n+1} = \{(x,t), x' \in R^{n-1}, x_n < 0, t > 0\}, \tag{6}$$

$$\frac{\partial u_2}{\partial t} = \Delta u_2 + f_2(x,t), \quad (x,t) \in D^+_{n+1} = \{(x,t), x' \in R^{n-1}, x_n > 0, t > 0\}, \tag{7}$$

with initial conditions

$$u_1(x,0) = \varphi_1(x), \quad u_2(x,0) = \varphi_2(x), \tag{8}$$

and with conjugation conditions

$$u_1\Big|_{x_n=-0} = u_2\Big|_{x_n=+0}, \tag{9}$$

$$k_1\frac{\partial u_1}{\partial x_n}\Big|_{x_n=-0} = k_2\frac{\partial u_2}{\partial x_n}\Big|_{x_n=+0}, \tag{10}$$

where $k_i > 0$, $(i = 1,2)$. Applying to problem (6)–(10) the Fourier transform with respect to variables $x' = (x_1, x_2, ..., x_{n-1})$ and the Laplace transform with respect to variable t, we obtain an inhomogeneous second-order differential equation

$$\frac{d^2\,\tilde{\tilde{u}}_1}{dx_n^2} - \left(p + |s'|^2\right)\tilde{\tilde{u}}_1 = -\tilde{\tilde{f}}_1(s', x_n, p) - \tilde{\varphi}_1(s', x_n), \quad x_n < 0, \tag{11}$$

$$\frac{d^2\,\tilde{\tilde{u}}_2}{dx_n^2} - \left(p + |s'|^2\right)\tilde{\tilde{u}}_2 = -\tilde{\tilde{f}}_2(s', x_n, p) - \tilde{\varphi}_2(s', x_n), \quad x_n > 0, \tag{12}$$

where $s' = (s_1, s_2, ..., s_{n-1})$. Conjugation conditions (9)–(10) take the following form:

$$\tilde{\tilde{u}}_1\Big|_{x_n=-0} = \tilde{\tilde{u}}_2\Big|_{x_n=+0}, \tag{13}$$

$$k_1\frac{d\tilde{\tilde{u}}_1}{dx_n}\Big|_{x_n=-0} = k_2\frac{d\tilde{\tilde{u}}_2}{dx_n}\Big|_{x_n=+0}, \tag{14}$$

The solutions to equations (11)–(12) have the form:

$$\tilde{\tilde{u}}_1(s', x_n, p) = \left(c_1 - \frac{1}{2\sqrt{p+|s'|^2}}\int_0^{x_n}\tilde{\tilde{F}}_1(s', \xi_n, p)e^{-\sqrt{p+|s'|^2}\xi_n}\,d\xi_n\right)e^{\sqrt{p+|s'|^2}x_n} +$$

$$+\left(c_2 + \frac{1}{2\sqrt{p+|s'|^2}}\int_0^{x_n}\tilde{\tilde{F}}_1(s', \xi_n, p)e^{\sqrt{p+|s'|^2}\xi_n}\,d\xi_n\right)e^{-\sqrt{p+|s'|^2}x_n}, \quad x_n < 0,$$

$$\tilde{\tilde{u}}_2(s', x_n, p) = \left(d_1 - \frac{1}{2\sqrt{p+|s'|^2}}\int_0^{x_n}\tilde{\tilde{F}}_2(s', \xi_n, p)e^{-\sqrt{p+|s'|^2}\xi_n}\,d\xi_n\right)e^{\sqrt{p+|s'|^2}x_n} +$$

$$+\left(d_2 + \frac{1}{2\sqrt{p+|s'|^2}}\int_0^{x_n}\tilde{\tilde{F}}_2(s', \xi_n, p)e^{\sqrt{p+|s'|^2}\xi_n}\,d\xi_n\right)e^{-\sqrt{p+|s'|^2}x_n}, \quad x_n > 0,$$

here $\widetilde{\widetilde{F}}_i(s', x_n, p) = \widetilde{\widetilde{f}}_i(s', x_n, p) + \widetilde{\varphi}_i(s', x_n)$, $(i = 1, 2)$. We obtain a solution to problem (11)-(14):

$$\widetilde{u}_1(s', x_n, p) = \int\limits_{-\infty}^{0} \frac{\widetilde{\widetilde{F}}_1(s', \xi_n, p)}{2\sqrt{p+|s'|^2}} \left(e^{-\sqrt{p+|s'|^2}|x_n-\xi_n|} + \lambda e^{\sqrt{p+|s'|^2}(x_n+\xi_n)} \right) d\xi_n +$$
$$+ \mu_2 \int\limits_{0}^{+\infty} \frac{\widetilde{\widetilde{F}}_2(s', \xi_n, p)}{2\sqrt{p+|s'|^2}} e^{-\sqrt{p+|s'|^2}(\xi_n-x_n)} d\xi_n, \quad x_n < 0,$$

$$\widetilde{u}_2(s', x_n, p) = \int\limits_{0}^{+\infty} \frac{\widetilde{\widetilde{F}}_2(s', \xi_n, p)}{2\sqrt{p+|s'|^2}} \left(e^{-\sqrt{p+|s'|^2}|x_n-\xi_n|} - \lambda e^{-\sqrt{p+|s'|^2}(x_n+\xi_n)} \right) d\xi_n +$$
$$+ \mu_1 \int\limits_{-\infty}^{0} \frac{\widetilde{\widetilde{F}}_1(s', \xi_n, p)}{2\sqrt{p+|s'|^2}} e^{-\sqrt{p+|s'|^2}(x_n-\xi_n)} d\xi_n, \quad x_n > 0,$$

here $\lambda = \frac{k_1-k_2}{k_1+k_2}$, $\quad \mu_i = \frac{2k_i}{k_1+k_2}$, $\quad (i = 1, 2)$.

The solutions to equations (6)–(10) have the form:

$$\begin{aligned}
u_1(x,t) = & \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} \left[\frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n} + \lambda \frac{e^{-\frac{|x'-\xi'|^2+(x_n+\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n} \right] \varphi_1(\xi', \xi_n)\, d\xi' d\xi_n + \\
& + \mu_2 \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n} \varphi_2(\xi', \xi_n)\, d\xi' d\xi_n + \\
& + \int\limits_{0}^{t} d\tau \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} \left[\frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4(t-\tau)}}}{\left(2\sqrt{\pi (t-\tau)}\right)^n} + \lambda \frac{e^{-\frac{|x'-\xi'|^2+(x_n+\xi_n)^2}{4(t-\tau)}}}{\left(2\sqrt{\pi (t-\tau)}\right)^n} \right] f_1(\xi', \xi_n, \tau)\, d\xi' d\xi_n + \\
& + \mu_2 \int\limits_{0}^{t} d\tau \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4(t-\tau)}}}{\left(2\sqrt{\pi (t-\tau)}\right)^n} f_2(\xi', \xi_n, \tau)\, d\xi' d\xi_n, \quad D_n^-, \\
u_2(x,t) = & \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \left[\frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n} - \lambda \frac{e^{-\frac{|x'-\xi'|^2+(x_n+\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n} \right] \varphi_2(\xi', \xi_n)\, d\xi' d\xi_n + \\
& + \mu_1 \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} \frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n} \varphi_1(\xi', \xi_n)\, d\xi' d\xi_n + \\
& + \int\limits_{0}^{t} d\tau \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \left[\frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4(t-\tau)}}}{\left(2\sqrt{\pi (t-\tau)}\right)^n} - \lambda \frac{e^{-\frac{|x'-\xi'|^2+(x_n+\xi_n)^2}{4(t-\tau)}}}{\left(2\sqrt{\pi (t-\tau)}\right)^n} \right] f_2(\xi', \xi_n, \tau)\, d\xi' d\xi_n + \\
& + \mu_1 \int\limits_{0}^{t} d\tau \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} \frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4(t-\tau)}}}{\left(2\sqrt{\pi (t-\tau)}\right)^n} f_1(\xi', \xi_n, \tau)\, d\xi' d\xi_n, \quad D_n^+,
\end{aligned} \tag{15}$$

where $d\xi' = d\xi_1 d\xi_2 \cdot ... \cdot d\xi_{n-1}$, $\quad |x' - \xi'| = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + ... + (x_{n-1}-\xi_{n-1})^2}$. We introduce the notation $G(x'-\xi', x_n \pm \xi_n, t) = \frac{e^{-\frac{|x'-\xi'|^2+(x_n\pm\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n}$. Then

$$u_1(x,t) = \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} \Big[G(x'-\xi', x_n-\xi_n, t) + \lambda\, G(x'-\xi', x_n+\xi_n, t)\Big]\varphi_1(\xi', \xi_n)\, d\xi' d\xi_n +$$
$$+\,\mu_2 \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} G(x'-\xi', x_n-\xi_n, t)\varphi_2(\xi', \xi_n)\, d\xi' d\xi_n +$$
$$+\int\limits_0^t d\tau \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} \Big[G(x'-\xi', x_n-\xi_n, t-\tau) + \lambda\, G(x'-\xi', x_n+\xi_n, t-\tau)\Big] f_1(\xi', \xi_n, \tau)\, d\xi' d\xi_n + \quad (16)$$
$$+\,\mu_2 \int\limits_0^t d\tau \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} G(x'-\xi', x_n-\xi_n, t-\tau) f_2(\xi', \xi_n, \tau)\, d\xi' d\xi_n, \quad D_n^-,$$

$$u_2(x,t) = \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \Big[G(x'-\xi', x_n-\xi_n, t) - \lambda\, G(x'-\xi', x_n+\xi_n, t)\Big]\varphi_2(\xi', \xi_n)\, d\xi' d\xi_n +$$
$$+\,\mu_1 \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} G(x'-\xi', x_n-\xi_n, t)\varphi_1(\xi', \xi_n)\, d\xi' d\xi_n +$$
$$+\int\limits_0^t d\tau \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \Big[G(x'-\xi', x_n-\xi_n, t-\tau) - \lambda\, G(x'-\xi', x_n+\xi_n, t-\tau)\Big] f_2(\xi', \xi_n, \tau)\, d\xi' d\xi_n + \quad (17)$$
$$+\,\mu_1 \int\limits_0^t d\tau \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} G(x'-\xi', x_n-\xi_n, t-\tau) f_1(\xi', \xi_n, \tau)\, d\xi' d\xi_n, \quad D_n^+.$$

We have obtained the solution to auxiliary problem (6)–(10) in the form (16)–(17).

Using [11], for the function $\Gamma(x,t) = \frac{e^{-\frac{|x|^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n}$, we obtain an estimate:

$$|D_x^k D_t^m \Gamma(x,t)| \le \frac{Ce^{-\delta\frac{|x|^2}{t}}}{t^{\frac{n+k}{2}+m}}.$$

This estimate is valid from [12]. Here $\delta < \frac{1}{4}$.

For the function $G(x'-\xi', x_{n-1}-\xi_{n-1}, t)$ the same estimate can be given:

$$|D_x^k D_t^m G(x'-\xi', x_{n-1}-\xi_{n-1}, t)| \le \frac{Ce^{-\delta\frac{|x-\xi|^2}{t}}}{t^{\frac{n+k}{2}+m}}.$$

Now consider the auxiliary problem B. Consider the Cauchy problem for a degenerate heat equation: in the domain $D_{n+1}^+ = \{(x,t), x \in R^n, t > 0\}$ to find a function $u(x,t)$ that satisfies the equation

$$t^p \frac{\partial u}{\partial t} = \Delta u + f(x,t), \quad (x,t) \in D_{n+1} = \{(x,t), x \in R^n, t > 0\}, \quad (18)$$

with initial condition

$$u(x,0) = \varphi(x). \quad (19)$$

By applying the Fourier transform in variables $x = (x_1, ..., x_n)$ to equation (18)

$$t^p \frac{\partial \widetilde{u}}{\partial t} + |s|^2 \widetilde{u} = \widetilde{f}(s,t), \tag{20}$$

we obtain a non-homogeneous differential equation of the first order. Here $s = (s_1, s_2, ..., s_n)$, $|s| = \sqrt{s_1^2 + s_2^2 + ... + s_n^2}$, $\quad p < 1$. The initial condition (19) takes the following form:

$$\widetilde{u}(s,0) = \widetilde{\varphi}(s). \tag{21}$$

Taking into account the initial condition (21), the solution to equation (20) has the form:

$$\widetilde{u}(s,t) = \widetilde{\varphi}(s) e^{-\frac{t^q}{q}|s|^2} + \int\limits_0^t \frac{\widetilde{f}(s,\tau)}{\tau^p} e^{-\frac{(t^q - \tau^q)}{q}|s|^2}\, d\tau, \tag{22}$$

here $q = 1 - p$.
Applying the inverse Fourier transform to equality (22), using the convolution formula, formulas [13] and (15), we obtain a solution to problem (18)–(19):

$$u(x,t) = \int\limits_{R^n} \frac{q^{\frac{n}{2}}}{\left(2\sqrt{\pi t^q}\right)^n} e^{-\frac{q|x-\xi|^2}{4t^q}} \varphi(\xi)\, d\xi + \int\limits_0^t \frac{d\tau}{\tau^p} \int\limits_{R^n} \frac{q^{\frac{n}{2}}}{\left(2\sqrt{\pi\left(t^q - \tau^q\right)}\right)^n} e^{-\frac{q|x-\xi|^2}{4(t^q-\tau^q)}} f(\xi,\tau)\, d\xi. \tag{23}$$

If we introduce the notation $\Gamma_q(x,t) = \frac{q^{\frac{n}{2}}}{\left(2\sqrt{\pi t^q}\right)^n} e^{-\frac{q|x|^2}{4t^q}}$, then formula (23) can be written in the form:

$$u(x,t) = \int\limits_{R^n} \Gamma_q(x-\xi,t)\varphi(\xi)\, d\xi + \int\limits_0^t \frac{d\tau}{\tau^p} \int\limits_{R^n} \Gamma_q(x-\xi,t-\tau) f(\xi,\tau)\, d\xi. \tag{24}$$

In [14], the function $\Gamma_q(x,t)$ was constructed in one-dimensional space. As shown in [12], for this function we can accept the following estimate:

$$|D_x^k D_t^m \Gamma_q(x,t)| \leq \frac{C e^{-\delta \frac{|x|^2}{t^q}}}{t^{\frac{q(n+k)}{2}+m}}, \tag{25}$$

where $\delta < \frac{1}{4}$.

The results of research.

Now let us solve the main problem (1)–(5). Using the solutions to auxiliary problems A and B, the solutions of which have the form (16)–(17) and (24), we can obtain the solution to problem (1)–(5) in the form:

$$u_1(x,t) = \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} \Big[G_q(x'-\xi', x_n-\xi_n, t) + \lambda\, G_q(x'-\xi', x_n+\xi_n, t)\Big]\varphi_1(\xi',\xi_n)\, d\xi' d\xi_n +$$
$$+ \mu_2 \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} G_q(x'-\xi', x_n-\xi_n, t)\varphi_2(\xi',\xi_n)\, d\xi' d\xi_n +$$
$$+ \int\limits_0^t \frac{d\tau}{\tau^p} \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} \Big[G_q(x'-\xi', x_n-\xi_n, t-\tau) + \lambda\, G_q(x'-\xi', x_n+\xi_n, t-\tau)\Big] f_1(\xi',\xi_n,\tau)\, d\xi' d\xi_n + \qquad (26)$$
$$+ \mu_2 \int\limits_0^t \frac{d\tau}{\tau^p} \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} G_q(x'-\xi', x_n-\xi_n, t-\tau) f_2(\xi',\xi_n,\tau)\, d\xi' d\xi_n, \quad D_n^-,$$

$$u_2(x,t) = \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \Big[G_q(x'-\xi', x_n-\xi_n, t) - \lambda\, G_q(x'-\xi', x_n+\xi_n, t)\Big]\varphi_2(\xi',\xi_n)\, d\xi' d\xi_n +$$
$$+ \mu_1 \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} G_q(x'-\xi', x_n-\xi_n, t)\varphi_1(\xi',\xi_n)\, d\xi' d\xi_n +$$
$$+ \int\limits_0^t \frac{d\tau}{\tau^p} \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \Big[G_q(x'-\xi', x_n-\xi_n, t-\tau) - \lambda\, G_q(x'-\xi', x_n+\xi_n, t-\tau)\Big] f_2(\xi',\xi_n,\tau)\, d\xi' d\xi_n + \qquad (27)$$
$$+ \mu_1 \int\limits_0^t \frac{d\tau}{\tau^p} \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} G_q(x'-\xi', x_n-\xi_n, t-\tau) f_1(\xi',\xi_n,\tau)\, d\xi' d\xi_n, \quad D_n^+,$$

where $G_q(x'-\xi', x_n \pm \xi_n, t) = \frac{q^{\frac{n}{2}} e^{-\frac{-q|x'-\xi'|^2+(x_n\pm\xi_n)^2}{4t^q}}}{\left(2\sqrt{\pi t^q}\right)^n}$. Thus, we have completely solved problem (1)-(5). It is easy to check that the obtained solutions (26)-(27) satisfy equations (1)-(2), initial conditions (3) and conjugation conditions (4)-(5). A similar estimate can be obtained for the function $G_q(x'-\xi', x_n-\xi_n, t)$:

$$|D_x^k D_t^m G_q(x'-\xi', x_n-\xi_n, t)| \le \frac{Ce^{-\delta\frac{|x-\xi|^2}{t^q}}}{t^{\frac{q(n+k)}{2}+m}}. \qquad (28)$$

The solution to problem (1)–(5) and estimates (25) and (28) can later be used in the study of differential properties and obtaining a priori estimates of initial-boundary value problems in the Sobolev and Holder classes for non-stationary heat equations.

Let us consider the following potential of the initial condition:

$$h_q(x,t) = \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \frac{q^{\frac{n}{2}}}{\left(2\sqrt{\pi t^q}\right)^n} e^{-\frac{q|x-\xi|^2}{4t^q}} \varphi(\xi',\xi_n)\, d\xi' d\xi_n,$$

$$h_q(x,t) = \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} G_q(x-\xi,t)\varphi(\xi',\xi_n)\, d\xi' d\xi_n = \int\limits_{R^n} G_q(x-\xi,t)\varphi^*(\xi',\xi_n)\, d\xi' d\xi_n,$$

here

$$\varphi^*(\xi', \xi_n) = \begin{cases} \varphi(\xi', \xi_n), & \xi_n > 0, \\ 0, & \xi_n < 0 \end{cases},$$

$$h_q(x,t) = \int\limits_{R^n} G_q(x-\xi, t)\varphi^*(\xi', \xi_n)\, d\xi = \Big| x - \xi = y \Big| = \int\limits_{R^n} G_q(y,t)\varphi^*(x-y)\, dy,$$

$$D_t h_q(x,t) = \int\limits_{R^n} D_t G_q(y,t)\varphi^*(x-y)\, dy.$$

As $D_tG_q(-y,t) = D_tG_q(y,t)$ is a even function, at the same time, for any $t > 0$ $\int\limits_{R^n} D_tG_q(y,t)dy = 0$. It can be written as follows: $D_t h_q(x,t) = \frac{1}{2}\int\limits_{R^n} D_tG_q(y,t)\Big[\varphi^*(x-y) - 2\varphi^*(x) + \varphi^*(x+y)\Big]\, dy$. Using Minkowski's inequality:

$$\left(\int\limits_{R^n} \Big| D_t h_q(x,t)\Big|^s dx\right)^{\frac{1}{s}} = \frac{1}{2}\int\limits_{R^n} \Big| D_tG_q(y,t)\Big| \left(\int\limits_{R^n} \Big| \Big[\varphi^*(x-y) - 2\varphi^*(x) + \varphi^*(x+y)\Big]\, dy \Big|^s dx\right)^{\frac{1}{s}},$$

$|D_tG_q(y,t)| \le \frac{Ce^{-\frac{|y|^2}{8t^q}}}{t^{\frac{qn}{2}+1}}$, taking into account the inequality, we obtain the following estimate:

$$\|D_t h_q(x,t)\|_{s,R^n} \le \frac{C}{t^{\frac{qn}{2}+1}} \int\limits_{R^n} e^{-\frac{|y|^2}{8t^q}} \cdot N(y)dy, \tag{29}$$

where $N(y) = \|\varphi^*(x-y) - 2\varphi^*(x) + \varphi^*(x+y)\|_{s,R^n}$. We write inequality (29) as follows:

$$\|D_t h_q(x,t)\|_{s,R^n} \le \frac{C}{t^{\frac{qn}{2}+1}} \int\limits_{R^n} e^{-\frac{|y|^2}{8t^q s}} \cdot e^{-\frac{|y|^2}{8t^q s'}} \cdot N(y)dy,$$

where $\frac{1}{s} + \frac{1}{s'} = 1$. Then using the Gelder inequality:

$$\|D_t h_q(x,t)\|_{s,R^n} \le \frac{C}{t^{\frac{qn}{2}+1}} \left(\int\limits_{R^n} e^{-\frac{|y|^2}{8t^q}} \cdot N^s(y)\, dy\right)^{\frac{1}{s}} \left(\int\limits_{R^n} e^{-\frac{|y|^2}{8t^q}}\, dy\right)^{\frac{1}{s'}},$$

taking into account $\frac{1}{s'} = 1 - \frac{1}{s}$ we get

$$\left(\int\limits_{R^n} e^{-\frac{|y|^2}{8t^q}}\, dy\right)^{1-\frac{1}{s}} \le C_1 t^{\frac{nq}{2} - \frac{nq}{2s}}.$$

So

$$\|D_t h_q(x,t)\|_{s,R^n} \le \frac{C_1}{t^{1+\frac{nq}{2s}}} \left(\int\limits_{R^n} e^{-\frac{|y|^2}{8t^q}} \cdot N^s(y)\, dy\right)^{\frac{1}{s}}.$$

Now let us take a norm $\|D_t h_q(x,t)\|_{s,D_{n+1}}$. Then from the last inequality we get:

$$\|D_t h_q(x,t)\|_{s,D_{n+1}} \le C_1 \left(\int\limits_0^{+\infty} \frac{dt}{t^{s+\frac{nq}{2}}} \int\limits_{R^n} e^{-\frac{|y|^2}{8t^q}} \cdot N^s(y)\, dy\right)^{\frac{1}{s}} = C_1 \left(\int\limits_{R^n} N^s(y)\, dy \int\limits_0^{+\infty} \frac{e^{-\frac{|y|^2}{8t^q}}}{t^{s+\frac{nq}{2}}}\, dt\right)^{\frac{1}{s}},$$

if we introduce $\frac{|y|^2}{8t} = z$ a replacement:

$$\|D_t h_q(x,t)\|_{s,D_{n+1}} \le C_2\Big(\int\limits_{R^n} \frac{N^s(y)}{|y|^{\frac{2s}{q}+n-\frac{2}{q}}}\,dy\Big)^{\frac{1}{s}} = C_2\cdot\Big(\int\limits_{R^n}\int\limits_{R^n} \frac{\Big|\varphi^*(x-y)-2\varphi^*(x)+\varphi^*(x+y)\Big|^s}{|y|^{\frac{2s}{q}+n-\frac{2}{q}}}\,dxdy\Big)^{\frac{1}{s}},$$

$$\ll \varphi \gg_{W_s^{\frac{2}{q}-\frac{2}{qs}}(R^n)} = \Big(\int\limits_{R^n} dx \int\limits_{R^n} \frac{\Big|\varphi(x-y)-2\varphi(x)+\varphi(x+y)\Big|^s}{|y|^{\frac{2s}{q}+n-\frac{2}{q}}}\,dy\Big)^{\frac{1}{s}}. \tag{30}$$

Given that (30), then

$$\|D_t h_q(x,t)\|_{s,D_{n+1}} \le C \ll \varphi^* \gg_{W_s^{\frac{2}{q}-\frac{2}{qs}}(R^n)}.$$

As estimates $D_x^s h_q(-x,t) = D_x^s h_q(x,t)$ and $D_x^s G_q(x,t)$ are consistent with estimate $D_t G_q(x,t)$, the estimate $\|D_x^2 h_q(x,t)\|_{s,D_{n+1}}$ is also taken similarly. Therefore, the following inequality is obtained:

$$\|D_x^2 h_q(x,t)\|_{s,D_{n+1}} \le C \ll \varphi^* \gg_{W_s^{\frac{2}{q}-\frac{2}{qs}}(R^n)}.$$

Theorem 1. The potential of the initial condition satisfies the estimate:

$$\ll h_q(x,t) \gg_{W_s^{2,1}(D_{n+1})} \le C \ll \varphi^* \gg_{W_s^{\frac{2}{q}-\frac{2}{qs}}(R^n)},$$

where

$$\ll h_q(x,t) \gg_{W_s^{2,1}(D_{n+1})} = \|\frac{\partial h_q}{\partial t}\|_{s,D_{n+1}} + \sum_{k,j=1}^{n} \|\frac{\partial^2 h_q}{\partial x_k \partial x_j}\|_{s,D_{n+1}}.$$

This notation $\ll . \gg$ means the main part of the norm in the Sobolev classes.

Consider the following volume potential:

$$g_q(x,t) = \int\limits_0^t \frac{d\tau}{\tau^p} \int\limits_{R^{n-1}} \int\limits_0^{+\infty} G_q(x-\xi, t-\tau) f(\xi', \xi_n, \tau)\, d\xi' d\xi_n.$$

Using the method [15], the following theorem can be proved.

Theorem 2. The following estimates are appropriate for the volume potential:

$$\ll g_q(x,t) \gg_{W_s^{2,1}(D_{n+1})} \le C\|f\|_{W_s^{2,1}(D_{n+1})}, \quad (1 < q < \infty),$$

where

$$\ll g_q(x,t) \gg_{W_s^{2,1}(D_{n+1})} = \|\frac{\partial g_q}{\partial t}\|_{s,D_{n+1}} + \sum_{k,j=1}^{n} \|\frac{\partial^2 g_q}{\partial x_k \partial x_j}\|_{s,D_{n+1}}.$$

This notation $\ll . \gg$ means the main part of the norm in the Sobolev classes.

References

1 Самарский А.А. Параболические уравнения с разрывными коэффициентами / А.А. Самарский // ДАН СССР. — 1958. — Т. 121. — №2. — С. 225–228.

2 Ким Е.И. О распределении температуры в кусочно-однородной полубесконечной пластинке / Е.И. Ким, Б.Б. Баймуханов // ДАН СССР. — 1961. — Т. 140. — №2. — С. 333–336.

3 Камынин Л.И. О решении краевых задач для параболического уравнения с разрывными коэффициентами / Л.И. Камынин // ДАН СССР. — 1961. — Т. 139. — №5. — С. 1048–1051.

4 Камынин Л.И. О решении IV и V краевых задач для одномерного параболического уравнения второго порядка в криволинейной области / Л.И. Камынин // Журн. вычисл. мат. и мат. физики. — 1969. — Т. 9. — №3. — С. 558–572.

5 Камынин Л.И. О методе потенциалов для параболического уравнения с разрывными коэффициентами / Л.И. Камынин // ДАН СССР. — 1962. — Т. 145. — №6. — С. 1213–1216.

6 Койлышов У.К. О дифференциальных свойствах решения задачи Коши для уравнения теплопроводности с разрывными коэффициентами в соболевских классах / У.К. Койлышов, М.А. Абдрахманов // Вестн. Казах. гос. ун-та. Серия мат., мех., инф. — 1998. — №14. — С. 102–108.

7 Ким Е.И. Решение задачи теории теплопроводности с разрывным коэффициентом и вырождающимися подвижными границами / Е.И. Ким, У.К. Койлышов // Изв. АН КазССР. Сер.физ.-мат. — 1984. — №3. — С. 35–39.

8 Koilyshov U. A conjugation problem for the heat equation in the field where the boundary moves in linear order / U. Koilyshov, K. Beisenbaeva // Bulletin of the Karaganda University-Mathematics. — 2019. — №3(95). — P. 26–32.

9 Ильин А.М. Вырождающиеся эллиптические и параболические уравнения / А.М. Ильин // Мат. сб. — 1960. — Т. 50(92). — №4. — С. 443–498.

10 Смирнова Г.Н. Линейные параболические уравнения, вырождающиеся на границе области / Г.Н. Смирнова // СМЖ. — 1963. — Т. 4. — №2. — С. 343-358.

11 Ладыженская О.А. Линейные и квазилинейные уравнения параболического типа / О.А. Ладыженская, В.А. Солонников, Н.Н. Уральцева. — М.: Наука, 1967. — 736 с.

12 Солонников В.А. Априорные оценки для уравнений второго порядка парабалического типа / В.А. Солонников // Тр. МИАН СССР им. В.А.Стеклова. — 1964. — Т. 70. — С. 133–212.

13 Диткин В.А. Интегральные преобразования и операционное исчисление / В.А. Диткин, А.П. Прудников. — М.: Наука, 1974. — С. 544.

14 Койлышов У.К. Решение одной краевой задачи для вырождающегося уравнения теплопроводности в области с подвижной границей / У.К. Койлышов, К.А. Бейсенбаева // Вестн. КазНИТУ. — 2020. — №3. — С. 623–626.

15 Абдрахманов М.А. Оценки тепловых потенциалов в гёльдеровских и соболевских классах (курс лекций) / М.А. Абдрахманов. — Алматы: Компьютерный центр ИТПМ МН-АН РК, 1997. — С. 51.

У.К. Койлышов[1,2], К.А. Бейсенбаева[3], С.Д. Жаппарова[1]

[1]*Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан;*
[2]*ҚР ҒБЖ БжҒМ ҒК Математика және математикалық модельдеу институты, Алматы, Қазақстан;*
[3]*Логистика және көлік академиясы, Алматы, Қазақстан*

Коэффициенті үзілісті жылуөткізгіштік теңдеу үшін Коши есебі шешімінің соболев класындағы априорлық бағасы

Коэффициенттері үзілісті параболалық типті дербес туындылы дифференциалдық теңдеулер және уақыт бойынша өзгешеленген жылуөткізгіштік теңдеулердің әрқайсысы жеке-жеке көптеген авторлармен жақсы зерттелген. Коэффициенті үзілісті уақыт бойынша өзгешеленген параболалық типті теңдеулер үшін түйіндес есептер іс жүзінде зерттелмеген. Мақалада n-өлшемді кеңістікте бастапқы уақыт мезетіндегі коэффициенттері үзілісті өзгешеленген жылуөткізгіштік теңдеу үшін бір түйіндес есеп қарастырылған. Қойылған есептің іргелі шешімі табылды және оның туындыларының бағасы алынды. Алынған нәтижені қолдана отырып, берілген есептің шешімінің соболев класындағы бағасы табылды.

Кілт сөздер: түйіндес есеп, жылуөткізгіштік теңдеу, өзгешеленген теңдеу, үзілісті коэффициенттер.

У.К. Койлышов[1,2], К.А. Бейсенбаева[3], С.Д. Жаппарова[1]

[1]*Казахский национальный университет имени Аль-Фараби, Алматы, Казахстан;*
[2]*Институт математики и математического моделирования КН МОН РК, Алматы, Казахстан;*
[3]*Академия логистики и транспорта, Алматы, Казахстан*

Априорная оценка решения задачи Коши для вырождающегося уравнения теплопроводности с разрывными коэффициентами в соболевских классах

Дифференциальные уравнения в частных производных параболического типа с разрывными коэффициентами и вырождающиеся по времени уравнения теплопроводности отдельности хорошо изучены многими авторами. Задачи сопряжения для вырождающегося по времени уравнениям параболического типа с разрывными коэффициентами практически не изучены. В статье рассмотрена одна задача сопряжения для уравнения теплопроводности с разрывными коэффициентами, вырождающегося в начальный момент времени в n-мерном пространстве. Построено фундаментальное решение поставленной задачи, и найдена оценка ее производных. С помощью этих оценок получена оценка решения поставленной задачи в соболевских классах.

Ключевые слова: задача сопряжения, уравнения теплопроводности, вырождающиеся уравнения, разрывные коэффициенты.

References

1 Samarskii, A.A. (1958). Parabolicheskie uravneniia s razryvnymi koeffitsientami [Parabolic equations with discontinuous coefficients]. *Doklady Akademii nauk SSSR — Reports of the Academy of Sciences of USSR, 121*(2), 225–228 [in Russian].

2 Kim, E.I., & Baimukhanov, B.B. (1961). O raspredelenii temperatury v kusochno-odnorodnoi polubeskonechnoi plastinke [The temperature distribution in a piecewise homogeneous semi-infinite plate]. *Doklady Akademii nauk SSSR — Reports of the Academy of Sciences of USSR, 140*(2), 333–336 [in Russian].

3 Kamynin, L.I. (1961). O reshenii kraevykh zadach dlia parabolicheskogo uravneniia s razryvnymi koeffitsientami [Solving boundary-value problems for a parabolic equation with discontinuous coefficients]. *Doklady Akademii nauk SSSR — Reports of the Academy of Sciences of USSR, 139*(5), 1048–1051 [in Russian].

4 Kamynin, L.I. (1969). O reshenii IV i V kraevykh zadach dlia odnomernogo parabolicheckogo uravneniia vtorogo poriadka v krivolineinoi oblasti [Solving IV and V boundary-value problems for a one-dimensional parabolic equation of the second order in a curved domain]. *Zhurnal vychiclitelnoi matematiki i matematicheskoi fiziki — Journal of Computational Mathematics and Mathematical Physics, 9*(3), 558–572 [in Russian].

5 Kamynin, L.I. (1962). O metode potentsialov dlia parabolicheskogo uravneniia s razryvnymi koeffitsientami [On the method of potentials for a parabolic equation with discontinuous coefficients]. *Doklady AN SSSR — Reports of the USSR Academy of Sciences, 145*(6), 1213–1216 [in Russian].

6 Koilyshov, U.K., & Abdrakhmanov, M.A. (1998). O differentsialnykh svoistvakh resheniia zadachi Koshi dlia uravneniia teploprovodnosti s razryvnymi koeffitsientami v sobolevskikh klassakh [Differential properties of the solution to the Cauchy problem for the equation of thermal conductivity with discontinuous coefficients in the Sobolev classes]. *Vestnik Kazakhskogo gosudarstvennogo universiteta. Seriia matematika, mekhanika, informatika — Bulletin of KazSU. Mathematics, Mechanics and Computer Science, 14*(6), 102–108 [in Russian].

7 Kim, E.I., & Koilyshov, U.K. (1984). Reshenie zadachi teorii teploprovodnosti s razryvnym koeffitsientom i vyrozhdaiushchimitsia podvizhnymi granitsami [Solving the problem of the theory of thermal conductivity with a discontinuous coefficient and degenerate mobile boundaries]. *Izvestiia Aksdemi Nauk KazSSR. Seriia fizika-matematika — News of Akademy of Sciences of KazSSSR. Physics and Mathematics series, 3*, 35–39 [in Russian].

8 Koilyshov, U., & Beisenbaeva, K. (2019). A conjugation problem for the heat equation in the field where the boundary moves in linear order. *Bulletin of the Karaganda University-Mathematics*, 3(95), 26–32.

9 Ilin, A.M. (1960). Vyrozhdaiushchiesia ellipticheskie i parabolicheskie uravneniia [Degenerate elliptic and parabolic equations]. *Matematicheskii sbornik — Mathematical collection, 50(92)*(4), 443–498 [in Russian].

10 Smirnova, G.N. (1963). Lineinye parabolicheskie uravneniia, vyrozhdaiushchiesia na granitse oblasti [Linear parabolic equations degenerating at the boundary of the domain]. *Sibirskii mathematicheskii zhurnal — Siberian Mathematical journal, 4*(2), 343–358 [in Russian].

11 Ladyzhenskaia, O.A., Solonnikov, V.A., & Uraltseva, N.N. (1967). *Lineinye i kvazilineinye uravneniia parabolicheskogo tipa [Linear and quasi-linear equations of parabolic type].* Moscow: Nauka [in Russian].

12 Solonnikov, V.A. (1964). Apriornye otsenki dlia uravnenii vtorogo poriadka parabalicheskogo tipa [A priori estimates for second-order equations of parabolic type]. *Trudy MIAN SSSR imeni V.A.Steklova — Proceedings V.A.Steklov Matematical Instute of Academy of Science of USSR, 70*, 133–212 [in Russian].

13 Ditkin, V.A., & Prudnikov, A.P. (1974). *Integralnye preobrazovaniia i operatsionnoe ischislenie [Integral transformations and operational calculus].* Moscow: Nauka [in Russian].

14 Koilyshov, U.K., & Beisenbaeva, K.A. (2020). Reshenie odnoi kraevoi zadachi dlia vyrozhdaiushchegosia uravneniia teploprovodnosti v oblasti s podvizhnoi granitsei [Solution of a boundary-value problem for a degenerate heat equation in a region with a movable boundary]. *Vestnik KazNITU — Bulletin of KazNRTU, 3*, 623–626 [in Russian].

15 Abdrakhmanov, M.A. (1997). *Otsenki teplovykh potentsialov v gelderovskikh i sobolevskikh klassakh (kurs lektsii) [Estimates of thermal potentials in the Helder and Sobolev classes (course of lectures)].* Almaty: Kompiuternyi tsentr ITPM MN-AN RK [in Russian].

DOI 10.31489/2022M3/70-84
UDC 517.956

B.D. Koshanov[1,2,*], N. Kakharman[3], R.U. Segizbayeva[4], Zh.B. Sultangaziyeva[1]

[1]*Al-Farabi Kazakh National University, Almaty, Kazakhstan;*
[2]*International University of Information Technology, Almaty, Kazakhstan;*
[3]*Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[4]*Civil Aviation Academy, Almaty, Kazakhstan*
(E-mail: koshanov@list.ru, n.kakharman@math.kz, segizbaeva-55@mail.ru, zhanat_87@mail.ru)

Two theorems on estimates for solutions of one class of nonlinear equations in a finite-dimensional space

The need to study boundary value problems for elliptic parabolic equations is dictated by numerous practical applications in the theoretical study of the processes of hydrodynamics, electrostatics, mechanics, heat conduction, elasticity theory and quantum physics. In this paper, we obtain two theorems on a priori estimates for solutions of nonlinear equations in a finite-dimensional Hilbert space. The work consists of four items. In the first subsection, the notation used and the statement of the main results are given. In the second subsection, the main lemmas are given. The third section is devoted to the proof of Theorem 1. In the fourth section, Theorem 2 is proved. The conditions of the theorems are such that they can be used in studying a certain class of initial-boundary value problems to obtain strong a priori estimates in the presence of weak a priori estimates. This is the meaning of these theorems.

Keywords: finite-dimensional Hilbert space, nonlinear equations, invertible operator, differentiable vector-functions, a priori estimate of solutions.

Introduction

The problem of describing the dynamics of an incompressible fluid, due to its theoretical and applied importance, attracts the attention of many researchers. In mid-2000, the Clay Mathematics Institute formulated this problem as The Millennium Prize Problems on the existence and smoothness of solutions to the Navier-Stokes equations for an incompressible viscous fluid [1].

Countless works were devoted to the solution of this problem even before it was declared the problem of the millennium. Since there are an infinite number of them, we simply do not list them. The given article provides an incomplete list of works [2].

Many first-class mathematicians who managed to solve other important mathematical problems, including those in problems of gas-hydrodynamics considered this problem. Such prominent mathematicians of the 20th century as A.N. Kolmogorov, J. Leray, E. Hopf, J.-L. Lions provided significant results in their works. Complete solution to the problem for two-dimensional case given by O.A. Ladyzhenskaya [3]. In [4], a complete analysis of the current state of the problem and a review of the available literature, as well as proposed methods for solving the problem, are given. In particular, the main problem of the global unique solvability of the three-dimensional Navier-Stokes problem is reduced to the question of finding a strong a priori estimate for all possible solutions. Works [5]–[12] are devoted to the study of the solvability in the whole of equations of the Navier-Stokes type, the continuous dependence of the solution to a parabolic equation and the smoothness of the solution. In papers [13], [14] questions about the formulation and their solvability of boundary value problems for high-order quasi-hyperbolic equations were studied.

In this article, we obtain two theorems on a priory estimates for solutions of nonlinear equations in a finite-dimensional space. These theorems are proved under certain conditions, which are borrowed

[*]Corresponding author.
E-mail: koshanov@list.ru

from the conditions that are satisfied by finite-dimensional approximations of one class of nonlinear initial-boundary value problems, rewritten in "restricted notation".

1 Used conditions and designations. Formulation of the main results

Let H be a finite-dimensional Hilbert space ($10 \le \dim H = N < \infty$) and G is an invertible operator in H such that $\|G\| \le 1$, $\|G^{-1}\| < \infty$. We will be interested in the following equation

$$f(u) := u + L(u) = g \in H.$$

Throughout this paper, $f(u)$ will mean an operation of the form $u + L(u)$, where $L(\cdot)$ is a non-linear transformation.

If ξ is a parameter from $[0, +\infty)$ and the vector $u(\xi)$ is a vector function that is continuously differentiable with respect to the parameter ξ, then we assume that the vector function $L(u(\xi))$ is also continuously differentiable (as well as the expressions arising from $L(u)$ and $f(u)$ below).

We introduce the notation L_u :

$$(L(u(\xi)))_\xi = L_u u_\xi.$$

It is obvious that L_u (for every $u \in H$) will be a linear operator

$$L_u v = (L(u(\xi)))_\xi \Big|_{u_\xi = v}.$$

We have

$$(f(u(\xi)))_\xi = u_\xi + L_u u_\xi = (E + L_u) u_\xi.$$

Here and throughout follows, E is the identity transformation.

Operator adjoint to L_u denote by L_u^*, that is $L_u^* = (L_u)^*$. Denote

$$D_u^* = E + L_u^* D_u^* f(u) = f(u) + L_u^* f(u).$$

If u is a differentiable vector function, then we set

$$(D_u^* f(u))_\xi = M_u u_\xi.$$

Here M_u is a linear operator for fixed is defined by the formula

$$M_u v = (M_u u_\xi) \big|_{u_\xi = v}.$$

We will use the following conditions C1–C4.

Condition 1. If $u, v \in H$, then the transformation L_u and L_u^* continuous in H, $L(0) = 0$ and the conditions are met

$$\|L(u) - L(v)\| \le \psi(\|u\|)\|u - v\|,$$

$$\|L_u - L_v\|_{H \to H} + \|L_u^* - L_v^*\|_{H \to H} \le \psi(\|u\|)\|u - v\|,$$

$$\|D_u^* v\| \le \psi(\|u\|)\|v\|,$$

$$\|M_u v\| \le \psi(\|u\|)\|v\|,$$

where $\|\cdot\| = \|\cdot\|_H$, $\psi(\cdot)$ strictly monotonously increasing on $[0, \infty)$ positive continuous function.

This condition is natural, since H is certainly, as a rule, it is performed. Therefore, we will use this condition often without stipulating. We will sometimes use the above designations without reservations. In addition to them, we give often the frequently used designations

$$\gamma(u) = \langle D_u^* f(u), u\rangle \|u\|^{-2},$$

$$\mu(u) = \|Gu\|^2 \|u\|^{-2},$$

$$S(u) = D_u^* f(u) - \gamma(u)u - K(u)R(u),$$

$$R(u) = G^*Gu - \mu(u)u,$$

$$K(u) = \frac{\left\langle G^*Gu - \mu(u)u, D_u^* f(u) - \frac{u}{\|u\|^2}\right\rangle}{\|G^*Gu\|^2},$$

$$J(u) = \|u\|^2 \exp(-\|f(u)\|^2).$$

Condition 2. If u-operator's own vector G^*G, then the inequality has been fulfilled

$$\|u\|^2 \leq \left(\|f(u)\|^2 + 2\right)^m,$$

where m–integer number, $m \geq 2$.

Condition 3. For any $u \in H$ evaluation is made

$$\|Gu\|^2 \leq d\,\|f(u)\|^2.$$

For some $0 \neq u \in H$

$$\widetilde{K} = \inf \frac{\langle M_u a, a\rangle - \frac{a}{\|u\|^2}}{\|Ga\|^2},$$

where the infinum takes on all such $a \in H$, that

$$\|a\| = 1,\ \langle G^*Gu, a\rangle = \langle u, a\rangle = 0.$$

Condition 4. If $0 \neq u \in H$, $S(u) = 0$, $K(u) \geq 0$, then

$$\widetilde{K}d < 1 - \delta,$$

fair, where $\delta \in (0, 1)$.

Theorem 1. If the conditions C1- C4 are met then for any $u \in H$ fair assessment

$$\|u\|^2 \leq C \exp(\|f(u)\|^2), \tag{1}$$

where C-does not depend on u and depends only on the conditions C2, C3, C4.

Remark 1. Since G is an invertible operator, we immediately have from condition C3 that the following estimate holds:

$$\|u\|^2 \leq \|G^{-1}\|^2\, \|Gu\|^2 \leq d\,\|f(u)\|^2. \tag{2}$$

When approximating an infinite-dimensional problem, the finite-dimensional quantity $\|G^{-1}\|$ can tend to ∞. Therefore, from (2) it is impossible to obtain the estimate for $\|u\|$.

Theorem 1 is extended to infinite-dimensional problems, and this is its meaning.

We present one more result.

Theorem 2. Let H be a finite-dimensional Hilbert space. Assume that $L(\cdot)$ is a continuous transformation in H and D is a linear invertible operator. Let us pretend that $L(0) = 0$ and for any H we have the inequality

$$\langle Du,\, DL\,(u)\rangle \geq -\delta \big\|Du\big\|^2$$

at some $0 < \delta < \frac{1}{2}$. Then for any $g \in H$ equation

$$u + L\,(u) = g$$

has a solution satisfying the estimate

$$\|Du\|^2 \leq (1 - 2\delta)^{-1}\,\|Dg\|^2\,.$$

Various forms of this theorem are well-known.

Basic lemmas

Lemma 1. If $0 \neq u \in H$, then the orthogonality equalities

$$\langle u, R(u)\rangle = \langle u, S(u)\rangle = \langle R(u), S(u)\rangle = 0.$$

Proof. These equalities are consequences of the definitions $R(u)$ and $S(u)$.

Lemma 2. For any $C > 0$

$$M_{C,\delta} = \{u:\ \|u\|^2\,e^{-\delta\|f(u)\|^2} \geq C\}.$$

Proof. Since G is an invertible operator and condition C3 is satisfied, then for $u \in M_{C,\delta}$ we have

$$C \leq \|u\|^2\,e^{-\delta\|f(u)\|^2} \leq \|u\|^2\,e^{-\delta d^{-1}\|Gu\|^2} \leq \|u\|^2\,e^{-\delta d^{-1}\|G^{-1}\|^{-2}\|u\|^2}.$$

This implies the boundedness of the set $M_{C,\delta}$. But then, since H is non-dimensional, we obtain the compactness of the set $M_{C,\delta}$. Lemma 2 is proved.

Let us put

$$b(\overset{\circ}{u}) = \sup\|Gu\|^2, \tag{3}$$

where the supremum is taken over all such $a \in H$, then

$$J(u) \geq J(\overset{\circ}{u}). \tag{4}$$

Lemma 3. If $0 \neq \overset{\circ}{u} \in H$. Then there is a vector $\widetilde{u}$, such that

$$\|G\widetilde{u}\|^2 = b(\overset{\circ}{u}) \geq \|Gu\|^2,\ \ J(\widetilde{u}) \geq J(\overset{\circ}{u}).$$

Proof. The existence of the vector $\widetilde{u}$ follows from Lemma 2, since over a compact set is achieved on some element of this compact space, and the suprenum set over which is taken is compact by Lemma 2 (see (3) suprenum and (4)). The lemma is proven.

We define a vector function as a solution to the problem

$$\begin{cases} u_\xi = x\,G^*\,Gu + y\,S(u), \\ u(\xi)|_{\xi=0} = \widetilde{u}. \end{cases} \tag{5}$$

here $\widetilde{u}$ is the vector constructed in Lemma 3 for some $0 \neq \overset{\circ}{u} \in H$. For functionality $J(u(\xi))$ and for the norm $\|Gu(\xi)\|$ using the orthogonality equalities of lemma 1 we have

$$(\|Gu(\xi)\|^2)_\xi = 2\,\langle G^*Gu, u_\xi\rangle = 2\,x\,\|G^*Gu\|^2, \tag{6}$$

$$\begin{aligned} J_\xi(u(\xi)) &= 2\,J(u(\xi))\left\langle \frac{u}{\|u\|^2} - D^*_{u(\xi)} f(u(\xi)), u_\xi \right\rangle = \\ &= -2\,J(u(\xi))\,\langle [K(u(\xi))G^*Gu(\xi) + S(u(\xi))], u_\xi\rangle = \\ &= 2\,J(u(\xi))\,\big[-x\,K(u(\xi))\|G^*Gu(\xi)\|^2 - y\,\|S(u(\xi))\|^2\big]. \end{aligned} \tag{7}$$

Lemma 4. Let $\overset{\circ}{u} \in H$, $\widetilde{u}$ be from Lemma 3. Then the conditions a), b), c), d), e):

$$a)\ S(\widetilde{u}) = 0,$$

$$b)\ K(\widetilde{u}) \geq 0,$$

$$c)\ K(\widetilde{u}) \leq \widetilde{K}(\widetilde{u}),$$

$$d)\ G(\widetilde{u}) = G(\overset{\circ}{u}),$$

$$e)\ J(\widetilde{u}) = J(\overset{\circ}{u}).$$

Proof. Let $S(\widetilde{u}) \neq 0$. Then in (5) we choose

$$x = 1,\ y = -\|S(\widetilde{u})\|^{-2}\big[1 + K(\widetilde{u})\|G^*Gu(\xi)\|^2\big].$$

Then from (6) and from (7) we find

$$(\|Gu(\xi)\|^2)_\xi > 0,\ J_\xi(u(\xi)) > 0. \tag{8}$$

This implies the existence of a number $\xi_0 > 0$ such that the strict inequalities

$$\|Gu(\xi_0)\| > \|G\widetilde{u}\|,\ \ J(u(\xi_0)) > J(\widetilde{u}). \tag{9}$$

These inequalities contradict the origin of the vector $\widetilde{u}$. Therefore, $S(\widetilde{u}) = 0$ and done a).

Let us pretend that $K(\widetilde{u}) < 0$. If we choose $x = 1$, $y = 0$, then it follows from (6) and (7) that (8) is satisfied. From (8) it follows that there is a small $\xi_0 > 0$, such that (9) is satisfied. We obtain a contradiction with the definition of $\widetilde{u}$. Therefore, b) is satisfied.

We define a vector function as a solution to the problem

$$\begin{cases} u_\xi = a - \frac{\langle a, G^*Gu\rangle}{\|G^*Gu\|^2}\, G^*Gu, \\ u(\xi)|_{\xi=0} = \widetilde{u}, \end{cases}$$

where $a \in H$ and $\langle a, G^*Gu\rangle = 0$. Because $\dim \widetilde{H} \geq 3$, such a vector $e \in \widetilde{H}$, $\|e\| = 1$ exists. Thus,

$$(\|Gu(\xi)\|^2)_\xi = 2\,\langle e, G^*Gu\rangle = 0 \tag{10}$$

$$\begin{aligned} J_\xi(u(\xi)) &= 2\,J(u(\xi))\,\langle -S(u(\xi)), u_\xi\rangle = 2\,J(u(\xi))\left\langle -\int_0^\xi S_\eta(u(\eta))d\eta, u_\xi \right\rangle = \\ &= -\,J(u(\xi))\Big[\xi\langle S_\eta(u(\eta)), u_\xi\rangle|_{\eta=0}\xi + \xi^2 \mathrm{O}(1)\Big] = \\ &= 2\,J(u(\xi))\Big[\Big\langle \frac{u_\eta}{\|u\|^2} - M_u u_\eta + K(u)G^*Guu_\eta, a\Big\rangle|_{\eta=0}\xi + \xi^2\mathrm{O}(1)\Big] = \\ &= 2\,J(u(\xi))\Big[\frac{\|a\|^2}{\|\widetilde{u}\|^2} - \langle M_{\widetilde{u}}a, a\rangle + K(\widetilde{u})\|Ga\|^2\Big]\xi + \xi^2\mathrm{O}(1). \end{aligned} \tag{11}$$

In the last transition, we used the condition C3 and the equality $u_\eta|_{\eta=0} = e$. By definition $K(\widetilde{u})$ from (10) and from (11) it follows that if $K(\widetilde{u}) > \widetilde{K}(\widetilde{u})$, then there exists a vector a and $\xi_0 > 0$ such that

$$\|Gu(\xi_0)\| = \|G\widetilde{u}\|,\ \ J(u(\xi)) > J(\widetilde{u}). \tag{12}$$

Now, we define the vector function $g(\xi)$ from the problem

$$\begin{cases} g_\xi(\xi) = G^*Gg(\xi), \\ g(\xi)|_{\xi=0} = u(\xi_0). \end{cases}$$

Then for $g(u(\xi))$ we have

$$\|Gg(\xi)\|^2 = \|Gu(\xi_0)\|^2 + 2\int_{\xi_0}^{\xi} \|G^*Gg(\eta)\|^2 d\eta.$$

From here and from (12) it follows that there exists ξ_1 such, that $\xi_1 > \xi_0$ as for $g(\xi_1)$ relations (9) are fulfilled, in which instead of ξ_0 taken ξ_1. We get a contradiction. Therefore item d) of the lemma is proved.

Suppose $J(\widetilde{u}) > J(\overset{\circ}{u})$ and define the vector function $g(\xi)$ how to solve a problem

$$\begin{cases} g_\xi(\xi) = G^*Gg(\xi), \\ g(\xi)|_{\xi=0} = \widetilde{u}. \end{cases}$$

For $\|Gg(\xi)\|$ we have

$$\|Gg(\xi)\|^2 = \|G\widetilde{u}\|^2 + 2\int_0^{\xi} \|G^*Gg(\eta)\|^2 d\eta \geq \|G\dot{u}\|^2 + 2\int_0^{\xi} \|G^*Gg(\eta)\|^2 d\eta.$$

Since for small ξ the strict inequality $J(\widetilde{u}) > J(\overset{\circ}{u})$ will not get spoiled, then from the inequality for $\|Gg(\xi)\|$ we obtain that there exists $\xi_0 > 0$ such that the strict inequalities $J(u(\xi_0)) > J(\overset{\circ}{u})$, $\|Gg(\xi)\| > 0$, which contradict the origin of the vector $\widetilde{u}$. That's why $J(\widetilde{u}) = J(\overset{\circ}{u})$. Item e) of the lemma is proved. The lemma is completely proved.

Lemma 5. Let $0 \neq \overset{\circ}{u} \in H$, $\widehat{u}$ be a vector constructed from $\overset{\circ}{u}$ according to Lemma 3. Let us pretend that $R(\widetilde{u}) \neq 0$ and define the vector function $u(\xi)$ as a solution to the problem

$$\begin{cases} u_\xi = R(u), \\ u(\xi)|_{\xi=0} = \widetilde{u}. \end{cases} \tag{13}$$

Then relations (15)–(17) are satisfied for $0 < \xi < 1$

$$e^{-4\xi}\|R(\widetilde{u})\|^2 \leq \|R(u(\xi))\|^2 \leq e^{4\xi}\|R(\widetilde{u})\|^2, \tag{14}$$

$$\|Gu\|^2 \geq \|G\widetilde{u}\|^2 + 2\int_0^{\xi} \|R(u(\eta))\|^2 d\eta \geq \|G\widetilde{u}\|^2 + 2\xi\|R(\widetilde{u})\|^2 - \xi^2 8e^{4\xi}\|R(\widetilde{u})\|, \tag{15}$$

$$\begin{gathered} J(\widetilde{u}) > J(u(\xi)) = J(\widetilde{u})exp\Big(-2\int_0^{\xi} K(u(\eta))\|R(u(\eta))\|^2 d\eta\Big) \\ J(\overset{\circ}{u})\exp\Big(-2\int_0^{\xi} K(u(\eta))\|R(u(\eta))\|^2 d\eta\Big) \geq \\ J(\overset{\circ}{u})\exp\Big(-2\,\widetilde{K}(\widetilde{u})\|R(\widetilde{u})\|^2 - \xi^2\|R(\widetilde{u})\|^2 C_1(\|\widetilde{u}\|)\Big), \end{gathered} \tag{16}$$

$$J(\widetilde{u}) \geq J(\overset{\circ}{u})\exp\Big[\widetilde{K}(\widetilde{u})(\|G(\dot{u})\|^2 - \|G(u(\xi))\|^2) - \xi_n^2 C_2(\|\widetilde{u}\|)\Big]. \tag{17}$$

where $C_1(\cdot)$, $C_2(\cdot)$ – functions continuous on $[0;\infty)$.

Proof. Let $R(u(\xi))$ then we have

$$\left|(\|R(u(\xi))\|^2)_\xi\right| = 2|\langle R(u(\xi)), (G^*G - \mu(u(\xi)))u_\xi - \mu_\xi(u(\xi))u(\xi)\rangle| = \\ 2|\langle R(u(\xi)), (G^*G - \mu(u(\xi)))Ru(\xi)\rangle| \le 4\|R(u(\xi))\|^2. \tag{18}$$

This implies estimates (14). Further

$$(\|G(u(\xi))\|^2))_\xi = 2\left\langle G^*Gu, u_\xi\right\rangle = 2\left\langle G^*Gu - \mu(u)u + \mu(u)u, R(u)\right\rangle = 2(\|R(u)\|^2 \tag{19}$$

$$J_\xi(u(\xi)) = 2J(u(\xi))\Big[\langle -K(u)G^*Gu - S(u), R(u)\rangle\Big] = \\ 2J(u(\xi))\Big[-K(u)\|R(u)\|^2 - \langle S(u), R(u)\rangle\Big]. \tag{20}$$

Integrating (19), using (18) and already proven inequalities (14), we obtain (15). Now we integrate (20), and then using the definitions $R(\cdot)$, $K(\cdot)$, $S(\cdot)$ and the results of Lemma 4, we get

$$\begin{gathered} J(\widetilde{u}) \ge J(\widetilde{u})\exp\Big[-2\widetilde{K}(\widetilde{u})\|R(\widetilde{u})\|^2\xi - \\ \int_0^\xi\int_0^\tau \Big(K(u(\eta))\|R(u(\eta))\|^2 + \langle S(u(\eta)), R(u(\eta))\rangle\Big)_\eta d\eta d\tau\Big] = \\ J(\widetilde{u})\exp\Big[-2\widetilde{K}(\widetilde{u})\|R(\widetilde{u})\|^2\xi - \int_0^\xi\int_0^\tau \Big(K(u(\eta))\|R(u(\eta))\|^2 + \\ \langle -\alpha u(\eta) + D^k_{u(\eta)}f(u(\eta)) - K(u(\eta))G^kGu(\eta), R(u(\eta))\rangle\Big)_\eta d\eta d\tau\Big] = \\ J(\widetilde{u})\exp\Big[-2\widetilde{K}(\widetilde{u})\|R(\widetilde{u})\|^2\xi - \int_0^\xi\int_0^\tau \langle D^k_{u(\eta)}f(u(\eta)), R(u(\eta))\rangle\Big] = \\ J(\widetilde{u})\exp\Big[-2\widetilde{K}(\widetilde{u})\|R(\widetilde{u})\|^2\xi - \int_0^\xi\int_0^\tau \langle M_{u(\eta)}R(u(\eta)), R(u(\eta))\rangle\Big] \ge \\ J(\widetilde{u})\exp 2\Big[-\widetilde{K}(\widetilde{u})\|R(\widetilde{u})\|^2\xi - \xi^2\|R(\widetilde{u})\|^2C_1(\|\widetilde{u}\|^2)\Big], \end{gathered} \tag{21}$$

where $C_1(\cdot)$ – functions continuous on $[0;\infty)$.

To estimate the factor at ξ^2, we used the equality $\|u(\xi)\| = \|\widetilde{u}\|$, which follows from the following equality

$$\left(\|u(\xi)\|^2\right)_\xi = 2\langle u(\xi), R(u(\xi))\rangle = 0.$$

From (21) follows (16), from (15) and (16) follows (17). The lemma is proven.

Lemma 6. Let $0 \ne \mathring{u} \in H$, $\widetilde{u}$ – the vector constructed from $\mathring{u}$ in accordance with Lemma 3. Let us pretend that $R(\widetilde{u}) = 0$ and define the vector function $u(\xi)$ as a solution to the problem

$$\begin{cases} u_\xi = G^*Gu, \\ u(\xi)|_{\xi=0} = \widetilde{u}. \end{cases} \tag{22}$$

Then at $0 < \xi < 1$ relations (23)-(26).

$$e^{-2\xi}\|G^*G\widetilde{u}\|^2 \le \|G^*Gu(\xi)\|^2 \le e^{2\xi}\|G^*G\widetilde{u}\|^2 \tag{23}$$

$$\|G\mathring{u}\|^2 \ge \|G\widetilde{u}\|^2 \ge \|G\widetilde{u}\|^2 = \|G\widetilde{u}\|^2 + \\ +2\int_0^\xi \|G^*Gu(\eta)\|^2 d\eta \le \|G\widetilde{u}\|^2 + 2\,\xi\|G^*G\widetilde{u}\|^2 + 2\xi^2 8e^{2\xi}\|G^*G\widetilde{u}\| \tag{24}$$

$$J(u(\xi)) \geq J(\widetilde{u}) \exp\left[-2\xi \widetilde{K}(\widetilde{u}) \|G^*G\widetilde{u}\|^2 - \xi^2 C_3(\|\widetilde{u}\|^2) l(\xi)\right] = \\ = J(\overset{\circ}{u}) \exp\left[-2\xi \widetilde{K}(\widetilde{u}) \|G^*G\widetilde{u}\|^2 - \xi^2 C_3(\|\widetilde{u}\|^2) l(\xi)\right], \tag{25}$$

$$J(u(\xi)) = J(\overset{\circ}{u}) \exp\left[\lambda_0^{-1}(1-\alpha)(\|G\widetilde{u}\|^2 - \|Gu\|^2) - \xi^2 C_4(\|\widetilde{u}\|^2) l(\xi)\right], \tag{26}$$

where $C_3(\cdot)$, $C_4(\cdot)$ functions continuous on $[0;\infty)$ a $l(\cdot)$ function with values from the interval $[-1;1]$.

Proof. For $\|G^*Gu\|$ and $\|Gu\|$ we have

$$\left|(\|G^*Gu\|^2)_\xi\right| = 2|\langle G^*Gu, G^*Gu_\xi\rangle| \leq 2\|G^*Gu\|^2, \\ (\|Gu\|^2|)_\xi = 2\langle G^*Gu, u_\xi\rangle = 2\|G^*Gu\|^2.$$

Integrating these inequalities and using Lemmas 3 and 4, we obtain (23) and (24).

For $J_\xi(u(\xi))$ we have

$$J_\xi(u(\xi)) = 2J(u(\xi))\Big[\langle -K(u)G^*Gu - S(u), G^*Gu\rangle\Big] = 2J(u(\xi))\Big[-K(u)\|G^*Gu\|^2\Big].$$

Hence, using Lemma 4, we find

$$J(u(\xi)) = J(\widetilde{u}) \exp\left(-2\int_0^\xi K(u(\eta))\|G^*Gu(\eta)\|^2 d\eta\right) \geq \\ \geq J(\widetilde{u}) \exp\left(-2\xi \widetilde{K}(\widetilde{u})\|G^*G\widetilde{u}\|^2 - \xi^2 C_3(\|\widetilde{u}\|^2) l(\xi)\right). \tag{27}$$

Here $C_3(\cdot)$ – continuous on $[0;\infty)$ functions and $l(\cdot)$ function with values from the segment $[-1;1]$. When estimating the factor at ξ^2, we used the equalities

$$|\left(\|u\|^2\right)_\xi| = 2|\langle u, u_\xi\rangle| = 2\langle u, G^*Gu\rangle = 2\|Gu\|^2 \leq 2\|u\|^2.$$

From which it follows that

$$e^{-2\xi}\|\widetilde{u}\|^2 \leq \|u\|^2 \leq e^{2\xi}\|\widetilde{u}\|^2.$$

From (27) and from $J(\widetilde{u}) = J(\dot{u})$ (25) follows, and (24) implies (26). The lemma is proven.

Proof Theorem 1. Let $0 \neq \overset{\circ}{u} \in H$. If $R(\overset{\circ}{u}) = 0$,, then $\overset{\circ}{u}$ will be an eigenvector of the operator G^*G. Therefore, from condition C2 we have

$$J(\dot{u}) = \|\dot{u}\|^2 \exp\left(-\|f(\dot{u})\|^2\right) = \left(\|f(\dot{u})\|^2 + 2\right)^m e^{-\|f(\dot{u})\|^2} \leq \\ \leq \sup_{x\geq 2} x^m e^{-x+2} = m^m e^{-m+2}. \tag{28}$$

If $R(\overset{\circ}{u}) \neq 0$, construct the vector $\overset{\circ}{u}$ then by the vector $\widetilde{u}_0$. If $R(\widetilde{u}_0) = 0$, then for $J(\widetilde{u}_0)$ we obtain an inequality similar to (28) $J(\widetilde{u}_0) \leq m^m e^{-m+2}$. Therefore, since by construction $J(\widetilde{u}_0) \geq J(\overset{\circ}{u})$ we have that (1) holds. From this we draw the following conclusion.

Thus, if at least one of the conditions $R(\overset{\circ}{u}) = 0$ and $R(\widetilde{u}) = 0$, is satisfied, then Theorem 1 will be proven.

If $R(\overset{\circ}{u}) \neq 0$ and $R(\widetilde{u}_0) \neq 0$, then we construct a sequence of pairs according to the following algorithm.

Let the pairs be built $(\widetilde{u}_0, \overset{0}{u}), ..., (\widetilde{u}_n, \overset{n}{u})$, $0 \leq n$, $R(\widetilde{u}_j) \neq 0$, $j = 0, ..., n$.

Let us build a pairs $(\widetilde{u}_{n+1}, \overset{n+1}{u})$.

In Lemma 3, instead of the vector $\overset{\circ}{u}$, we take the vector $\overset{n}{u}$ and construct the vector $\widetilde{u}$, which we take as $\widetilde{u}_{n+1}$.

For $\widetilde{u}_{n+1}$, two cases are possible:

$$(I)\ R(\widetilde{u}_{n+1}) = 0,$$

$$(II)\ R(\widetilde{u}_{n+1}) \neq 0.$$

If (I) is satisfied, then we construct the vector $\overset{n+1}{u}$ using Lemma 6. To do this, in (22) as $\widetilde{u}$ we take the vector $\widetilde{u}_{n+1}$ and for $\overset{n+1}{u}$ we take the value $u(\xi)$ at point $\widehat{\xi}_n$:

$$\widehat{\xi}_n = \widetilde{\xi},\ \overset{n+1}{u} = u(\xi_n).$$

Then from Lemma 6, using condition C4, we have

$$\begin{gathered} J(\overset{n+1}{u}) \geq J(\overset{n}{u}) \exp\left[d^{-1}(1-\delta)(\|G\widetilde{u}_{n+1}\|^2 - \|G\overset{n+1}{u}\|^2 - \xi_n^2 G(\widetilde{u}_{n+1}))\right] \geq \\ \geq J(\overset{n}{u}) \exp\left[d^{-1}(1-\delta)(\|G\overset{n}{u}\|^2 - \|G\overset{n+1}{u}\|^2) - \xi_n^2 C_4(\widetilde{u}_{n+1})\right]. \end{gathered} \tag{29}$$

When deriving (29), relations $J(u_{n+1}) = J(\overset{n}{u})$, $\|G\widetilde{u}_{n+1}\| \geq \|G\overset{n}{u}\|$ were used, which follow from Lemmas 3 and 4 and the definition of $\widetilde{u}_{n+1}$. Let's choose ξ_n in the right place.

In the case (I) pair $(\widetilde{u}_{n+1}, \overset{n+1}{u})$ is constructed.

In the this case, we stop the process of constructing a sequence of pairs.

In case (II), we construct $\overset{n+1}{u})$ using Lemma 5. To do this, in (13) as $\widetilde{u}$ we take the vector $\widetilde{u}_{n+1}$ and for $\overset{n+1}{u}$ we take the value (13) at point $\widehat{\xi}_n$:

$$\xi_n = \frac{\delta_0 \|R(\widetilde{u}_{n+1})\|^2}{(n+1)[C_1(\|\widetilde{u}_{n+1}\|) + C_2(\|\widetilde{u}_{n+1}\|) + 1]\,[\|R(\widetilde{u}_{n+1})\|^4 + 1]}, \tag{30}$$

here $C_1(\cdot)$, $C_2(\cdot)$– continuous on $[0;\infty)$ functions from Lemma 5, δ_0 is a small number. From Lemma 5 (17) and from (30) we find

$$\begin{cases} \|G\overset{n+1}{u}\|^2 \geq \|G\overset{n+1}{u}\|^2 + 2\xi\|R(\widetilde{u}_{n+1})\|^2 - 20\frac{\delta_0^2}{(n+1)^2}, \\ J(\overset{n+1}{u}) \geq J(\overset{n}{u}) \exp\left[d^{-1}(1-\delta)(\|G\overset{n}{u}\|^2 - \|G\overset{n+1}{u}\|^2) - 20\frac{\delta_0^2}{(n+1)^2}\right]. \end{cases} \tag{31}$$

Pair $(\widetilde{u}_{n+1}, \overset{n+1}{u})$ built. Let $1 \leq n_0$ be an integer number, which holds for all $j \leq n_0$

$$R(\widetilde{u}_j) \neq 0,\ \ R(\widetilde{u}_{n_0+1}) = 0.$$

Then from (31) we deduce

$$\begin{cases} \|G\overset{n_0}{u}\|^2 \geq \|G\overset{0}{u}\|^2 + 2\sum_{j=0}^{n_0} \xi_j \|R(\widetilde{u}_j)\|^2 - 10^2\delta_0^2, \\ J(\overset{n_0}{u}) \geq J(\overset{0}{u}) \exp\left[d^{-1}(1-\delta)(\|G\overset{0}{u}\|^2 - \|G\overset{n_0}{u}\|^2) - 10^2\delta_0^2\right]. \end{cases} \tag{32}$$

From the second inequality (32) we have

$$\begin{gathered} J(\overset{0}{u}) \exp\left[d^{-1}(1-\delta)\|G\overset{0}{u}\|^2 - 10^2\delta_0^2\right] \leq J(\overset{n_0}{u}) \exp\left[d^{-1}(1-\delta)\|G\overset{n_0}{u}\|^2\right] = \\ = \|\overset{n_0}{u}\|^2 \exp\left[-\|f(\overset{n_0}{u})\|^2 + d^{-1}(1-\delta)\|G\overset{n_0}{u}\|^2\right] \leq \|\overset{n_0}{u}\|^2 \exp\left[\delta\|f(\overset{n_0}{u})\|^2\right]. \end{gathered} \tag{33}$$

In the last transition, condition C3 is used.

From (33) and from Lemma 2 on compactness (see Lemma 2), since the left side of (33) does not depend on n_0 and the inequalities $0 < \delta < 1$, it follows that

$$\|\overset{n_0}{u}\| \leq J(\overset{0}{u}) < \infty, \tag{34}$$

where $J(\overset{0}{u})$ does not depend on n_0. From (34) and from the first inequality in (32), due to the choice of ξ_n, follows that only two cases (A) and (B) are possible:

(A) There is an $n \geq 0$ such that $R(\widetilde{u}_{n_1}) = 0$ and $R(\widetilde{u}_j) \neq 0$, if $0 < j < n_1$;

(B) For any $j = 0, 1, ...$ done $R(\widetilde{u}_j) \neq 0$ and $\underline{\lim}_{n\to\infty}\|R(\widetilde{u}_n)\| = 0$.

Here $\underline{\lim}$ means lower limit.

Indeed, if none of the conditions (A) is satisfied, then by virtue of (34) and the choice of A (see (30)) from the first inequality in (32) we obtain

$$J(\overset{0}{u}) \geq \|\overset{n+1}{u}\|^2 \geq \|G\overset{n+1}{u}\|^2 \geq \|G\overset{0}{u}\|^2 - 10^2\delta_0^2 + 2\sum_{j=0}^{n}\frac{\delta_0}{j+1}\inf_k \|R(\widetilde{u}_k)\|^2.$$

When $n \to \infty$, the right side tends to $+\infty$. So we got a contradiction. Therefore, at least one of conditions (A) and (B) is satisfied.

Let condition (B) be satisfied. Then, by virtue of (34), if necessary, passing to sequences can be considered

$$\lim_{j\to\infty}\overset{j}{u} = \widetilde{g},\ \lim_{j\to\infty} R(\overset{j}{u}) = R(\widetilde{g}) = 0,\ \lim_{j\to\infty} f(\overset{j}{u}) = f(\widetilde{g}).$$

When deriving the equality for $R(\widetilde{g})$ and $f(\widetilde{g})$, we used the estimates for $R(\overset{j}{u})$ in terms of $R(\widetilde{u}_j)$ from Lemma 5 and choice ξ_j (see (30)), as well as the divergence of the harmonic series.
Letting go to infinity and then using the conditions C2 and C3, we obtain

$$\begin{aligned} J(\overset{0}{u})\exp\left[d^{-1}(1-\delta)\|G\overset{0}{u}\|^2 - 10^2\delta_0^2\right] \leq \|\widetilde{g}\|^2\exp\left[-\delta\|f(\widetilde{g})\|^2\right] \leq \\ \leq \left(\|f(\widetilde{g})\|^2+2\right)^m\exp\left[-\delta(\|f(\widetilde{g})\|^2+2)+2\delta\right] \leq \sup_{x\geq 2} x^m e^{-x+2\delta} = \left(\frac{m}{\delta}\right)^m e^{-m+2\delta}. \end{aligned} \tag{35}$$

Now from the definition of A and from (35) we deduce

$$\|\overset{0}{u}\|^2 \leq \exp\left[\|f(\overset{0}{u})\|^2 - d^{-1}(1-\delta)\|G\overset{0}{u}\|^2 - 10^2\delta_0^2\right]\left(\frac{m}{\delta}\right)^m \leq \left(\frac{m}{\delta}\right)^m \exp\|f(\overset{0}{u})\|^2. \tag{36}$$

In the derivation, we used that the possibility of choosing δ_0 small and inequalities $0 < \delta < \frac{1}{2}, m \geq 1$. Theorem 1 follows from (36) in case (B).

If (A) is satisfied, then (29) is satisfied. From (29), since for all $j \leq n$ the inequalities $R(\widetilde{u}_j) \neq 0$, then choosing $\xi_n = \widetilde{\xi}$ small enough, we get

$$J(\overset{n+1}{u}) \geq J(\overset{0}{u})\exp\left[d^{-1}(1-\delta)(\|G\overset{0}{u}\|^2 - \|G\overset{n+1}{u}\|^2) - 10^2\delta_0^2\right]. \tag{37}$$

Since $\overset{n+1}{u}$ is defined in terms of $\widetilde{u}_{n+1}$ by the equation (see (22))

$$\begin{cases} u_\xi = G^*Gu, \\ u(\xi)|_{\xi=0} = \widetilde{u}_{n+1}. \end{cases}$$

And $\overset{n+1}{u} = u(\widetilde{\xi}_n)$, then choosing the number $\widetilde{\xi}_n = \widetilde{\xi}$ small enough from (37) we obtain

$$J(\widetilde{u}_{n+1}) \geq J(\overset{0}{u}) \exp\left[d^{-1}(1-\delta)(\|G\overset{0}{u}\|^2 - \|G\widetilde{u}_{n+1}\|^2) - 10^2\delta_0^2\right].$$

Hence follows

$$J(\overset{0}{u}) \exp\left[d^{-1}(1-\delta)\|G\overset{0}{u}\|^2 - 10^2\delta_0^2\right] \leq \|\widetilde{u}_{n+1}\| exp\left[-\|f(\widetilde{u}_{n+1})\|^2 + d^{-1}(1-\delta)\|G\widetilde{u}_{n+1}\|^2\right]. \quad (38)$$

From (38), since $\widetilde{u}_{n+1}$ is an eigenvector of the operator G^*G, we get the estimate

$$\|\overset{0}{u}\|^2 \leq \left(\frac{m}{\delta}\right)^m \exp\|f(\overset{0}{u})\|^2,$$

which are derived in the same way as we derived the estimate from (36). Theorem 1 is proved in case (A). Therefore, Theorem 1 is proved completely.

Proof Theorem 2. We use the notation of Theorem 2. If $g \in H$, then vector $u \equiv 0$ is a solution to the equation

$$u + L(u) = 0.$$

Let $0 \neq g \in H-$ arbitrary vector. Since D is an invertible operator, then $\|Dg\| > 0$.

Denote by M the set

$$M = \{u: \|Du\|^2 \leq \frac{2}{(1-2\delta)\delta}\|Dg\|^2\}.$$

Let's put

$$F(u) = -\frac{u + L(u) - g}{\|D(u + L(u) - g)\|}\eta,$$

where the number η is chosen as follows:

$$\eta = \sqrt{\frac{2}{(1-2\delta)\delta}\|Dg\|^2}.$$

Suppose equation $u + L(u) = g$ has no solution in M: Since equation $u + L(u) = g$ has no solution, the transformation $F(\cdot)$ continuously translates from M to M. But then by the Schauder fixed-point theorem u_0 we get that there exists such that

$$-\frac{u_0 + L(u_0) - g}{\|D(u_0 + L(u_0) - g)\|}\eta = u_0. \quad (39)$$

From here and the choice of η we have

$$\|Du_0\|^2 = \eta^2 = \frac{2}{(1-2\delta)\delta}\|Dg\|^2.$$

We act on (39) with the operator D, and then multiply scalarly by Du_0. Then, using (39) and the condition of the theorem, we obtain

$$\eta^{-1}\|Du_0\|^2\|D(u_0 + L(u_0) - g)\| = -\|Du_0\|^2 - \langle DL(u_0), Du_0\rangle + \langle Dg, Du_0\rangle \leq$$

$$\leq -\|Du_0\|^2 + \delta\|Du_0\|^2 + \frac{\varepsilon^{-1}}{2} + \frac{1}{2}\varepsilon\|Du_0\|^2.$$

Let us take $\varepsilon = 2\delta$. Then from the last inequality and from (39) we deduce

$$0 \le -\|Du_0\|^2(1-\delta) + \frac{1}{4\delta}\|Dg\|^2 = -\frac{2}{(1-2\delta)\delta}(1-\delta)\|Dg\|^2 + \frac{1}{4\delta}\|Dg\|^2 = -\frac{7}{4\delta}\|Dg\|^2.$$

We got a contradiction. Therefore, the equation $u + L(u) = g$ has a solution. We act on the equation $u + L(u) = g$ by the operator D:

$$Du + DL(u) = Dg.$$

Multiplying the resulting equality scalarly by Du, we obtain

$$\|Du\|^2 + \langle Du, DL(u)\rangle = \langle Dg, DL(u)\rangle \le \frac{1}{2}\|Dg\|^2 + \frac{1}{2}\|Du\|^2.$$

Now, using condition $\langle Du, DL(u)\rangle \ge -\delta\|Du\|^2$, , we obtain the desired evaluation

$$(1-2\delta)\|Du\|^2 \le \|Dg\|^2.$$

Theorem 2 is proven.

Remark 2. Note that in Lemma 1 we can take K non-linear transformations as K.

Acknowledgments

The authors express their gratitude for the formulation of the problem and attention to the work of Academician of the NAS of the Republic of Kazakhstan Mukhtarbai Otelbaevich Otelbaev.

This work was done by grant AP 08857604 of the Ministry of Education and Science of the Republic of Kazakhstan.

References

1 Fefferman Ch. Existence and smoothness of the Navier-Stokes equation / Ch. Fefferman. — Clay Mathematics Institute, 2000. — P. 1–5.

2 Отелбаев М. Существование сильного решения уравнения Навье-Стокса / М. Отелбаев // Мат. журн. — 2013. — 13. — № 4 (50). — С. 5–104.

3 Ладыженская О.А. Решение «в целом» краевой задачи Навье-Стокса в случае двух пространственных переменных / О.А. Ладыженская // Докл. АН СССР. — 1958. — 123. — №3. — С. 427–429.

4 Ладыженская О.А. Шестая проблема тысячелетия: уравнения Навье-Стокса, существование и гладкость / О.А. Ладыженская // Успехи математических наук. — 2003. — 58. — № 2(350). — С. 45–78.

5 Hopf E. Uber die Anfangswertaufgabe fur die Hydrodinamischen Grundgleichungen / E. Hopf // Math. Nachr. — 1951. — 4. — P. 213–231.

6 Отелбаев М. Примеры не сильно разрешимых в целом уравнений типа Навье-Стокса / М. Отелбаев // Математические заметки. — 2011. — 89. — № 5. — С. 771–779.

7 Отелбаев М. Условия существования сильного решения в целом одного класса нелинейных эволюционных уравнений в гильбертовом пространстве. II / М. Отелбаев, А.А. Дурмагамбетов, Е.Н. Сейткулов // Сиб. мат. журн. — 2008. — 49. — № 4. — С. 855–864.

8 Отелбаев М. Непрерывная зависимость решения параболического уравнения в гильбертовом пространстве от параметров и от начальных данных / М. Отелбаев, Л.К. Жапсарбаева // Дифференциальные уравнения. — 2009. — 45. — № 6. — С. 818–849.

9 Лионс Ж.-Л. Некоторые методы решения нелинейных краевых задач. Пер. с фр. / Ж.-Л. Лионс — М.: Мир, 1972. — 586 с.

10 Сакс Р.С. Задача Коши для уравнений Навье–Стокса, метод Фурье / Р.С. Сакс // Уфим. мат. журн. — 2011. — 3. — № 1. — С. 53–79.

11 Похожаев С.И. Гладкие решения уравнений Навье-Стокса / С.И. Похожаев // Мат. сб. — 2014. — 205. — № 2. — С. 131–144.

12 Koshanov B.D. Correct Contractions stationary Navier-Stokes equations and boundary conditions for the setting pressure / B.D. Koshanov, M.O. Otelbaev // AIP Conference Proceedings. — 2016. — 1759. — 020005. http://dx.doi.org/10.1063/1.4959619

13 Кожанов А.И. Новые краевые задачи для четвертого порядка квазигиперболического уравнения / А.И. Кожанов, Б.Д. Кошанов, Ж.Б. Султангазиева // Сиб. электрон. мат. изв. — 2019. — № 16. — С. 1410–1436

14 Kozhanov A.I. The spectral problem for nonclassical differential equations of the sixth order / A.I. Kozhanov, B.D. Koshanov, Zh.B. Sultangazieva, A.N. Emir Kady oglu, G.D. Smatova // Bulletin of the Karaganda University-Mathematics. — 2020. — № 1(97). — P.79–86. https://doi.org/10.31489/2020M1/79-86

Б.Д. Қошанов[1,2], Н. Кахарман[3], Р.У. Сегизбаева[4], Ж.Б. Султанғазиева[1]

[1] *Әл-Фараби атындағы Қазақ Ұлттық университеті, Алматы, Қазақстан;*
[2] *Халықаралық ақпараттық технологиялар университеті, Алматы, Қазақстан;*
[3] *Математика және математикалық модельдеу институты, Алматы, Қазақстан;*
[4] *Азаматтық авиация академиясы, Алматы, Қазақстан*

Ақырлыөлшемді кеңістіктегі сызықты емес теңдеулердің бір класының шешімін бағалаудың екі теоремасы

Эллиптикалық және параболалық теңдеулер үшін шеттік есептерді зерттеу қажеттілігі гидродинамика, электростатика, механика, жылу өткізгіштік, серпімділік теориясы, кванттық физика процестерін теориялық тұрғыдан зерттеуде көптеген практикалық қосымшалардың түсіндіруімен тікелей байланысты. Бұл жұмыста ақырлыөлшемді кеңістікте сызықтық емес теңдеулердің шешімдері үшін априорлық бағалаулары туралы екі теорема алынған. Жұмыс төрт бөлімнен тұрады. Бірінші бөлімде пайдаланылған белгілеулер мен негізгі нәтиженің тұжырымдамасы келтірілген. Екінші бөлімде негізгі леммалар берілген. Үшінші бөлім 1-ші теореманың дәлелдемесіне арналған. Төртінші бөлімде екінші теорема дәлелденген. Теореманың шарты мынадай, оны бастапқы-шекаралық есептердің белгілі бір класын зерттеу кезінде олардың шешімдеріне априорлық бағалау алу үшін қолдануға болады. Теореманың мәні осында.

Кілт сөздер: ақырлыөлшемді Гильберт кеңістігі, сызықтық емес теңдеулер, кері оператор, дифференциалданатын вектор-функциялар, шешімдерді априорлық бағалау.

Б.Д. Кошанов[1,2], Н. Кахарман[3], Р.У. Сегизбаева[4], Ж.Б. Султангазиева[1]

[1]*Казахский национальный университет имени Аль-Фараби, Алматы, Казахстан;*
[2]*Международный университет информационных технологий, Алматы, Казахстан;*
[3]*Институт математики и математического моделирования, Алматы, Казахстан;*
[4]*Академия гражданской авиации, Алматы, Казахстан*

Две теоремы об оценках решений одного класса нелинейных уравнений в конечномерном пространстве

Необходимость исследования краевых задач для эллиптических и параболических уравнений продиктована с многочисленными практическими приложениями при теоретическом изучении процессов гидродинамики, электростатики, механики, теплопроводности, теории упругости, квантовой физики. В этой работе мы получили две теоремы об априорных оценках решений нелинейных уравнений в конечномерном гильбертовом пространстве. Работа состоит из четырех пунктов. В первом пункте приведены используемые обозначения и формулировка основных результатов. Во втором — основные леммы. Третий пункт посвящен доказательству теоремы 1. В четвертом — доказана теорема 2. Условия теорем таковы, что можно использовать при изучении некоторого класса начально-краевых задач для получения сильных априорных оценок при наличии слабых априорных оценок. В этом и состоит смысл этих теорем.

Ключевые слова: конечномерное гильбертово пространство, нелинейные уравнения, обратимый оператор, дифференцируемые вектор-функции, априорная оценка решений.

References

1 Fefferman, Ch. (2000). Existence and smoothness of the Navier-Stokes equation. *Clay Mathematics Institute,* 1–5.

2 Otelbaev, M. (2013). Sushchestvovanie silnogo resheniia uravneniia Nave-Stoksa [Existence of a strong solution of the Navier-Stokes equation]. *Matematicheskii zhurnal — Mathematical journal, 13*(50), 5–104 [in Russian].

3 Ladyzhenskaya, O.A. (1958). Reshenie «v tselom» kraevoi zadachi Nave-Stoksa v sluchae dvukh prostranstvennykh peremennykh [Solution "in the large"of the boundary value problem for the Navier-Stokes equations in two space variables]. *Doklady AN SSSR — Reports of the Academy of Sciences of the USSR, 123*(3), 427–429 [in Russian].

4 Ladyzhenskaya, O.A. (2003). Shestaia problema tysiacheletiia: uravneniia Nave-Stoksa, sushchestvovanie i gladkost [The sixth problem of the millennium: the Navier-Stokes equations, existence and smoothness]. *Uspekhi matematicheskikh nauk — Advances in Mathematical Sciences, 58*(2), 45–78 [in Russian].

5 Hopf, E. (1951). Uber die Anfangswertaufgabe fur die Hydrodinamischen Grundgleichungen. *Math. Nachr.,* 4, 213–231.

6 Otelbaev, M. (2011). Primery ne silno razreshimykh v tselom uravnenii tipa Nave-Stoksa [Examples of equations of the Navier-Stokes type that are not strongly solvable in the large]. *Matematicheskie zametki — Math notes, 89*(5), 771–779 [in Russian].

7 Otelbaev, M.O., Durmagambetov, A.A., & Seitkulov E.N. (2008). Usloviia sushchestvovaniia silnogo resheniia v tselom odnogo klassa nelineinykh evoliutsionnykh uravnenii v gilbertovom prostranstve. II [Conditions for the existence of a strong solution in the large of one class of nonlinear evolution equations in a Hilbert space. II]. *Sibirskii matematicheskii zhurnal — Siberian Mathematical Journal, 49*(4), 855–864 [in Russian].

8 Otelbaev, M.O., & Zhapsarbaeva, L.K. (2009). Nepreryvnaia zavisimost resheniia paraboliches-kogo uravneniia v gilbertovom prostranstve ot parametrov i ot nachalnykh dannykh [Continuous dependence of the solution of a parabolic equation in a Hilbert space on parameters and initial data]. *Differentsialnye uravneniia — Differential Equations, 45*(6), 818–849 [in Russian].

9 Lions, J.-L. (1972). *Nekotorye metody resheniia nelineinykh kraevykh zadach [Some methods for solving nonlinear boundary value problems].* Translated from French. Moscow: Mir [in Russian].

10 Saks, R.S. (2011). Zadacha Koshi dlia uravnenii Nave–Stoksa, metod Fure [Cauchy problem for the Navier-Stokes equations, Fourier method]. *Ufimskii matematicheskii zhurnal — Ufa Mathematical Journal, 3*(1), 53–79 [in Russian].

11 Pokhozhayev, S.I. (2014). Gladkie resheniia uravnenii Nave-Stoksa [Smooth solutions of the Navier-Stokes equations]. *Matematicheskii sbornik — Mathematical collection, 205*(2), 131–144 [in Russian].

12 Koshanov, B.D, & Otelbaev M.O. (2016). Correct Contractions stationary Navier-Stokes equations and boundary conditions for the setting pressure. *AIP Conference Proceedings, 1759*, 020005. http://dx.doi.org/10.1063/1.4959619.

13 Kozhanov, A.I., Koshanov, B.D., & Sultangazieva, Zh.B. (2019). Novye kraevye zadachi dlia kvazigiperbolicheskikh uravnenii chetvertogo poriadka [New boundary value problems for fourth-order quasi-hyperbolic equations]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian Electronic Mathematical Reports, 16* 1410–1436. http://dx.doi.org/10.33048/semi.2019.16.098 [in Russian].

14 Kozhanov, A.I., Koshanov, B.D., Sultangazieva, Zh.B., Emir Kady oglu, A.N., & Smatova G.D. (2020). The spectral problem for nonclassical differential equations of the sixth order.*Bulletin of the Karaganda University-Mathematics, 97*(1), 79–86. https://doi.org/10.31489/2020M1/79-86

DOI 10.31489/2022M3/85-97

UDC 512

R. Mousarezaei, B. Davvaz*

Department of Mathematical Sciences, Yazd University, Yazd, Iran
(E-mail: rasoul.mousarezaei@stu.yazd.ac.ir, davvaz@yazd.ac.ir)

A different look at the soft topological polygroups

Soft topological polygroups are defined in two different ways. First, it is defined as a usual topology. In the usual topology, there are five equivalent definitions for continuity, but not all of them are necessarily established in soft continuity. Second it is defined as a soft topology including concepts such as soft neighborhood, soft continuity, soft compact, soft connected, soft Hausdorff space and their relationship with soft continuous functions in soft topological polygroups.

Keywords: soft set, soft continuous, soft topological polygroups, soft Hausdorff space, soft open covering, soft compact, soft connected.

1 Introduction

The real world is full of uncertainties. To support these uncertainties, we insert soft sets into mathematical structures. As polygroups have the closest properties to groups among all hyperstructures, we combine polygroups with soft sets and usual topologies, then introduce soft topological polygroups and provide their examples. We are interested in making connections between complete parts in polygroups with closure of soft topological polygroups, continuous function, and usual topology. Then we enter the soft topology and present a combination of the polygroups and the soft sets with the soft topology and another definition of the soft topological polygroups.

The efforts of many scientists were used in this direction, including G. Oguz [1], Heidari et al. [2], Cagman et al. [3], Wang et al. [4], Shah and Shaheen [5], Davvaz [6], Maji [7], Mousarezaei and Davvaz [8], Nuzmul [9], and Hida [10]. Figure 1 shows the relations between polygroups, topology and soft sets, where each item is studied and investigated by many authors.

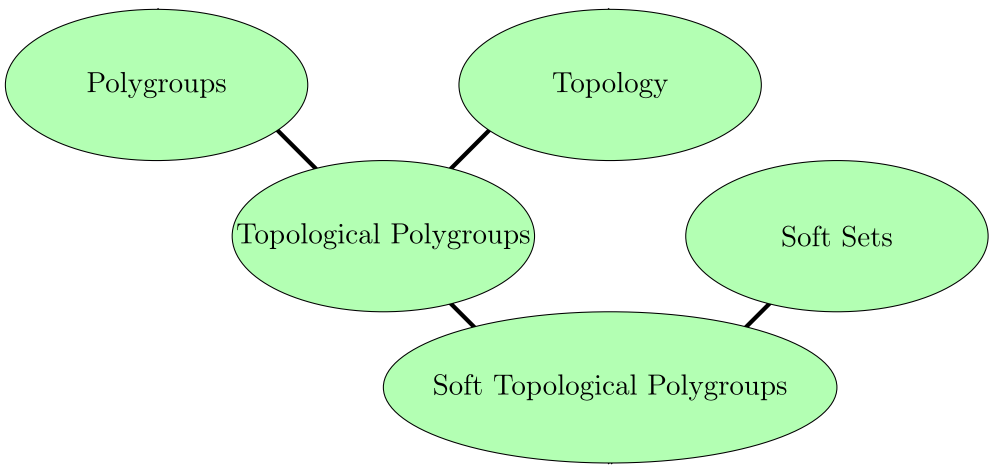

Figure 1. Relations between polygroups, topology, and soft sets

*Corresponding author.
E-mail: davvaz@yazd.ac.ir

In [8], R. Mousarezaei and B. Davvaz made a soft topological polygroup over a polygroup. The ideas presented in this article can be used to build more polygroups and more soft topological polygroup.

This paper aims to combine soft sets, topology, and polygroups from different point of view. Also, the concepts of soft neighborhood, soft continuity, soft compact, soft connected, soft Hausdorff space appear and their relationship with soft continuous functions in soft topological polygroups are studied.

To consider soft topological polygroups which represent a generalization of topological polygroups, this paper is constructed as follows: after an introduction, Section 2 contains a brief review of basic definitions related to soft sets and polygroups that are used throughout the paper. Section 3 studies two different definitions of the soft topological polygroup. Attributes are given for each definition along with examples. In continuing the connection between the complete parts and the soft continuous function, soft Hausdorff space, soft $\mathcal{T}_0$ space, soft $\mathcal{T}_1$ space, soft open covering, soft compact, soft connected in soft topological polygroups are studied.

2 Basic definitions

2.1 Soft Sets

Let U be an initial universe, $\mathcal{P}(U)$ denote the power set of U, and $\mathcal{P}^*(U)$ be power set without $\varnothing$. Suppose that E is a set of parameters and A is a non-empty subset of E. A pair $(\mathbb{F}, A)$ is said to be a soft set over U, if $\mathbb{F} : A \to \mathcal{P}(U)$ is a function.

Let $(\mathbb{F}, A)$ and $(\mathbb{G}, B)$ be soft sets over U. In this case, we have the following compliments:

• $(\mathbb{F}, A)$ is a soft subset of $(\mathbb{G}, B)$ and denoted by $(\mathbb{F}, A)\widehat{\subset}(\mathbb{G}, B))$ if $A \subseteq B$ and $\mathbb{F}(a) \subseteq \mathbb{G}(a)$ for all $a \in A$.

• $(\mathbb{F}, A)$ is soft super set of $(\mathbb{G}, B)$ and denoted by $(\mathbb{F}, A)\widehat{\supset}(\mathbb{G}, B)$ If $(\mathbb{G}, B)\widehat{\subset}(\mathbb{F}, A)$.

• $(\mathbb{F}, A)$ is soft equal $(\mathbb{G}, B)$ and denoted by $(\mathbb{F}, A)\widehat{=}(\mathbb{G}, B)$ if $(\mathbb{F}, A)\widehat{\subset}(\mathbb{G}, B)$ and $(\mathbb{G}, B)\widehat{\subset}(\mathbb{F}, A)$.

• $(\mathbb{F}, A)$ is a absolute soft set and denoted by $\widehat{U}$ If $\mathbb{F}(a) = U$ for all $a \in A$. The set $Supp(\mathbb{F}, A) = \{a \in A : \mathbb{F}(a) \neq \varnothing\}$ said to be the support of the soft set $(\mathbb{F}, A)$. A soft set is called non-null if its support is not equal to the empty set.

• $(\mathbb{F}, A)$ is a null soft set and denoted by $\widehat{\varnothing}$ if $\mathbb{F}(a) = \varnothing$ (null set) for all $a \in A$. If A is equal to E we write $\mathbb{F}$ instead of $(\mathbb{F}, A)$.

• Let $\theta : U \longmapsto U'$ be a function and $\mathbb{F}(resp.\mathbb{F}')$ be a soft set over $U(resp.U')$ with a parameter set E. Then $\theta(\mathbb{F})(resp.\theta^{-1}(\mathbb{F}'))$ is the soft set on $U'(resp.U)$ defined by $(\theta(\mathbb{F}))(e) = \theta(\mathbb{F}(e))(resp.(\theta^{-1}(\mathbb{F}'))(e) = \theta^{-1}(\mathbb{F}'(e)))$.

• Use hat $\widehat{(.)}$ to distinguish "soft"objects from usual ones. For example, for a subset X of U, $\widehat{X}$ denotes the soft set satisfying that $\widehat{X}(e) = X$ for all $e \in E$.

• We write $\mathbb{F}_1\widehat{\cap}\mathbb{F}_2$ for the soft intersection of $\mathbb{F}_1$ and $\mathbb{F}_2$, where it is defined by $(\mathbb{F}_1\widehat{\cap}\mathbb{F}_2)(e) = \mathbb{F}_1(e) \cap \mathbb{F}_2(e)$ for every $e \in E$.

• The soft union of $\mathbb{F}_1$ and $\mathbb{F}_2$, will be denoted by $\mathbb{F}_1\widehat{\cup}\mathbb{F}_2$, is defined by $(\mathbb{F}_1\widehat{\cup}\mathbb{F}_2)(e) = \mathbb{F}_1(e) \cup \mathbb{F}_2(e)$ for all $e \in E$.

• We will use the symbol $\mathbb{F}^{\widehat{c}}$ to denote soft complement of $\mathbb{F}$ and is defined by $\mathbb{F}^{\widehat{c}}(e) = U \setminus \mathbb{F}(e)(e \in E)$.

• Let $\mathbb{F}$ be a soft set over U and x be an element of U. We call x is a soft element of $\mathbb{F}$, if $x \in \mathbb{F}(e)$ for all parameters $e \in E$ and denoted by $x\widehat{\in}\mathbb{F}$.

2.2 Polygroups

• Let H be a non-empty set, the couple $(H, \circ)$ is called a hypergroupoid if

$\circ : H \times H \longmapsto \mathcal{P}(H^*)$ be a function, the combination of two subset A and B of H is defined as $A \circ B = \bigcup\limits_{a\in A} a \circ B$ and $a \circ B = \bigcup\limits_{b\in B} a \circ b$.

• A hypergroupoid $(H, \circ)$ is called a quasihypergroup if for every $h \in H$, $h \circ H = H = H \circ h$ and is called a semihypergroup if for every $t, u, w \in H$, $t \circ (u \circ w) = (t \circ u) \circ w$. The pair $(H, \circ)$ is called a hypergroup if it is a quasihypergroup and a semihypergroup [11, 12].

• Let $(H, \circ)$ be a semihypergroup and A be a subset of H. Say that A is a complete part of H if for any $n \in \mathbb{N}$ and for all $a_1, \ldots, a_n$ of H, the following implication true:

$$A \cap \prod_{i=1}^{n} a_i \neq \varnothing \Longrightarrow \prod_{i=1}^{n} a_i \subseteq A.$$

The complete parts were introduced for the first time by Koskas [13].

• Let $(P, \circ)$ be hypergroup and have other additional features. If there exist unitary operation $^{-1}$ on P and $e \in P$ with the property that for all $p, q, r \in P$, the following items be true;

(A) $(p \circ q) \circ r = p \circ (q \circ r)$,

(B) $e \circ p = p \circ e = p$,

(C) If $p \in q \circ r$, then $q \in p \circ r^{-1}$ and $r \in q^{-1} \circ p$.

In this case, hypergroup P is called polygroup.

• The following results follow from the above axioms:

$e \in p \circ p^{-1} \cap p^{-1} \circ p, e^{-1} = e, (p^{-1})^{-1} = p$, and $(p \circ q)^{-1} = q^{-1} \circ p^{-1}$. A nonempty subset Q of a polygroup P is called a subpolygroup of P if and only if for all $x, y \in Q$ follows that $x \circ y \subseteq Q$ and for all $x \in Q$ follows that $x^{-1} \in Q$.

• Let P be polygroup and $(\mathbb{F}, A)$ be a soft set on P. Then $(\mathbb{F}, A)$ is called a (normal)soft polygroup on P if $\mathbb{F}(x)$ is a (normal)subpolygroup of P for all $x \in Supp(\mathbb{F}, A)$.

Example 1. Let P be $\{e, a, b, c\}$ and multiplication table be:

$\circ$	e	a	b	c
e	e	a	b	c
a	a	$\{e, a\}$	c	$\{b, c\}$
b	b	c	e	a
c	c	$\{b, c\}$	a	$\{e, a\}$

Then P is a polygroup. Let $\mathbb{F}, A)$ be a soft set over P, where A equal with P and define $\mathbb{F} : A \longmapsto \mathcal{P}(P)$ by $\mathbb{F}(x) = \{y \in P | xRy \Leftrightarrow y \in x^2\}$ for all $x \in A$. In this case, we will have $\mathbb{F}(e) = \mathbb{F}(b) = \{e\}$ and $\mathbb{F}(a) = \mathbb{F}(c) = \{e, a\}$ are subpolygroups of P. In conclusion, $(\mathbb{F}, A)$ is a soft polygroup over P [4].

3 *Soft Topological polygroups*

Let $(P, \mathcal{T})$ be a topological space, where $(P, \circ, e, ^{-1})$ is a polygroup. Then the $(P, \mathcal{T})$ is called a topological polygroup if the following axioms hold:

(1) The mapping $\circ : P \times P \longmapsto \mathcal{P}(P)$ is continuous, where $\circ(x, y) = x \circ y$,

(2) The mapping $^{-1} : P \longmapsto P$ is continuous, where $^{-1}(x) = -x$.

Definition 1. [8] Let $\mathcal{T}$ be a topology on a polygroup P. Let $(\mathbb{F}, A)$ be a soft set over P. Then the system $(\mathbb{F}, A, \mathcal{T})$ said to be soft topological polygroup over P if the following axioms hold:

(a) $\mathbb{F}(a)$ is a subpolygroup of P for all $a \in A$,

(b) The mapping $(x, y) \longmapsto x \circ y$ of the topological space $\mathbb{F}(a) \times \mathbb{F}(a)$ onto $\mathcal{P}^*(\mathbb{F}(a))$ and mapping $x \longmapsto x^{-1}$ of the topological space $\mathbb{F}(a)$ onto $\mathbb{F}(a)$ are continuous for all $a \in A$.

Topology $\mathcal{T}$ on P induces topolgies on $\mathbb{F}(a)$, $\mathbb{F}(a) \times \mathbb{F}(a)$ and $\mathcal{P}^*(\mathbb{F}(a))$.

Example 2. Let P be $\{1, 2\}$ and its hyperoperation be as follows:

$\circledast$	1	2
1	1	2
2	2	$\{1,2\}$

Hyperoperation $\circledast : P \times P \longmapsto \mathcal{P}(P)$ and inverse operation $^{-1} : P \longmapsto P$ are continuous with topology $\mathcal{T}_1 = \{\varnothing, P, \{1\}\}$ but $\circledast : P \times P \longmapsto \mathcal{P}(P)$ is not continuous with topology $\mathcal{T}_2 = \{\varnothing, P, \{2\}\}$. Therefore, P with $\mathcal{T}_1$, $\mathcal{T}_{dis}$,$\mathcal{T}_{ndis}$ is topological polygroup. Subpolygroups of P are $\varnothing, P, \{1\}$. Let A be arbitrary set and $a_1, a_2 \in A$ and define soft set $\mathbb{F}$:

$$\mathbb{F}(x) = \begin{cases} \{1\} & \text{if } x = a_1, \\ P & \text{if } x = a_2, \\ \varnothing & \text{otherwise.} \end{cases}$$

Therefore, $(\mathbb{F}, A, \mathcal{T}_1)$ is a soft topological polygroup.

Example 3. Let $\widehat{D_4}$ be $\{1,2,3,4,5\}$ and its hyperoperation be as follows:

$*$	1	2	3	4	5
1	1	2	3	4	5
2	2	1	4	3	5
3	3	4	1	2	5
4	4	3	2	1	5
5	5	5	5	5	$\{1,2,3,4\}$

Hyperoperation $* : \widehat{D_4} \times \widehat{D_4} \longmapsto \mathcal{P}(\widehat{D_4})$ is not continuous with the following topologies:

$$\begin{array}{ll}
\mathcal{T}_1 = \{\varnothing, \widehat{D_4}, \{1\}\} & \mathcal{T}_2 = \{\varnothing, \widehat{D_4}, \{2\}\} \\
\mathcal{T}_3 = \{\varnothing, \widehat{D_4}, \{3\}\} & \mathcal{T}_4 = \{\varnothing, \widehat{D_4}, \{4\}\} \\
\mathcal{T}_5 = \{\varnothing, \widehat{D_4}, \{5\}\} & \mathcal{T}_6 = \{\varnothing, \widehat{D_4}, \{1,2\}\} \\
\mathcal{T}_7 = \{\varnothing, \widehat{D_4}, \{1,3\}\} & \mathcal{T}_8 = \{\varnothing, \widehat{D_4}, \{1,4\}\} \\
\mathcal{T}_9 = \{\varnothing, \widehat{D_4}, \{1,5\}\} & \mathcal{T}_{10} = \{\varnothing, \widehat{D_4}, \{2,3\}\} \\
\mathcal{T}_{11} = \{\varnothing, \widehat{D_4}, \{2,4\}\} & \mathcal{T}_{12} = \{\varnothing, \widehat{D_4}, \{2,5\}\} \\
\mathcal{T}_{13} = \{\varnothing, \widehat{D_4}, \{3,4\}\} & \mathcal{T}_{14} = \{\varnothing, \widehat{D_4}, \{3,5\}\} \\
\mathcal{T}_{15} = \{\varnothing, \widehat{D_4}, \{4,5\}\} & \mathcal{T}_{16} = \{\varnothing, \widehat{D_4}, \{1,2,3\}\} \\
\mathcal{T}_{17} = \{\varnothing, \widehat{D_4}, \{1,2,4\}\} & \mathcal{T}_{18} = \{\varnothing, \widehat{D_4}, \{1,2,5\}\} \\
\mathcal{T}_{19} = \{\varnothing, \widehat{D_4}, \{2,3,4\}\} & \mathcal{T}_{20} = \{\varnothing, \widehat{D_4}, \{2,3,5\}\} \\
\mathcal{T}_{21} = \{\varnothing, \widehat{D_4}, \{2,4,5\}\} & \mathcal{T}_{22} = \{\varnothing, \widehat{D_4}, \{3,4,5\}\} \\
\mathcal{T}_{23} = \{\varnothing, \widehat{D_4}, \{1,2,3,4\}\} & \mathcal{T}_{24} = \{\varnothing, \widehat{D_4}, \{1,2,3,5\}\} \\
\mathcal{T}_{25} = \{\varnothing, \widehat{D_4}, \{2,3,4,5\}\}. &
\end{array}$$

This means that $(\widehat{D_4}, \mathcal{T}_{dis})$ and $(\widehat{D_4}, \mathcal{T}_{ndis})$ are topological polygroups. Subpolygroups of $\widehat{D_4}$ are $\varnothing$, $\widehat{D_4}, \{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3,4\}$. Let A be a arbitrary set and $a_1, a_2, a_3, a_4, a_5 \in A$. Then we define a soft set $\mathbb{F}$ by

$$\mathbb{F}(x) = \begin{cases} \{1\} & \text{if } x = a_1 \\ \{1,2\} & \text{if } x = a_2 \\ \{1,3\} & \text{if } x = a_3 \\ \{1,4\} & \text{if } x = a_4 \\ \{1,2,3,4\} & \text{if } x = a_5 \\ \varnothing & \text{otherwise.} \end{cases}$$

Since restriction of topology $\mathcal{T}_5 = \{\varnothing, \widehat{D_4}, \{5\}\}$ to subspaces $\mathbb{F}(x)$ are discrete or anti-discrete topologies $(\mathbb{F}, A, \mathcal{T}_5)$ is a soft topological polygroup. With this method we can make many examples of soft topological polygroups.

Definition 2. Let $(\mathbb{F}, A, \mathcal{T})$ be a soft topological polygroup over P. Then the closure of $\overline{(\mathbb{F}, A, \mathcal{T})}$ denoted by $(\overline{\mathbb{F}}, A, \mathcal{T})$ and defined as $\overline{\mathbb{F}}(a) = \overline{\mathbb{F}(a)}$ where $\overline{\mathbb{F}(a)}$ is the closure of $\mathbb{F}(a)$ in topology defined on P.

Theorem 1. [2] Let A and B be subsets of a topological polygroup P with the property that every open subset of P is a complete part. Then
(1) $\overline{A} \circ \overline{B} \subseteq \overline{A \circ B}$,
(2) $(\overline{A})^{-1} = \overline{(A^{-1})}$.

Theorem 2. [2] Let P be a topological polygroup with the property that every open subset of P is a complete part. Then
(1) If K is a subsemihypergroup of P, then as well as $\overline{K}$,
(2) If K is a subpolygroup of P, then as well as $\overline{K}$.

Theorem 3. [8] Let $(\mathbb{F}, A, \mathcal{T})$ be a soft topological polygroup over a topological polygroup $(P, \mathcal{T})$ and every open subset of P is a complete part. Then the following are true.
(1) $(\overline{\mathbb{F}}, A, \mathcal{T})$ is also a soft topological polygroup over $(P, \mathcal{T})$,
(2) $(\mathbb{F}, A, \mathcal{T}) \widehat{\subset} (\overline{\mathbb{F}}, A, \mathcal{T})$.

• Now instead of the usual topology we use the soft topology to define the soft topological polygroup based on [10].
• A family θ of soft sets over U is called a soft topology on U if the following axioms hold:
(1) $\widehat{\varnothing}$ and $\widehat{U}$ are in θ,
(2) θ is closed under finite soft intersection,
(3) θ is closed under (arbitrary) soft union.
• We will use the symbol (U, θ, E) to denote a soft topological space and soft set $\mathbb{F}$ is called a soft close set if $\mathbb{F}^{\widehat{c}}$ is soft open set, where each member of θ said to be a soft open set.

Example 4. Let U be $\mathbb{Z}_2$ and θ be $\{\widehat{\varnothing}, \{e_2\} \times \mathbb{Z}_2, \widehat{\mathbb{Z}_2}\}$, where $E = \{e_1, e_2\}$ and $\{e_2\} \times \mathbb{Z}_2$ be soft set $\mathbb{F} : E \longmapsto P(\mathbb{Z}_2)$ with the property that $\mathbb{F}(e_1) = \varnothing; \mathbb{F}(e_2) = \mathbb{Z}_2$. Then $(\mathbb{Z}_2, \theta, E)$ is soft topological space.

Example 5. Let P be $\{1, 2\}$ and hyperoperation $\circledast$ be as follows:

$\circledast$	1	2
1	1	2
2	2	$\{1, 2\}$

polygroup P with topology $\theta = \{\widehat{\varnothing}, \{e_2\} \times P, \widehat{P}\}$ is a soft topological space.

Example 6. Let P be $\{e, a, b, c\}$ and hyperoperation $\circ$ be as follows:

$\circ$	e	a	b	c
e	e	a	b	c
a	a	$\{e, a\}$	c	$\{b, c\}$
b	b	c	e	a
c	c	$\{b, c\}$	a	$\{e, a\}$

polygroup P with topologies $\theta_1 = \{\widehat{\varnothing}, \{e_1\} \times P, \widehat{P}\}$, $\theta_2 = \{\widehat{\varnothing}, \{e_2\} \times P, \widehat{P}\}$ are soft topological spaces.

• The closure of $\mathbb{F}$ is denoted by $\overline{\mathbb{F}}$ and is defined by soft intersection of all soft closed supersets of $\mathbb{F}$, where $\mathbb{F}$ is soft set over U.

• A soft set $\mathbb{F}$ said to be a soft neighborhood of x if there exists a soft open set $\mathbb{G}$ with the property that $x\widehat{\in}\mathbb{G}\widehat{\subseteq}\mathbb{F}$, where x be an element of the universe U. The soft neighborhood system of x we will consider the collection of all soft neighborhoods of x.

• Let V be a subset of the universe U. A soft set $\mathbb{F}$ said to be a soft neighborhood of V if there exists a soft open set $\mathbb{G}$ with the property that $V\widehat{\subseteq}\mathbb{G}\widehat{\subseteq}\mathbb{F}$ (i.e. $\forall e \in E : V \subseteq \mathbb{G}(e) \subseteq \mathbb{F}(e)$).

In this section, we will define soft continuity and express its equivalent theorems, then define soft topological polygroups with the idea of Hida [10] and study the properties of soft topological polygroups.

Definition 3. Let P_1, P_2 be polygroups and $(P_1, \theta_1, E), (P_2, \theta_2, E)$ be soft topological spaces. The function $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ said to be a soft continuous function if for all $x \in P_1$ and for all soft neighborhood $\mathbb{F}_{\varphi(x)}$ of $\varphi(x)$, there exists a soft neighborhood $\mathbb{F}_x$ of x with the property that $\varphi(\mathbb{F}_x)\widehat{\subseteq}\mathbb{F}_{\varphi(x)}$.

Theorem 4. Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a function such that for all soft open set $\mathbb{F}' \in \theta_2$ the inverse image $\varphi^{-1}(\mathbb{F}')$ is soft open set if and only if for every soft closed set $\mathbb{F}'$ the inverse image $\varphi^{-1}(\mathbb{F}')$ is soft closed set.

Proof. This is easily seen to be an equivalence relation.

Theorem 5. Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be function. For every soft closed set $\mathbb{F}'$, the inverse image $\varphi^{-1}(\mathbb{F}')$ is also soft closed if and only if for all soft set $\mathbb{F}$, $\varphi(\overline{\mathbb{F}})\widehat{\subseteq}\overline{\varphi(\mathbb{F})}$.

Proof.

(i) $\Longleftarrow$ Let $\mathbb{F}'$ be a soft closed set. Then we have $\varphi(\varphi^{-1}(\mathbb{F}'))\widehat{\subseteq}\mathbb{F}'$. The soft closeness of $\mathbb{F}'$, together with the assumption (for all soft set $\mathbb{F}$, we have $\varphi(\overline{\mathbb{F}})\widehat{\subseteq}\overline{\varphi(\mathbb{F})}$, proves that

$$\varphi(\overline{(\varphi^{-1}(\mathbb{F}'))})\widehat{\subseteq}\overline{\varphi(\varphi^{-1}(\mathbb{F}'))}\widehat{\subseteq}\mathbb{F}'.$$

Therefore, it holds that $\overline{\varphi^{-1}(\mathbb{F}')}\widehat{\subseteq}\varphi^{-1}(\mathbb{F}')\widehat{\subseteq}\overline{\varphi^{-1}(\mathbb{F}')}$, which shows that $\varphi^{-1}(\mathbb{F}')$ is soft closed.

(ii) $\Longrightarrow$ We have $\mathbb{F}\widehat{\subseteq}\varphi^{-1}(\overline{\varphi(\mathbb{F})})$ for any soft set $\mathbb{F}$. Since (for every soft closed set $\mathbb{F}'$, the inverse image $\varphi^{-1}(\mathbb{F}')$ is also soft closed, we have $\overline{\mathbb{F}}\widehat{\subseteq}\varphi^{-1}(\overline{\varphi(\mathbb{F})})$. Thus, we have

$$\varphi(\overline{\mathbb{F}})\widehat{\subseteq}\varphi(\varphi^{-1}(\overline{\varphi(\mathbb{F})}))\widehat{=}\overline{\varphi(\mathbb{F})}.$$

Theorem 6. Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a function. If for all soft open set $\mathbb{F}' \in \theta_2$, the inverse image $\varphi^{-1}(\mathbb{F}')$ is also soft open set then φ is a soft continuous function.

Proof. For all $x \in P_1$ and a soft open neighborhood $\mathbb{F}'$ of $\varphi(x), \varphi^{-1}(\mathbb{F}')$ is a soft open set having x as a soft element. Since $\varphi(\varphi^{-1}(\mathbb{F}'))\widehat{\subseteq}\mathbb{F}'$, give $F = \varphi^{-1}(\mathbb{F}')$ in this case $\varphi(\mathbb{F})\widehat{\subseteq}\mathbb{F}'$.

Example 7. The opposite Theorem 6 is not true.

Let P_1 be $< \{e\}, \theta_1, \{a_1, a_2\} >$ and P_2 be $< \{e\}, \theta_2, \{a_1, a_2\} >$, where

$$\theta_1 = \{\widehat{\varnothing}, \{(a_1, e), (a_2, e)\}\},$$
$$\theta_2 = \{\widehat{\varnothing}, \{(a_2, e)\}, \{(a_1, e), (a_2, e)\}\}.$$

The unique soft neighborhood of the point e is $\{a_1, a_2\} \times \{e\}$ in θ_2. The inverse image of $\{a_1, a_2\} \times \{e\}$ under $id : P_1 \longmapsto P_2$ is $\{a_1, a_2\} \times \{e\}$. Thus $id : P_1 \longmapsto P_2$ satisisfies in the second part of Theorem 6, but $id^{-1}(\{(a_2, e)\})$ is not soft open in P_1.

Definition 4. A bijection $\varphi : P_1 \longmapsto P_2$ said to be a soft homeomorphism between (P_1, θ_1, E) and (P_2, θ_2, E) if φ and φ^{-1} are soft continuous.

Theorem 7. Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a soft continuous function. Then for all soft open set $\mathbb{F}_2 \in \theta_2$, there exists a soft open set $\mathbb{F}_1 \in \theta_1$ with the property that for all $x \in P_1$; $x \widehat{\in} \mathbb{F}_1$ if and only if $x \widehat{\in} \varphi^{-1}(\mathbb{F}_2)$.

Proof. For every $x \in P_1$ with $\varphi(x) \widehat{\in} \mathbb{F}_2$, choose a soft open $\mathbb{F}_x \in \theta_1$ with the property that $x \widehat{\in} \mathbb{F}_x$ and $\varphi(\mathbb{F}_x) \widehat{\subseteq} \mathbb{F}_2$. Then define $\mathbb{F}_1 = \widehat{\bigcup}\{\mathbb{F}_x | x \in P_1, \varphi(x) \widehat{\in} \mathbb{F}_2\}$, $\mathbb{F}_1$; has the required properties.

Definition 5. Let $(P, \circ, e, ^{-1})$ be a polygroup and θ be a soft topology on P with a parameter set E. Then (P, θ, E) is a soft topological polygroup if the following items are true:

(i) For each soft neighborhood $\mathbb{F}_{p \circ q}$ of $p \circ q$, where $(p, q) \in P \times P$ there exist soft neighborhoods $\mathbb{F}_p$ and $\mathbb{F}_q$ of p and q with the property that $\mathbb{F}_p \circ \mathbb{F}_q \widehat{\subseteq} \mathbb{F}_{p \circ q}$,

(ii) The inversion function $^{-1} : P \longmapsto P$ is soft continuous.

Every soft topological group is a soft topological polygroup.

Example 8. Let E be $\{e_1, e_2\}$ and θ be $\{\widehat{\varnothing}, \{(e_1, \overline{1})\}, \widehat{\mathbb{Z}_2}\}$. In conclusion, $(\mathbb{Z}_2, \theta, E)$ is a soft topological polygroup.

Example 9. Show that $(\mathbb{R}, \theta, E)$ is a soft topological group, where $E = \{e_1, e_2\}$ and θ is the soft topology generated by the following subbase:

$$\{\widehat{\varnothing}, \widehat{\mathbb{R}}\} \cup \{\{(e_1, r), (e_2, x) | r - \epsilon < x < r + \epsilon\} | r \in \mathbb{R}, \epsilon > 0\}.$$

Example 10. Every polygroup with discrete or anti-discrete topology is soft topological polygroup.

Example 11. Let P be $\{e, a, b, c\}$ and hyperoparation $\circ$ be as follow:

$\circ$	e	a	b	c
e	e	a	b	c
a	a	$\{e, a\}$	c	$\{b, c\}$
b	b	c	e	a
c	c	$\{b, c\}$	a	$\{e, a\}$

Let E be $\{e_1, e_2, e_3\}$. Then polygroup P with topologies

$$\theta_1 = \{\widehat{\varnothing}, \{e_1\} \times P, \widehat{P}\}$$
$$\theta_2 = \{\widehat{\varnothing}, \{e_2\} \times P, \widehat{P}\}$$
$$\theta_3 = \{\widehat{\varnothing}, \{e_3\} \times \{a, b\}, \widehat{P}\}$$
$$\theta_3 = \{\widehat{\varnothing}, \{e_3\} \times \{a, b\}, \{e_1\} \times \{e, b\}, \widehat{P}\}$$
$$\theta_4 = \{\widehat{\varnothing}, \{e_3\} \times \{a, b\}, \{e_1\} \times \{e, b\}, \{e_2\} \times \{e, b, c\}, \widehat{P}\}$$

are soft topological polygroups.

Theorem 8. (P, θ, E) is a soft topological polygroup if and only if for all $x, y \in P$ and for each soft open set $\mathbb{F}$ with $x \circ y^{-1} \widehat{\subseteq} \mathbb{F}$, there exist soft open sets $\mathbb{F}_x, \mathbb{F}_y$ with the property that $x \widehat{\in} \mathbb{F}_x, y \widehat{\in} \mathbb{F}_y$ and $\mathbb{F}_x \circ \mathbb{F}_y^{-1} \widehat{\subseteq} \mathbb{F}$.

Proof. $\Longrightarrow$ From item (i) of Definition 5 we know that there exist soft open sets $\mathbb{F}_x, \mathbb{F}_{y^{-1}}$ with the property that $x \widehat{\in} \mathbb{F}_x, y^{-1} \widehat{\in} \mathbb{F}_{y^{-1}}$ and $\mathbb{F}_x \circ \mathbb{F}_{y^{-1}} \widehat{\subseteq} \mathbb{F}$. From item (ii) Definition 5 we know that there exists a soft open set $\mathbb{F}_y$ satisfying $y \widehat{\in} \mathbb{F}_y$ and $(\mathbb{F}_y)^{-1} \widehat{\subseteq} \mathbb{F}_{y^{-1}}$. In conclusion, $x \widehat{\in} \mathbb{F}_x, y \widehat{\in} \mathbb{F}_y$ and $\mathbb{F}_x \circ (\mathbb{F}_y)^{-1} \widehat{\subseteq} \mathbb{F}_x \circ \mathbb{F}_{y^{-1}} \widehat{\subseteq} \mathbb{F}$. $\Longleftarrow$ Let $\mathbb{F}$ be a soft open set with the property that $x^{-1} \widehat{\in} \mathbb{F}$. Since $x^{-1} = e \circ x^{-1}$, there exist soft open sets $\mathbb{F}_e, \mathbb{F}_x$ with the property that $e \widehat{\in} \mathbb{F}_e, x \widehat{\in} \mathbb{F}_x$ and $\mathbb{F}_e \circ \mathbb{F}_x^{-1} \widehat{\subseteq} \mathbb{F}$. In conclusion, $\mathbb{F}_x^{-1} \widehat{\subseteq} \mathbb{F}_e \circ \mathbb{F}_x^{-1} \widehat{\subseteq} \mathbb{F}$, which that the item (ii) of Definition 5 is proved. Let $\mathbb{F}$ be a soft open set satisfying $x \circ y \widehat{\subseteq} \mathbb{F}$. Since $x \circ y = x \circ (y^{-1})^{-1}$, we can find soft open sets $\mathbb{F}_x, \mathbb{F}_{y^{-1}}$ with the property that $x \widehat{\in} \mathbb{F}_x, y^{-1} \widehat{\in} \mathbb{F}_{y^{-1}}$ and $\mathbb{F}_x \circ (\mathbb{F}_{y^{-1}})^{-1} \widehat{\subseteq} \mathbb{F}$. Since $^{-1} : P \longmapsto P$ is soft continuous, we can find a soft open set $\mathbb{F}_y$ with the property that $y \widehat{\in} \mathbb{F}_y$ and $\mathbb{F}_y^{-1} \widehat{\subseteq} \mathbb{F}_{y^{-1}}$. In conclusion, $\mathbb{F}_x \circ \mathbb{F}_y \widehat{=} \mathbb{F}_x \circ ((\mathbb{F}_y)^{-1})^{-1} \widehat{\subseteq} \mathbb{F}_x \circ (\mathbb{F}_{y^{-1}})^{-1} \widehat{\subseteq} \mathbb{F}$.

Definition 6. Let (P, θ, E) be a soft topological polygroup and for all $x, y \in P$ with $x \neq y$, there exists a soft open set $\mathbb{F}$ with the property that either $x \widehat{\in} \mathbb{F} \bigwedge \forall e \in E(y \notin \mathbb{F}(e))$ or $y \widehat{\in} \mathbb{F} \bigwedge \forall e \in E(x \notin \mathbb{F}(e))$ holds, then (P, θ, E) is a soft $\mathcal{T}_0 space$.

Theorem 9. Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a soft continuous injection and (P_2, θ_2, E) be a soft $\mathcal{T}_0 space$. Then (P_1, θ_1, E).

Proof. Let (P_2, θ_2, E) be a soft $\mathcal{T}_0 space$ and φ be a soft continuous injection, if $x, y \in P_1$ and $x \neq y$ then $\varphi(x), \varphi(y) \in P_2$ and $\varphi(x) \neq \varphi(y)$, on the other hand, (P_2, θ_2, E) is a soft $\mathcal{T}_0 space$ hence there exists a soft open set $\mathbb{F}$ with the property that $\varphi(x) \widehat{\in} \mathbb{F}$ or $\varphi(y) \widehat{\in} \mathbb{F}$. Without loss of generality let $\varphi(x) \widehat{\in} \mathbb{F}$ then since φ is continuous, there exists $\mathbb{F}_x \in \theta_1$ with the property that $x \widehat{\in} \mathbb{F}_x$ and $\varphi(\mathbb{F}_x) \widehat{\subseteq} \mathbb{F}$ hence $x \widehat{\in} \mathbb{F} \bigwedge \forall e \in E(y \notin \mathbb{F}(e))$.

Definition 7. Let (P, θ, E) be a soft topological polygroup and for every distinct points $x_1, x_2 \in P$, there exist soft open sets $\mathbb{F}_1, \mathbb{F}_2$ with the property that both $x_1 \widehat{\in} \mathbb{F}_1 \bigwedge \forall e \in E(x_2 \notin \mathbb{F}_1(e))$ and $x_2 \widehat{\in} \mathbb{F}_2 \bigwedge \forall e \in E(x_1 \notin \mathbb{F}_2(e))$ hold, then (P, θ, E) is a soft $\mathcal{T}_1 space$.

Theorem 10. Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a soft continuous injection and (P_2, θ_2, E) be a soft $\mathcal{T}_1 space$. Then (P_1, θ_1, E).

Proof. It is simillar to the proof of Theorem 9.

Definition 8. Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a soft continuous injection and for every distinct elements $x_1, x_2 \in P$, there exist soft open sets $\mathbb{F}_1, \mathbb{F}_2 \in \theta$ with $x_1 \widehat{\in} \mathbb{F}_1, x_2 \widehat{\in} \mathbb{F}_2$ and $\mathbb{F}_1 \widehat{\cap} \mathbb{F}_2 = \widehat{\varnothing}$, then (P, θ, E) is a soft Hausdorff (or soft $\mathcal{T}_2 space$ [14]).

Example 12. Let $\mathbb{R}$ be real number, E be $\{e_1, e_2\}$, and θ be the soft topology generated by the following subbase:

$$\{\widehat{\varnothing}, \widehat{\mathbb{R}}\} \cup \{\{(e_1, r), (e_2, x) | r - \epsilon < x < r + \epsilon\} | r \in \mathbb{R}, \epsilon > 0\}.$$

It can be shown that $(\mathbb{R}, \theta, E)$ is a soft Hausdorff space.

Theorem 11. Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a soft continuous injection and (P_2, θ_2, E) be a soft Hausdorff space. Then (P_1, θ_1, E).

Proof. Take distinct elements x and y from P_1. We can separate $\varphi(x)$ from $\varphi(y)$ by soft open sets, $\mathbb{F}_{\varphi(x)}, \mathbb{F}_{\varphi(y)}$. Since φ is soft continuous, we have soft open neighborhoods $\mathbb{F}_x, \mathbb{F}_y$ of x, y, respectively, satisfying that $\varphi(\mathbb{F}_x) \widehat{\subseteq} \mathbb{F}_{\varphi(x)}$ and $\varphi(\mathbb{F}_y) \widehat{\subseteq} \mathbb{F}_{\varphi(y)}$. Clearly $\mathbb{F}_x$ and $\mathbb{F}_y$ separate x from y.

Definition 9. Let (P, θ, E) be a soft topological polygroup and for every $x \in P$, every soft neighborhood of x contains a soft closed neighborhood of x, then (P, θ, E) is a soft regular space [14].

Example 13. Let P be $\mathbb{Z}_2$ and E be $\{e_1, e_2, e_3\}$. Define soft topology θ on P by $\theta = \{\widehat{\varnothing}, \{(e_2, \overline{1})\}, \widehat{P}\}$. Since in soft topological polygroup, every point $x \in P$ has only one soft clopen neighborhood $\widehat{P}$, it is obvious that soft space is soft regular.

Definition 10. A family C of soft open sets over P is said to be a soft open covering of P if for all $x \in P$ there exists an $\mathbb{F} \in C$ with the property that $x \widehat{\in} \mathbb{F}$.

Definition 11. A soft space (P, θ, E) is soft compact if for any soft open covering C of (P, θ, E), there exist $\mathbb{F}_1, ..., \mathbb{F}_n \in C$ with the property that $\{\mathbb{F}_1, ..., \mathbb{F}_n\}$ is a soft open covering.

Theorem 12. If V is soft compact with respect to θ_1, then so is $\varphi(V)$ with respect to θ_2, where $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a soft continuous function and $V \subseteq P_1$ be a subset.

Proof. Let C' be a soft open covering of $\varphi(V)$. For every $v \in V$, there exists an $\mathbb{G}'_{\varphi(v)} \in C'$ with the property that $\varphi(v) \widehat{\in} \mathbb{G}'_{\varphi(v)}$. Since φ is soft continuous, there exists a soft open neighborhood $\mathbb{G}_v$ of v with the property that $\varphi(\mathbb{G}_v) \widehat{\subseteq} \mathbb{G}'_{\varphi(v)}$. Then the family $\{\mathbb{G}_v | v \in V\}$ is a soft covering of V. V is soft compact with respect to θ_1, there exist $v_1, ..., v_n \in V$ with the property that $\{\mathbb{G}_{v_i}\}_{i=1}^n$ is a soft covering of V. Thus we have $\varphi(\mathbb{G}_{v_i}) \widehat{\subseteq} \mathbb{G}'_{\varphi(v_i)}$, in conclusion $\{\mathbb{G}'_{\varphi(v_i)}\}_{i=1}^n$ is a soft covering of $\varphi(V)$.

Definition 12. If for any soft open covering $\mathbb{F}_1, \mathbb{F}_2$ of X subject to the condition that $\nexists x \in X(x\widehat{\in}\mathbb{F}_1 \bigwedge x\widehat{\in}\mathbb{F}_2)$, either $\forall x \in X(x\widehat{\notin}\mathbb{F}_1)$ or $\forall x \in X(x\widehat{\notin}\mathbb{F}_2)$ holds, then subset X of P said to be soft connected.

Example 14. Let E be $\{e_1, e_2\}$ and θ be $\{\widehat{\varnothing}, \{(e_1, \overline{1}), (e_2, \overline{0})\}, \{(e_1, \overline{0}), (e_2, \overline{1})\}, \widehat{\mathbb{Z}_2}\}$ on $\mathbb{Z}_2$. In this case, since both $\overline{0}$ and $\overline{1}$ have only one soft neighborhood $\widehat{\mathbb{Z}_2}$ the soft topological polygroup $(\mathbb{Z}_2, \theta, E)$ is soft connected.

Example 15. Let $\mathbb{Z}$ be integer numbers and consider soft topological polygroup $(\mathbb{Z}, \theta, E)$ where $E = \{e_1, e_2\}$ and $\theta = \{\widehat{\varnothing}, \widehat{\mathbb{Z}}\} \cup \{\{e_1\} \times S | S \subseteq \mathbb{Z}\}$. Since $\widehat{\mathbb{Z}}$ is the unique soft neighborhood of $i \in \mathbb{Z}$ thus $(\mathbb{Z}, \theta, E)$ is soft compact and also soft connected.

Theorem 13. If both X_1 and X_2 are soft connected then $X_1 \cup X_2$ is soft connected, where X_1 and X_2 are subsets of P having non-empty intersection.

Proof. Suppose that $\{\mathbb{F}_1, \mathbb{F}_2\}$ be soft open covering of $X_1 \cup X_2$ Since $\{\mathbb{F}_1, \mathbb{F}_2\}$ is a soft open covering also of X_1. Without loss of generality assume $\forall z \in X_1(z\widehat{\notin}\mathbb{F}_1)$. In particular, $x\widehat{\notin}\mathbb{F}_1$ holds for every $x \in X_1 \cap X_2$. Suppose that for the contradiction that there were a $z \in X_2$ with the property that $z\widehat{\in}\mathbb{F}_1$. Since $\{\mathbb{F}_1, \mathbb{F}_2\}$ is a soft open covering of X_2, it would hold that $\forall z \in X_2(z\widehat{\notin}\mathbb{F}_2)$. In particular, $x\widehat{\notin}\mathbb{F}_2$ holds for every $x \in X_1 \cap X_2$, which gives a contradiction. Therefore, $\forall z \in X_2(z\widehat{\notin}\mathbb{F}_1)$ holds; so has obtained $\forall z \in X_1 \cup X_2(z\widehat{\notin}\mathbb{F}_1)$.

Theorem 14. If $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ is a soft continuous and X subset of P is soft connected, then $\varphi(X)$ is soft connected.

Proof. Let X be a soft connected subset of P_1. Select an arbitrary soft open covering $\{\mathbb{F}'_1, \mathbb{F}'_2\}$ of $\varphi(x)$. By Theorem 7, there exist soft open sets $\mathbb{F}_i \in \theta_1$ with the property that $\varphi(\mathbb{F}_i)\widehat{\subseteq}\mathbb{F}'_i$ and $\forall y \in P_2(y\widehat{\in}\mathbb{F}'_i$ if and only if $y\widehat{\in}\varphi(\mathbb{F}_i))(i = 1, 2)$. Thus for every $x \in X$, exactly one of $x\widehat{\in}\mathbb{F}_1$ and $x\widehat{\in}\mathbb{F}_2$ holds. Since X is soft connected, there is no loss of generality in assuming $\forall x \in X(x\widehat{\notin}\mathbb{F}_1)$. Therefore have $\forall y \in \varphi(X)(y\widehat{\notin}\varphi(\mathbb{F}_1))$, and $\forall y \in \varphi(X)(y\widehat{\notin}\mathbb{F}'_1)$.

Theorem 15. If $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ is a soft homeomorphism, then $X \subseteq P$ is soft connected if and only if $\varphi(X)$ is soft connected.

Proof. We have showed a left-to-right direction in the previous theorem. The converse direction follows from the same argument applied to φ^{-1}.

Definition 13. We say that Property $\mathcal{P}$ of soft topological polygroups is a soft topological property if the following condition holds for any soft space (P, θ, E).

A soft space (P, θ, E) has the property $\mathcal{P}$ if and only if every soft space which is soft homeomorphic to (P, θ, E) has the property $\mathcal{P}$.

Theorem 16. Soft compactness and soft connectedness are soft topological properties.

Definition 14. For every soft topological polygroups (P_1, θ_1, E) and (P_2, θ_2, E), the set $\{\mathbb{F}_1 \times \mathbb{F}_2 | \mathbb{F}_1 \in \theta_1, \mathbb{F}_2 \in \theta_2\}$ generates a soft topology $\theta^\times$ over $P_1 \times P_2$. The soft space $(P_1 \times P_2, \theta^\times, E)$ said to be the soft product of (P_1, θ_1, E) and (P_2, θ_2, E), where $\mathbb{F}_1 \times \mathbb{F}_2$ is the soft set on $P_1 \times P_2$ defined by $(\mathbb{F}_1 \times \mathbb{F}_2)(e) := \mathbb{F}_1(e) \times \mathbb{F}_2(e)$ for every $e \in E$.

Theorem 17. The soft product of every two soft $\mathcal{T}_0 spaces$ is a soft $\mathcal{T}_0 space$.

Proof. Suppose that (P, θ, E) and (P', θ', E) are soft $\mathcal{T}_0 spaces$. Take distinct points $(x, x'), (y, y') \in P \times P'$. Without loss of generality, suppose that $x \neq y$. Since (P, θ, E) is a soft $\mathcal{T}_0 space$, there exists a soft open set $\mathbb{F}$ with the property that either $x\widehat{\in}\mathbb{F} \wedge \forall e \in E(y\widehat{\notin}\mathbb{F}(e))$ or $y\widehat{\in}\mathbb{F} \wedge \forall e \in E(x\widehat{\notin}\mathbb{F}(e))$ holds. Thus we have:

$$(x, x')\widehat{\in}\mathbb{F} \times \widehat{P} \wedge \forall e \in E((y, y') \notin \mathbb{F}(e) \times P)$$

or

$$(y,y')\widehat{\in}\mathbb{F}\times\widehat{P}\wedge\forall e\in E((x,x')\notin\mathbb{F}(e)\times P).$$

Theorem 18. The soft product of every two soft $\mathcal{T}_1 spaces$ is a soft $\mathcal{T}_1 space$.

Proof. It is clear.

Theorem 19. The soft product of every two soft Hausdorff spaces is a soft Hausdorff space.

Proof. It is clear.

Theorem 20. Let h be an element of a polygroup P and (P,θ,E) be soft topological polygroup. Then:

(i) $\varphi_L(h):P\longmapsto\{h\circ x\}_{x\in P};x\longmapsto h\circ x(\ \varphi_R(h):P\longmapsto\{h\circ x\}_{x\in P};x\longmapsto x\circ h)$ is a soft homeomorphism.

(ii) $\varphi(h):P\longmapsto\{h\circ x\circ h^{-1}\}_{x\in P};x\longmapsto h\circ x\circ h^{-1}$ is a soft homeomorphism.

Proof. For every $h,x\in P$ and a soft neighborhood $\mathbb{F}$ of $h\circ x$, by the definition of soft topological polygroup, there exist soft neighborhood $\mathbb{F}_h$ and $\mathbb{F}_x$ of h and x with the property that $\mathbb{F}_h\circ\mathbb{F}_x\widehat{\subseteq}\mathbb{F}$. Thus, we have $\varphi_L(h)(\mathbb{F}_x)=(h\circ\mathbb{F}_x)\widehat{\subseteq}\mathbb{F}_h\circ\mathbb{F}_x\widehat{\subseteq}\mathbb{F}$, in conclusion $\varphi_L(h)$ is soft continuous. Since $\varphi_L(h)$ is soft continuous for each $h\in P$, for both h and h^{-1}, the first case follows at once by $(\varphi_L(h))^{-1}=\varphi_L(h^{-1})$. The second case can be proved similarly.

Theorem 21. For every soft topological polygroup (P,θ,E), the following items are equivalent:

(i) (P,θ,E) is a soft $\mathcal{T}_0 space$,

(ii) (P,θ,E) is a soft $\mathcal{T}_1 space$,

(iii) (P,θ,E) is a soft Hausdorff space.

Proof. $(i)\Longrightarrow(ii)$ We prove that $\widehat{\{e\}}$ is soft closed. For this, note that every $x\neq e$ can be separated from e by a soft open set. Take an $x\in p\setminus\{e\}$ arbitrarily. By the item (i), there exists a soft open set $\mathbb{F}$ with the property that either $x\widehat{\in}\mathbb{F}\wedge\forall e\in E(e\notin\mathbb{F}(e))$ or $e\widehat{\in}\mathbb{F}\wedge\forall e\in E(x\notin\mathbb{F}(e))$ holds. If the first happens, it is done. In the second property, the soft continuity of $\varphi_L(x)$ and the inversion $^{-1}:p\longmapsto P$ guarantees the existence of a soft set $\mathbb{F}'$ satisfying that $x\widehat{\in}\mathbb{F}'$ and $x\circ(\mathbb{F}')^{-1}\widehat{\subseteq}\mathbb{F}$. Thus, we have $\mathbb{F}'\widehat{\subseteq}\mathbb{F}^{-1}\circ x$. If e were in $\mathbb{F}'(e)$ for some $e\in E$, then we would have $e\in u^{-1}\circ x$ for some $u\in\mathbb{F}(e)$. Thus x is equal to $u(\in\mathbb{F}(e))$, contradicting the assumption that $\forall e\in E(x\notin\mathbb{F}(e))$. Therefore, e is not in $\mathbb{F}'(e)$ for any $e\in E$, in conclusion $\widehat{\{e\}}\widehat{\cap}\mathbb{F}'=\widehat{\varnothing}$ holds for this soft neighborhood $\mathbb{F}'$ of x. Take every distinct x,y from P. Since $x^{-1}\circ y$ is a soft subset of a soft open set $\{e\}^{\widehat{c}}$, the soft continuity of $\varphi_L(x^{-1})$ implies the existence of a soft open set $\mathbb{F}$ with the property that $y\widehat{\in}\mathbb{F}$ and $x^{-1}\circ\mathbb{F}\widehat{\subseteq}\{e\}^{\widehat{c}}$. In conclusion, this soft open set $\mathbb{F}$ satisfies $\forall e\in E(x\notin\mathbb{F}(e))$.

$(ii)\Longrightarrow(i)$ it is clear.

$(ii)\Longrightarrow(iii)$ Take $x\neq y$ from P. Since $e\neq x^{-1}\circ y$, item (ii) implies that $\widehat{\{x^{-1}\circ y\}}^{\widehat{c}}$ is soft open. Take a soft neighborhood $\mathbb{F}$ of e with the property that $\mathbb{F}\circ\mathbb{F}^{-1}\widehat{\subseteq}\widehat{\{x^{-1}\circ y\}}^{\widehat{c}}$. Suppose that for the contradiction, for some $e\in E$, the soft sets $x\circ\mathbb{F}(e)$ and $y\circ\mathbb{F}(e)$ had a common element, say g. Take $g\in x\circ h, g\in y\circ k$ for $h,k\in\mathbb{F}(e)$. However, then we would have

$g^{-1}\in h^{-1}\circ x^{-1}, g\in y\circ k$ Thus $e\in h^{-1}\circ x^{-1}\circ y\circ k$ then $h\in x^{-1}\circ y\circ k$ hence $h\circ k^{-1}\subseteq x^{-1}\circ y$ and $h\circ k^{-1}\subseteq\mathbb{F}(e)\circ\mathbb{F}(e)^{-1}$ and $\mathbb{F}(e)\circ\mathbb{F}(e)^{-1}\subseteq\{x^{-1}\circ y\}^{\widehat{c}}$ then $(h\circ k^{-1})\subseteq(x^{-1}\circ y)\cap\{x^{-1}\circ y\}^{\widehat{c}}$. This is a contradiction. In conclusion $x\circ\mathbb{F}\widehat{\cap}y\circ\mathbb{F}\widehat{=}\widehat{\varphi}$. The soft continuity of $\varphi_L(x^{-1})(resp.\varphi_L(y^{-1}))$, presents a soft open $\mathbb{F}_x(resp.\mathbb{F}_y)$ with the property that $x\widehat{\in}\mathbb{F}_x\widehat{\subseteq}x\circ\mathbb{F}(resp. y\widehat{\in}\mathbb{F}_y\widehat{\subseteq}y\circ\mathbb{F}$. Obviously, $\mathbb{F}_x$ and $\mathbb{F}_y$ are soft disjoint, as

$$\mathbb{F}_x\widehat{\cap}\mathbb{F}_y\widehat{\subseteq}x\circ\mathbb{F}\widehat{\cap}y\circ\mathbb{F}\widehat{=}\widehat{\varphi}.$$

$(iii)\Longrightarrow(ii)$ it is clear.

Definition 15. The soft connected component of x, is the largest soft connected subset of P containing x, for every $x \in P$.

Definition 16. The soft connected component of P is the soft connected component of $e \in P$.

Theorem 22. Let N_e be the soft connected component of P. Then, for each $q \in P$ the connected component of q is $q \circ N_e$.

Proof. If N_q be The soft connected component of q by Theorems 14 and 20 $q \circ N_e \subseteq N_q$ since $q \circ N_e$ is a soft connection containing of q. Notably $N_e \circ q \subseteq N_q$ Nonetheless, $N_e \subseteq N_q \circ q^{-1} \subseteq N_e$ then $N_e = N_q \circ q^{-1}$ in conclusion $N_e \circ q = N_q$.

Theorem 23. If N is a soft connected component of P, then N is normal subpolygroup of P.

Proof. Suppose that $a, b \in N$ Since $^{-1} : P \longmapsto P$ is soft homeomorphism and
$\varphi_L(h) : P \longmapsto \{h \circ x\}_{x \in P}; x \longmapsto h \circ x (resp. \varphi_R(h) : P \longmapsto \{x \circ h\}_{x \in P}; x \longmapsto x \circ h)$ are soft continuous. By Theorems 14 and 15 $a \circ N^{-1}$ is soft connected. Since $a \circ N^{-1}$ contains $e \in a \circ a^{-1}$, we have $a \circ N^{-1} \subseteq N$. Obviously, $a \circ b^{-1} \subseteq a \circ N^{-1}$, thus have $a \circ b^{-1} \subseteq a \circ N^{-1} \subseteq N$. This proves that N is a subpolygroup of P. Note that both $a^{-1} \circ N \circ a$ and $a \circ N \circ a^{-1}$ are soft connected, and contain e. Above all N is the largest soft connected subset containing e, we have $a^{-1} \circ N \circ a$ and $a \circ N \circ a^{-1}$ are a subset of N, in conclusion, N is normal subpolygroup of P.

Theorem 24. Let H and K be soft connected subsets of a soft topological polygroup P. Then $H \circ K$ subset of P is soft connected.

Proof. Suppose that $\{\mathbb{F}_1, \mathbb{F}_2\}$ is a soft open covering of $H \circ K$ with the property that $\nexists g \in H \circ K$ satisfies both $g \widehat{\in} \mathbb{F}_1$ and $g \widehat{\in} \mathbb{F}_2$. Due to the Theorem 14, $h \circ K = (\varphi_L(h))(K)$ is soft connected for each $h \in H$. Note that $\{\mathbb{F}_1, \mathbb{F}_2\}$ is a soft covering of $h \circ K$ for all $h \in H$. Take an $h \in H$ arbitrarily. We suppose that $\forall g \in h \circ K (g \widehat{\notin} \mathbb{F}_1)$ without loss of generality. Assume for the contradiction that $\exists g' \in h' \circ K (g' \widehat{\in} \mathbb{F}_1)$ holds for some $h' \in H$. Select a g' from $h' \circ K$, and deposit $g' \in h' \circ k' (k' \in K)$. In conclusion, both $(\forall t \in h \circ k') t \widehat{\notin} \mathbb{F}_1$ and $h' \circ k' \ni g' \widehat{\in} \mathbb{F}_1$ are true, contradicting the soft connectedness of $H \circ k'$. Thus, $\forall g \in h \circ K (g \widehat{\notin} \mathbb{F}_1)$ holds for each $h \in H$. Notably, $\forall g \in H \circ K (g \widehat{\notin} \mathbb{F}_1)$. Therefore, $H \circ K$ is soft connected.

Theorem 25. If H is a subpolygroup of P with the property that $\widehat{H}$ is soft open, then $\widehat{H}$ is soft closed.

Proof. Suppose that $P = H \cup (\bigcup_{\alpha \in \Omega} H \circ g_\alpha)$ is a right coset decomposition. First, prove that $\widehat{H \circ g_\alpha}$ is soft open for all $\alpha \in \Omega$. For all $h \in H$, from the soft continuity of $\varphi_R(g_\alpha^{-1})$, it can be select a soft neighborhood $\mathbb{F}_{h \circ g_\alpha}$ of $h \circ g_\alpha$ with the property that $\mathbb{F}_{h \circ g_\alpha} \circ g_\alpha^{-1} \widehat{\subseteq} \widehat{H}$. Above all for every $h \in H$, we have $h \widehat{\in} \widehat{\bigcup}_{h \in H} \mathbb{F}_{h \circ g_\alpha} \circ g_\alpha^{-1} \widehat{\subseteq} \widehat{H}$. Therefore $\widehat{H} \widehat{=} \widehat{\bigcup}_{h \in H} \mathbb{F}_{h \circ g_\alpha} \circ g_\alpha^{-1}$, and $\widehat{H \circ g_\alpha}$ is soft equal to $\widehat{\bigcup}_{h \in H} \mathbb{F}_{h \circ g_\alpha}$. As a soft union of soft open sets $\mathbb{F}_{h \circ g_\alpha}$, $\widehat{H \circ g_\alpha}$ is also soft open.

In summary, $\widehat{\bigcup}_{\alpha \in \Omega} \widehat{H \circ g_\alpha}$ is soft open as it is the soft union of soft open sets. Therefore $\widehat{H} = \widehat{G} \setminus \widehat{\bigcup}_{\alpha \in \Omega} \widehat{H \circ g_\alpha}$ is soft closed.

Theorem 26. Let H be a subpolygroup of G. $\widehat{H}$ is soft open if and only if there exist an $h \in H$ and a soft neighborhood $\mathbb{F}$ of h with the property that $\mathbb{F} \widehat{\subseteq} \widehat{H}$.

Proof. $\Longrightarrow$: Select h and $\mathbb{F}$ as above. For every $h' \in H$, there exists a soft neighborhood $\mathbb{F}'_{h'}$ of h' with the property that $h \circ (h')^{-1} \circ \mathbb{F}'_{h'} \widehat{\subseteq} \mathbb{F}$ as $\varphi_L(h \circ (h')^{-1}) : P \longmapsto \{h \circ (h')^{-1} \circ x\}_{x \in P}$ is soft continuous. Since $\mathbb{F}'_{h'} \widehat{\subseteq} h' \circ h^{-1} \circ \mathbb{F}$ and H is a subpolygroup we have $\mathbb{F}'_{h'} \widehat{\subseteq} h' \circ h^{-1} \circ \mathbb{F} \widehat{\subseteq} \widehat{H}$. Thus $\widehat{H} \widehat{=} \widehat{\bigcup}_{h \in H} \mathbb{F}'_h$ is soft open.

$\Longleftarrow$: It is clear.

Conclusion and Future Work

This study presented two different definitions of the soft topological polygroup. The authors provided attributes for each definition along with examples. The connection between the complete parts and the concepts such as soft continuous function, soft Hausdorff space, soft $\mathcal{T}_0$ space, soft $\mathcal{T}_1$ space, soft open covering, soft compact, soft connected in soft topological polygroups was examined. Lastly, necessary arrangements were made.

References

1 Oguz, G. (2020). A new view on topological polygroups. *Turk. J. Sci. 5*(2), 110–117.

2 Heidari, D., Davvaz, B., & Modarres, S. M. S. (2016). Topological polygroups. *Bull. Malays. Math. Sci. Soc. 39*, 707–721. https://doi.org/10.1007/s40840-015-0136-y

3 Cagman, N., Karatas, S., & Enginoglu, S. (2011). Soft topology. *Comput. Math. Appl. 62*(1), 351–358. https://doi.org/10.1016/j.camwa.2011.05.016

4 Wang, J., Yina, M., & Gu, W. (2011). Soft Polygroups. *Comput. Math. Appl. 62*, 3529–3537. https://doi.org/10.1016/j.camwa.2011.08.069

5 Shah, T., & Shaheen, S. (2014). Soft topological groups and rings. *Ann. fuzzy math. Inform. 7*(5), 725-–743.

6 Davvaz, B. (2013). *Polygroup theory and related systems.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ. https://doi.org/10.1142/8593

7 Maji, P. K., Biswas, R., & Roy, A. R. (2003). Soft set theory. *Comput. Math. Appl. 45*(4-5), 555-–562. https:/doi.org/10.1016/S0898-1221(03)00016-6

8 Mousarezaei, R., & Davvaz, B. (2021). On soft topological polygroups and their examples. *Int. J. Fuzzy Log. Intell. Syst. 21*(1), 29–37. https:/doi.org/10.5391/IJFIS.2021.21.1.29

9 Nazmul, S., & Samanta, S. K. (2013). Neighbourhood properties of soft topological spaces. *Ann. Fuzzy Math. Inform. 6*(1), 1–15.

10 Hida, T. (2014). Soft topological group. *Ann. Fuzzy Math. Inform. 8*(6), 1001–1025.

11 Corsini, P. (1993). *Prolegomena of hypergroup theory.* Aviani Editore, Tricesimo.

12 Marty, F. (1934). *Sur une généralization de la notion de groupe.* 8^{iem}, Congress Math. Scandinaves, Stockholm, 45–49.

13 Koskas, M. (1970). Groupoides, demi-hypergroupes et hypergroupes. *J. Math. Pures Appl. 49*(9), 155–192.

14 Shabir, M., & Naz, M. (2011). On soft topological spaces. *Comput. Math. Appl. 61*(7), 1786–1799. https:/doi.org/10.1016/j.camwa.2011.02.006

Р. Мусарезаи, Б.Давваз

Математика ғылымдары кафедрасы, Йезд университеті, Йезд, Иран

Жұмсақ топологиялық полигруппаларға тағы бір көзқарас

Жұмсақ топологиялық полигруппалар екі түрлі жолмен анықталады. Бірінші анықтамада тұрақты топология, ал екінші анықтамада жұмсақ топология бар. Екінші анықтамада жұмсақ маңай, жұмсақ үзіліссіздік, жұмсақ компакт, жұмсақ байланыс, жұмсақ хаусдорф кеңістігі сияқты ұғымдар пайда болады және олардың жұмсақ топологиялық полигруппалардағы жұмсақ үздіксіз функциялармен байланысы зерттеледі. Кәдімгі топологияда үздіксіздіктің бес баламалы анықтамасы бар, бірақ олардың барлығы міндетті түрде жұмсақ үзіліссіздікте анықталмаған.

Кілт сөздер: жұмсақ жиын, жұмсақ үзіліссіздік, жұмсақ топологиялық полигруппалар, жұмсақ хаусдорф кеңістігі, жұмсақ ашық жабын, жұмсақ компакт, жұмсақ байланыс.

Р. Мусарезаи, Б. Давваз

Йездский университет, Йезд, Иран

Другой взгляд на мягкие топологические полигруппы

Мягкие топологические полигруппы определяются двумя разными способами. Первое определение имеет обычную топологию, а второе — мягкую топологию. Во втором определении появляются такие понятия, как мягкая окрестность, мягкая непрерывность, мягкий компакт, мягкая связность, мягкое хаусдорфово пространство, и изучается их связи с мягкими непрерывными функциями в мягких топологических полигруппах. В обычной топологии есть пять эквивалентных определений непрерывности, но не все они обязательно установлены в мягкой непрерывности.

Ключевые слова: мягкое множество, мягкая непрерывность, мягкие топологические полигруппы, мягкое хаусдорфово пространство, мягкое открытое покрытие, мягкий компакт, мягкая связность.

DOI 10.31489/2022M3/98-104
UDC 517.9

M.B. Muratbekov*, Ye.N. Bayandiyev

M.Kh. Dulaty Taraz Regional University, Taraz, Kazakhstan
(E-mail: musahan_ m@mail.ru, erikn87@mail.ru)

Existence and smoothness of solutions of a singular differential equation of hyperbolic type

This paper investigates the question of the existence of solutions to the semiperiodic Dirichlet problem for a class of singular differential equations of hyperbolic type. The problem of smoothness of solutions is also considered, i.e., maximum regularity of solutions. Such a problem will be interesting when the coefficients are strongly growing functions at infinity. For the first time, a weighted coercive estimate was obtained for solutions to a differential equation of hyperbolic type with strongly growing coefficients.

Keywords: resolvent, hyperbolic type equation, maximal regularity, unbounded domain.

1 Introduction. Formulation of results

Considered on the strip

$$\overline{\Omega} = \{(x, y) : -\pi < x < \pi, -\infty < y < \infty\}$$

next problem:

$$(L + \lambda I)u = u_{xx} - u_{yy} + a(y)u_x + c(y)u + \lambda u = f(x, y) \in L_2(\Omega), \tag{1}$$

$$u(-\pi, y) = u(\pi, y), u_x(-\pi, y) = u_x(\pi, y), -\infty < y < \infty. \tag{2}$$

Further, we assume that the coefficients $a(y)$, $c(y)$ satisfy the conditions:

i) $|a(y)| \geq \delta_0 > 0$, $c(y) \geq \delta > 0$ are continuous functions in $R(-\infty, \infty)$;

ii) $\mu_0 = \sup\limits_{|y-t|\leq 1} \frac{a(y)}{a(t)} < \infty$, $\mu = \sup\limits_{|y-t|\leq 1} \frac{c(y)}{c(t)} < \infty$;

iii) $c(y) \leq c_0 \cdot c^2(y)\ \ for\ \ all\ \ y \in\ R, c_0 > 0$ is a constant number.

Here it has to be noted that $a(y)$ and $c(y)$ can be unlimited functions at infinity.

The existence and uniqueness, as well as the qualitative behavior, of solutions for differential equations of hyperbolic type, were studied in [1–14]. In these works, Darboux and Goursat problems and the Cauchy problem, periodic and some boundary value problems for differential equations of hyperbolic type with constant or variable bounded coefficients were examined.

In this paper, in the space $L_2(\Omega)$, we study questions about the existence, uniqueness of solutions, and also the smoothness of solutions to a periodic problem without initial conditions [13] for a differential equation of hyperbolic type with strongly increasing coefficients at infinity.

In our previous paper [14], we studied a differential operator of hyperbolic type in the space $L_2(R^2)$.

In contrast to [14], in this paper, on a strip, we consider the so-called periodic problem without initial conditions. Here we note that in the future, this work will allow us to study questions about the compactness of the resolvent, about estimates for the singular (s-numbers) and eigenvalues of a differential operator of hyperbolic type corresponding to problem (1)–(2).

*Corresponding author.
E-mail: musahan_ m@mail.ru

Definition 1. We say that function $u(x,y) \in L_2(\Omega)$ is a strong solution to problem (1)–(2), if there is a sequence $\{u_n\} \subset C^{\infty}_{0,\pi}(\Omega)$ such that

$$\|u_n - u\|_{L_2(\Omega)} \to 0, \ \ \|(L+\lambda I)u_n - f\|_{L_2(\Omega)} \to 0 \ \ for \ \ n \to \infty,$$

where $C^{\infty}_{0,\pi}(\Omega)$ is the set consisting of infinitely differentiable finite functions with respect to variable y and satisfying conditions (2) with respect to variable x.

Theorem 1. Let the condition $i)$ be fulfilled. Then for $\lambda \geq 0$ for any $f(x,y) \in L_2(\Omega)$ there is a unique strong solution to the problem (1)–(2), and the equality is true

$$u(x,y) = (L+\lambda I)^{-1} f = \sum_{n=-\infty}^{\infty} (l_n + \lambda I)^{-1} f_n(y) \cdot e^{inx},$$

where $f(x,y) \in L_2(\Omega),\ f(x,y) = \sum\limits_{n=-\infty}^{\infty} f_n(y) \cdot e^{inx},\ f_n(y) =< f(x,y), e^{inx} >$ are fourier coefficients, $i^2 = -1$, $< \cdot, \cdot >$ is the scalar product in $L_2(\Omega)$,

$$(l_n + \lambda I)u = -u^{''}(y) + (-n^2 + ina(y) + c(y) + \lambda)u(y), u \in D(l_n).$$

Theorem 2. Let the condition $i)$ be fulfilled. Then for $\lambda \geq 0$ for any $f(x,y) \in L_2(\Omega)$ there is a unique strong solution to the problem (1)–(2), and the equality is true

$$\|u_{xx} - u_{yy}\|_2 + \|u_y\|_2 + \|a(y)u_x\|_2 + \|c(y)u\|_2 \leq c \cdot \|f\|_2,$$

where $c > 0$ is constant number.

2 Proof of theorems 1-2

Using the Fourier method, we reduce problem (1)–(2) to the study of the following differential operator with negative discrete parameter n $(n = 0, \pm 1, \pm 2, ...)$:

$$(l_n + \lambda I)u = -u^{''}(y) + (-n^2 + ina(y) + c(y) + \lambda)u(y), u \in D(l_n),$$

where $D(l_n)$ is the domain of the operator l_n.

Consider two cases:

I. Let be $(n = 0)$. In this case, the operator l_0 is the Sturm–Liouville operator.

This operator has been studied thoroughly in [15–21].

II. Let be $n \neq 0$. In this case, it is easy to see that the first term in the coefficient $(-n^2 + ina(y) + c(y) + \lambda)$ tends to $-\infty$, i.e. $-n^2 \to -\infty$.

In this case, the l_n operator is not a semi-bounded operator. Consequently, the methods that have been worked out for the Sturm–Liouville operator $L = -\frac{d^2}{dx^2} + q(x)u$ turn out to be poorly adapted to the study of the Sturm–Liouville operator with a negative parameter.

Let us take $\{\varphi_j\}$ the set of non-negative functions from $C^{\infty}_0(R)$ such that

$$\sum_j \varphi_j^2 \equiv 1, \sup\ p\ \varphi_j \subset \triangle_j,\ \bigcup_j \triangle_j = R,$$

where $\triangle_j = (j-1, j+1)$, $j = 0, \pm 1, \pm 2, ...$ the multiplicity of the intersection of which is not higher than three. The existence of such a covering follows from the results of [22].

Continue $a(y)$, $c(y)$ from $\triangle_j$ for all R . The resulting functions will be denoted by $a_j(y)$ and $c_j(y)$.

These functions are bounded and periodic functions. Denote by $(l_{n,j,\alpha}+\lambda I)$ the closure of operator

$$(l_{n,j,\alpha}+\lambda I)u=-u^{''}(y)+(-n^2+in(a_j(y)+\alpha)+c_j(y)+\lambda)u$$

defined on $C_0^\infty(R)$. We introduced the real number α to evaluate the norm of the operator $D_y(l_{n,j,\alpha}+\lambda I)^{-1}$ in the space $L_2(\Omega)$, where $D_y=\frac{\partial}{\partial y}$. The sign of the number α is chosen as follows: $\alpha\cdot b(y)>0$ at $y\in R$.

In the course of the proof using Lemma 3, we will get rid of this number.

Lemma 1. Suppose that the coefficients of the operator $l_{n,j,\alpha}+\lambda I$ satisfy condition i). Then for $\lambda\geq 0$:

1) for the differential operator $l_{n,j,\alpha}+\lambda I$ at $\lambda\geq 0$, there is a bounded inverse operator $(l_{n,j,\alpha}+\lambda I)^{-1}$ defined at all $L_2(R)$.

2) the resolvent of the operator $l_{n,j,\alpha}+\lambda I$ satisfies the following estimates:

a) $\|(l_{n,j,\alpha}+\lambda I)^{-1}\|_{2\to 2}\leq\frac{c}{(\delta+\lambda)^{\frac{1}{2}}}$, $c>0$ — constant number independent of n,j,α;

b) $\|\frac{d}{dy}(l_{n,j,\alpha}+\lambda I)^{-1}\|_{2\to 2}\leq\frac{c}{(\delta+\lambda)^{\frac{1}{4}}}$, $c>0$ — constant number independent of n,j,α;

c) $\|(l_{n,j,\alpha}+\lambda I)^{-1}\|_{2\to 2}\leq\frac{1}{|n|\cdot|a(\tilde{y}_j)|}$, $n\neq 0$, $c>0$ — constant number independent of n,j,α;

d) $\|(l_{n,j,\alpha}+\lambda I)^{-1}\|_{2\to 2}\leq\frac{2\cdot c}{c(\overline{y}_j)+\lambda}$, $c>0$ — constant number independent of n,j,α;

where $\|\cdot\|_{2\to 2}$ — is the norm of the operator $(l_{n,j,\alpha}+\lambda I)^{-1}$ in space $L_2(R)$, $|a(\tilde{y}_j)|=\min\limits_{y\in\overline{\triangle}_j}|a(y)|$, $|c(\overline{y}_j)|=\min\limits_{y\in\overline{\triangle}_j}|c(y)|$.

Lemma 1 is proved using functionals $<(l_{n,j,\alpha}+\lambda I)u,u>$, $<(l_{n,j,\alpha}+\lambda I)u,-inu>$ $(n=0,\pm 1,\pm 2,...)$ and repeating the calculations and arguments for these functionals, which were used in the proof of Lemma 2.1 [22] and Lemmas 4–6 [23].

Now, consider the differential operator

$$(l_{n,\alpha}+\lambda I)u=-u^{''}+(-n^2+in(a(y)+\alpha)+c(y)+\lambda)\cdot u,$$

which is a closure in $L_2(R)$ of the following operator originally defined on $C_0^\infty(R)$:

$$(l_{n,\alpha}+\lambda I)u=-u^{''}+(-n^2+in(a(y)+\alpha)+c(y)+\lambda)\cdot u.$$

We introduce the operator

$$K_{\lambda,\alpha}f=\sum_{\{j\}}\varphi_j(l_{n,j,\alpha}+\lambda I)^{-1}\varphi_j f,\, f\in L_2(R).$$

The following lemma is proved with the help of calculations and arguments that were used in the proof of Theorem 1.1–1.3 in [22] and Theorem 1 in [23].

Lemma 2. Suppose that the coefficients of the operator $l_{n,j,\alpha}+\lambda I$ satisfy condition i). Then there is a number $\lambda_0>0$ such that for the operator $l_{n,j,\alpha}+\lambda I$ for $\lambda\geq\lambda_0$ there is a resolvent and the equality

$$(l_{n,j,\alpha}+\lambda I)^{-1}f=K_{\lambda,\alpha}(I-M_{\lambda,\alpha})^{-1}f,\, f\in L_2(R)$$

holds, where $M_{\lambda,\alpha}f=\sum\limits_{\{j\}}\varphi_j^{''}(l_{n,j,\alpha}+\lambda I)^{-1}\varphi_j f+2\sum\limits_{\{j\}}\varphi_j^{'}\frac{d}{dy}(l_{n,j,\alpha}+\lambda I)^{-1}\varphi_j f$, $f\in L_2(R)$.

Lemma 3. Suppose that the coefficients of the operator $l_{n,j,\alpha}+\lambda I$ satisfy condition i). Then there is a number $\lambda_0>0$ such that for the operator $l_n+\lambda I$ $(n=0,\pm 1,\pm 2,...)$ for $\lambda\geq\lambda_0$ there is a resolvent and the equality

$$(l_n+\lambda I)^{-1}f=(l_{n,\alpha}+\lambda I)^{-1}(I-M_{\lambda,\alpha})^{-1}f,\, f\in L_2(R)$$

holds, where $M_{\lambda,\alpha} = in\alpha(l_{n,\alpha}+\lambda I)^{-1}$ $(n = 0, \pm 1, \pm 2, ...)$ and the operator's norm $M_{\lambda,\alpha}$: $\|M_{\lambda,\alpha}\|_{2\to 2} < 1$.

Using the method used in the proof of Lemma 9 in [23], we obtain the proof of Lemma 3.

Proof of Theorem 1. Using the scalar product $< (L + \lambda I)u, u >$ for all $u \in D(L)$ and taking into account the condition i), we obtain that

$$\|(L + \lambda I)u\|_2 \geq c \cdot \|u\|_2,$$

where $c(\delta) > 0$ is a constant number. Further, repeating the calculations and arguments used in the proof of Theorem 1 in [23], we obtain the proof of Theorem 1.

Proof of Theorem 2. Taking into account conditions ii)–iii), and also using the method used in the proof of Theorems in [24–26], we obtain the proof of Theorem 2.

Acknowledgements

This research was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08856281).

References

1 Адамар Ж. Задача Коши для линейных уравнений с частными производными гиперболического типа / Ж. Адамар. — М.: Наука, 1978. — 352 с.

2 Friedrichs K. Symmetric hyperbolic linear differential equations / K. Friedrichs // Comm. Pure and Appl. Math. — 1954. — 7. — P. 345–392.

3 Соболев С.Л. Избранные труды / С.Л. Соболев. — Новосибирск: Изд-во Института математики, 2006. — Т. 2.

4 Garding L. Linear hyperbolic partial differential equations with constant coefficients /L. Garding // Acta Math. — 1951. — 85. — P. 1–62.

5 Ладыженская О.А. О разрешимости основных краевых задач для уравнений гиперболического и параболического типов / О.А. Ладыженская // ДАН СССР. — 1954. — 97. — № 3. — С. 395–398.

6 Бицадзе А.В. Некоторые классы уравнений в частных производных / А.В. Бицадзе. — М.: Наука, 1981.

7 Лере Ж. Гиперболические дифференциальные уравнения / Ж. Лере. — М.: Наука, 1984.

8 Нахушев А.М. Задачи со смещением для уравнений в частных производных / А.М. Нахушев. — М.: Наука, 2006.

9 Кальменов Т.Ш. К теории начально-краевых задач для дифференциальных уравнений / Т.Ш. Кальменов. — Алматы, 2013.

10 Кигурадзе Т.И. О периодических краевых задачах для линейных гиперболических уравнений / Т.И. Кигурадзе // Дифф. уравн. — 1993. — 29. – № 2. — С. 281–297.

11 Асанова А.Т. О корректной разрешимости нелокальной краевой задачи для систем гиперболических уравнений / А.Т. Асанова, Д.С. Джумабаев // Докл. РАН. — 2003. — 391. — № 3. — С. 295–297.

12 Filinovskii A.V. Hyperbolic equations with growing coefficients in unbounded domains / A.V. Filinovskii // J. Math. Sci. (N. Y.). — 2014. — 197. — № 3. — P. 435–446.

13 Tikhonov A.N. Equations of mathematical physics / A.N. Tikhonov, A.A. Samarskii. — New York: Dover Publications, Inc., 1990.

14 Muratbekov M.B. Existence and maximal regularity of solutions in $L_2(R^2)$ for a hyperbolic type differential equation with quickly growing coefficients / M.B. Muratbekov, Ye.N. Bayandiyev // Eurasian Mathematical Journal. — 2020. — 11. — № 1. — P. 95–100.

15 Като Т. Теория возмущений линейных операторов / Т. Като. — М.: Мир, 1972. — 740 с.

16 Рид М. Методы современной математической физики. Т. 2. Гармонический анализ. Самосопряженность / М. Рид, Б. Саймон. — М.: Мир, 1978. — 430 с.

17 Титчмарш Э.Ч. Разложения по собственным функциям, связанные с дифференциальными уравнениями второго порядка — Т. 1,2. / Э.Ч. Титчмарш. — М.: ИЛ, 1961. — 278 с.

18 Мазья В.Г. О (p,l)-емкости, теоремах вложения и спектре самосопряженного эллиптического оператора / В.Г. Мазья // Изв. АН СССР. Сер мат. — 1973. — Т. 37. — С. 356–385.

19 Отелбаев М. Теоремы вложения пространств с весом и их применения к изучению спектра оператора Шредингера / М. Отелбаев // Тр. Мат. ин-та им. В.А. Стеклова АН СССР. — 1979. — 150. — С. 265–305.

20 Lorenzi L. On Schrodinger type operators with unbounded coefficients: Generation and heat kernel estimates / L. Lorenzi, A. Rhandi // J. Evol. Equ. — 2015. — 15. — № 1. — P. 53–88.

21 Metafune G. Uniqueness for elliptic operators on with unbounded coefficients / G. Metafune, D. Pallara, P.J. Rabier, R. Schnaubelt // Forum Math. — 2010. — 22. — P. 583–601.

22 Muratbekov M. On the existence of a resolvent and separability for a class of singular hyperbolic type differential operators on an unbounded domain / M. Muratbekov, M. Otelbaev // Eurasian Mathematical Journal. — 2016. — 7. — No. 1. — P. 50–67.

23 Muratbekov M.B. On the compactness of the resolvent of a Schrodinger type singular operator with a negative parameter / M.B. Muratbekov, M.M. Muratbekov // Chaos, Solitons and Fractals. — 2021. — 151. — P. 111–248.

24 Muratbekov M.B. Estimates of Eigenvalues of a Semiperiodic Dirichlet Problem for a Class of Degenerate Elliptic Equations / M.B. Muratbekov, S. Igissinov // Symmetry. — 2022. — 14. — P. 692.

25 Muratbekov M.B. On the bounded invertibility of a Schrodinger operator with a negative parameter in the space $L_2(R^n)$ / M.B. Muratbekov, M.M. Muratbekov // Bulletin of the Karaganda University-Mathematics. — 2019. — 93. — № 1. — P. 36–47.

26 Muratbekov M.B. On the existence of the resolvent and separability of a class of the Korteweg-de Vriese type linear singular operators / M.B. Muratbekov, A.O. Suleimbekova // Bulletin of the Karaganda University-Mathematics. — 2021. — 101. — № 1. — P. 87–97.

М.Б. Муратбеков, Е.Н. Баяндиев

М.Х. Дулати атындағы Тараз өңірлік университеті, Тараз, Қазақстан

Гиперболалық типтегі сингулярлық дифференциалдық теңдеудің шешімдерінің бар болуы және тегістігі

Мақалада гиперболалық типті сингулярлық дифференциалдық теңдеулер класы үшін жартылай периодтық Дирихле есебінің шешімдерінің бар екендігі туралы мәселе зерттелген. Сонымен қатар шешімдердің тегістігі туралы мәселе, яғни шешімдердің максималды регулярлығы қарастырылған. Коэффициенттері шексіздікте жылдам өсетін функциялар болғанда мұндай есеп қызықты болады. Осы жұмыста бірінші рет коэффициенттері жылдам өсетін гиперболалық типті дифференциалдық теңдеудің шешімдері үшін салмақты коэрцитивті бағалаулар алынған.

Кілт сөздер: резольвента, гиперболалық типтес теңдеу, максималды регулярлық, шексіз облыс.

М.Б. Муратбеков, Е.Н. Баяндиев

Таразский региональный университет имени М.Х. Дулати, Тараз, Казахстан

Существование и гладкость решений сингулярного дифференциального уравнения гиперболического типа

В статье изучен вопрос о существовании решений полупериодической задачи Дирихле для одного класса сингулярных дифференциальных уравнений гиперболического типа. Также рассмотрена задача о гладкости решений, т.е. максимальная регулярность решений. Данная задача будет интересной, когда коэффициенты являются сильно растущими функциями на бесконечности. По-видимому, в настоящей работе впервые получена весовая коэрцитивная оценка решений дифференциального уравнения гиперболического типа с сильно растущими коэффициентами.

Ключевые слова: резольвента, уравнение гиперболического типа, максимальная регулярность, бесконечная область.

References

1 Adamar, Zh. (1978). *Zadacha Koshi dlia lineinykh uravnenii s chastnymi proizvodnymi giperbolicheskogo tipa [The Cauchy problem for hyperbolic linear partial differential equations]*. Moscow: Nauka [in Russian].

2 Friedrichs, K.O. (1954). Symmetric hyperbolic linear differential equations. *Communications on Pure and Applied Mathematics, 7,* 345–392.

3 Sobolev, S.L. (2006). *Izbrannye trudy [Selected works]*. Novosibirsk: Izdatelstvo Instituta matematiki [in Russian].

4 Gårding, L. (1951). Linear hyperbolic partial differential equations with constant coefficients. *Acta Mathematica, 85*, 1–62.

5 Ladyzhenskaia, O.A. (1951). O razreshimosti osnovnykh kraevykh zadach dlia uravnenii giperbolicheskogo i parabolicheskogo tipov [On the solvability of the basic boundary value problems for the equation of hyperbolic and parabolic types]. *Doklady Akademii Nauk SSSR — Reports of the Academy of Sciences of the USSR, 97*(3), 395–398 [in Russian].

6 Bitsadze, A.V. (1981). *Nekotorye klassy uravnenii v chastnykh proizvodnykh [Some classes of partial differential equations]*. Moscow: Nauka [in Russian].

7 Leray, J. (1984). *Giperbolicheskie differentsialnye uravneniia [Hyperbolic differential equations]*. Moscow: Nauka [in Russian].

8 Nakhushev, A.M. (2006). *Zadachi so smeshcheniem dlia uravnenii v chastnykh proizvodnykh [Offset problems for partial differential equations]*. Moscow: Nauka [in Russian].

9 Kalmenov, T.Sh. (2013). *K teorii nachalno-kraevykh zadach dlia differentsialnykh uravnenii [On the theory of initial boundary value problems for differential equations]*. Almaty [in Russian].

10 Kiguradze, T.I. (1993). O periodicheskikh kraevykh zadachakh dlia lineinykh giperbolicheskikh uravnenii [On periodic boundary value problems for linear hyperbolic equations]. *Differentsialnye uravneniia — Differential equations, 29*(2), 281–297 [in Russian].

11 Assanova, A.T., & Dzhumabaev, D.S. (2003). O korrektnoi razreshimosti nelokalnoi kraevoi zadachi dlia sistem giperbolicheskikh uravnenii [On the correct solvability of a nonlocal boundary value problem for systems of hyperbolic equations]. *Doklady RAN — RAS reports, 391*(3), 295–297 [in Russian].

12 Filinovskii, A.V. (2014). Hyperbolic Equations with Growing Coefficients in Unbounded Domains. *J. Math. Sci., 197*, 435–446.

13 Tikhonov, A.N., & Samarskii, A.A. (1990). *Equations of mathematical physics.* New York: Dover Publications, Inc.

14 Muratbekov, M.B., & Bayandiyev, Ye.N. (2020). Existence and maximal regularity of solutions in $L_2(R^2)$ for a hyperbolic type differential equation with quickly growing coefficients. *Eurasian Mathematical Journal, 11*(1), 95–100.

15 Kato, T. (1972). *Teoriia vozmushchenii lineinykh operatorov [Perturbation theory for linear operators].* Moscow: Mir [in Russian].

16 Reed, M., & Simon, B. (1978). *Metody sovremennoi matematicheskoi fiziki. Tom 2. Garmonicheskii analiz. Samosopriazhennost. [Methods of Modern Mathematical Physics II: Fourier Analysis. Self-Adjointness].* Moscow: Mir [in Russian].

17 Titchmarsh, E.Ch. (1961). *Razlozheniia po sobstvennym funktsiiam, sviazannye s differentsialnymi uravneniiami vtorogo poriadka [Eigenfunction expansions associated with second-order differential equations].* Moscow: Inostrannaia literatura [in Russian].

18 Mazia, V.G. (1973). O (p, l)-emkosti, teoremakh vlozheniia i spektre samosopriazhennogo ellipticheskogo operatora [On $O(p, l)$-capacity, inbedding theorems, and the spectrum of a selfadjoint elliptic operator]. *Izvestiia AN SSSR. Seriia matematika — News of Academy of Sciences of the USSR. Mathematics series, 37*, 356–385 [in Russian].

19 Otelbaev, M. (1979). Teoremy vlozheniia prostranstv s vesom i ikh primeneniia k izucheniiu spektra operatora Shredingera [Imdedding theorems for spaces with a weight and their application to the study of the spectrum of the Schrodinger operator]. *Part 7, Work collection, Trudy Matematicheskogo Instituta imeni V.A. Steklova — Proceedings of the Steklov Institute of Mathematics, 150*, 265–305 [in Russian].

20 Lorenzi, L., & Rhandi, A. (2012). On Schrödinger type operators with unbounded coefficients: generation and heat kernel estimates. *Journal of Evolution Equations, 15*, 53–88.

21 Metafune, G., Pallara, D., Rabier, P., & Schnaubelt, R. (2010). Uniqueness for elliptic operators on with unbounded coefficients. *Forum Math., 22*(3), 583–601.

22 Muratbekov, M., & Otelbaev, M. (2016). On the existence of a resolvent and separability for a class of singular hyperbolic type differential operators on an unbounded domain. *Eurasian Math. J., 7*, 1, 50–67.

23 Muratbekov, M.B., & Muratbekov, M.M. (2021). On the compactness of the resolvent of a Schrodinger type singular operator with a negative parameter. *Chaos, Solitons and Fractals, 151*, 111–248.

24 Muratbekov, M.B., & Igissinov, S. (2022). Estimates of Eigenvalues of a Semiperiodic Dirichlet Problem for a Class of Degenerate Elliptic Equations. *Symmetry, 14*, 692.

25 Muratbekov, M.B., & Muratbekov, M.M. (2019). On the bounded invertibility of a Schrodinger operator with a negative parameter in the space $L_2(R^n)$. *Bulletin of the Karaganda University-Mathematics, 93*, 1, 36–47.

26 Muratbekov, M.B., & Suleimbekova, A.O. (2021). On the existence of the resolvent and separability of a class of the Korteweg-de Vriese type linear singular operators. *Bulletin of the Karaganda University-Mathematics, 101*, 1, 87–97.

DOI 10.31489/2022M3/105-115

UDC 517.5

A. Myrzagaliyeva*

Astana IT University, Nur-Sultan, Kazakhstan
(E-mail: aigul.myrzagalieva@astanait.edu.kz)

Multipliers in weighted Sobolev spaces on the axis

This work establishes necessary and sufficient conditions for the boundedness of one variable differential operator acting from a weighted Sobolev space $W^l_{p,v}$ to a weighted Lebesgue space on the positive real half line. The coefficients of differential operators are often assumed to be pointwise multipliers of function spaces. The author introduces pointwise multipliers in weighted Sobolev spaces; obtains the description of the space of multipliers $M(W_1 \to W_2)$ for a pair of weighted Sobolev spaces (W_1, W_2) with weights of general type.

Keywords: Sobolev space, pointwise multiplier, weighted space, differential operator, admissible function, slow variation condition, Otelbaev function.

The results obtained in this paper can be regarded as a natural extension of certain results (in dimension one) of the monograph "Theory of multipliers in spaces of differentiable functions" by the authors V.G. Maz'ya and T.O. Shaposhnikova [1]. Such a book is currently the only work in which the theory of pointwise multipliers in unweighted spaces of differentiable functions is treated systematically. A part of the chapters of this work are devoted to multipliers in classical Sobolev spaces $W^k_p,\ k \geq 1 -$ integer, $1 \leq p < \infty$.

For the latest developments of pointwise multipliers we refer to the monographs [1], [2], which are entirely devoted to this topic. Let us point out some specific directions through the works [3–6].

Let X, Y be Banach spaces whose elements are functions $y\colon \Omega \to \mathbb{R}\ (\mathbb{C})$. We say that a function $z\colon \Omega \to \mathbb{R}\ (\mathbb{C})$ such that a multiplication operator

$$Ty = zy,\ y \in X,$$

is bounded from X to Y, is a multiplier for the pair (X, Y). We denote by $M(X \to Y)$ the space of all multipliers for the pair (X, Y). We introduce the norm

$$\|z;\, M(X \to Y)\| = \|T;\, X \to Y\|\,,$$

in $M(X \to Y)$ [1]. Different kinds of problems arise in the theory of multipliers. The first problem is the problem of describing the space $M(X \to Y)$ for the pair (X, Y). Further, there are problems with studying differential operators as operators acting in the space of multipliers such as the problem of norm evaluation.

We denote by $L_{q,\omega}(I),\ I = [0, \infty)$, the weighted Lebesgue space of all measurable functions in I with the norm

$$\|f\|_{q,\omega} = \|f;\, L_{q,\omega}\,(I)\| = \left(\int\limits_I |f\,(x)|^q\, \omega(x)\, dx\right)^{\frac{1}{q}} < \infty \quad (1 \leq q < \infty)\,,$$

$L_q(I) = L_{q,\omega}(I), \quad \omega \equiv 1$. Here $\omega(\cdot)$ is a weight in I, i.e., it is an almost everywhere positive locally integrable function.

*Corresponding author.
E-mail: aigul.myrzagalieva@astanait.edu.kz

Below $A^l(I)$ is a class of all functions y in I having absolutely continuous derivatives up to order $l-1$ in I.

Let $\omega_0,\ \omega_1$ be weighted functions in I. Let $l \geq 1$ be an integer. We denote by $W^l_{q,\omega_0,\omega_1}(I)$ the weighted Sobolev space of all functions $y \in A^l(I)$ equipped with the following weighted norm

$$\left\| y;\, W^l_{q,\omega_0,\omega_1}(I) \right\| = \left\| y^{(l)};\, L_{q,\omega_0}(I) \right\| + \left\| y;\, L_{q,\omega_1}(I) \right\|.$$

The purpose of this paper is to obtain the description of the space $M(W^l_{p,v}(I) \to W^m_{q,\omega_0,\omega_1}(I))$.

We define as a length function in I any positive and right-continuous function $h(\cdot)$ ($h(\cdot)$ is a l.f.). We denote by $\Delta(x)$ the segment $[x, x+h(x)]$ for the l.f. $h(\cdot)$.

Definition 1. A weighted function v in I is called admissible with respect to the length function $h(\cdot)$, if there exist $0 < \delta < 1$, $0 < \tau \leq 1$, such that the following inequality is true

$$h(x)^{l-\frac{1}{p}} \inf_{\{e\}} \left(\int\limits_{\Delta(x)\setminus e} v(t)\, dt \right)^{\frac{1}{p}} \geq \tau \tag{1}$$

for all $\Delta(x)$, $x \in I$. In (1) the infimum is taken over all measurable subset e of $\Delta(x)$ with Lebesgue measure $|e| \leq \delta\, |\Delta(x)|$. We denote by $\Pi_{l,p}(\delta,\tau)$ the set of admissible weights v with respect to the l.f. $h(\cdot)$.

Let us give some examples.

Example 1. Since

$$h(x)^{l-\frac{1}{p}} \inf_{\{e\}} \left(\int\limits_{\Delta(x)\setminus e} v(t)\, dt \right)^{\frac{1}{p}} \geq (1-\delta)^{\frac{1}{p}} = \tau,$$

the function $v \equiv 1$ is admissible with respect to the l.f. $h(\cdot) = 1$.

Definition 2. We say that a function $\omega(\cdot) > 0$ satisfies the slow variation condition with respect to the l.f. $h(\cdot)$, if there exist constants $0 < b_1 < 1 < b_2$ such that

$$b_1\omega(x) \leq \omega(t) \leq b_2\omega(x) \quad \text{for all} \quad t \in \Delta(x). \tag{2}$$

Example 2. Let $v(\cdot) > 0$ satisfy the slow variation condition (2) with respect to the l.f. $h(x) = v(x)^{-\frac{1}{lp}}$. Then v is admissible with respect to the l.f. $h(x) = v(x)^{-\frac{1}{lp}}$ with $\tau^p = b_1(1-\delta)$. The proof is trivial.

Every power function $v\,(x) = (1+x)^{\mu}$ $(x \geq 0)$, $0 < \mu < +\infty$ satisfies the slow variation condition with respect to the l.f. $h\,(x) = (1+x)^{-\frac{\mu}{lp}}$ in I. Indeed,

$$\left(\frac{1+t}{1+x}\right)^{\mu} \leq 2^{\mu} = b_2,\ \left(\frac{1+t}{1+x}\right)^{\mu} \geq 1 > 2^{-\mu} = b_1$$

for all $t \in \Delta(x)$.

Definition 3. We say that a weight v satisfies the condition $A_{(\delta,\beta)}$ $(0 < \delta, \beta < 1)$ with respect to the length function $h(\cdot)$ in I, if for any interval $\Delta = [a,b] \subset \Delta(x) = [x, x+h(x)]$ $(x \geq 0)$ and any measurable subset e of Δ with the Lebesgue measure $|e| \leq \delta|\Delta|$ the following inequality holds

$$\int\limits_{e} v(t)\, dt \leq \beta \int\limits_{\Delta} v(t)\, dt.$$

We denote by $A_{(\delta,\beta)}$ the set of all weights v which satisfy the condition $A_{(\delta,\beta)}$ with respect to the l.f. $h(\cdot)$. For example, if $b_2 b_1^{-1}\delta < 1$ in (2), then $v \in A_{(\delta,\beta)}$ with $\beta = b_2 b_1^{-1}\delta$.

Let v^* be an Otelbaev function [7]. Namely

$$v^*(x) = \sup\left\{ h > 0\colon h^{lp-1}\int\limits_x^{x+h} v(t)\,dt \le 1 \right\}.$$

We first show that $0 < v^*(x) < \infty$ for all $x \ge 0$. To do this, we note that

$$M(x,h;v) \overset{\text{def}}{=} h^{lp-1}\int\limits_x^{x+h} v(t)\,dt \xrightarrow[h\to 0+]{} 0$$

and that $M(x,h;v) \to \infty$ if $h \to \infty$. Hence, there exist $\delta_x > 0$ and $T_x > 0$, such that

$$M(x,h;v) \le 1, \quad \text{if} \quad 0 < h \le \delta_x, M(x,h;v) > 1, \quad \text{if} \quad h \ge T_x.$$

Therefore, we obtain

$$(0,\delta_x) \subset H_{x,v} = \{h > 0\colon M(x,h;v) \le 1\} \subset (0,T_x), \delta_x \le \sup H_{x,v} = v^*(x) \le T_x.$$

The function $v^*(\cdot)$ is right-continuous in I. By using absolute continuity property of the integral, we can imply that

$$v^*(x)^{lp-1}\int\limits_x^{x+v^*(x)} v(t)\,dt = 1.$$

Example 3. Any weight $v \in A_{(\delta,\beta)}$ (with respect to the l.f. $h(x) = v^*(x)$) in I is admissible with respect to the l.f. $h(x) = v^*(x)$. Thus, for all $e \subset \Delta^*(x) = [x, x + v^*(x)]$ with the measure $|e| \le \delta|\Delta^*(x)|$, we have

$$v^*(x)^{lp-1}\inf_{\{e\}}\int\limits_{\Delta^*(x)\setminus e} v(t)\,dt = v^*(x)^{lp-1}\inf_{\{e\}}\left(\int\limits_{\Delta^*(x)} v(t)\,dt - \int\limits_e v(t)\,dt\right) \ge$$

$$\ge (1-\beta)v^*(x)^{lp-1}\int\limits_{\Delta^*(x)} v(t)\,dt = 1-\beta = \tau.$$

Let $C^l[a,b]$ $(-\infty < a < b < \infty)$ be a space of all functions y, having continuous derivatives up to order l in $[a,b]$.

Lemma 1. [8] Let v belong to $\Pi_{l,p}(\delta,\tau)$ with respect to the l.f. $h(\cdot)$. Then there exists a constant $C^* = C^*(\delta,\tau) > 1$ such that

$$h(x)^{-lp}\int\limits_x^{x+h(x)} |y|^p\,dt \le C^*\int\limits_x^{x+h(x)}\left(\left|y^{(l)}\right|^p + |y|^p\,v(t)\right)dt \quad (x \ge 0)$$

for all $y \in C^l(\Delta)$, where $\Delta = [x, x + h(x)]$.

Lemma 2. Let $1 \le p, q < \infty$. Let $0 \le j < l$ be integers. Let $v \in \Pi_{l,p}(\delta,\tau)$ with respect to the l.f. $h(\cdot)$. Let $\omega \in L^+_{loc}(I)$, $d\omega(t) = \omega(t)\,dt$. Then

$$\max_{[x,x+h(x)]}|y^{(j)}(t)| \le (c^*+1)A(l,j,p)\times$$

$$\times h(x)^{l-j-1/p}\left(\int_x^{x+h(x)}|y^{(l)}(t)|^p\,dt+\int_x^{x+h(x)}|y(t)|^p v(t)\,dt\right)^{1/p}, \qquad (3)$$

$$\left(\int_x^{x+h(x)}|y^{(j)}(t)|^q\,d\omega(t)\right)^{1/q}\le (c^*+1)A(l,j,p,q)h(x)^{l-j-1/p}\times$$

$$\times\left(\int_x^{x+h(x)}\omega(t)\,dt\right)^{1/q}\left(\int_x^{x+h(x)}|y^{(l)}(t)|^p\,dt+\int_x^{x+h(x)}|y(t)|^p v(t)\,dt\right)^{1/p}.$$

Here we consider a differential operator of the form

$$Ly=\sum_{k=0}^{m}\rho_k(x)y^{(k)}\quad (x\ge 0), \qquad (4)$$

where $\rho_k(\cdot)\in L_{loc}(I),\ I=[0,\infty),\ m\ge 1$ is an integer. In the sequel, we assume that L is defined on a subspace $D(L)$ of $W^l_{p,v}$. Here we will investigate the boundedness of the operator $L\colon W^l_{p,v}\to L_{q,\omega}$, $l>m\ge 1$.

Theorem 1. Let $l>m\ge 1$ be integers. Let $1<p\le q<\infty$. Let v belong to $\Pi_{l,p}(\delta,\tau)$ with respect to the l.f. $h(\cdot)$. Let $(d\,\omega(t)=\omega(t)\,dt)$

$$R_k=\sup_{x\ge 0}h(x)^{l-k-\frac{1}{p}}\left\{\int_x^{x+h(x)}|\rho_k(t)|^q\ d\omega(t)\right\}^{\frac{1}{q}}<\infty$$

for $k=0,1,\ldots,m$. Then the operator L in (4) is bounded from $W^l_{p,v}(I)$ to $L_{q,\omega}(I)$. Here the norm satisfies the inequality

$$\left\|L;\,W^l_{p,v}(I)\to L_{q,\omega}(I)\right\|\le c\sum_{k=0}^{m}R_k.$$

Proof. Let $y\in D(L)\subset W^l_{p,v}$. For the k-th summand in (4), we have

$$\left\|\rho_k y^{(k)}\right\|^q_{q,\omega}=\int_0^\infty\left|\rho_k y^{(k)}\right|^q\,d\omega(t)=\sum_{j=0}^{\infty}\int_{\Delta_j}\left|\rho_k y^{(k)}\right|^q\,d\omega(t),$$

where the system of segments $\{\Delta_j\}$, $j\ge 0$, is constructed as follows

$$\Delta_{j+1}=[x_j,x_{j+1}]\,,\ x_{j+1}=x_j+h(x_j)\ \ (x_0=0)\,.$$

By virtue of (3), we obtain

$$\int_0^\infty\left|\rho_k(t)y^{(k)}\right|^q\,d\omega(t)=\sum_{j=0}^{\infty}\int_{\Delta_j}\left|\rho_k(t)y^{(k)}\right|^q\,d\omega(t)\le$$

$$\le\sum_{j=0}^{\infty}\left(\max_{\Delta_j}\left|y^{(k)}\right|\right)^q\int_{\Delta_j}|\rho_k(t)|^q\,d\omega(t)\le$$

$$\le\sum_{j=0}^{\infty}\left((1+c^*)^{\frac{1}{p}}A(l,k,p)\,|\Delta_j|^{l-k-\frac{1}{p}}\left[\int_{\Delta_j}\left(\left|y^{(l)}\right|^p+v(t)\,|y|^p\right)\,dt\right]^{\frac{1}{p}}\right)^q\times$$

$$\times \int_{\Delta_j} |\rho_k(t)|^q \, d\omega(t) \leq \widetilde{c}_{l,k,p}^q \sum_{j=0}^{\infty} \left[|\Delta_j|^{l-k-\frac{1}{p}} \left(\int_{\Delta_j} |\rho_k(t)|^q \, d\omega(t) \right)^{\frac{1}{q}} \right]^q \times$$

$$\times \left[\int_{\Delta_j} \left(\left| y^{(l)} \right|^p + v(t) \, |y|^p \right) \, dt \right]^{\frac{q}{p}} \leq \widetilde{c}_{l,k,p}^q R_k^q \left\| y; \, W_{p,v}^l(I) \right\|^q,$$

where $\widetilde{c}_{l,k,p} = A(l,k,p) \, (1 + c^*)^{\frac{1}{p}}$.

As a result, we have

$$\|Ly; \, L_{q,\omega} \, (I)\| \leq \sum_{k=0}^{m} \left\| \rho_k y^{(k)}; \, L_{q,\omega} \right\| \leq c \sum_{k=0}^{m} R_k \left\| y; \, W_{p,v}^l(I) \right\|.$$

Thus the proof of Theorem 1 is complete.

Let us assume that the operator L in (4) is bounded as an operator from $W_{p,v}^l$ to $L_{q,\omega}$, i.e., $D(L) \subset W_{p,v}^l$ and there exists a constant $b > 0$ such that

$$\left(\int_I |Ly|^q \, d\omega(t) \right)^{\frac{1}{q}} \leq b \, \|y; \, W_{p,v}^l\| \quad (y \in D(L)). \tag{5}$$

We take the function $\eta \in C_0^\infty(I)$, $0 \leq \eta \leq 1$, with $\operatorname{supp}(\eta) \subset [0,1]$, such that $\eta = 1$ in $\left[\frac{1}{4}, \frac{3}{4}\right]$. Let $\Delta = [x, x + h(x)]$, $h(x) = v^*(x)$, $\widetilde{\Delta} = [x + \frac{h}{4}, x + \frac{3h}{4}]$. We set $y_0(t) = \eta\left(\frac{t-x}{h}\right)$. Then $y_0(t) = 1$, $Ly_0(t) = \rho_0(t)$ for all $t \in \widetilde{\Delta}$. Therefore,

$$\left(\int_{\widetilde{\Delta}} |\rho_0|^q \, d\omega(t) \right)^{\frac{1}{q}} = \left(\int_{\widetilde{\Delta}} |Ly_0|^q \, d\omega(t) \right)^{\frac{1}{q}} \leq b \, \|y_0; \, W_{p,v}^l(\Delta)\|. \tag{6}$$

Moreover,

$$\|y_0; \, W_{p,v}^l(\Delta)\| = \left(\int_{\Delta} \left| y_0^{(l)}(t) \right|^p \, dt \right)^{\frac{1}{p}} + \left(\int_{\Delta} |y_0(t)|^p v(t) \, dt \right)^{\frac{1}{p}} =$$

$$= \left(\int_{\Delta} h^{-lp} \left| \eta^{(l)} \left(\frac{t-x}{h} \right) \right|^p \, dt \right)^{\frac{1}{p}} + \left(\int_{\Delta} v(t) \, dt \right)^{\frac{1}{p}} \leq \tag{7}$$

$$\leq h^{-l+\frac{1}{p}} \left(c_l^* + 1 \right),$$

where $c_l^* = \|\eta^{(l)}; \, C[0,1]\| = \max\limits_{t \in [0,1]} |\eta^{(l)}(t)|$.

Recall that the following equality holds in Δ

$$h^{l-\frac{1}{p}} \left(\int_{\Delta} v(t) \, dt \right)^{\frac{1}{p}} = 1.$$

By (6), (7), we obtain

$$\left(\int_{\widetilde{\Delta}} |\rho_0|^q \, d\omega(t) \right)^{\frac{1}{q}} \leq \widetilde{c}_0 \, b \, h^{-l+\frac{1}{p}}, \tag{8}$$

where $\widetilde{c}_0 = c_l^* + 1$. We take the function $y_1(t) = (t-x) y_0(t)$. We have $|y_1(t)| = \left| (t-x) \eta\left(\frac{t-x}{h}\right) \right| = |t-x| \leq h$, $|y_1'(t)| = |(t-x) y_0'(t) + y_0(t)| = 1$, $\left| y_1^{(k)}(t) \right| = 0$ for any $t \in \widetilde{\Delta}$, when $k \geq 2$. Therefore, from (5) it follows that

$$\left(\int_{\widetilde{\Delta}} |\rho_1|^q \, d\omega(t) \right)^{\frac{1}{q}} = \left\| Ly_1 - \rho_0 y_1; \, L_{q,\omega}(\widetilde{\Delta}) \right\| \leq$$

$$\leq \|Ly_1;\, L_{q,\omega}(\widetilde{\Delta})\| + \|\rho_0 y_1;\, L_{q,\omega}(\widetilde{\Delta})\| \leq b\,\|y_1;\, W^l_{p,v}(\Delta)\| + \widetilde{c}_0\, b\, h^{1-l+\frac{1}{p}}. \tag{9}$$

We have

$$\left|y_1^{(l)}(t)\right| = \left|\sum_{j=0}^{l}\binom{l}{j}(t-x)^{(j)}h^{-(l-j)}\eta^{(l-j)}\left(\frac{t-x}{h}\right)\right| \leq$$

$$\leq \left|\binom{l}{0}(t-x)h^{-l}\eta^{(l)}\left(\frac{t-x}{h}\right)\right| + \left|\binom{l}{1}h^{-(l-1)}\eta^{(l-1)}\left(\frac{t-x}{h}\right)\right| \leq h^{-l+1}\left[\binom{l}{0}c_l^* + \binom{l}{1}c_{l-1}^*\right].$$

Hence,

$$\left\|y_1;\, W^l_{p,v}(\Delta)\right\| = \left\|y_1^{(l)};\, L_p(\Delta)\right\| + \left(\int_\Delta \left|(t-x)\eta\left(\frac{t-x}{h}\right)\right|^p v(t)\,dt\right)^{\frac{1}{p}} \leq$$

$$\leq h^{1-l+\frac{1}{p}}\left\{\binom{l}{0}c_l^* + \binom{l}{1}c_{l-1}^* + 1\right\} = h^{1-l+\frac{1}{p}}\left\{1 + \sum_{j=0}^{1}\binom{l}{j}c_{l-j}^*\right\}. \tag{10}$$

By (8)–(10), we have

$$\left(\int_{\widetilde{\Delta}} |\rho_1|^q\, d\omega(t)\right)^{\frac{1}{q}} \leq b\,h^{1-l+\frac{1}{p}}\left\{1 + \sum_{j=0}^{1}\binom{l}{j}c_{l-j}^*\right\} + \widetilde{c}_0\, b\, h^{1-l+\frac{1}{p}} =$$

$$= b\,h^{1-l+\frac{1}{p}}\left\{\sum_{j=0}^{1}\binom{l}{j}c_{l-j}^* + \widetilde{c}_0 + 1\right\} = \widetilde{c}_1\, b\, h^{1-l+\frac{1}{p}}.$$

Let us assume that for any k $(1 \leq k < m)$ following estimates hold

$$\left\|\rho_j;\, L_{q,\omega}(\widetilde{\Delta})\right\| \leq b\,h^{j-l+\frac{1}{p}}\widetilde{c}_j \quad (0 \leq j \leq k-1).$$

Then we take $y_k(t) = (t-x)^k y_0(t)$, and we have

$$y_k(t) = (t-x)^k,$$

$$y_k^{(j)}(t) = k(k-1)\ldots(k-j+1)(t-x)^{k-j} \quad (1 \leq j \leq k),$$

$$y_k^{(j)}(t) = 0 \quad (j > k)$$

for all $t \in \widetilde{\Delta}$. Thus,

$$\left(\int_{\widetilde{\Delta}} |\rho_k(t)|^q\, d\omega(t)\right)^{\frac{1}{q}} = \frac{1}{k!}\left(\int_{\widetilde{\Delta}} \left|\rho_k(t) y_k^{(k)}\right|^q d\omega(t)\right)^{\frac{1}{q}} =$$

$$= \frac{1}{k!}\left(\int_{\widetilde{\Delta}} \left|Ly_k(t) - \sum_{j=0}^{k-1}\rho_j(t) y_k^{(j)}\right|^q d\omega(t)\right)^{\frac{1}{q}} \leq$$

$$\leq \frac{1}{k!}\left\{\left\|Ly_k;\, L_{q,\omega}(\widetilde{\Delta})\right\| + \sum_{j=0}^{k-1}\left\|\rho_j y_k^{(j)};\, L_{q,\omega}(\widetilde{\Delta})\right\|\right\} \leq$$

$$\leq \frac{1}{k!}\, b\left\|y_k;\, W^l_{p,v}(\Delta)\right\| + \frac{1}{k!}\sum_{j=0}^{k-1}\left\|\rho_j y_k^{(j)};\, L_{q,\omega}(\widetilde{\Delta})\right\| \leq$$

$$\le \frac{1}{k!}\, b\, \left\{ \left\| y_k^{(l)};\, L_p(\Delta) \right\| + \| y_k;\, L_{p,v}(\Delta) \| \right\} + \sum_{j=0}^{k-1} \frac{1}{(k-j)!} h^{k-j} \left\| \rho_j;\, L_{q,\omega}(\widetilde{\Delta}) \right\| \le$$

$$\le \frac{1}{k!}\, b \left\{ \left(\int_{\Delta} \left| \sum_{j=0}^{k} \binom{l}{j} \left((t-x)^k \right)^{(j)} h^{j-l} \eta^{(l-j)} \left(\frac{t-x}{h} \right) \right|^p dt \right)^{\frac{1}{p}} + \right.$$

$$\left. + \left(\int_{\Delta} |y_k|^p v(t)\, dt \right)^{\frac{1}{p}} \right\} +$$

$$+ \sum_{j=0}^{k-1} \frac{1}{(k-j)!} h^{k-j} \left(\int_{\widetilde{\Delta}} |\rho_j|^q\, d\omega(t) \right)^{\frac{1}{q}} \le b\, h^{k-l+\frac{1}{p}} \widetilde{c}_k.$$

So, we have

$$\left(\int_{x+h/4}^{x+3h/4} |\rho_k(t)|^q\, d\omega(t) \right)^{1/q} \ll b\, h^{k-l+\frac{1}{p}} \quad (h = v^*(x),\ 0 \le k \le m).$$

Theorem 2. Let $l > m \ge 1$ be integers, $1 < p < \infty,\ 1 \le q < \infty,\ lp > 1$. Let the operator L in (4) be bounded from $W^l_{p,v}$ to $L_{q,\omega}$. Then $(d\,\omega(t) = \omega(t)\, dt)$

$$\widetilde{R}_k = \sup_{x \ge 0} v^*(x)^{l-k-\frac{1}{p}} \left\{ \int\limits_{x+\frac{v^*(x)}{4}}^{x+\frac{3v^*(x)}{4}} |\rho_k(t)|^q\, d\omega(t) \right\}^{\frac{1}{q}} \le \widetilde{c}_k \left\| L;\, W^l_{p,v} \to L_{q,\omega} \right\|. \tag{11}$$

Proof. We have the fulfillment of condition (5) with $b = \left\| L;\, W^l_{p,v} \to L_{q,\omega} \right\|$. In this case, we have shown that the following inequality holds

$$v^*(x)^{l-k-\frac{1}{p}} \left\{ \int\limits_{x+\frac{v^*(x)}{4}}^{x+\frac{3v^*(x)}{4}} |\rho_k(t)|^q\, d\omega(t) \right\}^{\frac{1}{q}} \le \widetilde{c}_k \left\| L;\, W^l_{p,v} \to L_{q,\omega} \right\|$$

for all $x \ge 0$. Then it follows the validity of inequality (11). The proof of Theorem 2 is complete.

We set $R^* = \sum\limits_{k=0}^{m} R^*_k$, where $R^*_k = R_k$ with $h(x) = v^*(x)$, and $\widetilde{R}^* = \sum\limits_{k=0}^{m} \widetilde{R}_k$.

Theorem 3. Let $l > m \ge 1$ be integers, $1 < p \le q < \infty$. Let v be in $A_{(\delta,\beta)}$. Let $R^* < \infty$. Then the operator L in (4) is bounded from $W^l_{p,v}$ to $L_{q,\omega}$. Furthermore,

$$c_0 \widetilde{R}^* \le \left\| L;\, W^l_{p,v} \to L_{q,\omega} \right\| \le c_1 R^*.$$

The statements of Theorem 3 are direct consequences of Theorem 1 and Theorem 2.

Theorem 4. Let $l > m \ge 1$ be integers, $1 < p \le q < \infty$. Let $v \in \Pi_{l,p}(\delta, \tau)$ with respect to the l.f. $h(\cdot)$ in I. Let $\mu \in A^m(I)$. If

$$M_{k,\mu,\omega_0} = \sup_{x \ge 0} h(x)^{l-k-\frac{1}{p}} \left\{ \int\limits_{x}^{x+h(x)} \left| \mu^{(m-k)}(t) \right|^q d\omega_0(t) \right\}^{\frac{1}{q}} < \infty \quad (k = 0, 1, \ldots, m),$$

$$M_{0,\mu,\omega_1} = \sup_{x\geq 0} h(x)^{l-\frac{1}{p}} \left\{ \int\limits_x^{x+h(x)} |\mu(t)|^q \; d\omega_1(t) \right\}^{\frac{1}{q}} < \infty,$$

then $\mu \in M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})$. Moreover,

$$\left\| \mu;\ M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1}) \right\| \leq C \left[\sum_{k=0}^{m} M_{k,\mu,\omega_0} + M_{0,\mu,\omega_1} \right],$$

where $C = C(n,l,p,q) > 0$.

Proof. We have

$$\|\mu y; W^m_{q,\omega_0,\omega_1}\|^q = \int_0^\infty \left(|(\mu y)^{(m)}|^q \omega_0 + |\mu y|^q \omega_1 \right) dt.$$

Since $(\mu y)^{(m)}(t) = Ly$, $\rho_k = \frac{m!}{k!(m-k)!}\mu^{(m-k)}$, it follows that

$$\int_0^\infty |(\mu y)^{(m)}|^q \omega_0(t)\, dt = \int_0^\infty |Ly|^q \omega_0(t)\, dt = \|Ly; L_{q,\omega_0}\|^q$$

and

$$\int_0^\infty |\mu y|^q \omega_1(t)\, dt \leq c \sum_j h_j^{(l-1/p)q} \left(\int_{\Delta_j} |\mu|^q \omega_1 \right) \|y; W^l_{p,v}\|^q \leq$$

$$\leq \left(\sup_x h(x)^{l-1/p} \left(\int_{\Delta_j} |\mu|^q \omega_1 \right)^{1/q} \right)^q \|y; W^l_{p,v}\|^q.$$

Thus, the proof of Theorem 4 follows the lines of the proof of Theorem 1.

Theorem 5. Let $1 < p \leq q < \infty$. Let $l > m \geq 1$ be integers. If $\mu \in M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})$, then

$$\left\| \mu;\ M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1}) \right\| \geq C \left[\sum_{k=0}^{m} M^*_{k,\mu,\omega_0} + M^*_{0,\mu,\omega_1} \right],$$

where

$$M^*_{k,\mu,\omega_0} = \sup_{x\geq 0} v^*(x)^{l-k-\frac{1}{p}} \left\{ \int\limits_{x+\frac{v^*(x)}{4}}^{x+\frac{3v^*(x)}{4}} \left| \mu^{(m-k)}(t) \right|^q \; d\omega_0(t) \right\}^{\frac{1}{q}} < \infty,$$

$$M^*_{0,\mu,\omega_1} = \sup_{x\geq 0} v^*(x)^{l-\frac{1}{p}} \left\{ \int\limits_{x+\frac{v^*(x)}{4}}^{x+\frac{3v^*(x)}{4}} |\mu(t)|^q \; d\omega_1(t) \right\}^{\frac{1}{q}} < \infty.$$

The constant C does not depend on $h(\cdot), v$ and μ.

Proof. By $\mu \in M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})$ it follows that

$$\|\mu; M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})\| \geq \frac{\|(\mu y)^{(m)}; L_{q,\omega_0}\|}{\|y; W^l_{p,v}\|} + \frac{\|\mu y; L_{q,\omega_1}\|}{\|y; W^l_{p,v}\|}.$$

Then

$$\|\mu; M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})\| \geq \sup_{0\neq y \in W^l_{p,v}} \frac{\|(\mu y)^{(m)}; L_{q,\omega_0}\|}{\|y; W^l_{p,v}\|} = \|L; W^l_{p,v} \to L_{q,\omega_0}\|,$$

where $Ly = \sum\limits_{k=0}^{m} \rho_k y^{(k)}$, $\rho_k = c_k \mu^{(m-k)}$. By Theorem 3, we obtain

$$\|\mu; M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})\| \geq c_0 \sum_{k=0}^{m} \sup_{x\geq 0} M^*_{k,\mu,\omega_0}$$

Next, we take a function $y_0(t)$ defined as in Theorem 2. Then

$$\|\mu; M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})\| \geq \frac{\left(\int\limits_{x+v^*(x)/4}^{x+3v^*(x)/4} |\mu|^q \, d\omega_1\right)^{1/q}}{\left(\int\limits_{x}^{x+v^*(x)} |y_0^{(l)}|^p \, dt\right)^{1/p} + \left(\int\limits_{x}^{x+v^*(x)} |y_0|^p v(t) \, dt\right)^{1/p}} \geq$$

$$\geq \frac{\left(\int\limits_{x+v^*(x)/4}^{x+3v^*(x)/4} |\mu|^q \, d\omega_1\right)^{1/q}}{\left(h^{1-lp} \int\limits_{0}^{1} |\eta^{(l)}|^p \, d\xi\right)^{1/p} + \left(\int\limits_{x}^{x+v^*(x)} |y_0|^p v(t) \, dt\right)^{1/p}} \geq c_1 M^*_{0,\mu,\omega_1}.$$

Thus, the proof of Theorem 5 is complete.

Corollary 1. Let $l > m \geq 1,\ 1 < p \leq q < \infty$. Let $\mu \in C^m(I)$. Then $\mu \in M(W^l_p \to W^m_{q,\omega_0,\omega_1})$ if and only if

$$U_k = \sup_{x\geq 1} \int\limits_{x}^{x+1} |\mu^{(m-k)}|^q \, d\omega_1 < \infty \quad (k = 0, 1, \ldots, m),$$

$$V = \sup_{x\geq 1} \int\limits_{x}^{x+1} |\mu|^q \, d\omega_0 < \infty.$$

Acknowledgments

The author thanks Professor Leili Kussainova for providing cooperative support and valuable assistance.

References

1 Maz'ya V.G.Theory of multipliers in spaces of differentiable functions / V.G. Maz'ya, T.O. Shaposhnikova. — Boston: Pitman, 1985. — 344 p.

2 Maz'ya V.G.Theory of Sobolev multipliers: with applications to differential and integral operators / V.G. Maz'ya, T.O. Shaposhnikova. — Berlin: Grundlehren der mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), Springer-Verlag, 2009. — 337. — xiv+614 p.

3 Нейман-Заде М.И. Операторы Шрёдингера с сингулярными потенциалами из пространств мультипликаторов / М.И. Нейман-Заде, А.А. Шкаликов // Математические заметки. — 1999. — 66. — 5. — С. 723–733.

4 Беляев А.А. Мультипликаторы в пространствах бесселевых потенциалов: случай индексов гладкости разного знака / А.А. Беляев, А.А. Шкаликов // Алгебра и анализ. — 2018. — 30. — 2. — С. 76–96.

5 Кусаинова Л.К. Об ограниченности оператора Шрёдингера в весовых пространствах Соболева / Л.К. Кусаинова, А.Х. Мырзагалиева, Я.Т. Султанаев // Математические заметки. — 2016. — 99. — 6. — С. 945–949.

6 Kussainova L. On Multipliers from Weighted Sobolev Spaces to Lebesgue Spaces / L. Kussainova, A. Myrzagaliyeva // Springer Proceedings in Mathematics and Statisticsthis. — 2017. — 216. — P. 52–57.

7 Отелбаев М. Теоремы вложения пространств с весом и их приложения к изучению спектра оператора Шрёдингера / М. Отелбаев // Тр. Мат. ин-та им. В.А. Стеклова. — 1979. — 150. — С. 265–305.

8 Кусаинова Л.К. Теоремы вложения и интерполяции весовых пространств Соболева: дис. д-ра физ.-мат. наук: 01.01.01 – «Вещественный, комплексный и функциональный анализ (алгебра, логика и теория чисел)» / Лейли Кабиденовна Кусаинова. — Алматы, 1999. — 255 с.

А. Мырзағалиева

Astana IT University, Нұр-Сұлтан, Қазақстан

Осьтегі салмақты Соболев кеңістіктеріндегі мультипликаторлар

Жұмыста $W^l_{p,v}$ салмақты Соболев кеңістігінен салмақты Лебег кеңістігіне оң нақты жартылай түзуде әсер ететін бір айнымалы дифференциалдық оператордың шектелуі үшін қажетті және жеткілікті шарттар анықталған. Дифференциалдық операторлардың коэффициенттерін мультипликаторлар ретінде қарастыру заңды екені белгілі. Салмақты Соболев кеңістіктерінде нүктелік мультипликаторлар енгізілген. Жалпы типті салмақтары бар (W_1, W_2) салмақты Соболев кеңістіктерінің жұбы үшін $M(W_1 \to W_2)$ кеңістігінің сипаттамасы алынған.

Кілт сөздер: Соболев кеңістігі, нүктелік көбейткіш, салмақты кеңістік, дифференциалдық оператор, рұқсат етілген функция, баяу вариация шарты, Өтелбаев функциясы.

А. Мырзагалиева

Astana IT University, Нур-Султан, Казахстан

Мультипликаторы в весовых пространствах Соболева на оси

В статье установлены необходимые и достаточные условия ограниченности дифференциального оператора одной переменной, действующего из весового пространства Соболева $W^l_{p,v}$ в весовое пространство Лебега на положительной вещественной полупрямой. Хорошо известно, что коэффициенты дифференциальных операторов естественно рассматривать как мультипликаторы. Мы вводим точечные мультипликаторы в весовых пространствах Соболева. Получено описание пространства $M(W_1 \to W_2)$ для пары весовых пространств Соболева (W_1, W_2) с весами общего типа.

Ключевые слова: пространство Соболева, точечный мультипликатор, весовое пространство, дифференциальный оператор, допустимая функция, условие медленного колебания, функция Отельбаева.

References

1 Maz'ya, V.G., & Shaposhnikova, T.O. (1985). *Theory of multipliers in spaces of differentiable functions.* Boston: Pitman.

2 Maz'ya, V.G., & Shaposhnikova, T.O. (2009). *Theory of Sobolev multipliers: with applications to differential and integral operators.* Berlin: Grundlehren der mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), Springer-Verlag.

3 Neiman-Zade, M.I., & Shkalikov, A.A. (1999). Operatory Shredingera s singuliarnymi potentsialami iz prostranstv multiplikatorov [Schrödinger Operators with Singular Potentials from Spaces of Multipliers]. *Matematicheskie zametki — Mathematical Notes, 66 (5),* 723-733 [in Russian].

4 Beliaev, A.A., & Shkalikov, A.A. (2018). Multiplikatory v prostranstvakh besselevykh potentsialov: sluchai indeksov gladkosti raznogo znaka [Multipliers in Bessel potential spaces: the case of smoothness indices of different sign]. *Algebra i analiz – Algebra and Analysis, 30*(2), 76–96 [in Russian].

5 Kusainova, L.K., Myrzagalieva, A.Kh., & Sultanaev, Ya.T. (2016). Ob ogranichennosti operatora Shredingera v vesovykh prostranstvakh Soboleva [On the boundedness of the Shrödinger operator in weighted Sobolev spaces]. *Matematicheskie zametki — Mathematical Notes, 99*(6), 945–949 [in Russian].

6 Kussainova, L., & Myrzagaliyeva, A. (2017). On Multipliers from Weighted Sobolev Spaces to Lebesgue Spaces. *Springer Proceedings in Mathematics and Statisticsthis, 216,* 52–57.

7 Otelbaev, M. (1979). Teoremy vlozheniia prostranstv s vesom i ikh primeneniia k izucheniiu spektra operatora Shredingera [Imbedding theorems for spaces with a weight and their application to the study of the spectrum of a Schrödinger operator]. *Trudy Matematicheskogo instituta imeni V.A. Steklova — Proceedings of the V.A. Steklov Institute of Mathematics, 150,* 265–305 [in Russian].

8 Kusainova, L.K. (1999). Teoremy vlozheniia i interpoliatsii vesovykh prostranstv Soboleva [Embedding and interpolation theorems of weighted Sobolev spaces]. *Doctor's thesis.* Almaty [in Russian].

DOI 10.31489/2022M3/116-123

UDC 517

M.I. Qureshi[1], M.S. Baboo[2], A. Ahmad[1,*]

[1]*Jamia Millia Islamia (A Central University), New Delhi, India;*
[2]*School of Basic Sciences and Research, Sharda University, Greater Noida, India*
(E-mail: miqureshi_ delhi@yahoo.co.in, mesub007@gmail.com, ashfaqmaths@gmail.com)

Some Convergent Summation Theorems For Appell's Function F_1 Having Arguments $-1, \frac{1}{2}$

In this paper, we obtain some closed forms of hypergeometric summation theorems for Appell's function of first kind F_1 having the arguments $-1, \frac{1}{2}$ with suitable convergence conditions, by adjustment of parameters and arguments in generalized form of first, second and third summation theorems of Kümmer and others.

Keywords: generalized hypergeometric function, Appell's function of first kind, Kümmer's first, second and third summation theorems.

Introduction

A great interest in the theory of hypergeometric functions (that is, hypergeometric functions of several variables) is motivated essentially by the fact that the solutions to many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric functions (see, for details, [1; 47]; [2] and the references cited therein). For instance, the energy absorbed by some non-ferromagnetic conductor sphere included in an internal magnetic field can be calculated through such functions [3, 4]. Hypergeometric functions of several variables are used in physical and quantum chemical applications as well [5–7].

The extensive development of the theories of hypergeometric functions of a single variable has led to a full-scale investigation of corresponding theories in two or more variables. In 1880, Appell [8–10] considered the product of two Gauss's hypergeometric functions ${}_2F_1$ to obtain four Appell's functions F_1, F_2, F_3, and F_4 in two variables. Later in 1893, Lauricella [11] further generalized the four Appell functions F_i $(i = 1, 2, 3, 4)$ to give the functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, and $F_D^{(n)}$ in n-variables. It is noted that $F_A^{(1)} = F_B^{(1)} = F_C^{(1)} = F_D^{(1)} = {}_2F_1$, $F_A^{(2)} = F_2$, $F_B^{(2)} = F_3$, $F_C^{(2)} = F_4$ and $F_D^{(2)} = F_1$.

Over eight decades ago Chaundy [12], Burchnall-Chaundy [13], and recently several others [14–24], systematically, presented a number of expansion and decomposition formulas for some double hypergeometric functions, for example, the Appell's functions F_i, in series of simpler hypergeometric functions. Recently, Khan & Abukhammash [25] introduced and investigated 10 Appell type generalized functions M_i $(i = 1, ..., 10)$ by considering the product of two ${}_3F_2$ functions. Here, motivated by the above-mentioned works, Choi et al. [16] aim to introduce 18 Appell type generalized functions κ_i $(i = 1, ..., 18)$ by considering the product of two ${}_4F_3$ functions.

In the usual notation, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad , \qquad \mathbb{N} := \{1, 2, 3, \dots\} = \mathbb{N}_0 \backslash \{0\} \quad ,$$

$$\mathbb{Z}_0^- := \{0, -1, -2, \dots\} = \mathbb{Z}^- \cup \{0\} \quad , \qquad \mathbb{Z}^- := \{-1, -2, -3, \dots\}$$

and $\mathbb{Z} = \mathbb{Z}_0^- \cup \mathbb{N}$ being the sets of integers.

*Corresponding author.
E-mail: ashfaqmaths@gmail.com

For definitions of Pochhammer symbol, generalized hypergeometric function ${}_pF_q$ with convergence conditions and other useful results, we refer the monumental work of Abramowitz & Stegun [26], Andrews et al. [27], Erdélyi et al. [28], Prudnikov et al. [29], Rainville [30], and Srivastava & Manocha [31]. *Appell's Function of First Kind* is defined as :

$$F_1[A\ ;\ B,\ C\ ;\ D\ ;\ x,\ y] = \sum_{r,s=0}^{\infty} \frac{(A)_{r+s}(B)_r(C)_s}{(D)_{r+s}} \frac{x^r}{r!} \frac{y^s}{s!}.$$

Convergence conditions of Appell's double series F_1

(a) Appell's series F_1 is convergent when $|x| < 1,\ |y| < 1\ ;\ \ A, B, C, D \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

(b) Appell's series F_1 is absolutely convergent when $|x| = 1,\ |y| = 1\ ;\ \ A, B, C, D \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\Re(A+B-D) < 0$, $\Re(A+C-D) < 0$ and $\Re(A+B+C-D) < 0$.

(c) Appell's series F_1 is conditionally convergent when $|x| = 1,\ |y| = 1\ ;\ x \neq 1, y \neq 1$; $A, B, C, D \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\Re(A+B-D) < 1$, $\Re(A+C-D) < 1$ and $\Re(A+B+C-D) < 2$.

(d) Appell's series F_1 is a polynomial if A is a negative integer; $B, C, D \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

(e) Appell's series F_1 is a polynomial if B and C are negative integers; $A, D \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

For absolutely and conditionally convergence (b,c) of Appell's function F_1, interested readers may consult the paper of Hài et al. [32] related to the convergence of multiple hypergeometric functions of Kampé de Fériet.

A result of Appell and Kampé de Fériet[8], see also [31; 55, Equation 1.6(15)]:

$$F_1[a\ ;\ b,\ c\ ;\ d\ ;\ 1, 1] = \frac{\Gamma(d)\Gamma(d-a-b-c)}{\Gamma(d-a)\Gamma(d-b-c)}\ , \tag{1}$$

$$\left(\Re(d-a-b-c) > 0\ ;\ d \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)\ .$$

Motivated by the work in equation (1) of Appell and Kampé de Fériet , we obtain some summation theorems for Appell's function of first kind F_1 having equal argument other than unity, in section 1, by suitable adjustment of numerator and denominator parameters.

When the values of parameters leading to the results which do not make sense are tacitly excluded, then using series iteration technique, the Appell's function F_1 with equal argument can also be written as [8; 23, Equation (25)]

$$F_1[A; B, C; D; x, x] = {}_2F_1\left[\begin{array}{r} A, B+C; \\ D; \end{array} x\right], \qquad \left(|x| < 1\ ;\ A, B, C, D \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)\ . \tag{2}$$

1 Some new Summations using the function $F_1[A; B, C; D; x, x]$

Further by putting $x = 1$ in equation (2) and applying Gauss classical summation theorem [31; 30, Equation 1.2(7)], we get a known result (1) of Appell and Kampé de Fériet.

In equation (2), by putting $A = a, B = b,\ C = c,\ D = 1+a-b-c-m$ and $x = -1$, using a summation theorem [33; 1524, Equation (2.3)], we get

$$F_1[a; b, c; 1+a-b-c-m; -1, -1] = \frac{\Gamma(1+a-b-c-m)}{2\Gamma(a)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a+2-2b-2c-2m}{2})} \right\}\ ,$$

$$(\Re(b+c) < \frac{2-m}{2} \ ; \Re(2b+c) < 2-m, \Re(2c+b) < 2-m, a, b, c, b+c,$$

$$1+a-b-c-m \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ; \ m \in \mathbb{N}_0).$$

In equation (2), by putting $A = a, B = b,\ C = c,\ D = 1+a-b-c+m$ and $x = -1$, using another summation theorem [33; 1523, Equation (2.2)], we obtain

$$F_1[a; b, c; 1+a-b-c+m; -1, -1] = \frac{\Gamma(1+a-b-c+m)}{2\Gamma(a)(1-b-c)_m} \sum_{r=0}^{m} \left\{ \binom{m}{r} \frac{(-1)^r \Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a+2-2b-2c}{2})} \right\} ,$$

$$(\Re(b+c) < \frac{2+m}{2}, \Re(2b+c) < 2+m \ , \Re(2c+b) < 2+m; \ a, b, c, b+c, \ 1+a-b-c+m,$$

$$1-b-c \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ; \ m \in \mathbb{N}_0).$$

In equation (2), by putting $A = a, B = b,\ C = c,\ D = a-b-c-m$ and $x = -1$, using the summation theorem [34; 14, Equation (3.1)], we find

$$F_1[a \ ; \ b, \ c \ ; \ a-b-c-m \ ; -1, -1]$$

$$= \frac{\Gamma(a-b-c-m)}{2\Gamma(a)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \left[\frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a-2b-2c-2m}{2})} + \frac{\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{r+a-2b-2c-2m+1}{2})} \right] \right\} ,$$

$$(\Re(b+c) < \frac{1-m}{2}, \Re(2b+c) < 1-m, \Re(2c+b) < 1-m; \ a, b, c, b+c,$$

$$a-b-c-m \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ; \ m \in \mathbb{N}_0) \ .$$

In equation (2), by putting $A = a, B = b,\ C = c,\ D = a-b-c+m$ and $x = -1$, using another summation theorem [34; 14, Equation (3.2)], we have

$$F_1[a \ ; \ b, \ c \ ; \ a-b-c+m \ ; -1, -1]$$

$$= \frac{\Gamma(a-b-c+m)}{2\Gamma(a)(-b-c)_m} \sum_{r=0}^{m} \left\{ \binom{m}{r} \left[\frac{(-1)^r \Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a-2b-2c}{2})} + \frac{(-1)^r \Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{r+a-2b-2c+1}{2})} \right] \right\} ,$$

$$(\Re(b+c) < \frac{1+m}{2}, \Re(2b+c) < 1+m, \Re(2c+b) < 1+m \ ; \ a, b, c, b+c,$$

$$-b-c, \ a-b-c+m \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ; \ m \in \mathbb{N}_0) \ .$$

In equation (2), by putting $A = n,\ B = C = \frac{a}{2}\ ,\ D = -a-m$ and $x = -1$, using the summation theorem [34; 14, Equation (3.3)], we get

$$F_1\left[n; \frac{a}{2}, \frac{a}{2}; -a-m; -1, -1\right] = \frac{\Gamma(-m-a)}{2\Gamma(n)} \sum_{r=0}^{m+n+1} \left\{ \frac{(-1)^r (-m-n-1)_r \Gamma(\frac{r+n}{2})}{r! \Gamma(\frac{r-n-2a-2m}{2})} \right\} ,$$

$$\left(\Re(a) < \tfrac{2}{3}(1-m-n); \quad n, \ a, \ -m-a \in \mathbb{C}\backslash\mathbb{Z}_0^-; \quad m+n \in \mathbb{N}_0 \cup \{-1\} \right) .$$

In equation (2), $A = n$, $B = C = \frac{a}{2}$, $D = -a + m$ and $x = -1$, using another summation theorem [34; 14, Equation (3.4)], we have

$$F_1\left[n; \frac{a}{2}, \frac{a}{2}; -a+m; -1, -1\right] = \frac{\Gamma(1-a)\Gamma(m-a)}{2\Gamma(n)\Gamma(m-a-n)} \sum_{r=0}^{m-n-1} \left\{ \frac{(1+n-m)_r \Gamma(\frac{r+n}{2})}{r!\Gamma(\frac{n+r+2-2a}{2})} \right\} ,$$

$$\left(\Re(a) < \left(\tfrac{1+m-n}{2}\right);\ n, a,\ m-a-n,\ m-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \ m-n \in \mathbb{N} \right) .$$

In equation (2), by putting $A = a, B = b$, $C = c$, $D = \frac{1+a+b+c-m}{2}$ and $x = \frac{1}{2}$, using the summation theorem [29; 491, Entry (7.3.7.2)], we obtain

$$F_1\left[a; b, c; \frac{1+a+b+c-m}{2}; \frac{1}{2}, \frac{1}{2}\right] = \frac{2^{a-1}\Gamma(\frac{1+a+b+c-m}{2})}{\Gamma(a)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{1+b+c+r-m}{2})} \right\} ,$$

$$\left(a, b, c,\ \tfrac{1+a+b+c-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ;\ m \in \mathbb{N}_0\right) .$$

In equation (2), by putting $A = a, B = b$, $C = c$, $D = \frac{1+a+b+c+m}{2}$ and $x = \frac{1}{2}$, using the summation theorem [35; 827, Theorems (1)], we find

$$F_1\left[a; b, c; \frac{1+a+b+c+m}{2}; \frac{1}{2}, \frac{1}{2}\right] = \frac{2^{a-1}\Gamma(\frac{1+a+b+c+m}{2})}{\Gamma(a)(\frac{1-a+b+c-m}{2})_m} \sum_{r=0}^{m} \left\{ \binom{m}{r} \frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{1+b+c+r-m}{2})} \right\} ,$$

$$\left(a, b, c,\ \tfrac{1+a+b+c+m}{2},\ \tfrac{1-a+b+c-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ;\ m \in \mathbb{N}_0\right) .$$

In the equation (2), by putting $A = a, B = b, C = c, D = \frac{a+b+c-m}{2}$ and $x = \frac{1}{2}$, using the summation theorem [36; 48, Equation (3.1)], we have

$$F_1\left[a; b, c; \frac{a+b+c-m}{2}; \frac{1}{2}, \frac{1}{2}\right]$$

$$= \frac{2^{a-1}\Gamma(\frac{a+b+c-m}{2})}{\Gamma(a)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \left[\frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{b+c+r-m}{2})} + \frac{\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{b+c+r-m+1}{2})} \right] \right\},$$

$$\left(a, b, c, \tfrac{a+b+c-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\quad m \in \mathbb{N}_0 \right) .$$

In the equation (2), by putting $A = a, B = b$, $C = c$, $D = \frac{a+b+c+m}{2}$ and $x = \frac{1}{2}$, using another summation theorem [36; 48, Equation (3.3)], we get

$$F_1\left[a; b, c; \frac{a+b+c+m}{2}; \frac{1}{2}, \frac{1}{2}\right]$$

$$= \frac{2^{a-1}\Gamma(\frac{a+b+c+m}{2})}{\Gamma(a)(\frac{b+c-a-m}{2})_m} \sum_{r=0}^{m} \left\{ \binom{m}{r} \left[\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{b+c+r-m}{2})} + \frac{(-1)^r\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{b+c+r-m+1}{2})} \right] \right\},$$

$$\left(a, b, c, \tfrac{a+b+c+m}{2}, \tfrac{b+c-a-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\quad m \in \mathbb{N}_0 \right).$$

In equation (2), by putting $A = a, B = b, C = 1-a-b-m, D = d$ and $x = \frac{1}{2}$, using the summation theorem [35; 828, Theorem (6)], we find

$$F_1\left[a;b,1-a-b-m;d;\frac{1}{2},\frac{1}{2}\right] = \frac{\Gamma(d)}{2^{a+m}\Gamma(d-a)}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{d-a+r}{2})}{\Gamma(\frac{d+a+r}{2})}\right\},$$

$$\left(a,b,1-a-b-m,d,\ d-a\in\mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m\in\mathbb{N}_0\right).$$

In equation (2), by putting $A = a, B = b, C = 1-a-b+m, D = d$ and $x = \frac{1}{2}$, using another summation theorem [35; 828, Theorem (5)], we have

$$F_1\left[a;b,1-a-b+m;d;\frac{1}{2},\frac{1}{2}\right] = \frac{\Gamma(d)\Gamma(a-m)}{2^{a-m}\Gamma(a)\Gamma(d-a)}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(-1)^r\Gamma(\frac{d-a+r}{2})}{\Gamma(\frac{d+a+r-2m}{2})}\right\},$$

$$\left(a,b,1-a-b+m,a-m,\ d-a,\ d\in\mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m\in\mathbb{N}_0\right).$$

In equation (2), by putting $A = a, B = b, C = -a-b-m,\ D = d$ and $x = \frac{1}{2}$, using the summation theorem [37; 144, Equation (3.3)], we get

$$F_1[a;b,-a-b-m;d;\frac{1}{2},\frac{1}{2}] = \frac{\Gamma(d)2^{-a-m-1}}{\Gamma(d-a)}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{d-a+r}{2})}{\Gamma(\frac{d+a+r}{2})}+\frac{\Gamma(\frac{d-a+r+1}{2})}{\Gamma(\frac{d+a+r+1}{2})}\right]\right\},$$

$$\left(a,b,-a-b-m,d,\ d-a\in\mathbb{C}\backslash\mathbb{Z}_0^-;\quad m\in\mathbb{N}_0\right).$$

In equation (2), by putting $A = a, B = b, C = -a-b+m, D = d$ and $x = \frac{1}{2}$, using another summation theorem [37; 144, Equation (3.5)], we obtain

$$F_1\left[a;b,-a-b+m;d;\frac{1}{2},\frac{1}{2}\right] = \frac{2^{-a+m-1}\Gamma(d)\Gamma(a-m)}{\Gamma(a)\Gamma(d-a)}\times$$

$$\times\sum_{r=0}^{m}\left\{\binom{m}{r}(-1)^r\left[\frac{\Gamma(\frac{d-a+r}{2})}{\Gamma(\frac{d+a+r-2m}{2})}+\frac{\Gamma(\frac{d-a+r+1}{2})}{\Gamma(\frac{d+a+r+1-2m}{2})}\right]\right\},$$

$$\left(a,b,-a-b+m,\ d,\ a-m,\ d-a\in\mathbb{C}\backslash\mathbb{Z}_0^-;\quad m\in\mathbb{N}_0\right).$$

Remark

By the theory of analytic continuation some convergence conditions associated with each result can be relaxed.

Conclusion

We conclude our present analysis by observing that several interesting summation theorems for Appell function of first kind can be derived in an analogous manner. Moreover, presented summation theorems should be beneficial to those who are interested in the field of applied mathematics and applied physics.

Acknowledgement

The authors are highly thankful to the anonymous referee for their valuable suggestions and comments to improve the paper in present form.

References

1 Srivastava, H.M., & Karlsson, P.W. (1985). *Multiple Gaussian Hypergeometric Series.* Halsted Press (Ellis Horwood Limited, Chichester), Wiley, New York, Chichester, Brisbane, Toronto.

2 Opps, S.B., Saad, N., & Srivastava, H.M. (2005). Some reduction and transformation formulas for the Appell hypergeometric function F_2. *J. Math. Anal.Appl., 302,* 180–195.

3 Lohofer, G. (1989). Theory of an electromagnetically deviated metal sphere. 1: absorbed power. *SIAM J. Appl. Math., 49,* 567–581.

4 Niukkanen, A.W. (1983). Generalised hypergeometric series ${}^{N}F(x_1, \ldots, x_N)$ arising in physical and quantum chemical applications. *J. Phys. A: Math. Gen., 16*, 1813–1825.

5 Mathai, A.M., & Saxena, R.K. (1973). *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences.* Springer-Verlag, Berlin, Heidelberg, New York.

6 Sneddon, I.N. (1980). *Special Functions of Mathematical Physics and Chemistry.* Longman, London, New York.

7 Srivastava, H.M., & Kashyap, B.R.K. (1982). *Special Functions in Queuing Theory and Related Stochastic Processes.* Academic Prees, New York, London, San Francisco.

8 Appell, P., & Kampé de Fériet, J. (1926). *Fonctions Hypergéométriques et Hypersphériques; Polynômes d'Hermite.* Gauthier-Villars, Paris.

9 Appell, P. (1880). Sur une classe de polynômes. *Ann. Sci. École Norm. Sup., 9*(2), 119–144.

10 Appell, P. (1925). Sur les fonctions hypergèometriques de plusieurs variables. *Mèm. des Sciences Math. de Acad. des Sciences de Paris, 3.*

11 Lauricella, G. (1893). Sulle funzioni ipergeometriche a piu variabili. *Rend. Circ. Mat. Palermo, 7,* 111–158.

12 Chaundy, T.W. (1942). Expansions of Hypergeometric Functions. *Quart. J. Math. Oxford Ser., 13*, 159–171.

13 Burchnall, J.L., & Chaundy, T.W. (1940). Expansions of Appell's Double Hypergeometric Functions. *Quart. J. Math. Oxford Ser., 11*, 249–270.

14 Brychkov, Yu.A., & Saad, N. (2014). On some formulas for the Appell function $F_2(a, b, b'; c, c'; w, z)$. *Integral Transforms Spec. Funct., 25*(2), 111–123. http://dx.doi.org/10.1080/10652469.2013.822207

15 Brychkov, Yu.A., & Saad, N. (2012). Some formulas for the Appell function $F_1(a, b, b'; c; w, z)$. *Integral Transforms Spec. Funct., 23*(11), 793–802. http://dx.doi.org/10.1080/10652469.2011.636651

16 Choi, J., Nisar, K.S., Jain, S., & Agarwal, P. (2015). Certain Generalized Appell Type Functions and Their Properties. *Applied Mathematical Sciences, 9*(132), 6567–6581.

17 Nishiyama, S. (1987). Appell's hypergeometric function F_2 and periods of certain elliptic K_3 surfaces. *Tokyo J Math., 10*, 33–68.

18 Opps, S., Saad, N., & Srivastava, H.M. (2009). Recursion formulas for Appell's hypergeometric function F_2 with some applications to radiation field problems. *Appl. Math. Comp., 207*, 545–558.

19 Tarasov, V.F.W. (2003). Gordon's integral (1929) and its representations by means of Appell's functions F_2, F_1, and F_3. *J. Math. Phys., 44*, 1449–1453.

20 Opps, S.B., Saad, N., & Srivastava, H.M. (2005). Some reduction and transformation formulas for the Appell hypergeometric function F_2. *J. Math. Anal.Appl., 302*, 180–195.

21 Brychkov, Yu.A., & Saad, N. (2017). On some formulas for the Appell function $F_4(a, b; c, c'; w, z)$. *Integral Transforms and Special Functions*, 116. https://doi.org/10.1080/10652469.2017.1338276

22 Brychkov, Yu.A., & Saad, N. (2013). On some formulas for the Appell function $F_2(a; b, b'; c, c'; w, z)$. *Integral Transforms Spec Funct., 25*(2), 111–123.

23 Bailey, W.N. (1941). On the double-integral representation of Appell's function F_4. *Quart. J. Math. Oxford, 12*, 12–14.

24 Wang, X. (2012). Recursion formulas for Appell functions. *Integral Transforms Spec. Funct., 23*(6), 421–433.

25 Khan, M.A., & Abukhammash, G.S. (2002). On a generalizations of Appell's functions of two variables. *Pro Mathematica, XVI*(31-32), 61-83.

26 Abramowitz, M., & Stegun, I.A. (1964). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* National Bureau of Standards Applied Mathematics Series, Vol. 55, U.S. Government Printing Office, Washington.

27 Andrews, G.E., Askey, R., & Roy, R. (1999). *Special Functions, Encyclopedia of Mathematics and its Applications*, Vol. 71. Cambridge University Press, Cambridge.

28 Erdélyi, A., Magnus, W., Oberhettinger, F., & Tricomi, F.G. (1953). *Higher Transcendental Functions*, Vol. I. McGraw -Hill Book Company, New York, Toronto and London.

29 Prudnikov, A.P., Brychkov, Yu.A., & Marichev, O.I. (1990). *Integrals and Series, Volume 3: More Special Functions*, Nauka, Moscow, 1986 (In Russian); (Translated from the Russian by G.G.Gould) Gordon and Breach Science Publishers, New York.

30 Rainville, E.D. (1971). *Special Functions.* The Macmillan Company, New York, 1960; Reprinted by Chelsea Publ. Co., Bronx, New York.

31 Srivastava, H.M., & Manocha, H.L. (1984). *A Treatise on Generating Functions.* Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto.

32 Hài, N.T., Marichev, O.I., & Srivastava, H.M. (1992). A note on the convergence of certain families of multiple hypergeometric series. *J. Math. Anal. Appl., 164*, 104–115.

33 Choi, J., Rathie, A.K., & Malani, S. (2007). Kummer's Theorem and its Contiguous Identities. *Taiwanese Journal of Mathematics, 11*(5), 1521–1527.

34 Qureshi, M.I., & Baboo, M.S. (2016a). Some Unified and Generalized Kummer's First Summation Theorems with Applications in Laplace Transform Technique. *Asia Pacific Journal of Mathematics, 3*(1), 10–23.

35 Rakha, M.A., & Rathie, A.K. (2011). Generalizations of Classical Summation Theorems for the Series ${}_2F_1$ and ${}_3F_2$ with Applications. *Integral Transforms and Special Functions, 22*(11), 823–840.

36 Qureshi, M.I., & Baboo, M.S. (2016b). Some Unified and Generalized Kummer's Second Summation Theorems with Applications in Laplace Transform Technique. *International Journal of Mathematics and its Applications, 4* (1-C), 45–52.

37 Qureshi, M.I., & Baboo, M.S. (2016c). Some Unified and Generalized Kummer's Third Summation Theorems with Applications in Laplace Transform Technique. *First International Conference cum Exhibition on Building Utility,* Faculty of Engineering and Technology Jamia Millia Islamia, New Delhi, India, ICEBU, 1–3 , pp. 139–149. Enriched Publications Pvt. Ltd., New Delhi, India.

М.И. Куреши[1], М.С. Бабу[2], А. Ахмад[1]

[1]*Джамия Миллия Исламия (Орталық университет), Нью-Дели, Үндістан;*
[2]*Шарда университетінің Іргелі ғылыми және зерттеулер мектебі, Үлкен Нойда, Үндістан*

$-1, \frac{1}{2}$ аргументтері бар F_1 Апелль функциясына арналған кейбір конвергентті қосындылар теоремалары

Жұмыста параметрлер мен аргументтерді Куммердің бірінші, екінші және үшінші жиынтық теоремаларының жалпыланған түрінде сәйкес келтіру арқылы $-1, \frac{1}{2}$ аргументтері бар F_1 бірінші текті Аппель функциясы үшін гипергеометриялық қосындылар теоремаларының кейбір жабық формалары алынған.

Кілт сөздер: жалпыланған гипергеометриялық функция, бірінші текті Апелл функциясы, Куммердің бірінші, екінші және үшінші жиынтық теоремалары.

М.И. Куреши[1], М.С. Бабу[2], А. Ахмад[1]

[1]*Джамия Миллия Исламия (Центральный университет), Нью-Дели, Индия;*
[2]*Школа фундаментальных наук и исследований Университета Шарда, Большая Нойда, Индия*

Некоторые теоремы о сходящемся суммировании для функции Аппеля F_1 с аргументами $-1, \frac{1}{2}$

В статье мы получаем некоторые замкнутые формы гипергеометрических теорем суммирования для функции Аппеля первого рода F_1 с аргументами $-1, \frac{1}{2}$ с подходящими условиями сходимости путем подгонки параметров и аргументов в обобщенной форме первой, второй и третьей суммирующих теорем Кюммера и других.

Ключевые слова: обобщенная гипергеометрическая функция, функция Аппеля первого рода, первая, вторая и третья теоремы суммирования Кюммера.

DOI 10.31489/2022M3/124-131

UDC 512.54

V.I. Senashov[1,*], I.A. Paraschuk[2]

[1] *Institute of Computational Modelling of SB RAS, Krasnoyarsk, Russia;*
[2] *Siberian Federal University, Krasnoyarsk, Russia*
(E-mail: sen1112home@mail.ru, ivan-ia-95@mail.ru)

On recognizing groups by the bottom layer

The article discusses the possibility of recognizing a group by the bottom layer, that is, by the set of its elements of prime orders. The paper gives examples of groups recognizable by the bottom layer, almost recognizable by the bottom layer, and unrecognizable by the bottom layer. Results are obtained for recognizing a group by the bottom layer in the class of infinite groups under some additional restrictions. The notion of recognizability of a group by the bottom layer was introduced by analogy with the recognizability of a group by its spectrum (the set of orders of its elements). It is proved that all finite simple non-Abelian groups are recognizable by spectrum and bottom layer simultaneously in the class of finite simple non-Abelian groups.

Keywords: group, layer-finite group, bottom layer, spectrum, recognizability.

Introduction

The article discusses the possibility of restoring groups by the bottom layer under additional conditions.

The bottom layer of a group is the set of its elements of prime orders.

A group is called *recognizable by the bottom layer under additional conditions* if it is uniquely reconstructed by the bottom layer under these conditions.

A group G is called *almost recognizable by the bottom layer under additional conditions* if there exists a finite number of pairwise non-isomorphic groups satisfying the same conditions, with the same bottom layer as the group G.

A group G is called *unrecognizable by the bottom layer under additional conditions* if there exists an infinite number of pairwise non-isomorphic groups satisfying the same conditions, with the same bottom layer as the group G.

Many results for groups with a given bottom layer describe some of the properties of the groups. For example, V.D. Mazurov proved that a group with a bottom layer consisting of elements 2, 3, 5, in which all other non-identity elements are of order 4, is locally finite [1]. If the bottom layer of finite group consists of elements of orders 2, 3, 5 and the group contains no non-identity elements of other orders, then W. Shi proved that this is a group of even permutations on five elements [2].

The results on group recognition by the bottom layer were reported at the conferences [3–5] and published in journals [6–8].

Main part

Let us give an example of a group recognizable by the bottom layer in the class of finite groups. If the bottom layer of group G consists of elements of order 2 and the group contains no non-identity elements of other orders, then G is an elementary Abelian 2-group. That is, group G is recognizable by the bottom layer in the class of finite groups.

*Corresponding author.
E-mail: sen1112home@mail.ru

An example of unrecognizability by the bottom layer in the class of finite groups is given by the following infinite series of groups: in the infinite row of the cycle groups of the orders p, p^2, $p^3, \dots$ for some prime p the same bottom layer consisting of $p-1$ elements of order p. In this example, the groups are unrecognizable by the bottom layer in the class of finite groups.

Recall that a group G is called *layer-finite* if it has a finite number of elements of each order. This term was introduced by S.N. Chernikov. The definition of a layer-finite group arose in connection with the study of infinite locally finite p-groups provided that the center of the group $Z(G)$ has a finite index [9].

The groups in the following example are almost recognizable groups by the bottom layer in the class of infinite layer-finite groups. V.P. Shunkov proved that if the bottom layer in an infinite layer-finite group consists of one element of order 2, then the group G is either quasi-cyclic or an infinite generalized quaternion group. The groups from the result of V.P. Shunkov are almost recognizable by the bottom layer in the class of infinite layer-finite groups.

Earlier the recognizability of a complete group with a layer-finite center and a periodic quotient group by it is obtained in the class of infinite groups:

Let G be a complete group in which $Z(G)$ is layer-finite and $G/Z(G)$ is a periodic group. If the bottom layer of group G consists of an element p^{n-1} of order p, then group G is recognizable by the bottom layer in the class of groups satisfying these conditions [6].

Let us recall some results on the recognizability of groups in some classes of groups obtained earlier by the authors.

If G is a complete group in which $Z(G)$ is layer-finite and $G/Z(G)$ is a periodic group containing for each prime p only a finite number of p-elements, then group G is recognizable by the bottom layer among groups with such properties [6].

Definition 1. Layer-finite group is called a *thin layer-finite group* if all of its Sylow subgroups are finite.

Let G be a group in which the center contains a complete layer-finite subgroup R such that the factor group G/R is a thin layer-finite group. The group G is recognizable by the bottom layer among groups with such properties [6].

Let G be a complete nilpotent p-group with a finite bottom layer. Then group G is recognizable by the bottom layer among groups with such properties [6].

Let G be a complete periodic group in which for each prime p there is only a finite number of Sylow p-subgroups and for every prime p there is at least, one Sylow p-subgroup in G, which is a layer-finite group. Then the group is recognizable by the bottom layer among groups with such properties [6].

A complete nilpotent p-group with a finite bottom layer is recognizable by the bottom layer in the class of groups satisfying these conditions [6].

In articles [7, 8], the recognizability by the bottom layer of the complete group is considered under slightly different conditions: layer finiteness of the group or the existence of a layer finite subgroup S in the center of the group G such that G/S is layer finite group. In the same papers, it was proved that a group is recognizable by the bottom layer among locally solvable group without involutions with the minimality condition.

It is convenient to consider the recognition of groups by the bottom layer in the class of layer-finite groups. However, we can also consider other classes of groups.

Now we consider under which conditions it is possible to recognize groups by the bottom layer in the class of infinite groups.

Periodic complete Abelian groups are not necessarily layer-finite. The next theorem establishes the recognizability of a group by the bottom layer in this class of groups.

Theorem 1. Group G is recognizable by the bottom layer among periodic complete Abelian groups.

Proof. Indeed, let group G satisfy the indicated conditions. By Proposition 1, the complete Abelian group G decomposes into a direct sum of subgroups isomorphic to the additive group of rational numbers or to quasi-cyclic groups, possibly for different prime numbers. There can be no rational groups in this extension, since G is a periodic group and, therefore, there are no elements of infinite order in it. Since the direct product of quasi-cyclic groups is restored from the bottom layer, the group G is recognizable by the bottom layer among groups with the properties as in the theorem. The theorem is proved.

Definition 2. A group is called *radically complete* if for any of its elements a and for each natural number n the equation $x^n = a$ has at least one solution in it [10].

Theorem 2. Group G is recognizable by the bottom layer among periodic radically complete groups satisfying the normalizing condition.

Proof. Indeed, let group G satisfy the indicated conditions. By Proposition 2 the elements of finite order of the radically complete group satisfying the normalizing condition G form a complete Abelian group. As G is periodic, such a group G by Theorem 1 is recognizable by the bottom layer among periodic complete Abelian groups. So G is recognizable by the bottom layer among periodic radically complete groups satisfying the normalizing condition. The theorem is proved.

Radically complete groups are not necessaryily layer-finite. For example, direct product of infinite number of quasi-cyclic groups for the same prime number is radically complete, but it is not a layer-finite group.

The notion of recognizability of a group by the bottom layer was introduced by analogy with the recognizability of a group by its spectrum.

The spectrum of a finite group is a set of orders of its elements. A finite group G is called *recognizable by spectrum* if any finite group which has the spectrum coinciding with the spectrum of G is isomorphic to G. A group G is called *almost recognizable by its spectrum* if there are finitely many pairwise non-isomorphic groups with the same spectrum as the group G. A group G is called *spectrum-unrecognizable* if there are infinitely many pairwise non-isomorphic groups with the same spectrum as G.

Results on groups recognizable by spectrum could be found in the works of A.V. Vasil'ev, V.D. Mazurov, A.M. Staroletov, A.A. Buturlakin, M.A. Grechkoseeva, and others [11–21].

An example of a group that is not recognizable by spectrum is group A_6 with the spectrum 2, 3, 4, 5, 8, 9 (there are infinitely many groups, one of which is group A_6) [12]. Also the group $L_3(3)$ with the spectrum 2, 3, 4, 8, 9, 13, 16, 27 is unrecognizable by spectrum [12].

It is proved in [14] that the symmetric groups S_n are recognizable by spectrum for $n \notin \{2, 3, 4, 5, 6, 8, 10, 15, 16, 18, 21, 27, 33, 35, 39, 45\}$. In 1994, W. Shi and R. Brandl proved the recognizability of an infinite series of simple linear groups $L_2(q)$, $q \neq 9$ [15, 16].

A.V. Vasil'ev established the result on the almost spectrum recognition of the group $U_4(5)$ in the class of finite groups:

Let G be a finite simple group $U_4(5)$ and H be a finite group with the property $\omega(H) = \omega(G)$. Then $H \cong G$ or $H \cong G(\gamma)$, where γ is a field automorphism of the group G of order 2. In particular, $h(G) = 2$.

By $h(G)$ it is denoted the number of pairwise non-isomorphic finite groups G with the same spectrum [17].

Thus, the group $U_4(5)$ is almost recognizable by its spectrum in the class of finite groups.

We established previously [6] that the group $U_4(5)$ is recognizable by both the spectrum and the bottom layer in the class of finite groups:

If G is a finite simple group $U_4(5)$, H is a finite group with the property $\omega(H) = \omega(G)$ and the bottom layer is the same as for the group $U_4(5)$, then $H \cong G$. That is, $U_4(5)$ is the only finite group with such a spectrum and such a bottom layer.

Almost all finite simple non-Abelian groups are recognized by their spectrum in the class of finite simple non-Abelian groups. However, there are some exceptions: different groups of this set have the same spectra.

Theorem 3. All finite simple non-Abelian groups are simultaneously recognizable by spectrum and bottom layer in the class of finite simple non-Abelian groups.

Proof. Let us show the possibility of recognizing by the bottom layer such finite simple non-Abelian groups with the same spectrum using the example of the groups $S_6(2)$ and $O_8^+(2)$.

The group $O_8^+(2)$ is simple, has order $174182400 = 2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$. With the help of the GAP package, it was established that there are 69615 involutions (elements of order 2) and 24883200 elements of order 7 in it.

The group $S_6(2)$ has order $1451520 = 2^9 \cdot 3^4 \cdot 5 \cdot 7$. Using the GAP package, it was found that it contains 5103 involutions (elements of order 2) and 207360 elements of order 7.

Thus, the groups $S_6(2)$ and $O_8^+(2)$ have different numbers of elements of the second and seventh orders on the bottom layer and thus are recognized simultaneously by the spectrum and the bottom layer in the class of finite simple non-Abelian groups.

In paper [11], it was established that among the finite simple non-Abelian groups, apart from the groups $S_6(2)$ and $O_8^+(2)$, there is only one more pair of almost unrecognizable by spectrum groups $O_7(3)$ and $O_8^+(3)$.

The first group $O_7(3)$ from this pair is simple non-Abelian, has order $4585351680 = 2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$. Using the GAP package, it was established that there are 38211264 elements of the fifth order in it.

The second considered group $O_8^+(3)$ has the order $4952179814400 = 2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$. Using the GAP package, it was found that it contains 8253633024 elements of the fifth order.

Thus, the groups $O_7(3)$ and $O_8^+(3)$ have different numbers of fifth-order elements in the bottom layer and thus are recognized simultaneously by the spectrum and the bottom layer in the class of finite simple non-Abelian groups. The theorem is proved.

In proving the results of the paper, we used the following theorems, which were referred to as propositions with the corresponding number.

Proposition 1 (Theorem 9.1.6 from [22]). A nonzero complete Abelian group can be decomposed into a direct sum of subgroups isomorphic to the additive rational group or quasi-cyclic groups, may be for different prime numbers.

Proposition 2 (Theorem 2.8 from [10]). If a radically complete group satisfies the normalizing condition, then the elements of its finite order form a complete Abelian subgroup.

Conclusion

The possibilities of recognizing of some finite and infinite layer-finite groups by the bottom layer are considered. Results are obtained for recovering groups by the bottom layer in the class of infinite groups with some additional conditions. It is proved that all simple non-Abelian groups are simultaneously recognizable by spectrum and bottom layer in the class of finite simple non-Abelian groups.

Acknowledgments

The authors are grateful to A.V. Timofeenko for help in working with the GAP package.

This work was supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation (Agreement No. 075-02-2022-876).

References

1 Мазуров В.Д. О группах экспоненты 60 с точными порядками элементов / В.Д. Мазуров // Алгебра и логика. — 2000. — 9. — № 3. — С. 189–198.

2 Shi W. A characteristic property of A_5 / W. Shi // J. Southwest-China Teachers Univ. — 1986. — 3. — P. 11–14.

3 Сенашов В.И. О распознавании групп по нижнему слою / В.И. Сенашов, И.А. Паращук // Сб. материалов V Междунар. науч.-практ. конф. «Актуальные проблемы авиации и космонавтики», посвящ. Дню космонавтики. — 2019. — 2. — С. 291–293.

4 Сенашов В.И. О влиянии нижнего слоя группы на строение группы в различных классах групп / В.И. Сенашов, И.А. Паращук // Сб. материалов VI Междунар. науч.-практ. конф. «Актуальные проблемы авиации и космонавтики», посвящ. Дню космонавтики. — 2020. — 2. — С. 293–295.

5 Сенашов В.И. О распознавании слойно-конечных групп по нижнему слою / В.И. Сенашов, И.А. Паращук // Сб. материалов VII Междунар. науч.-практ. конф. «Актуальные проблемы авиации и космонавтики», посвящ. Дню космонавтики. — 2021. — 2. — С. 460–463.

6 Senashov V.I. On a bottom layer in a group / V.I. Senashov, I.A. Paraschuk // Bulletin of the Karaganda University-Mathematics. — 2020. — 100. — 4. — P. 136–142.

7 Сенашов В.И. Восстановление информации о группе по нижнему слою / В.И. Сенашов, И.А. Паращук // Сиб. журн. науки и технологий. — 2018. — 19. — 2. — С. 223–226.

8 Сенашов В.И.Восстановление группы по нижнему слою / В.И. Сенашов, И.А. Паращук// Вестн. Тув. гос. ун-та. — 2018. — 3. — С. 114–118.

9 Черников С.Н. К теории бесконечных специальных p-групп / С.Н. Черников // Докл. АН СССР. — 1945. — 50. — 1. — С. 71–74.

10 Черников С.Н. Группы с заданными свойствами системы подгрупп / С.Н. Черников. — М.: Наука, 1980. — 384 с.

11 Бутурлакин А.А. Изоспектральные конечные простые группы / А.А.Бутурлакин // Сиб. электрон. матем. изв. — 2010. — 7. — С. 111–114.

12 Васильев А.В. Почти распознаваемость по спектру простых исключительных групп лиева типа / А.В. Васильев, А.М. Старолетов // Алгебра и логика. — 2014. — 53. — 6. — С. 669–692.

13 Бутурлакин А.А. О конструктивном распознавании конечных простых групп по порядкам их элементов / А.А. Бутурлакин, А.В. Васильев // Алгебра и логика. — 2014. — 53. — № 4. — С. 541–544.

14 Горшков И.Б. Распознавание симметрических групп по спектру / И.Б. Горшков // Алгебра и логика. — 2014. — 53. — 6. — С. 693–703.

15 Shi W. A characteristic property of J_1 and $PSL_2(2^n)$ / W. Shi // Adv. Math. — 1987. — 16. — P. 397–401.

16 Brandl R. The characterization of $PSL(2, q)$ by its element orders / R. Brandl, W. Shi // J. Algebra. — 1994. — 163. — 1. — P. 109–114.

17 Васильев А.В. О распознавании всех конечных неабелевых простых групп, простые делители порядков которых не превосходят 13 / А.В. Васильев // Сиб. мат. журн. — 2005. — 46. — № 2. — С. 315–324.

18 Васильев А.В. Распознаваемость по спектру для простых классических групп в характеристике 2 / А.В. Васильев, М.А. Гречкосеева // Сиб. мат. журн. — 2015. — 56. — № 6. — С. 1264–1276.

19 Гречкосеева М.А. Почти распознаваемость по спектру конечных простых линейных групп простой размерности / М.А. Гречкосеева, Д.В. Лыткин // Сиб. мат. журн. — 2012. — 53. — 4. — С. 805—818.

20 Мазуров В.Д. Характеризация конечных групп множествами порядков их элементов / В.Д. Мазуров // Алгебра и логика. — 1997. — 36. — № 1. — С. 37–53.

21 Мазуров В.Д. Нераспознаваемость конечной простой группы ${}^3D_4(2)$ по спектру / В.Д. Мазуров // Алгебра и логика. — 2013. — 52. — № 5. — С. 601–605.

22 Каргаполов М.И. Основы теории групп. 3-е изд./ М.И. Каргаполов, Ю.И. Мерзляков. — М.: Наука, 1982. — 288 с.

В.И. Сенашов[1], И.А. Паращук[2]

[1] *РҒА СБ Есептік модельдеу институты, Красноярск, Ресей;*
[2] *Сібір федералды университеті, Красноярск, Ресей*

Төменгі қабат бойынша группаларды тану туралы

Мақалада группаны төменгі қабаттан, яғни оның элементтерінің жай реттері жиынын қалпына келтіру мүмкіндігі қарастырылған. Жұмыста төменгі қабат арқылы танылатын, төменгі қабат арқылы дерлік танылатын және төменгі қабат арқылы танылмайтын топтардың мысалдары келтірілген. Шексіз группалар класындағы төменгі қабаттан группаны қайта құру нәтижелері кейбір қосымша шектеулер бойынша алынды. Төменгі қабат бойынша группаны тану түсінігі спектр бойынша группаларды тануға (оның элементтерінің қатарларының жиынтығы) ұқсас енгізілді. Барлық ақырлы жай абельдікемес группалардың спектрі және төменгі қабаты бойынша танылуы ақырлы қарапайым абельдікемес группалар класында бір уақытта дәлелденген.

Кілт сөздер: группа, қабатты ақырлы группа, төменгі қабат, спектр, танымдылық.

В.И. Сенашов[1], И.А. Паращук[2]

[1] *Институт вычислительного моделирования СО РАН, Красноярск, Россия;*
[2] *Сибирский федеральный университет, Красноярск, Россия*

О распознавании групп по нижнему слою

В статье обсуждена возможность восстановления группы по нижнему слою, то есть по множеству её элементов простых порядков. Приведены примеры распознаваемых по нижнему слою, почти распознаваемых по нижнему слою и нераспознаваемых по нижнему слою групп. Получены результаты восстановления группы по нижнему слою в классе бесконечных групп при некоторых дополнительных ограничениях. Понятие распознаваемости группы по нижнему слою введено по аналогии с распознаваемостью группы по спектру (множеству порядков её элементов). Доказана распознаваемость всех конечных простых неабелевых групп по спектру и нижнему слою одновременно в классе конечных простых неабелевых групп.

Ключевые слова: группа, слойно конечная группа, нижний слой, спектр, распознаваемость, конечные простые неабелевы группы.

References

1 Mazurov, V.D. (2000). O gruppakh eksponenty 60 s tochnymi poriadkami elementov [On groups of exponent 60 with exact orders of elements]. *Algebra i logika — Algebra and Logic*, 39, 3, 189–198 [in Russian].

2 Shi, W. (1986). A characteristic property of A_5. *J. Southwest-China Teachers Univ.*, 3, 11–14.

3 Senashov, V.I., & Parashchuk, I.A. (2019). O raspoznavanii grupp po nizhnemu sloiu [On the recognition of groups by the bottom layer]. *Proceedings from «Actual problems of aviation and cosmonautics» dedicated to Cosmonautics Day: V Mezhdunarodnaia nauchno-prakticheskaia konferentsiia (2019 goda) — 5th International Scientific and Practical Conference, 2* (pp. 291–293) [in Russian].

4 Senashov, V.I., & Parashchuk, I.A. (2020). O vliianii nizhnego sloia gruppy na stroenie gruppy v razlichnykh klassakh grupp [On the influence of the bottom layer of the group on the structure of the group in different classes of groups]. *Proceedings from «Actual problems of aviation and cosmonautics» dedicated to Cosmonautics Day: VI Mezhdunarodnaia nauchno-prakticheskaia konferentsiia — 6th International Scientific and Practical Conference, 2*, (pp. 293–295) [in Russian].

5 Senashov, V.I., & Parashchuk, I.A. (2021). O raspoznavanii sloino-konechnykh grupp po nizhnemu sloiu [On recognition of layered finite groups by the bottom layer]. *Proceedings from «Actual problems of aviation and astronautics» dedicated to Cosmonautics Day: VII Mezhdunarodnaia nauchno-prakticheskaia konferentsiia (2021 goda) — 7th International Scientific and Practical Conference, 2*, (pp. 460–46)3 [in Russian].

6 Senashov, V.I., & Paraschuk, I.A. (2020). On a bottom layer in a group. *Bulletin of the Karaganda University-Mathematics*, 100, 4, 136–142.

7 Senashov V.I., & Parashchuk, I.A. (2018a). Vosstanovlenie informatsii o gruppe po nizhnemu sloiu [Restoration of information about the group by the bottom layer]. *Sibirskii zhurnal nauki i tekhnologii — Siberian Journal of Science and Technology, 19*, 2, 223–226 [in Russian].

8 Senashov, V.I., & Parashchuk, I.A. (2018b). Vosstanovlenie gruppy po nizhnemu sloiu [Restoration of the group by the bottom layer]. *Vestnik Tuvinskogo gosudarstvennogo universiteta — Bulletin of the Tuva State University,* 3, 114–118 [in Russian].

9 Chernikov, S.N. (1945). K teorii beskonechnykh spetsialnykh p-grupp [On the theory of infinite special p-groups]. *Doklady AN SSSR — Reports of the Academy of Sciences of the USSR, 50*, 1, 71–74 [in Russian].

10 Chernikov, S.N. (1980). *Gruppy s zadannymi svoistvami sistemy podgrupp [Groups with given properties of a system of subgroups].* Moscow: Nauka [in Russian].

11 Buturlakin, A.A. (2010). Izospektralnye konechnye prostye gruppy [Isospectral finite simple groups]. *Sibirskiie Electronnye Mathematicheskie Izvestiia — Siberian Electronic Mathematical Reports, 7*, 111–114 [in Russian].

12 Vasilev, A.V., & Staroletov, A.M. (2014). Pochti raspoznavaemost po spektru prostykh iskliuchitelnykh grupp lieva tipa [Almost recognizability by spectrum of simple exceptional groups of Lie type]. *Algebra i logika — Algebra and Logic, 53*, 6, 669–692 [in Russian].

13 Buturlakin, A.A., & Vasilev, A.V. (2014). O konstruktivnom raspoznavanii konechnykh prostykh grupp po poriadkam ikh elementov [On constructive recognition of finite simple groups by the orders of their elements]. *Algebra i logika — Algebra and Logic, 53*, 4, 541–544 [in Russian].

14 Gorshkov, I.B. (2014). Raspoznavanie simmetricheskikh grupp po spektru [Recognition of symmetric groups by spectrum]. *Algebra i logika — Algebra and Logic, 53*, 6, 693–703 [in Russian].

15 Shi, W. (1987). A characteristic property of J_1 and $PSL_2(2^n)$. *Adv. Math.*, 16, 397–401.

16 Brandl, R., & Shi, W. (1994). The characterization of $PSL(2,q)$ by its element orders. *J. Algebra*, 163, 1, 109–114.

17 Vasilev, A.V. (2005). O raspoznavanii vsekh konechnykh neabelevykh prostykh grupp, prostye deliteli poriadkov kotorykh ne prevoskhodiat 13 [On the recognition of all finite non-Abelian simple groups whose prime divisors of orders do not exceed 13]. *Sibirskii Mathematicheskii Zhurnal — Siberian Mathematical Journal, 46*, 2, 315–324 [in Russian].

18 Vasilev, A.V., & Grechkoseeva, M.A. (2015). Raspoznavaemost po spektru dlia prostykh klassicheskikh grupp v kharakteristike 2 [Recognizability by spectrum for simple classical groups in characteristic 2]. *Sibirskii Mathematicheskii Zhurnal — Siberian Mathematical Journal, 56*, 1264–1276 [in Russian].

19 Grechkoseeva, M.A., & Lytkin, D.V. (2012). Pochti raspoznavaemost po spektru konechnykh prostykh lineinykh grupp prostoi razmernosti [Almost recognizability by spectrum of finite simple linear groups of prime dimension]. *Sibirskii Mathematicheskii Zhurnal — Siberian Mathematical Journal, 53*, 4, 805–818 [in Russian].

20 Mazurov, V.D. (1997). Kharakterizatsiia konechnykh grupp mnozhestvami poriadkov ikh elementov [Characterization of finite groups by sets of orders of their elements]. *Algebra i logika — Algebra and Logic, 36*, 1, 37–53 [in Russian].

21 Mazurov, V.D. (2013). Neraspoznavaemost konechnoi prostoi gruppy ${}^3D_4(2)$ po spektru [Unrecognizability of a finite simple group ${}^3D_4(2)$ by spectrum]. *Algebra i logika — Algebra and Logic, 52*, 5, 601–605 [in Russian].

22 Kargapolov, M.I., & Merzlyakov, Yu.I. (1982). *Osnovy teorii grupp [Foundations of group theory]*. (3ed ed.). Moscow: Nauka [in Russian].

DOI 10.31489/2022M3/132-144

UDC 517.9

S. Shaimardan[1,*] , N.S. Tokmagambetov[1,2], Y. Aikyn[1]

[1]*L.N. Gumilyev Eurasian National University, Nur-Sultan, Kazakhstan;*
[2]*Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan*
(E-mail: shaimardan.serik@gmail.com, nariman.tokmagambetov@gmail.com, aikynyergen@gmail.com)

The Bessel equation on the quantum calculus

A large number of the most diverse problems related to almost all the most important branches of mathematical physics and designed to answer topical technical questions are associated with the use of Bessel functions. This paper introduces a h-difference equation analogue of the Bessel differential equation and investigates the properties of its solution, which is express using the Frobenius method by assuming a generalized power series. The authors find discrete analogue formulas for Bessel function and the h-Neumann function and these are solutions presented by a series with the h-fractional function $t_h^{(\alpha)}$. Lastly they obtain the linear dependencies between h-functions Bessel on T_a.

Keywords: Bessel function, modified Bessel function, Bessel difference equation, h-calculus, the h-derivative and h-fractional function.

1 Introduction and Preliminary

Now a days, the theory of transformation operators is a fully formed independent branch of mathematics, located at the junction of differential, integral, and integro-differential equations, functional analysis, function theory, complex analysis, the theory of special functions and fractional integro-differentiation, the theory of inverse problems and scattering problems, the theory of optimal control and dynamic systems. The special area of application of transformation operators has become the theory of differential equations with singularities in the coefficients, especially with Bessel operators.

The Bessel functions are widely used in solving problems in acoustics, radiophysics, hydrodynamics, problems of atomic and nuclear physics. There are numerous applications of Bessel functions to the theory of heat conduction and the theory of elasticity (problems of vibrations of plates, problems of the theory of shells, problems of determining the stress concentration near cracks) [1–5].

The theory of fractional h-calculus is a rapidly developing field of great interest from both a theoretical and an applied point of view. Especially we refer to [6–12] and the references in it. As for applications in various fields of mathematics, we refer to [13–20] and references in them. Let $h > 0$ and $T_a = \{a, a+h, a+2h...\}$, $\forall a \in \mathbb{R}$.

Definition 1. (see [9]) Let $f : T_a \to \mathbb{R}$. Then the h-derivative of the function $f = f(t)$ has the form and is defined as

$$D_h f\left(t\right)\left(x\right) = \frac{f(\delta_h(t)) - f(t)}{h}, t \in T_a, \tag{1}$$

where $\delta_h(t) = t + h$.

We assume $f \cdot g : T_a \to \mathbb{R}$. Then the product rule for h-derivation reads (see [9])

$$D_h\left(f\left(x\right)g(x)\right) = f(x+h)D_h g(x) + g(x)D_h f(x) \tag{2}$$

*Corresponding author.
E-mail: shaimardan.serik@gmail.com

and the h-integral (or the h-difference sum) is given by

$$\int_a^x f(t)d_h t = \sum_{k=a/h}^{x/h-1} f(kh)h, \quad x \in T_a. \tag{3}$$

Definition 2. (see [9]) Let $t, \alpha \in \mathbb{R}$. Then the h-fractional function $t_h^{(\alpha)}$ is defined as

$$t_h^{(\alpha)} = h^\alpha \frac{\Gamma\left(\frac{t}{h}+1\right)}{\Gamma\left(\frac{t}{h}+1-\alpha\right)},$$

where Γ is the gamma function of Euler, $\frac{t}{h} \geq 0$ and we use the convention that division at the pole gives zero. Notice that

$$\lim_{h\to 0} t_h^{(\alpha)} = t^\alpha.$$

Hence, from (1) we find that

$$t^{\alpha-1} = \frac{1}{\alpha} D_h \left[t_h^{(\alpha)}\right].$$

Let $t \in T_0$. Then, for $\alpha, \beta \in \mathbb{R}$,

$$t_h^{(\alpha+\beta)} = t_h^{(\alpha)}(t-\alpha h)_h^{(\beta)}, \tag{4}$$

Definition 3. (Foundamental theorem h-calculus) If $F(x)$ is an h-antiderivative of $f(x)$ is continus at $x = 0$, we get

$$\int_a^b f(x)d_h x = F(b) - F(a),$$

for $a, b \in T_a$.

2 The Bessel equation. Bessel functions.

2.1 The Bessel differential equation. We consider the h-difference equation in the following form:

$$t_h^{(2)} D_h^2 y(t-2h) + t_h^{(1)} D_h y(t-h) + t_h^{(2)} y(t-2h) - v^2 y(t) = 0 \tag{5}$$

which is called the h-Bessel equation of the indicator in v, where v is a real number. This equation has a special point $t = 0$ (the coefficient at the highest derivative in (5) vanishes at $t = 0$).

Theorem 2.1. Let $v \leq 0$. Then there is a particular solution to equation (5), given by a uniformly convergent series

$$J_{v,h}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{v+2k}}{k!\Gamma(v+k+1)2^{v+2k}} \tag{6}$$

which is the solution to the Bessel equation and is called the Bessel function of the first kind v-th order.

Proof. Following the classical methods (see, for example, [6], p. 379), we will look for a solution to this equation in the form of a series. Therefore, there is a solution to equation (5) in the form of a generalized power series

$$y(t) = t^{\alpha} \sum_{k=0}^{\infty} a_k (t - \alpha h)_h^{(k)}, a_0 \neq 0, \tag{7}$$

where α is the characteristic indicator to be determined. By (4) we can rewrite the expression (6) in the form

$$y(t) = \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k)}$$

and using Definition 2 and (1) we find the h-derivatives:

$$\begin{aligned} D_h^2 y(t-2h) &= D_h^2 \sum_{k=0}^{\infty} a_k (t-2h)_h^{(\alpha+k)} \\ &= (\alpha+k)(\alpha+k-1) \sum_{k=0}^{\infty} a_k (t-2h)_h^{(\alpha+k-2)} \end{aligned}$$

and

$$\begin{aligned} D_h y(t-h) &= D_h \sum_{k=0}^{\infty} a_k (t-h)_h^{(\alpha+k)} \\ &= (\alpha+k) \sum_{k=0}^{\infty} a_k (t-h)_h^{(\alpha+k-1)}. \end{aligned}$$

Therefore, substituting (7) and its first and second h-derivatives into the equation (5), we get that

$$t_h^{(2)}(\alpha+k)(\alpha+k-1) \sum_{k=0}^{\infty} a_k (t-2h)_h^{(\alpha+k-2)} + t_h^{(1)}(\alpha+k) \sum_{k=0}^{\infty} a_k (t-h)_h^{(\alpha+k-1)} +$$

$$t_h^{(2)} \sum_{k=0}^{\infty} a_k (t-2h)_h^{(\alpha+k)} - v^2 \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k)} = 0$$

so we can rewrite the equation:

$$(\alpha+k)(\alpha+k-1) \sum_{k=0}^{\infty} a_k t_h^{(2)} (t-2h)_h^{(\alpha+k-2)} + (\alpha+k) \sum_{k=0}^{\infty} a_k t_h^{(1)} (t-h)_h^{(\alpha+k-1)} +$$

$$\sum_{k=0}^{\infty} a_k t_h^{(2)} (t-2h)_h^{(\alpha+k)} - \sum_{k=0}^{\infty} a_k v^2 t_h^{(\alpha+k)} = 0$$

where $t_h^{(2)}(t-2h)_h^{(\alpha+k-2)} = t_h^{\alpha+k}$ and $t_h^{(1)}(t-h)_h^{(\alpha+k-1)} = t_h^{\alpha+k}$.

From here we get a general formula for all these series.

$$(\alpha+k)(\alpha+k-1) \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k)} + (\alpha+k) \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k)} + \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k+2)} - \sum_{k=0}^{\infty} a_k v^2 t_h^{(\alpha+k)} = 0$$

and

$$\sum_{k=0}^{\infty} a_k \left((\alpha+k)(\alpha+k-1)+(\alpha+k)-v^2\right) t_h^{(\alpha+k)} + \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k+2)} = 0$$

and, finally

$$\sum_{k=0}^{\infty} a_k \left((\alpha+k)^2-v^2\right) t_h^{(\alpha+k)} + \sum_{k=0}^{\infty} a_k t_h^{(\alpha+k+2)} = 0.$$

Next, for $\alpha, \alpha+1, ..., \alpha+k, ...,$, we are equating to zero the coefficients at the same powers of x, we lead at the following recurrent relations for the coefficients:

$$\begin{cases} t^{\alpha} \quad |(\alpha^2-v^2)a_0 = 0, \\ t^{\alpha+1}| \left((\alpha+1)^2-v^2\right) a_1 = 0, \\ t^{\alpha+2}| \left((\alpha+2)^2-v^2\right) a_2 + a_0 = 0, \\ \vdots \quad\quad |.. \\ t^{\alpha+k}| \left((\alpha+k)^2-v^2\right) a_k + a_{k-2} = 0, \ \ \forall k \geq 2. \end{cases} \tag{8}$$

Since $a_0 \neq 0$, it follows from the first equation (8) that $\alpha^2 - v^2 = 0$, or $\alpha = \pm v$. Now from the second equation (8) we will have $a_1 = 0$.

Let us consider the case $\alpha = v > 0$ first. Let us rewrite the $k - th$ $(k > 1)$ equation of system (8) in the following form

$$a_k = \frac{-a_{k-2}}{k(2v+k)}.$$

Considering that $a_1 = 0$, we get from here $a_3 = 0$ and $a_{2k+1} = 0$ in general. On the other hand, each even coefficient can be expressed in terms of the previous one by the formula

$$a_{2k} = -\frac{a_{2k-2}}{2^2 k(v+k)}.$$

Consistent application of this formula allows us to find an expression a_{2k} through a_0 :

$$\begin{aligned} a_2 \quad &= \quad -\frac{a_0}{2^2 \cdot 1 \cdot (v+1)} \\ &\Rightarrow \quad a_4 = -\frac{a_2}{2^2 \cdot 2 \cdot (v+2)} = \frac{a_0}{2^4 \cdot 1 \cdot 2 \cdot (v+1)(v+2)} \\ &\Rightarrow \quad a_6 = -\frac{a_4}{2^2 \cdot 3 \cdot (v+3)} = -\frac{a_0}{2^6 \cdot 3! \cdot (v+1)(v+2)(v+3)} \\ &\cdots \quad \cdots \quad \cdots \\ &\Rightarrow \quad a_{2k} = \frac{(-1)^k a_0}{2^{2k} k! \prod\limits_{r=1}^{k} (v+r)}. \end{aligned}$$

The coefficient a_0 has so far been left arbitrary. If $v \neq -n$, where $n > 0$ is an integer, then assuming

$$a_0 = \frac{1}{2^v \Gamma(v+1)}$$

we find

$$\begin{aligned} a_{2k} \quad &= \quad \frac{(-1)^k}{2^{2k} k!(v+1)(v+2)(v+3)...(v+k)} \cdot \frac{1}{2^v \Gamma(v+1)} \\ &= \quad \frac{(-1)^k}{2^{2k+v} k! \Gamma(v+k+1)}. \end{aligned}$$

Substituting this expression for the coefficients in (7), we get

$$y_1(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{v+2k}}{k!\Gamma(v+k+1)2^{v+2k}}.$$

The proof is complete.

Corollary 2.3. The equation (5) does not change when v is replaced by $-v$, then the function:

$$J_{-v,h}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{-v+2k}}{k!\Gamma(-v+k+1)2^{-v+2k}} \tag{9}$$

is also a solution to the equation (5).

Theorem 2.4. If $v \neq n$. Then the general solution to equation (5) has the form:

$$y(t) = C_1 J_{v,h}(t) + C_2 J_{-v,h}(t). \tag{10}$$

Proof. Now we prove that $y(t)$ in the following form is also a solution to equation (8):

$$\begin{aligned} y(t) &= C_1 J_{v,h}(t) + C_2 J_{-v,h}(t) \\ &= C_1 \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}} \\ &+ C_2 \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}}. \end{aligned}$$

Using (1) to find the h-derivatives from the formula (10):

- $$t_h^{(2)} D_h^2 \left(C_1 J_{v,h}(t-2h) + C_2 J_{-v,h}(t-2h)\right) = C_1 \sum_{k=0}^{\infty} \frac{(-1)^k (v+2k)(v+2k-1) t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}} + C_2 \sum_{k=0}^{\infty} \frac{(-1)^k (-v+2k)(-v+2k-1) t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}},$$
- $$t_h D_h \left(C_1 J_{v,h}(t-h) + C_2 J_{-v,h}(t-h)\right) = C_1 \sum_{k=0}^{\infty} \frac{(-1)^k (v+2k) t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}} + C_2 \sum_{k=0}^{\infty} \frac{(-1)^k (-v+2k) t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}},$$
- $$t_h^{(2)} \left(C_1 J_{v,h}(t-2h) + C_2 J_{-v,h}(t-2h)\right) = C_1 \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(v+2k+2)}}{k!\Gamma(v+k+1)2^{v+2k}} + C_2 \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(-v+2k+2)}}{k!\Gamma(-v+k+1)2^{-v+2k}},$$
- $$-v^2 \left(C_1 J_{v,h}(t) + C_2 J_{-v,h}(t)\right) = -v^2 C_1 \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}} + C_2 \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}}.$$

Now we substitute in equation (10):

$$C_1\left(\sum_{k=0}^{\infty}\frac{(-1)^k(v+2k)(v+2k-1)t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}}+\sum_{k=0}^{\infty}\frac{(-1)^k(v+2k)t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}}\right)+$$
$$+C_1\left(\sum_{k=0}^{\infty}\frac{(-1)^k t_h^{(v+2k+2)}}{k!\Gamma(v+k+1)2^{v+2k}}-v^2\sum_{k=0}^{\infty}\frac{(-1)^k t_h^{(v+2k)}}{k!\Gamma(v+k+1)2^{v+2k}}\right)+$$
$$+C_2\left(\sum_{k=0}^{\infty}\frac{(-1)^k(-v+2k)(-v+2k-1)t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}}+\sum_{k=0}^{\infty}\frac{(-1)^k(-v+2k)t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}}\right)+$$
$$+C_2\left(\sum_{k=0}^{\infty}\frac{(-1)^k t_h^{(-v+2k+2)}}{k!\Gamma(-v+k+1)2^{-v+2k}}-v^2\sum_{k=0}^{\infty}\frac{(-1)^k t_h^{(-v+2k)}}{k!\Gamma(-v+k+1)2^{-v+2k}}\right)=0.$$

If $C_1=-C_2$ then $y(t)=C_1J_{v,h}(t)+C_2J_{-v,h}(t)$ is a solution to the equation (5). The proof is complete.

Example 2.5. Find a general solution to the following equation:

$$t_h^{(2)}D_h^2y(t-2h)+t_h^{(1)}D_hy(t-h)+t_h^{(2)}y(t-2h)-2y(t)=0. \tag{11}$$

Proof. We consider two cases $v=1/2$ and $v=-1/2$. 1) According to the definition (see (6)) of the Bessel function $J_{\frac{1}{2},h}(t)$ we have:

$$J_{\frac{1}{2},h}(t)=\sum_{k=0}^{\infty}\frac{(-1)^k t_h^{\left(\frac{1}{2}+2k\right)}}{k!\Gamma\left(\frac{1}{2}+k+1\right)2^{\frac{1}{2}+2k}}.$$

Since

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^{\infty}e^{-t}t^{-\frac{1}{2}}dt=\int_0^{\infty}e^{-t}d(2\sqrt{t})=2\int_0^{\infty}e^{-\xi^2}d(2\sqrt{\xi})=\sqrt{\pi},\\ \Gamma(t+1) &= t\Gamma(t),\quad t>0,\end{aligned}$$

then

$$\begin{aligned}k!\Gamma\left(k+\frac{3}{2}\right) &= \Gamma(k+1)\Gamma\left(k+1+\frac{1}{2}\right)\\ &= k\left(k+\frac{1}{2}\right)\cdot\frac{2^{2k-1}}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2^{2k-1}}\cdot\Gamma(k)\Gamma\left(k+\frac{1}{2}\right)\\ &= \frac{\sqrt{\pi}}{2^{2k-1}}k\left(k+\frac{1}{2}\right)\Gamma(2k).\end{aligned}$$

Considering also that $\Gamma(k+1)=k!$ for $k\in\mathbb{N}$, we get

$$\begin{aligned}J_{\frac{1}{2},h}(t) &= \sum_{k=0}^{\infty}\frac{(-1)^k t_h^{\left(\frac{1}{2}+2k\right)}}{k!\Gamma\left(\frac{3}{2}+k\right)2^{\frac{1}{2}+2k}}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k t_h^{\left(-\frac{1}{2}\right)}\left(t+\frac{1}{2}h\right)^{(2k+1)}}{k!\Gamma\left(\frac{3}{2}+k\right)2^{\frac{1}{2}+2k}}\\ &= t_h^{\left(-\frac{1}{2}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k\left(t+\frac{1}{2}h\right)^{(2k+1)}}{\frac{\sqrt{\pi}}{2^{2k-1}}k\left(k+\frac{1}{2}\right)\Gamma(2k)2^{\frac{1}{2}+2k}}\\ &= \frac{\sqrt{2}t_h^{\left(-\frac{1}{2}\right)}}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k\left(t+\frac{1}{2}h\right)_h^{(2k+1)}}{(2k+1)!}.\end{aligned}$$

The row on the right hand side of the last equality represents the decomposition of the function $sin_h t$. Therefore, the following equality is true

$$J_{\frac{1}{2},h}(t) = \frac{\sqrt{2}t_h^{(-\frac{1}{2})}}{\sqrt{\pi}} sin_h\left(t+\frac{1}{2}h\right). \quad (12)$$

Now, using (1), (6), and (12) we find the h-derivatives of $sin_h t$:

$$\begin{aligned} D_h sin_h\left(t+\frac{1}{2}h\right) &= D_h \sum_{k=0}^{\infty} \frac{(-1)^k \left(t+\frac{1}{2}h\right)_h^{(2k+1)}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1) \left(t+\frac{1}{2}h\right)_h^{(2k)}}{(2k)!(2k+1)} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \left(t+\frac{1}{2}h\right)_h^{(2k)}}{(2k)!} \\ &= cos_h\left(t+\frac{1}{2}h\right). \end{aligned}$$

2) Let us now consider the case when $v = -\frac{1}{2}$. By using (9) we have that

$$J_{-\frac{1}{2},h}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(-\frac{1}{2}+2k)}}{k!\Gamma\left(-\frac{1}{2}+k+1\right) 2^{-\frac{1}{2}+2k}}.$$

Taking into account that

$$\begin{aligned} k!\Gamma\left(k-\frac{1}{2}+1\right) &= k\Gamma(k)\Gamma\left(k+\frac{1}{2}\right) \\ &= k\left(k+\frac{1}{2}\right) \cdot \frac{2^{2k-1}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2^{2k-1}} \cdot \Gamma(k)\Gamma\left(k+\frac{1}{2}\right) \\ &= \frac{\sqrt{\pi}}{2^{2k}} 2k\Gamma(2k), \end{aligned}$$

we get that

$$\begin{aligned} J_{-\frac{1}{2},h}(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(-\frac{1}{2}+2k)}}{k!\Gamma\left(k+\frac{1}{2}\right) 2^{-\frac{1}{2}+2k}} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k t_h^{(-\frac{1}{2})} \left(t+\frac{1}{2}h\right)_h^{(2k)}}{k!\Gamma\left(k+\frac{1}{2}\right) 2^{-\frac{1}{2}+2k}} \\ &= t_h^{(-\frac{1}{2})} \sum_{k=0}^{\infty} \frac{(-1)^k \left(t+\frac{1}{2}h\right)_h^{(2k)}}{k\Gamma(k)\Gamma\left(k+\frac{1}{2}\right) 2^{-\frac{1}{2}+2k}} \\ &= t_h^{(-\frac{1}{2})} \sum_{k=0}^{\infty} \frac{(-1)^k \left(t+\frac{1}{2}h\right)_h^{(2k)}}{\frac{\sqrt{\pi}}{2^{2k}} 2k\Gamma(2k) 2^{-\frac{1}{2}+2k}} \\ &= \frac{\sqrt{2}t_h^{(-\frac{1}{2})}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(t+\frac{1}{2}h\right)_h^{(2k)}}{2k!}. \end{aligned}$$

The row on the right side of the last equality is a function $\cos_h t$. Therefore,

$$J_{-\frac{1}{2},h}(t) = \frac{\sqrt{2}t_h^{-\frac{1}{2}}}{\sqrt{\pi}} \cos_h \left(t + \frac{1}{2}h\right). \tag{13}$$

Now, using (1) we find the h-derivatives of:

$$\begin{aligned} D_h \cos_h \left(t + \frac{1}{2}k\right) &= D_h \sum_{k=0}^{\infty} \frac{(-1)^k \left(t + \frac{1}{2}h\right)_h^{(2k)}}{2k!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k 2k \left(t + \frac{1}{2}h\right)_h^{(2k-1)}}{2k!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k \left(t + \frac{1}{2}h\right)_h^{(2k-1)}}{(2k-1)!} \\ &= -\sum_{n=0}^{\infty} \frac{(-1)^n \left(t + \frac{1}{2}h\right)_h^{(2n+1)}}{(2n+1)!} \\ &= -sin_h(t + \frac{1}{2}h). \end{aligned}$$

According to (12) and (13), we get a general solution to the equation (11):

$$y(t) = C_1 \frac{\sqrt{2}t_h^{(-\frac{1}{2})}}{\sqrt{\pi}} sin_h \left(t + \frac{1}{2}h\right) + C_2 \frac{\sqrt{2}t_h^{-\frac{1}{2}}}{\sqrt{\pi}} \cos_h \left(t + \frac{1}{2}h\right).$$

The proof is complete.

Theorem 2.6. We define the h-Neumann function for non-integers ν (complex constant) by the formula:

$$N_{\nu,h}(t) = \frac{\cos_h(\nu\pi) J_{\nu,h}(t) - J_{-\nu,h}(t)}{\sin_h(\nu\pi)} \tag{14}$$

and it is a solution to equation (5).

Proof. Now, by using (1) we obtain the h-derivatives of the function (14):

$$D_h N_{\nu,h}(t) = \frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)} D_h J_{\nu,h}(t) - \frac{1}{\sin_h(\nu\pi)} D_h J_{-\nu,h}(t)$$

$$D_h^2 N_{\nu,h}(t) = \frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)} D_h^2 J_{\nu,h}(t) - \frac{1}{\sin_h(\nu\pi)} D_h^2 J_{-\nu,h}(t).$$

Substitute equation (5) into

$$t_h^{(2)} \left(\frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)} D_h^2 J_{\nu,h}(t - 2h) - \frac{1}{\sin_h(\nu\pi)} D_h^2 J_{-\nu,h}(t - 2h) \right) +$$

$$+t_h\left(\frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)}D_hJ_{\nu,h}(t-h)-\frac{1}{\sin_h(\nu\pi)}D_hJ_{-\nu,h}(t-h)\right)+$$
$$+t_h^{(2)}\left(\frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)}J_{\nu,h}(t-2h)-\frac{1}{\sin_h(\nu\pi)}J_{-\nu,h}(t-2h)\right)-$$
$$-\nu^2\left(\frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)}J_{\nu,h}(t)-\frac{1}{\sin_h(\nu\pi)}J_{-\nu,h}(t)\right)=0.$$

Consequently:

$$\frac{\cos_h(\nu\pi)}{\sin_h(\nu\pi)}(t_h^{(2)}D_h^2J_{\nu,h}(t-2h)+t_hD_hJ_{\nu,h}(t-h)+t_h^{(2)}J_{\nu,h}(t-2h)-\nu^2J_{\nu,h}(t))-$$

$$-\frac{1}{\sin_h(\nu\pi)}(t_h^{(2)}D_h^2J_{-\nu,h}(t-2h)+t_hD_hJ_{-\nu,h}(t-h)+t_h^{(2)}J_{-\nu,h}(t-2h)-\nu^2J_{-\nu,h}(t))=0.$$

We know that functions (5) and (9) of the first kind in the form $J_{\nu,h}(t)$ and $J_{-\nu,h}(t)$ which is the solution to the Bessel equation. Thus, we can say that the h-Neumann function (15) is the solution to equation (9).

Let $\nu>0$ and

$$L^2_{\nu,h}\left[a,b\right]:=\{f:\left[\int_a^b |f(x)|^2\,|x|^{2(\nu+1/2)}\,d_hx\right]^{1/2}\},$$

for $\forall\in a,b\in T_a$.

The h-Bessel operator: In this article, we consider a discrete analogue of the Bessel operator, where the h-Bessel operator has the following form:

$$(B_hy)(t):=t_h^{(-2\nu-1)}D_h\left[D_hy(t)\frac{1}{t_h^{(-2\nu-1)}}\right].$$

In addition, B_h is a linear operator, that is

$$B_h(\alpha y+\beta f)=\alpha B_h(y)+\beta B_h(f),\forall y,f\in L^2_{\nu,h}(a,b).$$

Theorem 2.7. (Orthogonality of eigenfunctions). Let (λ_1,y) and (λ_2,f) two pairs of eigenvalues and eigenfunctions, and $\lambda_1\neq\lambda_2$. Then, for both regular and periodic problems, the corresponding eigenfunctions $y(t)$ and $f(t)$ are orthogonal with weight r (therefore $\langle y(t),f(t)\rangle=0$).

Proof. The first two statements follow from the definition 3 and (1)–(3) for $\forall y,\ f\in L_{\nu,2}(a,b)$, we get that

$$\begin{aligned}\frac{(f(t+h)B_hy(t)-y(t+h)B_hf(t))}{t_h^{(-2\nu-1)}} &= D_h\left[D_hy(t)\frac{1}{t_h^{(-2\nu-1)}}\right]f(t+h)\\ &- D_h\left[D_hf(t)\frac{1}{t_h^{(-2\nu-1)}}\right]y(t+h)\\ &= D_h\left[f(t)D_hy(t)\frac{1}{t_h^{(-2\nu-1)}}\right.\\ &- \left.y(t)D_hf(t)\frac{1}{t_h^{(-2\nu-1)}}\right] \qquad (15)\end{aligned}$$

and

$$\int_0^h \frac{(f(t+h)B_h y(t) - y(t+h)B_h f(t))}{t_h^{(-2\nu-1)}} d_h t = \left[f(t) D_h y(t) \frac{1}{t_h^{(-2\nu-1)}} - y(t) D_h f(t) \frac{1}{t_h^{(-2\nu-1)}} \right] \Bigg|_0^h. \quad (16)$$

And using (5), we see that

$$B_h y(t) = t_h^{(-2\nu-1)} D_h \left[D_h y(t) \frac{1}{t_h^{(-2\nu-1)}} \right] = -\lambda_1^2 y(t+h) \quad (17)$$

and

$$B_h f(t) = t_h^{(-2\nu-1)} D_h \left[D_h f(t) \frac{1}{t_h^{(-2\nu-1)}} \right] = -\lambda_2^2 f(t+h) \quad (18)$$

Now multiply the first of the obtained equations (17) and (18) by $f(t)$, the second by $y(t)$, and find the difference. The resulting equation is reduced to the following form

$$\begin{aligned} f(t+h)B_h y(t) - y(t+h)B_h f(t) &= t_h^{(-2\nu-1)} D_h \left[\frac{D_h y(t)}{t_h^{(-2\nu-1)}} \right] f(t+h) \\ &- t_h^{(-2\nu-1)} D_h \left[D_h f(t) \frac{1}{t_h^{(-2\nu-1)}} \right] y(t+h) \\ &= (\lambda_2^2 - \lambda_1^2) y(t+h) f(t+h). \end{aligned}$$

We can rewrite

$$\begin{aligned} \frac{(f(t+h)B_h y(t) - y(t+h)B_h f(t))}{t_h^{(-2\nu-1)}} &= D_h \left[D_h y(t) \frac{1}{t_h^{(-2\nu-1)}} \right] f(t+h) \\ &- D_h \left[D_h f(t) \frac{1}{t_h^{(-2\nu-1)}} \right] y(t+h) \\ &= (\lambda_2^2 - \lambda_1^2) y(t+h) f(t+h) \frac{1}{t_h^{(-2\nu-1)}}. \end{aligned} \quad (19)$$

From (2), (16), and (19), we may compute

$$\begin{aligned} (\lambda_2^2 - \lambda_1^2) \int_0^h \frac{y(t+h) f(t+h)}{t_h^{(-2\nu-1)}} d_h t &= \int_0^h D_h \left[f(t) D_h y(t) \frac{1}{t_h^{(-2\nu-1)}} - y(t) D_h f(t) \frac{1}{t_h^{(-2\nu-1)}} \right] d_h t \\ &= \int_0^h \frac{(f(t+h)B_h y(t) - y(t+h)B_h f(t))}{t_h^{(-2\nu-1)}} d_h t \\ &= \int_0^h D_h \left[f(t) D_h y(t) \frac{1}{t_h^{(-2\nu-1)}} - y(t) D_h f(t) \frac{1}{t_h^{(-2\nu-1)}} \right] d_h t \\ &= h \frac{1}{h_h^{(-2\nu-1)}} - h \frac{1}{h_h^{(-2\nu-1)}} - 0 + 0. \end{aligned}$$

Here:

$$\begin{bmatrix} t = 0 \Rightarrow j_{(\nu,h)}(0) = 1; D_h j_{(\nu,h)}(0) = 0 \\ t = h \Rightarrow j_{(\nu,h)}(h) = 1; D_h j_{(\nu,h)}(h) = h. \end{bmatrix}$$

Therefore,

$$\int_0^h \frac{(f(t+h)B_h y(t) - y(t+h)B_h f(t))}{t_h^{(-2\nu-1)}} d_h t = (\lambda_2^2 - \lambda_1^2) \int_0^h \frac{y(t+h)f(t+h)}{t_h^{(-2\nu-1)}} d_h t = 0$$

and

$$\lambda_2 \neq \lambda_1 \Rightarrow \langle y(t+h), f(t+h) \rangle = 0.$$

It proves the claim.

Acknowledgments

We thank the referee for some general advice, which improved the final version of the paper. The authors were supported by the Ministry of Education and Science of the Republic of Kazakhstan Grant AP08052208.

References

1 Andrews, G.E., Askey, R., & Roy, R. (1999). *Special functions.* Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, Cambridge.

2 Bell, W.W. (2004). *Special functions for scientists and engineers.* Dover Publications, Inc., Mineola, NY., Reprint of the 1968 original.

3 Lebedev, N.N. (1965). *Special functions and their applications, Revised English edition.* Translated and edited by Richard A. Silverman, Prentice-Hall, Inc., Englewood Cliffs, N.J.

4 Rainville, E.D. (1971). *Special functions.* 1st ed., Chelsea Publishing Co., Bronx, N.Y.

5 Watson, G.N. (1995). *A treatise on the theory of Bessel functions.* Cambridge Mathematical Library, Cambridge University Press, Cambridge, Reprint of the second (1944) edition.

6 Schwartz, L. (1965). *Methodes mathematiques pour les sciences physiques.* Collection Enseignement des sciences, (3), French.

7 Atici, F.M., & Eloe, P.W. (2007). A Transform Method in Discrete Fractional Calculus. *International Journal of Difference Equations, 2*(2), 165–176.

8 Atici, F.M., & Eloe, P.W. (2009). Initial value problems in discrete fractional calculus. *Proceedings of the American Mathematical Society, 137*(3), 981–989.

9 Bastos, N.R.O., Ferreira, R.A.C., & Torres, D.F.M. (2011). Necessary optimality conditions for fractional difference problems of the calculus of variations. *Discrete Contin. Dyn. Syst., 29*(2), 417–437.

10 Ferreira, R.A.C., & Torres, D.F.M. (2011). Fractional h-difference equations arising from the calculus of variations. *Appl. Anal. Discrete Math., 1*(5), 110–121.

11 Girejko, E., & Mozyrska, D. (2013). Overview of fractional h-difference operators. *Advances in harmonic analysis and operator theory, Oper. Theory Adv. Appl., 229.* Birkhauser/Springer, Basel AG, Basel, 253–268.

12 Kac, V., & Cheung, P. (2002). *Quantum Calculus.* Springer, New York.

13 Agrawal, O.P., Sabatier, J., & Tenreiro Machado, J.A. (2018). *Advances in Fractional Calculus.* Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht (2007) Persson et al. Journal of Inequalities and Applications, 2018(73), 14.

14 Almeida, R., & Torres, D.F.M. (2010). Leitmann's direct method for fractional optimization problems. *Appl. Math. Comput., 217*(3), 956–962.

15 Magin, R.L. (2006). *Fractional Calculus in Bioengineering.* Begell House, Redding.

16 Malinowska, A.B., & Torres, D.F.M. (2010). Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative. *Comput. Math. Appl., 59*(9), 3110–3116.

17 Mozyrska, D., Pawluszewicz, E., & Wyrwas, M. (2015). The h-difference approach to controllability and observability of fractional linear systems with Caputo-type operator. *Asian J. Control, 17*(4), 1163–1173.

18 Ortigueira, M.D., & Manuel, D. (2011). *Fractional Calculus for Scientists and Engineers.* Lecture Notes in Electrical Engineering, 84. Springer, Dordrecht.

19 Oinarov, R., Persson, L.-E., & Shaimardan, S. (2018). Hardy-type inequalities in fractional h-discrete calculus. *J.Inequal. Appl., 2018*(73), 14.

20 Shaimardan, S. (2019). Fractional order Hardy-type inequality in fractional h-discrete calculus. *Math. Inequal. Appl., 22*(2), 691–702.

С. Шаймардан[1], Н.С. Токмагамбетов[1,2], Е. Айқын[1]

[1] *Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Нұр-Сұлтан, Қазақстан;*
[2] *Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан*

Кванттық есептеудегі Бессель теңдеуі

Бессель функцияларын қолдану математикалық физиканың барлық дерлік маңызды салаларына қатысты және өзекті техникалық сұрақтарға жауап беруге арналған әртүрлі есептердің үлкен санымен байланысты. Жұмыста Бессель дифференциалдық теңдеуінің аналогы болып табылатын h-айырымдық теңдеуі енгізілген және жалпыланған дәрежелер қатарын алып, Фробениус әдісі арқылы өрнектейтін оның шешімінің қасиеттері зерттелген. Бессель функциясы мен h-Нейман функциясы үшін дискретті аналогтық формулалар табылды, олардың шешімдері h-бөлшек функциясы $t_h^{(\alpha)}$ бар қатармен берілген. Сонымен қатар, T_a бойынша h-Бессель функциялары арасындағы сызықтық тәуелділіктер алынған.

Кілт сөздер: Бессель функциясы, модификацияланған Бессель функциясы, Бессель айырымдық теңдеуі, h-есептеу, h-туынды және h-бөлшек функциясы.

С. Шаймардан[1], Н.С. Токмагамбетов[1,2], Е. Айкын[1]

[1] *Евразийский национальный университет имени Л.Н. Гумилева, Нур-Султан, Казахстан;*
[2] *Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан*

Уравнение Бесселя в квантовом исчислении

С использованием функций Бесселя связано большое количество самых разнообразных задач, относящихся практически ко всем важнейшим разделам математической физики и призванных ответить на актуальные технические вопросы. В статье мы вводим h-разностное уравнение, аналог дифференциального уравнения Бесселя, и исследуем свойства его решения, которые мы выражаем с помощью метода Фробениуса, предполагая обобщенный степенной ряд. Найдены дискретные формулы-аналоги для функции Бесселя и h-функции Неймана, решения которых представлены рядом с h-дробной функцией $t_h^{(\alpha)}$. Кроме того, мы получили линейные зависимости между h-функциями Бесселя на T_a.

Ключевые слова: функция Бесселя, модифицированная функция Бесселя, разностное уравнение Бесселя, h-исчисление, h-производная и h-дробная функции.

DOI 10.31489/2022M3/145-151

UDC 517.951

A.G. Smadiyeva*

Al-Farabi Kazakh National University, Almaty, Kazakhstan;
Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
(E-mail: smadiyeva@math.kz)

Well-posedness of the initial-boundary value problems for the time-fractional degenerate diffusion equations

This paper deals with the solving of initial-boundary value problems for the one-dimensional linear time-fractional diffusion equations with time-degenerate diffusive coefficients t^{β} with $\beta > 1 - \alpha$. The solutions to initial-boundary value problems for the one-dimensional time-fractional degenerate diffusion equations with Riemann-Liouville fractional integral $I^{1-\alpha}_{0+,t}$ of order $\alpha \in (0,1)$ and with Riemann-Liouville fractional derivative $D^{\alpha}_{0+,t}$ of order $\alpha \in (0,1)$ in the variable, are shown. The solutions to these fractional diffusive equations are presented using the Kilbas-Saigo function $E_{\alpha,m,l}(z)$. The solution to the problems is discovered by the method of separation of variables, through finding two problems with one variable. Rather, through finding a solution to the fractional problem depending on the parameter t, with the Dirichlet or Neumann boundary conditions. The solution to the Sturm-Liouville problem depends on the variable x with the initial fractional-integral Riemann-Liouville condition. The existence and uniqueness of the solution to the problem are confirmed. The convergence of the solution was evidenced using the estimate for the Kilbas-Saigo function $E_{\alpha,m,l}(z)$ from and by Parseval's identity.

Keywords: time-fractional diffusion equation, method of separation variables, Kilbas-Saigo function.

Introduction

Many mathematicians have attracted most interest to the fractional diffusion equations. Inverse source problems for degenerate time-fractional PDE were studied in [1]. In [2, 3], Al-Refai and Luchko analyzed the initial-boundary value problems for the linear and non-linear fractional diffusion equations with the Riemann-Liouville time-fractional derivative. Various types of fractional derivatives and their properties were investigated in the monograph [4–8]. Fractional calculus can be applied in mechanics, physics, mathematics, etc. [8–12]. Note that degenerate fractional evolutionary equations were investigated in [13, 14]. In [15], maximum and minimum principles for time-fractional diffusion equations involving fractional derivatives were proposed. Luchko studied initial-boundary value problems for a generalized diffusion equation with a distributed order [16].

In our previous work [17], we studied the Cauchy-Dirichlet and Cauchy-Neumann problems for the Caputo time-fractional diffusion equation. This paper considers the Cauchy-Dirichlet and Cauchy-Neumann problems for the diffusion equation with Riemann–Liouville time-fractional derivative. A solution is discovered by using the Kilbas-Saigo function and by the method of separation of variables. The existence, convergence, and uniqueness of the solution are proved.

1 Dirichlet problem

Let us consider the one-dimensional case of the time-fractional diffusion equation

$$D^{\alpha}_{0+,t}u(t,x) - t^{\beta}u_{xx}(t,x) = 0, \quad (t,x) \in (0,\infty) \times (0,1), \tag{1}$$

*Corresponding author.
E-mail: smadiyeva@math.kz

with the Dirichlet boundary condition

$$u(t,0) = u(t,1) = 0, \ \ t > 0, \ \ x \in [0,1], \tag{2}$$

and the Cauchy initial condition

$$I^{1-\alpha}_{0+,t} u(0,x) = \phi(x), \ \ x \in [0,1], \tag{3}$$

where $\beta > 1-\alpha$, $D^{\alpha}_{0+,t}$ is the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ defined by [5; 79]

$$D^{\alpha}_{0+,t} f(t) = \frac{d}{dt} I^{1-\alpha}_{0+,t} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)ds}{(t-s)^{\alpha}}.$$

Here $I^{1-\alpha}_{0+,t}$ is the Riemann-Liouville fractional integral given by [5; 80]

$$I^{1-\alpha}_{0+,t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(s)ds}{(t-s)^{\alpha}}.$$

Let $H^2(0,1)$ is a Hilbert space, defined by

$$H^2(0,1) = \{u: \ \ u \in L^2(0,1); \ \ u_{xx} \in L^2(0,1)\},$$

where the norm is

$$||u||^2_{H^2(0,1)} = \sum_{k=1}^{\infty} \lambda_k^2 |(u,e_k)|^2 < \infty.$$

Definition 1. The solution to problem (1)–(3) is $t^{1-\alpha}u \in C((0,\infty); L^2(0,1))$, which satisfies $t^{1-\alpha-\beta} D^{\alpha}_{0+,t} u$, $t^{1-\alpha} u_{xx} \in C((0,\infty); L^2(0,1))$.

Theorem 1. Let $\phi(x) \in H^2(0,1)$, then there exists a unique solution u to problem (1)–(3), which has the form

$$u(t,x) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \phi_k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \sin \pi k x, \ \ (t,x) \in (0,\infty) \times (0,1),$$

where $\phi_k = 2\int_0^1 \phi(x) \sin \pi k x dx$, $k \in N$ and $E_{\alpha,m,l}(z)$ is the Kilbas-Saigo function defined as [8, Remark 5.1]

$$E_{\alpha,m,l}(z) = \sum_{i=0}^{\infty} c_i z^i, \ \ c_0 = 1, \ \ c_i = \Pi_{j=0}^{i-1} \frac{\Gamma(\alpha(jm+l)+1)}{\Gamma(\alpha(jm+l+1)+1)}, \ \ i \geq 1. \tag{4}$$

For the function $E_{\alpha,m,m-\frac{1}{\alpha}}(-\lambda_k t^{\beta+\alpha})$ the following estimate holds [4, Proposition 3.6]

$$E_{\alpha,m,m-\frac{1}{\alpha}}(-\lambda_k t^{\beta+\alpha}) \leq \frac{1}{\left(1 + \frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha(m+1))} \lambda_k t^{\beta+\alpha}\right)^{1+\frac{1}{m}}}, \ \ m = \frac{\beta+\alpha}{\alpha}, \ \ t > 0. \tag{5}$$

Proof Theorem 1.

Existence of solution. Since the Sturm–Liouville operator has eigenvalues $\{\lambda_k > 0, \ k \in N\}$ on $L^2(0,1)$ and the corresponding orthonormal eigenfunctions $\{X_k(x), \ k \in N\}$ in $L^2(0,1)$ and $\phi(x) \in H^2(0,1)$, then we can write the solution to problem (1)–(3) as follows

$$u(t,x) = \sum_{k=1}^{\infty} T_k(t) X_k(x), \ \ (t,x) \in (0,\infty) \times (0,1), \tag{6}$$

$$\phi(x)=\sum_{k=1}^{\infty}\phi_k X_k(x),\ \ x\in(0,1),$$

where

$$\phi_k=2\int_0^1\phi(x)X_k(x)dx.$$

Substituting (6) to diffusion equation (1)–(3), we gain the next problem

$$D^{\alpha}_{0+,t}T_k(t)+\lambda_k t^{\beta}T_k(t)=0,\ \ t>0, \tag{7}$$

$$I^{1-\alpha}_{0+,t}T_k(0)=\phi_k, \tag{8}$$

$$X_k''(x)+\lambda_k X_k(x)=0, \tag{9}$$

$$X_k(0)=X_k(1)=0. \tag{10}$$

The orthonormal eigenfunctions and the corresponding eigenvalues of the Dirichlet problem (9)–(10) are $X_k(x)=\sin\pi kx$ and $\lambda_k=(\pi k)^2$, respectively. It is known that a unique solution to problem (7)–(8) is [5; 227]

$$T_k(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}\phi_k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2k^2t^{\beta+\alpha}). \tag{11}$$

Substituting (11) and the orthonormal eigenfunctions $X_k(x)=\sin\pi kx$ to (6), we can get the solution to problem (1)–(3) in the next form

$$u(t,x)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{k=1}^{\infty}\phi_k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})\sin\pi kx, (t,x)\in(0,\infty)\times(0,1). \tag{12}$$

Convergence of solution. Applying (5) to (11), we get

$$T_k(t)\le\frac{|\phi_k||t^{\alpha-1}|}{\Gamma(\alpha)\left(1+\frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha(m+1))}\pi^2k^2t^{\beta+\alpha}\right)^{1+\frac{1}{m}}},\ \ m=\frac{\beta+\alpha}{\alpha}.$$

By Parseval's identity, it follows from (12) that

$$\sup_{t\ge0}||t^{1-\alpha}u(t,\cdot)||^2_{L^2(0,1)}=\sup_{t\ge0}\frac{1}{|\Gamma(\alpha)|^2}\sum_{k=1}^{\infty}|\phi_k|^2\left|E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})\right|^2\le$$

$$\le\sup_{t\ge0}\frac{1}{|\Gamma(\alpha)|^2}\sum_{k=1}^{\infty}\frac{|\phi_k|^2}{\left(1+\frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha(m+1))}\pi^2k^2t^{\beta+\alpha}\right)^{2(1+\frac{1}{m})}}\le$$

$$\le\sup_{t\ge0}\frac{1}{|\Gamma(\alpha)|^2\left(1+\frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha(m+1))}\pi^2t^{\beta+\alpha}\right)^{2(1+\frac{1}{m})}}\sum_{k=1}^{\infty}|\phi_k|^2\le\sum_{k=1}^{\infty}|\phi_k|^2=||\phi(\cdot)||^2_{L^2(0,1)}. \tag{13}$$

Solving $D^{\alpha}_{0+,t}u$ and u_{xx} we get

$$D^{\alpha}_{0+,t}u(t,x)=\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{\infty}\phi_k D^{\alpha}_{0+,t}t^{\alpha-1}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})\sin\pi kx=$$

$$= -\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \pi^2 k^2 \phi_k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \sin \pi k x, \tag{14}$$

and

$$u_{xx}(t,x) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \phi_k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \sin'' \pi k x =$$

$$= -\frac{t^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \pi^2 k^2 \phi_k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \sin \pi k x. \tag{15}$$

Applying (13)–(15) we get

$$\sup_{t\geq 0} ||t^{1-\alpha-\beta} \mathcal{D}_t^{\alpha} u(t,\cdot)||^2_{L^2(0,1)} \leq \sum_{k=1}^{\infty} \pi^4 k^4 |\phi_k|^2 = ||\phi(\cdot)||^2_{H^2(0,1)} < \infty,$$

and

$$\sup_{t\geq 0} ||t^{1-\alpha} u_{xx}(t,\cdot)||^2_{L^2(0,1)} \leq \sum_{k=1}^{\infty} \pi^4 k^4 |\phi_k|^2 = ||\phi(\cdot)||^2_{H^2(0,1)} < \infty.$$

Uniqueness of the solution. Suppose that u_1 and u_2 are solutions to problem (1)–(3) and we choose $u = u_1 - u_2$ in such a way, that u satisfies the diffusion equation (1) and boundaries, initial conditions (2), (3). Define

$$T_k(t) = \int_0^1 u(t,x) \sin \pi k x dx, \;\; k \in N, \;\; t > 0. \tag{16}$$

Applying $D^{\alpha}_{0+,t}$ to left-side (16) equation by using (1) we obtain

$$\begin{aligned} D^{\alpha}_{0+,t} T_k(t) &= \int_0^1 D^{\alpha}_{0+,t} u(t,x) \sin \pi k x dx \\ &= t^{\beta} \int_0^1 u_{xx}(t,x) \sin \pi k x dx \\ &= t^{\beta} \int_0^1 u(t,x) \sin'' \pi k x dx \\ &= -t^{\beta} \pi^2 k^2 \int_0^1 u(t,x) \sin \pi k x dx \\ &= -t^{\beta} \pi^2 k^2 T_k(t), \;\; k \in N, \;\; t > 0. \end{aligned}$$

From (2), (3) we have

$$I^{1-\alpha}_{0+,t} T_k(0) = 0,$$

which means that $u(t,x) \equiv 0$. Hence $u_1(t,x) = u_2(t,x)$, therefore the problem (1)–(3) has a unique solution.

2 Cauchy-Neumann problem

Let us consider time-fractional diffusion equation

$$D^{\alpha}_{0+,t} u(t,x) - t^{\beta} u_{xx}(t,x) = 0, \;\; (t,x) \in (0,\infty) \times (0,1), \tag{17}$$

with the Neumann boundary condition

$$u_x(t,0) = u_x(t,1) = 0, \;\; t > 0, \;\; x \in [0,1], \tag{18}$$

and the Cauchy initial condition

$$I_{0+,t}^{1-\alpha}u(0,x) = \phi(x). \tag{19}$$

Definition 2. The solution to problem (17)–(19) is $t^{1-\alpha}u \in C((0,\infty);\ L^2(0,1))$, which satisfies $t^{1-\alpha-\beta}D_{0+,t}^{\alpha}u,\ t^{1-\alpha}u_x,\ t^{1-\alpha}u_{xx} \in C((0,\infty);L^2(0,1))$.

Theorem 2. Let $\phi(x) \in H^2(0,1)$. The unique solution to problem (17)–(19) is the function u, which has form

$$u(t,x) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\phi_k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})\cos\pi kx,\ \ (t,x)\in(0,\infty)\times(0,1),$$

where $\phi_0 = \int_0^1\phi(x)dx$ and $\phi_k = 2\int_0^1\phi(x)\cos\pi kxdx,\ \ k\in N$ and $E_{\alpha,m,l}(z)$ is a Kilbas-Saigo function, which is defined by the formula (4) and (5).

It can be easily proven by the idea of Theorem 1.

Acknowledgements

The author was supported by the Ministry of Education and Science of the Republic of Kazakhstan Grant AP08052046.

References

1 Al-Salti N. Inverse source problems for degenerate time-fractional PDE / N. Al-Salti, E. Karimov // Progr. Fract. Differ. Appl. — 2022. — 1. — № 8. — P. 39–52.

2 Al-Refai M. Maximum principle for the fractional diffusion equations with the Riemann-Liouville fractional derivative and its applications / M. Al-Refai, Yu. Luchko // Fract. Calc. Appl. Anal. — 2014. — 2. — № 17. — P. 483–498.

3 Al-Refai M. Maximum principle for the multi-term time-fractional diffusion equations with the Riemann-Liouville fractional derivatives / M. Al-Refai, Yu. Luchko // Appl. Math. Comput. — 2015. — № 257. — P. 40–51.

4 Boudabsa L. Some properties of the Kilbas-Saigo function / L. Boudabsa, T. Simon // ArXiv. — 2020. — P. 1–46.

5 Kilbas A.A. Theory and applications of fractional differential equations / A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. — Elsevier, 2006. — 523 p.

6 Нахушев А.М. Дробное исчисление и его применение / А.М. Нахушев. — М.: Физматлит, 2003. — 272 с.

7 Podlubny I. Fractional Differential Equations / I. Podlubny. — Academic Press, San Diego, 1998. — 340 p.

8 Kilbas A.A. On solution of integral equations of Abel–Volterra type / A.A. Kilbas, M. Saigo // Differ. Integral Equ. — 1995. — 5. — № 8. — P. 993–1011.

9 Carpinteri A. Fractals and Fractional Calculus in Continuum Mechanics / A. Carpinteri, F. Mainardi. — Springer, Berlin, 1997.

10 Hilfer R. Applications of Fractional Calculus in Physics / R. Hilfer. — World Sci. Publishing, River Edge, NJ, 2000. — 463 p.

11 Metzler R. The random walk's guide to anomalous diffusion: a fractional dinamics approach / R. Metzler, J. Klafter // Physics Reports. — 2000. — № 339. — P. 1–77.

12 Samko S.G. Fractional Integrals and Derivatives / S.G. Samko, A.A. Kilbas, O.I. Marichev. — Theory and Applications, Gordon and Breach, Amsterdam, 1993. — 976 p.

13 Dipierro S. Decay estimates for evolutionary equations with fractional time-diffusion / S. Dipierro, E. Valdinoci, V. Vespri // J. Evol. Equ. — 2019. — № 19. — P. 435–462.

14 Turmetov B.Kh. On a problem for nonlocal mixed-type fractional order equation with degeneration / B.Kh. Turmetov, B.J. Kadirkulov // Chaos Solitons Fractals. — 2021. — № 146. — P. 110-835.

15 Kadirkulov B.J. On a generalization of heat equations / B.J. Kadirkulov, B.Kh. Turmetov // Uzbek Math. Journal. — 2006. — 3. — P. 40–45.

16 Luchko Yu. Boundary value problems for the generalized time-fractional diffusion equation of distributed order / Yu. Luchko // Fract. Calc. Appl. Anal. — 2009. — 4. — № 12. — P. 409–422.

17 Smadiyeva A.G. Initial-boundary value problem for the time-fractional degenerate diffusion equation / A.G. Smadiyeva // JMMCS. — 2022. — 1. — № 113. — P. 32–41.

А.Г. Смадиева

Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан;
Математика және математикалық модельдеу институты, Алматы, Қазақстан

Бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін бастапқы шеттік есептің қисындылығы

Мақалада t^β, $\beta > 1 - \alpha$ диффузиялық коэффициенттері бар бір өлшемді сызықтық бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін бастапқы шеттік есептерді шешу қарастырылған. $\in (0,1)$ үшін бөлшек ретті Риман-Лиувилль интегралы $I_{0+,t}^{1-\alpha}$ және $\alpha \in (0,1)$ үшін бөлшек ретті Риман-Лиувилль туындысы $D_{0+,t}^{\alpha}$ бар бір өлшемді уақыт бойынша бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін бастапқы шеттік есептердің шешімдері көрсетілген. Бөлшек ретті диффузиялық теңдеулердің шешімдері $E_{\alpha,m,l}(z)$ Килбас-Сайго функциясы арқылы берілген. Есептердің шешімі айнымалыларды ажырату әдісі арқылы, бір айнымалысы бар екі есепті табу арқылы анықталады. Демек, Дирихле немесе Нейман шекаралық шарттарымен t параметріне тәуелді бөлшек ретті есебінің шешімін және x параметріне тәуелді Штурм-Лиувилл есебіне қойылған бастапқы шарты бөлшек ретті Риман-Лиувилл интегралы арқылы берілген есептің шешімін табу арқылы. Есептің шешімінің бар болуы мен жалғыздығы дәлелденген. Шешімнің жинақтылығы Kilbas-Saigo $E_{\alpha,m,l}(z)$ функциясының бағалауы көмегімен және Парсевал теңдігін қолдану арқылы дәлелденді.

Кілт сөздер: бөлшек ретті туындылы өзгешеленген диффузия теңдеуі, айнымалыларын ажырату әдісі, Килбас-Сайго функциясы.

А.Г. Смадиева

Казахский национальный университет имени Аль-Фараби, Алматы, Казахстан;
Институт математики и математического моделирования, Алматы, Казахстан

Корректность начально-краевых задач для дробных вырожденных диффузионных уравнений

Статья посвящена решению начально-краевых задач для одномерных дробных вырожденных линейных диффузионных уравнений коэффициентами диффузии t^β при $\beta > 1-\alpha$, начально-краевых задач для одномерных уравнений вырождающейся диффузии с дробным временем с дробным интегралом Римана-Лиувилля $I_{0+,t}^{1-\alpha}$ порядка $\alpha \in (0,1)$ и с дробной производной Римана-Лиувилля $D_{0+,t}^{\alpha}$ порядка $\alpha \in (0,1)$ по переменной. Решения этих дробных диффузионных уравнений представлены с помощью функции Килбаса-Сайго $E_{\alpha,m,l}(z)$, их получили методом разделения переменных, путем нахождения двух задач с одной переменной. Вернее, путем нахождения решения дробной задачи, зависящей от параметра t, с граничными условиями Дирихле или Неймана, и решение задачи Штурма–Лиувилля, зависящей от переменной x с начальным дробно-интегральным условием Римана-Лиувилля. Доказаны существование и единственность решения задачи. Сходимость в решения подтверждена с помощью оценки функции Килбаса-Сайго $E_{\alpha,m,l}(z)$ и тождества Парсеваля.

Ключевые слова: дробно-вырожденное диффузионное уравнение, метод разделения переменных, функция Килбаса-Сайго.

References

1 Al-Salti, N., & Karimov, E. (2022). Inverse source problems for degenerate time-fractional PDE. *Progr. Fract. Differ. Appl., 8*(1), 39–52.

2 Al-Refai, M., & Luchko, Yu. (2014). Maximum principle for the fractional diffusion equations with the Riemann-Liouville fractional derivative and its applications. *Fract. Calc. Appl. Anal., 17*(2), 483–498.

3 Al-Refai, M., & Luchko, Yu. (2015). Maximum principle for the multi-term time-fractional diffusion equations with the Riemann-Liouville fractional derivatives. *Appl. Math. Comput., 257*, 40–51.

4 Boudabsa, L., & Simon, T. (2020). Some properties of the Kilbas-Saigo function. *ArXiv.*, 1–46.

5 Kilbas, A.A., Srivastava, H.M., & Trujillo, J.J. (2006). *Theory and applications of fractional differential equations.* Elsevier.

6 Nakhushev, A.M. (2003). *Drobnoe ischislenie i ego primenenie [Fractional calculus and its applications].* Moscow: Fizmatlit [in Russian].

7 Podlubny, I. (1998). *Fractional Differential Equations.* San Diego: Academic Press.

8 Kilbas, A.A., & Saigo, M. (1995). On solution of integral equations of Abel–Volterra type. *Differ. Integral Equ., 8*(5), 993–1011.

9 Carpinteri, A., & Mainardi, F. (1997). *Fractals and Fractional Calculus in Continuum Mechanics.* Berlin: Springer.

10 Hilfer, R. (2000). *Applications of Fractional Calculus in Physics.* NJ: River Edge: World Sci. Publishing.

11 Metzler, R., & Klafter, J. (2000). The random walk's guide to anomalous diffusion: a fractional dinamics approach. *Physics Reports, 339*, 1–77.

12 Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). *Fractional Integrals and Derivatives.* Amsterdam: Gordon and Breach: Theory and Applications.

13 Dipierro, S., Valdinoci, E., & Vespri, V. (2019). Decay estimates for evolutionary equations with fractional time-diffusion. *J. Evol. Equ., 19*, 435–462.

14 Turmetov, B.Kh., & Kadirkulov, B.J. (2021). On a problem for nonlocal mixed-type fractional order equation with degeneration. *Chaos Solitons Fractals., 146*, 110-835.

15 Kadirkulov, B.J., & Turmetov, B.Kh. (2006). On a generalization of heat equations. *Uzbek Math. Journal., 3*, 40–45.

16 Luchko, Yu. (2009). Boundary value problems for the generalized time-fractional diffusion equation of distributed order. *Fract. Calc. Appl. Anal., 12*(4), 409–422.

17 Smadiyeva, A.G. (2022). Initial-boundary value problem for the time-fractional degenerate diffusion equation. *JMMCS., 113*(1), 32–41.

DOI 10.31489/2022M3/152-162

UDC 531.31, 519.21

M.I. Tleubergenov[1,2], G.K. Vassilina[1,3,*], S.R. Seisenbayeva[1,2]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *Al-Farabi Kazakh National University, Almaty, Kazakhstan;*
[3] *Almaty University of Power Engineering and Telecommunications named after G.Daukeev, Almaty, Kazakhstan*
(E-mail: marat207@mail.ru, v_gulmira@mail.ru, saramatal@mail.ru)

Construction of stochastic differential equations of motion in canonical variables

Galiullin proposed a classification of inverse problems of dynamics for the class of ordinary differential equations (ODE). Considered problem belongs to the first type of inverse problems of dynamics (of the three main types of inverse problems of dynamics): the main inverse problem under the additional assumption of the presence of random perturbations. In this paper Hamilton and Birkhoff equations are constructed according to the given properties of motion in the presence of random perturbations from the class of processes with independent increments. The obtained necessary and sufficient conditions for the solvability of the problem of constructing stochastic differential equations of both Hamiltonian and Birkhoffian structure by the given properties of motion are illustrated by the example of the motion of an artificial Earth satellite under the action of gravitational and aerodynamic forces.

Keywords: stochastic differential equation, class of processes with independent increments, stochastic equations of Hamiltonian and Birkhoffian structures, the main inverse problem.

Introduction

At present, the theory of inverse problems of dynamics in the class of ODEs is fully developed [1–9] and goes back to the Yerugin's fundamental work [10]. In [10], there is constructed a set of ODE that have a given integral curve.

Methods for solving inverse problems of dynamics are generalized to the class of Ito stochastic differential equations in [11–19].

Let the set

$$\Lambda(t) : \lambda(x, \dot{x}, t) = 0, \quad \lambda \in R^m, \;\; x \in R^n, \tag{1}$$

be given. It is required to construct a set of stochastic equations of Hamiltonian and Birkhoffian structure

$$\begin{cases} \dot{q}_k = \dfrac{\partial H}{\partial p_k}, \\ \dot{p}_k = -\dfrac{\partial H}{\partial q_k} + \sigma'_{kj}(q,p,t)\dot{\xi}^j, \quad \left(k = \overline{1,n}\right); \end{cases} \tag{2}$$

$$\left[\frac{\partial R_i(z,t)}{\partial z_l} - \frac{\partial R_l(z,t)}{\partial z_i}\right]\dot{z}_i - \left[\frac{\partial B(z,t)}{\partial z_l} + \frac{\partial R_l(z,t)}{\partial t}\right] = T_{l\mu}\dot{\psi}_\mu, \;\; (i,l = \overline{1,2n}, \; \mu = \overline{1,n+r}) \tag{3}$$

so that the set (1) is an integral manifold of the constructed stochastic equations of the Hamiltonian (2) and Birkhoffian structure (3).

Here $\{\xi_1(t,\omega), \ldots, \xi_k(t,\omega)\}$ and $\{\psi_1(t,\omega), ..., \psi_{n+r}(t,\omega)\}$ are systems of random processes with independent increments that can be represented as a sum of Wiener and Poisson processes [20]:

[*]Corresponding author.
E-mail: v_gulmira@mail.ru

1) $\xi = \xi_0 + \int c(y)P^0(t, dy)$, where ξ_0 is a Wiener process, P^0 is a Poisson process, $P^0(t, dy)$ is the number of the jumps of P^0 in the interval $[0, t]$ that fall onto the set dy, $c(y)$ is a vector function mapping the space R^{2n} into the space R^r of the values of the process $\xi(t)$ for all t;

2) $\psi = \psi_0 + \int \tilde{c}(y)\tilde{P}^0(t, d\gamma)$, ψ_0 is a Wiener process, $\tilde{P}^0$ is a Poisson process, $\tilde{P}^0(t, d\gamma)$ is the number of the jumps of $\tilde{P}^0$ in the interval $[0, t]$ that fall onto the set $d\gamma$, $\tilde{c}(y)$ $c(y)$ is a vector function mapping the space R^{2n} into the space R^{n+r} of the values of the process $\psi(t)$ for all t; $B = B(z, t)$ is a Birkhoff function, $W = (W_{il})$ Birkhoff tensor with components $W_{il} = \left[\frac{\partial R_i(z,t)}{\partial z_l} - \frac{\partial R_l(z,t)}{\partial z_i}\right]$.

The stated problem was solved for the class of ODEs in [21]. In particular, the stochastic Helmholtz problem, i.e., the problem of constructing equivalent stochastic equations of the Lagrangian, Hamiltonian, and Birkhoffian structures by a given second order stochastic Ito equation was considered in [22]. In [23, 24], the above problem of constructing stochastic equations of the form (2) and (3) by a given integral set (1) is considered under the assumption that systems $\{\xi_1(t, \omega), ..., \xi_r(t, \omega)\}$ and $\{\psi_1(t, \omega), ..., \psi_{n+r}(t, \omega)\}$ are systems of independent Wiener processes (as a special case of random processes with independent increments).

Let us give the scheme of solving the set problems: at the first step by the quasi-inversion method [3] in combination with Yerugin's method [10] and by virtue of stochastic differentiation of the complex function in the case of processes with independent increments [20] by the given set (1) the second order Ito differential equation

$$\ddot{x} = f(x, \dot{x}, t) + \sigma(x, \dot{x}, t)\dot{\xi} \tag{4}$$

is constructed so that the set $\Lambda(t)$ is an integral manifold of the constructed equation (4). Further, at the second step, equivalent stochastic equations of Hamiltonian and Birkhoffian structures are constructed by the constructed stochastic equation (4).

1 Construction of stochastic Hamiltonian equation (2) by the given properties of motion (1)

Previously, by virtue of the Ito formula for stochastic differentiation of a complex function, the equation of perturbed motion

$$\dot{\lambda} = \frac{\partial\lambda}{\partial t} + \frac{\partial\lambda}{\partial x}\dot{x} + \frac{\partial\lambda}{\partial\dot{x}}f + S_1 + S_2 + S_3 + \frac{\partial\lambda}{\partial\dot{x}}\sigma\dot{\xi}, \tag{5}$$

is compiled. Here $S_1 = \dfrac{1}{2}\dfrac{\partial^2\lambda}{\partial\dot{x}^2} : \sigma\sigma^T$; following [20], $\dfrac{\partial^2\lambda}{\partial\dot{x}^2} : D$, $D = \sigma\sigma^T$ is understood as a vector, the elements of which are the traces of the products of matrices of the second derivatives of the corresponding elements $\lambda_\mu(x, \dot{x}, t)$ of the vector $\lambda(x, \dot{x}, t)$ with respect to the components $\dot{x}$ and the matrix D

$$\frac{\partial^2\lambda}{\partial\dot{x}^2} : D = \left[tr\left(\frac{\partial^2\lambda_1}{\partial\dot{x}^2}D\right), \ \dots \ , tr\left(\frac{\partial^2\lambda_m}{\partial\dot{x}^2}D\right)\right]^T;$$

$$S_2 = \int \left\{\lambda(x, \dot{x} + \sigma c(y), t) - \lambda(x, \dot{x}, t) + \frac{\partial\lambda}{\partial\dot{x}}\sigma c(y)\right\} dy;$$

$$S_3 = \int [\lambda(x, \dot{x} + \sigma c(y), t) - \lambda(x, \dot{x}, t)]P^0(t, dy).$$

Further, in order for the set (1) to be an integral manifold of equation (4), we introduce arbitrary Yerugin functions [10]: a vector function $A = A(\lambda, x, \dot{x}, t)$ and a matrix $B = B(\lambda, x, \dot{x}, t)$ with properties $A(0, x, \dot{x}, t) \equiv 0$, $B(0, x, \dot{x}, t) \equiv 0$, and such that

$$\dot{\lambda} = A(\lambda, x, \dot{x}, t) + B(\lambda, x, \dot{x}, t)\dot{\xi}. \tag{6}$$

Equations (5) and (6) imply the following equalities

$$\begin{cases} \dfrac{\partial\lambda}{\partial\dot{x}} f = A - \frac{\partial\lambda}{\partial t} - \frac{\partial\lambda}{\partial x}\dot{x} - S_1 - S_2 - S_3, \\ \dfrac{\partial\lambda}{\partial\dot{x}}\sigma = B. \end{cases} \tag{7}$$

To determine the desired functions f and σ from relations (7), we use the following statement: *Lemma 1* [3; 12–13]. The set of all solutions of the linear system

$$Hv = g, H = (h_{\mu k}), v = (v_k), g = (g_\mu), \mu = \overline{1,m}, k = \overline{1,n}, m \leq n, \tag{8}$$

is determined by the expression

$$v = \alpha v^T + v^\upsilon, \tag{9}$$

where the rank of the matrix H equals to m. Here α is a scalar value,

$$v^T = [HC] = [h_1 ... h_m c_{m+1} ... c_{n-1}] = \begin{vmatrix} e_1 & ... & e_n \\ h_{11} & ... & h_{1n} \\ ... & ... & ... \\ h_{m1} & ... & h_{mn} \\ c_{m+1,1} & ... & c_{m+1,n} \\ ... & ... & ... \\ c_{n-1,1} & ... & c_{n-1,n} \end{vmatrix}$$

is the cross product of vectors $h_\mu = (h_{\mu k})$ and arbitrary vectors $c_\rho = (c_{\rho k}), \rho = \overline{m+1, n-1}$; e_k are unit vectors of space R^n, $v^T = (v_k^T)$

$$v_k^T = \begin{vmatrix} 0 & ... & 1 & ... & 0 \\ h_{11} & ... & h_{1k} & ... & h_{1n} \\ ... & ... & ... & ... & ... \\ h_{m1} & ... & h_{mk} & ... & h_{mn} \\ c_{m+1,1} & ... & c_{m+1,n} & ... & c_{m+1,n} \\ ... & ... & ... & ... & ... \\ c_{n-1,1} & ... & c_{n-1,k} & ... & c_{n-1,n} \end{vmatrix}, \quad v^\upsilon = H^+ g,$$

where $H^+ = H^T(HH^T)^{-1}, H^T$ is the matrix transposed to H.

By Lemma 1, using (8), (9) we determine the form of the vector function f and the columns σ_i of the matrix σ

$$f = s_1 \left[\frac{\partial\lambda}{\partial\dot{x}} C\right] + \left(\frac{\partial\lambda}{\partial\dot{x}}\right)^+ \left(A - \frac{\partial\lambda}{\partial t} - \frac{\partial\lambda}{\partial x}\dot{x} - S_1 - S_2 - S_3\right), \tag{10}$$

$$\sigma_i = s_{2i} \left[\frac{\partial\lambda}{\partial\dot{x}} C\right] + \left(\frac{\partial\lambda}{\partial\dot{x}}\right)^+ B_i, \left(i = \overline{1,r}\right). \tag{11}$$

Here $\sigma_i = (\sigma_{1i}, \sigma_{2i}, \ldots, \sigma_{ni})^T$ denotes the i-th column of the matrix $\sigma = (\sigma_{\nu j})$, $\left(\nu = \overline{1,n},\ \ j = \overline{1,r}\right)$. $B_i = (B_{1i}, B_{2i}, \ldots, B_{mi})^T$ is the i-th column of the matrix $B = (B_{\mu j})$, $\left(\mu = \overline{1,m},\ \ j = \overline{1,r}\right)$. By s_1, s_2 are denoted arbitrary scalar quantities.

The forms of the vector function f (10) and matrix σ (11) imply the general form of the set of second-order Ito differential equations (4) with a given integral manifold (1)

$$\ddot{x} = s_1 \left[\frac{\partial\lambda}{\partial\dot{x}} C\right] + \left(\frac{\partial\lambda}{\partial\dot{x}}\right)^+ \left(A - \frac{\partial\lambda}{\partial t} - \frac{\partial\lambda}{\partial x}\dot{x} - S_1 - S_2 - S_3\right) +$$

$$+\left(s_{21}\left[\frac{\partial\lambda}{\partial\dot{x}}C\right]+\left(\frac{\partial\lambda}{\partial\dot{x}}\right)^{+}B_1,\ \ldots\ ,\ s_{2r}\left[\frac{\partial\lambda}{\partial\dot{x}}C\right]+\left(\frac{\partial\lambda}{\partial\dot{x}}\right)^{+}B_r\right)\dot{\xi}.$$

To construct the Hamilton function, we first introduce a new variable $y_k = \dot{x}_k$ and rewrite the constructed equation (4) in the form

$$\begin{cases} \dot{x}_k = y_k, \\ y_k = f_k\left(x, y, t\right) + \sigma_{kj}\left(x, y, t\right)\dot{\xi}^j. \end{cases} \tag{12}$$

Here the vector function $f = (f_1, f_2, \ldots, f_n)^T$ and matrix columns $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_r)$ have the form (10), (11), respectively.

Then, using $z_k = \begin{cases} x_k \\ y_k \end{cases}$, $G_k = \begin{cases} y_k \\ f_k \end{cases}$, $\psi_j = \begin{cases} 0, \text{ for } j = \overline{1,n}\ , \\ \xi^{j-n}, \text{ for } j = n+1, n+2, \ldots, n+m\,, \end{cases}$
$\mu = (\mu_{kj}) = \begin{pmatrix} 0_{n\times n} & 0_{n\times m} \\ 0_{n\times n} & \sigma_{n\times m} \end{pmatrix}$, $\sigma = (\sigma_{kj})$ we rewrite equation (12) in the form

$$\dot{z}_k = G_k\left(z, t\right) + \mu_{kj}\left(z, t\right)\dot{\psi}_j. \tag{13}$$

Further, using $\nu_k = \begin{cases} q_k, & k = \overline{1,n} \\ p_{k-n}, & k = n+1, n+2, \ldots 2n \end{cases}$ and $\varphi = (\varphi_{k\nu}) = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix}$,
$p = (p_{kj}) = \begin{pmatrix} 0_{n\times n} & 0_{n\times m} \\ 0_{n\times n} & \sigma'_{n\times m} \end{pmatrix}$, and also taking into account $\begin{pmatrix} \frac{\partial H}{\partial p_k} \\ -\frac{\partial H}{\partial q_k} \end{pmatrix} = \left(\varphi_{k\nu}\frac{\partial H}{\partial \nu_\nu}\right)$, we rewrite the stochastic equation of the Hamiltonian structure (2) in the form

$$\dot{\nu}_k - \phi_{k\nu}\frac{\partial H}{\partial \nu_\nu} = p_{kj}\dot{\psi}_j. \tag{14}$$

If we introduce the inverse matrix $(\omega_{k\nu}) = (\varphi_{k\nu})^{-1} = \begin{pmatrix} 0_{n\times n} & -I_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{pmatrix}$ for $(\varphi_{k\nu})$ and vector $z_k \equiv \omega_{k\nu}\nu_\nu = \begin{pmatrix} -p_k, & k = \overline{1,n} \\ q_{k-n}, & k = \overline{n+1, 2n} \end{pmatrix}$, then equation (14) is transformed to the equivalent equation

$$\omega_{\nu k}\dot{z}_k - \frac{\partial H}{\partial z_k} = \omega_{\nu k}p_{\nu j}\dot{\psi}_j. \tag{15}$$

Consider the problem of indirect representation of equation (13) in the form of an equation of the Hamiltonian structure (15), i.e., using some matrix $\Gamma = \left(\gamma_\nu^k\right)$, consider the relation

$$\gamma_\nu^k\left(\dot{z}_k - G_k - \mu_{kj}\dot{\psi}_j\right) \equiv \omega_{\nu k}\dot{z}_k - \frac{\partial H}{\partial z_\nu} - \omega_{\nu k}p_{\nu j}\dot{\psi}_j$$

or

$$C_{\nu k}\dot{z}_k - D_\nu(z,t) - \gamma_\nu^k\mu_{kj}\dot{\psi}_j \equiv \omega_{\nu k}\dot{z}_k - \frac{\partial H}{\partial z_\nu} - \omega_{\nu k}p_{\nu j}\dot{\psi}_j\,, \tag{16}$$

where $C_{\nu k} = \gamma_\nu^k;\ D_\nu(z,t) = \gamma_\nu^k G_k$.

To fulfill the identity (16) it is necessary the conditions

$$C_{\nu k} = \omega_{\nu k},\ \ D_\nu(z,t) = -\frac{\partial H}{\partial z_\nu}\,, \tag{17}$$

$$\gamma_\nu^k\mu_{kj} = \omega_{\nu k}p_{\nu j},\ \ \left(\nu,\, k = \overline{1,2n},\ j = \overline{1, n+m}\right), \tag{18}$$

$$\gamma_\nu^k = \omega_{\nu k} \tag{19}$$

to be satisfied. From relations (16) and conditions (17)–(19), $\mu_{kj} = p_{\nu j}$ follows. This entails the fulfillment of the equality $\sigma_{kj} = \sigma'_{kj}$, $\left(k = \overline{1,n},\ j = \overline{1,m}\right)$.

Theorem 1. For the indirect construction of Hamiltonian structure stochastic equation (2) by the given set (1) so that the set (1) is an integral manifold of equation (15), it is necessary and sufficient that conditions (17)–(19) be satisfied.

2 Construction of the Birkhoffian structure stochastic equation (3)
by the given properties of motion (1)

To solve the problem, consider the relation

$$C_{\upsilon k}\dot z_k - D_\upsilon(z,t) - \mu_{\upsilon j}\dot\psi_j \equiv$$

$$\equiv \left[\frac{\partial R_k(z,t)}{\partial z_\nu} - \frac{\partial R_\nu(z,t)}{\partial z_\kappa}\right]\dot z_\kappa - \left[\frac{\partial B(z,t)}{\partial z_\nu} + \frac{\partial R_\nu(z,t)}{\partial t}\right] - T_{\nu j}\dot\psi_j,\ \ (\nu,\kappa = \overline{1,2n},\ j = \overline{1,n+m}). \tag{20}$$

(20) is fulfilled identically under the following conditions

$$C_{\nu k} = \left[\frac{\partial R_\kappa(z,t)}{\partial z_\upsilon} - \frac{\partial R_\upsilon(z,t)}{\partial z_\kappa}\right],\ \ D_\nu = \left[\frac{\partial B(z,t)}{\partial z_\upsilon} + \frac{\partial R_\upsilon(z,t)}{\partial t}\right],\ \ \mu = T. \tag{21}$$

Theorem 2. To construct the Birkhoffian structure stochastic equation (3) by the given set (1), so that set (1) is an integral manifold of equation (3), it is necessary and sufficient that conditions (21) are satisfied.

3 Example

Consider the stochastic problem of constructing Hamilton and Birkhoff functions for a given property of motion by the example of the motion of an artificial Earth satellite under the action of gravitational and aerodynamic forces [25].

Let the properties of motion

$$\Delta\,(t) : \lambda = \theta^2 + \alpha_1\dot\theta^2 + \alpha_2 = 0, \qquad \lambda \in R^1 \tag{22}$$

be given. Then the equation of perturbed motion (5) takes the form

$$\dot\lambda = 2\theta\dot\theta + 2\alpha_1\dot\theta\ddot\theta + S_1 + S_2 + S_3 = 2\theta\dot\theta + 2\alpha_1\dot\theta f + S_1 + S_2 + S_3 + 2\alpha_1\dot\theta\sigma\dot\xi, \tag{23}$$

where $S_1 = \alpha_1\sigma^2$, $\quad S_2 = \int\left\{2\alpha_1\sigma c(y)[4\dot\theta + \sigma c(y)]\right\}dy$, $\quad S_3 = \int\left\{2\alpha_1\sigma c(y)[4\dot\theta + \sigma c(y)]\right\}P^0(t,dy)$.

Let us introduce the scalar Yerugin functions $a = a(\lambda,\theta,\dot\theta,t)$, $b = b(\lambda,\theta,\dot\theta,t)$ with the property $a(0,\theta,\dot\theta,t) \equiv b(0,\theta,\dot\theta,t) \equiv 0$ and such that the relation

$$\dot\lambda = a\lambda(\theta,\dot\theta,t) + b\lambda(\theta,\dot\theta,t)\dot\xi \tag{24}$$

takes place. In our example from relations (23), (24), it follows that a set of equations (4) is written as $\ddot\theta = f(\theta,\dot\theta,t) + \sigma(\theta,\dot\theta,t)\dot\xi$ and it has the integral manifold (22) if f and σ have, respectively, the forms

$$f = \frac{a(\theta^2 + \alpha_1\dot\theta^2 + \alpha_2) - 2\theta\dot\theta - S_1 - S_2 - S_3}{2\alpha_1\dot\theta},\quad \sigma = \frac{b(\theta^2 + \alpha_1\dot\theta^2 + \alpha_2)}{2\alpha_1\dot\theta}. \tag{25}$$

Following [25], we write the equation of motion of an artificial Earth satellite under the action of gravitational and aerodynamic forces in the form

$$\ddot{\theta} = \tilde{f}(\theta, \dot{\theta}) + \tilde{\sigma}(\theta, \dot{\theta})\dot{\xi}, \tag{26}$$

where θ is pitch angle, $\tilde{f}$and $\tilde{\sigma}$ have the forms

$$\tilde{f} = Ql \sin 2\theta - Q[g(\theta) + \eta\dot{\theta}], \quad \tilde{\sigma} = Q\delta[g(\theta) + \eta\dot{\theta}].$$

Before constructing the Hamilton and Birkhoff functions, we first construct the Lagrangian by (26). In (26), relations (25) should be taken into account, which ensure the integrality of the given set (22). From $f = \tilde{f}$, $\sigma = \tilde{\sigma}$, it follows that the four parameters Q, δ, η, l, determining the dynamics of satellite motion, must satisfy the following relations

$$\begin{cases} a(\theta^2 + \alpha_1\dot{\theta}^2 + \alpha_2) - 2\theta\dot{\theta} - S_1 - S_2 - S_3 = 2\alpha_1\dot{\theta}\left\{Ql\sin 2\theta - Q[g(\theta) + \eta\dot{\theta}]\right\}, \\ b(\theta^2 + \alpha_1\dot{\theta}^2 + \alpha_2) = 2\alpha_1\dot{\theta}Q\delta[g(\theta) + \eta\dot{\theta}]. \end{cases}$$

Then, considering definition [26] and the action of random perturbations, equation (26) admits an indirect analytic representation in terms of a stochastic equation with a Lagrangian structure if there exists a function h such that the identity

$$d\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} - \sigma'(\theta, \dot{\theta}, t)\dot{\xi} \equiv h[\ddot{\theta} - f - \sigma\dot{\xi}] \tag{27}$$

takes place. We find the function $h = h(t)$ from the Helmholtz condition [26; 107] $\dfrac{\partial l_2}{\partial \dot{\theta}} = \dfrac{\partial l_1}{\partial t} + \dot{\theta}\dfrac{\partial l_1}{\partial \theta}$, which is necessary and sufficient for constructing the Lagrange equation equivalent to the scalar equation $l_1(\theta, \dot{\theta}, t)\ddot{\theta} + l_2(\theta, \dot{\theta}, t) = 0$. In particular, function $h = e^{-Q\eta t}$ satisfies this condition. Substituting h in (27), we get

$$e^{-Q\eta t}[\ddot{\theta} - f - \sigma\dot{\xi}] = \frac{\partial^2 L}{\partial \dot{\theta}^2}\ddot{\theta} + \frac{\partial^2 L}{\partial \dot{\theta}\partial\theta}\dot{\theta} + \frac{\partial^2 L}{\partial\theta\partial t} - \frac{\partial L}{\partial t}\sigma'\dot{\xi}.$$

Then we construct the desired Lagrangian in the form

$$L = e^{-Q\eta t}[\frac{1}{2}\dot{\theta}^2 - Q(\frac{1}{2}l\cos 2\theta + G)], \quad G = \int g(\theta)d\theta, \tag{28}$$

which provides an indirect representation of equation (27) in the form of the Lagrangian structure equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = e^{-Q\eta t}\sigma(\theta, \dot{\theta})\dot{\xi}.$$

Using the Lagrange function (28) and the Legendre transform, we define the Hamilton function as $H = \chi\dot{\theta} - L(\theta, \dot{\theta}, t)\Big|_{\dot{\theta} = \dot{\theta}(\theta, \chi, t)}$. Since $\chi = \dfrac{\partial L}{\partial \dot{\theta}}$, then $\chi = e^{-Q\eta t}\dot{\theta}$ and therefore $\dot{\theta} = e^{Q\eta t}\chi$. Then the canonical equation corresponding to the stochastic Lagrangian structure equation (27) will take the form

$$\begin{cases} \dot{\theta} = \dfrac{\partial H}{\partial \chi}, \\ \dot{\chi} = -\dfrac{\partial H}{\partial \theta} + \widehat{\sigma}\,(\theta, \chi, t)\dot{\xi}, \end{cases} \tag{29}$$

where $\widehat{\sigma} = \sigma'(\theta, \dot{\theta}, t)\Big|_{\dot{\theta}=\dot{\theta}(\theta,\chi,t)}$, and the Hamilton function is defined as

$$H = \frac{1}{2} e^{Q\eta t} \chi^2 e^{-Q\eta t} b(\theta). \tag{30}$$

To solve the problem of representing the Birkhoffian by a given equation (26), we use Theorem 2. According to the above constructed equation (29) and Hamilton function (30) from relations (21) for $C = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix}$ functions $R_\upsilon(\upsilon = 1, 2)$, $R = (R_1, R_2)$ and B are defined as $R = \{\chi, (1 + \varphi)\theta\}, B = \frac{1}{2}\varphi e^{Q\eta t}\chi^2 - \varphi e^{-Q\eta t} b(\theta)$, where φ is an arbitrary constant.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP 09258966).

References

1 Галиуллин А.С. Методы решения обратных задач динамики / А.С. Галиуллин. — М.: РУДН, 1986. — 224 с.

2 Галиуллин А.С. Избранные труды: [В 2-х т.]. — Т. II / А.С. Галиуллин. — М.: РУДН, 2009. — 462 с.

3 Мухаметзянов И.А. Уравнения программных движений / И.А. Мухаметзянов, Р.Г. Мухарлямов. — М.: РУДН, 1986. — 88 с.

4 Mukharlyamov R.G. Differential-algebraic equations of programmed motions of Lagrangian dynamical systems / R.G. Mukharlyamov // Mechanics of Solids. — 2011. — 46. — No. 4. — P. 534–543.

5 Mukharlyamov R.G. Control of system dynamics and constrains stabilization / R.G. Mukharlyamov, M.I. Tleubergenov // Communications in Computer and Information Science. — 2017. — No. 700. — P. 431–442. DOI:10.1007/978-3-319-66836-936.

6 Llibre J. Inverse Problems in Ordinary Differential Equations and Applications / J. Llibre, R. Ramirez. — Switzerland: Springer International Publishing, 2016. — 266 p.

7 Zhumatov S.S. Asymptotic Stability of Implicit Differential Systems in the Vicinity of Program Manifold / S.S. Zhumatov // Ukrainian Mathematical Journal. — 2014. — 66. — No. 4. — P. 625–632. DOI: 10.1007/s11253-014-0959-y.

8 Zhumatov S.S. Exponential Stability of a Program Manifold of Indirect Control Systems / S.S. Zhumatov // Ukrainian Mathematical Journal. — 2010. — 62. — No. 6. — P. 907–915. DOI: 10.1007/s11253-010-0399-2.

9 Zhumatov S.S. Stability of a Program Manifold of Control Systems with Locally Quadratic Relations / S.S. Zhumatov // Ukrainian Mathematical Journal. — 2009. — 61. No. 3. — P. 500–509. DOI: 10.1007/s11253-009-0224-y.

10 Еругин Н.П. Построение всего множества систем дифференциальных уравнений, имеющих заданную интегральную кривую / Н.П. Еругин // Прикладная математика и механика. — 1952. — 10. — Вып. 6. — С. 659–670.

11 Ibraeva G.T. Main Inverse Problem for Differential Systems With Degenerate Diffusion / G.T. Ibraeva, M.I. Tleubergenov // Ukrainian Mathematical Journal. — 2013. — 65. — No. 5. — P. 787–792.

12 Tleubergenov M.I. On the inverse stochastic reconstruction problem / M.I. Tleubergenov // Differential equations. — 2014. — 50. – Issue 2. — P. 274–278. DOI:10.1134/S0012266114020165.

13 Tleubergenov M.I. Stochastic Inverse Problem with Indirect Control / M.I. Tleubergenov, G.T. Ibraeva // Differential equations. — 2017. — 53. — Issue 10. — P. 1387–1391. DOI:10.1134 /S0012266117100172.

14 Vassilina G.K. Solution of the Problem of Stochastic Stability of an Integral Manifold by the Second Lyapunov Method / G.K. Vassilina, M.I. Tleubergenov // Ukrainian Mathematical Journal. – 2016. — 68. — No. 1. – P. 14–28. DOI:10.1007/s11253-016-1205-6.

15 Tleubergenov M.I. On the Solvability of the Main Inverse Problem for Stochastic Differential Systems / M.I. Tleubergenov, G.T. Ibraeva // Ukrainian Mathematical Journal. — 2019. — 71. — No. 1. — P. 157–165. DOI:10.1007/s11253-019-01631-w.

16 Tleubergenov M.I. On the Closure of Stochastic Differential Equations of Motion / M.I. Tleubergenov, G.T. Ibraeva // Eurasian Mathematical Journal. — 2021. — 12. — No. 2. — P. 82–89.

17 Tleubergenov M.I. An Inverse Problem for Stochastic Differential Systems / M.I. Tleubergenov // Differential equations. — 2001. — 37. — Issue 5. — P. 751—753. https://doi.org/10.1023/A:1019 285119532.

18 Tleubergenov M.I. On Stochastic Inverse Problem of Construction of Stable Program Motion / M.I. Tleubergenov, G.K. Vassilina // Open Mathematics. — 2021. — 19. — P. 157–162. https://doi.org/10.1515/math-2021-0005.

19 Tleubergenov M.I. On Construction of a Field of Forces along Given Trajectories in the Presence of Random Perturbations / M.I. Tleubergenov, G.K. Vassilina, G.A. Tuzelbaeva // Bulletin of Karaganda University. Mathematics series. — 2021. — No. 1(101). — P. 98–103. DOI 10.31489/ 2021M1/98-10.

20 Пугачев В.С. Стохастические дифференциальные системы. Анализ и фильтрация / В.С. Пугачев, И.Н. Синицын. — М.: Наука, 1990. — 632 с.

21 Туладхар Б.М. Построение уравнений в форме Лагранжа, Гамильтона и Биркгофа по заданным свойствам движения: автореф. ... канд. физ.-мат. наук / Б.М. Туладхар. — М.: РУДН им. П. Лумумбы, 1983.

22 Тлеубергенов М.И. Обратные задачи стохастических дифференциальных систем: автореф. ... канд. физ.-мат. наук / М.И. Тлеубергенов. — Алматы: Институт математики, 1999.

23 Тлеубергенов М.И. О построении дифференциального уравнения по заданным свойствам движения при наличии случайных возмущений / М.И. Тлеубергенов, Д.Т. Ажымбаев // Изв. НАН РК. Сер. физ.-мат. — 2007. — № 3. — С. 15–20.

24 Tleubergenov M.I. On the construction of a set of stochastic differential equations on the basis of a given integral manifold independent of velocities / M.I. Tleubergenov, D.T. Azhymbaev // Ukrainian Mathematical Journal. — 2010. — 62. – No. 7. — P. 1163–1173. DOI:10.1007/s11253-010-0421-8

25 Сагиров П. Стохастические методы в динамике спутников / П. Сагиров // Механика. Период. сб. переводов ин. ст. — 1974. — № 5 (147). — С. 28–47.

26 Santilli R.M. Foundations of Theoretical Mechanics. 1. The Inverse Problem in Newtonian Mechanics / R.M. Santilli. — New-York: Springer-Verlag, 1978. — 266 p.

М.Ы. Тілеубергенов[1,2], Г.Қ. Василина[1,3], С.Р. Сейсенбаева[1,2]

[1] *Математика және математикалық модельдеу институты, Алматы, Қазақстан;*
[2] *Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан;*
[3] *Ғ. Дәукеев атындағы Алматы энергетика және байланыс университеті, Алматы, Қазақстан*

Канондық айнымалылардағы қозғалыстың стохастикалық дифференциалдық теңдеулерін құру

А.С. Галиуллин динамиканың кері есептерінің классификациясын қарапайым дифференциалдық теңдеулер класында ұсынды. Мақалада қарастырылатын есеп динамиканың кері есептерінің бірінші түріне жатады (динамиканың кері есептерінің негізгі үш түрінің ішінде) кездейсоқ түрткілердің бар болуы туралы қосымша болжамдағы негізгі кері есепке. Сонымен бірге Гамильтон және Бирхофф теңдеулері тәуелсіз өсушелері бар процестер класынан кездейсоқ түрткілер бар болған кезде қозғалыстың берілген қасиеттерінен құрастырылған. Ал алынған қозғалыстың берілген қасиеттері үшін Гамильтондық та, Бирхоффтық та құрылымды стохастикалық теңдеулерін құру есебінің шешімін табу үшін алынған қажетті және жеткілікті шарттары ауырлық күштерінің және аэродинамикалық күштерінің әсерінен Жердің жасанды серігінің қозғалысы мысалында көрсетілген.

Кілт сөздер: стохастикалық дифференциалдық теңдеу, тәуелсіз өсімшелі үдерістер класы, Гамильтондық және Бирхоффтық құрылымды стохастикалық теңдеулер, негізгі кері есеп.

М.И. Тлеубергенов[1,2], Г.К. Василина[1,3], С.Р. Сейсенбаева[1,2]

[1] *Институт математики и математического моделирования, Алматы, Казахстан;*
[2] *Казахский национальный университет имени Аль-Фараби, Алматы, Казахстан;*
[3] *Алматинский университет энергетики и связи имени Г. Даукеева, Алматы, Казахстан*

Построение стохастических дифференциальных уравнений движения в канонических переменных

А.С. Галиуллиным была предложена классификация обратных задач динамики в классе обыкновенных дифференциальных уравнений. И рассматриваемая в настоящей работе задача относится к первому типу обратных задач динамики (из трех основных типов обратных задач динамики) — основной обратной задаче при дополнительном предположении о наличии случайных возмущений. В статье строятся уравнения Гамильтона и Биркгофа по заданным свойствам движения при наличии случайных возмущений из класса процессов с независимыми приращениями. И полученные необходимые и достаточные условия разрешимости задачи построения стохастических уравнений как гамильтоновой, так и биркгофиановой структуры по заданным свойствам движения проиллюстрированы на примере движения искусственного спутника Земли под действием сил тяготения и аэродинамических сил.

Ключевые слова: стохастическое дифференциальное уравнение, класс процессов с независимыми приращениями, стохастическое уравнения гамильтоновой и биркгофиановой структур, основная обратная задача.

References

1 Galiullin, A.S. (1986). *Metody resheniia obratnykh zadach dinamiki [Methods for solving inverse problems of dynamics]*. Moscow: RUDN [in Russian].

2 Galiullin, A.S. (2009). *Izbrannye trudy [Selected works]*. (Vols. 1-2; Vol. II). Moscow: RUDN [in Russian].

3 Mukhametzyanov, I.A., & Mukharlyamov, R.G. (1986). *Uravneniia programmnykh dvizhenii [Equations of program motions]*. Moscow: RUDN [in Russian].

4 Mukharlyamov, R.G. (2011). Differential-algebraic equations of programmed motions of Lagrangian dynamical systems. *Mechanics of Solids, 46*(4), 534–543.

5 Mukharlyamov, R.G., & Tleubergenov, M.I. (2017). Control of system dynamics and constrains stabilization. *Communications in Computer and Information Science, 700*, 431–442. DOI:10.1007/978-3-319-66836-936.

6 Llibre, J., & Ramizer, R. (2016). *Inverse Problems in Ordinary Differential Equations and Applications*. Switzerland: Springer International Publishing.

7 Zhumatov, S.S. (2014). Asymptotic Stability of Implicit Differential Systems in the Vicinity of Program Manifold. *Ukrainian Mathematical Journal, 66*(4), 625–632. DOI: 10.1007/s11253-014-0959-y.

8 Zhumatov, S.S. (2010). Exponential Stability of a Program Manifold of Indirect Control Systems. *Ukrainian Mathematical Journal, 62*(6), 907–915. DOI: 10.1007/s11253-010-0399-2.

9 Zhumatov, S.S. (2009). Stability of a Program Manifold of Control Systems with Locally Quadratic Relations. *Ukrainian Mathematical Journal, 61*(3), 500–509. DOI: 10.1007/s11253-009-0224-y.

10 Yerugin, N.P. (1952). Postroenie vsego mnozhestva sistem differentsialnykh uravnenii, imeiushchikh zadannuiu integralnuiu krivuiu [Construction of the entire set of systems of differential equations with a given integral curve]. *Prikladnaia matematika i mekhanika — Applied Mathematics and Mechanics, 10*(6), 659–670 [in Russian].

11 Ibraeva, G.T., & Tleubergenov, M.I. (2013). Main Inverse Problem for Differential Systems With Degenerate Diffusion. *Ukrainian Mathematical Journal, 65*(5), 787–792.

12 Tleubergenov, M.I. (2014). On the inverse stochastic reconstruction problem. *Differential equations, 50*(2), 274–278. DOI:10.1134/S0012266114020165.

13 Tleubergenov, M.I., & Ibraeva, G.T. (2017). Stochastic Inverse Problem with Indirect Control. *Differential equations, 53*(10), 1387–1391. DOI:10.1134/S0012266117100172.

14 Vassilina, G.K., & Tleubergenov, M.I. (2016). Solution of the Problem of Stochastic Stability of an Integral Manifold by the Second Lyapunov Method. *Ukrainian Mathematical Journal, 68*(1), 14–28. DOI:10.1007/s11253-016-1205-6.

15 Tleubergenov, M.I., & Ibraeva, G.T. (2019). On the Solvability of the Main Inverse Problem for Stochastic Differential Systems. *Ukrainian Mathematical Journal, 71*(1), 157–165. DOI:10.1007/s11253-019-01631-w.

16 Tleubergenov, M.I., & Ibraeva, G.T. (2021). On the Closure of Stochastic Differential Equations of Motion. *Eurasian Mathematical Journal, 12*(2), 82–89.

17 Tleubergenov, M.I. (2001). An Inverse Problem for Stochastic Differential Systems. *Differential equations, 37*(5), 751-–753. https://doi.org/10.1023/A:1019285119532.

18 Tleubergenov, M.I., & Vassilina, G.K. (2021). On Stochastic Inverse Problem of Construction of Stable Program Motion. *Open Mathematics, 19*, 157–162. https://doi.org/10.1515/math-2021-0005

19 Tleubergenov, M.I., Vassilina, G.K., & Tuzelbaeva, G.A. (2021). On construction of a field of forces along given trajectories in the presence of random perturbations. *Bulletin of Karaganda University-Mathematics, 1*(101), 98–103. DOI 10.31489/2021M1/98-10.

20 Pugachev, V.S., & Sinitsyn, I.N. (1990). *Stokhasticheskie differentsialnye sistemy. Analiz i filtratsiia [Stochastic differential systems. Analysis and filtering]*. Moscow: Nauka [in Russian].

21 Tuladkhar, B.M. (1983). *Postroenie uravnenii v forme Lagranzha, Gamiltona i Birkgofa po zadannym svoistvam dvizheniia [Construction of Equations of Lagrange, Hamilton, and Birkhoff Types on the Basis of Given Properties of Motion]. Extended abstract of candidate's thesis.* Moscow [in Russian].

22 Tleubergenov, M.I. (1999). *Obratnye zadachi stokhasticheskikh differentsialnykh sistem [Inverse problems of stochastic differential systems]. Extended abstract of candidate's thesis.* Almaty [in Russian].

23 Tleubergenov, M.I., & Azhymbaev, D.T. (2007). O postroenii differentsialnogo uravneniia po zadannym svoistvam dvizheniia pri nalichii sluchainykh vozmushchenii [On construction of a differential equation on the basis of given properties of motion under random perturbations]. *Izvestiia NAN RK. Seriia fiziko-matematicheskaia — News of the NAS RK. Physico-mathematical series, 3*, 15–20 [in Russian].

24 Tleubergenov, M.I., & Azhymbaev, D.T. (2010). On the construction of a set of stochastic differential equations on the basis of a given integral manifold independent of velocities *Ukrainian Mathematical Journal, 62*(7), 1163–1173. DOI:10.1007/s11253-010-0421-8

25 Sagirov, P. (1974). Stokhasticheskie metody v dinamike sputnikov [Stochastic methods in dynamics of satellites]. *Mekhanika. Periodicheskii sbornik perevodov inostrannykh statei — Mechanics. Periodical Collection of Translations of Foreign Articles, 5*(147), 28–47 [in Russian].

26 Santilli, R.M. (1978). *Foundations of Theoretical Mechanics. 1. The Inverse Problem in Newtonian Mechanics.* New-York: Springer-Verlag.

DOI 10.31489/2022M3/152-163
UDC 510.67

A.R. Yeshkeyev[1], I.O. Tungushbayeva[1,*], M.T. Omarova[1,2]

[1]*Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan;*
[2]*Karaganda University of Kazpotrebsoyuz, Karaganda, Kazakhstan*
(E-mail: aibat.kz@gmail.com, intng@mail.ru, omarovamt_963@mail.ru)

Forcing companions of Jonsson AP-theories

This article is devoted to the study of the forcing companions of the Jonsson AP-theories in the enriched signature. It is proved that the forcing companion of the theory does not change when expanding the theories under consideration, which have some properties, by adding new predicate and constant symbols to the language. The model-theoretic results obtained in this paper in general form are supported by examples from differential algebra. An approach in combining a Jonsson and non-Jonsson theories is demonstrated. In this paper, for the first time in the history of Model Theory. This will allow us to further develop the methods of research of Jonsson theories and expand the apparatus for studying incomplete theories.

Keywords: Jonsson theory, perfect Jonsson theory, AP-theory, forcing, forcing companion, enrichment of a signature, expanding theory, differential field, differentially closed field, differentially perfect field.

Introduction

In recent years, Model Theory has increasingly revealed its potential in solving important problems from various areas of mathematics. Thus, many significant facts concerning differential algebras, namely differential fields of zero and positive characteristic, were established through the use of model-theoretic methods in the studies of D. Marker, L. Blum, K. Wood, and others. At the same time, there is an increasing need to develop their own apparatus of Model Theory, especially in the study of incomplete theories. In the 1980s, among inductive theories, a special subclass of Jonsson theories was singled out, which are incomplete. Examples of Jonsson theories are the theories of well-known classical algebras, such as group theory, fixed characteristic field theory, linear order theory, etc. are provided. The methods used in the study of this class largely demonstrate their usefulness due to the numerous results obtained by B. Poizat, T.G. Mustafin, A.R. Yeshkeyev, E.T. Mustafin.

In [1], the authors began the study of the Jonsson differential algebras: results were obtained for differential fields of characteristic 0 and p. Here we continue to develop this direction while expanding the language of these theories and considering forcing companions in a new enrichment.

In the framework of the study of Jonsson theories, earlier works [2–4] considered theories obtained as constructions of Jonsson theories. In this paper, we work with a theory that is a union of two theories, where the first one is Jonsson and the other is not.

1 Preliminary information

We start with the main definitions and facts concerning the subject of the study. Recall the definitions of a model companion and a forcing companion.

Definition 1. [5; 156] Let T and T_{MC} be some L-theories. The theory T_{MC} is called a model completion of the theory T if:

1) T and T_{MC} are mutually model consistent, i.e., any model of the theory T is embedded in the model of the theory T_{MC} and vice versa;

*Corresponding author.
E-mail: intng@mail.ru

2) T_{MC} is a model complete theory;

3) if $A \models T$, then $T_{MC} \cup D(A)$ is a complete theory. The theory T_{MC} is called a model companion if conditions 1) and 2) hold.

Definition 2. [6; 129] Let T be a theory of the language L. A forcing companion of the theory T is a theory T^f which is the set of all sentences of the language L weakly forced by $\emptyset$.

The following results were proved by J. Barwise and A. Robinson:

Theorem 1. [6; 133] Let T_1 and T_2 be the theories of the language L. Then T_1 and T_2 are mutually model consistent if and only if $T_1^f = T_2^f$.

Theorem 2. [6; 134] Let T be mutually model consistent with some inductive theory T'. Then $T' \subseteq T^f$. Therefore, if T is an inductive theory then $T \subseteq T^f$.

Definition 3. [5; 80] A theory T has the joint embedding property (JEP) if for any models U, B of the theory T there exists a model M of the theory T and isomorphic embeddings $f : U \to M$, $g : B \to M$.

Definition 4. [5; 68] A theory T has the amalgam property (AP) if for any models U, B_1, B_2 of the theory T and isomorphic embeddings $f_1 : U \to B_1$, $f_2 : U \to B_2$ there are $M \models T$ and isomorphic embeddings $g_1 : B_1 \to M$, $g_2 : B_2 \to M$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Since the work relates mainly to the study of the Jonsson theories, we will give the main definitions concerning them. More detailed information about the Jonsson theories can be found mainly in [7]. In works [8–11], newer and more specific results have been published, and the apparatus for studying Jonsson theories has been expanded.

We are working within the framework of the following definition of Jonsson theory published in the Russian edition of [5].

Definition 5. [5; 80] A theory T is called Jonsson if:
1. the theory T has at least one infinite model;
2. T is an inductive theory;
3. T has the amalgam property (AP);
4. T has the joint embedding property (JEP).

Many classical objects from Algebra satisfied such conditions, and these theories are Jonsson
1) group theory;
2) theory of abelian groups;
3) theory of boolean algebras;
4) theory of linear orders;
5) field theory of characteristic p, where p is zero or a prime number;
6) theory of ordered fields;
7) theory of modules.

The following concepts and facts play a crucial role in the construction of a model-theoretic apparatus associated with the study of Jonsson theories.

Definition 6. [7; 155] Let T be a Jonsson theory. A model C_T of power $2^{|T|}$ is called to be a semantic model of the theory T if C_T is a $|T|^+$-homogeneous $|T|^+$-universal model of the theory T.

Theorem 3. [7; 155] T is Jonsson if it has a semantic model C_T.

The following definition was introduced by T.G. Mustafin.

Definition 7. [7; 155] A Jonsson theory T is called perfect if its semantic model C_T is saturated.

Definition 8. [7; 161] The elementary theory of a semantic model of the Jonsson theory T is called the center of this theory. The center is denoted by T^*, i.e. $Th(C) = T^*$.

Theorem 4. [7; 158] Let T be an arbitrary Jonsson theory. Then the following conditions are equivalent:

1) the theory T is perfect;
2) $T^* = Th(C)$ is the model companion of the theory T.

The following theorem is of particular importance for this study:

Theorem 5. [7; 162] Let T be a perfect Jonsson theory. Then the following statements are equivalent:
1) T^* is the model companion of T;
2) $ModT^* = E_T$;
3) $T^* = T^f$, where T^f is a forcing companion of the theory T.

Theorem 6. [12; 1243] Let T be a Jonsson theory. Then for any model $A \in E_T$ theory $T^0(A)$ is Jonsson, where $T^0(A) = Th_{\forall\exists}(A)$.

We can see that in the case of the perfectness of T its center T^* is also a Jonsson theory.

The following definition will help us to specify the class of Jonsson theories which we will deal with in this paper.

Definition 9. [13; 120] A Jonsson theory is said to be hereditary if, in any of its permissible enrichment, it preserves the Jonssonness.

As mentioned before, Jonsson theories should have joint embedding and amalgam properties. At the same time, it is known from [14; 270] that these two properties are generally independent of each other. However, theories with AP and JEP form special subclasses among inductive theories that are of interest for studying the internal structure of their model classes. In work [1], A.R. Yeshkeyev introduced the following concepts:

Definition 10. [1; 130] A theory T is called an AP-theory if, from the fact that it has the amalgam property, it follows that T also has the joint embedding property, i.e. $AP \to JEP$.

Definition 11. [1; 130] A theory T is called a JEP-theory if T has the joint embedding property and this implies the presence of the amalgam property, i.e. $JEP \to AP$.

Definition 12. [1; 130] We call a theory T an AJ-theory if the properties of the amalgam and the joint embedding are equivalent for T, i.e. $AP \leftrightarrow JEP$.

Examples are various classes of unars [14; 270]. In addition, in [1], it is shown that the theory of differential fields of characteristic 0 and the theory of differentially perfect fields of characteristic p, which will be discussed later, are AP-theories.

2 Forcing companions of theories in an enriched signature

Now we move on to the problem statement. We consider the theories Δ_1, Δ_2, Δ_3 satisfy the following conditions:

1) Δ_1 is an inductive theory that is not a Jonsson theory but has a model companion which is the theory Δ_3,
2) Δ_2 is a hereditary Jonsson AP-theory that has a model companion, which is also Δ_3.

Based on the conditions set, we can draw the following conclusions. All three theories are mutually model consistent, because Δ_3 is mutually model consistent with both Δ_1 and Δ_2, for which Δ_3 is the model companion, which means that Δ_1 and Δ_2 are mutually model consistent with each other. At the same time, according to Theorem 1, the forcing companions of mutually model consistent theories must coincide, which means that $\Delta_1^f = \Delta_2^f$. Δ_2 is a perfect Jonsson theory, while $\Delta_2^* = Th(C) = \Delta_3$, C is a semantic model of Δ_2, which follows from Theorem 4. In addition, Theorem 5 gives us reason to assert that Δ_3 is also a forcing companion of Δ_2, i.e. $\Delta_3 = \Delta_2^f$. So we get $\Delta_1^f = \Delta_2^f = \Delta_3$.

Consider the following extensions of the theories Δ_1, Δ_2, Δ_3 in various language enrichment L by adding new constant and predicate symbols c and P. Let $\overline{\Delta_1}$ be a theory extending Δ_1 by enriching the language L with the predicate symbol P as follows:

$$\overline{\Delta_1} = \Delta_1 \cup \Delta_1^f \cup \{P, \subseteq\},$$

where $\{P, \subseteq\}$ is an infinite list of $\exists$-sentences and interpretation of P is an existentially closed submodel in model of Δ_1.

Let $\overline{\Delta_2}$ be a theory that extends Δ_2 when a new constant symbol c is added to the language L and defined as follows:

$$\overline{\Delta_2} = \Delta_2 \cup \Delta_2^f \cup Th_{\forall\exists}(C, c),$$

where C is a semantic model of Jonsson theory Δ_2. Since Δ_2 is a hereditary Jonsson theory, $\overline{\Delta_2}$ is also a Jonsson theory.

Here we pose two questions:

1) How will the addition of new symbols P and c to the language L and the subsequent expansion of Δ_1 and Δ_2 affect the forcing companion of the received theories?

2) When combining the theories $\overline{\Delta_1}$ and $\overline{\Delta_2}$, can a consistent theory be obtained and what will be its forcing companion?

The answer to the first question is the following theorem.

Theorem 7. $\overline{\Delta_1}^f = \Delta_1^f$.

Proof. According to Theorem 2, because Δ_1 is an inductive theory, $\Delta_1 \subseteq \Delta_1^f$. This means that $\Delta_1 \cup \Delta_1^f = \Delta_1^f = \Delta_3$. Therefore, $\overline{\Delta_1}$ can be written as $\Delta_3 \cup \{P, \subseteq\}$. Since the set $\{P, \subseteq\}$ consists only of existential formulas, theories Δ_3 and $\overline{\Delta_1}$ do not differ in universal formulas, which means they are mutually model consistent. As is known from Theorem 1, the forcing companions in this case of these two theories must be equal. At the same time, Δ_3, which is a forcing companion of Δ_1 and Δ_2, is forcing-complete, because $\Delta_3^f = (\Delta_1^f)^f = \Delta_1^f = \Delta_3$. Hence, $\Delta_3^f = \overline{\Delta_1}^f = \Delta_3$, and $\overline{\Delta_1}^f = \Delta_1^f$.

Thus, we can conclude that the forcing companion of the inductive theory Δ_1 does not change when enriching the language of this theory with a new predicate symbol P.

Theorem 8. $\overline{\Delta_2}^f = \Delta_2^f$.

Proof. The proof is similar to the proof of Theorem 7. Since Δ_2 is a Jonsson theory, it is inductive, which means by Theorem 2 $\Delta_2 \subseteq \Delta_2^f$ and $\Delta_2 \cup \Delta_2^f = \Delta_2^f = \Delta_3$. So $\overline{\Delta_2} = \Delta_3 \cup Th_{\forall\exists}(C, c)$. All the sentences in $Th_{\forall\exists}(C, c)$ are $\forall\exists$-formulas, which means that theories Δ_3 and $\overline{\Delta_2}$ do not differ in universal formulas, i.e., they are mutually model consistent. We can conclude from this that their forcing companions are equal, with $\Delta_3^f = \overline{\Delta_2}^f = \Delta_3$, and $\overline{\Delta_2}^f = \Delta_2^f$.

This means that the addition of the new constant c to language L did not affect the forcing companion when expanding theory Δ_2 to $\overline{\Delta_2}$.

To answer the second question, we recall the Robinson's consistency theorem.

Theorem 9. [5; 77] Let T be a complete theory of language L, languages L_1 and L_2 are extensions of language L such that $L_1 \cap L_2 = L$, and theories T_1 and T_2 are consistent extensions of theory T in languages L_1 and L_2 respectively. Then $T_3 = T_1 \cup T_2$ is a consistent theory.

Now we can formulate and prove the following result.

Theorem 10. i) The theory $\overline{\Delta_1} \cup \overline{\Delta_2}$ is consistent.

ii) $(\overline{\Delta_1} \cup \overline{\Delta_2})^f = \Delta_1^f = \Delta_2^f$

Proof. i) As noted above, $\overline{\Delta_1} = \Delta_3 \cup \{P, \subseteq\}$ and $\overline{\Delta_2} = \Delta_3 \cup Th_{\forall\exists}(C, c)$. Applying Theorem 9, we will consider Δ_3 as the theory T, $\overline{\Delta_1}$ as the theory T, acting as an extension of Δ_3 by adding a new predicate symbol P to the language, and T_2 as the theory $\overline{\Delta_2}$, which is an extension of Δ_3 by adding the constant symbol c to the language. In this case, $L = L_1 \cap L_2$, where L_1 is the language of theory $\overline{\Delta_1}$, L_2 is the language of theory $\overline{\Delta_2}$. Therefore, the theory obtained as the union of $\overline{\Delta_1} \cup \overline{\Delta_2}$ is consistent.

ii) Obviously, $\overline{\Delta_1} \cup \overline{\Delta_2} = \Delta_3 \cup \{P, \subseteq\} \cup Th_{\forall\exists}(C, c)$. Theorems 7 and 8 allow us to assert that the forcing companion of theories $\Delta_3 \cup \{P, \subseteq\} \cup Th_{\forall\exists}(C, c)$ and Δ_3 is theory Δ_3. Hence, $(\overline{\Delta_1} \cup \overline{\Delta_2})^f = \Delta_1^f = \Delta_2^f$.

3 Application of the result to differential algebra

The results formulated above, described for the general situation in model theory, can be interpreted using examples of differential algebra, namely, when considering the theory of differential fields of characteristic 0, the theory of differentially closed fields of characteristic 0, the theory of differential fields of characteristic p, the theory of differentially closed fields of characteristic p. First, we will give the basic definitions and theorems concerning these theories. All concepts whose definitions are not given here can be found in [1].

We use the following notation: DF for the theory of differential fields, DPF for the theory of differentially perfect fields, DCF for the theory of differentially closed fields. The lower index 0 or p indicates the corresponding characteristic of the underlying field.

Definition 13. [15; 7] The differentiation of the ring A is called the mapping $a \to D(a)$ rings A into itself satisfying the relations

$$D(x + y) = D(x) + D(y),$$

$$D(xy) = xDy + yDx.$$

Definition 14. [15; 8] A differential ring is a commutative ring with a unit in which some differentiation is given.

In the case where the differential ring is a field F, we will talk about a differential field. Differential fields are models of the theory of differential fields DF, given by the axioms of field theory and the following two sentences:

$$\forall x \forall y \;\; D(x + y) = D(x) + D(y),$$

$$\forall x \forall y \;\; D(xy) = xD(y) + yD(x),$$

where $x, y \in F$.

The language used to study differential fields is the language $L = \{+, -, \cdot, D, 0, 1\}$. Here the differentiation operator D plays the role of a single functional symbol.

The concept of a differentially closed field was first proposed by A. Robinson [16; p. 2]. However, A. Robinson did not formulate axioms for the theory of differentially closed fields, which was corrected later by L. Blum for the case of characteristic 0. The situation with characteristic p was studied in detail by C. Wood and looks similar.

Definition 15. [17; 9] A differential field F is called differentially closed if whenever $f(x), g(x) \in F\{X\}$, $g(x)$ is nontrivial, has a nonzero value and the order of $f(x)$ is greater than the order of $g(x)$, there exists $a \in F$ such that $f(a) = 0$ and $g(a) \neq 0$.

Thus, the theory of differentially closed fields DCF is a theory consisting of the axioms DF and the following two axioms:

1) Each nonconstant polynomial from one variable has a solution.

2) If $f(x)$ and $g(x)$ are differential polynomials such that the order of $f(x)$ is greater than the order of $g(x)$, $g(x)$ is nontrivial, then $f(x)$ has a solution not being the solution of $g(x)$.

The following are some basic facts about the theories of differential fields and differentially closed fields of various characteristics.

Theorem 11. [18; 581] DCF_p is complete and model-complete.

Theorem 12. [19; 131] DF_0 has the joint embedding and amalgam properties.

Theorem 13. [19; 128] The DCF_0 theory is a model completion of the DF_0 theory.

Theorem 14. [18; 578] The theory DF_p of differential fields of characteristic p does not admit the amalgam property.

The author notes that the main reason is the absence of roots of the p-th degree in some constant elements of the field in the general case.

Theorem 15. [20; 92] DF_p has a model companion, but does not have a model completion.

Definition 16. [20; 92] A differential field F is called differentially perfect if any of its extensions is separable.

Theorem 16. [20; 92] In order for the differential field F of characteristic p to be differentially perfect, it is necessary and sufficient that $p = 0$ or $p > 0$ and $F^p = C$.

Thus, the theory DPF differentially perfect fields of characteristic p is given by the axioms DF and the following axiom:

$$\forall x \exists y \ \ (D(x) = 0 \to y^p = x).$$

Theorem 17. [18; 579] DPF_p is a model consistent extension of DF_p.

Based on this fact, it is easy to see that theories DPF_p and DF_p are mutually model consistent, since each differentially perfect field is a model of theory DF_p and there will always be some model of theory DPF_p, in which any differential field of characteristic p can be embedded.

Theorem 18. [18; 578] The theory DPF_p of differentially perfect fields of characteristic p admits the amalgam property.

Theorem 19. [18; 581] The theory DCF_p of differentially closed fields of characteristic p is the model companion of the theory DF_p differential fields of characteristic p and the model completion for the theory DPF_p of differentially perfect fields of characteristic p.

In work [1], the following statements related to the theories described above were proved.

Theorem 20. [1; 131] DF_0 is a perfect Jonsson theory.

Theorem 21. [1; 131] DCF_0 is the center of the Jonsson theory DF_0.

Theorem 22. [1; 131] DF_p is not a Jonsson theory.

Theorem 23. [1; 132] DPF_p is a perfect Jonsson theory.

Theorem 24. [1; 132] DCF_p is the center of the Jonsson theory DPF_p.

In addition, DF_0 and DPF_p are strongly convex theories in the classical Robinson sense, which allows us to state the following:

Theorem 25. [1; 132] DF_0 and DPF_p are Jonsson AP-theories.

Due to the above facts, we can project the results described in the previous paragraph to the case of differentially closed fields of zero and positive characteristic. However, while in the case of characteristic 0 the results are trivial by virtue of Theorem 20, the situation with differential fields of characteristic p is of greater interest. As the theory Δ_1, we can consider DF_p, which is not Jonsson, as stated in Theorem 22, but inductive (because of universality) and has a model companion according to Theorem 19, which is DCF_p. The role of the theory Δ_2 will be played by the Jonsson AP-theory DPF_p, whose model completion (and, consequently, model companion) is DCF_p. Δ_3 is replaced by

DCF_p, which is the center and the forcing companion of DPF_p. We additionally impose a condition on DCF_p, considering it to be hereditary Jonsson theories with respect to enrichment with a new constant symbol c. Since DCF_p is the center of DPF_p, and also due to the saturation of the semantic model C of DPF_p, the heredity of DCF_p is sufficient for DPF_p to be a hereditary Jonsson theory as well. According to Theorem 17, DF_p and DPF_p are mutually model consistent (which is also clear from the fact that they have a common model companion). We obtain that, by virtue of mutual model consistency, the forcing companions of the theories of differential fields and differentially perfect fields of the characteristics of p are equal and represent DCF_p:

$$DF_p^f = DPF_p^f = DCF_p.$$

Since we are going to add a new predicate symbol P later, it will not affect the mutual model compatibility of these theories in any way, because P does not generate new elements in the models DPF_p and DCF_p. The situation is similar with the new constant c: since the constant can be represented as a single predicate symbol, mutual model compatibility is preserved for the new specified theories.

Finally, by enriching the language of differential field theory with the new predicate symbol and constant, as was done in Section 2, we can obtain the following theories:

$$\overline{DF_p} = DF_p \cup DF_p^f \cup \{P, \subseteq\}, \quad (1)$$

$$\overline{DPF_p} = DPF_p \cup DPF_p^f \cup Th_{\forall\exists}(C, c). \quad (2)$$

Note that the equalities (1) and (2) can be written as

$$\overline{DF_p} = DCF_p \cup \{P, \subseteq\},$$

$$\overline{DPF_p} = DCF_p \cup Th_{\forall\exists}(C, c).$$

Thus, based on the reasoning and conclusions of the previous section, we can draw the following conclusions:

Theorem 26. $\overline{DF_p}^f = DF_p^f$.

Theorem 27. $\overline{DPF_p}^f = DPF_p^f$.

Theorem 28. i) $\overline{DF_p} \cup \overline{DPF_p}$ is consistent.
ii) $(\overline{DF_p} \cup \overline{DPF_p})^f = DF_p^f = DPF_p^f$.

In the future, the authors plan to continue the study of theory $\overline{\Delta_1} \cup \overline{\Delta_2}$ obtained within the framework of constructing the central types in the Jonsson theory and the Jonsson spectrum in the sense of the works [21–23].

Acknowledgments

This research was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP09260237).

References

1 Yeshkeyev A.R. Connection between the amalgam and joint embedding properties / A.R. Yeshkeyev, I.O. Tungushbayeva, M.T. Kassymetova // Bulletin of the Karaganda University-Mathematics. — 2022. — 105. — No. 1. — P. 127–135. DOI 10.31489/2022M1/127-135

2 Yeshkeyev A.R. Properties of hybrids of Jonsson theories / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University-Mathematics. — 2018. — 92. — No. 4. — P. 99–104. DOI 10.31489/2018M4/99-104

3 Yeshkeyev A.R. Small models of hybrids for special subclasses of Jonsson theories / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University-Mathematics. — 2019. — 95. — No. 3. — P. 74–78. DOI 10.31489/2019M2/74-78

4 Yeshkeyev A.R. On Jonsson varieties and quasivarieties / A.R. Yeshkeyev // Bulletin of the Karaganda University-Mathematics. — 2021. — 104. — No. 4. — P. 151–157. DOI 10.31489 /2021M4/151-157

5 Барвайс Дж. Теория моделей: справочная книга по математической логике. — Ч.1. / Дж. Барвайс. — М: Наука, 1982. — 392 с.

6 Barwise J. Completing Theories by Forcing / J. Barwise, A. Robinson // Annals of Mathematical Logic. — 1970. — 2. — No. 2. — P. 119-142.

7 Ешкеев А.Р. Йонсоновские теории и их классы моделей / А.Р. Ешкеев, М.Т. Касыметова. — К.: Изд-во Караганд. гос. ун-та, 2016. — 370 с.

8 Yeshkeyev A.R. Companions of the fragments in the Jonsson enrichment / A.R. Yeshkeyev // Bulletin of the Karaganda University-Mathematics. — 2017. — 85. — No. 1. — P. 41–45.

9 Yeshkeyev A.R. The atomic definable subsets of semantic model / A.R. Yeshkeyev, Issayeva A.K., N.M. Mussina // Bulletin of the Karaganda University-Mathematics. — 2019. — 94. — No. 2. — P. 84–91. DOI 10.31489/2019M2/84-91

10 Yeshkeyev A.R. Companions of (n_1, n_2)-Jonsson theory / A.R. Yeshkeyev, M.T. Omarova // Bulletin of the Karaganda University-Mathematics. — 2019. — 96. — No. 4. — P. 75–80.

11 Yeshkeyev A.R. Method of the rheostat for studying properties of fragments of theoretical sets / A.R. Yeshkeyev // Bulletin of the Karaganda University-Mathematics. — 2020. — 100. — No. 4. — P. 152–159. DOI 10.31489/2020M4/152-159

12 Ешкеев А.Р. JSp-косемантичность R-модулей / А.Р. Ешкеев, О.И. Улбрихт // Сибирские электронные математические известия. — 2019. — 16. — P. 1233–1244.

13 Yeshkeyev A.R. An essential base of the central types of the convex theory / A.R. Yeshkeyev, M.T. Omarova // Bulletin of the Karaganda University-Mathematics. — 2021. — 101. — No. 1. — P. 119–126. DOI 10.31489/2021M1/119-126

14 Forrest W.K. Model Theory for Universal Classes with the Amalgamation Property: a Study in the Foundations of Model Theory and Algebra / W.K. Forrest // Annals of Mathematical Logic. — 1977. — 11. — P. 263–366.

15 Капланский И. Введение в дифференциальную алгебру / И. Капланский; под ред. М.М. Постникова; пер. с англ. — М.: ИЛ, 1959. — 85 с.

16 Robinson A. On the Concept of a Differentially Closed Field / A. Robinson. — Jerusalem: The Hebrew University. — 1959.

17 Marker D. Model Theory of Differential Fields / D. Marker // Published online by Cambridge University Press. — 2017. — 2. — P. 38–113.

18 Wood C. The Model Theory of Differential Fields of Characteristic $p \neq 0$ / C. Wood // Proceedings of the American Mathematical Society. — 1973. — 40. — No. 2. — P. 577–584.

19 Blum L.C. Generalized Algebraic Theories: a Model Theoretic Approach: submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy: 06.00.1963 / Blum Lenore Carol. — MIT, 1968. — 180 p.

20 Kolchin E.R. Differential Algebra and Algebraic Groups / E.R. Kolchin. — NY and Lond.: Academic Press, 1973. — 446 p.

21 Yeshkeyev A.R. The properties of central types with respect to enrichment by Jonsson set / A.R. Yeshkeyev // Bulletin of the Karaganda University-Mathematics. — 2017. — 85. — No. 1. — P. 36–40.

22 Yeshkeyev A.R. The J-minimal sets in the hereditary theories / A.R. Yeshkeyev, M.T. Omarova, G.E. Zhumabekova // Bulletin of the Karaganda University-Mathematics. — 2019. — 94. — No. 2. — P. 92–98. DOI 10.31489/2019M2/92-98

23 Yeshkeyev A.R. Model-theoretical questions of the Jonsson spectrum / A.R. Yeshkeyev // Bulletin of the Karaganda University-Mathematics. — 2020. — 98. — No. 2. — P. 165–173. DOI 10.31489/2020M2/165-173

А.Р. Ешкеев[1], И.О. Тунгушбаева[1], М.Т. Омарова[1,2]

[1] *Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан;*
[2] *Қазтұтынуодағы Қарағанды университеті, Қарағанды, Қазақстан*

Йонсондық AP-теориялардың форсинг-компаньондері

Мақала йонсондық AP-теорияларының форсинг компаньондерін байытылған сигнатурада зерттеуге арналған. Теорияның форсинг-компаньоны тілге жаңа предикаттық және тұрақты символдарын қосу арқылы белгілі бір қасиеттері бар қарастырылып отырған теориялардың кеңеюінде өзгермейтіні дәлелденді. Осы жұмыста жалпы түрде алынған модельді-теоретикалық нәтижелер дифференциалды алгебраның мысалдарымен расталады. Сонымен қатар модельдер теориясының тарихында алғаш рет йонсондық және йонсондық емес теорияларды біріктіруге деген көзқарас көрсетілген. Бұл йонсондық теорияларды зерттеу әдістерін одан әрі дамытуға және толық емес теорияларды зерттеуге арналған аппаратты кеңейтуге мүмкіндік береді.

Кілт сөздер: йонсондық теория, кемел йонсондық теория, AP-теория, форсинг, форсинг-компаньон, сигнатураны байыту, теорияны кеңейту, дифференциалдық өріс, дифференциалды тұйық өріс, дифференциалды кемел өріс.

А.Р. Ешкеев[1], И.О. Тунгушбаева[1], М.Т. Омарова[1,2]

[1]*Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан;*
[2]*Карагандинский университет Казпотребсоюза, Караганда, Казахстан*

Форсинг-компаньоны йонсоновских AP-теорий

Статья посвящена изучению форсинг-компаньонов йонсоновских AP-теорий в обогащённой сигнатуре. Доказано, что форсинг-компаньон теории не меняется при расширении рассматриваемых теорий, обладающих некоторыми свойствами, с помощью добавления в язык новых предикатного и константного символов. Теоретико-модельные результаты, полученные в данной работе в общем виде, подкреплены примерами из дифференциальной алгебры. Авторами статьи впервые в истории теории моделей продемонстрированы подход к комбинированию йонсоновской и нейонсоновской теорий. Это позволит в дальнейшем развить методы исследования йонсоновских теорий и расширить аппарат для изучения неполных теорий.

Ключевые слова: йонсоновская теория, совершенная йонсоновская теория, AP-теория, форсинг, форсинг-компаньон, обогащение сигнатуры, расширение теории, дифференциальное поле, дифференциально замкнутое поле, дифференциально совершенное поле.

References

1 Yeshkeyev, A.R., Tungushbayeva, I.O., & Kassymetova, M.T. (2022). Connection between the amalgam and joint embedding properties. *Bulletin of the Karaganda University-Mathematics, 105*(1), 127-135. DOI 10.31489/2022M1/127-135

2 Yeshkeyev, A.R., & Mussina, N.M. (2018). Properties of hybrids of Jonsson theories. *Bulletin of the Karaganda University-Mathematics, 92*(4), 99–104. DOI 10.31489/2018M4/99-104

3 Yeshkeyev, A.R., & Mussina, N.M. (2019). Small models of hybrids for special subclasses of Jonsson theories. *Bulletin of the Karaganda University-Mathematics, 95*(3), 74–78. DOI 10.31489/2019M2/74-78

4 Yeshkeyev, A.R. (2021). On Jonsson varieties and quasivarieties. *Bulletin of the Karaganda University-Mathematics, 104*(4), 151–157. DOI 10.31489/2021M4/151-157

5 Barwise, J. (1982). *Teoriia modelei: spravochnaia kniga po matematicheskoi logike. Chast 1 [Model theory: Handbook of mathematical logic. Part 1]*. Moscow: Nauka [in Russian].

6 Barwise, J., & Robinson, A. (1970). Completing Theories by Forcing. *Annals of Mathematical Logic, 2*(2), 119-142.

7 Yeshkeyev, A.R., & Kassymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei [Model Theory and their Classes of Models]*. Karaganda: Izdatelstvo Karagandinskogo gosudarstvenogo universiteta [in Russian].

8 Yeshkeyev, A.R. (2017). Companions of the fragments in the Jonsson enrichment. *Bulletin of the Karaganda University-Mathematics, 85*(1), 41–45.

9 Yeshkeyev, A.R., Issayeva, A.K., & Mussina, N.M. (2019). The atomic definable subsets of semantic model. *Bulletin of the Karaganda University-Mathematics, 94*(2), 84–91. DOI 10.31489/2019M2/84-91

10 Yeshkeyev, A.R., & Omarova, M.T. (2019). Companions of (n_1, n_2)-Jonsson theory. *Bulletin of the Karaganda University-Mathematics, 96*(4), 75–80. DOI 10.31489/2019M4/75-80

11 Yeshkeyev, A.R. (2020). Method of the rheostat for studying properties of fragments of theoretical sets. *Bulletin of the Karaganda University-Mathematics, 100*(4), 152–159. DOI 10.31489/2020M4/152-159

12 Yeshkeyev, A.R., & Ulbrikht, O.I. (2019). JSp-kosemantichnost R-modulei [JSp-cosemanticness of R-modules]. *Siberian Electronic Mathematical Reports, 16*, 1233–1244 [in Russian].

13 Yeshkeyev, A.R., & Omarova, M.T. (2021). An essential base of the central types of the convex theory. *Bulletin of the Karaganda University-Mathematics, 101*(1), 119–126. DOI 10.31489/ 2021M1/ 119-126

14 Forrest, W.K. (1977). Model Theory for Universal Classes with the Amalgamation Property: a Study in the Foundations of Model Theory and Algebra. *Annals of Mathematical Logic, 11*, 263–366.

15 Kaplansky, I. (1959). *Vvedenie v differentsialnuiu algebru [An introduction to differential algebra]*. M.M. Postnikov (Ed.). Moscow: Inostrannaiia literatura [in Russian].

16 Robinson, A. (1959). *On the Concept of a Differentially Closed Field*. Jerusalem: The Hebrew University.

17 Marker, D. (2017). Model Theory of Differential Fields. *Published online by Cambridge University Press, 2*, 38–113.

18 Wood, C. (1973). The Model Theory of Differential Fields of Characteristic $p \neq 0$. *Proceedings of the American Mathematical Society, 40*(2), 577–584.

19 Blum, L.C. (1968). Generalized Algebraic Theories: a Model Theoretic Approach. *Doctor's thesis.* Cambridge.

20 Kolchin, E.R. (1973). *Differential Algebra and Algebraic Groups*. New York and London: Academic Press.

21 Yeshkeyev, A.R. (2017). The properties of central types with respect to enrichment by Jonsson set. *Bulletin of the Karaganda University-Mathematics, 85*(1), 36–40.

22 Yeshkeyev, A.R., Omarova, M.T., & Zhumabekova, G.E. (2019). The J-minimal sets in the hereditary theories. *Bulletin of the Karaganda University-Mathematics, 94*(2), 92–98. DOI 10. 31489/ 2019M2/92-98

23 Yeshkeyev, A.R. (2020). Model-theoretical questions of the Jonsson spectrum. *Bulletin of the Karaganda University-Mathematics, 98*(2), 165–173. DOI 10.31489/2020M2/165-173

INFORMATION ABOUT AUTHORS

Ahmad, A. — Research Scholar, Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi, India.

Aikyn, Y. — PhD doctoral student of the L.N.Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan.

Akysh (Akishev), A.Sh. — Doctor of physical and mathematical sciences, Pensioner, Almaty, Kazakhstan.

Ala, V. — Researcher, Mersin University, Science and Letters Faculty, Mathematics Department, Mersin, Turkey.

Ashyralyev, A. — Doctor of Physical and Mathematical Sciences, Professor, Bahcesehir University, Istanbul, Turkey; Peoples's Friendship University of Russia, Moscow, Russian Federation; Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan.

Baboo, M.S. — Assistant Professor, Ph.D., School of Basic Sciences and Research, Sharda University, Greater Noida, Uttar Pradesh, India.

Bayandiyev, Ye.N. — PhD student, Senior teacher, M.Kh. Dulaty Taraz Regional University, Taraz, Kazakhstan.

Beisenbaeva, K.A. — Candidate of physical and mathematical sciences, Associate Professor, Academy of Logistics and Transport, Almaty, Kazakhstan.

Bekmaganbetov, K.A. — Doctor of physical and mathematical sciences (habilitation), Associate professor, M.V. Lomonosov Moscow State University, Kazakhstan Branch, Nur-Sultan, Kazakhstan.

Chechkin, G.A. — Doctor of physical and mathematical sciences (habilitation), Full Professor, M.V. Lomonosov Moscow State University, Moscow, Russia.

Davvaz, B. — Professor of Mathematics, Ph.D., Department of Mathematical Sciences, Yazd University, Yazd, Iran.

Haso, B. — Master of Mathematics Degree, Near East University, Nicosia, TRNC, Mersin 10, Turkey; Lecturer of High School, Duhok, Iraq.

Igisinov, S.Zh. — PhD, M.Kh. Dulaty Taraz Regional University, Taraz, Kazakhstan.

Kakharman, N. — PhD doctor of the Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan.

Koilyshov, U.K. — Candidate of physical and mathematical sciences, Professor, Al-Farabi Kazakh National University, Almaty, Kazakhstan.

Koshanov, B.D. — Doctor of physical and mathematical sciences, Head of the Department of Mathematics Al-Farabi Kazakh National University, Chief Researcher at the International University of Information Technology, Almaty, Kazakhstan.

Mousarezaei, R. — Doctor of Mathematics, Ph.D., Department of Mathematical Sciences, Yazd University, Yazd, Iran.

Muratbekov, M.B. — Professor, M.Kh. Dulaty Taraz Regional University, Taraz, Kazakhstan.

Myrzagaliyeva, A. — PhD, Assistant Professor, Astana IT University, Nur-Sultan, Kazakhstan.

Omarova, M.T. — Master of Mathematics, Master of Management, Senior Lecturer, Karaganda University of Kazpotrebsoyuz; Research Assistant, Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan.

Parashchuk, I.A. — Postgraduate student, Siberian Federal University, Krasnoyarsk, Russia.

Qureshi, M.I. — Professor, Ph.D., Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi, India.

Segizbayeva, R.U. — Candidate of Physical and Mathematical Sciences, Professor of the Civil Aviation Academy, Almaty, Kazakhstan.

Seisenbayeva, S.R. — Master's student, Institute of Mathematics and Mathematical Modeling, Al-Farabi Kazakh National University, Almaty, Kazakhstan.

Senashov, V.I. — Professor, Doctor of physical and mathematical sciences, Professor of Siberian Federal University, Leading Researcher of Institute of Computational Modelling of Siberian Branch of Russian Academy of Sciences, Krasnoyarsk, Russia.

Shaimardan, S. — PhD, Senior teacher of the Department of Fundamental and Applied Mathematics Faculty at L.N. Gumilyov National Eurasian University, Nur-Sultan, Kazakhstan.

Smadiyeva, A.G. — PhD doctoral student of the Al-Farabi Kazakh National University; Junior researcher of the Institute of mathematics and mathematical modeling, Almaty, Kazakhstan.

Suleimbekova, A.O. — M.Kh. Dulaty Taraz Regional University, Taraz, Kazakhstan.

Sultangaziyeva, Zh.B. — Lecturer of the Al-Farabi Kazakh National University, Almaty, Kazakhstan.

Tleubergenov, M.I. — Professor, Doctor of phys.-math.sci., Chief Researcher, Institute of Mathematics and Mathematical Modeling, Al-Farabi Kazakh National University, Almaty, Kazakhstan.

Tokmagambetov, N.S. — Master of Science, Senior teacher, Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan; PhD doctoral student of the L.N. Gumilyov National Eurasian University, Nur-Sultan, Kazakhstan.

Toleubay, A.M. — PHD student, L.N. Gumilev Eurasian National University, Nur-Sultan, Kazakhstan.

Tungushbayeva, I.O. — Master of Science, Junior Research Assistant, Doctoral student, Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan.

Vassilina, G.K. — PhD, Senior Researcher, Institute of Mathematics and Mathematical Modeling, Almaty University of Power Engineering and Telecommunications named after G.Daukeev, Almaty, Kazakhstan.

Yeshkeyev, A.R. — Doctor of Physical and Mathematical Sciences, Professor of the Department of Algebra, Mathematical Logic and Geometry named after Professor T.G. Mustafin, Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan.

Zhapparova, S.D. — Master degree, Al-Farabi Kazakh National University, Almaty, Kazakhstan.

Zhumaliyeva, L.D. — PhD, M.Kh. Dulaty Taraz Regional University, Taraz, Kazakhstan.