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Partial best approximations and the absolute Cesaro summability of multiple Fourier series

The article is devoted to the problem of absolute Cesaro summability of multiple trigonometric Fourier series. Taking a central place in the theory of Fourier series this problem was developed quite widely in the one-dimensional case and the fundamental results of this theory are set forth in the famous monographs by N.K. Bari, A. Zigmund, R. Edwards, B.S. Kashin and A.A. Saakyan [1–4]. In the case of multiple series, the corresponding theory is not well developed. The multidimensional case has own specifics and the analogy with the one-dimensional case does not always be unambiguous and obvious. In this article, we obtain sufficient conditions for the absolute summability of multiple Fourier series of the function $f \in L_q(I_s)$ in terms of partial best approximations of this function. Four theorems are proved and four different sufficient conditions for the $|C;\bar{\beta}|_\lambda$-summability of the Fourier series of the function f are obtained. In the first theorem, a sufficient condition for the absolute $|C;\bar{\beta}|_\lambda$- summability of the Fourier series of the function f is obtained in terms of the partial best approximation of this function which consists of s conditions, in the case when $\beta_1 = ... = \beta_s = \frac{1}{q'}$. Other sufficient conditions are obtained for double Fourier series. Sufficient conditions for the $|C;\beta_1;\beta_2|_\lambda$-summability of the Fourier series of the function $f \in L_q(I_2)$ are obtained in the cases $\beta_1 = \frac{1}{q'}$, $-1 < \beta_2 < \frac{1}{q'}$ (in the second theorem), $\frac{1}{q'} < \beta_1 < +\infty$, $\beta_2 = \frac{1}{q'}$ (in the third theorem), $-1 < \beta_1 < \frac{1}{q'}$, $\frac{1}{q'} < \beta_2 < +\infty$ (in the fourth theorem).

Keywords: trigonometric series, Fourier series, Lebesgue space, partial best approximation of a function, absolute summability of the series.

Introduction

Let R^s be a s-dimensional Euclidean space of points $\bar{x} = (x_1, x_2, ..., x_s)$ with real coordinates; $I_s = \{\bar{x} \in R^s : 0 \le x_j \le 2\pi, \quad j = 1, 2, ..., s\}$ is a s-dimensional cube.

We put $\gamma_i(nx) = \begin{cases} \cos nx, & i = 1, \\ \sin nx, & i = 2 \end{cases}$.

We will consider the following multiple series

$$\sum_{\bar{n} \ge 1} B_{\bar{n}}(\bar{x}) = \sum_{n_1=1}^{\infty} \dots \sum_{n_s=1}^{\infty} B_{n_1,...,n_s}(x_1, ..., x_s), \tag{1}$$

where $\bar{n} \ge \bar{\alpha} = (\alpha_1, ..., \alpha_s)$ means $n_j \ge \alpha_j$ for all $j = 1, 2, ..., s$;

$$B_{\bar{n}}(\bar{x}) = \sum_{\bar{1} \le \bar{i} \le \bar{2}} a_{\bar{n}}^{(\bar{i})} \cdot \prod_{\nu=1}^{s} \gamma_{i_\nu}(n_\nu x_\nu).$$

Assume that $A_n^{(\beta)} = \frac{(\beta+1)(\beta+2)...(\beta+n)}{n!}, \beta \in R$, where n is natural number.

The sum

$$\sigma_{\bar{n}}^{(\bar{\beta})}(\bar{x}) = \sum_{\bar{1} \le \bar{k} \le \bar{n}} \prod_{j=1}^{s} A_{n_j - k_j}^{(\beta_j - 1)} \left(A_{n_j}^{(\beta_j)} \right)^{-1} B_{\bar{k}}(\bar{x})$$

is called $(C;\bar{\beta}) \equiv (C;\beta_1, ..., \beta_s)$ average of the series (1).

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For a given number $b_{\bar{n}}$ we define the mixed difference as follows:

$$\Delta b_{\bar{n}} = \sum_{\bar{0} \le \bar{\varepsilon} \le \bar{1}} (-1)^{s - \sum\limits_{i=1}^{s} \varepsilon_i} \cdot b_{\bar{n} - \bar{1} + \bar{\varepsilon}}.$$

Here $\bar{\varepsilon} = (\varepsilon_1, ..., \varepsilon_s)$. The series (1) is called $\left|C; \bar{\beta}\right|_\lambda$ - summable (or absolutely Cesaro summable), $\lambda \ge 1$, at the point $\bar{x} \in I_s$, if [5]:

$$\sum_{\bar{n} \ge \bar{1}} \left| \Delta \sigma_{\bar{n}}^{(\bar{\beta})}(\bar{x}) \right|^\lambda \cdot \prod_{j=1}^{s} n_j^{\lambda - 1} < +\infty.$$

Further, we put $\rho_{\bar{k}} = \sqrt{\sum\limits_{\bar{1} \le \bar{i} \le \bar{2}} \left| a_{\bar{k}}^{(\bar{i})} \right|^2}$.

By $L_q(I_s)$ we denote the space of all Lebesgue measurable, 2π-periodic in each variable functions $f(\bar{x})$, for which

$$\|f\|_q = \left(\int\limits_{I_s} |f(\bar{x})|^q d\bar{x} \right)^{\frac{1}{q}} < +\infty, \quad 1 \le q < +\infty.$$

Let

$$E_{n_1, ..., n_s}(f)_q = \inf_{T_{n_1, ..., n_s}} \|f(\cdot) - T_{n_1, ..., n_s}(\cdot)\|_q$$

be the full best approximation [6] of the function f by trigonometric polynomials of order not exceeding n_j in variables $x_j (j = 1, .., s)$. Let us also consider the partial best approximation of the function f which is determined by the formula [6]:

$$E_{n,\infty}^{(j)}(f)_q = \inf_{T_n} \|f(x_1, ..., x_s) - T_n(x_1, ..., x_{j-1}, (x_j), x_{j+1}, ..., x_s)\|_q,$$

where $T_n(x_1, ..., x_{j-1}, (x_j), x_{j+1}, ..., x_s)$ is a trigonometric polynomial in the variable x_j of order not exceeding n with coefficients from the space $L_q(I_{s-1})$.

It is known that [6; 44]:

$$E_{n_1, ..., n_s}(f)_q \le C \cdot \sum_{j=1}^{s} E_{n_j, \infty}^{(j)}(f)_q, \quad 1 \le q < \infty.$$

Conditions for the absolute summability of series (1) in the case $\lambda = 1$, $s = 2$, $0 < \beta_j < \frac{1}{2}$, $j = 1, 2$ were investigated by I.E. Zhak and M.F. Timan [7], and the questions of $|C; \bar{\beta}|_\lambda$ summability of the Fourier series of the function $f \in L_2(I_s)$ were studied by Yu.A. Ponomarenko, M.F. Timan [5], and in the one-dimensional case these questions were studied by I. Szalay [8]. Questions of the absolute summability of multiple trigonometric series were also investigated in [9–18].

Results

Theorem 1. Let $1 < q \le 2$, $1 \le \lambda \le q$, $\beta_1 = ... = \beta_s = \frac{1}{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$. If $f \in L_q(I_s)$ and

$$\sum_{n=3}^{\infty} n^{\frac{s\lambda}{q}(2-q)-1} (\ln n)^{\frac{s\lambda}{q}} \left(E_{n,\infty}^{(j)}(f)_q \right)^\lambda < +\infty, \quad j = 1, ..., s,$$

then the Fourier series of the function $f \in L_q(I_s)$ will be $|C; \bar{\beta}|_\lambda$ -summable almost everywhere on I_2.

Proof. We prove the theorem for $s = 2$ (the methods are similar for the higher dimensions). In [9] it was proved that if $1 < q \le 2$, $1 \le \lambda \le q$, $\beta_1 = \beta_2 = \frac{1}{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$ and

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}} \sum_{k_2=2^{n_2}+1}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \ln k_2 \right)^{\frac{\lambda}{q}} < +\infty,$$

then a double series of the form (1) will be $|C; \beta_1; \beta_2|_\lambda$-summable almost everywhere on I_2.

Let's estimate the last series. For this we use the following inequality

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^q\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}\le$$

$$\le\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^q\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}+$$

$$+\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{n_2}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^q\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}. \quad (2)$$

To estimate the first term, we apply Holder's inequalities to the sum over n_2 for $\theta=\frac{q}{\lambda}$, $\frac{1}{\theta}+\frac{1}{\theta'}=1$. We get

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^q\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}\le$$

$$\le C\cdot\sum_{n_1=0}^{\infty}2^{n_1(2-q)\frac{\lambda}{q}}(n_1+1)^{\frac{\lambda}{q}}\cdot\sum_{n_2=0}^{n_1}2^{n_2(2-q)\frac{\lambda}{q}}(n_2+1)^{\frac{\lambda}{q}}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^q\right)^{\frac{\lambda}{q}}\le$$

$$\le C\cdot\sum_{n_1=0}^{\infty}2^{n_1(2-q)\frac{\lambda}{q}}(n_1+1)^{\frac{\lambda}{q}}\cdot\left(\sum_{n_2=0}^{n_1}\left(2^{n_2(2-q)\frac{\lambda}{q}}(n_2+1)^{\frac{\lambda}{q}}\right)^{\theta'}\right)^{\frac{1}{\theta'}}\times$$

$$\times\left(\sum_{n_2=0}^{n_1}\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^q\right)^{\frac{\lambda}{q}}\le$$

$$\le C\cdot\sum_{n_1=0}^{\infty}2^{n_1(2-q)\frac{\lambda}{q}}(n_1+1)^{\frac{\lambda}{q}}\cdot\left(\sum_{n_2=0}^{n_1}2^{n_2(2-q)\frac{\lambda}{q-\lambda}}(n_2+1)^{\frac{\lambda}{q-\lambda}}\right)^{\frac{q-\lambda}{q}}\times$$

$$\times\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2}^{2^{n_1+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^q\right)^{\frac{\lambda}{q}}. \quad (3)$$

Now let's estimate the following sum:

$$\sum_{n_2=0}^{n_1}2^{n_2(2-q)\frac{\lambda}{q-\lambda}}(n_2+1)^{\frac{\lambda}{q-\lambda}}\le(n_1+1)^{\frac{\lambda}{q-\lambda}}\sum_{n_2=0}^{n_1}2^{n_2(2-q)\frac{\lambda}{q-\lambda}}\le C(n_1+1)^{\frac{\lambda}{q-\lambda}}2^{n_1(2-q)\frac{\lambda}{q-\lambda}}.$$

Using this inequality from (3), we obtain

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^q\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}\le$$

$$\le C\cdot\sum_{n_1=0}^{\infty}2^{2n_1(2-q)\frac{\lambda}{q}}(n_1+1)^{2\frac{\lambda}{q}}\cdot\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2}^{2^{n_1+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^q\right)^{\frac{\lambda}{q}}.$$

It is similarly proved

$$\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{n_2}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^q\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}\le$$

$$\leq C\cdot\sum_{n_2=0}^{\infty}2^{2n_2(2-q)\frac{\lambda}{q}}(n_2+1)^{2\frac{\lambda}{q}}\cdot\left(\sum_{k_1=2^{n_2}+1}^{2^{n_2+1}}\sum_{k_2=2}^{2^{n_2+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^q\right)^{\frac{\lambda}{q}}.$$

Taking into account the last two inequalities from (2), we obtain

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^q\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}\leq$$

$$\leq C\cdot\sum_{n=0}^{\infty}2^{2n(2-q)\frac{\lambda}{q}}(n+1)^{2\frac{\lambda}{q}}\cdot\left(\sum_{k_1=1}^{2^{n+1}}\sum_{k_2=2^n+1}^{2^{n+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^q\right)^{\frac{\lambda}{q}}. \tag{4}$$

Further, applying the Hardy-Littlewood theorem [19] and using the monotonicity of the logarithmic function and the best approximation from (4), we have

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^q\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}\leq$$

$$\leq C\cdot\sum_{n=0}^{\infty}2^{2n(2-q)\frac{\lambda}{q}}(n+1)^{2\frac{\lambda}{q}}\cdot E^{\lambda}_{2^n,2^n}(f)_q<C\cdot E^{\lambda}_{1,1}(f)_q+C\cdot E^{\lambda}_{2,2}(f)_q+$$

$$+C\cdot\sum_{n=2}^{\infty}(2^n)^{\frac{2\lambda}{q}(2-q)-1}(n-1)^{\frac{2\lambda}{q}}\cdot E^{\lambda}_{2^n,2^n}(f)_q\cdot\sum_{m=2^{n-1}+1}^{2^n}1\leq$$

$$\leq C\cdot E^{\lambda}_{1,1}(f)_q+C\cdot E^{\lambda}_{2,2}(f)_q+C\sum_{m=3}^{\infty}m^{\frac{2\lambda}{q}(2-q)-1}(\ln m)^{\frac{2\lambda}{q}}\cdot E^{\lambda}_{m,m}(f)_q.$$

The theorem is proved.

Theorem 2. Let $1<q\leq 2$, $1\leq\lambda\leq q$, $\beta_1=\frac{1}{q'}$, $-1<\beta_2<\frac{1}{q'}$, $\frac{1}{q}+\frac{1}{q'}=1$. If $f\in L_q(I_2)$ and

$$\sum_{n=3}^{\infty}n^{\frac{\lambda}{q}(3-q(1+\beta_2))-1}(\ln n)^{\frac{\lambda}{q}}\left(E^{(j)}_{n,\infty}(f)_q\right)^{\lambda}<+\infty,\quad j=1,2,$$

then the Fourier series of function $f\in L_q(I_2)$ will be $|C;\beta_1;\beta_2|_\lambda$ -summable almost everywhere on I_2.

Proof. Since $1<q\leq 2$, $1\leq\lambda\leq q$, $\beta_1=\frac{1}{q'}$, $-1<\beta_2<\frac{1}{q'}$, $\frac{1}{q}+\frac{1}{q'}=1$, then for $|C;\beta_1;\beta_2|_\lambda$- summability almost everywhere on I_2 of a double series of the form (1) is sufficient that [12]

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^q\cdot\ln k_1\cdot k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda}{q}}<+\infty.$$

It is known that

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^q\cdot\ln k_1\cdot k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda}{q}}\leq$$

$$\leq\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^q\cdot\ln k_1\cdot k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda}{q}}+$$

$$+\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{n_2}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^q\cdot\ln k_1\cdot k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda}{q}}. \tag{5}$$

Let us estimate the first term. To do this, we apply Holder's inequalities to the sum over n_2 for $\theta = \frac{q}{\lambda}$, $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. We get

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}} \sum_{k_2=2^{n_2}+1}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \cdot k_2^{q(1-\beta_2)-1} \right)^{\frac{\lambda}{q}} \leq$$

$$\leq C \cdot \sum_{n_1=0}^{\infty} 2^{n_1(2-q)\frac{\lambda}{q}} (n_1+1)^{\frac{\lambda}{q}} \cdot \sum_{n_2=0}^{n_1} 2^{n_2(1-q\beta_2)\frac{\lambda}{q}} \left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}} \sum_{k_2=2^{n_2}+1}^{2^{n_2+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}} \leq$$

$$\leq C \cdot \sum_{n_1=0}^{\infty} 2^{n_1(3-q(1+\beta_2))\frac{\lambda}{q}} (n_1+1)^{\frac{\lambda}{q}} \left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}} \sum_{k_2=2}^{2^{n_1+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}}.$$

For the second term (5), we similarly obtain

$$\sum_{n_2=0}^{\infty} \sum_{n_1=0}^{n_2} \left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}} \sum_{k_2=2^{n_2}+1}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \cdot k_2^{q(1-\beta_2)-1} \right)^{\frac{\lambda}{q}} \leq$$

$$\leq C \cdot \sum_{n_2=0}^{\infty} 2^{n_2(3-q(1+\beta_2))\frac{\lambda}{q}} (n_2+1)^{\frac{\lambda}{q}} \cdot \left(\sum_{k_2=2^{n_1}+1}^{2^{n_2+1}} \sum_{k_1=2}^{2^{n_2+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}}.$$

Therefore

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}} \sum_{k_2=2^{n_2}+1}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \cdot k_2^{q(1-\beta_2)-1} \right)^{\frac{\lambda}{q}} \leq$$

$$\leq C \cdot \sum_{n=0}^{\infty} 2^{n(3-q(1+\beta_2))\frac{\lambda}{q}} (n+1)^{\frac{\lambda}{q}} \cdot \left(\sum_{k_2=2^{n}+1}^{2^{n+1}} \sum_{k_1=2}^{2^{n+1}} (k_1 k_2)^{q-2} \rho_{k_1 k_2}^q \right)^{\frac{\lambda}{q}}. \qquad (6)$$

Now, applying the Hardy-Littlewood theorem [19] and using the monotonicity of the logarithmic function and the best approximation from (6), we obtain

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}} \sum_{k_2=2^{n_2}+1}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_1 \cdot k_2^{q(1-\beta_2)-1} \right)^{\frac{\lambda}{q}} \leq$$

$$\leq C \cdot E_{1,1}^{\lambda}(f)_q + C \cdot E_{2,2}^{\lambda}(f)_q + C \sum_{m=3}^{\infty} m^{\frac{\lambda}{q}(3-q(1+\beta_2))-1} (\ln m)^{\frac{\lambda}{q}} \cdot E_{m,m}^{\lambda}(f)_q.$$

The theorem is proved.

Theorem 3. Let $1 < q \leq 2$, $1 \leq \lambda \leq q$, $\frac{1}{q'} < \beta_1 < +\infty$, $\beta_2 = \frac{1}{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$. If $f \in L_q(I_2)$ and

$$\sum_{n=3}^{\infty} n^{2\frac{\lambda}{q}(2-q)-1} (\ln n)^{\frac{\lambda}{q}} \left(E_{n,\infty}^{(j)}(f)_q \right)^{\lambda} < +\infty, \quad j = 1, 2,$$

then the Fourier series of function $f \in L_q(I_2)$ will be $|C; \beta_1; \beta_2|_\lambda$ -summable almost everywhere on I_2.

Proof. It was proved in [12] that, if $1 < q \leq 2$, $1 \leq \lambda \leq q$, $\frac{1}{q'} < \beta_1 < +\infty$, $\beta_2 = \frac{1}{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$ and

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}} \sum_{k_2=2^{n_2}+1}^{2^{n_2+1}} \rho_{k_1 k_2}^q \cdot \ln k_2 \right)^{\frac{\lambda}{q}} < +\infty,$$

then a double series of the form (1) is $|C; \beta_1; \beta_2|_\lambda$- summable almost everywhere on I_2.

Similarly to the previous theorem, we obtain

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_2\right)^{\frac{\lambda}{q}}\leq$$

$$\leq C\cdot\sum_{n=0}^{\infty}2^{2n(2-q)\frac{\lambda}{q}}(n+1)^{\frac{\lambda}{q}}\cdot\left(\sum_{k_2=2^{n}+1}^{2^{n+1}}\sum_{k_1=2}^{2^{n+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}}.$$

Hence, applying the Hardy-Littlewood theorem [19] and using the monotonicity of the logarithmic function and the best approximation, we have:

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_2\right)^{\frac{\lambda}{q}}\leq$$

$$\leq C\cdot E_{1,1}^{\lambda}(f)_q+C\cdot E_{2,2}^{\lambda}(f)_q+C\sum_{m=3}^{\infty}m^{\frac{2\lambda}{q}(2-q)-1}(\ln m)^{\frac{\lambda}{q}}\cdot E_{m,m}^{\lambda}(f)_q.$$

The theorem is proved.

Theorem 4. Let $1<q\leq 2$, $1\leq\lambda\leq q$, $-1<\beta_1<\frac{1}{q'}$, $\frac{1}{q'}<\beta_2<+\infty$, $\frac{1}{q}+\frac{1}{q'}=1$. If $f\in L_q(I_2)$ and

$$\sum_{n=2}^{\infty}n^{\frac{\lambda}{q}(3-q(1+\beta_1))-1}\left(E_{n,\infty}^{(j)}(f)_q\right)^{\lambda}<+\infty,\quad j=1,2,$$

then the Fourier series of function $f\in L_q(I_2)$ will be $|C;\beta_1;\beta_2|_\lambda$ -summable almost everywhere on I_2.

Proof. It was proved in [12] that, if $1<q\leq 2$, $1\leq\lambda\leq q$, $-1<\beta_1<\frac{1}{q'}$, $\frac{1}{q'}<\beta_2<+\infty$, $\frac{1}{q}+\frac{1}{q'}=1$ and

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot {k_1}^{q(1-\beta_1)-1}\right)^{\frac{\lambda}{q}}<+\infty,$$

then a double series of the form (1) is $|C;\beta_1;\beta_2|_\lambda$-summable almost everywhere on I_2.

For the last converging series, similarly to Theorem 2, we obtain

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot {k_1}^{q(1-\beta_1)-1}\right)^{\frac{\lambda}{q}}\leq$$

$$\leq C\cdot\sum_{n=0}^{\infty}2^{n(3-q(1+\beta_1))\frac{\lambda}{q}}\cdot\left(\sum_{k_2=2^{n}+1}^{2^{n+1}}\sum_{k_1=2}^{2^{n+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}}.$$

Hence, applying the Hardy-Littlewood theorem [19] and the monotonicity of the best approximation, we have:

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot {k_1}^{q(1-\beta_1)-1}\right)^{\frac{\lambda}{q}}\leq$$

$$\leq C\cdot E_{1,1}^{\lambda}(f)_q+C\sum_{m=2}^{\infty}m^{\frac{\lambda}{q}(3-q(1+\beta_1))-1}\cdot E_{m,m}^{\lambda}(f)_q.$$

The theorem is proved.

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Дербес ең жақсы жуықтаулар және еселі Фурье қатарының абсолютті чезаролық қосындылануы

Мақала еселі тригонометриялық Фурье қатарының Чезаро бойынша абсолютті қосындылану сұрағына арналған. Фурье қатарлары теориясында ерекше орны бар бұл сұрақ бір өлшемді жағдайда жеткілікті кең зерттелген және бұл теорияның іргелі нәтижелері Н.К. Бари, А. Зигмунд, Р. Эдвардс, Б.С. Кашин және А.А. Саакянның [1–4] белгілі монографияларында келтірілген. Еселі қатарлар

жағдайына сәйкес теория соншалықты күшті жасалмаған. Еселі жағдайдың өз ерекшеліктері бар және бір өлшемді жағдаймен ұқсастық әрқашан бірмәнді емес және айқын бола бермейді. Мақалада $f \in L_q(I_s)$ функциясының еселі Фурье қатарының абсолютті қосындылануының жеткілікті шарттары осы функцияның дербес ең жақсы жуықтаулары тілінде алынған. Төрт теорема дәлелденіп, $f \in L_q(I_s)$ функциясының Фурье қатарының $|C;\bar{\beta}|_\lambda$ қосындылануының әртүрлі төрт жеткілікті шарты нақтыланған. Бірінші теоремада $f \in L_q(I_s)$ функциясының Фурье қатарының $|C;\bar{\beta}|_\lambda$ қосындылануы осы функцияның дербес ең жақсы жуықтаулары тіліндегі s шарттан тұратын жеткілікті шарты $\beta_1 = ... = \beta_s = \frac{1}{q'}$ жағдайында алынған. Басқа жеткілікті шарттар екі еселі Фурье қатарлары үшін алынды. $f \in L_q(I_2)$ функциясының Фурье қатарының $|C;\beta_1;\beta_2|_\lambda$ қосындылану шарттары мына жағдайларда алынды: $\beta_1 = \frac{1}{q'}$, $-1 < \beta_2 < \frac{1}{q'}$ (екінші теоремада), $\frac{1}{q'} < \beta_1 < +\infty$, $\beta_2 = \frac{1}{q'}$ (үшінші теоремада), $-1 < \beta_1 < \frac{1}{q'}$, $\frac{1}{q'} < \beta_2 < +\infty$ (төртінші теоремада).

Кілт сөздер: тригонометриялық қатар, Фурье қатары, Лебег кеңістігі, функцияның дербес ең жақсы жуықтауы, қатардың абсолютті қосындылануы.

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Частные наилучшие приближения и абсолютная чезаровская суммируемость кратных рядов Фурье

Статья посвящена вопросу абсолютной суммируемости по Чезаро кратных тригонометрических рядов Фурье. Этот вопрос, занимая центральное место в теории рядов Фурье, в одномерном случае разработан достаточно широко и фундаментальные результаты этой теории изложены в известных монографиях Н.К. Бари, А. Зигмунда, Р. Эдвардса, Б.С. Кашина и А.А. Саакяна [1–4]. В случае кратных рядов соответствующая теория разработана не столь сильно. Многомерный случай имеет свою специфику, и аналогия с одномерным случаем далеко не всегда однозначна и очевидна. В статье получены достаточные условия абсолютной суммируемости кратных рядов Фурье функции $f \in L_q(I_s)$ в терминах частных наилучших приближений данной функции. Доказаны четыре теоремы и получены четыре разных достаточных условий $|C;\bar{\beta}|_\lambda$ суммируемости ряда Фурье функции $f \in L_q(I_s)$. В первой теореме получено достаточное условие абсолютной $|C;\bar{\beta}|_\lambda$ суммируемости ряда Фурье функции $f \in L_q(I_s)$ в терминах частного наилучшего приближения этой функции, которое состоит из s условий, в случае когда $\beta_1 = ... = \beta_s = \frac{1}{q'}$. Другие достаточные условия получены для двойных рядов Фурье. Достаточные условия $|C;\beta_1;\beta_2|_\lambda$ суммируемости ряда Фурье функции $f \in L_q(I_2)$ получены в случаях $\beta_1 = \frac{1}{q'}$, $-1 < \beta_2 < \frac{1}{q'}$ (во второй теореме), $\frac{1}{q'} < \beta_1 < +\infty$, $\beta_2 = \frac{1}{q'}$ (в третьей теореме), $-1 < \beta_1 < \frac{1}{q'}$, $\frac{1}{q'} < \beta_2 < +\infty$ (в четвертой теореме).

Ключевые слова: тригонометрический ряд, ряд Фурье, пространство Лебега, частное наилучшее приближение функции, абсолютная суммируемость ряда.

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Solving fully fuzzy linear programming problems by controlling the variation range of variables

This paper deals with a fully fuzzy linear programming problem (FFLP) in which the coefficients of decision variables, the right-hand coefficients and variables are characterized by fuzzy numbers. A method of obtaining optimal fuzzy solutions is proposed by controlling the left and right sides of the fuzzy variables according to the fuzzy parameters. By using fuzzy controlled solutions, we avoid unexpected answers. Finally, two numerical examples are solved to demonstrate how the proposed model can provide a better optimal solution than that of other methods using several ranking functions.

Keywords: fully fuzzy linear programming, fuzzy linear programming, fuzzy number, ranking function.

Introduction

The general idea about fuzzy set theory was introduced by Zadeh [1]. Bellman and Zadeh [2] postulated the very idea of decision making in fuzzy environment. The aforementioned concept has been utilized by Tanaka et al. [3] as a solution for mathematical programming problems. Fuzzy linear programming was first generated and suggested by Zimmerman [4] . Using the parametric programming method, Chanas [5] suggested the contingency of the identification of a complete fuzzy decision in fuzzy linear programming. Programming models with crisp or fuzzy constraints as well as crisp or fuzzy goals can be solved using interactive system proposed by Werners [6].

Using a technique introduced by Fang et al. [7], linear programming problems with fuzzy coefficients in constraints could be solved. In an attempt to find solution for fully fuzzified linear programming problems with fuzzy numbers in place of all the parameters and variables, Buckley and Feuring [8] proposed substitution of the objective function for multiobjective fuzzy linear programming problem. Maleki et al. [9] solved linear programming problems by comparing fuzzy numbers as decision parameters.

The ranking function used and suggested by Maleki [10] is a technique in solving linear programming that has vague and uncertain constraints. By transforming fuzzy linear programming problems into multi-objective linear programming problems, Nehi et al. [11] proposed and defined the concept of optimality for linear programming problems. Ganesan and Veeramani [12] proposed and advocated an approach in which fuzzy linear programming cannot be converted into crisp linear programming in solving a fuzzy linear programming problem using symmetric trapezoidal fuzzy numbers. Hashemi et al. [13] suggested a two-phase method in which fuzzy numbers replace decision parameters and the variables determine the optimal solutions of FFLP problems. Rommelfanger [14] also introduced a new approach in solving stochastic linear programming problems with fuzzy parameters.

Allahviranloo [15] suggested a new method in solving FFLP problems using ranking function. Nasseri [16] proposed a method for solving fuzzy linear programming problems using a classical linear programming model. Lotfi et al. [17] discussed FFLP problems by identifying all parameters and variables as triangular fuzzy numbers. Ebrahimnejad and Nasseri [18] attempted to solve fuzzy linear programming problem with fuzzy parameters by applying the complementary slackness theorem without necessitating the use of the simplex tableau. Kumar et al. [19] proposed a new method in finding the fuzzy optimal solution of FFLP problems with inequality constraints.

In this paper a new method is proposed to find the fuzzy optimal solutions of an FFLP problem with equality and inequality constraints, as well as with triangular and trapezoidal fuzzy numbers, by solving one linear programming problem only. Moreover, by controlling the left and the right sides of the fuzzy variables, this model prevents the generation of broad fuzzy solutions that do not conform to other fuzzy parameters.

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One of the problems exhibited by a number of models for FFLP problems is their incapability to simultaneously solve equality and inequality constraints. Buckley and Feuring [8], Hosseinzadeh et al. [17], and Kumar et al. [19] proposed three different models in solving FFLP problems. The model, proposed by Buckley and Feuring, can only be used to solve FFLP problems with inequality constraints, whereas the models proposed by Hosseinzadeh et al. and Kumar et al. can only be used to solve FFLP problems with equality constraints. Another problem is the inability of some models to control the left and the right sides of fuzzy numbers, thus generating broad answers and causing difficulties in the decision-making process. Therefore, this paper attempts to overcome the shortcomings of previous approaches and to provide further insight.

In general, the narrow fuzzy parameters in an FFLP application indicate that these parameters have a low range of variation. Thus, the fuzzy parameters of FFLP in this case have slight flexibility. Based on this approach, a new model is proposed in solving FFLP problems, with equality and inequality constraints being handled simultaneously by identifying the maximum and the minimum. Fuzzy solutions can also be determined by this model, specifically when a number or all parameters and variables are triangular and trapezoidal fuzzy numbers.

Preliminary Concepts

In this section we begin by recalling some basic definitions from fuzzy set theory and introduce the main concepts needed in this paper.

Definition 1. Let X be a collection of objects denoted generically by x. The fuzzy set $\widetilde{A}$ in the set of real numbers is called a fuzzy set in X if $\widetilde{A}$ is a set of ordered pairs:

$$\widetilde{A} = \left\{ \left(x, \mu_{\widetilde{A}}(x)\right) \mid x \in X \right\}$$

where $\mu_{\widetilde{A}}(x)$ is membership function of x in $\mu_{\widetilde{A}}$ such that $\mu_{\widetilde{A}}(x) : X \to [0,1]$.

Definition 2. A fuzzy set $\widetilde{A}$ is called normal if there is a real number x such that $\mu_{\widetilde{A}}(x) = 1$.

Definition 3. Let $\widetilde{A} = \left(m^l, m^u, \alpha, \beta\right)_{LR}$ denote the LR-fuzzy number if its membership function is defined as

$$\mu_{\widetilde{A}}(x) = \begin{cases} L\left(\dfrac{m^l - x}{\alpha}\right), & x < m^l, \\ 1, & m^l \le x \le m^u, \\ R\left(\dfrac{x - m^u}{\beta}\right), & x > m^u, \\ 0, & otherwise, \end{cases}$$

where L and R are reference functions, i.e., $L, R : [0, +\infty) \to [0,1]$ are non-increasing that $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$ and $\alpha, \beta \ge 0$. The membership function of a LR-type triangular fuzzy number, $\widetilde{A} = (m, \alpha, \beta)_{LR}$, can be also defined as

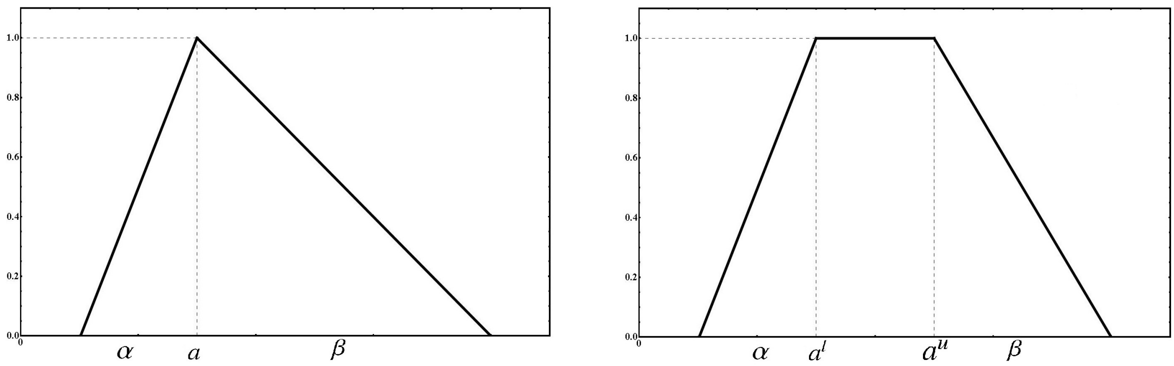
$$\mu_{\widetilde{A}}(x) = \begin{cases} L\left(\dfrac{m - x}{\alpha}\right), & x \le m. \\ R\left(\dfrac{x - m}{\beta}\right), & x > m. \\ 0, & otherwise. \end{cases}$$

Definition 4. A fuzzy number $\widetilde{A} = \left(m^l, m^u, \alpha, \beta\right)$ is a trapezoidal fuzzy number if (see Figure 1(a))

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x - (m^l - \alpha)}{\alpha}, & m^l - \alpha \le x < m^l. \\ 1, & m^l \le x \le m^u. \\ \dfrac{(m^u + \beta) - x}{\beta}, & m^u < x \le m^u + \beta. \\ 0, & otherwise. \end{cases}$$

Definition 5. A fuzzy number $\widetilde{A} = (m, \alpha, \beta)$ is a triangular fuzzy number if (see Figure 1(b))

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x-(m-\alpha)}{\alpha}, & m-\alpha \le x < m. \\ \dfrac{(m+\beta)-x}{\beta}, & m < x \le m+\beta. \\ 0, & otherwise. \end{cases}$$



(a) Triangular fuzzy number (b) Trapezoidal fuzzy number

Figure 1. The LR-fuzzy numbers.

Definition 6. Let $\widetilde{A} = \left(m^l, m^u, \alpha, \beta\right)$ be trapezoidal fuzzy number, then

$$\begin{array}{l} \widetilde{A} \succ 0 \quad if \quad m^l - \alpha > 0 \\ \widetilde{A} \prec 0 \quad if \quad m^u + \beta < 0. \end{array}$$

Definition 7. Let $\widetilde{A}_1 = \left(m_1^l, m_1^u, \alpha_1, \beta_1\right)$ and $\widetilde{A}_2 = \left(m_2^l, m_2^u, \alpha_2, \beta_2\right)$ be trapezoidal fuzzy numbers, then we define arithmetic on fuzzy numbers:

$$\widetilde{A}_1 + \widetilde{A}_2 = \left(m_1^l + m_2^l, m_1^u + m_2^u, \alpha_1 + \alpha_2, \beta_1 + \beta_2\right)$$

$$\widetilde{A}_1 - \widetilde{A}_2 = \left(m_1^l - m_2^u, m_1^u - m_2^l, \alpha_1 + \beta_2, \beta_1 + \alpha_2\right)$$

$$\widetilde{A}_1.\widetilde{A}_2 = \begin{cases} \left(m_1^l m_2^l, m_1^u m_2^u, m_1^l \alpha_2 + m_2^l \alpha_1, m_1^u \beta_2 + m_2^u \beta_1\right), & \widetilde{A}_1 \succ 0, \quad \widetilde{A}_2 \succ 0. \\ \left(m_1^u m_2^l, m_1^l m_2^u, m_1^u \alpha_2 + m_2^l \beta_1, m_1^l \beta_2 + m_2^u \alpha_1\right), & \widetilde{A}_1 \succ 0, \quad \widetilde{A}_2 \prec 0. \\ \left(m_1^u m_2^u, m_1^l m_2^l, -m_1^u \beta_2 - m_2^u \beta_1, -m_1^l \alpha_2 - m_2^l \alpha_1\right), & \widetilde{A}_1 \prec 0, \quad \widetilde{A}_2 \prec 0. \end{cases}$$

Remark 1. We denote the set of all LR-fuzzy numbers by $\mathcal{F}(\mathbb{R})$.

Definition 8. [20] A linear ranking function is a function $\mathfrak{R} : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$, which maps each fuzzy number into a real line, where there is a natural order

$$\mathfrak{R}\left(\widetilde{A}\right) = \frac{1}{2}\int_0^1 (L_\alpha(x) + R_\alpha(x))dx$$

such that

$$\mathfrak{R}\left(k\widetilde{A} + \widetilde{B}\right) = k\mathfrak{R}\left(\widetilde{A}\right) + \mathfrak{R}\left(\widetilde{B}\right).$$

Let $\widetilde{A} = \left(m^l, m^u, \alpha, \beta\right)$ be a trapezoidal fuzzy number, then

$$\mathfrak{R}\left(\widetilde{A}\right) = \frac{1}{2}\left(m^l + m^u\right) + \frac{1}{4}(\beta - \alpha).$$

If $\widetilde{A} = (a, \alpha, \beta)$ is a triangular fuzzy number, then

$$\Re\left(\widetilde{A}\right) = a + \frac{1}{4}(\beta - \alpha).$$

Definition 9. [21] Let $\widetilde{m}, \widetilde{n} \in \mathcal{F}(\mathbb{R})$ be fuzzy numbers then

- $\widetilde{A} \succeq \widetilde{B}$ if and only if $\Re\left(\widetilde{A}\right) \geq \Re\left(\widetilde{B}\right)$;
- $\widetilde{A} \preceq \widetilde{B}$ if and only if $\Re\left(\widetilde{A}\right) \leq \Re\left(\widetilde{B}\right)$;
- $\widetilde{A} = \widetilde{B}$ if and only if $\Re\left(\widetilde{A}\right) = \Re\left(\widetilde{B}\right)$.

Definition 10. A fully fuzzy linear programming (FFLP) is defined as follows:

$$\begin{aligned} \widetilde{Z}^* = Min \quad & \widetilde{c}\,\widetilde{\mathbf{x}}, \\ s.t. \quad & \widetilde{A}\,\widetilde{\mathbf{x}} \succeq \widetilde{b}, \\ & \widetilde{\mathbf{x}} \succeq 0, \end{aligned} \tag{1}$$

where $\widetilde{c} = (\widetilde{c}_1, \ldots, \widetilde{c}_n)$, $\widetilde{A} = (\widetilde{a}_{ij})_{(m\times n)}$, $\widetilde{b} = \left(\widetilde{b}_1, \ldots, \widetilde{b}_m\right)^t$, $\widetilde{\mathbf{x}} = (\widetilde{x}_1, \ldots, \widetilde{x}_n)^t$ and $\widetilde{a}_{ij}, \widetilde{b}_i, \widetilde{c}_j, \widetilde{x}_j \in \mathcal{F}(\mathbb{R})$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$.

The decision variables are fuzzy numbers while the coefficients of decision variables and the right hand coefficients are characterized by uncertainties.

Definition 11. The fuzzy vector $\widetilde{\mathbf{x}} \in (\mathcal{F}(\mathbb{R}))^n$ is a fuzzy feasible solution to (1) if

$$\widetilde{A}\,\widetilde{\mathbf{x}} \succeq \widetilde{b} \qquad \text{and} \qquad \widetilde{\mathbf{x}} \succeq 0;$$

as a result $\widetilde{\mathbf{x}}$ is a fuzzy feasible solution to (1) if

$$\Re\left(\widetilde{A}\,\widetilde{\mathbf{x}}\right) \geq \Re\left(\widetilde{b}\right) \qquad \text{and} \qquad \Re\left(\widetilde{\mathbf{x}}\right) \geq 0.$$

Hence, S be a set of fuzzy numbers which satisfies the set of constraints are defined as follows:

$$S = \left\{\widetilde{\mathbf{x}} \middle| \Re\left(\widetilde{A}\,\widetilde{\mathbf{x}}\right) \geq \Re\left(\widetilde{b}\right) \quad \text{and} \quad \Re\left(\widetilde{\mathbf{x}}\right) \geq 0\right\}.$$

Definition 12. A fuzzy feasible solution $\widetilde{\mathbf{x}}^* \in S$ is a fuzzy optimal solution for (1) if for all fuzzy feasible solution $\widetilde{\mathbf{x}} \in S$ we have

$$\widetilde{c}\,\widetilde{\mathbf{x}}^* \preceq \widetilde{c}\,\widetilde{\mathbf{x}};$$

as a result $\widetilde{\mathbf{x}}^*$ is a optimal fuzzy optimal feasible solution to (1) if

$$\Re\left(\widetilde{c}\,\widetilde{\mathbf{x}}^*\right) \leq \Re\left(\widetilde{c}\,\widetilde{\mathbf{x}}\right).$$

Solving Fully Fuzzy Linear Programming

In this section a new method in finding the fuzzy optimal solution of the FFLP model (1) is proposed. This method can be used to solve FFLP problems with equality and inequality constraints by identifying the maximum and the minimum. This method can also be used in solving FFLP problems with triangular and trapezoidal fuzzy parameters. Furthermore, by controlling the left and the right side values of fuzzy parameters to make them proportional with other fuzzy parameters, the generation of broad fuzzy variables is prevented.

For this purpose we obtained an upper bound for the left (α_{x_j}) and the right (β_{x_j}) sides of the fuzzy values. The upper bound is prevented from exceeding α_{x_j} and β_{x_j}. Consequently, the obtained fuzzy solutions are not considerably broad but are controlled and thinner fuzzy solutions. In this paper, the upper bounds for α_{x_j} and β_{x_j} are proportional to the ratio of the left and the right side values to the central value of fuzzy parameters. Thus, we calculated the values of α_{x_j} , β_{x_j} and m_{x_j}, such that the maximum value of the ratio of α_{x_j} and β_{x_j} to m_{x_j} is equal to M. By adding the following constraints to the problem fuzzy variables can be prevented from becoming exceedingly broad:

$$\frac{\alpha_{x_j}}{|m_{x_j}|} \leq M, \qquad \frac{\beta_{x_j}}{|m_{x_j}|} \leq M, \qquad j = 1, 2, \cdots, n. \tag{2}$$

Two solutions for the calculation of M are proposed. Firstly, M is calculated by acquiring the maximum amount of the ratio of α and β to m for all fuzzy parameters of FFLP. Secondly, M is calculated by their mean.

The fully fuzzy linear programming (1) can be rewritten as follows:

$$\begin{aligned} &Min \quad \sum_{j=1}^{n} \widetilde{c}_j \widetilde{x}_j, \\ &s.t. \quad \sum_{j=1}^{n} \widetilde{a}_{ij}\, \widetilde{x}_j \succeq \widetilde{b}_i \quad , \quad i = 1, 2, \cdots, m, \\ &\qquad \widetilde{x}_j \succeq 0 \qquad\qquad , \quad j = 1, 2, \cdots, n, \end{aligned} \tag{3}$$

where $\widetilde{c}_j = \left(m_{c_j}, \alpha_{c_j}, \beta_{c_j}\right)$, $\widetilde{a}_{ij} = \left(m_{a_{ij}}, \alpha_{a_{ij}}, \beta_{a_{ij}}\right)$, $\widetilde{b}_i = (m_{b_i}, \alpha_{b_i}, \beta_{b_i})$, $\widetilde{x}_j = \left(m_{x_j}, \alpha_{x_j}, \beta_{x_j}\right)$ are triangular fuzzy numbers and $\widetilde{a}_{ij}, \widetilde{b}_i, \widetilde{c}_j, \widetilde{x}_j \in \mathcal{F}(\mathbb{R})$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

We use the ranking function that was defined in Definition 8 for the defuzzification of the fully fuzzy linear programming (3) and by adding constraints (2) then we have

$$\begin{aligned} &Min \quad \mathfrak{R}\left(\sum_{j=1}^{n} \widetilde{c}_j \widetilde{x}_j\right) \\ &s.t. \quad \mathfrak{R}\left(\sum_{j=1}^{n} \widetilde{a}_{ij}\, \widetilde{x}_j\right) \geq \mathfrak{R}\left(\widetilde{b}_i\right) \quad , \quad i = 1, 2, \cdots, m, \\ &\qquad \frac{\alpha_{x_j}}{\left|m_{x_j}\right|} \leq M \qquad , \quad j = 1, 2, \cdots, n, \\ &\qquad \frac{\beta_{x_j}}{\left|m_{x_j}\right|} \leq M \qquad , \quad j = 1, 2, \cdots, n, \\ &\qquad \mathfrak{R}\left(\widetilde{x}_j\right) \geq 0 \qquad , \quad j = 1, 2, \cdots, n, \end{aligned} \tag{4}$$

where

$$M = \max_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,n}} \left\{ \frac{\alpha_{a_{ij}}}{\left|m_{a_{ij}}\right|}, \frac{\beta_{a_{ij}}}{\left|m_{a_{ij}}\right|}, \frac{\alpha_{b_i}}{\left|m_{b_i}\right|}, \frac{\beta_{b_i}}{\left|m_{b_i}\right|}, \frac{\alpha_{c_j}}{\left|m_{c_j}\right|}, \frac{\beta_{c_j}}{\left|m_{c_j}\right|} \;\middle|\; m_{a_{ij}} \neq 0\ ,\ m_{b_i} \neq 0\ ,\ m_{c_j} \neq 0 \right\} \tag{5}$$

or

$$M = \operatorname*{mean}_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,n}} \left\{ \frac{\alpha_{a_{ij}}}{\left|m_{a_{ij}}\right|}, \frac{\beta_{a_{ij}}}{\left|m_{a_{ij}}\right|}, \frac{\alpha_{b_i}}{\left|m_{b_i}\right|}, \frac{\beta_{b_i}}{\left|m_{b_i}\right|}, \frac{\alpha_{c_j}}{\left|m_{c_j}\right|}, \frac{\beta_{c_j}}{\left|m_{c_j}\right|} \;\middle|\; m_{a_{ij}} \neq 0\ ,\ m_{b_i} \neq 0\ ,\ m_{c_j} \neq 0 \right\}. \tag{6}$$

According to $\widetilde{x}_j \succeq 0$ we have

$$\frac{\alpha_{x_j}}{m_{x_j}} \leq M \quad \text{and} \quad \frac{\beta_{x_j}}{m_{x_j}} \leq M, \quad j = 1, 2, \cdots, n.$$

Then

$$Mm_{x_j} - \alpha_{x_j} \geq 0 \quad \text{and} \quad Mm_{x_j} - \beta_{x_j} \geq 0, \quad j = 1, 2, \cdots, n.$$

Thus, we can write model (4) as follows:

$$\begin{aligned} &Min \quad \sum_{j=1}^{n} \mathfrak{R}\left(\widetilde{c}_j \widetilde{x}_j\right) \\ &s.t. \quad \sum_{j=1}^{n} \mathfrak{R}\left(\widetilde{a}_{ij}\, \widetilde{x}_j\right) \geq \mathfrak{R}\left(\widetilde{b}_i\right) \quad , \quad i = 1, 2, \cdots, m, \\ &\qquad Mm_{x_j} - \alpha_{x_j} \geq 0 \qquad , \quad j = 1, 2, \cdots, n, \\ &\qquad Mm_{x_j} - \beta_{x_j} \geq 0 \qquad , \quad j = 1, 2, \cdots, n, \\ &\qquad m_{x_j} - \alpha_{x_j}, \alpha_{x_j}, \beta_{x_j} \geq 0 \quad , \quad j = 1, 2, \cdots, n. \end{aligned} \tag{7}$$

A standard form on a fully fuzzy linear programming (FFLP) is also defined as follows:

$$\begin{aligned} \widetilde{Z}^* = Min \quad & \widetilde{c}\,\widetilde{\mathbf{x}}; \\ s.t. \quad & \widetilde{A}\,\widetilde{\mathbf{x}} = \widetilde{b}; \\ & \widetilde{\mathbf{x}} \succeq 0; \end{aligned}$$

then we have

$$\begin{aligned} Min \quad & \sum_{j=1}^{n} \Re\left(\widetilde{c}_j \widetilde{x}_j\right) & & \\ s.t. \quad & \sum_{j=1}^{n} \Re\left(\widetilde{a}_{ij}\,\widetilde{x}_j\right) = \Re\left(\widetilde{b}_i\right) & , & \quad i = 1, 2, \cdots, m, \\ & Mm_{x_j} - \alpha_{x_j} \geq 0 & , & \quad j = 1, 2, \cdots, n, \\ & Mm_{x_j} - \beta_{x_j} \geq 0 & , & \quad j = 1, 2, \cdots, n, \\ & m_{x_j} - \alpha_{x_j}, \alpha_{x_j}, \beta_{x_j} \geq 0 & , & \quad j = 1, 2, \cdots, n. \end{aligned} \tag{8}$$

Numerical Examples

In this section, two numerical examples are presented to describe the proposed method. The first application is the standard diet problem formulated by Buckley and Feuring [8]. The first example presents FFLP with triangular fuzzy numbers and inequality constraints, while in the second example FFLP with triangular fuzzy numbers is also considered, but with equality constraints. We compared the proposed fuzzy objective optimal solution with that from other models by Fortemps and Roubens [22], Chang [23] and Liou and Wang [24] that use ranking function.

Example 1. Let us start by considering the following fully fuzzy linear program problem.

A farmer has three products P1, P2 and P3 to mix together to feed his pigs. He knows that the pigs need a certain amount of foods $F1$ and $F2$ per day. Table 1 illustrates the units of $F1$ and $F2$ that are available per gram of $P1$, $P2$ and $P3$. Also, each pig should have at least (54, 4, 4) units of $F1$ and at least (60, 4, 4) units of $F2$ per day. The costs of $P1$, $P2$ and $P3$ vary slightly from day to day but the average costs are $(8, 1, 1)$ ¢ per gram of $P1$, $P2$ is $(9, 1, 1)$ ¢ per gram, and $(10, 1, 1)$ ¢ gram for $P3$. The farmer therefore wants to know how many grams of $P1$, $P2$ and $P3$ he should mix together each day, so that his pigs could get the approximate minimum requirement and his costs reduce.

Table 1

Approximate units of food F_j and product P_i Example 1

	Foods		
Products	F1	F2	Average costs
P1	(2.5, 0.5, 0.5)	(5, 0.5, 0.5)	$(8, 1, 1)$
P2	(4.5, 0.5, 0.5)	(3, 0.5, 0.5)	$(9, 1, 1)$
P3	(5, 0.5, 0.5)	(10, 1.0, 1.0)	$(10, 1, 1)$

Buckley and Feuring [8] have formulated a standard diet problem as follows:

$$\begin{aligned} Min \quad & (8,1,1)\widetilde{x}_1 + (9,1,1)\widetilde{x}_2 + (10,1,1)\widetilde{x}_3 \\ s.t. \quad & (2.5,0.5,0.5)\widetilde{x}_1 + (4.5,0.5,0.5)\widetilde{x}_2 + (5,0.5,0.5)\widetilde{x}_3 \succeq (54,4,4) \\ & (5,0.5,0.5)\widetilde{x}_1 + (3,0.5,0.5)\widetilde{x}_2 + (10,1,1)\widetilde{x}_3 \succeq (60,4,4) \\ & \widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \succeq 0, \end{aligned} \tag{9}$$

where $\widetilde{x}_j$ is triangular fuzzy numbers as follows:

$$\widetilde{x}_j = (m_{x_j}, \alpha_{x_j}, \beta_{x_j}) \quad , \quad j = 1, 2, 3.$$

In this example, we used formula (6) to calculate M for control the left and the right side of fuzzy variables proportional to other fuzzy parameters.

$$M = \underset{\substack{i=1,2 \\ j=1,2,3}}{\text{mean}} \left\{ \frac{\alpha_{a_{ij}}}{|m_{a_{ij}}|}, \frac{\beta_{a_{ij}}}{|m_{a_{ij}}|}, \frac{\alpha_{b_i}}{|m_{b_i}|}, \frac{\beta_{b_i}}{|m_{b_i}|}, \frac{\alpha_{c_j}}{|m_{c_j}|}, \frac{\beta_{c_j}}{|m_{c_j}|} \;\middle|\; m_{a_{ij}} \neq 0\ ,\ m_{b_i} \neq 0\ ,\ m_{c_j} \neq 0 \right\}$$

$$= \text{mean} \left\{ \frac{0.5}{|2.5|}, \frac{0.5}{|2.5|}, \frac{0.5}{|4.5|}, \frac{0.5}{|4.5|}, \frac{0.5}{|5|}, \frac{0.5}{|5|}, \frac{0.5}{|5|}, \frac{0.5}{|5|}, \frac{0.5}{|3|}, \frac{0.5}{|3|}, \frac{1}{|10|}, \frac{1}{|10|}, \frac{4}{|54|}, \frac{4}{|54|}, \frac{4}{|60|}, \frac{4}{|60|}, \frac{1}{|8|}, \frac{1}{|8|}, \frac{1}{|9|}, \frac{1}{|9|}, \frac{1}{|10|}, \frac{1}{|10|} \right\} = 0.11.$$

By using the model (7) we convert the FFLP (9) to following model:

$$\begin{aligned}
Min\quad & 8m_{x_1} + 9m_{x_2} + 10m_{x_3} - 2\alpha_{x_1} - 2.25\alpha_{x_2} - 2.5\alpha_{x_3} + 2\beta_{x_1} + 2.25\beta_{x_2} + 2.5\beta_{x_3} \\
s.t.\quad & 2.5m_{x_1} + 4.5m_{x_2} + 5m_{x_3} - 0.625\alpha_{x_1} - 1.125\alpha_{x_2} - 1.25\alpha_{x_3} + 0.625\beta_{x_1} + 1.125\beta_{x_2} + 1.25\beta_{x_3} \geq 54 \\
& 5m_{x_1} + 3m_{x_2} + 10m_{x_3} - 1.25\alpha_{x_1} - 0.75\alpha_{x_2} - 2.5\alpha_{x_3} + 1.25\beta_{x_1} + 0.75\beta_{x_2} + 2.5\beta_{x_3} \geq 60 \\
& 0.11m_{x_j} - \alpha_{x_j} \geq 0, \qquad j = 1,2,3 \\
& 0.11m_{x_j} - \beta_{x_j} \geq 0, \qquad j = 1,2,3 \\
& m_{x_j} - \alpha_{x_j} \geq 0, \quad \alpha_{x_j} \geq 0, \quad \beta_{x_j} \geq 0, \qquad j = 1,2,3.
\end{aligned}$$

As a result, the fuzzy optimal solutions obtained are as follows:

$$\widetilde{x}_1^* = (0,0,0), \quad \widetilde{x}_2^* = (0,0,0), \quad \widetilde{x}_3^* = (10.5,0,1.2), \quad \widetilde{Z}^* = (105,10.5,22.5).$$

The following is the fuzzy optimal solution obtained from Buckley and Feuring [8]:

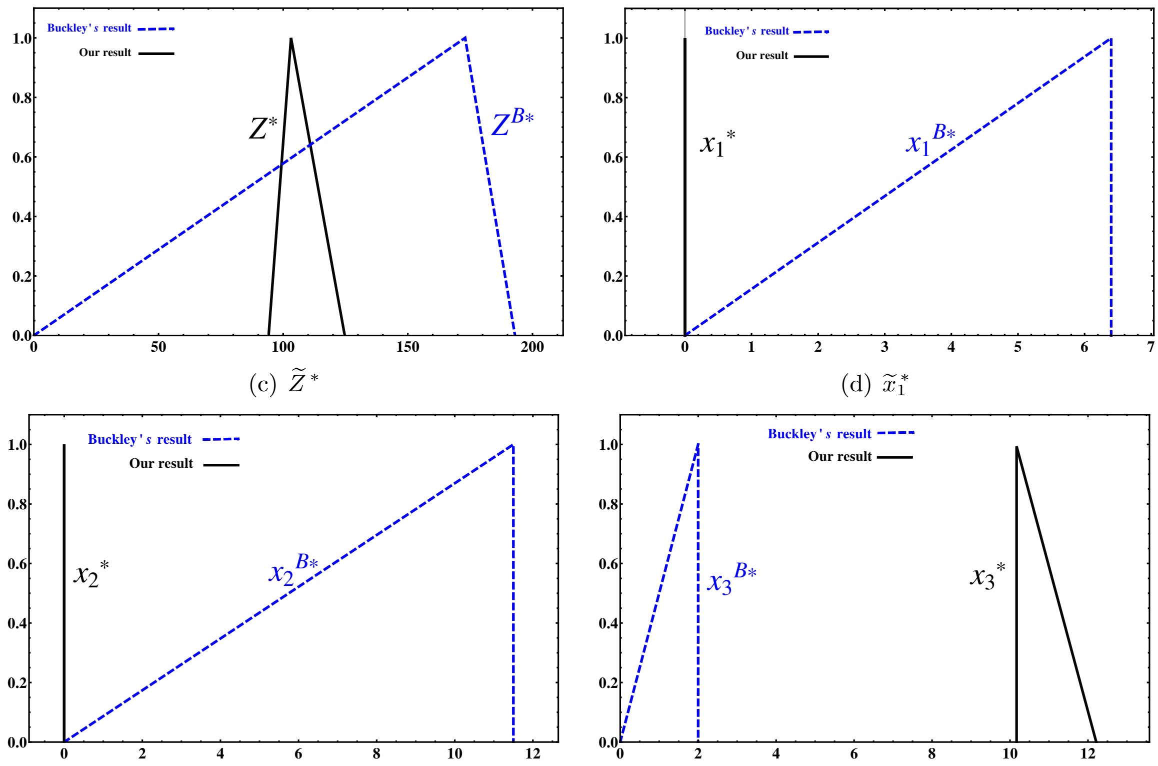
$$\widetilde{x}_1^{B*} = (6.4,6.4,0), \quad \widetilde{x}_2^{B*} = (11.5,11.5,0), \quad \widetilde{x}_3^{B*} = (2,2,0), \quad \widetilde{Z}^{B*} = (173,173,20).$$

The solutions of the method by Buckley were considerably broad, whereas all fuzzy parameters in this example are narrow. In addition, the range of variation in these parameters is low, implying that the solutions obtained by the method by Buckley are not homogeneous with the fuzzy parameters. Comparison between the obtained optimal fuzzy variable solutions by the proposed method and those by the method of Buckley revealed that the optimal fuzzy variable solutions ($\widetilde{x}_j^*$) by the former are narrower than those ($\widetilde{x}_j^{B*}$) by the latter (Figures 2(d), 2(e) and 2(f)). Specifically, $\widetilde{x}_1^{B*}$ and $\widetilde{x}_2^{B*}$ were wider than $\widetilde{x}_1^*$ and $\widetilde{x}_2^*$. Thus, the fuzzy solutions acquired by the proposed method are homogeneous with the fuzzy parameters of the fuzzy problem. We subsequently compared the fuzzy objective optimal solutions obtained by both methods by performing several methods of comparison of fuzzy numbers, and the results are shown in Table 2. All ranking functions indicated that the fuzzy objective optimal solution by the proposed method is better than that by the method of Buckley, except for the Liou and Wang ranking function with $\lambda = 0$ that exhibits the fuzzy objective optimal solution by the method of Buckley as much broader than that by the proposed method. Therefore, a farmer can produce food for his pigs at a reduced price, given that all conditions are satisfied. With regards to decisions related to the narrow fuzzy variable, the farmer must best speculate the amounts of $P1$, $P2$, and $P3$ products to be mixed together.

Table 2

Comparison of our optimal solution with optimal solution of Buckley on the FFLP in Example 1

Ranking Functions	$\widetilde{Z}^*$	$\widetilde{Z}^{B*}$	Ranking results
Fortemps and Roubens	108	134.8	$\widetilde{Z}^* \preceq \widetilde{Z}^{B*}$
Chang	1797	11773	$\widetilde{Z}^* \preceq \widetilde{Z}^{B*}$
Liou and Wang $\lambda = 0$	99.7	86.5	$\widetilde{Z}^* \succeq \widetilde{Z}^{B*}$
Liou and Wang $\lambda = 0.5$	108	134.8	$\widetilde{Z}^* \preceq \widetilde{Z}^{B*}$
Liou and Wang $\lambda = 1$	116.2	183	$\widetilde{Z}^* \preceq \widetilde{Z}^{B*}$



(c) $\widetilde{Z}^*$ (d) $\widetilde{x}_1^*$

(e) $\widetilde{x}_2^*$ (f) $\widetilde{x}_3^*$

Figure 2. The graphical comparison of the results obtained from this study with the results represented by Buckley and Feuring [8] in Example 1

Example 2. Consider the following fuzzy linear program [17].

$$\begin{aligned} Max \quad & (15,5,2)\widetilde{x}_1 + (16,6,4)\widetilde{x}_2 + (14,4,3)\widetilde{x}_3 + (12,2,2)\widetilde{x}_4 \\ s.t. \quad & (10,2,3)\widetilde{x}_1 + (11,1,2)\widetilde{x}_2 + (12,3,1)\widetilde{x}_3 + (15,4,2)\widetilde{x}_4 = (411.75,140,162) \\ & (14,2,2)\widetilde{x}_1 + (18,4,1)\widetilde{x}_2 + (17,3,3)\widetilde{x}_3 + (14,1,4)\widetilde{x}_4 = (539.5,154,220) \\ & \widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4 \succeq 0, \end{aligned} \qquad (10)$$

where $\widetilde{x}_j$ is triangular fuzzy numbers as follows:

$$\widetilde{x}_j = (m_{x_j}, \alpha_{x_j}, \beta_{x_j}), \quad j = 1,2,3 \text{ and } 4.$$

In this example, we used Formula (5) in order to calculate M and control the left and the right side of fuzzy variable proportional to other fuzzy parameters.

$$\begin{aligned} M = & \max_{\substack{i=1,2 \\ j=1,2,3,4}} \left\{ \frac{\alpha_{a_{ij}}}{|m_{a_{ij}}|}, \frac{\beta_{a_{ij}}}{|m_{a_{ij}}|}, \frac{\alpha_{b_i}}{|m_{b_i}|}, \frac{\beta_{b_i}}{|m_{b_i}|}, \frac{\alpha_{c_j}}{|m_{c_j}|}, \frac{\beta_{c_j}}{|m_{c_j}|} \;\middle|\; m_{a_{ij}} \neq 0 \;,\; m_{b_i} \neq 0 \;,\; m_{c_j} \neq 0 \right\} \\ & = \max \left\{ \frac{2}{|10|}, \frac{3}{|10|}, \frac{1}{|11|}, \frac{2}{|11|}, \frac{3}{|12|}, \frac{1}{|12|}, \frac{4}{|15|}, \frac{2}{|15|}, \frac{2}{|14|}, \frac{2}{|14|}, \frac{4}{|18|}, \frac{1}{|18|}, \frac{3}{|17|}, \frac{3}{|17|}, \frac{1}{|14|}, \frac{4}{|14|}, \right. \\ & \left. \frac{140}{|411.75|}, \frac{162}{|411.75|}, \frac{154}{|539.5|}, \frac{220}{|539.5|}, \frac{5}{|15|}, \frac{2}{|15|}, \frac{6}{|16|}, \frac{4}{|16|}, \frac{4}{|14|}, \frac{3}{|14|}, \frac{2}{|12|}, \frac{2}{|12|} \right\} = 0.408. \end{aligned}$$

By using Model (8) for maximization form of fully fuzzy linear programming, we convert the FFLP (10) to the following model:

$$\begin{aligned}
Max\quad & 15m_{x_1}+16m_{x_2}+14m_{x_3}+12m_{x_4}+\frac{1}{4}(-3m_{x_1}-2m_{x_2}-m_{x_3}\\
& -15\alpha_{x_1}-16\alpha_{x_2}-14\alpha_{x_3}-12\alpha_{x_4}+15\beta_{x_1}+16\beta_{x_2}+14\beta_{x_3}+12\beta_{x_4})\\
s.t.\quad & 10m_{x_1}+11m_{x_2}+12m_{x_3}+15m_{x_4}+\frac{1}{4}(m_{x_1}+m_{x_2}-2m_{x_3}-2m_{x_4}\\
& -10\alpha_{x_1}-11\alpha_{x_2}-12\alpha_{x_3}-15\alpha_{x_4}+10\beta_{x_1}+11\beta_{x_2}+12\beta_{x_3}+15\beta_{x_4})=417.25\\
& 14m_{x_1}+18m_{x_2}+17m_{x_3}+14m_{x_4}+\frac{1}{4}(-3m_{x_2}+3m_{x_4}\\
& -14\alpha_{x_1}-18\alpha_{x_2}-17\alpha_{x_3}-14\alpha_{x_4}+14\beta_{x_1}+18\beta_{x_2}+17\beta_{x_3}+14\beta_{x_4})=556\\
& 0.408m_{x_j}-\alpha_{x_j}\geq 0,\qquad j=1,2,3,4\\
& 0.408m_{x_j}-\beta_{x_j}\geq 0,\qquad j=1,2,3,4\\
& m_{x_j}-\alpha_{x_j}\geq 0,\quad \alpha_{x_j}\geq 0,\quad \beta_{x_j}\geq 0,\qquad j=1,2,3,4.
\end{aligned}$$

As a result, the fuzzy optimal solutions are as follows:

$$\widetilde{x}_1^* = (33.23, 0, 13.57),\quad \widetilde{x}_2^* = (0,0,0),\quad \widetilde{x}_3^* = (0,0,0),\quad \widetilde{x}_4^* = (2.63, 0, 1.07)$$

$$\widetilde{Z}^* = (531, 172, 228)$$

This example has been solved by Lotfi et al. [17]. They have solved this example by solving two linear programming problem. The fuzzy solutions of first and second linear programming problem are $\widetilde{x}_j^{H1*}$ and $\widetilde{x}_j^{H2*}$ $(j = 1, 2, 3, 4)$, respectively (Figure 3). Then, fuzzy objective solution has been obtained by using the fuzzy solutions of these linear programming problem. The following results have been obtained:

$$\widetilde{x}_1^{H1*} = (38.14, 10.25, 0),\quad \widetilde{x}_2^{H1*} = (0, 0, 3.31),\quad \widetilde{x}_3^{H1*} = (0,0,0),\quad \widetilde{x}_4^{H1*} = (2.65, 0, 0)$$

$$\widetilde{x}_1^{H2*} = (37.47, 8.33, 0),\quad \widetilde{x}_2^{H2*} = (0, 0, 3.82),\quad \widetilde{x}_3^{H2*} = (0,0,0),\quad \widetilde{x}_4^{H2*} = (2.97, 1.18, 0)$$

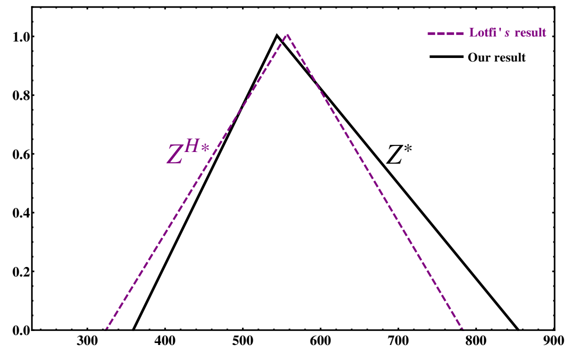
$$\widetilde{Z}^{H*} = (560, 226.3, 226.3)$$

In this example, we show that the proposed method can also solve FFLP problems with equality constraints by focusing on one linear programming problem only. Comparison of the optimal fuzzy variable solutions of the proposed method and those of the method by Lofti et al. revealed that $\widetilde{x}_2^{H1*}$ and $\widetilde{x}_2^{H2*}$ are wider than $\widetilde{x}_2^*$. (Figure 3(i)) Moreover, $\widetilde{x}_3^{H1*} = \widetilde{x}_3^{H2*} = \widetilde{x}_3^*$ and the widths of $\widetilde{x}_1^{H1*}$ (or $\widetilde{x}_1^{H2*}$) and $\widetilde{x}_4^{H2*}$ are almost identical to that of $\widetilde{x}_1^*$ and $\widetilde{x}_4^*$, respectively (Figures 3(h) and 3(j)). Furthermore, Lotfi et al. obtained two fuzzy solutions corresponding to each fuzzy variable. However, the fuzzy solutions were difficult to determine. Table 3 displays the comparison between the obtained fuzzy objective optimal solutions of the proposed method and those of Lotfi et al. who employed various methods in comparing fuzzy numbers. The obtained solutions by the proposed methods are narrower and more controlled than those of Lotfi et al. All ranking functions indicated that the obtained fuzzy objective optimal solution of the proposed method is better than that of Lotfi et al.

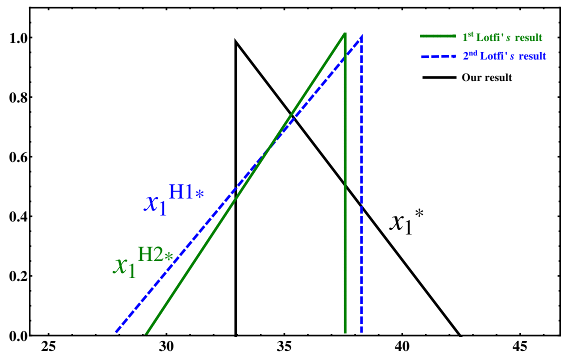
Table 3

Comparison of our optimal solution with optimal solution of F. H. Lotfi [17] on the FFLP in Example 2

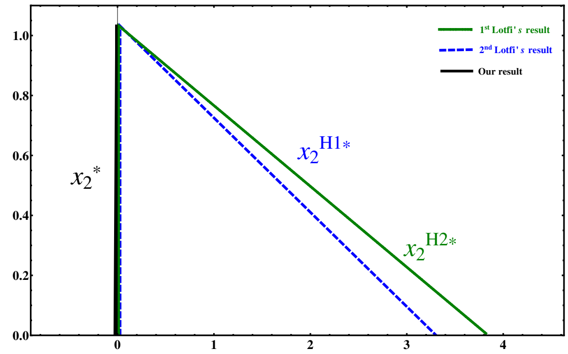
Ranking Functions	$\widetilde{Z}^*$	$\widetilde{Z}^{H*}$	Ranking results
Fortemps and Roubens	560	560	$\widetilde{Z}^* \succeq \widetilde{Z}^{H*}$
Chang	130997	126728	$\widetilde{Z}^* \succeq \widetilde{Z}^{H*}$
Liou and Wang $\lambda = 0$	444.95	444.85	$\widetilde{Z}^* \succeq \widetilde{Z}^{H*}$
Liou and Wang $\lambda = 0.5$	560	560	$\widetilde{Z}^* \succeq \widetilde{Z}^{H*}$
Liou and Wang $\lambda = 1$	674.9	673.1	$\widetilde{Z}^* \succeq \widetilde{Z}^{H*}$



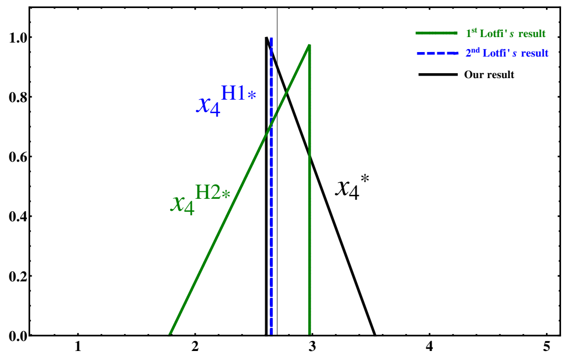
(g) $\widetilde{Z}^*$



(h) $\widetilde{x}_1^*$



(i) $\widetilde{x}_2^*$



(j) $\widetilde{x}_4^*$

Figure 3. The graphical comparison of the results obtained from this study with the results represented by Lotfi et al. [17] in Example 2

Conclusion

In this paper a new method of solving fuzzy linear programming problems is proposed. The suggested method can solve fuzzy linear programming models with equality and inequality constraints by maximization and minimization, and by solving one linear programming problem only. By considering the constraints $Mm_{x_j} - \alpha_{x_j} \geq 0$ and $Mm_{x_j} - \beta_{x_j} \geq 0$, the proposed model was able to control the left and the right sides of fuzzy variables with respect to other fuzzy parameters of the problem. Therefore, the homogeneity among fuzzy solutions and other fuzzy parameters of the problem can be achieved. In addition, this model prevents the generation of broad fuzzy solutions, therefore, making the appropriate decision-making process easier and more convenient. Moreover, the fuzzy objective optimal solutions obtained by the proposed method are better than those by other methods, whereas the fuzzy solutions obtained by the proposed method are also more limited and narrower.

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Айнымалылардың өзгеру диапазонын басқару арқылы сызықтық бағдарламалаудың толық анық емес мәселелерін шешу

Мақалада шешім айнымалыларының коэффициенттері, оң коэффициенттер және айнымалылары анық емес сандармен сипатталатын толық анық емес сызықтық бағдарламалау (FFLP) мәселесі қарастырылған. Анық емес параметрлерге сәйкес анық емес айнымалылардың сол және оң бөліктерін басқару арқылы оңтайлы анық емес шешімдерді алу әдісі ұсынылған. Анық емес басқарылатын шешімдерді қолданса, авторлар күтпеген жауаптардан аулақ болады. Соңында, ұсынылған модель бірнеше саралау функцияларын қолданатын басқа әдістерге қарағанда жақсы оңтайлы шешім бере алатындығын көрсететін екі сандық мысал келтірілді.

Кілт сөздер: толық анық емес сызықтық бағдарламалау, анық емес сызықтық бағдарламалау, анық емес сан, саралау функциясы.

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Решение полностью нечетких задач линейного программирования путем управления диапазоном изменения переменных

В статье рассмотрена задача полностью нечеткого линейного программирования (FFLP), в которой коэффициенты переменных решения, правые коэффициенты и переменные характеризуются нечеткими числами. Предложен метод получения оптимальных нечетких решений путем управления левой и правой частями нечетких переменных в соответствии с нечеткими параметрами. Используя нечеткие контролируемые решения, авторы избежали неожиданных ответов. Таким образом, приведены решения двух численных примеров, демонстрирующих как предлагаемая модель может обеспечить лучшее оптимальное решение, чем другие методы, использующие несколько функций ранжирования.

Ключевые слова: полностью нечеткое линейное программирование, нечеткое линейное программирование, нечеткое число, функция ранжирования.

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Central Limit Theorem in View of Subspace Convex-Cyclic Operators

In our work, we have defined an operator called subspace convex-cyclic operator. The property of this newly defined operator relates eigenvalues which have eigenvectors of modulus one with kernels of the operator. We have also illustrated the effect of the subspace convex-cyclic operator when we let it function in linear dynamics and joining it with functional analysis. The work is done on infinite dimensional spaces which may make linear operators have dense orbits. Its property of measure preserving puts together probability space with measurable dynamics and widens the subject to ergodic theory. We have also applied Birkhoff's Ergodic Theorem to give a modified version of subspace convex-cyclic operator. To work on a separable infinite Hilbert space, it is important to have Gaussian invariant measure from which we use eigenvectors of modulus one to get what we need to have. One of the important results that we have got from this paper is the study of Central Limit Theorem. We have shown that providing Gaussian measure, Central Limit Theorem holds under the certain conditions that are given to the defined operator. In general, our work is theoretically new and is combining three basic concepts dynamical system, operator theory, and ergodic theory under the measure and statistics theory.

Keywords: Central limit theory, Subspace convex-cyclic operator, Gaussian measures.

Introduction

Linear dynamics is a branch of functional analysis. It studies the dynamics of linear operators connecting functional analysis with dynamics. Linear dynamics is mostly dealing with the behaviour of iterates of linear transformations. Linear transformations designated by their Jordan canonical form makes linear dynamics easier to understand when on finite-dimensional spaces. However, when infinite-dimensional space taken into account, linear operators may have dense orbits. One of the focal branches of dynamical system is ergodic theory [1], which relates analysis with probability theory and deals with measurable dynamics. It exerts measure theory to the study of the behavior of dynamical systems. Measure-preserving transformations and measure spaces are the main study subjects in ergodic theory. In probability theory, one of the most substantial results is Central limit theorem, in which under specific conditions the sum of a large number of random variables approaches the normal distribution. This distribution is important since it is suitable for a lot of natural phenomena and social sciences.

In this paper each section demonstrates some of the concepts described above whilst the operator subspace convex-cyclic operator is working on them and illustrates the connections between those notions as follows. In section two, subspace convex-cyclic operator is defined with such a property that correlates eigenvalues having eigenvectors of modulus 1 with kernels of the operators. In section three we have shown that operators with eigenvectors of modulus 1 are subspace convex-cyclic operators. It is also shown that those operators having measure 1 Section four is to come up with a connection between linear dynamics and measurable dynamics with the help of subspace convex-cyclic operators. In this section we have spelled some basic definitions in order to be able to clarify the ergodicity of a transformation. The modified version of subspace convex-cyclic operator is given by applying Birkhoff's Ergodic Theorem on the given operator. In section five Gaussian measure is studied. Here eigenvectors of modulus 1 are used to get Gaussian invariant measure which is crucial for working on a separable infinite Hilbert space. We have given a result that connects the concepts described together. In the last section we have shown that Central Limit Theorem holds under the certain conditions that are given to the defined operator after providing Gaussian measure.

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New subspace convex-cyclic operators

We try to define a new result for showing operators are subspace convex-cyclic operators for subspace $\mathcal{M}$. In this section we try to find subspace convex-cyclic operators in a new point of view, that relates eigenvalues of modulus 1 and kernels of operators. They play an important role in the following sections.

Theorem 1. Let $T \in B(\mathcal{H})$. Suppose $\bigcup_{|\lambda|>1} ker(T-\lambda)$ and $\bigcup_{|\lambda|<1} ker(T-\lambda)$ both span a dense subspace $\mathcal{M}$ of $\mathcal{H}$. Then T is subspace convex-cyclic operator for subspace $\mathcal{M}$.

Proof. We show that T satisfies the subspace convex-cyclic criterion by letting

$$X := Span\left(\bigcup_{|\lambda|<1} ker(T-\lambda)\right) \quad and$$

$$Y := Span\left(\bigcup_{|\lambda|>1} ker(T-\lambda)\right).$$

The sequence of functions $x_k : Y \to H$ are defined as $x_k(y) = \frac{1}{\lambda^k}y$. Also, we define $P_k(T)y = \lambda^k y$. and we use technique Theorem 2.8 in [2] for extending x_k to Y by linear functional. This makes sense because the subspace Y is linearly independent. Thus for any $y \neq 0$ and $y \in Y$, it may uniquely be written as $y = y_1 + \cdots + y_k$, with $y_i \in ker(T-\lambda_i) \setminus \{0\}$ and $|\lambda_i| > 1$. These vectors x and y can be expressed as the following form

$$x := \sum_{i=1}^{k} \alpha_i x_i \qquad and \qquad y := \sum_{i=1}^{k} \beta_i y_i.$$

where $P_k(T)x_i = \lambda x_i$ and $P_k(T)y_i = \mu y_i$ and the scalars $\alpha_i, \beta_i, \lambda, \mu \in \mathbb{C}$ such that $|\lambda_i| < 1$ and $|\mu_i| > 1$ for $i = 1, 2, \cdots, k$. Since

$$P_k(T)(x) = \sum_{i=1}^{k} \alpha_i \lambda^m x_i \to 0 \qquad and$$

$$x_k(y) = \sum_{i=1}^{k} \beta_i \frac{1}{\mu^m} y_i \to 0 \quad as\ m \to \infty$$

$$and \qquad P_k(T)x_k(y) = y,$$

then the first and the second conditions of Theorem 3 in [3] are hold, for showing the third, the space $\mathcal{H}$ is an infinite dimensional (Real or Complex) separable Hilbert space. We observe that

$$x \in \bigcap_{j=1}^{\infty} \bigcup_{P_k \in \mathcal{P}} P_k(T)^{-1}(B_j)$$

if and only if for all $j \in \mathbb{N}$ there exist a convex polynomials P_k such that $x \in P_k(T)^{-1}(B_j)$ which implies $P_k(T)(x) \in B_j$. But since $\{B_j\}$ is a basis for the relatively topology of $\mathcal{M}$, this occurs if and only if $\widehat{Orb(T,x)} \cap \mathcal{M}$ is dense in $\mathcal{M}$, which means T is subspace convex-cyclic transitive and by definition of $\mathcal{M}$ convex-cyclic transitive, there exist $U\ and\ V$ relatively open subsets in $\mathcal{M}$ such that

$$W := P(T)^{-1}(U) \cap V \neq \phi \quad [3].$$

In particular, non-empty subset W relatively in $\mathcal{M}$, and $W \subset P(T)^{-1}(U)$. Then $P(T)(W) \subset U$ and $U \subset \mathcal{M}$, so we get that

$$P(T)(W) \subset \mathcal{M}.$$

Let $x \in \mathcal{M}$, we must show that $P(T)(\mathcal{M}) \subset \mathcal{M}$. Take $w_0 \in W$, since W is relatively open in $\mathcal{M}$ and $x \in \mathcal{M}$ so there exist $r > 0$ such that $w_0 + rx \in W$. But $P(T)(W) \subseteq \mathcal{M}$, that is,

$$P(T)(w_0 + rx) = P(T)(w_0) + rP(T)x \in \mathcal{M},$$

then $P(T)(w_0) \in \mathcal{M}$ and $\mathcal{M}$ is subspace. So,

$$r^{-1}\left(-P(T)(w_0) + P(T)(w_0) + rP(T)(x)\right) \in \mathcal{M},$$

that is, $P(T)(x) \in \mathcal{M}$. This is true for any $x \in \mathcal{M}$, hence for $P(T)(x) \in \mathcal{M}$, that is $P(T)(\mathcal{M}) \subseteq \mathcal{M}$. All conditions are satisfied. We get that T is subspace convex-cyclic for $\mathcal{M}$.

Remark 1. The inverse of the above theorem is not true, see Proposition 2.

Eigenvalue measure

We aim to show that operators with sufficiently many eigenvectors of modulus 1 are subspace convex-cyclic operators. Let us first construct the following definition from Theorems 3.1 and 3.9 in [2] and Theorem 1. Here we have not mentioned any density or properties related to density for details you can see [4], we just join it with collection of eigenvalues and associated eigenvectors of the subspace convex-cyclic orbits that we have named them orbit eigen-spaces. Such operator with unit measure property will help us for proofing the next outcomes.

Definition 1. Let $T \in B(\mathcal{H})$ be subspace convex-cyclic operator for a nontrivial subspace $\mathcal{M}$ of $\mathcal{H}$. Then for any scalar $\lambda \in \mathbb{T}$. We define $\mathfrak{A}$ as an orbit eigen-spaces if

$$\mathfrak{A} := Span\left[ker(\lambda I - T)Or\widehat{b(T, \mathcal{M}})\right].$$

Definition 2. Let $T \in B(\mathcal{H})$ be subspace convex-cyclic operator for a nontrivial subspace $\mathcal{M}$ of $\mathcal{H}$. Then for any scalar $\lambda \in \mathbb{T}$. We define unit measure μ if for every measurable subset $\Omega \subset \mathbb{T}$, $\mu(\Omega) = 1$. And $\mu\{ker(\lambda I - T)\} = 0$.

For imagining the Definition 1 and Definition 2 see the following example. For skipped steps we refer the reader to review Example 4 in [3].

Example 1. Let $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$, and consider $T := \lambda B$ where B is the backward shift on ℓ^2. Let $\mathcal{M}$ be the subspace of ℓ^2 consisting of all sequences with zeros on the even entries as $\mathcal{M} = \left\{\{a_n\}_{n=0}^{\infty} \in \ell^2 : a_{2k} = 0 \text{ for all } k\right\}$ [3].

Solution. For operators like T defined above, surely we have an eigenvalue γ under the condition $|\lambda| > |\gamma|$, so in this situation we have $ker(\gamma I - T) = ker(\gamma I - \lambda B)$ such vector like $x_\lambda \in \mathbb{R}$ will span them as

$$x_\gamma = \sum_{i=0}^{\infty} \left(\frac{\gamma}{\lambda}\right)^i e_i,$$

where (e_i) is a canonical basis for $i = 1, \cdots$ Let μ be the measure on the unit circle that normalized $\mathbb{T}$, and suppose that a measurable set $\Omega \subset \mathbb{T}$ let $h \in \mathcal{H}$ be an orthogonal vector such that $\langle h, x_\gamma\rangle = 0$ for every $\gamma \in \Omega$ by Hahn-Banach Theorem we have a well known linear functional defined as

$$\zeta(\gamma) = \sum_{i=0}^{\infty} \langle x, e_i\rangle \left(\frac{\gamma}{\lambda}\right)^i$$

$\zeta(\gamma) \to 0$, since ω is any subset of $\mathbb{T}$, so there are two choices, if it is countable then we get contradiction for been T as a subspace convex-cyclic operators, then it should be uncountable and in that case we have a limit points around the circle center and that leads to $\mu\{ker(\lambda I - T)\} = 0$ and its obvious that taking $\mu(\Omega) = max\{\rho(\Omega, \mathbb{T}), 1\} = 1$, where ρ is the metric that defined on the space depends on ℓ^2 space. Then the conditions in Definition 2 are satisfied.

Now, depending on Example 1 in [3] we can define $\mathcal{M} = \ell^2 \oplus \{0\}$. Consequently we get that

$$\left[\widehat{Orb}(T \oplus I, (x \oplus 0))\right] \cap \left[\ell^2 \oplus \{0\}\right] = \ell^2 \oplus \{0\} = \mathcal{M}$$

finally we can define

$$\mathfrak{A} := Span\left[ker(\lambda I - T)\mathcal{M}\right].$$

Subspace Conv-Cyc and Ergodic Theory

We previously obtained Subspace Convex-Cyclic Operators in Section 1 by another way. The aim of this section is to provide a bridge between linear dynamics and measurable dynamics. The most important concept to start with it is invariant measure, because it has the direct connection with Subspace Convex-Cyclic Transitive Operators (see Definition 2 and Theorem 1 in [3]), bounded $T : \mathcal{H} \to \mathcal{H}$, where $\mathcal{M}$ is subspace of $\mathcal{H}$. If for all non-empty open sets $U \subset \mathcal{M}$ and $V \subset \mathcal{M}$, there exist a convex P such that $U \cap P(T)(V) \neq \phi$ or $P(T)^{-1}(U) \cap V \neq \phi$ contains a relatively open non-empty subset of $\mathcal{M}$ if and only if $\mathcal{M}$ is an invariant subspace for $P_k(T)$ for all $k \geq 0$ [3].

We first start by recalling some basic definitions of Ergodic theory. For more details see [5] which is very useful related to that branch. In this section $\mathfrak{B}$ is Borel $\sigma - algebra$.

Definition 3. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space. We recall a measurable transformation $T : (\mathcal{H}, \mathfrak{B}, \mu) \to$ $\to (\mathcal{H}, \mathfrak{B}, \mu)$ is a measure-preserving transformation, or μ is $T - invariant$, if $\mu(f^{-1}(\mathfrak{U})) = \mu(\mathfrak{U})$ for all $\mathfrak{U} \in \mathfrak{B}$.

Definition 4. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space and T is measure-preserving transformation. For any non-empty subset G of $\mathcal{H}$, We say μ is positive measure if $\mu(G) > 0$, as well For any non-empty open subset U of $\mathcal{H}$, We say μ is fully support if $\mu(U) > 0$.

Definition 5. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space. We recall a measurable transformation $T : (\mathcal{H}, \mathfrak{B}, \mu) \to$ $\to (\mathcal{H}, \mathfrak{B}, \mu)$ is Ergodic if it is a measure-preserving transformation and satisfies one of the following equivalent conditions [5]:

1 Given any measurable sets U and V with positive measures, one can find an integer $n \geq 0$ such that $T^n(U) \cap V \neq \phi$,

2 if $U \in \mathfrak{B}$ satisfies $T(U) \subset U$ then $\mu(U) = 0$ or $\mu(U) = 1$.

The following Theorem is known as Birkhoff's Ergodic Theorem.

Theorem 2. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space and $T : (\mathcal{H}, \mathfrak{B}, \mu) \to (\mathcal{H}, \mathfrak{B}, \mu)$ is measure-preserving and Ergodic transformation. For any non-zero function $f \in L^1(\mathcal{H}, \mu)$,

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \to \int_{\mathcal{H}} f d\mu$$

as $N \to \infty$, *almost everywhere.*

In the following proposition, we demonstrate that Birkhoff's Ergodic Theorem can also be applied to our operator. So, this version will be our modified version with Subspace Convex-Cyclic Transitive Operators. The proof will depend on the Mahlar measure as worked in [6] of polynomial measure.

Proposition 1. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space and $T : (\mathcal{H}, \mathfrak{B}, \mu) \to (\mathcal{H}, \mathfrak{B}, \mu)$ is measure-preserving and Ergodic transformation. Given any measurable sets U and V. Then

$$\frac{1}{N} \sum_{k=0}^{N-1} \mu(P(T)^{-1}(U) \cap V) = \lambda \mu(V)$$

as $N \to \infty$, *almost everywhere.*

Proof. Since $P(T)x$ as $P(T) = a_0 + a_1 T + a_2 T^2 + \cdots + a_n T^n, n \in \mathbb{N}$, so we write it as

$$P(T) = a_n \prod_{i=1}^{n} (T - a_i)$$

is defined by the formula

$$\mu(P) = a_n \prod_{i=1}^{n} max\{1, a_i\}$$

and was first considered by Mahler. If P and Q are non-zero polynomials, note that

$$\mu(P.Q) = \mu(P).\mu(Q).$$

The result directly comes after evaluating $P(T)^{-1}$ and letting $\lambda = a_n \prod_{i=1}^{n} max\{1, a_i\}$ for measure intersection from de Morgan's laws, a collection of subsets is $\sigma - algebra$ under the operations of taking complements and countable intersections.

Now one of our results can be stated. We add fully support measure because here we depend on open sets for taking any measure. The proof will depend the previous proposition.

Theorem 3. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space with fully support measure and $T : (\mathcal{H}, \mathfrak{B}, \mu) \to (\mathcal{H}, \mathfrak{B}, \mu)$ is measure-preserving and Ergodic transformation. Then T is Subspace Convex-Cyclic Operator.

Proof. Let (V_j) for $j \in \mathbb{N}$ be a countable basis of open sets for $\mathcal{H}$. Applying Proposition 1 to the collection of constant function that are convex, a_i where $\sum_{i=0}^{n} a_i = 1$ for any V_j, we get a sequence of sets $\{U_j\}$ for all j, such that

$$\frac{1}{N}\sum_{k=0}^{N-1} a_j(P(T)^{-1}(U_j) \cap V_j) = \lambda\mu(V_j)$$

as $N \to \infty$, *almost everywhere.*

Now, we have two cases depends on the measure of V_j and Ergodic definition. $\mu(V_j) = 1,\ j \in \mathbb{N}$, then we set $U := \bigcap_{j\in\mathbb{N}} U_j$, let U_1 and U_2 be open sets. Since by given

$$\frac{1}{N}\sum_{k=0}^{N-1} (P(T)^{-1}(U_1) \cap U_2) = \lambda$$

[3] which leads to $P(T)^{-1}(U) \cap V \neq \phi$ is Subspace Convex-Cyclic Transitive Operator and we can deduce that T is Subspace Convex-Cyclic Operator. The same thing is true for $\mu(V_j) = 0$.

Gaussian measure

We claimed that there is a special measure under an invariant bounded transformation preserving with its measure being ergodic T on $\mathcal{H}$. We are interested to add an additional tool that supports measure space $\mathcal{H}$. Gaussian measure is an atmosphere space that should be studied. As we know, working on an infinite Hilbert space without using Gaussian measure is not an easy way. To work with such a situation, we need Subspace convex-cyclic on Borel $\sigma - algebra$ that has sufficiently many eigenvectors of modulus 1.

For that purpose we need this section. Eigenvectors of modulus 1 are the fundamental tools we use to get Gaussian invariant measures. We need other definitions here. You can find more details in [7].

Definition 6. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space and $f : (\mathcal{H}, \mathfrak{B}, \mu) \to \mathbb{C}$ is a complex valued measurable function. Then f is said to have complex symmetric Gaussian distribution if the real and imaginary parts $\mathfrak{Re} f$ and $\mathfrak{Im} f$ of f have independent centered Gaussian distribution with the same variance.

This is equivalent to saying that $\mathfrak{Re} f$ and $\mathfrak{Im} f$ are jointly normal and that f and λf have the same distribution for any λ of modulus 1 [8].

Definition 7. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space a Gaussian measure on $\mathcal{H}$ is a probability measure μ on $\mathcal{H}$ such that for every $x \in \mathcal{H}$, the function $f_x : y \to \langle x, y\rangle$ has symmetric complex Gaussian distribution.

In particular with this terminology, such a measure is centered:

$$\int_{\mathcal{H}} \langle x, y\rangle d\mu(y) = \int_{\mathcal{H}} f_x(y) d\mu(y) = \int_{\mathcal{H}} y d(f_x(\mu))(y) \quad [9].$$

Remark 2. A Gaussian measure is determined exactly by the operator S defined on $\mathcal{H}$ by the relation

$$\langle Sx, y\rangle = \int_{\mathcal{H}} \langle x, z\rangle \overline{\langle y, z\rangle} d\mu(z)$$

The operator S in probability books are called as covariance.

Proposition 2. Let T be a subspace convex-cyclic operator on a separable Hilbert space, and $(\mathcal{M}, \mathfrak{B}, \mu)$ be a probability space with Gaussian measure on $\mathcal{M}$ for any $\mathcal{M} \subset \mathcal{H}$. Then $T(\mu)$ also is a Gaussian measure on $\mathcal{M}$. Such that $ker(T - \lambda I)$ span $(f_k(\lambda)),\ k \geq 1$ for all $\lambda \in \mathbb{T}$.

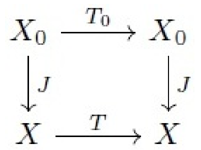
Proof. Since T is a subspace convex-cyclic operator, then each iteration of T will be a new element in $\mathcal{M}$, so directly by definition of subspace convex-cyclic operator we get dense set that itself is Gaussian measure. Now define a sequence of Borel $\sigma - algebra$ that sufficiently many eigenvectors points with certain eigenvectors of modulus 1 and polynomial operators as $f_k : \mathbb{T} \to \mathfrak{B}$ by $f_k(\lambda) = \tau_\lambda(x_k)$ such that for every $\lambda \in \mathbb{T}$, where $\tau_\lambda(x_k)$ be orthogonal onto $ker(T - \lambda I)$. Then each element of the sequence f_k is a Borel measure, so

$$\tau_\lambda(x_k) = x \Leftrightarrow T(x) = \lambda(x),$$

which means τ roll as a projection invariant and will span vectors $(f_k(\lambda))$ $k \geq 1$ is dense in $ker(T - \lambda I)$.

Quasi-factor is one of the most important concepts that has great application in dynamical system and operator theory you can find details and application in both [9] and [10].

Definition 8. Let $T_0 : X_0 \to X_0$ and $T : X \to X$ be two continuous maps acting on topological spaces X_0 and X. The map T is said to be a quasi-factor of T_0 if there exists a continuous map with dense range $J : X_0 \to X$ such that the diagram commutes, such that $TJ = JT_0$. When this can be achieved with a homeomorphism $J : X_0 \to X$, so that $T = JT_0J^{-1}$ we say that T_0 and T are topologically conjugate. Finally, when T_0 and T are linear operators and the factoring map (resp. the homeomorphism) J can be taken as linear, we say that T is a linear quasi-factor of T_0 (resp. that T_0 and T are linearly conjugate) [10].



We have similar results for the following theorem but with more conditions because they are defined with other operators. Our operator makes this easier. Now, it remains to state the theorem that connect all concepts to gather. Subspace convex-cyclic on Borel $\sigma - algebra$ that sufficiently many eigenvectors points with certain eigenvectors of modulus 1, and invariant Gaussian measure.

Theorem 4. If T is a subspace convex-cyclic operator on a separable Hilbert space X, then T admits a symmetry Gaussian invariant measure which is quasi-factor.

Proof. Let $(f_k(\lambda))$, $k \geq 1$ with the property that we deal with it in Proposition 2 for all $\lambda \in \mathbb{T}$ as defined in Definition 2, Let φ be defined on $\ell^2(\mathbb{T}, \sigma)$ of sequences (g_k) $k \geq 1$ of functions converges of $\ell^2(\mathbb{T}, \sigma)$

$$\varphi \sum_{k=1}^{\infty} g_k(\lambda) = \sum_{k=1}^{\infty} \lambda g_k(\lambda).$$

You can note that φ behaves as an operator of multiplication by λ on each component, now it is time to define κ as $\kappa : \sum\limits_{k=1}^{\infty} \ell^2(\mathbb{T}, \sigma) \to \mathcal{H}$ by Definition 7 we have,

$$\kappa \sum_{k=1}^{\infty} g_k = \sum_{k=1}^{\infty} \frac{1}{2^k} \int\limits_{\mathcal{H}} g_k(\lambda) f_k(\lambda) d\sigma(\lambda).$$

Then κ is a well defined operator on Hilbert space; each $\kappa_k : \ell^2(\mathbb{T}, \sigma) \to \mathcal{H}$, which maps g_k onto $\int\limits_{\mathcal{H}} g_k(\lambda) f_k(\lambda) d\sigma(\lambda)$ is a kernel operator. So, for any element in this sequence to be 0 without one which is arbitrary, this implies that for every $x \in \mathcal{H}$

$$\langle x, \int\limits_{\mathcal{H}} g_k(\lambda) f_k(\lambda) d\sigma(\lambda) \rangle = 0.$$

This implies that $\langle x, f_k(\lambda) \rangle = 0$ in the sense $\sigma - algebra$ which means that x is orthogonal to $ker(T - \lambda I)$. This implies that T has a $\sigma - algebra$ set of eigenvectors that spanned. Which leads to κ having dense range.

Now, if we want to show that T is quasi-factor, we make a choice of the operators pair as κ, φ, use the fact that $f_k(\lambda) \in ker(T - \lambda I)$, we get that for every $g_k(\lambda) \in \sum\limits_{i=1}^{\infty} \ell^2(\mathbb{T}, \sigma)$,

$$
\begin{aligned}
T\kappa\sum_{i=1}^{\infty} g_k(\lambda) &= \sum_{i=1}^{\infty}\frac{1}{2^i}\int\limits_{\mathbb{T}} g_k(\lambda)Tf_k(\lambda)d\sigma(\lambda)\\
&= \sum_{i=1}^{\infty}\frac{1}{2^i}\int\limits_{\mathbb{T}} g_k(\lambda)\lambda f_k(\lambda)d\sigma(\lambda)\\
&= \sum_{i=1}^{\infty}\frac{1}{2^i}\int\limits_{\mathbb{T}} \varphi\sum_{i=1}^{\infty} g_k(\lambda)f_k(\lambda)d\sigma(\lambda)\\
&= \kappa\varphi\sum_{i=1}^{\infty} g_k(\lambda)
\end{aligned}
$$

Central Limit Theorem

There are many important applications of central limit theorem, which are related to many branch of mathematics such as probability, dynamical system, operator system and many others. Our focus is on dynamical system that deals with Gaussian measure. How far a dynamical system is form an independed under conditions that added to such operator T. We prove that the central limit theorem also holds after providing Gaussian measure. The concepts of central limit theorem can be defined as follows without losing generality and modifying the definition in [11].

Definition 9. Let A be a $\sigma - algebra$ with a Subspace Convex-Cyclic operator T and some operator f where $f \in L^1(\mathcal{H},\mu)$. Then by assumption of Theorem 2 we say that

$$
\frac{f + f + \cdots + f\circ T^{n-1}}{\sqrt{n}} \to \int\limits_{\mathcal{H}} f d\mu
$$

converges in distribution to a $\sigma - algebra$ random variable.

Theorem 5. Let $T \in B(\mathcal{H})$. Suppose that T satisfies the following assertion:

1 Theorem 3 in [3], and then T be Subspace Convex-Cyclic operator.

2 Definition 2

3 Suppose that there exists $\mathcal{P}$ collection of polynomials, also $\alpha \in (1,\infty)$ such that for any $f, g \in \mathcal{P}$

$$
\langle f\circ T^n, g\rangle < C\frac{1}{n^{\alpha}} .
$$

Then T and the sequence of function $\frac{1}{\sqrt{n}}(f+f+\cdots+f\circ T^{n-1})$ converges in distribution to a Gaussian measure spaces.

Proof. Let ω be any non-decreasing function. Let $(x_k)_{k\in\mathbb{N}}$ be a dense sequence in $\mathcal{D}$ and $T^k x_k \leq \omega(k)$. By the first assumption we have that any $k \in \mathbb{N}$,

$$
T^k x_k \leq \frac{\omega(k)}{(1+|k|)^{\alpha}}.
$$

We construct the measure exactly as we did in Proposition 2, with additional properties that the sequence (p_l) also satisfies:

$$
\forall k \geq 1, \quad \forall l \geq q, \quad \omega(l)^q \leq p_l^{-1/2}
$$
$$
\forall k \geq 1, \quad \sum_{l\geq 1}(N_{l+1} - N_l)p_l^{\frac{1}{2q}},
$$

where the nature of l can be review in Lemma 2 [3] for choosing N_l and N_{l+1}, to prove that $P \subset \ell^2(\mathcal{H},\mathfrak{B},\mu)$, it enough to show that, for any $k \geq 1$, $\langle x,y\rangle^k \in \ell^2(\mathcal{H},\mathfrak{B},\mu)$

$$
\int\limits_{\mathcal{H}} \langle x,y\rangle^q = \int\limits_{\mathcal{M}} \left\langle \sum_{l\geq 1}\sum_{|k|=N_l}^{N_l+1} T^k x_{n_k}, y\right\rangle^q d\mu((n_k))
$$

$$\leq \sum_{l_1,\ldots,l_q\geq 1} \int_{\mathcal{M}} \left\langle \sum_{|k|=N_{l_i}}^{N_l+1} T^k x_{n_k}, y \right\rangle d\mu((n_k)).$$

We then apply Hölder's inequality to get

$$\int_{\mathcal{H}} \langle x, y\rangle^q \leq \sum_{l_1,\ldots,l_q\geq 1} \left(\int_{\mathcal{M}} \left\langle \sum_{|k|=N_{l_i}}^{N_l+1} T^k x_{n_k}, y \right\rangle^q d\mu((n_k)) \right)^{\frac{1}{q}}.$$

We fix $l \geq 1$ and we want to calculate

$$\int_{\mathcal{M}} \left\langle \sum_{|k|=N_l}^{N_l+1} T^k x_{n_k}, y \right\rangle^q d\mu((n_k)).$$

Let $(n_k) \subset \mathcal{M}$ and let us write

$$\left\langle \sum_{|k|=N_l}^{N_l+1} T^k x_{n_k}, y \right\rangle^q \leq 2^q \left(\left\langle \sum_{|k|=N_l}^{N_l+1} ((n_k \leq l)) T^k x_{n_k}, y \right\rangle^q \right.$$
$$\left. + \left\langle \sum_{|k|=N_l}^{N_l+1} ((n_k \geq l)) T^k x_{n_k}, y \right\rangle^q \right)$$
$$\leq \frac{2^q}{2^{ld}} + 2^{2q-1}(N_{l+1} - N_l)^{q-1}$$
$$\sum_{|k|=N_l}^{N_l+1} \left\langle T^k x_{n_k}, y \right\rangle^q.$$

We take integral to this inequality over $\mathcal{M}$ for getting

$$\int_{\mathcal{M}} \left\langle \sum_{|k|=N_l}^{N_l+1} T^k x_{n_k}, y \right\rangle^q d\mu((n_k)) \leq \frac{2^q}{2^{ld}} +$$
$$2^{2q-1}(N_{l+1} - N_l)^{q-1} \sum_{|k|=N_l}^{N_l+1} \sum_{m>l} p_m \left\langle T^k x_m, y \right\rangle^q$$
$$\leq \frac{2^q}{2^{ld}} + 2^{2q}(N_{l+1} - N_l)^q \sum_{m>l} p_m \omega(m)^q$$
$$\leq \frac{2^q}{2^{ld}} + 2^{2q}(N_{l+1} - N_l)^q p_l \max(\omega(l), \omega(q))^q$$
$$\leq \frac{2^q}{2^{ld}} + 2^{2q}(N_{l+1} - N_l)^q p_l^{1/2} \omega(q)^q.$$

Since we assumed $\omega(l)^q p_l \leq p_l^{1/2}$, we take the exponent $1/q$ and we collect the inequalities to get

$$\int_X \langle X, y\rangle^q d\mu(x) \leq$$
$$C \left(\sum_{l\geq 1} \left(\frac{2}{2^{1/2}} + 4(N_{l+1} - N_l) p_l^{\frac{1}{2q}} \right) \right)^q \omega(q)^q$$
$$\leq C_n.$$

Thus, $\langle x, y\rangle^k \in \ell^2(\mathcal{H}, \mathfrak{B}, \mu)$ by constant C_d which can be much bigger.

Because our operator is subspace convex-cyclic operator so such a polynomial, we choose a derivative function, as $f \in \omega(k)$ and we observe that

$$\left\langle D^k f(x_k), y_k\right\rangle \leq \left\langle D^k, f\right\rangle \cdot \langle x_k, y_k\rangle^k$$

so that, from the proof of the first point, we deduce

$$\left\langle D^k, f\right\rangle \leq C^k \omega(k)^k D^k(f).$$

Since $D^k f$ is convergent, then the series $\sum_k \frac{D^k f(x_k)}{k!}$ is convergent in $\ell^2(\mathcal{H}, \mathfrak{B}, \mu)$ converges in distribution to a Gaussian measure spaces.

The following example will help us to understand the above theorem more. We apply Theorem 5, because B_w satisfied all conditions that we had in the statement. It remains to show that how the inequality in the $3rd$ condition will happen.

Example 2. Let B_w be a bounded backward weighted shift on $\ell^2(\mathbb{N})$. Suppose moreover that $\sum\limits_{n=1}^{\infty} \frac{1}{(w_1 \ldots w_n)^2}$ converge. Then B_w converges in distribution to a Gaussian measure spaces.

Proof. Let ω be any non-decreasing function. Let $(\alpha_n)_n \in \mathbb{N}$ be a dense sequence in $\mathbb{N}$ with $|\alpha_n^p| \leq \omega(n)$. We set $\mathcal{D} := (x_n)\ n \geq 1$, with $x_n = \alpha_n e_l$, where $(e_l)\ l \geq 1$ is the standard basis of $\ell^2(\mathbb{N})$. We define S_n on $\mathcal{D}$ by $S_n(e_l) = \frac{1}{(w_1 \ldots w_n)^2} e_n$.

Since $(\alpha_n)_n$ is dense in $\mathbb{N}$ then $span(S_n x_n\ n \geq 1)$ is dense in $\mathcal{H}$.

As we get $\sum\limits_{k \geq 1}^{\infty} B_w^k x_k = \alpha_n e_l$, also $\sum\limits_{k<1}^{\infty} S_k x_k = \alpha_n e_l$, now

$$\begin{aligned} \left\langle \sum_{k<1}^{\infty} S_k x_k, e_k \right\rangle &= \left\langle \sum_{k<1}^{\infty} \alpha_n e_k, e_k \right\rangle \\ &= \left(\sum_{k<1}^{\infty} \frac{|\alpha_n|}{(w_1 \ldots w_n)^2} \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k<1}^{\infty} \frac{\omega(n)}{(w_1 \ldots w_n)^2} \right)^{\frac{1}{2}}. \end{aligned}$$

Since $\frac{\omega(n-1)}{w_1} \geq \frac{1}{w_1}$, this yields to

$$\left\langle \sum_{k<1}^{\infty} S_k x_k, e_k \right\rangle \leq C_w leq \left(\sum_{k<1}^{\infty} \frac{\omega(n)}{(w_1 \ldots w_n)^2} \right).$$

So, B_w satisfied all conditions. We get the result.

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Ішкі кеңістіктікті дөңес-циклдік операторларды ескере отырып, орталық шекті теорема

Мақалада дөңес-циклдік ішкі кеңістікті оператор деп аталатын оператор анықталған. Бұл жаңадан анықталған оператордың қасиеті өзінің векторлары бар меншікті мәндерді, бір модульді оператордың ядроларымен байланыстырады. Авторлар сызықтық динамикада жұмыс жасаған, оны функционалды талдаумен біріктірген кезде дөңес-циклдік ішкі кеңістік операторының әсерін суреттеген. Жұмыс шексіз өлшемді кеңістіктерде орындалды, бұл сызықтық операторлардың тығыз орбиталарға ие болуына әкелуі мүмкін. Оның өлшемді сақтау қасиеті ықтималдық кеңістігін өлшенетін динамикамен біріктіреді және эргодтық теорияның тақырыбын кеңейтеді. Сондай-ақ, дөңес-циклдік ішкі кеңістік операторының модификацияланған нұсқасын беру үшін Биркгоф эргодтық теоремасы қолданылған. Сепарабелді шексіз Гильберт кеңістігімен жұмыс істеу үшін Гаусстың инвариантты өлшемі болуы керек, оның көмегімен қажет нәрсені алу үшін модульдің жеке векторлары пайданылған. Осы мақалада алынған маңызды нәтижелердің бірі - орталық шекті теореманы зерттеу. Гаусс өлшемін қамтамасыз ете отырып, белгілі бір операторға берілген белгілі бір жағдайларда орталық шекті теорема дұрыс екені көрсетілген. Жалпы, жұмыс теориялық тұрғыдан жаңа және үш негізгі ұғымды біріктіреді: динамикалық жүйе, операторлар теориясы және өлшеу теориясы мен статистика шеңберіндегі эргодтық теория.

Кілт сөздер: орталық шекті теорема, ішкі кеңістікті дөңес-циклдік оператор, Гаусс өлшемдері.

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Центральная предельная теорема с учетом подпространственных выпукло-циклических операторов

В статье определен оператор, называемый выпукло-циклическим оператором подпространства. Свойство этого вновь определенного оператора связывает собственные значения, имеющие собственные векторы, модуля один с ядрами оператора. Авторами проиллюстрирован эффект выпукло-циклического оператора подпространства, в случае когда показаны функции в линейной динамике и объединены с функциональным анализом. Работа выполнена в бесконечномерных пространствах, которые могут привести к тому, что линейные операторы будут иметь плотные орбиты. Его свойство сохранения меры объединяет вероятностное пространство с измеримой динамикой и расширяет предмет эргодической теории. Авторами статьи использована эргодическая теорема Биркгофа, дающая модифицированную версию выпукло-циклического оператора подпространства. Чтобы работать с сепарабельным бесконечным гильбертовым пространством, важно иметь гауссову инвариантную меру, из которой применяются собственные векторы модуля один, чтобы получить то, что необходимо. Одним из важных результатов, полученных в этой статье, является изучение центральной предельной теоремы. Показано, что, обеспечивая гауссову меру, центральная предельная теорема верна при определенных условиях, которые задаются определенному оператору. В целом, данная работа является теоретически новой и объединяет три основных понятия: динамическую систему, теорию операторов и эргодическую теорию в рамках теории меры и статистики.

Ключевые слова: центральная предельная теорема, выпукло-циклический оператор подпространства, гауссовы меры.

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Cohomology of simple modules for $\mathfrak{sl}_3(k)$ in characteristic 3

In this paper we calculate cohomology of a classical Lie algebra of type A_2 over an algebraically field k of characteristic $p = 3$ with coefficients in simple modules. To describe their structure, we will consider them as modules over an algebraic group $SL_3(k)$. In the case of characteristic $p = 3$, there are only two peculiar simple modules: a simple that module isomorphic to the quotient module of the adjoint module by the center, and a one-dimensional trivial module. The results on the cohomology of simple nontrivial module are used for calculating the cohomology of the adjoint module. We also calculate cohomology of the simple quotient algebra Lie of A_2 by the center.

Keywords: Lie algebra, simple module, restricted module cohomology, exact sequence.

Introduction

The cohomology theory of modular Lie algebras is one of the interesting questions in the theory of Lie algebras. Many significant results are devoted to the study of the cohomology of classical modular Lie algebras. Their restricted cohomology with coefficients in the dual Weyl modules was studied in [1–3]. Central extensions are described in [4, 5]. In [6, 7] the outer derivations are calculated. As the second cohomology, local deformations are calculated in [8–10].

Among the classical modular Lie algebras, the cohomology of simple modules is completely described only for a three-dimensional Lie algebra of type A_1 [11]. It is known that for other classical modular Lie algebras a complete description of the cohomology of simple modules has not yet been obtained. In this paper we give a complete description of such cohomology for the Lie algebra of type A_2 over an algebraically closed field of characteristic $p = 3$. The first cohomology groups of simple modules for A_2 was computed in [12]. A similar result for the second cohomology groups was obtained in [13]. In all other cases, the computation of the cohomology structure of simple modules for A_2 is close to completion. The results will be published in the next works of the second author.

Let us introduce the basic definitions and notation. Let $\mathfrak{g}$ be a Lie algebra over a field k characteristics of p and M be a $\mathfrak{g}$-module. We denote the n-th exterior power of the space $\mathfrak{g}$ by $\Lambda^n(\mathfrak{g})$ and let

$$C^n(\mathfrak{g}, M) = Hom(\Lambda^n, M) = \langle\, \psi \,:\, \mathfrak{g} \times \cdots \times \mathfrak{g} \to M \,\rangle_k,\ n > 0$$

is a space of multilinear skew-symmetric mappings in n arguments with coefficients in M. We put

$$C^n(\mathfrak{g}, M) = 0,\ n < 0,\ \ C^0(\mathfrak{g}, M) = M,\ \ C^*(\mathfrak{g}, M) = \bigoplus_{n=-\infty}^{+\infty} C^n(\mathfrak{g}, M).$$

Define the coboundary operator

$$d \,:\, C^*(\mathfrak{g}, M) \longrightarrow C^*(\mathfrak{g}, M)$$

as follows:

$$d\psi(l_1, l_2, \cdots, l_{n+1}) =$$
$$\sum_{i<j}(-1)^{i+j}\psi([l_i, l_j], \cdots, \hat{l_i}, \cdots, \hat{l_j}, \cdots, l_{n+1}) + \sum_{i}(-1)^{i+1}[l_i, \psi(l_1, \cdots, \hat{l_i}, \cdots, l_{n+1})],$$

where $\psi \in C^n(\mathfrak{g}, M)$. Then $d^2 = 0$, therefore $B^*(\mathfrak{g}, M) \subseteq Z^*(\mathfrak{g}, M)$, where

$$Z^*(\mathfrak{g}, M) = \langle\, \psi \in C^*(\mathfrak{g}, M) \,:\, d\psi = 0 \,\rangle_k,$$

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$$B^*(\mathfrak{g}, M) = \langle\, d\psi \;:\; \psi \in C^*(\mathfrak{g}, M)\,\rangle_k.$$

So, we can introduce the factor-space

$$H^*(\mathfrak{g}, M) = Z^*(\mathfrak{g}, M)/B^*(\mathfrak{g}, M).$$

The spaces $C^*(\mathfrak{g}, M),\ Z^*(\mathfrak{g}, M),\ B^*(\mathfrak{g}, M),\ H^*(\mathfrak{g}, M)$ are called *space of cochains*, *space of cocycles*, *spaces of coboundaries*, and *space of cohomologies* of the Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module M respectively.

Similarly, the spaces

$$C^n(\mathfrak{g}, M),\ Z^n(\mathfrak{g}, M) = Z^*(\mathfrak{g}, M) \cap C^n(\mathfrak{g}, M),$$
$$B^n(\mathfrak{g}, M) = B^*(\mathfrak{g}, M) \cap C^n(\mathfrak{g}, M) \text{ и } H^n(\mathfrak{g}, M) = H^*(\mathfrak{g}, M) \cap C^n(\mathfrak{g}, M)$$

are called *space of n-cochains*, *space of n-cocycles*, *spaces of n-coboundaries*, and *space of n-cohomologies* of the Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module M respectively.

We say that the $\mathfrak{g}$-module M is *peculiar* if $H^*(\mathfrak{g}, M) \neq 0$. We say that the M is *n-peculiar* module over $\mathfrak{g}$ if $H^n(\mathfrak{g}, M) \neq 0$.

Now let $\mathfrak{g}$ be a classical Lie algebra of type A_2 over algebraically closed field k of positive characteristic $p > 0$ and M is a $\mathfrak{g}$-module. We decompose $C^*(\mathfrak{g}, M)$ into a direct sum of weight subspaces with respect to the maximal torus T of the group $G = SL_3(k)$:

$$C^*(\mathfrak{g}, M) = \bigoplus_{\mu \in X(T)} C^*_\mu(\mathfrak{g}, M),$$

where $X(T)$ is the additive character group of T. Then

$$H^n(\mathfrak{g}, M) = \bigoplus_{\mu \in X(T)} H^n_\mu(\mathfrak{g}M).$$

Identify the space $C^n(\mathfrak{g}, M)$ with the space $\bigwedge^n \mathfrak{g}^* \bigotimes M$ and denote by $\prod(V)$ the set of weights of the G-module subspace V of $H^*(\mathfrak{g}, M)$.

Since $\prod(H^n(\mathfrak{g}, M)) \subseteq pX(T) \bigcap \prod(\bigwedge^n \mathfrak{g}^* \bigotimes M)$, then we can consider only the elements of the subspace $\overline{C}^n(\mathfrak{g}, M)$ of $C^n(\mathfrak{g}, M)$ with weights contained in the set $pX(T) \bigcap \prod(\bigwedge^n \mathfrak{g}^* \bigotimes M)$. The corresponding subspaces of cocycles and cohomologies are denoted by $\overline{Z}^n(\mathfrak{g}, M)$ and $\overline{H}^n(\mathfrak{g}, M)$. Note that

$$H^n(\mathfrak{g}, M) = \overline{H}^n(\mathfrak{g}, M).$$

We will use the following well known formulas:

$$\dim\, H^n(\mathfrak{g}, M) = \dim\, \overline{Z}^n(\mathfrak{g}, M) + \dim\, \overline{Z}^{n-1}(\mathfrak{g}, M) - \dim\, \overline{C}^{n-1}(\mathfrak{g}, M), \tag{1}$$

$$\dim\, H^n(\mathfrak{g}, M) = \dim\, H^{\dim\, \mathfrak{g} - n}(\mathfrak{g}, M^*). \tag{2}$$

The weight subspaces are invariant under the action of the coboundary operator, therefore the formula (1) is also holds for weight subspaces:

$$\dim\, H^n_\mu(\mathfrak{g}, M) = \dim\, \overline{Z}^n_\mu(\mathfrak{g}, M) + \dim\, \overline{Z}^{n-1}_\mu(\mathfrak{g}, M) - \dim\, \overline{C}^{n-1}_\mu(\mathfrak{g}, M). \tag{3}$$

Let $L(r, s)$ be a simple $\mathfrak{g}$-module with the highest weight $r\omega_1 + s\omega_2$, where ω_1, ω_2 are fundamental weights.

It is known that the composition of a representation of $SL_3(k)$ on a vector space L with a d-th power of the Frobenius map defines a new representation, on which the Lie algebra $\mathfrak{g}$ acts trivially. We denote the resulting module by $L^{(d)}$. To each weight μ of the space L there corresponds a weight $p^d\mu$ of the space $L^{(d)}$. The cohomology group $H^n(\mathfrak{g}, M)$, as a $SL_3(k)$-module, consists of either a twisted module $L^{(d)}$ for some d, or a one-dimensional trivial module k. For the multiplicity of a $SL_3(k)$-module $L^{(d)}$ in $H^n(\mathfrak{g}, M)$, we use the notation $[H^n(\mathfrak{g}, M) : L^{(d)}]$. Further, for convenience we use the following abbreviations: $H^n(\mathfrak{g}, k) := H^n(\mathfrak{g})$, $\bigoplus_{i=1}^m V := mV$, where V is a $SL_3(k)$-module.

Let's formulate the main result of this paper:

Theorem 1. Let $\mathfrak{g}$ be a classical Lie algebra of type A_2 over an algebraically closed field k of characteristic $p = 3$ and M ba a simple $\mathfrak{g}$-module. Then there are the following isomorphisms of $SL_3(k)$-modules:

(a) $H^0(\mathfrak{g}) \cong H^8(\mathfrak{g}) \cong k$, $H^2(\mathfrak{g}) \cong H^6(\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)}$, $H^3(\mathfrak{g}) \cong H^5(\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)} \oplus k$;

(b) $H^1(\mathfrak{g}, L(1,1)) \cong H^7(\mathfrak{g}, L(1,1)) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)} \oplus k$, $H^3(\mathfrak{g}, L(1,1)) \cong H^5(\mathfrak{g}, L(1,1)) \cong H^0(1,1)^{(1)}$, $H^4(\mathfrak{g}, L(1,1)) \cong 2H^0(1,1)^{(1)}$.

In other cases $H^n(\mathfrak{g}, M) = 0$.

Proof of the Theorem 1

As the basis vectors for $\mathfrak{g}$ we choose the special derivations of the algebra of divided powers $O_3(\mathbf{1})$:

$$h_1 = x_1\partial_1 - x_2\partial_2, h_2 = x_2\partial_2 - x_3\partial_3, e_1 = x_1\partial_2, e_2 = x_2\partial_3, e_3 = x_1\partial_3, f_1 = x_2\partial_1, f_2 = x_3\partial_2, f_3 = x_3\partial_1.$$

Over a field of characteristic $p = 3$, the Lie algebra $\mathfrak{g}$ is not simple, it has a one-dimensional center $\langle h_1 - h_2\rangle_k$. The quotient algebra by the center is a simple Lie algebra; we denote it by $\overline{\mathfrak{g}}$ or $\overline{A_2}$.

It is known that the peculiar modules of the Lie algebra $\mathfrak{g}$ are restricted [11]. According to Lemma 3.1 in [13], only the following two simple restricted modules are peculiar: $L(0,0) \cong k$ and $L(1,1) \cong \overline{\mathfrak{g}}$. For $L(1,1)$ we get the following description:

$$L(1,1) \cong \langle h_1,\, h_2, e_1,\, e_2,\, e_3,\, f_1,\, f_2,\, f_3\, :\, h_1 - h_2 = 0\rangle_k.$$

Consider each of these modules separately.

Let $M = L(0,0) \cong k$.

Lemma 1. There are the following isomorphisms of $SL_3(k)$-modules:
(a) $H^0(\mathfrak{g}) \cong k$;
(b) $H^2(\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)}$;
(c) $H^3(\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)} \oplus k$;
(d) $H^5(\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)} \oplus k$;
(e) $H^6(\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)}$;
(f) $H^8(\mathfrak{g}) \cong k$.
In other cases $H^n(\mathfrak{g}) = 0$.

Proof. The statements (a) and (f) are obvious. The triviality of $H^1(\mathfrak{g})$ in characteristic $p = 3$ was proved in [12].

(b) The set $\prod(\overline{C}^2(\mathfrak{g}))$ only consists of the following weights: $0 \pm 3\omega_1$, $\pm 3(\omega_1 - \omega_2)$, $\pm 3\omega_2$. Therefore, only the trivial one-dimensional module and the twisted simple modules $L(1,0)^{(1)}$, $L(0,1)^{(1)}$, can be as nonzero composition factors of $H^2(\mathfrak{g})$. They are generated by the classes of cocycles with dominant weights 0, $3\omega_1$, and $3\omega_2$ respectively.

The subspace $\overline{C}^2_0(\mathfrak{g})$ is 4-dimensional and spans by the cochains $h_1^* \wedge h_2^*$, $e_1^* \wedge f_1^*$, $e_2^* \wedge f_2^*$, $e_3^* \wedge f_3^*$. If $a_1 h_1^* \wedge h_2^* + a_2 e_1^* \wedge f_1^* + a_3 e_2^* \wedge f_2^* + a_4 e_3^* \wedge f_3^* \in \overline{Z}^2(\mathfrak{g})$ then, by cocycle condition, $a_1 = 0$, $a_4 = a_2 + a_3$. Therefore $\dim \overline{Z}^2_0(\mathfrak{g}) = 2$. Since $\dim \overline{C}^1_0(\mathfrak{g}) = 2$ and $\dim \overline{Z}^1_0(\mathfrak{g}) = 0$, by (2), $\dim \overline{H}^2_0(\mathfrak{g}) = 2 + 0 - 2 = 0$.

The subspace $\overline{C}^2_{3\omega_1}(\mathfrak{g})$ is one-dimensional and spans by the cochain $f_1^* \wedge f_3^*$. Notice that $a f_1^* \wedge f_2^* \in \overline{Z}^2(\mathfrak{g})$ for all $a \in k$. Therefore $\dim \overline{Z}^2_{3\omega_1}(\mathfrak{g}) = 1$. Since $\dim \overline{C}^1_{3\omega_1}(\mathfrak{g}) = 0$, by (3), $\dim \overline{H}^2_{3\omega_1}(\mathfrak{g}) = 1$. So $[H^2(\mathfrak{g}) : L(1,0)^{(1)}] = 1$.

Arguing as in the previous case, we obtain $[H^2(\mathfrak{g}) : L(0,1)^{(1)}] = 1$. Thus $H^2(\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)}$.

(c) The sets of weights $\prod(\overline{C}^3(\mathfrak{g}))$ and $\prod(\overline{C}^2(\mathfrak{g}))$ coincide. Therefore, we consider only the weight subspaces of 3-cochains corresponding to the dominant weights 0, $3\omega_1$, and $3\omega_2$.

The subspace $\overline{C}^3_0(\mathfrak{g})$ is 8-dimensional and spans by the cochains

$$h_1^* \wedge e_1^* \wedge f_1^*,\ h_2^* \wedge e_1^* \wedge f_1^*,\ h_1^* \wedge e_2^* \wedge f_2^*,\ h_2^* \wedge e_2^* \wedge f_2^*,$$

$$h_1^* \wedge e_3^* \wedge f_3^*,\ h_2^* \wedge e_3^* \wedge f_3^*,\ e_3^* \wedge f_1^* \wedge f_2^*,\ e_1^* \wedge e_2^* \wedge f_3^*.$$

Suppose that a linear combination of these vectors with coefficients b_i, $i = 1, \cdots, 8$ respectively, is a 3-cocycle. Then the cocycle condition implies that

$$b_1+b_2+b_5+b_7-b_8 = 0,\ b_2+b_3-b_7+b_8 = 0,\ b_3+b_4+b_6+b_7-b_8 = 0,\ 2b_4+2b_7-2b_8 = 0,\ 2b_5+2b_6+2b_7-2b_8 = 0.$$

Whence it follows that $\dim Z^3_0(\mathfrak{g}) = 3$. By (3),

$$\dim H^3_0(\mathfrak{g}) = \dim \overline{Z}^3_0(\mathfrak{g}) + \dim \overline{Z}^2_0(\mathfrak{g}) - \dim \overline{C}^2_0(\mathfrak{g}) = 3 + 2 - 4 = 1.$$

Therefore $[H^3_0(\mathfrak{g}) : k] = 1$.

The weight subspaces $\overline{C}^3_{3\lambda_1}(\mathfrak{g})$, $\overline{C}^3_{3\lambda_2}(\mathfrak{g})$ are two-dimensional and span respectively with 3-cochains: $h_1^* \wedge f_1^* \wedge f_3^*$, $h_2^* \wedge f_1^* \wedge f_3^*$, and $h_1^* \wedge f_2^* \wedge f_3^*$, $h_2^* \wedge f_2^* \wedge f_3^*$. Using the cocycle condition, we get $\dim \overline{Z}^3_{3\lambda_1}(\mathfrak{g}) = \dim \overline{Z}^3_{3\lambda_2}(\mathfrak{g}) = 1$. So, $H^3(\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)} \oplus k$.

Now we prove that $H^4(\mathfrak{g}) = 0$. It's obvious that $\prod(\overline{C}^4(\mathfrak{g})) = \prod(\overline{C}^3(\mathfrak{g}))$. Therefore we consider only the weight subspaces

$$\overline{C}_0^4(\mathfrak{g}),\ \overline{C}_{3\omega_1}^4(\mathfrak{g}),\ \overline{C}_{3\omega_2}^4(\mathfrak{g}).$$

The subspace $\overline{C}_0^4(\mathfrak{g})$ is 10-dimensional and spans by the cochains

$$h_1^* \wedge h_2^* \wedge e_1^* \wedge f_1^*,\ h_1^* \wedge h_2^* \wedge e_2^* \wedge f_2^*,\ h_1^* \wedge h_2^* \wedge e_3^* \wedge f_3^*,\ h_1^* \wedge e_1^* \wedge e_2^* \wedge f_3^*,$$

$$h_2^* \wedge e_1^* \wedge e_2^* \wedge f_3^*,\ h_1^* \wedge e_3^* \wedge f_1^* \wedge f_2^*,\ h_2^* \wedge e_3^* \wedge f_1^* \wedge f_2^*,$$

$$e_1^* \wedge e_2^* \wedge f_1^* \wedge f_2^*,\ e_1^* \wedge e_3^* \wedge f_1^* \wedge f_3^*,\ e_2^* \wedge e_3^* \wedge f_2^* \wedge f_3^*.$$

Suppose that the linear combination of these vectors coefficients b_i, $i = 1, \cdots, 10$ respectively is a 4-cocycle. Then $b_1 = b_2 = b_3 = 0$, $b_4 = b_6$, $b_5 = b_7$. Whence it follows that $\dim \overline{z}_0^4(\mathfrak{g}) = 5$. By (3), $\dim H_0^4(\mathfrak{g}) = 5+3-8 = 0$. Therefore, $[H^4(\mathfrak{g}) : k] = 0$.

It's obvious that $\overline{Z}_{3\omega_1}^4(\mathfrak{g}, M)) = \overline{Z}_{3\omega_1}^4(\mathfrak{g}, M)) = 1$. Then by (3), $\overline{H}_{3\omega_1}^4(\mathfrak{g}, M)) = \overline{H}_{3\omega_1}^4(\mathfrak{g}, M)) = 1+1-2 = 0$. So, $H^4(\mathfrak{g}) = 0$.

Using (2) and the statements (b), (c), we get the statements (e), (f) respectively. The proof of Lemma 1 is complete.

Now let $M = L(1, 1)$.

Lemma 2. There are the following isomorphisms of $SL_3(k)$-modules:

(a) $H^1(\mathfrak{g}, L(1, 1)) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus k$;

(b) $H^3(\mathfrak{g}, L(1, 1)) \cong H^0(1, 1)^{(1)}$;

(c) $H^4(\mathfrak{g}, L(1, 1)) \cong 2H^0(1, 1)^{(1)}$;

(d) $H^5(\mathfrak{g}, L(1, 1)) \cong H^0(1, 1)^{(1)}$;

(e) $H^7(\mathfrak{g}, L(1, 1)) \cong L(1, 0)^{(1)} \oplus L(0, 1)^{(1)} \oplus k$.

In other cases $H^n(\mathfrak{g}, L(1, 1)) = 0$.

Proof. The calculations similar to the previous Lemma 1 yield:

1) $\prod(\overline{C}^0(\mathfrak{g}, L(1, 1))) = \prod(\overline{C}^8(\mathfrak{g}, L(1, 1))) = \{0\}$,
$\prod(\overline{C}^i(\mathfrak{g}, L(1, 1))) = \{0, \pm 3\omega_1, \pm 3(\omega_1 - \omega_2), \pm 3\omega_2\}$ for $i = 1, 2, 6, 7$,
$\prod(\overline{C}^j(\mathfrak{g}, L(1, 1))) = \prod(\overline{C}^1(\mathfrak{g}, L(1, 1))) \cup \{\pm 3(\omega_1 + \omega_2), \pm 3(2\omega_1 - \omega_2), \pm 3(-\omega_1 + 2\omega_2)\}$ for $j = 3, 4$;

2) $\dim \overline{C}_0^0(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_0^8(\mathfrak{g}, L(1, 1)) = 1$, $\dim \overline{C}_0^1(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_0^7(\mathfrak{g}, L(1, 1)) = 8$,
$\dim \overline{C}_0^2(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_0^6(\mathfrak{g}, L(1, 1)) = 22$, $\dim \overline{C}_0^3(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_0^5(\mathfrak{g}, L(1, 1)) = 38$,
$\dim \overline{C}_0^4(\mathfrak{g}, L(1, 1)) = 44$;

3) $\dim \overline{C}_{3\omega_i}^0(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3\omega_i}^8(\mathfrak{g}, L(1, 1)) = 0$, $\dim \overline{C}_{3\omega_i}^1(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3\omega_i}^7(\mathfrak{g}, L(1, 1)) = 2$,
$\dim \overline{C}_{3\omega_i}^2(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3\omega_i}^6(\mathfrak{g}, L(1, 1)) = 7$, $\dim \overline{C}_{3\omega_i}^3(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3\omega_i}^5(\mathfrak{g}, L(1, 1)) = 14$,
$\dim \overline{C}_{3\omega_i}^4(\mathfrak{g}, L(1, 1)) = 18$ for $i = 1, 2$;

4) $\dim \overline{C}_{3(\omega_1+\omega_2)}^0(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3(\omega_1+\omega_2)}^8(\mathfrak{g}, L(1, 1)) = 0$,
$\dim \overline{C}_{3(\omega_1+\omega_2)}^1(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3(\omega_1+\omega_2)}^7(\mathfrak{g}, L(1, 1)) = 0$,
$\dim \overline{C}_{3(\omega_1+\omega_2)}^2(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3(\omega_1+\omega_2)}^6(\mathfrak{g}, L(1, 1)) = 0$,
$\dim \overline{C}_{3(\omega_1+\omega_2)}^3(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3(\omega_1+\omega_2)}^5(\mathfrak{g}, L(1, 1)) = 1$,
$\dim \overline{C}_{3(\omega_1+\omega_2)}^4(\mathfrak{g}, L(1, 1)) = 2$;

5) $\dim \overline{Z}_0^0(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_0^8(\mathfrak{g}, L(1, 1)) = 0$, $\dim \overline{Z}_0^1(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_0^7(\mathfrak{g}, L(1, 1)) = 2$,
$\dim \overline{Z}_0^2(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_0^6(\mathfrak{g}, L(1, 1)) = 6$, $\dim \overline{Z}_0^3(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_0^5(\mathfrak{g}, L(1, 1)) = 18$,
$\dim \overline{Z}_0^4(\mathfrak{g}, L(1, 1)) = 24$;

6) $\dim \overline{Z}_{3\omega_i}^0(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3\omega_i}^8(\mathfrak{g}, L(1, 1)) = 0$, $\dim \overline{Z}_{3\omega_i}^1(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3\omega_i}^7(\mathfrak{g}, L(1, 1)) = 1$,
$\dim \overline{Z}_{3\omega_i}^2(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3\omega_i}^6(\mathfrak{g}, L(1, 1)) = 1$, $\dim \overline{C}_{3\omega_i}^3(\mathfrak{g}, L(1, 1)) = \dim \overline{C}_{3\omega_i}^5(\mathfrak{g}, L(1, 1)) = 6$,
$\dim \overline{C}_{3\omega_i}^4(\mathfrak{g}, L(1, 1)) = 8$ for $i = 1, 2$;

7) $\dim \overline{Z}_{3(\omega_1+\omega_2)}^0(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3(\omega_1+\omega_2)}^8(\mathfrak{g}, L(1, 1)) = 0$,
$\dim \overline{Z}_{3(\omega_1+\omega_2)}^1(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3(\omega_1+\omega_2)}^7(\mathfrak{g}, L(1, 1)) = 0$,
$\dim \overline{Z}_{3(\omega_1+\omega_2)}^2(\mathfrak{g}, L(1, 1)) = \dim \overline{Z}_{3(\omega_1+\omega_2)}^6(\mathfrak{g}, L(1, 1)) = 0$,

$\dim\, \overline{Z}^3_{3(\omega_1+\omega_2)}(\mathfrak{g}, L(1,1)) = \dim\, \overline{Z}^5_{3(\omega_1+\omega_2)}(\mathfrak{g}, L(1,1)) = 1$,
$\dim\, \overline{Z}^4_{3(\omega_1+\omega_2)}(\mathfrak{g}, L(1,1)) = 2$.

Then, by (3), $\dim\, \overline{H}^n_\mu(\mathfrak{g}, L(1,1)) = 0$ except in the following cases:

i) $\dim\, \overline{H}^1_0(\mathfrak{g}, L(1,1)) = \dim\, \overline{H}^7_0(\mathfrak{g}, L(1,1)) = 1$, $\dim\, \overline{H}^3_0(\mathfrak{g}, L(1,1)) = \dim\, \overline{H}^5_0(\mathfrak{g}, L(1,1)) = 2$, $\dim\, \overline{H}^4_0(\mathfrak{g}, L(1,1)) = 4$;

ii) $\dim\, \overline{H}^1_{3\omega_i}(\mathfrak{g}, L(1,1)) = \dim\, \overline{H}^7_{3\omega_i}(\mathfrak{g}, L(1,1)) = 1$ for $i = 1, 2$;

iii) $\dim\, \overline{H}^3_{3(\omega_1+\omega_2)}(\mathfrak{g}, L(1,1)) = \dim\, \overline{H}^5_{3(\omega_1+\omega_2)}(\mathfrak{g}, L(1,1)) = 1$, $\dim\, \overline{H}^4_{3(\omega_1+\omega_2)}(\mathfrak{g}, L(1,1)) = 2$.

Analyzing the dimensions of the weight subspaces of the corresponding cohomology groups, we obtain the required statements of Lemma 2. The proof of Lemma 2 is complete.

Combining the results of Lemmas l and 2, we obtain all the statements of Theorem 1.

Cohomlogy of the adjoint module

Using Theorem 1, we can easily compute the cohomology of the adjoint module for $\mathfrak{g}$. There is the following short exact sequence of $\mathfrak{g}$-modules:

$$0 \to k \to \mathfrak{g} \to L(1,1) \to 0.$$

Consider the corresponding long exact cohomological sequence of $SL_3(k)$-modules

$$\cdots \to H^{n-1}(\mathfrak{g}, L(1,1)) \to H^n(\mathfrak{g}) \to H^n(\mathfrak{g},\mathfrak{g}) \to H^n(\mathfrak{g}, L(1,1)) \to H^{n+1}(\mathfrak{g}) \to \cdots$$

It is known that $H^2(\mathfrak{g},\mathfrak{g}) = 0$ [14]. Then, according to Theorem 1, the last long exact cohomological sequence splits into the following five exact sequences:

$$0 \to H^0(\mathfrak{g}) \to H^0(\mathfrak{g},\mathfrak{g}) \to 0,$$

$$0 \to H^1(\mathfrak{g},\mathfrak{g}) \to H^1(\mathfrak{g}, L(1,1)) \to H^2(\mathfrak{g}) \to 0,$$
$$0 \to H^3(\mathfrak{g}) \to H^3(\mathfrak{g},\mathfrak{g}) \to H^3(\mathfrak{g}, L(1,1)) \to 0,$$
$$0 \to H^4(\mathfrak{g},\mathfrak{g}) \to H^4(\mathfrak{g}, L(1,1)) \to H^5(\mathfrak{g}) \to H^5(\mathfrak{g},\mathfrak{g}) \to H^5(\mathfrak{g}, L(1,1)) \to H^6(\mathfrak{g}) \to H^6(\mathfrak{g},\mathfrak{g}) \to 0,$$
$$0 \to H^7(\mathfrak{g},\mathfrak{g}) \to H^7(\mathfrak{g}, L(1,1)) \to H^8(\mathfrak{g}) \to H^8(\mathfrak{g},\mathfrak{g}) \to 0.$$

The first three short exact sequences yield the following isomorphisms of A-modules respectively:

$$H^0(\mathfrak{g},\mathfrak{g}) \cong k,\ H^1(\mathfrak{g},\mathfrak{g}) \cong k,\ H^3(\mathfrak{g},\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)} \oplus H^0(1,1)^{(1)} \oplus k.$$

Since $3(\omega_1 + \omega_2) \notin \prod(H^i(\mathfrak{g}))$ for $i = 5, 6$, then the fourth exact sequence splits and yields the following isomorphisms:

$$H^4(\mathfrak{g},\mathfrak{g}) \cong H^4(\mathfrak{g}, L(1,1)) \cong H^0(1,1)^{(1)}, H^5(\mathfrak{g},\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)} \oplus H^0(1,1)^{(1)} \oplus k,$$

$$H^6(\mathfrak{g},\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)}.$$

Similarly to the previous case, from the last exact sequence we obtain

$$H^7(\mathfrak{g},\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)} \oplus k, H^8(\mathfrak{g},\mathfrak{g}) \cong k.$$

Thus, we get the following

Proposition 1. Let $\mathfrak{g}$ be a classical Lie algebra of type A_2 over an algebraically closed field k of characteristic $p = 3$. Then there are the following isomorphisms of $SL_3(k)$-modules:

(a) $H^0(\mathfrak{g},\mathfrak{g}) \cong H^1(\mathfrak{g},\mathfrak{g}) \cong H^8(\mathfrak{g},\mathfrak{g}) \cong k$;
(b) $H^3(\mathfrak{g},\mathfrak{g}) \cong H^5(\mathfrak{g},\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)} \oplus H^0(1,1)^{(1)} \oplus k$;
(c) $H^4(\mathfrak{g},\mathfrak{g}) \cong 2H^0(1,1)^{(1)}$;
(d) $H^6(\mathfrak{g},\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)}$;
(e) $H^7(\mathfrak{g},\mathfrak{g}) \cong L(1,0)^{(1)} \oplus L(0,1)^{(1)} \oplus k$.

In other cases $H^n(\mathfrak{g},\mathfrak{g}) = 0$.

Cohomlogy for $\overline{A_2}$

Recall that $\overline{A_2}$ is the quotient algebra of the classical Lie algebra of type A_2 over an algebraically closed field of characteristic $p = 3$ by the center. In this section we compute cohomology of the simple Lie algebra $\overline{A_2}$ with coefficients in the simple modules.

First, we consider an arbitrary Lie algebra $\mathfrak{g}$ with the center $C_\mathfrak{g}$ such that the corresponding quotient algebra is a simple algebra. The following result immediately leads to our goal.

Lemma 3. Let $\overline{\mathfrak{g}}$ be a simple quotient Lie algebra of a Lie algebra $\mathfrak{g}$ by the center $C_\mathfrak{g}$. Then $H^n(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}) \cong H^n(\mathfrak{g}, \overline{\mathfrak{g}})$ for all $n > 0$.

Proof. The space $\overline{\mathfrak{g}}$ can be equipped with the structure of a module over each of the Lie algebras $C_\mathfrak{g}$, $\mathfrak{g}$ and $\overline{\mathfrak{g}}$:

$C_\mathfrak{g} \times \overline{\mathfrak{g}} \to \overline{\mathfrak{g}},\ (c, \overline{a}) \mapsto \mu(c)\overline{a}$, where μ is a nonzero linear form on $C_\mathfrak{g}$;

$\mathfrak{g} \times \overline{\mathfrak{g}} \to \overline{\mathfrak{g}},\ (a_1, \overline{a}_2) \mapsto \overline{[a_1, a_2]},\ a_1 \in \mathfrak{g},\ \overline{a}_2 \in \overline{\mathfrak{g}}$;

$\overline{\mathfrak{g}} \times \overline{\mathfrak{g}} \to \overline{\mathfrak{g}},\ (\overline{a}_1, \overline{a}_2) \mapsto \overline{[a_1, a_2]},\ \overline{a}_1,\ \overline{a}_2 \in \overline{\mathfrak{g}}$.

The short exact sequence of cochain complexes

$$0 \to (C^*(C_\mathfrak{g}, \overline{\mathfrak{g}}), d) \to (C^*(\mathfrak{g}, \overline{\mathfrak{g}}), d) \to (C^*(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}), d) \to 0$$

gives a long exact cohomological sequence

$$\cdots \to H^{n-1}(C_\mathfrak{g}, \overline{\mathfrak{g}}) \to H^n(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}) \to H^n(\mathfrak{g}, \overline{\mathfrak{g}}) \to H^n(C_\mathfrak{g}, \overline{\mathfrak{g}}) \to \cdots .$$

Since $H^n(C_\mathfrak{g}, \overline{\mathfrak{g}}) = 0$ for all $n \geq 0$ [15, Lemma 4.2], it follows from the fact that last cohomological sequence is exact that $H^n(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}) \cong H^n(\mathfrak{g}, \overline{\mathfrak{g}})$ for all $n > 0$. The proof of Lemma 3 is complete.

Remark. A special case of Lemma 3 for $n = 1$ was proved in [7]. Using Lemma 3 to Theorem 1, we obtain a complete description of the cohomology of a simple Lie algebra $\overline{A_2}$ with coefficients in simple modules.

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Сипаттамасы 3 $\mathfrak{sl}_3(k)$ үшін жәй модульдердің когомологиялары

Мақалада сипаттамасы $p = 3$ алгебралық тұйық k өрісіне қатысты A_2 түріндегі классикалық Ли алгебрасы коэффициенттерінің жәй модульдердегі когомологиялары есептелген. Құрылымын беру үшін оларды $SL_3(k)$ алгебралық группасының модульдері ретінде қарастырған. Өріс сипаттамасы $p = 3$ болғанда тек екі арнайы жәй модуль бар: кіріктірілген модульдің центр бойынша фактор-модуліне изоморфты модуль және бір өлшемді тривиаль модуль. Жәй тривиаль емес модульдің когомологиялары туралы алынған нәтижелер кіріктірілген модульдің когомологияларын есептеуге қолданылды. Сонымен қатар, A_2-нің центр бойынша жәй фактор-алгебрасының да когомологиялары есептелді.

Кілт сөздер: Ли алгебрасы, жәй модуль, шектелген модуль, когомология, дәл тізбек.

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Когомологии простых модулей для $\mathfrak{sl}_3(k)$ в характеристике 3

В статье вычислены когомологии классической алгебры Ли типа A_2 над алгебраически замкнутым полем k характеристики $p = 3$ с коэффициентами в простых модулях. Для описания структуры авторы рассмотрели их как модули над алгебраической группой $SL_3(k)$. В случае характеристики $p = 3$ существуют только два простых особых модулей: простой модуль, изоморфный фактор-модулю присоединенного модуля по центру, и одномерный тривиальный модуль. Результаты, полученные для когомологии простого нетривиального модуля, применены для вычисления когомологии присоединенного модуля. Кроме того, рассчитаны когомологии простой фактор-алгебры Ли, алгебры A_2 по центру.

Ключевые слова: алгебра Ли, простой модуль, ограниченный модуль, когомология, точная последовательность.

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On zeros of an entire function coinciding with exponential typequasi-polynomials, associated with a regular third-order differential operator on an interval

In this paper, we consider the question on study of zeros of an entire function of one class, which coincides with quasi-polynomials of exponential type. Eigenvalue problems for some classes of differential operators on a segment are reduced to a similar problem. In particular, the studied problem is led by the eigenvalue problem for a linear differential equation of the third order with regular boundary value conditions in the space $W_2^3(0,1)$. The studied entire function is adequately characteristic determinant of the spectral problem for a third-order linear differential operator with periodic boundary value conditions. An algorithm to construct a conjugate indicator diagram of an entire function of one class is indicated, which coincides with exponential type quasi-polynomials with comparable exponents according to the monograph by A.F. Leontyev. Existence of a countable number of zeros of the studied entire function in each series is proved, which are simultaneously eigenvalues of the above-mentioned third-order differential operator with regular boundary value conditions. We determine distance between adjacent zeros of each series, which lies on the rays perpendicular to sides of the conjugate indicator diagram, that is a regular hexagon on the complex plane. In this case, zero is not an eigenvalue of the considered operator, that is, zero is a regular point of the operator. Fundamental difference of this work is finding the corresponding eigenfunctions of the operator. System of eigenfunctions of the operator corresponding in each series is found. Adjoint operator is constructed.

Keywords: entire function, zeros, quasi-polynomials, indicator diagram, series, operator, regular periodic boundary value conditions, eigenvalues, system of eigenfunctions.

Introduction and Formulation of the problem

We consider the question on distribution of zeros of an entire function of the following form:

$$\Delta(\lambda) = \sqrt[3]{\lambda}((k_2 - k_3)e^{k_1\sqrt[3]{\lambda}} + (k_1 - k_2)e^{(k_2+k_1)\sqrt[3]{\lambda}} +$$

$$+(k_3 - k_1)e^{k_2\sqrt[3]{\lambda}} + (k_3 - k_1)e^{(k_3+k_1)\sqrt[3]{\lambda}} + (k_1 - k_2)e^{k_3\sqrt[3]{\lambda}} + (k_2 - k_3)e^{(k_2+k_3)\sqrt[3]{\lambda}}),$$

where $k_1 = 1$, $k_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $k_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Eigenvalue problems for some classes of differential operators on a segment are reduced to a similar problem. In particular, the following problem on eigenvalues in the space $W_2^3(0,1)$ leads to the studied question:

$$L_0u \equiv l(u) = u'''(x) = -\lambda u(x),\ 0 < x < 1, \tag{1}$$

$$U_1(u) = u(0) = 0,\ U_2(u) = u(1) = 0,\ U_3(u) = u'(0) = u'(1), \tag{2}$$

where $U_1(u)$, $U_2(u)$, $U_3(u)$ are linear forms, which are regular, according to J.D. Birkhoff [1, 2]. An important result established by Birkhoff was to estimate resolvent of a regular differential operator and to establish asymptotics of the spectrum. In the monograph by M.A. Naimark [3; 67], a subclass of regular boundary conditions, so-called strongly regular boundary conditions, was singled out, where it was noted that for an odd order of the equation all regular conditions are strongly regular.

Connection between zeros of quasi-polynomials and spectral problems was reflected in [3–15]. Zeros of entire functions having an integral representation were studied in [16–23].

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Main Results

We consider the question on distribution of zeros of the entire function $\Delta_1(\lambda) = \frac{\Delta(\lambda)}{\sqrt[3]{\lambda}}$ on the complex plane λ.

$$\Delta_1(\lambda) = (k_2 - k_3)e^{k_1 \sqrt[3]{\lambda}} + (k_1 - k_2)e^{(k_2+k_1)\sqrt[3]{\lambda}} +$$

$$+(k_3 - k_1)e^{k_2 \sqrt[3]{\lambda}} + (k_3 - k_1)e^{(k_3+k_1)\sqrt[3]{\lambda}} + (k_1 - k_2)e^{k_3 \sqrt[3]{\lambda}} + (k_2 - k_3)e^{(k_2+k_3)\sqrt[3]{\lambda}} = 0. \quad (3)$$

In [11, 14] the following was proved:

Proposition 1.

1. There are infinitely many zeros of an entire function $\Delta_1(\lambda)$;

2. Distance between two adjacent zeros of the same series ($j - const$) is exactly $\frac{2\pi}{|d|}$;

3. Zeros of each series lie on the rays perpendicular to the segment, that is, perpendicular to sides of the hexagon containing

$$(\overline{k_1}, \overline{(k_3 + k_1)}); (\overline{k_3}, \overline{k_3 + k_1}); (\overline{k_2 + k_3}; \overline{k_3}); (\overline{k_2}, \overline{k_2 + k_3}); (\overline{k_2}, \overline{k_2 + k_1}); (\overline{k_2 + k_1}, \overline{k_1}).$$

The rays which are perpendicular to the indicator diagram are called critical. According to the result of the monograph [6], there are exactly six critical rays on the plane λ, that is $arg\sqrt[3]{\lambda} = \frac{\pi}{6} + \frac{\pi n}{3}$, $n = 0, 1, 2, 4, 5$;

In [11, 14] the zeros of the entire function $\Delta(\lambda)$:

$$\lambda_{jk} = \frac{(ln|z_j| + i(Arg(z_j) + 2\pi k))^3}{d^3}, \quad k = 0, \pm 1, \pm 2, ...; \; j = \overline{1, m} \quad (4)$$

were found, and conjugate indicator diagram-hexagon was constructed on the complex plane λ.

Taking $k_1 = 1, k_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, k_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, into account, due to the formula (4) and Proposition 1, we have that along the ray perpendicular to the segment passing through the points $\overline{2}; \overline{1 - i\sqrt{3}}$ there are zeros of the quasi-polynomial $(\sqrt{3} + 3i) \cdot e^{(1+i\sqrt{3})\lambda} - 2\sqrt{3} \cdot e^{2\lambda}$, they are majorizing exponents. In this case, other exponents from (3) do not contribute along this ray. Let's find zeros of the quasi-polynomial:

$$(\sqrt{3} + 3i) \cdot e^{(1+i\sqrt{3})\lambda} - 2\sqrt{3} \cdot e^{2\lambda} = 0$$

$$(\sqrt{3} + 3i) \cdot e^{(1+i\sqrt{3})\lambda} = 2\sqrt{3} \cdot e^{2\lambda}$$

$$\lambda_{k1} = \frac{2ik\pi}{-1 + i\sqrt{3}} + \frac{\ln|\frac{2\sqrt{3}}{\sqrt{3}+3i}| + iarg(\frac{2\sqrt{3}}{\sqrt{3}+3i})}{-3 + i\sqrt{3}}, \quad k = 1, 2, 3, ...$$

Which are zeroes of the first series, where $\ln|\frac{2\sqrt{3}}{\sqrt{3}+3i}| + iArg(\frac{2\sqrt{3}}{\sqrt{3}+3i}) = const$. Similar procedure is performed on the other sides of the hexagon, and along other perpendicular rays we have the corresponding series of zeros of the quasi-polynomials from (3):

- segment $[\overline{-1 - i\sqrt{3}}; \overline{1 - i\sqrt{3}}]$, 2-nd series of zeroes $\lambda_{k2} = \frac{ik\pi}{1+i\sqrt{3}} + \frac{const}{2(1+i\sqrt{3})}, \quad k = 1, 2, ..., (1 + i\sqrt{3})$
- segment $[\overline{-1 + i\sqrt{3}}; \overline{1 + i\sqrt{3}}]$, 3-rd series of zeroes $\lambda_{k3} = ik\pi + const, \quad k = 1, 2, ...,$
- segment $[\overline{-2}; \overline{-1 - i\sqrt{3}}]$, 4-th series of zeroes $\lambda_{k4} = \frac{2ik\pi}{1+i\sqrt{3}} + \frac{const}{1+i\sqrt{3}}, \quad k = 1, 2, ...,$
- segment $[\overline{2}; \overline{1 + i\sqrt{3}}]$, 5-th series of zeroes $\lambda_{k5} = -\frac{2ik\pi}{1+i\sqrt{3}} - \frac{const}{1+i\sqrt{3}}, \quad k = 1, 2, ...,$
- segment $[\overline{-1 + i\sqrt{3}}; \overline{-2}]$, 6-th series of zeroes $\lambda_{k6} = \frac{2ik\pi}{1+i\sqrt{3}} + \frac{const}{2(1+i\sqrt{3})}, \quad k = 1, 2, ...,$

The zeros that were found are adequately eigenvalues of the operator L_0 [11].

Fundamental difference of this section from [11, 14, 22, 23] is the determination of eigenfunctions of the operator L_0. The following theorem takes place.

Theorem. Let the entire function $\Delta_1(\lambda) = \frac{\Delta(\lambda)}{\sqrt[3]{\lambda}}$ in (3), according to [11, 14], be a characteristic polynomial of the spectral problem (1), (2) and all points of Proposition 1. be satisfied, as well as zeros of the characteristic polynomial (4) be the corresponding eigenvalues of the operator L_0. Then the system of eigenfunctions of the operator L_0 of each series:

$$u_{k1}(x) = C_1 e^{-2\frac{k\sqrt{3}}{2}x} \cdot e^{2\frac{\pi}{4\sqrt{3}}x}\left[\cos 2\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x - i\sin 2\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\right] + \left\{C_2\left[\cos\left(\frac{k\pi}{2}-\right.\right.\right.$$

$$\left.-\frac{\pi}{6}\right)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot \mathrm{ch}\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x - \sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot$$

$$\cdot\,\mathrm{sh}\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\cdot\sin\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x - i\left(\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi}{2}-\right.\right.$$

$$\left.-\frac{\pi}{6}\right)x\sin\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x + \cos\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\cdot\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{sh}\sqrt{3}\left(\frac{k\pi}{2}-\right.$$

$$\left.\left.\left.-\frac{\pi}{6}\right)x\right)\right] + C_3\left[\cos\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\cdot\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x-\right.$$

$$-\sin\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{sh}\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x - i\left(\sin\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\cdot\right.$$

$$\cdot\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x + \cos\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\right.$$

$$\left.\left.\left.\left.+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{sh}\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\right)\right]\right\}e^{-\frac{k\sqrt{3}}{2}x}\cdot e^{\frac{\pi}{4\sqrt{3}}x};$$

$$u_{k2}(x) = C_1 e^{k\sqrt{3}x}\cdot e^{-\frac{\pi}{4}x}\left[\cos\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x + i\sin\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\right] + \left\{C_2\left[\cos\left(\frac{k\pi}{4}-\right.\right.\right.$$

$$\left.-\frac{\pi}{24}\right)x\cdot\cos\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi}{8}x\right)\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi}{4}-\frac{\pi}{24}\right)x + \sin\left(\frac{k\pi}{4}-\frac{\pi}{24}\right)x\cdot\sin\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\right.$$

$$\left.-\frac{\pi}{8}\right)x\cdot\mathrm{sh}\sqrt{3}\left(\frac{k\pi}{4}-\frac{\pi}{24}\right)x - i\left(\cos\left(\frac{k\pi}{4}-\frac{\pi}{24}\right)x\cdot\sin\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi}{8}\right)x\cdot\mathrm{sh}\sqrt{3}\left(\frac{k\pi}{4}-\right.\right.$$

$$\left.\left.\left.-\frac{\pi}{24}\right)x - \sin\left(\frac{k\pi}{4}-\frac{\pi}{24}\right)x\cdot\cos\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi}{8}\right)x\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi}{4}-\frac{\pi}{24}\right)x\right)\right]+$$

$$+C_3\left[\cos\left(\frac{k\pi}{4}-\frac{\pi}{24}\right)x\cdot\sin\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi}{8}\right)\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi}{4}-\frac{\pi}{24}\right)x + \sin\left(\frac{k\pi}{4}-\right.\right.$$

$$\left.-\frac{\pi}{24}\right)x\cos\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi}{8}\right)x\cdot\mathrm{sh}\sqrt{3}\left(\frac{k\pi}{4}-\frac{\pi}{24}\right)x - i\left(\cos\left(\frac{k\pi}{4}-\frac{\pi}{24}\right)x\cos\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\right.\right.$$

$$\left.-\frac{\pi}{8}\right)x\cdot\mathrm{sh}\sqrt{3}\left(\frac{k\pi}{4}-\frac{\pi}{24}\right)x + \sin\left(\frac{k\pi}{4}-\frac{\pi}{24}\right)x\cdot\sin\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi}{8}\right)\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi}{4}-\right.$$

$$\left.\left.\left.\left.-\frac{\pi}{24}\right)x\right)\right]\right\}e^{\frac{k\pi\sqrt{3}}{2}x}\cdot e^{-\frac{\pi}{8}x};$$

$$u_{k3}(x) = C_1\left(\cos 2\left(k\pi-\frac{\pi}{3}\right)x + i\sin 2\left(k\pi-\frac{\pi}{3}\right)x\right) + \left[C_2\,\mathrm{ch}\sqrt{3}\left(k\pi-\frac{\pi}{3}\right)x\cdot\right.$$

$$\cdot\cos\left(k\pi-\frac{\pi}{3}\right)x + C_3\,\mathrm{sh}\sqrt{3}\left(k\pi-\frac{\pi}{3}\right)x\cdot\sin\left(k\pi-\frac{\pi}{3}\right)x + i\left(C_3\,\mathrm{sh}\sqrt{3}\left(k\pi-\frac{\pi}{3}\right)x\cdot\right.$$

$$\cdot\,\cos\left(k\pi-\frac{\pi}{3}\right)x-C_2\operatorname{ch}\sqrt{3}\left(k\pi-\frac{\pi}{3}\right)x\cdot\sin\left(k\pi-\frac{\pi}{3}\right)x\Big)\Big];$$

$$u_{k4}(x)=C_1e^{2\frac{k\pi\sqrt{3}}{2}x}\cdot e^{-2\frac{\pi}{12}x}\left[\cos 2\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x+i\sin 2\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\right]+\Big\{C_2\Big[\cos\Big(\frac{k\pi}{2}+$$

$$+\frac{\pi\sqrt{3}}{12}\Big)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\operatorname{ch}\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x-\sin\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\cdot$$

$$\cdot\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\operatorname{sh}\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x-i\Big(\sin\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\cdot\cos\sqrt{3}\Big(-\frac{k\sqrt{3}}{2}+$$

$$+\frac{\pi}{4\sqrt{3}}\Big)x\cdot\operatorname{ch}\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x+\cos\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\cdot\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot$$

$$\cdot\,\operatorname{sh}\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\Big)\Big]+C_3\Big[\cos\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\cdot\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\operatorname{ch}\sqrt{3}\Big(\frac{k\pi}{2}-$$

$$-\frac{\pi}{69}\Big)x-\sin\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\operatorname{sh}\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x-i\Big(\sin\Big(\frac{k\pi}{2}-$$

$$-\frac{\pi}{6}\Big)x\cdot\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\operatorname{ch}\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x+\cos\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\cdot\cos\sqrt{3}\Big(-\frac{k\sqrt{3}}{2}+$$

$$+\;\frac{\pi}{4\sqrt{3}}\Big)x\cdot\operatorname{sh}\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\Big)\Big]\Big\}e^{-\frac{k\pi\sqrt{3}}{2}x}\cdot e^{\frac{\pi}{12}x};$$

$$u_{k5}(x)=C_1e^{-2\frac{k\pi\sqrt{3}}{2}x}\cdot e^{2\frac{\pi}{12}x}\left[\cos 2\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x+i\sin 2\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\right]+$$

$$+\Big\{C_2\Big[\cos\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\operatorname{ch}\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x+$$

$$+\sin\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\operatorname{sh}\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x+i\Big(\sin\Big(\frac{k\pi}{2}+$$

$$+\frac{\pi\sqrt{3}}{12}\Big)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\operatorname{ch}\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x-$$

$$-\;\cos\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\Big)\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\operatorname{sh}\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\Big]-C_3\Big[\cos\Big(\frac{k\pi}{2}+$$

$$+\frac{\pi\sqrt{3}}{12}\Big)x\cdot\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\operatorname{ch}\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x+\sin\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\cdot$$

$$\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\operatorname{sh}\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x+i\Big(\sin\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\sin\sqrt{3}\Big(-\frac{k\sqrt{3}}{2}+$$

$$+\frac{\pi}{4\sqrt{3}}\Big)x\cdot\operatorname{ch}\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x-\cos\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot$$

$$\cdot\ \mathrm{sh}\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\Big)\Big]\Big]\Big\}e^{-\frac{k\pi\sqrt{3}}{2}x}\cdot e^{-\frac{\pi}{12}x};$$

$$u_{k6}(x)=C_1e^{-2\frac{k\pi\sqrt{3}}{2}x}\cdot e^{2\frac{\pi}{12}x}\left[\cos 2\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x+i\sin 2\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\right]+$$

$$+\Big\{C_2\Big[\cos\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\cdot\cos\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x+$$

$$+\sin\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\sin\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot\mathrm{sh}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x+i\Big(\sin\Big(\frac{k\pi\sqrt{3}}{2}-$$

$$-\frac{\pi\sqrt{3}}{12}\Big)x\cdot\cos\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x-$$

$$-\cos\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\sin\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot\mathrm{sh}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\Big)\Big]+C_3\Big[\cos\Big(\frac{k\pi\sqrt{3}}{2}-$$

$$-\frac{\pi\sqrt{3}}{12}\Big)x\cdot\sin\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x+\sin\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\cdot$$

$$\cdot\cos\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot\mathrm{sh}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x+i\Big(\sin\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\sin\sqrt{3}\Big(\frac{k\pi}{2}-$$

$$-\frac{\pi}{12}\Big)x\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x-\cos\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\cdot\cos\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot$$

$$\cdot\ \mathrm{sh}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\Big)\Big]\Big\}e^{-\frac{k\pi\sqrt{3}}{2}x},$$

for each series, where $k = 1, 2, 3, \ldots$.

Here is the scheme of the proof:

General solution of the equation (1) has the form:

$$u(x) = C_1e^{2\lambda x} + (C_2cos\sqrt{3}\lambda x + C_3sin\sqrt{3}\lambda x)e^{-\lambda x}. \tag{5}$$

Substituting the zeros of each series into (5) in order and satisfying the equation (1), as well as the boundary value conditions (2), we obtain the corresponding eigenfunctions of the operator L_0.

Remark. Questions of completeness, uniform minimality, and basis property of systems of eigenfunctions of the operator L_0 remain open. Note that questions of the basis property of systems of root vectors of the multiple differentiation operator with regular, but not strongly regular boundary value conditions were studied in [24–30].

Conjugate problem

By using integration by parts, we obtain the Lagrange formula: $\int\limits_0^1 l(u)\overline{v(x)}dx + \int\limits_0^1 u(x)\overline{l^*(v)}dx = u''(1)\overline{v(1)}-$ $-u''(0)\overline{v(0)} - [\overline{v'(0)} - \overline{v'(1)}]\cdot u'(0) + u(1)\overline{v''(1)} - u(0)\overline{v''(0)}$.

Here $l^*(v)$ is an adjoint differential expression:

$$l^*(v) = -v'''(x), \quad 0 < x < 1. \tag{6}$$

Consequently, an operator L_0^*, adjoint to the operator L_0 is given by the differential expression (6) and boundary value conditions:

$$V_1(v) \equiv v(1) = 0, V_2(v) \equiv v(0) = 0, V_3(v) = v'(0) - v'(1) = 0.$$

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Кесіндідегі үшінші ретті регулярлы дифференциалдық оператормен байланысқан, экспоненциалды типтегі квазикөпмүшеліктермен сәйкес келетін бүтін функцияның нөлдері жайлы

Мақалада көрсеткіштері өлшемді экспоненциалды типтегі квазикөпмүшеліктермен сәйкес келетін бір кластағы бүтін функциялардың нөлдерін зерттеу мәселесі қарастырылды. Мұндағы қарастырылатын мәселе, көп жағдайларда, кейбір кластардағы кесіндідегі дифференциалдық операторлардың меншікті мәндерін зерттеуге берілген есептерден туындайды. Дәлірек айтқанда, қарастырылатын мәселеге $W_2^3(0,1)$ кеңістігіндегі регулярлы шеттік шарттармен берілген үшінші ретті сызықтық дифференциалдық теңдеудің меншікті мәндерін зерттеуге арналған есепке алып келеді. Зерттелетін бүтін функция, тікелей периодтық шеттік шарттармен берілген сызықтық дифференциалдық үшінші ретті оператор үшін аталған спетралдық есептің характеристикалық анықтауышы болып табылады. А.Ф. Леонтьевтің монографиясындағы нәтижесінің негізінде, қарастырылып отырған бір кластағы өлшемді көрсеткіштері бар экспоненциалды типтегі квазикөпмүшеліктермен сәйкес келетін бүтін функцияның түйіндес индикаторлық диаграммасын құрудың алгоритмі көрсетілген. Бүтін функцияның әрбір сериядағы саналымды нөлдерінің бар болуы дәлелденген және олардың кесіндідегі регулярлы периодтық шеттік шарттармен берілген сызықтық үшінші ретті дифференциалдық оператордың меншікті мәндері екендігі сипатталған. Бүтін функцияның әр сериядағы көршілес жатқан нөлдерінің арақашықтығы анықталған және әр серия комплексті жазықтықтағы түйіндес индикаторлық диаграмманың, яғни дұрыс алтыбұрыштың қабырғаларына перпендикуляр, координаталар бас нүктесінен шығатын сәулелер болатындығы көрсетілген. Алайда, нөл нүктесі жоғарыда айтылған қарастырылатын оператордың меншікті мәні болмайтындығы, яғни нөл оператордың регулярлы нүктесі екендігі сипатталған. Бұл жұмыстағы алынған нәтиженің ерекшелігі, оператордың әр сериядағы меншікті мәндеріне сәйкес меншікті функциялар жүйесінің табылуында. Сондай-ақ, осы жұмыстың зерттеу нысанына айналып отырған оператордың түйіндес операторы құрылған.

Кілт сөздер: бүтін функцияның нөлдері, квазикөпмүшеліктер, индикаторлық диаграмма, серия, оператор, регулярлы периодтық шеттік шарттар, меншікті мәндер, меншікті функциялардың жүйесі.

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О нулях целой функции, совпадающей с квазиполиномами экспоненциального типа, связанной с регулярным дифференциальным оператором третьего порядка на отрезке

В статье рассмотрен вопрос распределения нулей целой функции одного класса, которые являются квазиполиномами экспоненциального типа. К подобной проблеме редуцированы задачи на собственные значения для некоторых классов дифференциальных операторов на отрезке. В частности, к изучаемому вопросу приводит задача на собственные значения линейного дифференциального уравнения третьего порядка с регулярными краевыми условиями в пространстве $W_2^3(0,1)$. Исследуемая целая функция адекватно является характеристическим определителем спектральной задачи для линейного дифференциального оператора третьего порядка с периодическими краевыми условиями. Построена сопряженная индикаторная диаграмма целой функции экспоненциального типа соизмеримыми показателями. Доказано существование счетного числа нулей исследуемой целой функции в каждой серии, которые являются одновременно собственными значениями рассматриваемого дифференциального оператора третьего порядка с периодическими краевыми условиями. Определено расстояние между соседними нулями каждой серии, лежащее на лучах, перпендикулярных сторонам сопряженной индикаторной диаграммы, то есть правильного шестиугольника на комплексной плоскости. При этом нуль не является собственным значением рассматриваемого оператора. Принципиальным отличием настоящей работы является нахождение соответствующих собственных функций рассматриваемого оператора. Построен сопряженный оператор.

Ключевые слова: целая функция, нули, квазиполиномы, индикаторная диаграмма, серия, оператор, регулярные периодические краевые условия, собственные значения, система собственных функций.

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Upper Estimates of the angle best approximations of generalized Liouville-Weyl derivatives

In this article we consider continuous functions f with period 2π and their approximation by trigonometric polynomials. This article is devoted to the study of estimates of the best angular approximations of generalized Liouville-Weyl derivatives by angular approximation of functions in the three-dimensional case. We consider generalized Liouville-Weyl derivatives instead of the classical mixed Weyl derivative. In choosing the issues to be considered, we followed the general approach that emerged after the work of the second author of this article. Our main goal is to prove analogs of the results of in the three-dimensional case. The concept of general monotonic sequences plays a key role in our study. Several well-known inequalities are indicated for the norms, best approximations of the r-th derivative with respect to the best approximations of the function f. The issues considered in this paper are related to the range of issues studied in the works of Bernstein. Later Stechkin and Konyushkov obtained an inequality for the best approximation $f^{(r)}$. Also, in the works of Potapov, using the angle approximation, some classes of functions are considered. In subsection 1 we give the necessary notation and useful lemmas. Estimates for the norms and best approximations of the generalized Liouville-Weyl derivative in the three-dimensional case are obtained.

Keywords: Lebesgue space, best approximation by three-dimensional angle, trigonometric polynomial, Liouville-Weyl derivative.

Introduction

Let us mention several well-known inequalities for norms and best approximations of the r-th derivative in terms of best approximations of the function f.

The following result was proved by Bernstein for $p = \infty$ (for $1 \le p < \infty$, see [2]) if $f \in L_p$, $1 \le p \le \infty$, and $\sum_{k=0}^{\infty}(k+1)^{r-1}E_k(f)_p < \infty, r \in \mathbb{N}$, then $\|f^{(r)}\|_p \le C(r)\sum_{k=0}^{\infty}(k+1)^{r-1}E_k(f)_p$ [1].

Later on, Stechkin [3] for $p = \infty$ and Konyushkov [4] for $1 < p < \infty$ obtained the following inequality for the best approximations of $f^{(r)}$:

$$E_n(f^{(r)})_p \le C(r,p)\left(n^r E_n(f)_p + \sum_{k=n+1}^{\infty} k^{r-1}E_k(f)_p\right) \quad r, n \in \mathbb{N}.$$

The last inequality was extended by the formula Timan [5] for the case of $1 < p < \infty$ as follows:

$$E_n(f^{(r)})_p \le C(r)\left(n^r E_n(f)_p + (\sum_{k=n+1}^{\infty} k^{\theta r-1}E_k^{\theta}(f)_p)^{\frac{1}{\theta}}\right), \quad \theta = min(2,p) \quad r, n \in \mathbb{N}.$$

Also, A. Jumabayeva and B. Simonov obtained estimates of norms and the angle best approximations of the generalized Liouville-Weyl derivatives by the angle approximation of functions in the two-dimensional case [6, 7].

Let $L_p(T^3)$, $1 < p < \infty$ be the space of measurable functions of three variables that are 2π periodic in each variable and such that

$$\|f\|_p = \left(\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \mid f(x_1,x_2,x_3) \mid^p dx_1dx_2dx_3\right)^{1/p} < \infty.$$

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L_p^0 is the set of functions $f \in L_p$ such that $\int\limits_0^{2\pi} f(x_1,x_2,x_3)dx_1 = 0$ for almost everyone x_2,x_3, $\int\limits_0^{2\pi} f(x_1,x_2,x_3)dx_2 = 0$ for almost everyone x_1,x_3 and $\int\limits_0^{2\pi} f(x_1,x_2,x_3)dx_3 = 0$ for almost everyone x_1,x_2.

Let $Y_{m_1,m_2,m_3}(f)_p$ be the best approximation by a three-dimensional angle of the function $f \in L_p(\mathbb{T}^3)$, i.e.

$$Y_{m_1,m_2,m_3}(f)_p = \inf_{T_{m_1,\infty,\infty},T_{\infty,m_2,\infty},T_{\infty,\infty,m_3}} \|f - T_{m_1,\infty,\infty} - T_{\infty,m_2,\infty} - T_{\infty,\infty,m_3}\|_p,$$

where the function $T_{m_1,\infty,\infty}(x_1,x_2,x_3) \in L_p(\mathbb{T}^3)$ is a trigonometric polynomial of order at most m_1 in x_1, the function $T_{\infty,m_2,\infty}(x_1,x_2,x_3) \in L_p(\mathbb{T}^3)$ is a trigonometric polynomial of order at most m_2 in x_2 and the function $T_{\infty,\infty,m_3}(x_1,x_2,x_3) \in L_p(\mathbb{T}^3)$ is a trigonometric polynomial of order at most m_3 in x_3. In the work of Potapov using the angle approximation, some classes of functions are considered [8, 9].

By $\sigma(f)$ we denote the Fourier series of a function $f \in L_p(T^3)$, that is

$$\sigma(f) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} c_{k_1,k_2,k_3} e^{i(k_1x_1+k_2x_2+k_3x_3)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} A_{n_1,n_2,n_3}(x_1,x_2,x_3), \tag{1}$$

where $c_{k_1,k_2,k_3} = \frac{1}{8\pi^3} \int\limits_0^{2\pi}\int\limits_0^{2\pi}\int\limits_0^{2\pi} f(x_1,x_2,x_3)e^{-i(k_1x_1+k_2x_2+k_3x_3)}dx_1dx_2dx_3$.

The transformed Fourier series of $\sigma(f)$ is given by

$$\sigma(f,\lambda,\beta_1,\beta_2,\beta_3) =$$

$$= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} \lambda_{n_1,n_2,n_3}[c_{n_1n_2,n_3} e^{i(n_1x_1+\beta_1\frac{\pi}{2})} e^{i(n_2x_2+\beta_2\frac{\pi}{2})} e^{i(n_3x_3+\beta_3\frac{\pi}{2})} \mid n_1 \mid^{\beta_1} \mid n_2 \mid^{\beta_2} \mid n_3 \mid^{\beta_3}],$$

where $\beta_1,\beta_2,\beta_3 \in R$ and $\lambda = \{\lambda_{n_1,n_2,n_3}\}_{n_1n_2n_3 \in N}$ is a sequence of real numbers.

Let $\varphi(x_1x_2x_3) \sim \sigma(f,\lambda,\beta_1,\beta_2,\beta_3)$ is the $(\lambda,\beta_1,\beta_2,\beta_3)$ is the mixed derivative of the function f (or Liouville–Weyl derivative) and denote it by $f^{(\lambda,\beta_1,\beta_2,\beta_3)}(x_1x_2x_3)$. For example, if $\lambda_{n_1,n_2,n_3} = n_1^{r_1}n_2^{r_2}n_3^{r_3}$, $r_i > 0$, $\beta_i = r_i$ $(i = 1,2,...) \Rightarrow f^{(\lambda,\beta_1,\beta_2,\beta_3)} = f^{(r_1,r_2,r_3)}$, where $f^{(r_1r_2r_3)}$-mixed derivative of the function f in the sense of Weyl.

Definition 1.1. [10, 11] A sequence $\lambda := \{\lambda_n\}_{n=1}^{\infty}$ is said to be general monotone, written $\lambda \in GM^3$, if the relations

$$\sum_{k_1=n_1}^{2n_1} \mid \lambda_{k_1,n_2,n_3} - \lambda_{k_1+1,n_2,n_3} \mid \leq C \mid \lambda_{n_1,n_2n_3,} \mid, \quad \sum_{k_2=n_2}^{2n_2} \mid \lambda_{n_1,k_2,n_3} - \lambda_{n_1k_2+1,n_3} \mid \leq C \mid \lambda_{n_1,n_2,n_3} \mid,$$

$$\sum_{k_3=n_3}^{2n_3} \mid \lambda_{n_1,n_2,k_3} - \lambda_{n_1,n_2,k_3+1} \mid \leq C \mid \lambda_{n_1,n_2,n_3,} \mid,$$

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} \mid \lambda_{k_1,k_2,n_3} - \lambda_{k_1+1,k_2,n_3} - \lambda_{k_1,k_2+1,n_3} + \lambda_{k_1+1,k_2+1,n_3} \mid \leq C \mid \lambda_{n_1,n_2,n_3} \mid,$$

$$\sum_{k_2=n_2}^{2n_2} \sum_{k_3=n_3}^{2n_3} \mid \lambda_{n_1,k_2,k_3} - \lambda_{n_1,k_2+1,k_3} - \lambda_{n_1,k_2,k_3+1} + \lambda_{n_1,k_2+1,k_3} \mid \leq C \mid \lambda_{n_1,n_2,n_3} \mid,$$

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_3=n_3}^{2n_3} \mid \lambda_{k_1,n_2,k_3} - \lambda_{k_1+1,n_2,k_3} - \lambda_{k_1,n_2,k_3+1} + \lambda_{k_1+1,n_2,k_3+1} \mid \leq C \mid \lambda_{n_1,n_2,n_3} \mid,$$

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} \sum_{k_3=n_3}^{2n_3} \mid \lambda_{k_1,k_2,k_3} - \lambda_{k_1+1,k_2,k_3} - \lambda_{k_1,k_2+1,k_3} - \lambda_{k_1,k_2,k_3+1} +$$

$$+\lambda_{k_1,k_2+1,k_3+1} + \lambda_{k_1+1,k_2,k_3+1} + \lambda_{k_1+1,k_2+1,k_3} - \lambda_{k_1+1,k_2+1,k_3+1} \mid \leq C \mid \lambda_{n_1,n_2,n_3} \mid$$

hold for all integers n_1,n_2 and n_3, where the constant C is independent of n_1, n_2 and n_3.

Auxiliary results

In order to prove the main result, we formulate auxiliary statements. We denote

$$\Delta_{m_1,m_2,m_3} := \sum_{n_1=2^{m_1-1}}^{2^{m_1}-1} \sum_{n_2=2^{m_2-1}}^{2^{m_2}-1} \sum_{n_3=2^{m_3-1}}^{2^{m_3}-1} A_{n_1,n_2,n_3}(x_1,x_2,x_3), \quad m_1,m_2,m_3 = 1,2,...$$

Lemma 2.1. [11] $\{\lambda_n\} \in GM$ if and only if there exists $C > 0$, such that

$$(i) \mid \lambda_k \mid \le C \mid \lambda_n \mid \text{ for } n \le k \le 2n; \ (ii) \sum_{k=n}^{N} \mid \Delta\lambda_k \mid \le C(\mid \lambda_n \mid + \sum_{k=n+1}^{N} \frac{\mid \lambda_k \mid}{k}) \text{ for any } n < N.$$

By [11], it follows that if $\{\lambda_{n_1 n_2 n_3}\} \in GM^3$, then

$$\mid \lambda_{k_1,k_2,k_3} \mid \le C \mid \lambda_{n_1,n_2,n_3} \mid \quad \text{for } n_1 \le k_1 \le 2n_1, n_2 \le k_2 \le 2n_2, n_3 \le k_3 \le 2n_3.$$

This implies that the condition

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} \sum_{k_3=n_3}^{2n_3} \mid \lambda_{k_1,k_2,k_3} - \lambda_{k_1+1,k_2,k_3} - \lambda_{k_1,k_2+1,k_3} - \lambda_{k_1,k_2,k_3+1} + \lambda_{k_1,k_2+1,k_3+1} +$$

$$+\lambda_{k_1+1,k_2,k_3+1} + \lambda_{k_1+1,k_2+1,k_3} - \lambda_{k_1+1,k_2+1,k_3+1} \mid \le C(\mid \lambda_{n_1,n_2,n_3} \mid + \mid \lambda_{2n_1,2n_2,2n_3} \mid)$$

is equivalent to the condition

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} \sum_{k_3=n_3}^{2n_3} \mid \lambda_{k_1,k_2,k_3} - \lambda_{k_1+1,k_2,k_3} - \lambda_{k_1,k_2+1,k_3} - \lambda_{k_1,k_2,k_3+1} +$$

$$+\lambda_{k_1,k_2+1,k_3+1} + \lambda_{k_1+1,k_2,k_3+1} + \lambda_{k_1+1,k_2+1,k_3} - \lambda_{k_1+1,k_2+1,k_3+1} \mid \le C \mid \lambda_{n_1,n_2,n_3} \mid .$$

Lemma 2.2. (Minkowskii inequality [12]) Let $1 \le p < \infty$ and $a_{\nu k} \ge 0$, then

$$(a) \Big(\sum_{k=1}^{\infty}(\sum_{\nu=1}^{k} a_{\nu k})^p\Big)^{\frac{1}{p}} \le \sum_{\nu=1}^{\infty}\Big(\sum_{k=\nu}^{\infty} a_{\nu k}^p\Big)^{\frac{1}{p}}, (b) \Big(\sum_{k=1}^{\infty}(\sum_{\nu=k}^{\infty} a_{\nu k})^p\Big)^{\frac{1}{p}} \le \sum_{\nu=1}^{\infty}\Big(\sum_{k=1}^{\nu} a_{\nu k}^p\Big)^{\frac{1}{p}}.$$

Lemma 2.3. [12] For a function $f(u,y)$ defined on measurable set $E = E_1 \times E_2 \subset \mathbb{R}_n$, where $x = (u,y)$, $u = (x_1,...,x_m), y = (x_{m+1},...,x_n)$, the following inequality holds

$$\Big(\int_{E_1} \mid \int_{E_2} f(u,y)dy \mid^p du\Big)^{\frac{1}{p}} \le \int_{E_2}\Big(\int_{E_1} \mid f(u,y) \mid^p du\Big)^{\frac{1}{p}} dy.$$

Lemma 2.4. [8] Let $f \in L_p(\mathbb{T}^2), 1 < p < \infty, m_i \in \mathbb{N} \cup 0 \ (i = 1,2)$. Then

$$\|f - S_{m_1,\infty}(f) - S_{\infty,m_2}(f) + S_{m_1,m_2}(f)\|_p \asymp Y_{m_1,m_2}(f)_p,$$

where S_{m_1,m_2} are the partial sums of the Fourier series of a function f.

Lemma 2.5. [8] a) Let $1 < p < \infty$ and (1) be the Fourier series of $f \in L_{p^0}(\mathbb{T}^3)$, then

$$C_1(p)\|f\|_p \le \Big(\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}(\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}\Delta^2_{m_1m_2m_3})^{\frac{p}{2}} dx_1dx_2dx_3\Big)^{\frac{1}{p}} \le C_2(p)\|f\|_p.$$

b) Let $1 < p < \infty$. If (1) satisfies the following inequality

$$I_p = \Big(\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}(\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}\Delta^2_{m_1m_2m_3})^{\frac{p}{2}} dx_1dx_2dx_3\Big)^{\frac{1}{p}} < \infty.$$

Then (1) is the Fourier series of a function $f = (x_1,x_2,x_3) \in L_p(\mathbb{T}^3)$ and $\|f\|_p \le C(p)I_p$.

Main result

The aim of this paper is to prove the following theorem.

Theorem 3.1. Let $1 < p < \infty$, $0 < \theta \leq \min(p, 2)$, $\lambda := \{\lambda_{n_1,n_2,n_3}\}_{n_1,n_2,n_3}$ be sequences of positive numbers such that $\lambda \in GM^3$, $\alpha_i \in \mathbb{R}_+$, $r_i \in \mathbb{R}_+ \cup \{0\}$ and $\beta_i \in \mathbb{R}(i = 1, 2)$. If for $f \in L_p^0(\mathbb{T}^3)$ the series

$$\sum_{n_1=1}^{\infty} \mid \lambda^{\theta}_{n_1+1,1,1} - \lambda^{\theta}_{n_1,1,1} \mid Y^{\theta}_{n_1,0,0}(f)_p +$$

$$+ \sum_{n_2=1}^{\infty} \mid \lambda^{\theta}_{1,n_2+1,1} - \lambda^{\theta}_{1,n_2,1} \mid Y^{\theta}_{0,n_2,0}(f)_p + \sum_{n_3=1}^{\infty} \mid \lambda^{\theta}_{1,1,n_3+1} - \lambda^{\theta}_{1,1,n_3} \mid Y^{\theta}_{0,0,n_3}(f)_p +$$

$$+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \mid \lambda^{\theta}_{n_1+1,n_2+1,1} - \lambda^{\theta}_{n_1+1,n_2,1} - \lambda^{\theta}_{n_1,n_2+1,1} + \lambda^{\theta}_{n_1,n_2,1} \mid Y^{\theta}_{n_1,n_2,0}(f)_p +$$

$$+ \sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty} \mid \lambda^{\theta}_{n_1,1,n_3} - \lambda^{\theta}_{n_1+1,1,n_3} - \lambda^{\theta}_{n_1,1,n_3+1} + \lambda^{\theta}_{n_1+1,1,n_3+1} \mid Y^{\theta}_{n_1,0,n_3}(f)_p + \tag{2}$$

$$+ \sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} \mid \lambda^{\theta}_{1,n_2,n_3} - \lambda^{\theta}_{1,n_2+1,n_3} - \lambda^{\theta}_{1,n_2,n_3+1} + \lambda^{\theta}_{1,n_2+1,n_3+1} \mid Y^{\theta}_{0,n_2,n_3}(f)_p +$$

$$+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} \mid \lambda^{\theta}_{n_1,n_2,n_3} - \lambda^{\theta}_{n_1+1,n_2,n_3} - \lambda^{\theta}_{n_1,n_2+1,n_3} - \lambda^{\theta}_{n_1,n_2,n_3+1} +$$

$$+\lambda^{\theta}_{n_1,n_2+1,n_3+1} + \lambda^{\theta}_{n_1+1,n_2,n_3+1} + \lambda^{\theta}_{n_1+1,n_2+1,n_3} - \lambda^{\theta}_{n_1+1,n_2+1,n_3+1} \mid Y^{\theta}_{n_1,n_2,n_3}(f)_p$$

converges, then there exists a function $\varphi \in L_p^0(\mathbb{T}^3)$, with the Fourier series $\sigma(f, \lambda, \beta_1, \beta_2, \beta_3)$ and

$$\|\varphi\|_p \leq \Bigg(\lambda^{\theta}_{1,1,1}\|f\|^{\theta}_p + \sum_{n_1=1}^{\infty} \mid \lambda^{\theta}_{n_1+1,1,1} - \lambda^{\theta}_{n_1,1,1} \mid Y^{\theta}_{n_1,0,0}(f)_p +$$

$$+ \sum_{n_2=1}^{\infty} \mid \lambda^{\theta}_{1,n_2+1,1} - \lambda^{\theta}_{1,n_2,1} \mid Y^{\theta}_{0,n_2,0}(f)_p + \sum_{n_3=1}^{\infty} \mid \lambda^{\theta}_{1,1,n_3+1} - \lambda^{\theta}_{1,1,n_3} \mid Y^{\theta}_{0,0,n_3}(f)_p +$$

$$+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \mid \lambda^{\theta}_{n_1,n_2,1} - \lambda^{\theta}_{n_1+1,n_2,1} - \lambda^{\theta}_{n_1,n_2+1,1} + \lambda^{\theta}_{n_1+1,n_2+1,1} \mid Y^{\theta}_{n_1,n_2,0}(f)_p + \tag{3}$$

$$+ \sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty} \mid \lambda^{\theta}_{n_1,1,n_3} - \lambda^{\theta}_{n_1+1,1,n_3} - \lambda^{\theta}_{n_1,1,n_3+1} + \lambda^{\theta}_{n_1+1,1,n_3+1} \mid Y^{\theta}_{n_1,0,n_3}(f)_p +$$

$$+ \sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} \mid \lambda^{\theta}_{1,n_2,n_3} - \lambda^{\theta}_{1,n_2+1,n_3} - \lambda^{\theta}_{1,n_2,n_3+1} + \lambda^{\theta}_{1,n_2+1,n_3+1} \mid Y^{\theta}_{0,n_2,n_3}(f)_p +$$

$$+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} \mid \lambda^{\theta}_{n_1,n_2,n_3} - \lambda^{\theta}_{n_1+1,n_2,n_3} - \lambda^{\theta}_{n_1,n_2+1,n_3} - \lambda^{\theta}_{n_1,n_2,n_3+1} +$$

$$+\lambda^{\theta}_{n_1,n_2+1,n_3+1} + \lambda^{\theta}_{n_1+1,n_2,n_3+1} + \lambda^{\theta}_{n_1+1,n_2+1,n_3} - \lambda^{\theta}_{n_1+1,n_2+1,n_3+1} \mid Y^{\theta}_{n_1,n_2,n_3}(f)_p \Bigg)^{\frac{1}{\theta}},$$

$$Y_{2^{m_1}-1,2^{m_2}-1\ ,2^{m_3}-1}(\varphi)_p \lesssim \Bigg(\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{m_3-1}} Y_{2^{m_1}-1,2^{m_2}-1\ ,2^{m_3}-1}(f)_p +$$

$$+ \sum_{\nu_1=m_1}^{\infty} \mid \lambda^{\theta}_{2^{\nu_1},2^{m_2-1},2^{m_3-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{m_3-1}} \mid \ Y^{\theta}_{2^{\nu_1-1}-1,2^{m_2}-1,2^{m_3}-1}(f)_p +$$

$$+\sum_{\nu_2=m_2}^{\infty} \mid \lambda^{\theta}_{2^{m_1-1},2^{\nu_2},2^{m_3-1}} - \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{m_3-1}} \mid \; Y^{\theta}_{2^{m_1}-1,2^{\nu_2-1}-1,2^{m_3}-1}(f)_p + \tag{4}$$

$$+\sum_{\nu_3=m_3}^{\infty} \mid \lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{\nu_3}} - \lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{\nu_2-1}} \mid \; Y^{\theta}_{2^{m_1}-1,2^{m_2}-1,2^{\nu_3-1}-1}(f)_p +$$

$$+\sum_{\nu_1=m_1}^{\infty}\sum_{\nu_2=m_2}^{\infty} \mid \lambda^{\theta}_{2^{\nu_1},2^{\nu_2},2^{m_3-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2},2^{m_3-1}} -$$

$$-\lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1},2^{m_3-1}} + \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{m_3-1}} \mid Y^{\theta}_{2^{\nu_1-1}-1,2^{\nu_2-1}-1,2^{m_3}-1}(f)_p +$$

$$+\sum_{\nu_2=m_2}^{\infty}\sum_{\nu_3=m_3}^{\infty} \mid \lambda^{\theta}_{2^{m_1-1},2^{\nu_2},2^{\nu_3}} - \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{\nu_3}} -$$

$$-\lambda^{\theta}_{2^{m_1-1},2^{\nu_2},2^{\nu_3-1}} + \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{\nu_3-1}} \mid Y^{\theta}_{2^{m_1}-1,2^{\nu_2-1}-1,2^{\nu_3-1}-1}(f)_p +$$

$$+\sum_{\nu_1=m_1}^{\infty}\sum_{\nu_3=m_3}^{\infty} \mid \lambda^{\theta}_{2^{\nu_1},2^{m_2-1},2^{\nu_3}} - \lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{\nu_3}} -$$

$$-\lambda^{\theta}_{2^{\nu_1},2^{m_2-1},2^{\nu_3-1}} + \lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{\nu_3-1}} \mid Y^{\theta}_{2^{\nu_1-1}-1,2^{m_2}-1,2^{\nu_3-1}-1}(f)_p +$$

$$+\sum_{\nu_1=m_1}^{\infty}\sum_{\nu_2=m_2}^{k_2}\sum_{\nu_3=m_3}^{\infty} \mid \lambda^{\theta}_{2^{\nu_1},2^{\nu_2},2^{\nu_3}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2},2^{\nu_3}} - \lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1},2^{\nu_3}} - \lambda^{\theta}_{2^{\nu_1},2^{\nu_2},2^{\nu_3-1}} +$$

$$\left. +\lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1},2^{\nu_3-1}} + \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2},2^{\nu_3-1}} + \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-1}} \mid Y^{\theta}_{2^{\nu_1-1}-1,2^{\nu_2-1}-1,2^{\nu_3-1}-1}(f)_p \right)^{\frac{1}{\theta}}.$$

Proof. Let the series (2) be convergent and $f \in L^0_p(T^3)$. We use the following inequality

$$\lambda^{\theta}_{2^{n_1-1},2^{n_2-1},2^{n_3-1}} \le \lambda^{\theta}_{111} + \sum_{m_3=2}^{n_3} \mid \lambda^{\theta}_{1,1,2^{m_3-1}} - \lambda^{\theta}_{1,1,2^{m_3-2}} \mid +$$

$$+\sum_{m_2=2}^{n_2} \mid \lambda^{\theta}_{1,2^{m_2-1},1} - \lambda^{\theta}_{1,2^{m_2-2},1} \mid + \sum_{m_1=2}^{n_1} \mid \lambda^{\theta}_{2^{m_1-1},1,1} - \lambda^{\theta}_{2^{m_1-2},1,1} \mid +$$

$$+\sum_{m_1=2}^{n_1}\sum_{m_2=2}^{n_2} \mid \lambda^{\theta}_{2^{m_1-1},2^{m_2-1},1} - \lambda^{\theta}_{2^{m_1-2},2^{m_2-1},1} - \lambda^{\theta}_{2^{m_1-1},2^{m_2-2},1} + \lambda^{\theta}_{2^{m_1-2},2^{m_2-2},1} \mid + \tag{5}$$

$$+\sum_{m_2=2}^{n_2}\sum_{m_3=2}^{n_3} \mid \lambda^{\theta}_{1,2^{m_2-1},2^{m_3-1}} - \lambda^{\theta}_{1,2^{m_2-2},2^{m_3-1}} - \lambda^{\theta}_{1,2^{m_2-1},2^{m_3-2}} + \lambda^{\theta}_{1,2^{m_2-2},2^{m_3-2}} \mid +$$

$$+\sum_{m_1=2}^{n_1}\sum_{m_3=2}^{n_3} \mid \lambda^{\theta}_{2^{m_1-1},1,2^{m_3-1}} - \lambda^{\theta}_{2^{m_1-2},1,2^{m_3-1}} - \lambda^{\theta}_{2^{m_1-1},1,2^{m_3-2}} + \lambda^{\theta}_{2^{m_1-2},1,2^{m_3-2}} \mid +$$

$$+\sum_{m_1=2}^{n_1}\sum_{m_2=2}^{n_2}\sum_{m_3=2}^{n_3} \mid \lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{m_3-1}} - \lambda^{\theta}_{2^{m_1-1},2^{m_2-2},2^{m_3-2}} - \lambda^{\theta}_{2^{m_1-2},2^{m_2-1},2^{m_3-2}} -$$

$$-\lambda^{\theta}_{2^{m_1-2},2^{m_2-2},2^{m_3-1}} - \lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{m_3-2}} + \lambda^{\theta}_{2^{m_1-2},2^{m_2-1},2^{m_3-1}} - \lambda^{\theta}_{2^{m_1-1},2^{m_2-2},2^{m_3-1}} +$$

$$+\lambda^{\theta}_{2^{m_1-2},2^{m_2-2},2^{m_3-2}} \mid .$$

Let us denote $\Delta_{n_1,n_2,n_3} = \sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1}\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1}\sum_{\nu_3=2^{n_3-1}}^{2^{n_3}-1} A_{\nu_1,\nu_2,\nu_3}(f,x_1,x_2,x_3)(n_1,n_2,n_3=1,2...)$. Using (5) and property of GM (Lemma 2.1), we get

$$I_1 = \left\{ \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \left[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} \lambda^2_{2^{n_1-1},2^{n_2-1},2^{n_3-1}} \Delta^2_{n_1,n_2,n_3} \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}} =$$

$$=\Big\|\Big[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\lambda^2_{2^{n_1-1},2^{n_2-1},2^{n_3-1}}\Delta^2_{n_1,n_2,n_3}\Big]^{\frac{1}{2}}\Big\|_p=$$

$$=\Big\|\Big[\lambda^2_{1,1,1}\Delta^2_{1,1,1}+\sum_{n_1=2}^{\infty}\lambda^2_{2^{n_1-1},1,1}\Delta^2_{n_1,1,1}+\sum_{n_2=2}^{\infty}\lambda^2_{1,2^{n_2-1},1}\Delta^2_{1,n_2,1}+\sum_{n_3=2}^{\infty}\lambda^2_{1,1,2^{n_3-1}}\Delta^2_{1,1,n^3}+$$

$$+\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\lambda^2_{2^{n_1-1},2^{n_2-1},1}\Delta^2_{n_1,n_2,1}+\sum_{n_1=2}^{\infty}\sum_{n_3=2}^{\infty}\lambda^2_{2^{n_1-1},1,2^{n_3-1}}\Delta^2_{n_1,1,n_3}+$$

$$+\sum_{n_2=2}^{\infty}\sum_{n_3=2}^{\infty}\lambda^2_{1,2^{n_2-1},2^{n_3-1}}\Delta^2_{1,n_2,n_3}+\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\sum_{n_3=2}^{\infty}\lambda^2_{2^{n_1-1},2^{n_2-1},2^{n_3-1}}\Delta^2_{n_1,n_2,n_2}\Big]^{\frac{1}{2}}\Big\|_p\lesssim$$

$$\lesssim\lambda_{1,1,1}\Big\|\Big[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\Delta^2_{n_1,n_2,n_3}\Big]^{\frac{1}{2}}\Big\|_p+\Big\|(\sum_{n_1=2}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\Delta^2_{n_1,n_2,n_3}[\sum_{\nu_1=2}^{n_1}\mid\lambda^\theta_{2^{\nu_1-1},1,1}-\lambda^\theta_{2^{\nu_1-2},1,1}\mid]^{\frac{2}{\theta}})^{\frac{1}{2}}\Big\|_p+$$

$$+\Big\|\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=2}^{\infty}\sum_{n_3=1}^{\infty}\Delta^2_{n_1,n_2,n_3}\Big[\sum_{\nu_2=2}^{n_2}\mid\lambda^\theta_{1,2^{\nu_2-1},1}-\lambda^\theta_{1,2^{\nu_2-2},1}\mid\Big]^{\frac{2}{\theta}}\Big)^{\frac{1}{2}}\Big\|_p+$$

$$+\Big\|\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=2}^{\infty}\Delta^2_{n_1,n_2,n_3}\Big[\sum_{\nu_3=2}^{n_2}\mid\lambda^\theta_{1,1,2^{\nu_3-1}}-\lambda^\theta_{1,1,2^{\nu_3-2}}\mid\Big]^{\frac{2}{\theta}}\Big)^{\frac{1}{2}}\Big\|_p+$$

$$+\Big\|(\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\sum_{n_3=1}^{\infty}\Delta^2_{n_1,n_2,n_3}(\sum_{\nu_1=2}^{n_1}\sum_{\nu_2=2}^{n_2}\mid\lambda^\theta_{2^{\nu_1-1},2^{\nu_2-1},1}-\lambda^\theta_{2^{\nu_1-2},2^{\nu_2-1},1}-\lambda^\theta_{2^{\nu_1-1},2^{\nu_2-2},1}+$$

$$+\lambda^\theta_{2^{\nu_1-2},2^{\nu_2-2},1}\mid)^{\frac{2}{\theta}})^{\frac{1}{2}}\Big\|_p+\Big\|(\sum_{n_1=2}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=2}^{\infty}\Delta^2_{n_1,n_2,n_3}(\sum_{\nu_1=2}^{n_1}\sum_{\nu_3=2}^{n_3}\mid\lambda^\theta_{2^{\nu_1-1},1,2^{\nu_3-1}}-\lambda^\theta_{2^{\nu_1-2},1,2^{\nu_3-1}}-$$

$$-\lambda^\theta_{2^{\nu_1-1},1,2^{\nu_3-2}}+\lambda^\theta_{2^{\nu_1-2},1,2^{\nu_3-2}}\mid)^{\frac{2}{\theta}})^{\frac{1}{2}}\Big\|_p+\Big\|(\sum_{n_1=1}^{\infty}\sum_{n_2=2}^{\infty}\sum_{n_3=2}^{\infty}\Delta^2_{n_1,n_2,n_3}(\sum_{\nu_2=2}^{n_2}\sum_{\nu_3=2}^{n_3}\mid\lambda^\theta_{1,2^{\nu_2-1},2^{\nu_3-1}}-$$

$$-\lambda^\theta_{1,2^{\nu_2-2},2^{\nu_3-1}}-\lambda^\theta_{1,2^{\nu_2-1},2^{\nu_3-2}}+\lambda^\theta_{1,2^{\nu_2-2},2^{\nu_3-2}}\mid)^{\frac{2}{\theta}})^{\frac{1}{2}}\Big\|_p+$$

$$+\Big\|\Big(\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\sum_{n_3=2}^{\infty}\Delta^2_{n_1,n_2,n_3}\Big[\sum_{\nu_1=2}^{n_1}\sum_{\nu_2=2}^{n_2}\sum_{\nu_3=2}^{n_3}\mid\lambda^\theta_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-1}}-\lambda^\theta_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-2}}-$$

$$-\lambda^\theta_{2^{\nu_1-2},2^{\nu_2-1},2^{\nu_3-1}}-\lambda^\theta_{2^{\nu_1-1},2^{\nu_2-2},2^{\nu_3-1}}+\lambda^\theta_{2^{\nu_1-1},2^{\nu_2-2},2^{\nu_3-2}}+\lambda^\theta_{2^{\nu_1-2},2^{\nu_2-1},2^{\nu_3-2}}+$$

$$+\lambda^\theta_{2^{\nu_1-2},2^{\nu_2-2},2^{\nu_3-1}}-\lambda^\theta_{2^{\nu_1-2},2^{\nu_2-2},2^{\nu_3-2}}\mid\Big]^{\frac{2}{\theta}}\Big)^{\frac{1}{2}}\Big\|_p=:H_1+H_2+H_3+H_4+H_5+H_6+H_7+H_8.$$

Let us estimate H_1. Applying Lemma 2.5, we have $H_1\leq C\lambda_{1,1,1}\|f\|_p<\infty$. Now we estimate H_2 :

$$H_2=\Big\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=2}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\Delta^2_{n_1,n_2,n_3}\Big[\sum_{\nu_1=2}^{n_1}\mid\lambda^\theta_{2^{\nu_1-1},1,1}-\lambda^\theta_{2^{\nu_1-2},1,1}\mid\Big]^{\frac{2}{\theta}}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{1}{p}}.$$

Using Minkowski's inequality and Lemma 2.2 (a) for $\frac{2}{\theta}\geq 1$, we derive

$$\sum_{n_3=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_1=2}^{\infty}\Delta^2_{n_1,n_2,n_3}\Big[\sum_{\nu_1=2}^{n_1}\mid\lambda^\theta_{2^{\nu_1-1},1,1}-\lambda^\theta_{2^{\nu_1-2},1,1}\mid\Big]^{\frac{2}{\theta}}=$$

$$=\sum_{n_1=2}^{\infty}\Big(\Big(\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\Big[\sum_{\nu_1=2}^{n_1}\mid\Delta_{n_1,n_2,n_3}\mid^\theta\mid\lambda^\theta_{2^{\nu_1-1},1,1}-\lambda^\theta_{2^{\nu_1-2},1,1}\mid\Big]^{\frac{2}{\theta}}\Big)^{\frac{\theta}{2}}\Big)^{\frac{2}{\theta}}\leq$$

$$\leq \sum_{n_1=2}^{\infty}\left(\sum_{\nu_1=2}^{n_1}\left[\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} | \Delta_{n_1,n_2,n_3}|^2 | \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}|^{\frac{2}{\theta}}\right]^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}=$$

$$=\left(\left(\sum_{n_1=2}^{\infty}\left\{\sum_{\nu_1=2}^{n_1}\left[\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} | \Delta_{n_1,n_2,n_3}|^2 | \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}|^{\frac{2}{\theta}}\right]^{\frac{\theta}{2}}\right\}^{\frac{2}{\theta}}\right)^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}\leq$$

$$\leq\left(\sum_{\nu_1=2}^{\infty}\left\{\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} | \Delta_{n_1,n_2,n_3}|^2 | \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}|^{\frac{2}{\theta}}\right\}^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}=$$

$$=\left(\sum_{\nu_1=2}^{\infty} | \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}|\left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} | \Delta_{n_1,n_2n_3}|^2\right)^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}.$$

Applying this inequality, we obtain

$$H_2\leq\left\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left[\left\{\sum_{\nu_1=2}^{\infty} | \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}|\left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} | \Delta_{n_1,n_2n_3}|^2\right)^{\frac{\theta}{2}}\right\}^{\frac{2}{\theta}}\right]^{\frac{p}{2}}dx_1,dx_2,dx_3\right\}^{\frac{1}{p}}=$$

$$=\left(\left\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left(\sum_{\nu_1=2}^{\infty} | \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}|\left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} | \Delta_{n_1,n_2,n_3}|^2\right)^{\frac{\theta}{2}}\right)^{\frac{p}{\theta}}dx_1,dx_2,dx_3\right\}^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}}.$$

Further, using Minkowski's inequality for $\frac{p}{\theta}\geq 1$, Lemmas 2.4 and 2.5, we have

$$H_2\leq\left(\sum_{\nu_1=2}^{\infty} | \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}|\left\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left(\left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} | \Delta_{n_1,n_2,n_3}|^2\right)^{\frac{\theta}{2}}\right)^{\frac{p}{\theta}}dx_1,dx_2,dx_3\right\}^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}}=$$

$$=\left(\sum_{\nu_1=2}^{\infty} | \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-2},1,1}|\left\|\left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} | \Delta_{n_1,n_2,n_3}|^2\right)^{\frac{1}{2}}\right\|_p^{\theta}\right)^{\frac{1}{\theta}}\lesssim$$

$$\lesssim\left(\sum_{\nu_1=2}^{\infty} | \lambda^{\theta}_{2^{\nu_1-1},1,1}-\lambda^{\theta}_{2^{\nu_1-1},1,1}| Y^{\theta}_{2^{\nu_1}-1,0,0}(f)_p\right)^{\frac{1}{\theta}}.$$

Thus, we obtain $H_2\lesssim\left(\sum_{\nu_1=2}^{\infty} | \lambda^{\theta}_{2^{\nu_1},1,1}-\lambda^{\theta}_{2^{\nu_1-1},1,1}| Y^{\theta}_{2^{\nu_1}-1,0,0}(f)_p\right)^{\frac{1}{\theta}}$. From (2) it follows that $H_2<\infty$, H_3,H_4 can be estimated similarly to H_2 and we have

$$H_3\lesssim\left(\sum_{\nu_2=1}^{\infty} | \lambda^{\theta}_{1,2^{\nu_2},1}-\lambda^{\theta}_{1,2^{\nu_2-1},1}| Y^{\theta}_{0,2^{\nu_2}-1,0}(f)_p\right)^{\frac{1}{\theta}},H_4\lesssim\left(\sum_{\nu_3=1}^{\infty} | \lambda^{\theta}_{1,1,2^{\nu_2}}-\lambda^{\theta}_{1,1,2^{\nu_2-1}}| Y^{\theta}_{0,0,2^{\nu_3}-1}(f)_p\right)^{\frac{1}{\theta}}.$$

To estimate H_5, we apply the method of estimate for H_4 as in article [9]. First, we obtain the upper estimate of the following sum. Applying Lemmas 2.2 and 2.3 twice for $\frac{2}{\theta}\geq 1$, we get

$$\sum_{n_3=1}^{\infty}\sum_{n_2=2}^{\infty}\sum_{n_1=2}^{\infty}\Delta^2_{n_1,n_2,n_3}\left[\sum_{\nu_1=2}^{n_1}\sum_{\nu_2=2}^{n_2} | \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},1}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},1}|\right]^{2/\theta}\leq$$

$$\leq\left(\sum_{\nu_2=2}^{\infty}\sum_{\nu_1=2}^{\infty} | \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},1}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},1}|\left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=\nu_2}^{\infty}\sum_{n_3=1}^{\infty} | \Delta_{n_1,n_2,n_3}|^2\right)^{\frac{\theta}{2}}\right)^{\frac{2}{\theta}}.$$

Hence, Lemma 2.3 with $\frac{p}{\theta}\geq 1$ implies that

$$H_5\leq\left(\left\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\nu_2=2}^{\infty}\sum_{\nu_1=2}^{\infty} | \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},1}-\right.\right.\right.$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},1}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},1}\mid\left(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=\nu_2}^{\infty}\sum_{n_3=1}^{\infty}\mid\Delta_{n_1,n_2,n_3}\mid^2\right)^{\frac{\theta}{2}}\Bigg]^{\frac{p}{\theta}}dx_1,dx_2,dx_3\Bigg\}^{\frac{\theta}{p}}\Bigg)^{\frac{1}{\theta}}\leq$$

$$\leq\Bigg(\sum_{\nu_2=2}^{\infty}\sum_{\nu_1=2}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},1}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},1}+$$

$$+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},1}\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=\nu_2}^{\infty}\sum_{n_3=1}^{\infty}\mid\Delta_{n_1,n_2,n_3}\mid^2\Bigg]^{\frac{p}{\theta}}dx_1,dx_2,dx_3\Bigg\}^{\frac{\theta}{p}}\Bigg)^{\frac{1}{\theta}}.$$

By Lemmas 2.4 and 2.5, we obtain

$$H_5\lesssim\Bigg(\sum_{\nu_2=1}^{\infty}\sum_{\nu_1=1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1},2^{\nu_2},1}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2},1}-\lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1},1}+\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},1}\mid Y^{\theta}_{2^{\nu_1}-1,2^{\nu_2}-1,0}(f)_p\Bigg)^{\frac{1}{\theta}}.$$

From (2), it follows that $H_5<\infty$. H_6,H_7,H_8 can be estimated similarly to H_5 and we have

$$H_6\lesssim\Bigg(\sum_{\nu_3=1}^{\infty}\sum_{\nu_1=1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1},1,2^{\nu_3}}-\lambda^{\theta}_{2^{\nu_1-1},1,2^{\nu_3}}-\lambda^{\theta}_{2^{\nu_1},1,2^{\nu_3-1}}+\lambda^{\theta}_{2^{\nu_1-1},1,2^{\nu_3-1}}\mid Y^{\theta}_{2^{\nu_1}-1,0,2^{\nu_3}-1}(f)_p\Bigg)^{\frac{1}{\theta}},$$

$$H_7\lesssim\Bigg(\sum_{\nu_3=1}^{\infty}\sum_{\nu_2=1}^{\infty}\mid\lambda^{\theta}_{1,2^{\nu_2},2^{\nu_3}}-\lambda^{\theta}_{1,2^{\nu_2-1},2^{\nu_3}}-\lambda^{\theta}_{1,2^{\nu_2},2^{\nu_3-1}}+\lambda^{\theta}_{1,2^{\nu_2-1},2^{\nu_3-1}}\mid Y^{\theta}_{0,2^{\nu_2}-1,2^{\nu_3}-1}(f)_p\Bigg)^{\frac{1}{\theta}},$$

$$H_8\lesssim\Bigg(\sum_{\nu_3=1}^{\infty}\sum_{\nu_2=1}^{\infty}\sum_{\nu_1=1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1},2^{\nu_2},2^{\nu_3}}-\lambda^{\theta}_{2^{\nu_1},2^{\nu_2},2^{\nu_3-1}}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2},2^{\nu_3}}-\lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1},2^{\nu_3}}+$$

$$+\lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1},2^{\nu_3-1}}+\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2},2^{\nu_3-1}}+\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3}}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-1}}\mid Y^{\theta}_{2^{\nu_1}-1,2^{\nu_2}-1,2^{\nu_3}-1}(f)_p\Bigg)^{\frac{1}{\theta}}.$$

Collecting estimates of H_1-H_8 we get $I_1<\infty$. Hence, by Lemma 2.5 (b), there exists a function $g(x_1,x_2,x_3)\in L^0_p$, with the Fourier series

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\lambda_{2^{n_1-1},2^{n_2-1},2^{n_3-1}}\Delta_{n_1,n_2,n_3} \tag{6}$$

and

$$\|g\|_p\leq C(p)I_1. \tag{7}$$

We rewrite series (6) in the form of $\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\gamma_{n_1,n_2,n_3}A_{n_1,n_2,n_3}(x_1,x_2,x_3)$, where

$$\gamma_{1,1,1}=\lambda_{1,1,1},\gamma_{1,\nu_2,\nu_3}=\lambda_{1,2^{n_2-1},2^{n_3-1}}\ \text{ for }\ 2^{n_2-1}\leq\nu_2\leq 2^{n_2}-1,\,2^{n_3-1}\leq\nu_3\leq 2^{n_3}-1\quad(n_2=2,3...),$$

$$\gamma_{\nu_1,1,1}=\lambda_{2^{n_1-1},1,1}\ \text{ for }\ 2^{n_1-1}\leq\nu_1\leq 2^{n_1}-1\quad(n_2=2,3...),$$

$$\gamma_{\nu_1,1,\nu_3}=\lambda_{2^{n_1-1},1,2^{n_3-1}}\ \text{ for }\ 2^{n_1-1}\leq\nu_1\leq 2^{n_1}-1,\ 2^{n_3-1}\leq\nu_3\leq 2^{n_3}-1,\quad(n_1,n_3=2,3...),$$

$$\gamma_{\nu_1,\nu_2,1}=\lambda_{2^{n_1-1},2^{n_2-1},1}\ \text{ for }\ 2^{n_1-1}\leq\nu_1\leq 2^{n_1}-1,2^{n_2-1}\leq\nu_2\leq 2^{n_2}-1,\quad(n_1,n_2=2,3...),$$

$$\gamma_{1,1,\nu_3}=\lambda_{1,1,2^{n_3-1}}\ \text{ for }\ 2^{n_3-1}\leq\nu_1\leq 2^{n_3}-1,\quad(n_3=2,3...),$$

$$\gamma_{1,\nu_2,1}=\lambda_{1,2^{n_2-1},1}\ \text{ for }\ 2^{n_2-1}\leq\nu_2\leq 2^{n_2}-1,\quad(n_2=2,3...),$$

$$\gamma_{\nu_1,\nu_2,\nu_3}=\lambda_{2^{n_1-1},2^{n_2-1},2^{n_3-1}}\ \text{ for }\ 2^{n_1-1}\leq\nu_1\leq 2^{n_1}-1,$$

$$2^{n_2-1}\leq\nu_2\leq 2^{n_2}-1,\,2^{n_3-1}\leq\nu_3\leq 2^{n_3}-1\quad(n_1,n_2,n_3=2,3...).$$

Now we consider the following series

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\lambda_{n_1,n_2,n_3}A_{n_1,n_2,n_3}(x_1,x_2,x_3)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\gamma_{n_1,n_2,n_3}\Lambda_{n_1,n_2,n_3}A_{n_1,n_2,n_3}(x_1,x_2,x_3) \qquad (8),$$

where

$$\Lambda_{1,1,1}=1, \Lambda_{1,\nu_2,\nu_3}=\frac{\lambda_{1,\nu_2,\nu_3}}{\gamma_{1,\nu_2,\nu_3}}=\frac{\lambda_{1,\nu_2,\nu_3}}{\lambda_{1,2^{n_2-1},2^{n_3-1}}}\text{for } 2^{n_2-1}\le\nu_2\le 2^{n_2}-1,$$

$$2^{n_3-1}\le\nu_3\le 2^{n_3}-1 \quad (n_2,n_3=2,3...),$$

$$\Lambda_{\nu_1,1,1}=\frac{\lambda_{\nu_1,1,1}}{\gamma_{\nu_1,1,1}}=\frac{\lambda_{\nu_1,1,1}}{\lambda_{2^{n_1-1},1,1}} \text{ for } 2^{n_1-1}\le\nu_1\le 2^{n_1}-1, \quad (n_2=2,3...),$$

$$\Lambda_{\nu_1,\nu_2,1}=\frac{\lambda_{\nu_1,\nu_2,1}}{\gamma_{\nu_1,\nu_2,1}}=\frac{\lambda_{\nu_1,\nu_2,1}}{\lambda_{2^{n_1-1},2^{n_2-1},1}} \text{ for } 2^{n_1-1}\le\nu_1\le 2^{n_1}-1,\ 2^{n_2-1}\le\nu_2\le 2^{n_2}-1, \quad (n_1,n_2=2,3...),$$

$$\Lambda_{\nu_1,1,\nu_3}=\frac{\lambda_{\nu_1,1,\nu_3}}{\gamma_{\nu_1,1,\nu_3}}=\frac{\lambda_{\nu_1,1,\nu_3}}{\lambda_{2^{n_1-1},1,2^{n_3-1}}} \text{ for } 2^{n_1-1}\le\nu_1\le 2^{n_1}-1,\ 2^{n_3-1}\le\nu_3\le 2^{n_3}-1, \quad (n_1,n_2=2,3...),$$

$$\Lambda_{1,1,\nu_3}=\frac{\lambda_{1,1,\nu_3}}{\gamma_{1,1,\nu_3}}=\frac{\lambda_{1,1,\nu_3}}{\lambda_{1,1,2^{n_3-1}}} \text{ for } 2^{n_3-1}\le\nu_3\le 2^{n_3}-1,$$

$$\Lambda_{1,\nu_2,1}=\frac{\lambda_{1,\nu_2,1}}{\gamma_{1,\nu_2,1}}=\frac{\lambda_{1,\nu_2,1}}{\lambda_{1,2^{n_2-1},1}} \text{ for } 2^{n_2-1}\le\nu_2\le 2^{n_2}-1,$$

$$\Lambda_{\nu_1,\nu_2,\nu_3}=\frac{\lambda_{\nu_1,\nu_2,\nu_3}}{\gamma_{\nu_1,\nu_2,\nu_3}}=\frac{\lambda_{\nu_1,\nu_2,\nu_3}}{\lambda_{2^{n_1-1},2^{n_2-1},2^{n_3-1}}} \text{ for } 2^{n_1-1}\le\nu_1\le 2^{n_1}-1,\ 2^{n_2-1}\le\nu_2\le 2^{n_2}-1,$$

$$2^{n_3-1}\le\nu_3\le 2^{n_3}-1 \quad (n_2,n_3=2,3...).$$

As shown in [6], the sequence $\{\Lambda_{n_1=1,n_2=1,n_3=1}\}_{n_1=1,n_2=1,n_3=1}^{\infty,\infty,\infty}$ satisfies the conditions of the Marcinkiewicz multiplier theorem [12], then the series (8) is the Fourier series of a function $\varphi(x_1,x_2,x_2)\in L_p$ and $\|\varphi\|_p\le C(\rho,\lambda)\|g\|_p$.

Taking into account (7) and the estimates of H_1-H_8 we get (3).

Let us estimate $Y_{2^{m_1}-1,2^{m_2}-1,2^{m_3}-1}(\varphi)_p$. Using Lemma 2.4, we get

$$Y_{2^{m_1}-1,2^{m_2}-1,2^{m_3}-1}(\varphi)_p\le C\|\varphi-S_{2^{m_1}-1,\infty,\infty}(\varphi)-S_{\infty,2^{m_2}-1,\infty}(\varphi)-$$

$$-S_{\infty,\infty,2^{m_3}-1}(\varphi)+2S_{2^{m_1}-1,2^{m_2}-1,2^{m_3}-1}(\varphi)\|_p.$$

We consider the series (see (8))

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\lambda_{n_1,n_2,n_3}A^*_{n_1,n_2,n_3}(x_1,x_2,x_3)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\gamma_{n_1,n_2,n_3}\Lambda_{n_1,n_2,n_3}A^*_{n_1,n_2,n_3}(x_1,x_2,x_3),$$

where $A^*_{n_1,n_2,n_3}(x_1,x_2,x_3)=0$, if $n_1\le 2^{m_1}-1$ and $n_2\le 2^{m_2}-1$, $n_3\le 2^{m_3}-1$ also $A^*_{n_1,n_2,n_3}(x_1,x_2,x_3)=$ $=A_{n_1,n_2,n_3}(x_1,x_2,x_3)$ otherwise. Since the sequence $\{\Lambda_{n_1=1,n_2=1,n_3=1}\}$ satisfies the conditions of the Marcinkiewicz multiplier theorem, then

$$\Big\|\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\lambda_{n_1,n_2,n_3}A^*_{n_1,n_2,n_3}(x_1,x_2,x_3)\Big\|_p\le C\Big\|\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\lambda_{2^{n_1-1},2^{n_2-1},2^{n_3-1}}\Delta^*_{n_1,n_2,n_3}\Big\|_p,$$

where $\Delta^*_{n_1,n_2,n_3}=0$, if $n_1\le m_1$ and $n_2\le m_2$, $n_3\le m_3$ $\Delta^*_{n_1,n_2,n_3}=\Delta_{n_1n_2,n_3}$ otherwise.

By Lemma 2.5, we have

$$Y_{2^{m_1}-1,2^{m_2}-1,2^{m_3}-1}(\varphi)_p\lesssim \tag{9}$$

$$\lesssim\Big\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\lambda^2_{2^{k_1-1},2^{k_2-1},2^{k_3-1}}\Delta^2_{k_1,k_2,k_3}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{1}{p}}.$$

For the sequence $\lambda^{\theta}_{2^{k_1-1},2^{k_2-1},2^{k_3-1}}$, we use inequality (5) where the index of the first element starts with $2^{m_1-1},2^{m_2-1},2^{m_3-1}$, and we take the sum from $2^{m_1-1},2^{m_2-1},2^{m_3-1}$ to $2^{k_1-1},2^{k_2-1},2^{k_3-1}$ respectively. The resulting inequality is substituted into inequality (9).

$$Y_{2^{m_1}-1,2^{m_2}-1,2^{m_3}-1}(\varphi)_p\lesssim$$

$$\lesssim \Bigg(\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{m_3-1}}\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}]^{\frac{p}{2}}dx_1,dx_2,dx_3\}^{\frac{1}{p}}+$$

$$+\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Bigg(\sum_{\nu_1=m_1+1}^{k_1}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{m_3-1}}\mid\Bigg)^{\frac{2}{\theta}}\Bigg]^{\frac{p}{2}}dx_1,dx_2,dx_3\Bigg\}^{\frac{1}{p}}+$$

$$+\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times(\sum_{\nu_2=m_2+1}^{k_2}\mid\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2},2^{m_3-1}}\mid\Bigg)^{\frac{2}{\theta}}\Bigg]^{\frac{p}{2}}dx_1,dx_2,dx_3\Bigg\}^{\frac{1}{p}}+$$

$$+\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Bigg(\sum_{\nu_3=m_3+1}^{k_3}\mid\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{\nu_3-1}}-\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{\nu_2-2}}\mid\Bigg)^{\frac{2}{\theta}}\Bigg]^{\frac{p}{2}}dx_1,dx_2,dx_3\Bigg\}^{\frac{1}{p}}+$$

$$+\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Bigg(\sum_{\nu_1=m_1+1}^{k_1}\sum_{\nu_2=m_2+1}^{k_2}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{m_3-1}}-$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{m_3-1}}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{m_3-1}}\mid\Bigg)^{\frac{2}{\theta}}\Bigg]^{\frac{p}{2}}dx_1,dx_2,dx_3\Bigg\}^{\frac{1}{p}}+$$

$$+\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Bigg(\sum_{\nu_2=m_2+1}^{k_2}\sum_{\nu_3=m_3+1}^{k_3}\mid\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{\nu_3-1}}-\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2},2^{\nu_3-1}}-$$

$$-\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{\nu_3-2}}+\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2},2^{\nu_3-2}}\mid\Bigg)^{\frac{2}{\theta}}\Bigg]^{\frac{p}{2}}dx_1,dx_2,dx_3\Bigg\}^{\frac{1}{p}}+$$

$$+\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Bigg(\sum_{\nu_1=m_1+1}^{k_1}\sum_{\nu_3=m_3+1}^{k_3}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{\nu_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{\nu_3-1}}-$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{\nu_3-2}}+\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{\nu_3-2}}\mid\Bigg)^{\frac{2}{\theta}}\Bigg]^{\frac{p}{2}}dx_1,dx_2,dx_3\Bigg\}^{\frac{1}{p}}+$$

$$+\Bigg\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Bigg[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Bigg(\sum_{\nu_1=m_1+1}^{k_1}\sum_{\nu_2=m_2+1}^{k_2}\sum_{\nu_3=m_3+1}^{k_3}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{\nu_3-1}}-$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{\nu_3-1}}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-2}}+\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{\nu_3-2}}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{\nu_3-2}}+$$

$$+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{\nu_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{\nu_3-2}}\mid\Big)^{\frac{2}{\theta}}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{1}{p}}\Big)=:L_1+L_2+L_3+L_4+L_5+L_6+L_7+L_8.$$

We estimate L_1 as H_1, to get

$$L_1\le\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{m_3-1}}\Big\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{1}{p}}\lesssim$$

$$\lesssim\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{m_3-1}}Y_{2^{m_1}-1,2^{m_2}-1\ ,2^{m_3}-1}(f)_p.$$

We also have

$$L_2=\Big\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times$$

$$\times\Big(\sum_{\nu_1=m_1+1}^{k_1}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{m_3-1}}\mid\Big)^{\frac{2}{\theta}}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{1}{p}}\lesssim$$

$$\lesssim\Big(\sum_{\nu_1=m_1+1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{m_3-1}}\mid\times$$

$$\times\Big\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=\nu_1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{\theta}{p}}\Big)^{\frac{1}{\theta}}\lesssim$$

$$\lesssim\Big(\sum_{\nu_1=m_1+1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{m_3-1}}\mid\ Y^{\theta}_{2^{\nu_1-1}-1,2^{m_2}-1,2^{m_3}-1}(f)_p\Big)^{\frac{1}{\theta}}.$$

Similarly, we obtain the estimates for L_3,L_4

$$L_3\lesssim\Big(\sum_{\nu_2=m_2+1}^{\infty}\mid\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2},2^{m_3-1}}\mid\ Y^{\theta}_{2^{m_1}-1,2^{\nu_2-1}-1,2^{m_3}-1}(f)_p\Big)^{\frac{1}{\theta}},$$

$$L_4\lesssim\Big(\sum_{\nu_3=m_3+1}^{\infty}\mid\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{\nu_3-1}}-\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},2^{\nu_2-2}}\mid\ Y^{\theta}_{2^{m_1}-1,2^{m_2}-1,2^{\nu_3-1}-1}(f)_p\Big)^{\frac{1}{\theta}}.$$

We estimate L_5 as follows:

$$L_5=\Big\{\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\times\Big(\sum_{\nu_1=m_1+1}^{k_1}\sum_{\nu_2=m_2+1}^{k_2}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{m_3-1}}-$$

$$-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{m_3-1}}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{m_3-1}}\mid\Big)^{\frac{2}{\theta}}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{1}{p}}\lesssim$$

$$\lesssim\Big(\sum_{\nu_1=m_1+1}^{\infty}\sum_{\nu_2=m_2+1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{m_3-1}}+$$

$$+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{m_3-1}}\mid\times\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=\nu_1}^{\infty}\sum_{k_2=\nu_2}^{\infty}\sum_{k_3=m_3+1}^{\infty}\Delta^2_{k_1,k_2,k_3}\Big]^{\frac{p}{2}}dx_1,dx_2,dx_3\Big\}^{\frac{\theta}{p}}\Big)^{\frac{1}{\theta}}\lesssim$$

$$\lesssim\Big(\sum_{\nu_1=m_1+1}^{\infty}\sum_{\nu_2=m_2+1}^{\infty}\mid\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{m_3-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{m_3-1}}-$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{m_3-1}}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{m_3-1}}\mid Y^{\theta}_{2^{\nu_1-1}-1,2^{\nu_2-1}-1,2^{m_3}-1,}(f)_p\Big)^{\frac{1}{\theta}}.$$

Similarly, we obtain the estimates for L_6, L_7 :

$$L_6 \lesssim \Bigg(\sum_{\nu_2=m_2+1}^{\infty} \sum_{\nu_3=m_3+1}^{\infty} \mid \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{\nu_3-1}} - \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2},2^{\nu_3-1}} -$$

$$-\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1},2^{\nu_3-2}} + \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2},2^{\nu_3-2}} \mid Y^{\theta}_{2^{m_1}-1,2^{\nu_2-1}-1,2^{\nu_3-1}-1}(f)_p \Bigg)^{\frac{1}{\theta}}.$$

$$L_7 \lesssim \Bigg(\sum_{\nu_1=m_1+1}^{\infty} \sum_{\nu_3=m_3+1}^{\infty} \mid \lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{\nu_3-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{\nu_3-1}} -$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},2^{\nu_3-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},2^{\nu_3-2}} \mid Y^{\theta}_{2^{\nu_1-1}-1,2^{m_2}-1,2^{\nu_3-1}-1}(f)_p \Bigg)^{\frac{1}{\theta}}.$$

Finally, we estimate L_8 as follows:

$$L_8 \lesssim \Bigg(\sum_{\nu_1=m_1+1}^{\infty} \sum_{\nu_2=m_2+1}^{k_2} \sum_{\nu_3=m_3+1}^{\infty} \mid \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{\nu_3-1}} -$$

$$-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{\nu_3-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},2^{\nu_3-2}} + \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},2^{\nu_3-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},2^{\nu_3-2}} +$$

$$+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{\nu_3-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},2^{\nu_3-2}} \mid Y^{\theta}_{2^{\nu_1-1}-1,2^{\nu_2-1}-1,2^{\nu_3-1}-1}(f)_p \Bigg)^{\frac{1}{\theta}}.$$

Taking into account the estimates for $L_1 - L_8$, we obtain (4). The theorem is proved.

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Жалпыланған Лиувилл-Вейл туындыларының бұрыштық ең жақын жуықтауларының жоғарғы бағалаулары

Мақалада 2π периодты f үзіліссіз функциялар және оларды тригонометриялық көпмүшеліктермен жуықтауқа жалпыланған Лиувиль-Вейл туындылары арқылы берілген үш өлшемді функциялардың бұрыштық ең жақын жуықтауын бастапқы берілген функциялардың бұрыштық ең жақын жуықтауы арқылы бағалауы қарастырылған. Авторлар классикалық Вейл аралас туындыларының орнына жалпыланған Лиувиль-Вейл туындыларын зерттеген. Қарастырылатын мәселелерді таңдағанда осы мақаладағы екінші автордың жұмысынан кейін қалыптасқан жалпы тәсілді ұстанған. Басты мақсат жұмыстың нәтижелерін үш өлшемді жағдайда дәлелдеу. Жалпы монотонды тізбектер туралы түсінік осы зерттеуде басты рөл атқарады. Функцияның ең жақын жуықтауларына қатысты r-туындысының ең жақын жуықтаулары, норма үшін бірнеше белгілі теңсіздіктер көрсетілген. Мақалада қарастырылған мәселелер Бернштейннің зерттелген еңбектерінің мәселелеріне жатады. Кейінірек Стечкин және Конюшков $f^{(r)}$ ең жақын жуықтау үшін теңсіздік алынды. Сонымен қатар Потаповтың еңбектерінде бұрыштарды жақындату арқылы функциялардың кейбір кластары қарастырылған. Бірінші бөлімде қажетті түсініктермен және пайдалы леммалар берілген. Үш өлшемді жағдайда жалпыланған Лиувилль-Вейл туындысы арқылы берілген функциялардың нормасының және бұрыштық ең жақын жуықтауының бағалауы алынды.

Кілт сөздер: Лебег кеңістігі, үш өлшемді бұрышпен ең жақын жуықтау, тригонометриялық көпмүшелік, Лиувилл-Вейл туындысы.

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Верхние оценки угловых наилучших приближений обобщенных производных Лиувилля-Вейля

В статье рассмотрены непрерывные функции f с периодом 2π и их приближения тригонометрическими полиномами. Изучены оценки наилучших угловых приближений обобщены производных Лиувилля-Вейля угловым приближением функций в трехмерном случае. Авторами обобщенные производные Лиувилля-Вейля вместо классического смешанного производного Вейля. При выборе рассматриваемых вопросов они следовали общему подходу, сформировавшемуся после работы второго автора настоящей статьи. Главная цель — доказать аналоги результатов работы в трехмерном случае. Понятие общих монотонных последовательностей играет ключевую роль в исследовании. Указаны несколько известных неравенств для норм, наилучших приближений r-го производного по наилучшим приближениям функции f. Вопросы, рассмотренные в настоящей работе, относятся к кругу проблем, изученных в работах Бернштейном. Позднее Стечкин и Конюшков получили неравенство для наилучшего приближения $f^{(r)}$. Также в работах Потапова при помощи приближения углом изучены некоторые классы функций. В подразделе 1 авторами даны необходимые обозначения и полезные леммы. Получены оценки норм и наилучшие приближения обобщенного производного Лиувилля-Вейля в трехмерном случае.

Ключевые слова: пространство Лебега, наилучшее приближение трехмерным углом, тригонометрический полином, производная Лиувилля-Вейля.

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Unique solvability of boundary value problem for functional differential equations with involution

In this paper, we consider a boundary value problem for systems of Fredholm type integral-differential equations with involutive transformation, containing derivative of the required function on the right-hand side under the integral sign. Applying properties of an involutive transformation, original boundary value problem is reduced to a boundary value problem for systems of integral-differential equations, containing derivative of the required function on the right side under the integral sign. Assuming existence of resolvent of the integral equation with respect to the kernel $\tilde{K}_2(t,s)$ (this is the kernel of the integral equation that contains the derivative of the desired function) and using properties of the resolvent, integral-differential equation with a derivative on the right-hand side is reduced to a Fredholm type integral-differential equation, in which there is no derivative of the desired function on the right side of the equation. Further, the obtained boundary value problem is solved by the parametrization method created by Professor D. Dzhumabaev. Based on this method, the problem is reduced to solving a special Cauchy problem with respect to the introduced new functions and to solving systems of linear algebraic equations with respect to the introduced parameters. An algorithm to find a solution is proposed. As is known, in contrast to the Cauchy problem for ordinary differential equations, the special Cauchy problem for systems of integral-differential equations is not always solvable. Necessary conditions for unique solvability of the special Cauchy problem were established. By using results obtained by Professor D. Dzhumabaev, necessary and sufficient conditions for the unique solvability of the original problem were established.

Keywords: system of integral-differential equations, boundary value conditions, parametrization method, integral equation, resolvent, involution, unique solvability, Special Cauchy Problem.

Introduction

Boundary value problems for integral-differential equations have been studied by many authors [1–7], however, with the development of computer technology, the question of creating constructive methods for solving the problem arises. In connection with this, Professor D. Dzhumabaev proposed a method for parameterizing the solution of a linear two-point boundary value problem for systems of differential equations [8]. This method was applied to study various boundary value problems [9–14].

On the segment $[0,T]$ we consider the following boundary value problem:

$$\frac{dx(t)}{dt} + \operatorname{diag}(a_1,\ a_2,\ \ldots,\ a_n)\frac{dx(\alpha(t))}{dt} = \int\limits_0^T K_1(t,s)x\,(s)\,ds + \int\limits_0^T K_2(t,s)\dot{x}\,(s)\,ds + f(t), \quad t \in [0,T]\,, \tag{1}$$

$$Bx(0) + C(T) = d, d \in R^n, \tag{2}$$

where the matrices $K_1(t,s)$, $K_2(t,s)$ are continuous on $[0,T]\times[0,T]$, respectively, n-dimensional vector-function $f(t)$ is continuous on $[0,T]$. $\alpha(t)$ is a reorientation homeomorphism $\alpha : [0,T] \to [0,T]$ such that $\alpha^2(t) = \alpha(\alpha(t)) = t$. It is known that the homeomorphism $\alpha(t)$ is called the involutive transformation. On the segment $[0,T]$ as such a transformation, we can consider the transformation $\alpha(t) = T - t$. Properties of the involutive transformation were studied by G.S. Litvinchuk [14], N.K. Karapetyants and S.G. Samko [15] and others.

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We consider a value of equation (1) at the point $t = \alpha(t)$

$$\frac{dx(\alpha(t))}{dt} + \mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)\frac{dx(t)}{dt} = \int_0^T K_1(\alpha(t), s)x\,(s)\,ds + \int_0^T K_2(\alpha(t), s)\dot{x}\,(s)\,ds + f(\alpha(t)).$$

From the system

$$\begin{cases} \frac{dx(t)}{dt} + \mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)\frac{dx(\alpha(t))}{dt} = \int_0^T K_1(t, s)x\,(s)\,ds + \int_0^T K_2(t, s)\dot{x}\,(s)\,ds + f(t), \\ \frac{dx(\alpha(t))}{dt} + \mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)\frac{dx(t)}{dt} = \int_0^T K_1(\alpha(t), s)x\,(s)\,ds + \int_0^T K_2(\alpha(t), s)\dot{x}\,(s)\,ds + f(\alpha(t)) \end{cases}$$

we define

$$\mathrm{diag}(1 - a_1^2, 1 - a_2^2,\ \ldots, 1 - a_n^2)\frac{dx(t)}{dt} = \int_0^T \left[K_1(t, s) - \mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)K_1(\alpha(t), s)\right] x(s)ds +$$

$$+ \int_0^T \left[K_2(t, s) - \mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)K_2(\alpha(t), s)\right] \dot{x}(s)ds + \left[f(t) - \mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)f(\alpha(t))\right].$$

Suppose that the matrix $\mathrm{diag}(1 - a_1^2, 1 - a_2^2,\ \ldots, 1 - a_n^2)$ is not degenerate, then it is invertible, and boundary value problem (1)–(2) can be written in the form

$$\frac{dx}{dt} = \int_0^T \tilde{K}_1(t, s)x(s)ds + \int_0^T \tilde{K}_2(t, s)\dot{x}(s)ds + \tilde{f}(t), \quad t \in [0, T]\,, \tag{3}$$

$$Bx(0) + Cx(T) = d, \ \ d \in R^n, \tag{4}$$

where

$$\tilde{K}_1(t, s) = \mathrm{diag}(1/(1 - a_1^2),\ 1/(1 - a_2^2), \ldots, 1/(1 - a_n^2))\left[K_1(t, s) - \mathrm{diag}(a_1,\ a_2,\ \ldots, a_n)K_1(\alpha(t), s)\right],$$

$$\tilde{K}_2(t, s) = \mathrm{diag}(1/(1 - a_1^2),\ 1/(1 - a_2^2), \ldots,\ 1/(1 - a_n^2))\left[K_2(t, s) - \mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)K_2(\alpha(t), s)\right],$$

$$\tilde{f}(t) = \mathrm{diag}(1/(1 - a_1^2),\ 1/(1 - a_2^2), \ldots,\ 1/(1 - a_n^2))\left[f(t) - \mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)f(\alpha(t))\right].$$

Condition A. Let the following Fredholm integral equation of the second kind

$$z(t) = \int_0^T \tilde{K}_2(t, s)z\,(s)\,ds + \Phi(t)$$

has a unique solution for any function $\Phi(t) \in C\left([0, T]\,, R^n\right)$.

If Condition A holds, then there exists $\Gamma_2(t, s; 1)$ – resolvent of the Fredholm integral equation of the second kind with the kernel $\tilde{K}_1(t, s)$ and a solution of the integral equation can be written as

$$z^*(t) = \Phi(t) + \int_0^T \Gamma_2(t, s; 1)\Phi\,(s)\,ds.$$

By using Condition A, problem (3) – (4) can be rewritten as

$$\frac{dx}{dt} = \int_0^T \tilde{K}_1(t, s)x(s)ds + \tilde{f}(t) + \int_0^T \Gamma_2(t, \tau; 1)\left[\int_0^T \tilde{K}_1(\tau, s)x(s)ds + \tilde{f}(\tau)\right]d\tau, \quad t \in [0, T]\,, \tag{5}$$

$$Bx(0) + Cx(T) = d, \ \ d \in R^n. \tag{6}$$

Changing the order of integration in the integral term we obtain

$$\int_0^T \Gamma_2(t,s;1)\int_0^T \tilde{K}_1(s,\tau)x(\tau)d\tau ds = \int_0^T\left(\int_0^T \Gamma_2(t,\tau;1)\tilde{K}_1(\tau,s)d\tau\right)x(s)ds = \int_0^T K^*{}_1(t,s)x(s)ds.$$

We denote

$$\hat{K}_1(t,s) = K^*{}_1(t,s) + \tilde{K}_1(t,s),$$

$$\hat{f}(t) = \tilde{f}(t) + \int_0^T \Gamma_2(t,\tau;1)\tilde{f}(\tau)d\tau.$$

Then we rewrite problem (5) – (6) in the form:

$$\frac{dx}{dt} = \int_0^T \hat{K}_1 t,s)x(s)ds + \hat{f}(t), \quad t\in[0,T], \tag{7}$$

$$Bx(0) + Cx(T) = d, \ \ d\in R^n. \tag{8}$$

We take the step $h>0$, that fits N times on the segment $[0,T]$ and along it we consider the partition $[0,T) = \bigcup\limits_{r=1}^{N}[(r-1)h, rh)$.

We denote restriction of the function $x(t)$ on the r-th interval $[(r-1)h, rh)$ by $x_r(t)$, i.e., $x_r(t)$ is a system of vector functions defined and coinciding with $x(t)$ on $[(r-1)h, rh)$. Then, the original two-point boundary value problem for systems of integral-differential equations is reduced to the equivalent multipoint boundary value problem

$$\frac{dx_r}{dt} = \sum_{j=1}^{N}\int_{(j-1)h}^{jh} \hat{K}_1(t,s)x_j(s)ds + \hat{f}(t), t\in[(r-1)h, rh), \tag{9}$$

$$Bx_1(0) + C\lim_{t\to T-0} x_N(t) = d, \tag{10}$$

$$\lim_{t\to sh-0} x_s(t) = x_{s+1}(sh), s=\overline{1,N-1}. \tag{11}$$

Here (11) are gluing conditions at the interior points of the partition $t=jh$, $j=\overline{1,N-1}$.

If the function $x(t)$ is a solution to problem (7)–(8), then the system of its restrictions $x[t]=(x_1(t), x_2(t), ..., x_N(t))'$ will be a solution of multipoint boundary value problem (9)–(11). And in inverse, if the system of vector functions $\tilde{x}[t]=(\tilde{x}_1(t), \tilde{x}_2(t), ..., \tilde{x}_N(t))'$ is a solution to problem (9)–(11), then the function $\tilde{x}(t)$, defined by the equalities $\tilde{x}(t)=\tilde{x}_r(t)$, $t\in[(r-1)h, rh)$, $r=\overline{1,N}$, $\tilde{x}(T) = \lim\limits_{t\to T-0}\tilde{x}_N(t)$ will be a solution of original boundary value problem (7)–(8). By λ_r we denote a value of the function $x_r(t)$ at the point $t=(r-1)h$ and on each interval $[(r-1)h, rh)$ we change $x_r(t) = u_r(t)+\lambda_r$, $r=\overline{1,N}$. Then problem (9)–(11) is reduced to the equivalent multipoint boundary value problem with parameters

$$\frac{du_r}{dt} = \sum_{j=1}^{N}\int_{(j-1)h}^{jh} \hat{K}_1(t,s)[u_j(s)+\lambda_j]ds + \hat{f}(t), \tag{12}$$

$$u_r[(r-1)h] = 0, t\in[(r-1)h, rh), r=\overline{1,N}, \tag{13}$$

$$B\lambda_1 + C\lambda_N + C\lim_{t\to T-0} u_N(t) = d, \tag{14}$$

$$\lambda_s + \lim_{t\to sh-0} u_s(t) = \lambda_{s+1}, s=\overline{1,N-1}. \tag{15}$$

Problems (9)–(11) and (12)–(15) are equivalent in the sense that if the system of functions $x[t]=(x_1(t), x_2(t), ..., x_N(t))'$ is a solution of problem (9)–(11), then pair $(\lambda, u[t])$ will be a solution of the problem (12)–(15), where $\lambda = (x_1(0), x_2(h), ..., x_N(N-1)h))'$, $u[t] = (x_1(t)-x_1(0), x_2(t)-x_2(h), ..., x_N(t)-x_N((N-1)h))$. And in inverse, if pair $(\tilde{\lambda}, \tilde{u}[t])$ is a solution of the problem (12)–(15), where $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_N)'$,

$\tilde{u}[t] = (\tilde{u}_1(t), \tilde{u}_2(t), ..., \tilde{u}_N(t))'$, then the system of functions $\tilde{x}[t] = (\tilde{\lambda}_1 + \tilde{u}_1(t), \tilde{\lambda}_2 + \tilde{u}_2(t), ..., \tilde{\lambda}_N + \tilde{u}_N(t))'$ will be a solution of problem (9)–(11).

Appearance of the initial conditions $u_r[(r-1)h] = 0, r = \overline{1,N}$, allows us to determine functions $u_r(t)$, $r = \overline{1,N}$, from the systems of integral equations for fixed values $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N)'$:

$$u_r(t) = \int\limits_{(r-1)h}^{t} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} \hat{K}_1(\tau,s)\,[u_j(s) + \lambda_j]\,ds d\tau + \int\limits_{(r-1)h}^{t} \hat{f}(\tau)d\tau, \qquad t \in [(r-1)h, rh). \tag{16}$$

From (16) defining $\lim\limits_{t\to Nh-0} u_N(t)$, $\lim\limits_{t\to sh-0} u_s(t)$, $s = \overline{1,N}-1$, putting the corresponding expressions into the conditions (14), (15), and multiplying both sides of (14) to $h > 0$, we get the system of linear equations concerning to the unknown parameters λ_r, $r = \overline{1,N}$:

$$hB\lambda_1 + hC\lambda_N + hC \int\limits_{(N-1)h}^{Nh} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} \hat{K}_1(\tau,s)\lambda_j ds d\tau =$$

$$= hd - hC \int\limits_{(N-1)h}^{Nh} \hat{f}(\tau)(\tau)d\tau - hC \int\limits_{(N-1)h}^{Nh} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} \hat{K}_1(\tau,s)u_j(s)dsd\tau \tag{17}$$

$$\lambda_s + \int\limits_{(s-1)h}^{sh} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} \hat{K}_1(\tau,s)\lambda_j ds\, d\tau - \lambda_{s+1} =$$

$$= -\int\limits_{(s-1)h}^{sh} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} \hat{K}_1(\tau,s)\, u_j\,(s)\, ds\, d\tau - \int\limits_{(s-1)h}^{sh} \hat{f}(\tau)d\tau, \qquad s = \overline{1,N-1.} \tag{18}$$

We denote the $nN \times nN$ dimensional matrix corresponding to the left side of the system of linear equations (17), (18) by $Q(h)$. Then the system of linear equations (17), (18) can be written in the form:

$$Q(h)\lambda = -F(h) - G(u,h), \lambda \in R^{nN}, \tag{19}$$

where

$$F(h) = \left(-hd + hC \int\limits_{(N-1)h}^{Nh} f_1(\tau)d\tau,\ \int\limits_{0}^{h} f_1(\tau)d\tau,\ \ldots,\ \int\limits_{(N-2)h}^{(N-1)h} f_1(\tau)d\tau \right),$$

$$G(u,h) = \left(hC \int\limits_{(N-1)h}^{Nh} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K_1(\tau,s)u_j(s)dsd\tau\ ,\ \int\limits_{0}^{h} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K_2(\tau,s)u_j(s)dsd\tau, \ldots, \right.$$

$$\left. \int\limits_{(N-2)h}^{(N-1)h} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K(\tau,s)u_j(s)dsd\tau \right).$$

Therefore, to find unknown pairs $(\lambda, u[t])$, solutions of the problem (12)–(15)... we have a closed system of equations (16), (19). We find solution of the multipoint boundary value problem (12)–(15) as a limit of the sequence of pairs $(\lambda'^{(k)}, u^{(k)}[t])$, k =0,1,2, ..., defined by the following algorithm:

Step 0. a) Assuming, that the matrix $Q(h)$ is invertible, from the equation $Q(h)\lambda = -F(h)$ we define the initial approximation by the parameter $\lambda^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_N^{(0)}) \in R^{nN}$: $\lambda^{(0)} = -[Q(h)]^{-1}F(h)$.

b) Putting the found $\lambda_r^{(0)}$, $r = \overline{1,N}$ into the right side of the system of integral-differential equations (12) and solving the special Cauchy problem with conditions (13), we find $u^{(0)}[t] = (u_1^{(0)}(t), u_2^{(0)}(t), \ldots, u_N^{(0)}(t))'$.

Step 1. a) Putting the found values $u_r^{(0)}(t)$, $r = \overline{1,N}$ into the right side of (19), from the equation $[Q(h)]\lambda = -F(h) - G(u^{(0)}, h)$ we define $\lambda^{(1)} = (\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots \lambda_N^{(1)})$.

b) Putting the found $\lambda_r^{(1)}$, $r = \overline{1,N}$ into the right side of the system of integral-differential equations (12) and solving the special Cauchy problem with conditions (13), we find $u^{(1)}[t] = (u_1^{(1)}(t), u_2^{(1)}(t), \ldots, u_N^{(1)}(t))'$ and etc.

Continuing the process, at the k-step of the algorithm we find the system of pairs $(\lambda^{(k)}, u^{(k)}[t])$, k =0,1,2, ...

Unknown functions $u[t] = (u_1(t), u_2(t), \ldots, u_N(t))$ are determined from the special Cauchy problem for systems of integral-differential equations (12) with initial conditions (13). In contrast to the Cauchy problem for ordinary differential equations, the special Cauchy problem for systems of integral-differential equations is not always solvable.

Sufficient conditions for unique solvability of the special Cauchy problem (12), (13) for known values of the parameters λ are established by

Theorem 1. Let the partition step $h = T/N$ satisfy the inequality

$$\delta(h) = \beta T h < 1,$$

where $\beta = \max\limits_{(t,s)\in[0,T]\times[0,T]} ||\hat{K}_1(t,s)||$.

Then, the special Cauchy problem (12), (13) has a unique solution.

Sufficient conditions for feasibility and convergence of the proposed algorithm, as well as existence of a unique solution to problem (1), (2) are established by

Theorem 2. Let the following conditions hold:

1) Condition A,

2) matrix $\text{diag}(1 - a_1^2, 1 - a_2^2, \ldots, 1 - a_n^2)$ is invertible,

3) conditions of Theorem 1 hold,

4) matrix $Q(h)$ is invertible and the following inequalities hold:

$$||[Q(h)]^{-1}|| \leq \gamma(h),$$

$$q(h) = \frac{\delta(h)}{1 - \delta(h)} \gamma(h) \max(1, h\,\|C\|)\delta(h) < 1.$$

Then the two-point boundary value problem for systems of integral-differential equations (1), (2) has a unique solution.

Proof of Theorem 1 and Theorem 2 is similar to the scheme of the proof of Theorem 1 and Theorem 3 from [16] and is carried out according to the above algorithm, taking into account the specifics of the system (1).

In [5], necessary and sufficient conditions for unique solvability of a linear boundary value problem for the following systems of differential equations were obtained

$$\frac{dx}{dt} = \int_0^T K(t,s)x(s)ds + f(t), \quad t \in [0,T],$$

$$Bx(0) + Cx(T) = d, \;\; d \in R^n.$$

Theorem ([8; 1216]). For unique solvability of the problem (14), (15) it is necessary and sufficient existence of $h \in (0, h_0] : Nh = T$, where the matrix $Q_{*,*}(h)$ is invertible.

The above theorem implies

Corollary. For unique solvability of the problem (1), (2) it is necessary and sufficient the conditions 1 and 2 of Theorem 2, as well as existence of $h \in (0, h_0] : Nh = T$, where the matrix $Q_{*,*}(h)$ is invertible.

Where h_0 is defined from the condition $q(h_0) = \beta T h_0 < 1$, and the matrix $Q_{*,*}(h)$ is defined in the same way as in [8].

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Инволюциялы функционалды-дифференциалдық теңдеулер үшін шеттік есептің бірмәнді шешілімділігі

Мақалада теңдеудің оң жағының құрамында интеграл таңбасының астында ізделінді функциядан туындысы бар инволютивті түрлендірумен Фредгольм типтес интегралдық-дифференциалдық теңдеулер жүйесі үшін шеттік есеп қарастырылды. Инволютивті түрлендірудің қасиетін пайдаланудан бастапқы есеп оң жақ бөлігінде интеграл таңбасының астында ізделінді функциядан туындысы бар интегралдық-дифференциалдық теңдеу үшін шеттік есепке және интегралдық теңдеудің ядросы $\tilde{K}_2(t,s)$-ке (ізделінді функциядан туындысы бар интегралдық теңдеудің ядросы) байланысты резольвентасы бар деп жорамалдап, интегралдық-дифференциалдық теңдеу оң жақ бөлігінде ізделінді функциядан туындысы жоқ теңдеуге келтіріледі. Алынған шеттік есеп профессор Д.С. Джумабаев ұсынған параметрлеу әдісімен шығарылған. Осы әдістің негізінде есеп жаңа енгізілген функцияларға байланысты арнайы Коши есебін және енгізілген параметрлерге байланысты сызықты алгебралық теңдеулер жүйесі шешуге келтіріледі. Есептің шешімін табу алгоритмі ұсынылған. Белгілі болғандай, жәй дифференциалдық теңдеулер үшін Коши есебіне қарағанда интегралдық-дифференциалдық теңдеулер жүйесі үшін арнайы Коши есебінің барлық уақытта шешімі бар бола бермейді. Профессор Д.С. Джумабаевтың алған нәтижелерін қолдана отырып, арнайы Коши есебінің бірмәнді шешілімділігінің қажетті шарттары тағайындалды.

Кілт сөздер: интегралдық-дифференциалдық теңдеулер жүйесі, шеттік шарттар, параметрлеу әдісі, интегралдық теңдеу, резольвента, инволюция, бірмәнді шешілімділік, арнайы Коши есебі.

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Однозначная разрешимость краевой задачи для функционально-дифференциальных уравнений с инволюцией

В статье рассмотрена краевая задача для систем интегро-дифференциальных уравнений типа Фредгольма с инволютивным преобразованием, содержащая в правой части производную от искомой функций под знаком интеграла. Пользуясь свойством инволютивного преобразования, задача сведена к краевой задаче для систем интегро-дифференциальных уравнений, содержащей в правой части производную от искомой функции под знаком интеграла. Предполагая существование резольвенты интегрального уравнения относительно ядра $\tilde{K}_2(t,s)$ (ядро интегрального уравнения, которое содержит производную от искомой функции) и используя резольвенту, интегро-дифференциальное уравнение сведено к уравнению, не содержащему производную от искомой функции в правой части интегро-дифференциального уравнения. Далее полученная краевая задача решается методом параметризации, предложенным профессором Д. Джумабаевым. На основе данного метода задача сведена к решению специальной задачи Коши относительно введенных новых функций и к решению систем линейных алгебраических уравнений относительно введенных параметров. Предложен алгоритм нахождения решений. Как известно, в отличие от задачи Коши для обыкновенных дифференциальных уравнений, специальная задача Коши для систем интегро-дифференциальных уравнений не всегда разрешима. Авторами были установлены необходимые условия однозначной разрешимости специальной задачи Коши. Используя результаты, полученные профессором Д. Джумабаевым, были найдены необходимые и достаточные условия однозначной разрешимости исходной задачи.

Ключевые слова: система интегро-дифференциальных уравнений, краевые условия, метод параметризации, интегральное уравнение, резольвента, инволюция, однозначная разрешимость, специальная задача Коши.

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Factorization method for solving nonlocal boundary value problems in Banach space

This article deals with the factorization and solution of nonlocal boundary value problems in a Banach space of the abstract form

$$B_1 u = \mathcal{A}u - S\Phi(u) - G\Psi(A_0 u) = f, \quad u \in D(B_1),$$

where $\mathcal{A}, A_0$ are linear abstract operators, S, G are vectors of functions, Φ, Ψ are vectors of linear bounded functionals, and u, f are functions. It is shown that the operator B_1 under certain conditions can be factorized into a product of two simpler lower order operators as $B_1 = BB_0$. Then the solvability and the unique solution of the equation $B_1 u = f$ easily follow from the solvability conditions and the unique solutions of the equations $Bv = f$ and $B_0 u = v$. The universal technique proposed here is essentially different from other factorization methods in the respect that it involves decomposition of both the equation and boundary conditions and delivers the solution in closed form. The method is implemented to solve ordinary and partial Fredholm integro-differential equations.

Keywords: boundary value problems, nonlocal conditions, factorization, linear operators, integro-differential equations, closed-form solutions.

Introduction

Let X be a complex Banach space and X^* the adjoint space of X, i.e., the set of all complex-valued linear bounded functionals ϕ on X. Let $\mathcal{A}$, $A_0 : X \to X$ be linear operators with boundary conditions incorporated, $\Phi = \mathrm{col}(\phi_1, \phi_2, ..., \phi_m)$, $\Psi = \mathrm{col}(\psi_1, \psi_2, ..., \psi_m)$ vectors of linear bounded functionals ϕ_i, ψ_i, $i = 1, 2, ..., m$, and $S(s_1, s_2, ..., s_m)$, $G = (g_1, g_2, ..., g_m)$ vectors of functions $s_i, g_i \in X$, $i = 1, 2, ..., m$. Let the operator $B_1 : X \to X$ be defined by

$$B_1 = \mathcal{A} - S\Phi - G\Psi(A_0),$$

and consider the boundary value problem

$$B_1 u = \mathcal{A}u - S\Phi(u) - G\Psi(A_0 u) = f, \quad u \in D(B_1),$$

where $f \in X$ is a given forcing function and u is the unknown function.

The primary objective of the paper is to establish factorization conditions under which this problem can be decomposed into two simpler lower order boundary value problems and derive the unique solution in closed form. The second goal is to implement this procedure to solve boundary value problems for ordinary and partial Fredholm integro-differential equations with nonlocal boundary conditions. In this case B_1 is an integro-differential operator, $\mathcal{A}$ is a differential operator of order n with nonlocal boundary conditions incorporated, and the functionals ϕ_i, ψ_i, $i = 1, ..., m$ are integrals with constant limits.

Integro-differential equations model many situations in biology, physics, economics, engineering and applied mathematics. Boundary value problems involving an integro-differential equation and nonlocal boundary conditions are very difficult to solve analytically and therefore very often numerical methods are employed. Factorization methods, where they can be applied, can reduce the problem to simpler lower order problems which can be solved and thus construct the solution of the initial complex problem [1–20].

The novelty of the factorization method presented here differs from other factorization methods in the literature in the respect that it involves decomposition of both the equation and boundary conditions and

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delivers the solution in closed form. The technique is new development in Banach spaces and an extension of a procedure used successfully by the authors to solve various other boundary value problems [21–24] and [25–27].

The method is simple to program to any Computer Algebra System.

The rest of the paper is organized as follows. In Section 1 some preliminary results are quoted. In Section 2 the solvability, uniqueness and decomposition conditions are established and the factorization solution method is explicated. In Section 3 two example problems are solved to show the implementation and efficiency of the method.

Preliminaries

Let X, Y be complex Banach space and $A : X \to Y$ a linear operator with $D(A)$ and $R(A)$ denoting its domain and range, respectively. We recall that A is said to be *injective* (or *uniquely solvable*) if for all $u_1, u_2 \in D(A)$ such that $Au_1 = Au_2$, follows that $u_1 = u_2$; alternatively, A is injective if and only if $\ker A = \{0\}$. The operator A is called *surjective* (or *everywhere solvable*) if $R(A) = Y$. The operator A is called *bijective* if A is both injective and surjective. Lastly, A is said to be *correct* if A is bijective and its inverse A^{-1} is bounded on Y. The problem $Au = f$ is called *correct* if the operator A is correct.

An operator $B_1 : X \to X$ is said to be factorable if there exist two operators $B_0, B : X \to X$ such that B_1 can be written as a product $B_1 = BB_0$. In this case, BB_0 is *a factorization (decomposition)* of B_1.

Throughout the paper, we will use the notation $\Phi(g)$ to denote the $m \times m$ matrix whose i, j-th entry $\phi_i(g_j)$ is the value of the functional ϕ_i on element g_j, where $i, j = 1, ..., m$. Note that $\Phi(gC) = \Phi(g)C$, where C is a $m \times k$ constant matrix. We will also denote by $\mathbf{c}$ the column vector $\mathbf{c} = \mathrm{col}(c_1, ..., c_m)$ and by $0_m, I_m$ the zero and identity $m \times m$ matrices, respectively.

We recall Corollary 3.11 from [25] which will need to prove the theorems below.

Corollary 1. Let A be a correct operator on a Banach space X and the components of the vectors $G = (g_1, ..., g_m)$ and $F = \mathrm{col}(F_1, ...F_m)$ are arbitrary elements of X and X^*, respectively. Then the operator $B : X \to X$ defined by

$$Bu = Au - GF(Au) = f, \quad D(B) = D(A), \quad f \in X \tag{1}$$

is correct if and only if

$$\det\, L = \det[I_m - F(G)] \neq 0. \tag{2}$$

If B is correct, then the unique solution of (1) for every $f \in X$ is given by

$$u = B^{-1}f = A^{-1}f + A^{-1}G[I_m - F(G)]^{-1}F(f). \tag{3}$$

The following theorem is the generalization of Theorem 1 in [28] and here we prove it without requiring the correctness of the operator A and the linear independence of the components of the functional vector $\Psi = \mathrm{col}(\psi_1, ..., \psi_m)$.

Theorem 2. Let X, Y and Z be Banach spaces and $A : X \to Y$ be a linear injective operator with $D(A) \subset Z \subseteq X$. Further let the vector $G = (g_1, ..., g_m) \in Y^m$ and the column vector $\Psi = \mathrm{col}(\psi_1, ..., \psi_m)$, where $\psi_1, ..., \psi_m \in Z^*$. Then:

(i) The operator $B : X \to Y$ defined by

$$Bu = Au - G\Psi(u) = f, \quad D(B) = D(A), \quad f \in X, \tag{4}$$

is injective if and only if

$$\det W = \det[I_m - \Psi(A^{-1}G)] \neq 0. \tag{5}$$

(ii) If B is injective and A is bijective, then B is bijective and for any $f \in Y$, the unique solution of (4) is given by

$$u = B^{-1}f = A^{-1}f + A^{-1}GW^{-1}\Psi(A^{-1}f). \tag{6}$$

(iii) If B is injective and A is correct, then B is correct.

Proof. (i) The sufficient injectiveness condition of the operator B is proved as in [28].

Now, we prove the converse statement "if the operator B is injective, then $\det W \neq 0$" or equivalently "if $\det W = 0$, then the operator B is not injective". Suppose $\det W = 0$. Then there exists a nonzero vector $\mathbf{c} = \mathrm{col}(c_1, ..., c_m)$ such that $W\mathbf{c} = \mathbf{0}$. Consider the element $u_0 = A^{-1}G\mathbf{c}$. This element is nonzero, because otherwise we would have

$$W\mathbf{c} = [I_m - \Psi(A^{-1}G)]\mathbf{c} = \mathbf{c} - \Psi(A^{-1}G\mathbf{c}) = \mathbf{c} \neq \mathbf{0},$$

which is a contradiction. Further,

$$Bu_0 = Au_0 - G\Psi(u_0) = G\mathbf{c} - G\Psi(A^{-1}G)\mathbf{c} = G[I_m - G\Psi(A^{-1}G)]\mathbf{c} = GW\mathbf{c} = 0,$$

which means that $u_0 \in \ker B$ and thus B is not injective.

(ii) Let B is injective and A is bijective. Then (5) holds ($\det W \neq 0$) and for any $f \in Y$ from (4) follows that

$$u = A^{-1}G\Psi(u) + A^{-1}f, \tag{7}$$

and

$$\Psi(u) = \Psi(A^{-1}G)\Psi(u) + \Psi(A^{-1}f),$$

$$[I_m - \Psi(A^{-1}G)]\Psi(u) = \Psi(A^{-1}f),$$

$$\Psi(u) = [I_m - \Psi(A^{-1}G)]^{-1}\Psi(A^{-1}f). \tag{8}$$

Substituting (8) into (7), we obtain the unique solution (6). Since this solution is given for arbitrary $f \in Y$, then $R(B) = Y$, i.e., B is surjective. Hence B is a bijective operator.

(iii) If B is injective and A is correct, then from (6) follows that B^{-1} is bounded since A^{-1} and Ψ are bounded. Hence B is correct. □

Main results

Theorem 3. Let X and Z_0, Z be Banach spaces, $Z_0, Z \subseteq X$, the vectors $G_0 = (g_{10}, ..., g_{m0})$, $G = (g_1, ..., g_m)$, $S = (s_1, ..., s_m) \in X^m$, the components of the column vectors $\Phi = col(\phi_1, ..., \phi_m)$ and $\Psi = col(\psi_1, ..., \psi_m)$ belong to Z_0^* and Z^*, respectively, and the operators $B_0, B, B_1 : X \to X$ be defined by

$$B_0u = A_0u - G_0\Phi(u) = f, \qquad D(B_0) = D(A_0) \subset Z_0, \tag{9}$$

$$Bu = Au - G\Psi(u) = f, \qquad D(B) = D(A) \subset Z, \tag{10}$$

$$B_1u = AA_0u - S\Phi(u) - G\Psi(A_0u) = f, \quad D(B_1) = D(AA_0), \tag{11}$$

where A_0 and A are linear correct operators on X and $G_0 \in D(A)^m$. Then the following statements are satisfied:

(i) If

$$S \in R(B)^m \quad \text{and} \quad S = BG_0 = AG_0 - G\Psi(G_0), \tag{12}$$

then the operator B_1 can be factorized as $B_1 = BB_0$.

(ii) If (12) holds, then the operator $B_1 = BB_0$ is correct if and only if the operators B_0 and B are correct which means that

$$\det L_0 = \det[I_m - \Phi(A_0^{-1}G_0)] \neq 0 \quad \text{and} \quad \det L = \det[I_m - \Psi(A^{-1}G)] \neq 0, \tag{13}$$

and the unique solution of (11) is

$$u = B_1^{-1}f = A_0^{-1}A^{-1}f + \left[A_0^{-1}A^{-1}G + A_0^{-1}G_0L_0^{-1}\Phi(A_0^{-1}A^{-1}G)\right]L^{-1}\Psi(A^{-1}f)$$

$$+A_0^{-1}G_0L_0^{-1}\Phi(A_0^{-1}A^{-1}f). \tag{14}$$

Proof. (i) Taking into account that $G_0 \in D(A)^m$ and (9)-(11) we get

$$D(BB_0) = \{u \in D(B_0) : \ B_0u \in D(B)\}$$

$$= \{u \in D(A_0) : \ A_0u - G_0\Phi(u) \in D(A)\}$$

$$= \{u \in D(A_0) : \ A_0u \in D(A)\} = D(AA_0) = D(B_1).$$

So $D(B_1) = D(BB_0)$. Let $y = B_0u$. Then for each $u \in D(AA_0)$ since (10) and (9) we have

$$BB_0u = By = Ay - G\Psi(y)$$

$$= A[A_0u - G_0\Phi(u)] - G\Psi\left(A_0u - G_0\Phi(u)\right)$$

$$= AA_0u - AG_0\Phi(u) - G\Psi(A_0u) + G\Psi(G_0)\Phi(u)$$

$$= AA_0u - [AG_0 - G\Psi(G_0)]\Phi(u) - G\Psi(A_0u)$$
$$= AA_0u - BG_0\Phi(u) - G\Psi(A_0u), \tag{15}$$

where the relation $BG_0 = AG_0 - G\Psi(G_0)$ follows from (10) if instead of u we take G_0. By comparing (15) with (11) it is easy to verify that $B_1u = BB_0u$ for each $u \in D(AA_0)$ if a vector S satisfies (12).

(ii) Let the operator B_1 be defined by (11), where $S = BG_0$. Then Equation (11) can be equivalently presented in the matrix form:

$$B_1u = AA_0u - (BG_0, G)\begin{pmatrix}\Phi(A_0^{-1}A^{-1}AA_0u)\\ \Psi(A^{-1}AA_0u)\end{pmatrix} = f$$

or

$$B_1u = \mathcal{A}u - \mathcal{G}\mathcal{F}(\mathcal{A}u) = f, \quad D(B_1) = D(\mathcal{A}),$$

where $\mathcal{A} = AA_0$, $\mathcal{G} = (BG_0, G)$, $\mathcal{F} = \mathrm{col}(\hat\Phi, \hat\Psi)$, and

$$\mathcal{F}(v) = \begin{pmatrix}\hat\Phi(v)\\ \hat\Psi(v)\end{pmatrix} = \begin{pmatrix}\Phi(A_0^{-1}A^{-1}v)\\ \Psi(A^{-1}v)\end{pmatrix}.$$

Notice that the operator $\mathcal{A} = AA_0$ is correct, because of A and A_0 are correct, and that the vector $\mathcal{F}$ is bounded, since the vector $\hat\Phi$ (resp.$\hat\Psi$) is bounded as a superposition of a bounded functional Φ (resp.Ψ) and a bounded operator $A_0^{-1}A^{-1}$ (resp. A^{-1}). Then we apply Corollary 1. In accordance to (2), (3), the operator B_1 is correct if and only if

$$\det L_1 = \det[I_{2m} - \mathcal{F}(\mathcal{G})] = \det\left[\begin{pmatrix}I_m & 0_m\\ 0_m & I_m\end{pmatrix} - \begin{pmatrix}\hat\Phi(BG_0) & \hat\Phi(G)\\ \hat\Psi(BG_0) & \hat\Psi(G)\end{pmatrix}\right]$$
$$= \det\begin{pmatrix}I_m - \hat\Phi(AG_0 - G\Psi(G_0)) & -\hat\Phi(G)\\ -\hat\Psi(AG_0 - G\Psi(G_0)) & I_m - \hat\Psi(G)\end{pmatrix}$$
$$= \det\begin{pmatrix}I_m - \Phi(A_0^{-1}G_0 - A_0^{-1}A^{-1}G\Psi(G_0)) & -\Phi(A_0^{-1}A^{-1}G)\\ -\Psi(G_0 - A^{-1}G\Psi(G_0)) & I_m - \Psi(A^{-1}G)\end{pmatrix}$$
$$= \det\begin{pmatrix}I_m - \Phi(A_0^{-1}G_0) + \Phi(A_0^{-1}A^{-1}G)\Psi(G_0) & -\Phi(A_0^{-1}A^{-1}G)\\ -\Psi(G_0) + \Psi(A^{-1}G)\Psi(G_0) & I_m - \Psi(A^{-1}G)\end{pmatrix} \neq 0. \tag{16}$$

Multiplying by $\Psi(G_0)$ from the left the second column of the matrix in (16) and then adding to the first column, we get

$$\det L_1 = \det\begin{pmatrix}I_m - \Phi(A_0^{-1}G_0) & -\Phi(A_0^{-1}A^{-1}G)\\ 0_m & I_m - \Psi(A^{-1}G)\end{pmatrix}$$
$$= \det[I_m - \Phi(A_0^{-1}G_0)]\det[I_m - \Psi(A^{-1}G)] = \det L_0 \det L \neq 0.$$

So we proved that the operator B_1 is correct if and only if (13) is fulfilled. From (13), by Theorem 2, follows that the operators B and B_0 are correct.

Let now $u \in D(AA_0)$ and $B_1u = BB_0u = f$. Then, by Theorem 2 (ii), since B, B_0 are correct operators, we obtain

$$B_0u = B^{-1}f = A^{-1}f + A^{-1}GL^{-1}\Psi(A^{-1}f),$$
$$u = B_0^{-1}\left(A^{-1}f + A^{-1}GL^{-1}\Psi(A^{-1}f)\right).$$

Denote $g = A^{-1}f + A^{-1}GL^{-1}\Psi(A^{-1}f)$. By using Theorem 2 (ii) again, with A_0, G_0, Φ, L_0, g in place of A, G, Ψ, L, f respectively, we get

$$u = B_0^{-1}g = A_0^{-1}g + A_0^{-1}G_0L_0^{-1}\Phi(A_0^{-1}g) = A_0^{-1}\big(A^{-1}f + A^{-1}GL^{-1}\Psi(A^{-1}f)\big)$$
$$+A_0^{-1}G_0L_0^{-1}\Phi\left(A_0^{-1}(A^{-1}f + A^{-1}GL^{-1}\Psi(A^{-1}f))\right) = A_0^{-1}A^{-1}f + A_0^{-1}A^{-1}GL^{-1}\Psi(A^{-1}f)$$
$$+A_0^{-1}G_0L_0^{-1}\big[\Phi(A_0^{-1}A^{-1}f) + \Phi(A_0^{-1}A^{-1}G)L^{-1}\Psi(A^{-1}f)\big],$$

which implies (14). The theorem is proved. □

The next theorem is useful for applications and is proved by using Theorem 3.

Theorem 4. Let the spaces X, Z_0, Z, the vectors S, G, Φ, Ψ be defined as in Theorem 3 and the operator $B_1 : X \to X$ by

$$B_1 u = \mathcal{A}u - S\Phi(u) - G\Psi(A_0 u) = f, \quad u \in D(B_1), \tag{17}$$

where A_0 is a correct m-order differential operator with $D(A_0) \subset Z_0$ and $\mathcal{A}$ is a n-order differential operator, $m < n$. Then the next statements are fulfilled:

(i) If there exists an $n - m$ order differential bijective operator $A : X \to X$ such that

$$\mathcal{A} = AA_0, \quad D(B_1) = D(AA_0), \quad D(A) \subset Z \subseteq X, \tag{18}$$

$$\det L = \det[I_m - \Psi(A^{-1}G)] \neq 0, \tag{19}$$

then the operator B_1 is factorized as $B_1 = BB_0$, where B_0, B are defined by (9), (10),

$$G_0 = A^{-1}S + A^{-1}GL^{-1}\Psi(A^{-1}S), \tag{20}$$

the operator A and vectors G, Ψ are determined from (18) and (17), respectively, and the operator A_0 and a vector Φ from (17).

(ii) If in addition to (i) A is correct, then the operator $B_1 = BB_0$ is correct if and only if

$$\det L_0 = \det[I_m - \Phi(A_0^{-1}G_0)] \neq 0 \tag{21}$$

and the unique solution of (17), (18) is given by

$$u = B_0^{-1}B^{-1}f = B_0^{-1}v = A_0^{-1}v + A_0^{-1}G_0L_0^{-1}\Phi(A_0^{-1}v), \tag{22}$$

where

$$v = A^{-1}f + A^{-1}GL^{-1}\Psi(A^{-1}f). \tag{23}$$

Proof. (i) Suppose that there exist the operators A, B, B_0, defined in (i). Acting by the operator B on the vector G_0, defined by (20), we get.

$$\begin{aligned} BG_0 &= AG_0 - G\Psi(G_0) \\ &= A\left(A^{-1}S + A^{-1}GL^{-1}\Psi(A^{-1}S)\right) - G\Psi\left(A^{-1}S + A^{-1}GL^{-1}\Psi(A^{-1}S)\right) \\ &= S + GL^{-1}\Psi(A^{-1}S) - G\Psi(A^{-1}S) - G\Psi(A^{-1}G)L^{-1}\Psi(A^{-1}S) \\ &= S + G[I_m - \Psi(A^{-1}G)]L^{-1}\Psi(A^{-1}S) - G\Psi(A^{-1}S) = S. \end{aligned}$$

So $BG_0 = S$. From (17) for $\mathcal{A} = AA_0$ and $BG_0 = S$ we get

$$B_1 u = AA_0 u - BG_0\Phi(u) - G\Psi(A_0 u) = f, \quad u \in D(AA_0). \tag{24}$$

Denote $y = A_0 u$. Then from (24) for any $u \in D(AA_0)$ follows that

$$B_1 u = Ay - G\Psi(y) - BG_0\Phi(u) = By - BG_0\Phi(u) = B\left(A_0 u - G_0\Phi(u)\right) = BB_0 u.$$

In Theorem 3 (i) we proved that $D(BB_0) = D(AA_0) = D(B_1)$. Consequently, B_1 is factorized in $B_1 = BB_0$.

(ii) Let A be a correct operator. Then by Theorem 2, since (19), (21), the operators B, B_0 are correct too. Remind that for G_0, defined by (20), we proved in (i) that $BG_0 = S$. Then by Theorem 3 (i), (iii), we have the factorization $B_1 = BB_0$ and B_1 is correct if and only if $\det L \neq 0$ and $\det L_0 \neq 0$. But by assumption $\det L \neq 0$. Thus B_1 is correct if and only if (21) holds. Let $BB_0 u = f$ for any $f \in X$. Then because of the operators B, B_0 are correct, we obtain

$$B_0 u = B^{-1}f = A^{-1}f + A^{-1}GL^{-1}\Psi(A^{-1}f).$$

From the above, denoting $v = A^{-1}f + A^{-1}GL^{-1}\Psi(A^{-1}f)$, follows that

$$B_0 u = v, \quad u = B_0^{-1}v = A_0^{-1}v + A_0^{-1}G_0L_0^{-1}\Phi(A_0^{-1}v),$$

which give (23) and (22). So the theorem is proved. □

Remark 5. Usually in applications X is the space $C[a, b]$ or $L_p(a, b)$, $p = 1, 2, ...,$ and Z_0, Z are the spaces $C^k[a, b]$ or $W_p^k(a, b)$, $k = 1, ..., n$, respectively. Problem (17) can be solved by factorization method if it is possible to determine from (17) the vectors S, G, Φ, Ψ and the operators A_0, A such that

$$\mathcal{A} = AA_0, \quad D(B_1) = D(AA_0), \quad D(A) \subset Z, \quad D(A_0) \subset Z_0, \quad \det L \neq 0, \quad \det L_0 \neq 0.$$

If the above conditions are fulfilled, then a unique solution to (17) can be found by (22), (23), where G_0 is given by (20).

Illustrative Examples

To explain the implementation of the factorization method and to show its efficiency, we solve two example problems.

Example 1. Let us find the solution of the nonlocal boundary value problem

$$u''(t)-(t+1)\int_0^1 (t-1)u(t)dt - t^2\int_0^1 t^3u'(t)dt = 2-3t, \quad 0<t<1,$$

$$u(0)+u(1)=0, \quad u'(0)-4u'(1)=0. \tag{25}$$

The operator $B_1 : C[0,1]\to C[0,1]$ corresponding to the problem is correct. The unique solution to problem (25) is given by the formula

$$u(t) = -\frac{5(1204t^4+402256t^3-811850t^2+549488t-70549)}{4037236}. \tag{26}$$

Proof. First we need to find the operators A, A_0 and check the condition $D(B_1)=D(AA_0)$. If we compare equation (25) with Problem (17), (18), it is natural to take $X=C[0,1]$, $m=1$, $I_m=1$,

$$\mathcal{A}u = AA_0u = u''(t), \tag{27}$$

$$D(B_1)=\{u(t)\in C^2[0,1] : u(0)+u(1)=0, \quad u'(0)-4u'(1)=0\},$$

$$A_0u(t)=u'(t), \quad D(A_0)=\{u(t)\in C^1[0,1] : u(0)=-u(1)\},$$

$$\Phi(u)=\int_0^1 (t-1)u(t)dt, \quad \Psi(A_0u)=\int_0^1 t^3u'(t)dt, \tag{28}$$

$S=t+1, G=t^2$. Let us denote $A_0u(t)=u'(t)=y(t)=y$. Then from (27) we have $y\in D(A)$, $AA_0u=(u'(t))'=y'(t)=Ay(t), y(0)-4y(1)=0$. So we proved that

$$Ay=y'(t), \quad D(A)=\{y(t)\in C^1[0,1] : y(0)-4y(1)=0\}.$$

Further by definition we find

$$D(AA_0)=\{u(t)\in D(A_0) : A_0u(t)\in D(A)\}$$

$$=\{u(t)\in C^1[0,1] :\ u(0)=-u(1),\ u'(t)\in C^1[0,1],\ u'(0)-4u'(1)=0\}$$

$$=\{u(t)\in C^2[0,1] : u(0)+u(1)=0, \quad u'(0)-4u'(1)=0\}=D(B_1).$$

So $D(B_1)=D(AA_0)$. It is easy to verify that the operators A, A_0 are correct on $C[0,1]$ and that for every $f(t)\in C[0,1]$ the following formulae hold true

$$A^{-1}f(t)=\int_0^t f(x)dx-\frac{4}{3}\int_0^1 f(x)dx, \tag{29}$$

$$A_0^{-1}f(t)=\int_0^t f(x)dx-\frac{1}{2}\int_0^1 f(x)dx. \tag{30}$$

From (28) we have

$$\Phi(f)=\int_0^1 (x-1)f(x)dx, \quad \Psi(f)=\int_0^1 x^3f(x)dx. \tag{31}$$

Then $|\Phi(f)|\leq \frac{1}{2}||f(x)||_C$, $\quad |\Psi(f)|\leq \frac{1}{4}||f(x)||_C$, that is $\Phi,\Psi\in C^*[0,1]$ and $Z_0=Z=C[0,1]$. Using (29), (31) and (19), we obtain

$$A^{-1}G=\int_0^t x^2dx-\frac{4}{3}\int_0^1 x^2dx=\frac{t^3}{3}-\frac{4}{9},$$

$$\Psi(A^{-1}G)=\int_0^1 x^3\left(\frac{x^3}{3}-\frac{4}{9}\right)dx=-\frac{4}{63},$$

$$\det L=\det[I_m-\Psi(A^{-1}G)]=1+4/63=67/63, \quad L^{-1}=63/67.$$

So (19) is fulfilled. Further using (20), (23), (29), (31) for $S = t+1, G = t^2$ and $f(t) = 2-3t$ we find

$$A^{-1}f = -\frac{3t^2}{2} + 2t - \frac{2}{3}, \quad \Psi(A^{-1}f) = -\frac{1}{60},$$

$$v = A^{-1}f + A^{-1}GL^{-1}\Psi(A^{-1}f) = -\frac{7t^3 + 2010t^2 - 2680t + 884}{1340}, \tag{32}$$

$$A^{-1}S = \int_0^t (x+1)dx - \frac{4}{3}\int_0^1 (x+1)dx = \frac{t^2}{2} + t - 2,$$

$$\Psi(A^{-1}S) = \int_0^1 x^3\left(\frac{x^2}{2} + x - 2\right)dx = -\frac{13}{63},$$

$$G_0 = A^{-1}S + A^{-1}GL^{-1}\Psi(A^{-1}S) = -\frac{273t^3 - 2010t^2 - 4020t + 7676}{4020}.$$

Taking into account (30), (31), we obtain

$$A_0^{-1}G_0 = -\frac{546t^4 - 5360t^3 - 16080t^2 + 61408t - 20257}{32160}, \quad \Phi(A_0^{-1}G_0) = -\frac{44509}{964800}.$$

Since

$$\det L_0 = \det[I_m - \Phi(A_0^{-1}G_0)] = \frac{1009309}{964800} \neq 0, \quad \text{then} \quad L_0^{-1} = \frac{964800}{1009309},$$

and by Theorem 4 (ii), Problem (25) is correct. By (30)-(32) we calculate

$$A_0^{-1}v = -\frac{14t^4 + 5360t^3 - 10720t^2 + 7072t - 863}{10720}, \quad \Phi(A_0^{-1}v) = \frac{1223}{107200}.$$

Substituting these values into (22), i.e.,

$$u = A_0^{-1}v + A_0^{-1}G_0L_0^{-1}\Phi(A_0^{-1}v),$$

we obtain the unique solution to (25), which is given by (26).

Example 2. Let $\overline{\Pi} = \{(t,s) \in \mathbb{R}^2 : 0 \le t, s \le 1\}$, $u = u(t,s), u'_t, u''_{ts} \in C(\overline{\Pi})$. The operator $B_1 : C(\overline{\Pi}) \to C(\overline{\Pi})$ corresponding to the problem:

$$u''_{ts}(t,s) - (2t-s)\int_0^1\int_0^1 u(t,s)dtds - (t+s)\int_0^1\int_0^1 tsu'_t(t,s)dtds \tag{33}$$

$$= -\frac{213s + 149t - 600}{220},$$

$$u(0,s) = s\int_0^1\int_0^1 t^2u(t,s)dtds,$$

$$u'_t(t,0) = (2t-1)\int_0^1\int_0^1 (s+3)u'_t(t,s)dtds$$

is correct. The unique solution to Problem (33) is given by the formula

$$u(t,s) = \frac{6s(25t+1) + 275t(t-1)}{55}. \tag{34}$$

Proof. First we need to find the operators A, A_0 and check the condition $D(B_1) = D(AA_0)$. If we compare (33) with Problem (17), (18), it is natural to take $X = C(\overline{\Pi})$, $m = 1$, $I_m = 1$,

$$AA_0x = u''_{ts}(t,s), \tag{35}$$

$$D(B_1) = \{u(t,s) \in C(\overline{\Pi}),\ u'_t, u''_{ts} \in C(\overline{\Pi}), \quad u(0,s) = s\int_0^1\int_0^1 t^2u(t,s)dtds,$$

$$u'_t(t,0) = (2t-1)\int_0^1\int_0^1 (s+3)u'_t(t,s)dtds\}, \tag{36}$$

$$A_0u(t,s) = u'_t(t,s), \tag{37}$$

$$D(A_0) = \{u(t,s) \in C(\overline{\Pi}) : u'_t(t,s) \in C(\overline{\Pi}), \quad u(0,s) = s\int_0^1\int_0^1 t^2u(t,s)dtds\},$$

$$\Phi(u) = \int_0^1\int_0^1 u(t,s)dtds, \quad \Psi(A_0u) = \int_0^1\int_0^1 tsu'_t(t,s)dtds, \tag{38}$$

$S = 2t-s$, $G = t+s$, $f = -(213s+149t-600)/220$. In (37), denote $A_0u(t,s) = u'_t(t,s) = y(t,s) = y$. Then from (35), (36) we have $y \in D(A)$, $AA_0u = (u'_t(t,s))'_s = y'_s(t,s) = Ay(t,s)$ and $y(t,0) = (2t-1)\int_0^1\int_0^1(s+3)y(t,s)dtds$. So we proved that

$$Ay = y'_s(t,s), \quad D(A) = \{y(t,s) \in C(\overline{\Pi}) : y'_s \in C(\overline{\Pi}),\ y(t,0) = (2t-1)\int_0^1\int_0^1 (s+3)y(t,s)dtds\}.$$

Then by definition

$$D(AA_0) = \{u(t,s) \in D(A_0) : A_0u(t,s) \in D(A)\}$$

$$= \{u(t,s) \in C(\overline{\Pi}) :\ u'_t \in C(\overline{\Pi}), \quad u(0,s) = s\int_0^1\int_0^1 t^2u(t,s)dtds,$$

$$u'_t(t,0) = (2t-1)\int_0^1\int_0^1 (s+3)u'_t(t,s)dtds, \quad u''_{ts}(t,s) \in C(\overline{\Pi})\}$$

$$= \{u(t,s) \in C(\overline{\Pi}),\ u'_t, u''_{ts} \in C(\overline{\Pi}), \quad u(0,s) = s\int_0^1\int_0^1 t^2u(t,s)dtds,$$

$$u'_t(t,0) = (2t-1)\int_0^1\int_0^1 (s+3)u'_t(t,s)dtds\} = D(B_1).$$

Thus $D(B_1) = D(AA_0)$. It is easy to verify that the operators A, A_0 are correct on $C(\overline{\Pi})$ and for every $f(t,s) \in C(\overline{\Pi})$ hold true

$$A^{-1}f(t,s) = \int_0^s f(t,x)dx + (2t-1)\int_0^1\int_0^1\int_0^s (s+3)f(t,x)dxdtds, \tag{39}$$

$$A_0^{-1}f(t,s) = \int_0^t f(z,s)dz + \frac{6s}{5}\int_0^1\int_0^1\int_0^t t^2f(z,s)dzdtds. \tag{40}$$

From (38) we get

$$\Phi(f) = \int_0^1\int_0^1 f(t,s)dtds, \quad \Psi(f) = \int_0^1\int_0^1 tsf(t,s)dtds. \tag{41}$$

Then $\Phi, \Psi \in C^*(\overline{\Pi})$ and $Z_0 = Z = C(\overline{\Pi})$. Using (39), (41) and (19) we obtain

$$A^{-1}G = \frac{s^2}{2} + st + \frac{37(2t-1)}{24}, \quad \Psi(A^{-1}G) = \frac{29}{96}, \quad L = 1 - \Psi(A^{-1}G) = 67/96, \quad L^{-1} = 96/67.$$

So (19) is fulfilled. Further, using (39), (41), (23), (20) for $S = 2t-s, G = t+s$ and $f(t) = -(213s + 149t - -600)/220$ we find

$$A^{-1}f = -\frac{2556s^2 + 24s(149t-600) - 19927(2t-1)}{5280}, \quad \Psi(A^{-1}f) = \frac{2675}{4224},$$

$$v = A^{-1}f + A^{-1}GL^{-1}\Psi(A^{-1}f) = -\frac{336s^2 - 6s(424t+5025) - 57187(2t-1)}{11055}, \tag{42}$$

$$A^{-1}S = -\frac{s^2}{2} + 2st + \frac{29(2t-1)}{24}, \quad \Psi(A^{-1}S) = \frac{25}{96},$$

$$G_0 = A^{-1}S + A^{-1}GL^{-1}\Psi(A^{-1}S) = -\frac{42s^2 - 318st - 239(2t-1)}{134}.$$

Taking into account (40), (41) we obtain

$$A_0^{-1}G_0 = -\frac{2100s^2t - 3s(2650t^2+9) - 11950t(t-1)}{6700}, \quad \Phi(A_0^{-1}G_0) = -\frac{6019}{40200}.$$

Since

$$\det L_0 = \det[I_m - \Phi(A_0^{-1}G_0)] = \frac{46219}{40200} \neq 0,$$

then $L_0^{-1} = \frac{40200}{46219}$, and hence by Theorem 4 (ii), problem (33) is correct. By (40)-(42) we calculate

$$A_0^{-1}v = -\frac{8400s^2t - 6s(5300t^2 + 125625t + 5043) - 1429675t(t-1)}{276375},$$

$$\Phi(A_0^{-1}v) = -\frac{92438}{829125}.$$

Substituting the above values into (22), we obtain, by Theorem 4 (ii), the unique solution of (33)

$$u = A_0^{-1}v + A_0^{-1}G_0L_0^{-1}\Phi(A_0^{-1}v) = \frac{6s(25t+1) + 275t(t-1)}{55},$$

which is (34).

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Банах кеңістігінде локальді емес шекаралық есептерді шешуге арналған факторизация әдісі

Мақала банах кеңістігінде абстрактілі операторлары бар

$$B_1 u = \mathcal{A}u - S\Phi(u) - G\Psi(A_0 u) = f, \quad u \in D(B_1),$$

түріндегі локалды емес шектік есептерді факторизациялау және шешуге арналған, мұндағы $\mathcal{A}, A_0$ сызықтық дерексіз операторлар, S, G функция векторлары, Φ, Φ сызықтық шектеулі функционалды векторлар және u, f функциялар. B_1 операторы белгілі бір жағдайларда $B_1 = Bb_0$ кіші екі қарапайым оператордың көбейтіндісіне факторлануы мүмкін екендігі көрсетілген. Содан кейін $B_1 u = f$ теңдеуінің шешімі мен жалғыз шешімі $Bv = f$ және $b_0 u = v$ теңдеулер шешімдерінің шешімділігі мен бірегейлігі шарттарынан оңай туындайды. Ұсынылған әмбебап әдіс басқа факторизация әдістерінен айтарлықтай ерекшеленеді, өйткені оған теңдеу мен шекаралық шарттардың факторизациясы кіреді және шешімді жабық түрде ұсынады. Бұл әдіс Фредгольмның қарапайым және жартылай интегро-дифференциалдық теңдеулерін шешуге арналған.

Кілт сөздер: шекаралық есептер, жергілікті емес жағдайлар, факторизация, сызықтық операторлар, интегро-дифференциалдық теңдеулер, жабық түрдегі шешімдер.

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Метод факторизации для решения нелокальных краевых задач в банаховом пространстве

Статья посвящена факторизации и решению нелокальных краевых задач с операторами абстрактного вида

$$B_1 u = \mathcal{A}u - S\Phi(u) - G\Psi(A_0 u) = f, \quad u \in D(B_1),$$

в банаховом пространстве, где $\mathcal{A}, A_0$ — линейные абстрактные операторы; S, G — векторы функций; Φ, Ψ — векторы линейных ограниченных функционалов; а u, f — функции. Показано, что оператор B_1 при определенных условиях может быть факторизован в произведение двух более простых операторов меньшего порядка $B_1 = BB_0$. Тогда разрешимость и единственное решение уравнения $B_1 u = f$ легко следует из условий разрешимости и единственности решений уравнений $Bv = f$ и $B_0 u = v$. Предлагаемый универсальный метод существенно отличается от других методов факторизации, поскольку он включает факторизацию уравнения и граничных условий и предоставляет решение в замкнутой форме. Метод разработан для решения обыкновенных и частных интегро-дифференциальных уравнений Фредгольма.

Ключевые слова: краевые задачи, нелокальные условия, факторизация, линейные операторы, интегро-дифференциальные уравнения, решения в замкнутой форме.

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Summation of some infinite series by the methods of Hypergeometric functions and partial fractions

In this article, we obtain the summations of some infinite series by partial fraction method and by using certain hypergeometric summation theorems of positive and negative unit arguments, Riemann Zeta functions, polygamma functions, lower case beta functions of one-variable and other associated functions. We also obtain some hypergeometric summation theorems for:

$${}_8F_7\left[\ \tfrac{9}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},3,3,1;\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{1}{2},2,2;1\ \right],\ {}_5F_4\left[\ \tfrac{5}{3},\tfrac{4}{3},\tfrac{4}{3},\tfrac{1}{3},\tfrac{1}{3};\tfrac{2}{3},1,2,2;1\ \right],\ {}_5F_4\left[\ \tfrac{9}{4},\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2};\tfrac{5}{4},2,3,3;1\ \right]$$

$${}_5F_4\left[\ \tfrac{13}{8},\tfrac{5}{4},\tfrac{5}{4},\tfrac{1}{4},\tfrac{1}{4};\tfrac{5}{8},2,2,1;1\ \right],\quad {}_5F_4\left[\ \tfrac{1}{2},\tfrac{1}{2},\tfrac{5}{2},\tfrac{5}{2},1;\tfrac{3}{2},\tfrac{3}{2},\tfrac{7}{2},\tfrac{7}{2};-1\ \right],\ {}_4F_3\left[\ \tfrac{3}{2},\tfrac{3}{2},1,1;\tfrac{5}{2},\tfrac{5}{2},2;1\ \right],$$

$${}_4F_3\left[\ \tfrac{2}{3},\tfrac{1}{3},1,1;\tfrac{7}{3},\tfrac{5}{3},2;1\ \right],\ {}_4F_3\left[\ \tfrac{7}{6},\tfrac{5}{6},1,1;\tfrac{13}{6},\tfrac{11}{6},2;1\ \right]\ \text{and}\ {}_4F_3\left[\ 1,1,1,1;3,3,3;-1\ \right].$$

Keywords: Riemann Zeta functions, Polygamma functions, Dougall's theorem, Bernoulli polynomials, Catalan's constant.

Introduction and preliminaries

In this paper, we shall use the following standard notations:

$$\mathbb{N} :=\ \{1,2,3,\cdots\};\quad \mathbb{N}_0 := \mathbb{N}\bigcup\{0\};\quad \mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0,-1,-2,-3,\cdots\}.$$

The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

The classical Pochhammer symbol $(\alpha)_p$ $(\alpha, p \in \mathbb{C})$ is defined by ([1; 22, Eq.(1), p.32, Q.N.(8) and Q.N.(9)], see also [2; 23, Eq.(22) and Eq.(23)]).

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters [2; 42, Eq.(1)].

The Riemann Zeta function $\zeta(z)$([3; 19, 4; 1037]) is defined as:

$$\zeta(z) = \sum_{k=1}^{\infty}\frac{1}{k^z};\quad \mathfrak{Re}(z) > 1,$$

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^z} = (2^{1-z}-1)\zeta(z);\qquad \mathfrak{Re}(z) > 0.,$$

The Catalan constant is defined as:

$$\mathbf{G} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = {}_3F_2\left[\begin{array}{c} 1,\ \frac{1}{2},\ \frac{1}{2};\\ \qquad -1\\ \frac{3}{2},\ \frac{3}{2}; \end{array}\right] = 0.9159655942...$$

The logarithmic derivative of the Gamma function also known as psi function or Digamma function ([1; 10, Eq.(1)], [5; 24, Eq.(2)], [6; 12, Eq.(1)]), is defined as:

$$\psi(z) = \frac{d}{dz}\ell n\{\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)};\quad z \neq 0,-1,-2,-3,...$$

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$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+n)}; \quad z \neq 0, -1, -2, -3, ...,$$

$$\psi(z) = -\gamma - \sum_{n=0}^{\infty} \left\{ \frac{1}{(z+n)} - \frac{1}{(n+1)} \right\}; \quad z \neq 0, -1, -2, -3, ...,$$

where γ is Euler-Mascheroni constant and $\gamma \cong 0.5772156649015328606065 12....$

$$\psi(1) = -\gamma, \quad \psi\left(\frac{2}{3}\right) = -\gamma + \frac{\pi\sqrt{3}}{6} - \frac{3}{2}\ell n\ 3, \quad \psi\left(\frac{3}{2}\right) = 2 - 2\ \ell n\ 2 - \gamma, \tag{1}$$

$$\psi\left(\frac{5}{6}\right) = -\gamma + \frac{\pi\sqrt{3}}{2} - \frac{3}{2}\ell n\ 3 - 2\ell n\ 2, \quad \psi\left(\frac{7}{6}\right) = 6 - \gamma - \frac{\pi\sqrt{3}}{2} - \frac{3}{2}\ \ell n\ 3 - 2\ \ell n\ 2. \tag{2}$$

$$\psi^{(1)}\left(\frac{3}{2}\right) = \frac{\pi^2}{2} - 4, \quad \psi^{(1)}\left(\frac{5}{2}\right) = \frac{\pi^2}{2} - 4.4,$$

$$\psi^{(2)}\left(\frac{3}{2}\right) = -\frac{14\pi^3}{25.79436} + 16, \quad \psi^{(2)}\left(\frac{5}{2}\right) = -\frac{14\pi^3}{25.79436} + \frac{448}{27}.$$

The polygamma function $\psi^{(n)}(z)$ ([5; 33, Eq.(52), Eq.(53), p.34, Eq.(58)], see also ([7; 260, Eq.(6.4.10), Eq.(6.4.4)], [8; 45, Eq.(9)], [3; 15]), is defined as:

$$\psi^{(n)}(z) = \frac{d^{n+1}}{d\ z^{n+1}} \ell n\,(\Gamma(z)) = \frac{d^n}{d\ z^n}\psi(z); \quad n \in \mathbb{N}_0, \ z \neq 0, -1, -2, ...$$

$$\psi^{(n)}(z) = (-1)^{n+1}\ n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}; \quad n \in \mathbb{N}, \ z \neq 0, -1, -2, ...$$

Lower case beta function of one variable:

$$\beta(z) = \frac{1}{2}\left[\psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right)\right] = \frac{G(z)}{2}, \ z \neq 0, -1, -2, -3, ...$$

$$\beta(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(z+k)} = \frac{1}{z}\ {}_2F_1\left[\begin{array}{c} 1, \ z; \\ \\ 1+z; \end{array} -1\right], \quad z \neq 0, -1, -2, -3, ...$$

$$\beta^{(n)}(z) = \frac{d^n}{dz^n}\beta(z) = (-1)^n\ n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(z+k)^{n+1}}; \qquad -z \in \mathbb{N}_0.$$

$$\beta(1) = \ell n\ 2, \ \beta^{(1)}(1) = -\frac{\pi^2}{12}, \quad \beta(2) = 1 - \ell n\ 2, \ \beta^{(1)}(2) = \frac{\pi^2}{12} - 1, \tag{3}$$

$$\beta\left(\frac{1}{2}\right) = \frac{\pi}{2}, \ \beta^{(1)}\left(\frac{1}{2}\right) = -4\mathbf{G}, \ \beta\left(\frac{3}{2}\right) = \frac{4-\pi}{2}, \ \beta^{(1)}\left(\frac{3}{2}\right) = 4\mathbf{G} - 4, \tag{4}$$

$$\beta\left(\frac{5}{2}\right) = \frac{\pi}{2} - \frac{4}{3}, \ \beta^{(1)}\left(\frac{5}{2}\right) = -4\mathbf{G} + \frac{32}{9}, \ \ \beta^{(2)}(1) = \frac{3\pi^3}{51.58872}, \ \ \beta^{(2)}(2) = 2 - \frac{3\pi^3}{51.58872}. \tag{5}$$

Some hypergeometric summation theorems in terms of Digamma $\psi(b)$, trigamma $\psi^{(1)}(b)$, tetragamma $\psi^{(2)}(b)$ functions and derivatives of lower case beta function of one-variable are given below ... [9; 489, Entry (7.3.6.(9))]

$${}_2F_1\left[\begin{array}{c} 1, \ \ a; \\ \\ a+1; \end{array} -1\right] = a\beta(a); \quad 1+a \in \mathbb{C}\backslash\mathbb{Z}_0^-. \tag{6}$$

See ref. [9; 536, Entry (7.4.4.(33))]

$${}_3F_2\left[\begin{array}{c} 1, \ \ a, \ \ b; \\ \\ 1+a, \ 1+b; \end{array} 1\right] = \frac{ab}{(b-a)}\left[\psi(b) - \psi(a)\right], \tag{7}$$

where $1+a, \ 1+b \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $b \neq a$.

See ref. [9; 536, Entry (7.4.4.(34))]

$${}_3F_2\left[\begin{array}{c} 1, \quad b, \quad b; \\ \quad\quad\quad\quad\quad\quad 1 \\ b+1, \ b+1; \end{array}\right] = b^2 \ \psi^{(1)}(b), \tag{8}$$

where $1+b \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $b = a$.

See ref. [9; 546, Entry (7.4.5.(5))]

$${}_3F_2\left[\begin{array}{c} 1, \quad a, \quad a; \\ \quad\quad\quad\quad\quad\quad -1 \\ a+1, \ a+1; \end{array}\right] = -a^2 \ \beta^{(1)}(a), \tag{9}$$

where $1+a \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $b = a$.

See ref. [9; 554, Entry (7.5.3.(3))]

$${}_4F_3\left[\begin{array}{c} 1, \quad a, \quad b, \quad c; \\ \quad\quad\quad\quad\quad\quad 1 \\ 1+a, 1+b, 1+c; \end{array}\right] = -abc\left[\frac{\psi(a)}{(b-a)(c-a)} + \frac{\psi(b)}{(a-b)(c-b)} + \frac{\psi(c)}{(a-c)(b-c)}\right], \tag{10}$$

where $1+a, \ 1+b, \ 1+c \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $a \neq b, b \neq c, a \neq c$.

See ref. [9; 554, Entry (7.5.3.(5))]

$${}_4F_3\left[\begin{array}{c} 1, \quad b, \quad b, \quad b; \\ \quad\quad\quad\quad\quad\quad 1 \\ b+1, \ b+1, \ b+1; \end{array}\right] = \frac{-b^3}{2} \ \psi^{(2)}(b), \tag{11}$$

where $1+b \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $a = b = c$.

See ref. [9; 561, Entry (7.5.4.(5))]

$${}_4F_3\left[\begin{array}{c} 1, \quad a, \quad a, \quad a; \\ \quad\quad\quad\quad\quad\quad -1 \\ a+1, \ a+1, \ a+1; \end{array}\right] = \frac{a^3}{2} \ \beta^{(2)}(a), \tag{12}$$

where $1+a \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $a = b = c$.

Gauss' classical summation theorem [1; 49, Th.(18)] in terms of Gamma function is given by:

$${}_2F_1\left[\begin{array}{c} \alpha, \quad \beta; \\ \quad\quad\quad\quad 1 \\ \gamma; \end{array}\right] = \frac{\Gamma(\gamma) \ \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \tag{13}$$

where $\mathfrak{Re}(\gamma-\alpha-\beta) > 0$ and $\gamma \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Dougall's theorem ([10; 71, Eq.(2.2.10), p.147, Entry(3.5.2)],[11],[9; 564, Entry(7.6.2(3))], [12; 56, Eq.(2.3.4.5), p.244, Entry(III.12)]), see also [13; 27, Eq.(4.4(1))] in terms of Gamma function is given as:

$${}_5F_4\left[\begin{array}{c} a, \quad 1+\frac{a}{2}, \quad b, \quad c, \quad d; \\ \quad\quad\quad\quad\quad\quad\quad\quad 1 \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d; \end{array}\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}, \tag{14}$$

provided $\mathfrak{Re}(a-b-c-d) > -1$ and $\frac{a}{2}, \ 1+a-b, \ 1+a-c, \ 1+a-d \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

The present article is organized as follows. In section 2, we have shown that the difference of two divergent series may be convergent. In section 3, we have obtained the summation of some infinite series whose general terms are rational functions of n, by using some summation theorems of positive and negative unit arguments and section 4 is related to the hypergeometrical representations of the involved infinite series.

The difference of two divergent series

Consider the two positive terms infinite series $\sum_{n=0}^{\infty} \frac{1}{(3+2n)}$ and $\sum_{n=0}^{\infty} \frac{1}{(5+2n)}$, which are divergent in nature by using the comparison test.

Taking the difference of the above two series, we get

$$\sum_{n=0}^{\infty} \frac{1}{(3+2n)} - \sum_{n=0}^{\infty} \frac{1}{(5+2n)} = \sum_{n=0}^{\infty} \frac{2}{(3+2n)\,(5+2n)}. \tag{15}$$

The right hand side of equation (15) is convergent by using the Raabe's higher ratio test.
In terms of hypergeometric function, the equation (15) can be written as

$$\frac{1}{3}\sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n} - \frac{1}{5}\sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n}{\left(\frac{7}{2}\right)_n} = \frac{2}{15}\sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n}{\left(\frac{7}{2}\right)_n},$$

$$\frac{1}{3}\,{}_2F_1\left[\begin{matrix} \frac{3}{2}, & 1; \\ & & 1 \\ & \frac{5}{2}; \end{matrix}\right] - \frac{1}{5}\,{}_2F_1\left[\begin{matrix} \frac{5}{2}, & 1; \\ & & 1 \\ & \frac{7}{2}; \end{matrix}\right] = \frac{2}{15}\,{}_2F_1\left[\begin{matrix} \frac{3}{2}, & 1; \\ & & 1 \\ & \frac{7}{2}; \end{matrix}\right]. \tag{16}$$

Since both the Gauss' series having the positive unit argument on left hand side of equation (16) are divergent. On using Gauss' classical summation theorem (13) on right hand side of equation (16), we get

$$\frac{1}{3}\,{}_2F_1\left[\begin{matrix} \frac{3}{2}, & 1; \\ & & 1 \\ & \frac{5}{2}; \end{matrix}\right] - \frac{1}{5}\,{}_2F_1\left[\begin{matrix} \frac{5}{2}, & 1; \\ & & 1 \\ & \frac{7}{2}; \end{matrix}\right] = \frac{2}{15}\,\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{7}{2}-\frac{3}{2}-1\right)}{\Gamma\left(\frac{7}{2}-\frac{3}{2}\right)\Gamma\left(\frac{7}{2}-1\right)},$$

$$\frac{1}{3}\,{}_2F_1\left[\begin{matrix} \frac{3}{2}, & 1; \\ & & 1 \\ & \frac{5}{2}; \end{matrix}\right] - \frac{1}{5}\,{}_2F_1\left[\begin{matrix} \frac{5}{2}, & 1; \\ & & 1 \\ & \frac{7}{2}; \end{matrix}\right] = \frac{1}{3}, \tag{17}$$

which is convergent.

Multiplying both sides of equation (17) by $\frac{3}{16}$, for application point of view in next section, we get the difference of two divergent Gauss' series having the positive unit argument may be convergent

$$\frac{1}{16}\,{}_2F_1\left[\begin{matrix} \frac{3}{2}, & 1; \\ & & 1 \\ & \frac{5}{2}; \end{matrix}\right] - \frac{3}{80}\,{}_2F_1\left[\begin{matrix} \frac{5}{2}, & 1; \\ & & 1 \\ & \frac{7}{2}; \end{matrix}\right] = \frac{1}{16}. \tag{18}$$

Summation of some infinite series

The following summation formulas of some infinite series are derived:

$$\sum_{n=0}^{\infty} \frac{(4n^4+32n^3+87n^2+92n+28)}{(64n^6+768n^5+3792n^4+9856n^3+14220n^2+10800n+3375)} = \frac{5}{27} - \frac{\pi^2}{64}. \tag{19}$$

$$\sum_{n=0}^{\infty} \frac{(27n^3+36n^2+15n+2)\left\{\left(\frac{1}{3}\right)_n\right\}^4}{(n!)^4\;(1+n)^2} = \frac{27}{4\left[\Gamma\left(\frac{2}{3}\right)\right]^3}. \tag{20}$$

$$\sum_{n=0}^{\infty} \frac{(32n^4+120n^3+156n^2+82n+15)\;\left\{\left(\frac{1}{2}\right)_n\right\}^4}{(n!)^4\;(n^5+7n^4+19n^3+25n^2+16n+4)} = \frac{128}{3\pi^2}. \tag{21}$$

$$\sum_{n=0}^{\infty} \frac{(128n^3+144n^2+48n+5)\left\{\left(\frac{1}{4}\right)_n\right\}^4}{(n!)^4\;(n^2+2n+1)} = \frac{32\sqrt{2}}{3\sqrt{\pi}\left[\Gamma\left(\frac{3}{4}\right)\right]^2}. \tag{22}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(16n^4+96n^3+184n^2+120n+25)} = \frac{\mathbf{G}}{8} - \frac{11}{144}. \tag{23}$$

$$\sum_{n=0}^{\infty} \frac{1}{(4n^3+16n^2+21n+9)} = 4 - 2\ell n\ 2 - \frac{\pi^2}{4}. \tag{24}$$

$$\sum_{n=0}^{\infty} \frac{1}{(81n^4+270n^3+315n^2+150n+24)} = \frac{1}{6} + \frac{\pi}{12\sqrt{3}} - \frac{1}{4}\ell n\ 3. \tag{25}$$

$$\sum_{n=0}^{\infty} \frac{1}{(36n^3+108n^2+107n+35)} = \ell n\ 12 + \ell n\ \sqrt{3} - 3. \tag{26}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n^6+9n^5+33n^4+63n^3+66n^2+36n+8)} = 10 - 12\ \ell n\ 2 - \frac{3}{2}\ \zeta(3). \tag{27}$$

Proof of the result (19):

On factorizing the general term of equation (19) and making use of partial fractions, we have

$$\frac{(4n^4+32n^3+87n^2+92n+28)}{(64n^6+768n^5+3792n^4+9856n^3+14220n^2+10800n+3375)} =$$

$$= \frac{\frac{3}{16}}{(3+2n)} + \frac{\frac{-1}{16}}{(3+2n)^2} + \frac{\frac{-1}{4}}{(3+2n)^3} + \frac{\frac{-3}{16}}{(5+2n)} + \frac{\frac{-1}{16}}{(5+2n)^2} + \frac{\frac{1}{4}}{(5+2n)^3}. \tag{28}$$

Now taking summation on both sides of equation (28) and n varying from 0 to ∞, we get

$$\sum_{n=0}^{\infty} \frac{(4n^4+32n^3+87n^2+92n+28)}{(64n^6+768n^5+3792n^4+9856n^3+14220n^2+10800n+3375)} =$$

$$= \sum_{n=0}^{\infty} \frac{\frac{3}{16}}{(3+2n)} + + \frac{\frac{-1}{16}}{(3+2n)^2} + \frac{\frac{-1}{4}}{(3+2n)^3} + \frac{\frac{-3}{16}}{(5+2n)} + \frac{\frac{-1}{16}}{(5+2n)^2} + \frac{\frac{1}{4}}{(5+2n)^3} =$$

$$= \frac{1}{16} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n} - \frac{1}{144} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n} - \frac{1}{108} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n} -$$

$$- \frac{3}{80} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n}{\left(\frac{7}{2}\right)_n} - \frac{1}{400} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n}{\left(\frac{7}{2}\right)_n \left(\frac{7}{2}\right)_n} + \frac{1}{500} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n}{\left(\frac{7}{2}\right)_n \left(\frac{7}{2}\right)_n \left(\frac{7}{2}\right)_n}.$$

Using the definition of generalized hypergeometric function, we get

$$\sum_{n=0}^{\infty} \frac{(4n^4+32n^3+87n^2+92n+28)}{(64n^6+768n^5+3792n^4+9856n^3+14220n^2+10800n+3375)} =$$

$$= \frac{1}{16}\ {}_2F_1\left[\begin{matrix} \frac{3}{2},\ 1; \\ \frac{5}{2}; \end{matrix}\ 1\right] - \frac{1}{144}\ {}_3F_2\left[\begin{matrix} \frac{3}{2},\ \frac{3}{2},\ 1; \\ \frac{5}{2},\ \frac{5}{2}; \end{matrix}\ 1\right] - \frac{1}{108}\ {}_4F_3\left[\begin{matrix} \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2},\ 1; \\ \frac{5}{2},\ \frac{5}{2},\ \frac{5}{2}; \end{matrix}\ 1\right] -$$

$$- \frac{3}{80}\ {}_2F_1\left[\begin{matrix} \frac{5}{2},\ 1; \\ \frac{7}{2}; \end{matrix}\ 1\right] - \frac{1}{400}\ {}_3F_2\left[\begin{matrix} \frac{5}{2},\ \frac{5}{2},\ 1; \\ \frac{7}{2},\ \frac{7}{2}; \end{matrix}\ 1\right] + \frac{1}{500}\ {}_4F_3\left[\begin{matrix} \frac{5}{2},\ \frac{5}{2},\ \frac{5}{2},\ 1; \\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2}; \end{matrix}\ 1\right].$$

Using summation theorems (8), (11) and the result (18), we get

$$\sum_{n=0}^{\infty} \frac{(4n^4+32n^3+87n^2+92n+28)}{(64n^6+768n^5+3792n^4+9856n^3+14220n^2+10800n+3375)} =$$

$$= \frac{1}{16} - \frac{1}{64}\ \psi^{(1)}\left(\frac{3}{2}\right) - \frac{1}{64}\ \psi^{(1)}\left(\frac{5}{2}\right) + \frac{1}{64}\ \psi^{(2)}\left(\frac{3}{2}\right) - \frac{1}{64}\ \psi^{(2)}\left(\frac{5}{2}\right) =$$

$$= \frac{1}{16} - \frac{1}{64}\ \left(\frac{\pi^2}{2} - 4\right) - \frac{1}{64}\ \left(\frac{\pi^2}{2} - \frac{40}{9}\right) + \frac{1}{64}\ \left(\frac{-14\pi^3}{25.79436} + 16\right) - \frac{1}{64}\ \left(\frac{-14\pi^3}{25.79436} + \frac{448}{27}\right).$$

On simplifying further, we arrive at the result (19).

Proof of the results (20) to (22):

The proof of the results (20) and (22) can be obtained in an analogous manner by following the same steps as in the proof of the result (19) and making use of the summation theorem (14).

Proof of the result (23):

The proof of the result (23) can be obtained by following the same procedure as in the proof of the result (19) and making use of the summation theorems (6), (9) and using the equations (4) and (5). So we omit the details here.

Proof of the result (24):

On factorizing the general term of equation (24) and making use of partial fractions, we have

$$\frac{1}{(4n^3+16n^2+21n+9)} = \frac{1}{(1+n)} + \frac{-2}{(3+2n)} + \frac{-2}{(3+2n)^2}. \tag{29}$$

Now taking summation on both sides of equation (29) and n varying from 0 to ∞, we get

$$\sum_{n=0}^{\infty} \frac{1}{(4n^3+16n^2+21n+9)} = \sum_{n=0}^{\infty} \left\{ \frac{1}{(1+n)} + \frac{-2}{(3+2n)} \right\} - 2\sum_{n=0}^{\infty} \frac{1}{(3+2n)^2} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1+n)\ (3+2n)} - 2\sum_{n=0}^{\infty} \frac{1}{(3+2n)^2} =$$

$$= \frac{1}{3}\sum_{n=0}^{\infty} \frac{(1)_n \ \left(\frac{3}{2}\right)_n}{(2)_n \ \left(\frac{5}{2}\right)_n} - \frac{2}{9}\sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \ \left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n \ \left(\frac{5}{2}\right)_n}.$$

Using the definition of generalized hypergeometric function, we get

$$\sum_{n=0}^{\infty} \frac{1}{(4n^3+16n^2+21n+9)} = \frac{1}{3}\ {}_3F_2\left[\begin{array}{c} \frac{3}{2},\ 1,\ 1; \\ \frac{5}{2},\ 2; \end{array} 1\right] - \frac{2}{9}\ {}_3F_2\left[\begin{array}{c} \frac{3}{2},\ \frac{3}{2},\ 1; \\ \frac{5}{2},\ \frac{5}{2}; \end{array} 1\right].$$

Using summation theorems (7) and (8), we get

$$\sum_{n=0}^{\infty} \frac{1}{(4n^3+16n^2+21n+9)} = \psi\left(\frac{3}{2}\right) - \psi(1) - \frac{1}{2}\ \psi^{(1)}\left(\frac{3}{2}\right).$$

On simplifying further, we arrive at the result (24).

Proof of the result (25):

The proof of the result (25) can be obtained in an analogous manner by following the same steps as in the proof of the result (19) and (24) and making use of Gauss' classical summation theorem (13), the summation theorem (7) and using the equation (1). So, we omit the details here.

Proof of the result (26):

The proof of the result (26) can be obtained by following the same procedure as in the proof of the result (19) and (24) and making use of the summation theorem (10) and using the equations (1) and (2). So, we omit the details here.

Proof of the result (27):

Similarly for the proof of the result (27), we make use of the summation theorems (6), (9), (12) and the equations (3) and (5). So, we omit the details here.

Representation of infinite series (19) to (27) in Hypergeometric forms

The following hypergeometric representation formulas are derived:

$${}_8F_7\left[\begin{array}{c} \frac{9}{2},\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2},\ 3,\ 3,\ 1; \\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{1}{2},\ 2,\ 2; \end{array} 1\right] = \frac{625}{28} - \frac{3375\pi^2}{1792}. \tag{30}$$

$$ {}_5F_4\left[\begin{array}{c} \frac{5}{3},\ \frac{4}{3},\ \frac{4}{3},\ \frac{1}{3},\ \frac{1}{3}; \\ \\ \frac{2}{3},\ 1,\ 2,\ 2; \end{array} 1\right] = \frac{27}{8\left[\Gamma\left(\frac{2}{3}\right)\right]^3}. \tag{31} $$

$$ {}_5F_4\left[\begin{array}{c} \frac{9}{4},\ \frac{5}{2},\ \frac{3}{2},\ \frac{1}{2},\ \frac{1}{2}; \\ \\ \frac{5}{4},\ 2,\ 3,\ 3; \end{array} 1\right] = \frac{512}{45\pi^2}. \tag{32} $$

$$ {}_5F_4\left[\begin{array}{c} \frac{13}{8},\ \frac{5}{4},\ \frac{5}{4},\ \frac{1}{4},\ \frac{1}{4}; \\ \\ 2,\ 2,\ 1,\ \frac{5}{8}; \end{array} 1\right] = \frac{32\sqrt{2}}{15\sqrt{\pi}\left[\Gamma\left(\frac{3}{4}\right)\right]^2}. \tag{33} $$

$$ {}_5F_4\left[\begin{array}{c} 1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{5}{2},\ \frac{5}{2}; \\ \\ \frac{3}{2},\ \frac{3}{2},\ \frac{7}{2},\ \frac{7}{2}; \end{array} -1\right] = \frac{25\ \mathbf{G}}{8} - \frac{275}{144}. \tag{34} $$

$$ {}_4F_3\left[\begin{array}{c} \frac{3}{2},\ \frac{3}{2},\ 1,\ 1; \\ \\ \frac{5}{2},\ \frac{5}{2},\ 2; \end{array} 1\right] = 36 - 18\ell n\ 2 - \frac{9\pi^2}{4}. \tag{35} $$

$$ {}_4F_3\left[\begin{array}{c} \frac{2}{3},\ \frac{1}{3},\ 1,\ 1; \\ \\ \frac{7}{3},\ \frac{5}{3},\ 2; \end{array} 1\right] = 4 - 6\ell n\ 3 + \frac{2\pi}{\sqrt{3}}. \tag{36} $$

$$ {}_4F_3\left[\begin{array}{c} \frac{7}{6},\ \frac{5}{6},\ 1,\ 1; \\ \\ \frac{13}{6},\ \frac{11}{6},\ 2; \end{array} 1\right] = 35\ell n\ 12 + 35\ell n\ \sqrt{3} - 105. \tag{37} $$

$$ {}_4F_3\left[\begin{array}{c} 1,\ 1,\ 1,\ 1; \\ \\ 3,\ 3,\ 3; \end{array} -1\right] = 96\ \ell n\ 2 - 80 + 12\ \zeta(3). \tag{38} $$

Proof of the result (30):

$$ \sum_{n=0}^{\infty} \frac{(4n^4+32n^3+87n^2+92n+28)}{(64n^6+768n^5+3792n^4+9856n^3+14220n^2+10800n+3375)} = $$

$$ = \sum_{n=0}^{\infty} \frac{(7+2n)\ (1+2n)\ (2+n)^2}{(3+2n)^3\ (5+2n)^3} = \frac{28}{3375} \sum_{n=0}^{\infty} \frac{\left(\frac{9}{2}\right)_n \left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n (3)_n\ (3)_n}{\left(\frac{7}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{7}{2}\right)_n \left(\frac{7}{2}\right)_n \left(\frac{7}{2}\right)_n (2)_n\ (2)_n}. $$

Using the definition of generalized hypergeometric function, we get

$$ \sum_{n=0}^{\infty} \frac{(4n^4+32n^3+87n^2+92n+28)}{(64n^6+768n^5+3792n^4+9856n^3+14220n^2+10800n+3375)} = $$

$$ = \frac{28}{3375}\ {}_8F_7\left[\begin{array}{c} \frac{9}{2},\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2},\ 3,\ 3,\ 1; \\ \\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{1}{2},\ 2,\ 2; \end{array} 1\right]. \tag{39} $$

Using equation (19) in equation (39), we arrive at the result (30).

Proof of the results (31) to (38):

The proof of the results (31) to (38) can be obtained in an analogous manner by following the same steps as in the proof of the above result (30). So we omit the details here.

Conclusion

In this paper, we have obtained the summation of some infinite series by using some summation theorems of positive and negative unit arguments, Riemann Zeta functions, polygamma functions, lower case beta functions of one-variable and other associated functions. We have also obtained some new hypergeometric summation theorems, which are not found in the literature. We conclude this paper with the remark that the summation of various other infinite series can be derived in an analogous manner. Moreover, the results deduced above are expected to lead to some potential applications in several fields of Applied Mathematics, Statistics and Engineering Sciences.

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Гипергеометриялық функциялар мен жартылай бөлшек әдістерімен кейбір шексіз серияларды жинақтау

Мақалада кейбір шексіз қатарлардың жартылай бөлшек әдісімен оң және теріс сингулярлық дәлелдерді, Риманның Зета функцияларын, полигамма функцияларын, кіші регистрдегі бір айнымалының бета функцияларын және басқа да байланысты функцияларды жинақтаудың кейбір гипергеометриялық теоремалары жинақталған. Сондай-ақ кейбір гипергеометриялық жиынтық теоремалар алынған:

$$ {}_8F_7\left[\tfrac{9}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},3,3,1;\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{1}{2},2,2;1\right],\ {}_5F_4\left[\tfrac{5}{3},\tfrac{4}{3},\tfrac{4}{3},\tfrac{1}{3},\tfrac{1}{3};\tfrac{2}{3},1,2,2;1\right],\ {}_5F_4\left[\tfrac{9}{4},\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2};\tfrac{5}{4},2,3,3;1\right] $$

$$ {}_5F_4\left[\tfrac{13}{8},\tfrac{5}{4},\tfrac{5}{4},\tfrac{1}{4},\tfrac{1}{4};\tfrac{5}{8},2,2,1;1\right],\quad {}_5F_4\left[\tfrac{1}{2},\tfrac{1}{2},\tfrac{5}{2},\tfrac{5}{2},1;\tfrac{3}{2},\tfrac{3}{2},\tfrac{7}{2},\tfrac{7}{2};-1\right],\quad {}_4F_3\left[\tfrac{3}{2},\tfrac{3}{2},1,1;\tfrac{5}{2},\tfrac{5}{2},2;1\right], $$

$$ {}_4F_3\left[\tfrac{2}{3},\tfrac{1}{3},1,1;\tfrac{7}{3},\tfrac{5}{3},2;1\right],\ {}_4F_3\left[\tfrac{7}{6},\tfrac{5}{6},1,1;\tfrac{13}{6},\tfrac{11}{6},2;1\right]\ \text{and}\ {}_4F_3\left[1,1,1,1;3,3,3;-1\right]. $$

Кілт сөздер: Риманның Зета функциялары, полигамма функциялары, Дугалл теоремасы, Бернулли көпмүшелері, Каталан константасы.

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Суммирование некоторых бесконечных рядов методами гипергеометрических функций и частных дробей

В статье получено суммирование некоторых бесконечных рядов методом частичных дробей и с помощью некоторых гипергеометрических теорем суммирования положительных и отрицательных единичных аргументов, дзета-функций Римана, полигамма-функций, бета-функций одной переменной в нижнем регистре и других связанных функций. Кроме того, авторами получены некоторые гипергеометрические теоремы суммирования для:

$$ {}_8F_7\left[\tfrac{9}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},3,3,1;\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{1}{2},2,2;1\right],\ {}_5F_4\left[\tfrac{5}{3},\tfrac{4}{3},\tfrac{4}{3},\tfrac{1}{3},\tfrac{1}{3};\tfrac{2}{3},1,2,2;1\right],\ {}_5F_4\left[\tfrac{9}{4},\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2};\tfrac{5}{4},2,3,3;1\right] $$

$$ {}_5F_4\left[\tfrac{13}{8},\tfrac{5}{4},\tfrac{5}{4},\tfrac{1}{4},\tfrac{1}{4};\tfrac{5}{8},2,2,1;1\right],\quad {}_5F_4\left[\tfrac{1}{2},\tfrac{1}{2},\tfrac{5}{2},\tfrac{5}{2},1;\tfrac{3}{2},\tfrac{3}{2},\tfrac{7}{2},\tfrac{7}{2};-1\right],\quad {}_4F_3\left[\tfrac{3}{2},\tfrac{3}{2},1,1;\tfrac{5}{2},\tfrac{5}{2},2;1\right], $$

$$ {}_4F_3\left[\tfrac{2}{3},\tfrac{1}{3},1,1;\tfrac{7}{3},\tfrac{5}{3},2;1\right],\ {}_4F_3\left[\tfrac{7}{6},\tfrac{5}{6},1,1;\tfrac{13}{6},\tfrac{11}{6},2;1\right]\ \text{и}\ {}_4F_3\left[1,1,1,1;3,3,3;-1\right]. $$

Ключевые слова: дзета-функции Римана, полигамма-функции, теорема Дугалла, многочлены Бернулли, константа Каталана.

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"Dedicated to Professor Filippo CAMMAROTO for his 70th Birthday"

Applications of operations on generalized topological spaces

In this paper, γ_μ-open sets and γ_μ-closed sets in a GTS (X, μ) have been studied, where γ_μ is an operation from μ to $\mathcal{P}(X)$. In general, collection of γ_μ-open sets is smaller than the collection of μ-open sets. The condition under which both are same are also established here. Some properties of such sets have been discussed. Some closure as operators are also defined and their properties are discussed. The relation between similar types of closure operators on the GTS (X, μ) has been established. The condition under which the newly defined closure like operator is a Kuratowski closure operator is given. We have also defined a generalized type of closed sets termed as γ_μ-generalized closed set with the help of this newly defined closure operator and discussed some basic properties of such sets. As an application, we have introduced some weak separation axioms and discussed some of their properties. Finally, we have shown some preservation theorems of such generalized concepts.

Keywords: operation, μ-open set, γ_μ-open set, $\gamma_\mu g$-closed set.

Introduction

In 1979, Kasahara [1] introduced the notion of an operation on a topological space and introduced the concept of an α-closed graph of a function. After then Janković defined [2] the concept of α-closed sets and investigated some properties of functions with α-closed graphs. In 1991 Ogata [3] introduced the notion of γ-open sets to investigate some new separation axioms of a topological space. Recently, Krishnan et al. [4] and Van An et al. [5] investigated the notion of operations on the family of all semi-open sets and pre-open sets.

In this paper our aim is to study an operation based on open like sets, where the operation is defined on a collection of generalized open sets instead of a topology. The family of open sets plays an important role in topology. For this, different open like sets or weakly open sets have been introduced by mathematicians to study different weak forms of continuous functions and covering properties of topological spaces. But the most common properties of these open like sets or weakly open sets are that they are closed under arbitrary union and contain the empty set. Observing these, Császár introduced the concept of generalized open sets. We now recall some notions defined in [6]. Let X be a non-empty set. A subcollection $\mu \subsetneqq \mathcal{P}(X)$ (where $\mathcal{P}(X)$ denotes the power set of X) is called a generalized topology [6], (briefly, GT) if $\varnothing \in \mu$ and any union of elements of μ belongs to μ. A set X with a GT μ on the set X is called a generalized topological space (briefly, GTS) and is denoted by (X, μ). If for a GTS (X, μ) $X \in \mu$, then (X, μ) is known as a strong GTS. The elements of μ are called μ-open sets and μ-closed sets are their complements. The μ-closure of a set $A \subsetneqq X$ is denoted by $c_\mu(A)$ and defined as the smallest μ-closed set containing A which is equivalent to the intersection of all μ-closed sets containing A. It is also known from [7, 8] that for a GTS (X, μ), $A \subsetneqq X$ and $x \in X$, $x \in c_\mu(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in \mu$ containing x. We use the symbol $i_\mu(A)$ to mean the μ-interior of A and it is defined as the union of all μ-open sets contained in A i.e., the largest μ-open set contained in A (see [6, 7]). We observe that $x \in i_\mu(A)$ if and only if there exists some μ-open set U containing x such that $U \subsetneqq A$ and $A \subsetneqq X$ is μ-open (resp. μ-closed) if and only if $A = i_\mu(A)$ (resp. $A = c_\mu(A)$). It is well known that i_μ and c_μ both are monotonic and idempotent. For any subset A of a GTS (X, μ), $i_\mu(X \setminus A) = X \setminus c_\mu(A)$ holds.

Császár continued to try to find a more general structure from general topology, generalized topology, and minimal structure. In 2010, he introduced the notion of weak structures [9] and proved that it can replace the

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already defined structures in some cases. A sub-collection $w \subseteqq \mathcal{P}(X)$ is said to be a weak structure on X if and only if it contains the empty set. Its properties have been investigated intensively in [10–13]. In Section 2 we have introduced the concept of a type of generalized open sets termed as γ_μ-open sets, the class of which is smaller than that of generalized open sets, by an operator defined on a GT. We have then studied some properties of such sets in detail. In section 3 we have defined a new type of generalized closed sets and studied some separation properties with the help of the idea developed in Section 2.

γ_μ-*open sets and operations*

Definition 2.1. [14] Let (X, μ) be a GTS. An operation γ_μ on a generalized topology μ is a mapping from μ to $\mathcal{P}(X)$ (where $\mathcal{P}(X)$ is the power set of X) with $G \subseteqq \gamma_\mu(G)$ for each $G \in \mu$. This operation is denoted by $\gamma_\mu : \mu \to \mathcal{P}(X)$. Note that $\gamma_\mu(A)$ and A^{γ_μ} are two different notation for the same set.

Definition 2.2. [14] Let (X, μ) be a GTS and γ_μ an operation on μ. A subset G of a GTS (X, μ) is called γ_μ-open if for each point x of G, there exists a μ-open set U containing x such that $\gamma_\mu(U) \subseteqq G$.

A subset of a GTS (X, μ) is called γ_μ-closed if its complement is γ_μ-open in (X, μ). We shall use the symbol γ_μ to mean the collection of all γ_μ-open sets of the GTS (X, μ).

Remark 2.3. (a) We observe that every γ_μ-open set is a μ-open set i.e., $\gamma_\mu \subseteqq \mu$. Let $G \in \gamma_\mu$. If $G = \varnothing$ then $G \in \mu$. If $G \neq \varnothing$, let $x \in G$. Then there exists a μ-open set U containing x such that $\gamma_\mu(U) \subseteqq G$. Thus for each $x \in G$ there exists a μ-open set U containing x such that $x \in U \subseteqq G$. Thus x is a μ-interior point of G i.e., $x \in i_\mu(G)$ i.e., $G \subseteqq i_\mu(G)$ proving G to be a μ-open set.

(b) We note that γ_μ is a GT on X i.e., $\varnothing \in \gamma_\mu$ and arbitrary unions of γ_μ-open sets are also γ_μ-open. For let $\{G_\alpha : \alpha \in I\}$ be a family of γ_μ-open subsets of X. We shall show that $\cup\{G_\alpha : \alpha \in I\}$ is also a γ_μ-open set. In fact, let $x \in \cup\{G_\alpha : \alpha \in I\}$. Then $x \in G_{\alpha_0}$ for some $\alpha_0 \in I$. Thus by γ_μ-openness of G_{α_0}, there exists a μ-open set U containing x such that $\gamma_\mu(U) \subseteqq G_{\alpha_0} \subseteqq \cup\{G_\alpha : \alpha \in I\}$.

Example 2.4. (a) Let $X = \{1, 2, 3\}$ and $\mu = \{\varnothing, \{1, 2\}, \{1, 3\}, X\}$. Then μ is a GT on X. Consider the mapping $\gamma_\mu : \mu \to \mathcal{P}(X)$ defined by $\gamma_\mu(A) = c_\mu(A)$ for each subset A of X. It can be easily checked that $\{1, 2\}$ is a μ-open set but not a γ_μ-open set.

(b) Let $X = \{1, 2, 3\}$ and $\mu = \{\varnothing, \{1\}, \{1, 2\}, \{2, 3\}, X\}$. Then (X, μ) is a GTS. Now $\gamma_\mu : \mu \to \mathcal{P}(X)$ defined by

$$\gamma_\mu(A) = \begin{cases} A, & \text{if } 1 \in A \\ \{2, 3\}, & \text{otherwise} \end{cases}$$

is an operation. It can be easily checked that $\{1, 2\}$ and $\{2, 3\}$ are two γ_μ-open sets but their intersection $\{2\}$ is not so.

Definition 2.5. A GTS (X, μ) is said to be a γ_μ-regular space if for each point x of X and each μ-open set V containing x, there exists a μ-open set U containing x such that $\gamma_\mu(U) \subseteqq V$.

Theorem 2.6. Let (X, μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation on a GTS X. Then (X, μ) is a γ_μ-regular space if and only if $\mu = \gamma_\mu$.

Proof. Let (X, μ) be a γ_μ-regular space. In view of Remark 2.3 it is sufficient to show that $\mu \subseteqq \gamma_\mu$. Let G be a μ-open set of X. If $G = \varnothing$, then $G \in \gamma_\mu$. Thus we may assume that $G \neq \varnothing$. Since (X, μ) is γ_μ-regular, then G is a γ_μ-open set. Therefore, we have $\mu \subseteqq \gamma_\mu$.

Conversely, let $x \in X$ and V be a μ-open set containing x. Then V is a γ_μ-open set containing x (as $\mu = \gamma_\mu$). Thus by definition of γ_μ-open sets, there exists a μ-open set U containing x such that $\gamma_\mu(U) \subseteqq V$. Hence (X, μ) is a γ_μ-regular space.

Theorem 2.7. A GTS (X, μ) is a γ_μ-regular space if and only if for each point $x \in X$ and every μ-open set U containing x, there exists a γ_μ-open set W containing x such that $W \subseteqq U$.

Proof. First let us assume that (X, μ) be a γ_μ-regular space. Let $x \in X$ and U be a μ-open set containing x. Then by Definition 2.5, there exists a μ-open set W containing x such that $W \subseteqq \gamma_\mu(W) \subseteqq U$. Thus by Theorem 2.6, W is a γ_μ-open set. Hence there exists a γ_μ-open set W such that $x \in W \subseteqq U$.

Conversely, suppose that for each point $x \in X$ and every μ-open set U containing x there exists a γ_μ-open set W containing x such that $W \subseteqq U$. In view of Theorem 2.6 and Remark 2.3(a) it is now sufficient to show that $\mu \subseteqq \gamma_\mu$. Let $U \in \mu$ and $x \in U$. Then by the given condition there exists a γ_μ-open set W_x containing x such that $W_x \subseteqq U$. Thus $U = \cup\{W_x : x \in U \text{ and } W_x \text{ is } \gamma_\mu\text{-open}\}$. Thus by Remark 2.3(b), U is γ_μ-open.

Definition 2.8. Let (X, μ) be a GTS. An operation $\gamma_\mu : \mu \to \mathcal{P}(X)$ is said to be regular if for each point $x \in X$ and any two μ-open sets U and V of X containing x there exists a μ-open set W containing x such that $\gamma_\mu(W) \subseteqq \gamma_\mu(U) \cap \gamma_\mu(V)$.

Theorem 2.9. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be a regular operation. Then the intersection of two γ_μ-open sets is also a γ_μ-open set. Furthermore, γ_μ is a topology if $X \in \mu$.

Proof. Let G and H be two γ_μ-open sets in a GTS (X, μ). We shall show that $G \cap H$ is also a γ_μ-open set. If $G \cap H = \varnothing$ then the proof is done. Let $x \in G \cap H$. Then by Definition 2.2, there exist two μ-open sets U and V with $x \in U \cap V$ such that $\gamma_\mu(U) \subseteqq G$ and $\gamma_\mu(V) \subseteqq H$. Since $\gamma_\mu : \mu \to \mathcal{P}(X)$ is a regular operation, there exists a μ-open set W containing x such that $\gamma_\mu(W) \subseteqq \gamma_\mu(U) \cap \gamma_\mu(V) \subseteqq G \cap H$. Thus by Definition 2.2, $G \cap H$ is γ_μ-open.
If $X \in \mu$, then for each $x \in X$, there exists a μ-open set X (as $X \in \mu$) containing x such that $X \subseteqq \gamma_\mu(X) \subseteqq X$. Thus X is a γ_μ-open set. It follows from Remark 2.3(b) that arbitrary union of γ_μ-open sets is a γ_μ-open set. Thus γ_μ is a topology on X.

Example 2.10. (a) Let $X = \{1, 2, 3\}$ and $\mu = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. Then $\gamma_\mu : \mu \to \mathcal{P}(X)$ defined by $\gamma_\mu(A) = c_\mu(A)$ is an operation on the GTS (X, μ) where μ is not strong. It can be easily checked the X is not a γ_μ-open set. We note that $\gamma_\mu : \mu \to \mathcal{P}(X)$ is a regular operation.
(b) Let $X = \{1, 2, 3\}$, $\mu = \{\varnothing, X, \{2\}, \{1, 3\}, \{2, 3\}\}$. Then $\gamma_\mu : \mu \to \mathcal{P}(X)$ defined by

$$\gamma_\mu(A) = \begin{cases} A \cup \{1\}, \text{ if } A \text{ is any singleton subset of } X \\ A, \text{otherwise} \end{cases}$$

is an operation on the GTS (X, μ). We note that γ_μ is not a regular operation. It can be checked easily that γ_μ is not a topology on X.

We now define the following two types of closure operators : one follows from the GT γ_μ on X and the second one is defined in the sense of Jankovič.

Definition 2.11. Let (X, μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation.

(a) It follows from Remark 2.3(b) that γ_μ is a GT. Thus the γ_μ-closure of a set A is denoted by $c_{\gamma_\mu}(A)$ and is defined as $c_{\gamma_\mu}(A) = \cap\{F : F \text{ is a } \gamma_\mu\text{-closed set and } A \subseteqq F\}$.

(b) γ_μ^*-closure of A is denoted by γ_μ-$c(A)$ and defined by γ_μ-$c(A) = \{x : A \cap \gamma_\mu(U) \neq \varnothing$ for every μ-open set U containing $x\}$.

A subset $A (\subseteqq X)$ is called γ_μ^*-closed if γ_μ-$c(A) = A$.

Proposition 2.12. Let (X, μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. For each $x \in X$, $x \in c_{\gamma_\mu}(A)$ if and only if $V \cap A \neq \varnothing$ for any $V \in \gamma_\mu$ with $x \in V$.

Proof. The proof follows from the fact that γ_μ is a GT on X (by Remark 2.3(b)) and the fact that for any GT μ on X, $x \in c_\mu(A)$ [7, 8] if and only if $U \cap A \neq \varnothing$ for each μ-open set U containing x.

Remark 2.13. It can be checked easily that for any subset A of a GTS (X, μ), $A \subseteqq c_\mu(A) \subseteqq \gamma_\mu$-$c(A) \subseteqq c_{\gamma_\mu}(A)$.

Definition 2.14. An operation $\gamma_\mu : \mu \to \mathcal{P}(X)$ is said to be μ-open if for each point x of X and for every μ-open set U containing x there exists a γ_μ-open set V containing x such that $V \subseteqq \gamma_\mu(U)$.

The next theorem gives the relation between the three types of closure operators.

Theorem 2.15. Let (X, μ) be a GTS, $\gamma_\mu : \mu \to \mathcal{P}(X)$ an operation and A a subset of X.

(i) The subset γ_μ-$c(A)$ is μ-closed in (X, μ).

(ii) If (X, μ) is γ_μ-regular, then γ_μ-$c(A) = c_\mu(A)$.

(iii) If γ_μ is μ-open, then γ_μ-$c(A) = c_{\gamma_\mu}(A)$ and γ_μ-$c[\gamma_\mu$-$c(A)] = \gamma_\mu$-$c(A)$.

Proof. (i) We shall only show that $c_\mu[\gamma_\mu$-$c(A)] \subseteqq \gamma_\mu$-$c(A)$. Let $x \in c_\mu[\gamma_\mu$-$c(A)]$ and U be any μ-open set in X containing x. Then $U \cap \gamma_\mu$-$c(A) \neq \varnothing$. Let $y \in U \cap \gamma_\mu$-$c(A)$. Then $y \in U$ and $y \in \gamma_\mu$-$c(A)$. Thus $\gamma_\mu(U) \cap A \neq \varnothing$ i.e., $x \in \gamma_\mu$-$c(A)$ (by Definition 2.11).

(ii) In view of Remark 2.13 we need only to show that in a γ_μ-regular GTS (X, μ), γ_μ-$c(A) \subseteqq c_\mu(A)$. Let $x \in \gamma_\mu$-$c(A)$ and G be any μ-open set containing x. Then there exists a μ-open set U containing x such that $\gamma_\mu(U) \subseteqq G$ (as (X, μ) is γ_μ-regular). Since $x \in \gamma_\mu$-$c(A)$ we have $\gamma_\mu(U) \cap A \neq \varnothing$ and hence $G \cap A \neq \varnothing$. Thus it follows that $x \in c_\mu(A)$.

(iii) Suppose that $x \notin \gamma_\mu$-$c(A)$. Then there exists a μ-open set U containing x such that $\gamma_\mu(U) \cap A = \varnothing$. Since γ_μ is μ-open, for the μ-open set U containing x, there exists a γ_μ-open set V containing x such that $V \subseteqq \gamma_\mu(U)$. Hence $V \cap A = \varnothing$. This shows that $x \notin c_{\gamma_\mu}(A)$. Thus $c_{\gamma_\mu}(A) \subseteqq \gamma_\mu$-$c(A)$. Also from Remark 2.13, γ_μ-$c(A) \subseteqq c_{\gamma_\mu}(A)$. Thus we have γ_μ-$c(A) = c_{\gamma_\mu}(A)$. Hence γ_μ-$c[\gamma_\mu$-$c(A)] = c_{\gamma_\mu}[c_{\gamma_\mu}(A)] = c_{\gamma_\mu}(A)$ (as γ_μ is a GT on X and c_{γ_μ} is idempotent) $= \gamma_\mu$-$c(A)$.

Example 2.16. (a) Let $X = \{1, 2, 3\}$, $\mu = \{\varnothing, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$. Then $\gamma_\mu : \mu \to \mathcal{P}(X)$ defined by

$$\gamma_\mu(A) = \begin{cases} A \cup \{3\}, \text{ if } A \neq \{1\} \\ A, \text{otherwise} \end{cases}$$

is an operation. It can be easily checked that $c_\mu(\{3\}) = \{3\} \neq \gamma_\mu\text{-}c(\{3\}) = \{2,3\}$ and thus from Theorem 2.15 it follows that (X,μ) is not γ_μ-regular.

(b) Let $X = \{1,2,3,4\}$, $\mu = \{\varnothing,\{1,2\},\{2,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{2,3,4\},X\}$. Then $\gamma_\mu : \mu \to \mathcal{P}(X)$ defined by

$$\gamma_\mu(A) = \begin{cases} A, \text{ if } 1 \in A \\ A \cup \{1\}, \text{if } \ 1 \notin A \end{cases}$$

is an operation. It can be checked that $\gamma_\mu\text{-}c(\{2\}) = \{2,3,4\}$ but $\gamma_\mu\text{-}c[\gamma_\mu\text{-}c(\{2\})] = X \neq \gamma_\mu\text{-}c(\{2\})$. Thus it follows from Theorem 2.15 that γ_μ is not μ-open.

Theorem 2.17. Let μ be a GT on a set X and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. For any subset A of X the followings are equivalent :

(i) A is γ_μ-open in (X,μ).

(ii) $X \setminus A$ is γ_μ^*-closed in (X,μ).

(iii) $c_{\gamma_\mu}(X \setminus A) = X \setminus A$ holds.

(iv) $X \setminus A$ is γ_μ-closed in (X,μ).

Proof. (i) $\Rightarrow$ (ii): Let $x \notin X \setminus A$. Then $x \in A$. Thus there exists a μ-open set U containing x such that $\gamma_\mu(U) \subseteqq A$ i.e., $\gamma_\mu(U) \cap (X \setminus A) = \varnothing$. This shows that $x \notin \gamma_\mu\text{-}c(X \setminus A)$. Thus it follows that $\gamma_\mu\text{-}c(X \setminus A) \subseteqq X \setminus A$.

(ii) $\Rightarrow$ (iii): We have to show that $c_{\gamma_\mu}(X \setminus A) \subseteqq X \setminus A$. Let $x \notin X \setminus A$. It then follows from (ii) that there exists a μ-open set U containing x such that $\gamma_\mu(U) \cap (X \setminus A) = \varnothing$. Then A is a γ_μ-open set containing x. Therefore $A \cap (X \setminus A) = \varnothing$ and hence $x \notin c_{\gamma_\mu}(X \setminus A)$.

(iii) $\Rightarrow$ (iv): We shall show that A is γ_μ-open. Let $x \in A$. Then by Proposition 2.12 and (iii), there exists a γ_μ-open set U containing x such that $U \cap (X \setminus A) = \varnothing$. Since U is γ_μ-open and $x \in U$, there exists a μ-open set V containing x such that $\gamma_\mu(V) \subseteqq U$. Thus we have $x \in \gamma_\mu(V) \subseteqq U \subseteqq A$ and hence A is γ_μ-open.

(iv) $\Rightarrow$ (i) : The proof follows from the definition.

Theorem 2.18. Let (X,μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. If γ_μ is regular, then $\gamma_\mu\text{-}c(A \cup B) = \gamma_\mu\text{-}c(A) \cup \gamma_\mu\text{-}c(B)$ for any two subsets A and B of X.

Proof. Let $x \notin \gamma_\mu\text{-}c(A) \cup \gamma_\mu\text{-}c(B)$. Then $x \notin \gamma_\mu\text{-}c(A)$ and $x \notin \gamma_\mu\text{-}c(B)$. Hence there exist two μ-open sets U and V containing x such that $\gamma_\mu(U) \cap A = \gamma_\mu(V) \cap B = \varnothing$. Since γ_μ is regular, there exists a μ-open set W containing x such that $\gamma_\mu(W) \subseteqq \gamma_\mu(U) \cap \gamma_\mu(V)$. Therefore, we have $(A \cup B) \cap \gamma_\mu(W) \subseteqq (A \cup B) \cap [\gamma_\mu(U) \cap \gamma_\mu(V)] \subseteqq$ $\subseteqq [A \cap \gamma_\mu(U)] \cup [B \cap \gamma_\mu(V)] = \varnothing$. Hence $x \notin \gamma_\mu\text{-}c(A \cup B)$. Therefore, we obtain $\gamma_\mu\text{-}c(A \cup B) \subseteqq \gamma_\mu\text{-}c(A) \cup \gamma_\mu\text{-}c(B)$.

Corollary 2.19. Let μ be a GT on a set X and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. If γ_μ is regular and μ-open, then the mapping defined by $\psi(A) = \gamma_\mu\text{-}c(A)$ for $A \subseteqq X$ is a Kuratowski closure operator.

Proof. This follows from Theorem 2.15, Theorem 2.18 and Definition 2.11.

γ_μ-generalized closed sets and γ_μ-T_i spaces ($i = 0, 1/2, 1, 2$)

Definition 3.1. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. A subset A of a GTS (X,μ) is said to be γ_μ-generalized closed (briefly $\gamma_\mu g$-closed) if $\gamma_\mu\text{-}c(A) \subseteqq U$ whenever $A \subseteqq U$ and U is γ_μ-open.

The complement of a $\gamma_\mu g$-closed set is called a $\gamma_\mu g$-open set.

We observe that every γ_μ^*-closed set is $\gamma_\mu g$-closed. The converse is false as shown in the next example.

Example 3.2. Consider $X = \{1,2,3\}$, $\mu = \{\varnothing,\{1\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$. Then $\gamma_\mu : \mu \to \mathcal{P}(X)$ defined by

$$\gamma_\mu(A) = \begin{cases} A \cup \{2\}, \text{ if } A \neq \{1\} \\ A, \text{otherwise} \end{cases}$$

is an operation. It can be checked easily that $\{1,3\}$ is $\gamma_\mu g$-closed but not γ_μ-closed.

The following theorem gives the characterizations of $\gamma_\mu g$-closed sets.

Theorem 3.3. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. Then for any $A \subseteqq X$, the following are equivalent:

(i) A is $\gamma_\mu g$-closed.

(ii) For each $x \in \gamma_\mu\text{-}c(A)$, $c_{\gamma_\mu}(\{x\}) \cap A \neq \varnothing$.

(iii) $\gamma_\mu\text{-}c(A) \subseteqq Ker_{\gamma_\mu}(A)$ (where $Ker_{\gamma_\mu}(A) = \cap\{V : A \subseteqq V$ and V is γ_μ-open$\}$ see [15] for detail).

Proof. (i) $\Rightarrow$ (ii) : Suppose that A be a $\gamma_\mu g$-closed subset and also suppose that there exists a point $x \in \gamma_\mu\text{-}c(A)$ for which $c_{\gamma_\mu}(\{x\}) \cap A = \varnothing$. Then $c_{\gamma_\mu}(\{x\})$ is γ_μ-closed (by Remark 2.3(b) and Definition 2.11(a)). Put $U = X \setminus c_{\gamma_\mu}(\{x\})$. Then $A \subseteqq U$ and $x \notin U$ with U a γ_μ-open set in (X,μ). Since A is $\gamma_\mu g$-closed, $\gamma_\mu\text{-}c(A) \subseteqq U$. Thus $x \notin \gamma_\mu\text{-}c(A)$ which is a contradiction.

(ii) $\Rightarrow$ (iii) : Let $x \in \gamma_\mu$-$c(A)$. We have only to show that $x \in Ker_{\gamma_\mu}(A)$. By (ii) there exists a point $z \in A$ such that $z \in c_{\gamma_\mu}(\{x\})$. Let U be any γ_μ-open subset of X such that $A \subseteqq U$. Since $z \in U$ and $z \in c_{\gamma_\mu}(\{x\})$, by Proposition 2.12 we have $U \cap \{x\} \neq \varnothing$ i.e., $x \in U$. Thus $x \in Ker_{\gamma_\mu}(A)$.

(iii) $\Rightarrow$ (i) : Let $A \subseteqq U$, where U be any γ_μ-open set. Let $x \in \gamma_\mu$-$c(A)$. It then follows from (iii) that $x \in Ker_{\gamma_\mu}(A)$. Thus $x \in U$ i.e., γ_μ-$c(A) \subseteqq U$.

Theorem 3.4. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation, where (X, μ) is a GTS. For each point x of X, $\{x\}$ is a γ_μ-closed set or $X \setminus \{x\}$ is a $\gamma_\mu g$-closed set in (X, μ).

Proof. Let $\{x\}$ be not a γ_μ-closed set. Then the complement $X \setminus \{x\}$ is not a γ_μ-open set. Let U be any γ_μ-open set with $X \setminus \{x\} \subseteqq U$. Then U must be equal to X. Thus γ_μ-$c(X \setminus \{x\}) \subseteqq U$. Thus $X \setminus \{x\}$ is $\gamma_\mu g$-closed.

Proposition 3.5. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation and A be a subset of a GTS (X, μ). If A is $\gamma_\mu g$-closed, then γ_μ-$c(A) \setminus A$ does not contain any non-empty γ_μ-closed set. If the operation $\gamma_\mu : \mu \to \mathcal{P}(X)$ is μ-open, then the converse part is also true.

Proof. If possible, let F be any γ_μ-closed set contained in γ_μ-$c(A) \setminus A$. Then $A \subseteqq X \setminus F$ where $X \setminus F$ is a γ_μ-open set. Thus γ_μ-$c(A) \subseteqq X \setminus F$ (as A is $\gamma_\mu g$-closed). Thus $F \subseteqq X \setminus \gamma_\mu$-$c(A)$. Also $F \subseteqq \gamma_\mu$-$c(A)$. Thus $F \subseteqq \gamma_\mu$-$c(A) \cap (X \setminus \gamma_\mu$-$c(A)) = \varnothing$, which is a contradiction. Thus $F = \varnothing$.

Conversely, let $A \subseteqq U$ where U be any γ_μ-open set. Since the operation γ_μ is μ-open, by Theorem 2.15 γ_μ-$c(A)$ is γ_μ-closed. Thus γ_μ-$c(A) \cap (X \setminus U) = F$ (say) is a γ_μ-closed set (by Remark 2.3(b) and Definition 2.11(a)). Since $X \setminus U \subseteqq X \setminus A$, $F \subseteqq \gamma_\mu$-$c(A) \setminus A$. Thus by the assumption it follows that $F = \varnothing$ and hence we have γ_μ-$c(A) \subseteqq U$.

Definition 3.6. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation, where μ is a GT on X. Then (X, μ) is said to be a γ_μ-$T_{1/2}$ space if every $\gamma_\mu g$-closed set is a γ_μ-closed set.

The next theorem characterizes a γ_μ-$T_{1/2}$ GTS.

Theorem 3.7. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation, where μ is a GT on X. Then (X, μ) is γ_μ-$T_{1/2}$ if and only if for each $x \in X$, $\{x\}$ is either γ_μ-open or γ_μ-closed.

Proof. Suppose that that (X, μ) is γ_μ-$T_{1/2}$ and $\{x\}$ is not γ_μ-closed. Then by Theorem 3.4, $X \setminus \{x\}$ is $\gamma_\mu g$-closed. Since (X, μ) is γ_μ-$T_{1/2}$, $X \setminus \{x\}$ is γ_μ-closed. Thus $\{x\}$ is γ_μ-open.

Conversely, let F be a $\gamma_\mu g$-closed set in (X, μ). By Theorem 2.17, it is sufficient to show that γ_μ-$c(F) \subseteqq F$. If possible, let there exist a point $x \in \gamma_\mu$-$c(F) \setminus F$. Then by the given condition $\{x\}$ is either γ_μ-open or γ_μ-closed.
Case -1 : $\{x\}$ is γ_μ-closed : For this case we have a γ_μ-closed set $\{x\}$ such that $\{x\} \subseteqq \gamma_\mu$-$c(F) \setminus F$. This is contrary to Proposition 3.5.
Case -2 : $\{x\}$ is γ_μ-open : Then by Remark 2.13, $x \in c_{\gamma_\mu}(F)$. Thus $\{x\} \cap F \neq \varnothing$. This is a contradiction. Thus we have γ_μ-$c(F) \subseteqq F$.

Definition 3.8. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation, where μ is a GT on X. Then (X, μ) is said to be
(a) γ_μ-T_0 if for each pair of distinct points $x, y \in X$, there exists a μ-open set G such that either $x \in G$ and $y \notin \gamma_\mu(G)$, or $y \in G$ and $x \notin \gamma_\mu(G)$.
(b) γ_μ-T_1 if for each pair of distinct points $x, y \in X$, there exist μ-open sets G and H containing x and y, respectively, such that either $y \notin \gamma_\mu(G)$ and $x \notin \gamma_\mu(H)$.
(c) γ_μ-T_2 if for each pair of distinct points $x, y \in X$, there exist μ-open sets G and H containing x and y, respectively, such that $\gamma_\mu(G) \cap \gamma_\mu(H) = \varnothing$.

A γ_μ-T_1 GTS is characterized by the following theorem.

Theorem 3.9. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation, where μ is a GT on X. Then the following are equivalent:

(i) (X, μ) is γ_μ-T_1.

(ii) For each $x \in X$, $\{x\}$ is a γ_μ^*-closed set.

(iii) For each pair of distinct points $x, y \in X$ there exist γ_μ-open sets U and V containing x and y, respectively, such that either $y \notin U$ and $x \notin V$.

Proof. (i) $\Rightarrow$ (ii) : Let $x \in X$. We shall show that $\{x\}$ is γ_μ^*-closed. Let $y \notin \{x\}$. Then by (i) there exists a μ-open set U_y such that $y \in U_y$, $x \notin \gamma_\mu(U_y)$. Thus $\gamma_\mu(U_y) \cap \{x\} = \varnothing$. Thus $y \notin \gamma_\mu$-$c(\{x\})$. Thus $\{x\}$ is γ_μ^*-closed.

(ii) $\Rightarrow$ (iii) Let x and y be two points of X with $x \neq y$. Then by (ii) $\{x\}$ and $\{y\}$ are two γ_μ-closed sets and hence by Theorem 2.17, $X \setminus \{y\}$ and $X \setminus \{x\}$ are two γ_μ-open sets containing x and y, respectively, such that $x \in (X \setminus \{y\})$ and $y \in (X \setminus \{x\})$.

(iii) $\Rightarrow$ (i) : Obvious.

Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation, where μ is a GT on X. Then it follows from Definitions 3.6 and 3.8 that γ_μ-$T_2 \Rightarrow \gamma_\mu$-$T_1 \Rightarrow \gamma_\mu$-$T_{1/2} \Rightarrow \gamma_\mu$-$T_0$. None of the implications are reversible as shown in the next example.

Example 3.10. (a) Let $X = \{1,2,3\}$ and $\mu = \mathcal{P}(X)$. Then μ is a GT on X. Then $\gamma_\mu : \mu \to \mathcal{P}(X)$ defined by

$$\gamma_\mu(A) = \begin{cases} A \cup \{2\}, \text{ if } A = \{1\} \\ A \cup \{3\}, \text{ if } A = \{2\} \\ A \cup \{1\}, \text{ if } A = \{3\} \\ A, \text{otherwise} \end{cases}$$

is an operation. It can be checked that (X,μ) is γ_μ-T_1 but not a γ_μ-T_2 space.

(b) Let $X = \{1,2,3\}$ and $\mu = \mathcal{P}(X)$. Then μ is a GT on X. Then $\gamma_\mu : \mu \to \mathcal{P}(X)$ defined by

$$\gamma_\mu(A) = \begin{cases} A \cup \{3\}, \text{ if } A \neq \{1\} \\ A, \text{otherwise} \end{cases}$$

is an operation. It can be checked that (X,μ) is γ_μ-$T_{1/2}$ but not a γ_μ-T_1 space.

(c) Let $X = \{1,2,3\}$ and $\mu = \{\varnothing, \{1\}, \{1,2\}, \{1,3\}, X\}$. Then μ is a GT on X. Then $\gamma_\mu : \mu \to \mathcal{P}(X)$ defined by

$$\gamma_\mu(A) = \begin{cases} A, \text{ if } A \neq \{1\} \\ \{1,2\}, \text{otherwise} \end{cases}$$

is an operation. It can be checked that (X,μ) is γ_μ-T_0 but not a γ_μ-$T_{1/2}$ space.

Throughout the rest of the paper (X,μ) and (Y,λ) will denote GTS's and $\gamma_\mu : \mu \to \mathcal{P}(X)$ and $\beta_\lambda : \lambda \to \mathcal{P}(Y)$ will denote two operations on μ and λ respectively.

Definition 3.11. A function $f : (X,\mu) \to (Y,\lambda)$ is said to be (γ,β)-continuous if for each $x \in X$ and each λ-open set V with $f(x) \in V$ there exists a μ-open set U containing x such that $f(\gamma_\mu(U)) \subseteqq \beta_\lambda(V)$.

Theorem 3.12. A (γ,β)-continuous mapping $f : (X,\mu) \to (Y,\lambda)$ satisfies the following properties:

(i) $f(\gamma_\mu\text{-}c(A)) \subseteqq \beta_\lambda\text{-}c(f(A))$ for every subset A of X.

(ii) $f^{-1}(W)$ is γ_μ-open for every β_λ-open set W of Y, i.e., the inverse image of any β_λ-closed set of (Y,β) is γ_μ-closed in (X,μ).

Proof. (i) Let y be a point of $f(\gamma_\mu\text{-}c(A))$ and V be any λ-open set containing y. Then there exists a point x in X such that $f(x) = y$ and $x \in \gamma_\mu\text{-}c(A)$. Thus by (γ,β)-continuity of f there exists a μ-open set U containing x such that $f(\gamma_\mu(U)) \subseteqq \beta_\lambda(V)$. As $x \in \gamma_\mu\text{-}c(A)$, we have $\gamma_\mu(U) \cap A \neq \varnothing$, and hence $\varnothing \neq f(\gamma_\mu(U) \cap A) \subseteqq$ $\subseteqq f(\gamma_\mu(U)) \cap f(A) \subseteqq \beta_\lambda(V) \cap f(A)$. This shows that $y \in \beta_\lambda\text{-}c(f(A))$.

(ii) Let W be a β_λ-open set in (Y,λ) and x any point of $f^{-1}(W)$. We have to show that $f^{-1}(W)$ is γ_μ-open. There exists a β-open set V containing $f(x)$ such that $\beta_\lambda(V) \subseteqq W$. Thus by (γ,β)-continuity of f, there exists a μ-open set U containing x such that $f(\gamma_\mu(U)) \subseteqq \beta_\lambda(V)$. Thus $\gamma_\mu(U) \subseteqq f^{-1}(\beta_\lambda(V)) \subseteqq f^{-1}(W)$. Thus $f^{-1}(W)$ is γ_μ-open.

Definition 3.13. A function $f : (X,\mu) \to (Y,\lambda)$ is said to be (γ,β)-closed if for any γ_μ-closed set A of X, $f(A)$ is a β_λ-closed set in Y.

Let $id_\mu : \mu \to \mathcal{P}(X)$ be the identity operation, where (X,μ) is a GTS. We note that id_μ-open sets and μ-open sets are identical.

Proposition 3.14. Let $f : (X,\mu) \to (Y,\lambda)$ be a (γ,β)-continuous function and f be a (id,β)-closed mapping. The following properties hold:

(i) For each $\gamma_\mu g$-closed set A of X, $f(A)$ is $\beta_\lambda g$-closed in Y.

(ii) For each $\beta_\lambda g$-closed set B of Y, $f^{-1}(B)$ is $\gamma_\mu g$-closed.

Proof. (i) Let V be any β_λ-open set of (Y,λ) with $f(A) \subseteqq V$. Then by Theorem 3.12 (ii), $f^{-1}(V)$ is a γ_μ-open set. Now as A is a $\gamma_\mu g$-closed set and $A \subseteqq f^{-1}(V)$, we have $\gamma_\mu\text{-}c(A) \subseteqq f^{-1}(V)$, and thus $f(\gamma_\mu\text{-}c(A)) \subseteqq V$. From the assumption and Theorem 2.15(i) it follows that, $f(\gamma_\mu\text{-}c(A))$ is β_λ-closed. Thus by Remark 2.13, we have $\beta_\lambda\text{-}c(f(A)) \subseteqq c_{\beta_\lambda}((f(\gamma_\mu\text{-}c(A)))) = f(\gamma_\mu\text{-}c(A)) \subseteqq V$. This shows that $f(A)$ is $\beta_\lambda g$-closed in Y.

(ii) Let U be any γ_μ-open set of (X,μ) such that $f^{-1}(B)$ is contained in U. Let $F = \gamma_\mu\text{-}c(f^{-1}(B)) \cap (X \setminus U)$. Then F is μ-closed in (X,μ) (by Theorem 2.15(i) and Remark 2.3(a)). Since f is a (id,β)-closed function, $f(F)$ is a β_λ-closed set in (Y,λ). Then by Proposition 3.5 and the relation $f(F) \subseteqq \beta_\lambda\text{-}c(B) \setminus B$, it follows that $f(F) = \varnothing$ and thus $F = \varnothing$. Thus $\gamma_\mu\text{-}c(f^{-1}(B)) \subseteqq U$ i.e., $f^{-1}(B)$ is $\gamma_\mu g$-closed.

Theorem 3.15. Let $f : (X,\mu) \to (Y,\lambda)$ be a (γ,β)-continuous and (id,β)-closed function.

(i) If f is an injective function and (Y,λ) is a β_λ-$T_{1/2}$ space, then (X,μ) is a γ_μ-$T_{1/2}$ space.

(ii) If f is a surjective function and (X,μ) is a γ_μ-$T_{1/2}$ space, then (Y,λ) is a β_λ-$T_{1/2}$ space.

(iii) If f is bijective,, then (X,μ) is a γ_μ-$T_{1/2}$ space if and only if (Y,λ) is a β_λ-$T_{1/2}$ space.

Proof. (i) We need only to show that every $\gamma_\mu g$-closed set is γ_μ-closed. Let A be a $\gamma_\mu g$-closed set of (X,μ). It then follows from Proposition 3.14(i) that $f(A)$ is $\beta_\lambda g$-closed and thus $f(A)$ is β_λ-closed (as (Y,λ) is β_λ-$T_{1/2}$). Now by Theorem 3.12(ii), $f^{-1}(f(A))$ is γ_μ-closed (as f is (γ,β)-continuous) i.e., A is γ_μ-closed.

(ii) Let B be a $\beta_\lambda g$-closed set of (Y,λ). We have to show that B is a β_λ-closed set. By Theorem 3.14(ii), $f^{-1}(B)$ is a $\gamma_\mu g$-closed set in (X,μ). Thus $f^{-1}(B)$ is γ_μ-closed (as (X,μ) is γ_μ-$T_{1/2}$). Thus from the assumption it follows that $B(=ff^{-1}(B))$ is β_λ-closed in (Y,λ). Thus it follows that (Y,λ) is a β_λ-$T_{1/2}$ space.

(iii) The proof follows from (i) and (ii).

Theorem 3.16. Suppose that $f:(X,\mu)\to(Y,\lambda)$ is a (γ,β)-continuous bijection and $f^{-1}:(Y,\lambda)\to(X,\mu)$ is (β,γ)-continuous. Then (X,μ) is a γ_μ-$T_{1/2}$ space if and only if (Y,λ) is a β_λ-$T_{1/2}$ space.

Proof. Let (X,μ) be a γ_μ-$T_{1/2}$ space. In view of Theorem 3.7 it is sufficient to show that any singleton set of (Y,λ) is either β_λ-closed or β_λ-open. Let $\{y\}$ be any subset of (Y,λ). Then, since f is surjective, there exists $x\in X$ such that $f(x)=y$. Then by Theorem 3.7 it follows that $\{x\}$ is γ_μ-closed or γ_μ-open (as (X,μ) is γ_μ-$T_{1/2}$). Then by Theorem 3.12, $\{y\}(=f(\{x\}))$ is β_λ-closed or β_λ-open. Thus (Y,λ) is a β_λ-$T_{1/2}$ space. The proof of the converse is similar.

Proposition 3.17. Let $f:(X,\mu)\to(Y,\lambda)$ be a (γ,β)-continuous injection and (Y,λ) be a β_λ-T_2 (resp. β_λ-T_1) space. Then (X,μ) is a γ_μ-T_2 (resp. γ_μ-T_1) space.

Proof. Let (Y,λ) be a β_λ-T_2 space. Let x,y be any two points of X with $x\neq y$. Then there exist λ-open sets V and W of Y contaning $f(x)$ and $f(y)$ respectively such that $\beta_\lambda(V)\cap\beta_\lambda(W)=\varnothing$. Now by (γ,β)-continuity of f, there exist μ-open sets G and H containing x and y respectively such that $f(\gamma_\mu(G))\subseteqq\beta_\lambda(V)$ and $f(\gamma_\mu(H))\subseteqq\beta_\lambda(W)$. Thus $\gamma_\mu(G)\cap\gamma_\mu(H)=\varnothing$. Thus (X,μ) is a γ_μ-T_2 space.

The proof of the case of β_λ-T_1 can be done similarly.

Lemma 3.18. Let $\gamma_\mu:\mu\to\mathcal{P}(X)$ be a regular, μ-open operation and $X\in\mu$. If (X,μ) is a γ_μ-T_2 GTS, then (X,γ_μ) is a T_2 space.

Proof. We first note that since $\gamma_\mu:\mu\to\mathcal{P}(X)$ is regular and $X\in\mu$, by Theorem 2.9, γ_μ is a topology on X. Let x,y be two distinct points of X. Then there exist μ-open sets U and V containing x and y, respectively, such that $\gamma_\mu(U)\cap\gamma_\mu(V)=\varnothing$. Since γ_μ is μ-open, there exist γ_μ-open sets U^* and V^* containing x and y, respectively, such that $U^*\subseteqq\gamma_\mu(U)$ and $V^*\subseteqq\gamma_\mu(V)$. Thus $U^*\cap V^*=\varnothing$ and (X,γ_μ) is a T_2 space.

Theorem 3.19. Let $\gamma_\mu:\mu\to\mathcal{P}(X)$ be a μ-open regular operation and $\beta_\lambda:\lambda\to\mathcal{P}(Y)$ be a λ-open regular operation such that $X\in\mu$ and $Y\in\lambda$. If $f,g:(X,\mu)\to(Y,\lambda)$ are (γ,β)-continuous and (Y,λ) is β_λ-T_2, then the set $A=\{x\in X: f(x)=g(x)\}$ is γ_μ-closed in (X,μ).

Proof. We observe first by Lemma 3.18 that, γ_μ and β_λ are two topologies on X and Y, respectively. We shall now show that if $f:(X,\mu)\to(Y,\lambda)$ is (γ,β)-continuous, then $f:(X,\gamma_\mu)\to(Y,\beta_\lambda)$ is continuous. Let $x\in X$ and V be any β_λ-open set containing $f(x)$. Then there exists a λ-open set V' such that $f(x)\in V'$ and $\beta_\lambda(V')\subseteqq V$. Since f is (γ,β)-continuous, there exists a μ-open set W such that $x\in W$ and $f(\gamma_\mu(W))\subseteqq\beta_\lambda(V')\subseteqq V$. Then by μ-openness of γ_μ there exists a γ_μ-open set W' containing x such that $W'\subseteqq\gamma_\mu(W)$. Thus $f(W')\subseteqq V$. Thus $f:(X,\gamma_\mu)\to(Y,\beta_\lambda)$ is continuous and similarly $g:(X,\gamma_\mu)\to(Y,\beta_\lambda)$ is continuous. By Lemma 3.18, (Y,β_λ) is a T_2 space. Therefore the set $A=\{x\in X: f(x)=g(x)\}$ is closed in (X,γ_μ) and hence $X\setminus A$ is γ_μ-open. Thus A is γ_μ-closed in (X,μ).

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Жалпыланған топологиялық кеңістіктерге операцияларды қолдану

Мақалада γ_μ — ашық жиындар және γ_μ-GTS-тегі жабық жиындар (X, μ), мұнда γ_μ - μ-дан $\mathcal{P}(X)$-ға операция зерттелген. Жалпы, γ_μ — ашық жиындар жиынтығы μ-ашық жиындар жиынтығынан аз. Сонымен қатар, авторлар екі жиын бірдей болатынын анықтаған. Мұндай жиындардың кейбір қасиеттері талқыланды. Сондай-ақ, жабу түрінің кейбір операторлары анықталып, олардың қасиеттері анықталды. GTS (X, μ)-да ұқсас жабу операторларының түрлері арасында байланыс орнатылған. Белгілі бір тұйықталу түрінің операторы Куратовскийдің тұйықталу операторы болып табылатын шарт беріледі. Сондай-ақ, γ_μ деп аталатын жабық жиындардың жалпыланған түрі анықталған-жалпыланған жабық жиын, осы жаңадан анықталған жабу операторының көмегімен және осындай жиындардың кейбір негізгі қасиеттері талқыланған. Қосымша ретінде бөлімнің әлсіз аксиомалары енгізіліп, олардың кейбір қасиеттері талқыланды. Соңында осындай жалпыланған ұғымдарды сақтаудың кейбір теоремалары көрсетілген.

Кілт сөздер: операция, μ — ашық жиын, γ_μ — ашық жиын, $\gamma_\mu g$ — жабық жиын.

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Приложения операций над обобщенными топологическими пространствами

В статье изучены γ_μ-открытые и γ_μ-замкнутые множества в GTS (X, μ), где γ_μ — операция из μ в $\mathcal{P}(X)$. В общем случае набор γ_μ-открытых множеств меньше, чем набор μ-открытых множеств. Кроме того, авторами установлено условие, при котором оба множества являются одинаковыми. Обсуждены и некоторые свойства таких множеств. Определены некоторые операторы типа замыкания и их свойства. Установлена связь между аналогичными типами операторов замыкания на GTS (X, μ). Дано условие, при котором по-новому определенный оператор типа замыкания является оператором замыкания Куратовского. Выявлен обобщенный тип замкнутых множеств, названный γ_μ-обобщенным замкнутым множеством, с помощью этого вновь определенного оператора замыкания и обсуждены некоторые основные свойства таких множеств. В качестве приложения авторами введены несколько слабых аксиом отделения и определены некоторые их свойства. Таким образом, показаны некоторые теоремы сохранения таких обобщенных понятий.

Ключевые слова: операция, μ-открытое множество, γ_μ-открытое множество, $\gamma_\mu g$-замкнутое множество.

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Analytical solution of a fractional differential equation in the theory of viscoelastic fluids

The aim of this paper is to present analytical solutions of fractional delay differential equations (FDDEs) of an incompressible generalized Oldroyd-B fluid with fractional derivatives of Caputo type. Using a modification of the method of separation of variables the main equation with non-homogeneous boundary conditions is transformed into an equation with homogeneous boundary conditions, and the resulting solutions are then expressed in terms of Green functions via Laplace transforms. This results presented in two condition, in first step when $0 \leq \alpha, \beta \leq \frac{1}{2}$ and in the second step we considered $\frac{1}{2} \leq \alpha, \beta \leq 1$, for each step 1,2 for the unsteady flows of a generalized Oldroyd-B fluid, including a flow with a moving plate, are considered via examples.

Keywords: Oldroyd-B fluid, fractional-order partial differential equations, analytical solutions, Delay differential equation, modified separation of variables method, Caputo fractional derivatives.

Introduction

Many real-world processes can be cast generally in the form of fractional differential systems with integer order (i.e., ordinary differential equations and systems) but there is a growing number of researchers that believe that fractional-differential equations can describe and model and complex physical processes more accurately than the corresponding ordinary differential equations. So, in recent decades the search for analytical and numerical solutions to fractional differential equations has been of considerable interest [1–4]. Fractional differential equations can be applied to the dynamic modeling of non-Newtonian fluids: for example, in the modeling of melting plastics and in the study of emulsion plastics or soft tissue. Practically speaking, there are few Newtonian fluids in reality, so most fluids are of the non-Newtonian type, which means there is no linear relationship between the stress tensor and the deformation tensor [5].

Viscoelastic fluids form an important class of non-Newtonian fluids, which exhibit both elastic and viscous properties. Among them the so-called *Oldroyd-B* fluid can be used to describe the response of fluids that have a small memory. This means that whenever they flow, these fluids will spend less time to find the first state and stability [6–7]. Due to the wide range of applications of these fluids, considerable attention has been paid to the prediction of the behavior of non-Newtonian fluids. Structural equations that are presented in a constitutive rheological fashion have a fractional calculation, so they are very effective for working with viscoelastic properties [8–9]. The viscoelastic fluid equations in fractional models are obtained by replacing ordinary derivatives with one of many possible definitions of fractional derivatives in the defining equations. In the study of fluids we deal with a phenomenon called delay, which is due to the distance between the sensor and the source of changes arising from e.g., plumbing, measurement slowness, or complex dynamics. Different methods for finding analytical solutions of these type of equations are proposed: an analytical solution for unsteady helical flows is presented by Tong *et al* in [10]. In Haitao and Mingyu [11] there is a discussion of an Oldroyd-B fluid between two parallel plates. In addition, Fetecau [12–13] developed a generalization of the flow of viscoelastic fluids between two-sided walls. Then Shah [14], Qi [15], Zheng *et al* [16] and Hayat [17] discussed the generalized flow of an Oldroyd-B fluid under varying conditions. In closing this brief review we mention that Javidi and Heris [18] gave analytical solutions of various forms of such delay equations.

Many events in the natural world can be modeled to form of fractional delay differential equations (FDDEs). FDDEs have important applications in many fields for example technology, economics, biology, medical science,

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physics and finance [19]. Some numerical methods for FDDEs are introduced in [20-23] and etc. Heris and Javidi [24] proposed a numerical method based on fractional backward differential formulas (FBDF) for solving fractional delay differential equations. Also they found the Green's functions for this equation corresponding to periodic/anti-periodic conditions in terms of the functions of Mittag Leffler type.

In this paper we present analytical solutions for unsteady flows of a generalized Oldroyd-B fluid with constant delay time using Riemann-Liouville fractional derivatives as the defining derivatives. A new separation of variables method [25] and use of Laplace transforms for the Riemann-Liouville fractional derivative are adapted to solve the new governing equation for fractional differential equations with constant delay when applied to viscoelastic fluids.

The paper is structured as follows: in section 2 we recall some basic definitions of fractional calculus; in section 3 we give the derivation of the governing equation; section 4 deals with the method of separation of variables, the Laplace transformation applied to fractional derivatives in two steps $0 \le \alpha,\ \beta \le \frac{1}{2}$ and $\frac{1}{2} \le \alpha$, $\beta \le 1$, and the method of solution for each two steps separately. Finally, in section 5 we give the examples dealing with varying initial conditions by considering two condition for α and β.

Preliminaries

In this section we will introduce some of the fundamental definitions.

Defenition 1.1 ([1]). Euler's gamma function is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \ \ Re(z) > 0.$$

$C(J, R)$ denotes the Banach space of all continuous functions from $J = [0, T]$ into R with the norm

$$\|u\|_\infty = \sup\{|u(t)| : t \in J\}, \quad T > 0.$$

$C^n(J, R)$ denotes the class of all real valued functions defined on $J = [0, T]$, $T > 0$ which have continuous n-th order derivatives.

Defenition 1.2 [4]. The fractional integral of order $\alpha > 0$ of the function $f \in C(J, R)$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad 0 < t < T.$$

Defenition 1.3 [4]. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of the function $f \in C(J, R)$ is defined as

$$^{RL}D^\alpha f(t) = \begin{cases} D^n I^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d^n}{dt^n}) \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, & \\ \qquad\qquad n-1 < \alpha < n, \ n \in N, & \\ f^{(n)}(t), \qquad\qquad \alpha = n. & \end{cases}$$

Defenition 1.4 [4]. The Caputo fractional derivative of order $\alpha > 0$ of the function $f \in C^n(J, R)$ is defined as

$$^{C}D^\alpha f(t) = \begin{cases} I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, & \\ \qquad\qquad n-1 < \alpha < n, \ n \in N, & \\ f^{(n)}(t), \qquad\qquad \alpha = n. & \end{cases}$$

Defenition 1.5 [4]. Mittag-leffler functions are defined by

$$E_{\alpha,\beta}(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(\alpha k + \beta)}\ , \ \ x, \beta \in C, Re(\alpha) > 0, E_\alpha(x) = E_{\alpha,1}.$$

Defenition 1.6 [22]. The generalized delay exponential function (of Mittag–Leffler type) is given by

$$G^{\lambda,\tau,m}_{\alpha,\beta}(t) = \sum_{j=0}^\infty \begin{pmatrix} j+m \\ j \end{pmatrix} \frac{\lambda^j (t-(m+j)\tau)^{\alpha(m+j)+\beta-1}}{\Gamma(\alpha(m+j)+\beta)} H(t-(m+j)\tau), \, t > 0,$$

where $\lambda \in C$, $\alpha, \beta, \tau \in R$ and $m \in Z$ and $H(z)$ is the Heaviside step function. If $\lambda \in C$, $\alpha, \beta, \tau \in R$ and $m \in Z$ then laplace transform of $G_{\alpha,\beta}^{\lambda,\tau,m}(t)$ is:

$$L(G_{\alpha,\beta}^{\lambda,\tau,m}(t))(s) = \frac{s^{\alpha-\beta}\exp\{-ms\tau\}}{(s^{\alpha} - \lambda\exp\{-s\tau\})^{m+1}}, \quad s > 0.$$

Governing equations

The fundamental equations governing the unsteady motion of an incompressible fluid are

$$\operatorname{div} V = 0, \tag{1}$$

$$\rho\frac{dV}{dt} = -\nabla p + \operatorname{div} S + F_b. \tag{2}$$

The constitutive equation for a generalized Oldroyd-B fluid is given by [15-16],

$$(1 + \lambda^{\alpha}\frac{D^{\alpha}}{Dt^{\alpha}})S = \mu(1 + \lambda^{\beta}\frac{D^{\beta}}{Dt^{\beta}})A_1 \ , \tag{3}$$

where $V = (u, v, w)$ is the fluid velocity, $S = (S_{i,j})$ is the extra-stress tensor, $A_1 = (\nabla V) + (\nabla V)^T$ present the first Rivlin-Ericksen tensor, ∇ is the gradient operator, and p is the pressure. Here $F_b = (F_{bx}, F_{by}, F_{bz})$ is the body force, ρ, μ are the density and the dynamic viscosity coefficient of the fluid respectively, λ_{α} and λ_{β} are the material constants that represent the relaxation time and retardation time, respectively, and α, β denote the orders of the fractional derivatives, i.e., real numbers that satisfy $0 \leq \alpha, \beta \leq 1$. Furthermore, $\frac{D^{\alpha}}{Dt^{\alpha}}$ and $\frac{D^{\beta}}{Dt^{\beta}}$ are fractional material derivatives that can be expressed as

$$\frac{D^{\alpha}S}{Dt^{\alpha}} = D_t^{\alpha}S + (V.\nabla)S - (\nabla.V)S - S(\nabla V)^T, \tag{4}$$

$$\frac{D^{\beta}S}{Dt^{\beta}} = D_t^{\beta}S + (V.\nabla)S - (\nabla.V)S - S(\nabla V)^T. \tag{5}$$

In Eq. (3), (5), the fractional derivative operator D^{α} is taken in the Caputo.

We consider unidirectional flow, that is the case where the velocity and the stress take the form

$$V = u(y,t)i, \qquad S = S(y,t),$$

where i is the unit vector along the x-direction of the Cartesian coordinate system x, y and z. Using Eq. (6) below, the continuity Eq.(1) is satisfied automatically while Eq. (4), bearing in mind the initial condition $S(y, 0) = 0$, leads to the following relationships for the constitutive equation

$S_{xz} = S_{zy} = S_{yz} = S_{zz} = S_{yy} = 0, \qquad S_{yx} = S_{xy}, S_{zx} = S_{xz},$

$$(1 + \lambda_{\alpha}D_t^{\alpha})\, S_{xy} = \ \mu\left(1 + \lambda_{\beta}D_t^{\beta}\right)\frac{\partial u}{\partial y} \ , \tag{6}$$

$$(1 + \lambda_{\alpha}D_t^{\alpha})\, S_{xx} - 2\lambda_{\alpha}S_{xy}\tfrac{\partial u}{\partial y} = -2\mu\lambda_{\beta}\left(\tfrac{\partial u}{\partial y}\right)^2 .$$

Substituting Eqs.(6) into momentum equation (2), we have the following equation in x-direction:

$$(1 + \lambda_{\alpha}D_t^{\alpha})\frac{\partial u}{\partial t} = v\left(1 + \lambda_{\beta}D_t^{\beta}\right)\frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho}(1 + \lambda_{\alpha}D_t^{\alpha})\left(F_{bx} - \frac{\partial p}{\partial x}\right), \tag{7}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity coefficient of fluid.
The constitutive equation of a generalized Burgers fluid is

$$\left(1 + \lambda_{\alpha}\frac{D^{\alpha}}{Dt^{\alpha}} + \theta\frac{D^{2\alpha}}{Dt^{2\alpha}}\right)\ S = \ \mu\left(1 + \lambda_{\beta}\frac{D^{\beta}}{Dt^{\beta}}\right)A_1 \ , \quad (0 < \alpha \ , \beta \leq 1)\, , \tag{8}$$

where θ is the material constant.

Combining the constitutive equation (8) with the equation (2) we get the following fractional Burgers fluid model

$$\left(1+\lambda_\alpha D_t^\alpha+\theta D_t^{2\alpha}\right)\frac{\partial u}{\partial t}=v\left(1+\lambda_\beta D_t^\beta\right)\frac{\partial^2 u}{\partial y^2}+\frac{1}{\rho}\left(1+\lambda_\alpha D_t^\alpha+\theta D_t^{2\alpha}\right)\left(F_{bx}-\frac{\partial p}{\partial x}\right), \tag{9}$$

where $\nu = {}^{\mu}/_{\rho}$. Eqs. (7)and (9) have the following form:

$$a_0 {D_t}^{2\alpha+1}u(y,t)+a_1 {D_t}^{\alpha+1}u(y,t)+a_2 {D_t}^{2\alpha}u(y,t)+a_3 {D_t}^{1}u(y,t)$$

$$+a_4 {D_t}^{\alpha}u(y,t)+a_5 u(y,t)=b_1 D_t^\beta\frac{\partial^2 u(y,t)}{\partial y^2}+b_2\frac{\partial^2 u(y,t)}{\partial y^2}+\bar{f}\left(y,t\right), \tag{10}$$

the delay form of Eqs (10) is

$$a_0 {D_t}^{2\alpha+1}u(y,t)+a_1 {D_t}^{\alpha+1}u(y,t)+a_2 {D_t}^{2\alpha}u(y,t)+a_3 {D_t}^{1}u(y,t)$$

$$+a_4 {D_t}^{\alpha}u(y,t)+a_5 u(y,t-\tau)=b_1 D_t^\beta\frac{\partial^2 u(y,t)}{\partial y^2}+b_2\frac{\partial^2 u(y,t)}{\partial y^2}+\bar{f}\left(y,t\right).$$

The associated initial and boundary conditions are as follows:

$$u\left(y,t\right)=\psi_1\left(y,t\right),\qquad u\left(0,t\right)=\varphi_1\left(t\right),-\tau\le t\le 0,$$

$$u_t\left(y,t\right)=\psi_2\left(y,t\right),\qquad u\left(L,t\right)=\varphi_1\left(t\right),\qquad 0<\alpha,\beta<1.$$

A method of separation of variables

At first, the problem involves non-homogeneous boundary conditions. We want to transform it into a problem with homogeneous boundary conditions. So, consider

$$u\left(y,t\right)=W\left(y,t\right)+V\left(y,t\right), \tag{11}$$

where

$$V\left(y,t\right)=\left(1-\frac{y}{L}\right)\varphi_1\left(t\right)+\frac{y}{L}\varphi_2\left(t\right), \tag{12}$$

which satisfies the boundary conditions

$$V\left(0,t\right)=\varphi_1\left(t\right),V\left(L,t\right)=\varphi_2\left(t\right).$$

Using Eqs.(11) and Eqs.(12) along with the associated initial and boundary conditions above, we have

$W\left(y,t\right)+\left(1-\frac{y}{L}\right)\varphi_1\left(t\right)+\frac{y}{L}\varphi_2\left(t\right)=\psi_1\left(y,t\right),\qquad -\tau\le t\le 0,$
$W_t\left(y,t\right)+\left(1-\frac{y}{L}\right){\varphi'}_1\left(t\right)+\frac{y}{L}{\varphi'}_2\left(t\right)=\psi_2\left(y,t\right),$
$W\left(L,t\right)+V\left(L,t\right)=\varphi_2\left(t\right),$
$W\left(L,t\right)+V\left(L,t\right)=\varphi_2(t),$
$W\left(y,t\right)=\psi_1\left(y,t\right)-\left(1-\frac{y}{L}\right)\varphi_1\left(t\right)-\frac{y}{L}\varphi_2\left(t\right)=\overline{\psi_1}\left(y,t\right),$
$W_t\left(y,t\right)=\psi_1\left(y,t\right)-\left(1-\frac{y}{L}\right){\varphi'}_1\left(t\right)-\frac{y}{L}{\varphi'}_2\left(t\right)=\overline{\psi_2}(y,t).$

Now main problem is solving

$$a_0 {D_t}^{2\alpha+1}W\left(y,t\right)+a_1 {D_t}^{\alpha+1}W\left(y,t\right)+a_2 {D_t}^{2\alpha}W\left(y,t\right)+a_3 {D_t}^{1}W\left(y,t\right)+$$

$$+a_4 {D_t}^{\alpha}W\left(y,t\right)+a_5 W\left(y,t-\tau\right)-b_1 D_t^\beta\frac{\partial^2 w\left(y,t\right)}{\partial y^2}-b_2\frac{\partial^2 w\left(y,t\right)}{\partial y^2}=$$

$$=-a_0 {D_t}^{2\alpha+1}V\left(y,t\right)-a_1 {D_t}^{\alpha+1}V\left(y,t\right)-a_2 {D_t}^{2\alpha}V\left(y,t\right)-a_3 {D_t}^{1}V\left(y,t\right),$$

where the initial condition is

$$\sum_{n=1}^{\infty}B_n\left(0\right)\sin\frac{n\pi y}{L}=\sum_{n=1}^{\infty}d_n^{(1)}\left(0\right)\sin\frac{n\pi y}{L}-\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[\varphi_1\left(0\right)-\left(-1\right)^n\varphi_2\left(0\right)\right]\sin\frac{n\pi y}{L},$$

$$\sum_{n=1}^{\infty} B'_n(0)\sin\frac{n\pi y}{L} = \sum_{n=1}^{\infty} d_n^{(2)}(0)\sin\frac{n\pi y}{L} - \sum_{n=1}^{\infty}\frac{2}{n\pi}\left[\varphi'_1(0) - (-1)^n\varphi'_2(0)\right]\sin\frac{n\pi y}{L},$$

and

$$d_n^{(i)} = \frac{2}{L}\int_0^L \bar{\psi}_i\ (y,0)\ sin\frac{n\pi y}{L}dy, \quad i = 1,2.$$

Let

$$W(y,t) = \sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi y}{L},$$

$$\overline{\psi_i}\ (y) = \sum_{n=1}^{\infty} d_n^{(i)}\sin\frac{n\pi y}{L} \quad (i = 1,2,\ldots,m).$$

Then, we have

$$a_0 {D_t}^{2\alpha+1}\sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi y}{L} + a_1 {D_t}^{\alpha+1}\sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi y}{L} + a_2 {D_t}^{2\alpha}\sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi y}{L} +$$

$$+ a_3 {D_t}^{1}\sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi y}{L} + a_4 {D_t}^{\alpha}\sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi y}{L} + a_5\sum_{n=1}^{\infty} B_n(t-\tau)\sin\frac{n\pi y}{L} -$$

$$-b_1\left(\frac{n\pi}{L}\right)^2 D_t^{\beta}\sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi y}{L} - b_2\left(\frac{n\pi}{L}\right)^2\sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi y}{L} =$$

$$= -a_0\frac{2}{n\pi}{D_t}^{2\alpha+1}\sum_{n=1}^{\infty}\left[\varphi_1(t) - (-1)^n\varphi_2(t)\right]\sin\frac{n\pi y}{L} - a_1\frac{2}{n\pi}{D_t}^{\alpha+1}\sum_{n=1}^{\infty}\left[\varphi_1(t) - (-1)^n\varphi_2(t)\right]\sin\frac{n\pi y}{L} -$$

$$-a_2\frac{2}{n\pi}{D_t}^{2\alpha}\sum_{n=1}^{\infty}\left[\varphi_1(t) - (-1)^n\varphi_2(t)\right]\sin\frac{n\pi y}{L} - a_3\frac{2}{n\pi}{D_t}^{1}\sum_{n=1}^{\infty}\left[\varphi_1(t) - (-1)^n\varphi_2(t)\right]\sin\frac{n\pi y}{L} -$$

$$-a_4\frac{2}{n\pi}{D_t}^{\alpha}\sum_{n=1}^{\infty}\left[\varphi_1(t) - (-1)^n\varphi_2(t)\right]\sin\frac{n\pi y}{L} - a_5\frac{2}{n\pi}\sum_{n=1}^{\infty}\left[\varphi_1(t-\tau) - (-1)^n\varphi_2(t-\tau)\right]\sin\frac{n\pi y}{L} +$$

$$+\sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi y}{L}.$$

Equating coefficients leads to

$$a_0 {D_t}^{2\alpha+1}B_n(t) + a_1 {D_t}^{\alpha+1}B_n(t) + a_2 {D_t}^{2\alpha}B_n(t) + a_3 {D_t}^{1}B_n(t) + \tag{13}$$

$$+a_4 {D_t}^{\alpha}B_n(t) + a_5 B_n(t-\tau) - b_1\left(\frac{n\pi}{L}\right)^2 D_t^{\beta}B_n(t) - b_2\left(\frac{n\pi}{L}\right)^2 B_n(t) =$$

$$= -a_0\frac{2}{n\pi}{D_t}^{2\alpha+1}\left[\varphi_1(t) - (-1)^n\varphi_2(t)\right] - a_1\frac{2}{n\pi}{D_t}^{\alpha+1}\left[\varphi_1(t) - (-1)^n\varphi_2(t)\right] -$$

$$-a_2\frac{2}{n\pi}{D_t}^{2\alpha}\left[\varphi_1(t) - (-1)^n\varphi_2(t)\right] - a_3\frac{2}{n\pi}{D_t}^{1}\left[\varphi_1(t) - (-1)^n\varphi_2(t)\right] -$$

$$-a_4\frac{2}{n\pi}{D_t}^{\alpha}\left[\varphi_1(t) - (-1)^n\varphi_2(t)\right] - a_5\frac{2}{n\pi}\left[\varphi_1(t-\tau) - (-1)^n\varphi_2(t-\tau)\right] + f_n(t),$$

with the boundary conditions

$$B_n(0) = d_n^{(1)}(0) - \frac{2}{n\pi}\varphi_1(0) + (-1)^n\frac{2}{n\pi}\varphi_2(0),$$

$$B'_n(0) = d_n^{(2)}(0) - \frac{2}{n\pi}\varphi'_1(0) + (-1)^n\frac{2}{n\pi}\varphi'_2(0).$$

In this part we divide the main problem in two part

3.1 $(0 \le \alpha, \beta \le \frac{1}{2})$

when $\frac{1}{2} \le \alpha, \beta \le 1$ and applying the Laplace transform with respect to t defined by

$$\bar{B}_n(s) = \int_0^\infty e^{-st} B_n(t)\, dt.$$

In Eq.(13), we obtain

$$a_0 s^{2\alpha+1}\bar{B}_n(s) - a_0 s^{2\alpha} B_n(0) + a_1 s^{\alpha+1}\bar{B}_n(s) - a_1 s^{\alpha} B_n(0) + a_2 s^{2\alpha}\bar{B}_n(s) - a_2 s^{2\alpha-1} B_n(0) +$$

$$+a_3 s\bar{B}_n(s) - a_3 B_n(0) + a_4 s^{\alpha}\bar{B}_n(s) - a_4 s^{\alpha-1} B_n(0) + a_5 e^{-s\tau}\left[\int_{-\tau}^{0} e^{-sp} B_n(p)\, dp\right] -$$

$$-a_5 e^{-s\tau}\bar{B}_n(s) - b_1\left(\frac{n\pi}{L}\right)^2 s^{\beta}\bar{B}_n(s) + b_1\left(\frac{n\pi}{L}\right)^2 s^{\beta-1} B_n(0) - b_2\left(\frac{n\pi}{L}\right)^2 \bar{B}_n(s) =$$

$$= -a_0\frac{2}{n\pi} s^{2\alpha+1}\left[\overline{\varphi_1}(s) - (-1)^n\overline{\varphi_2}(s)\right] + a_0\frac{2}{n\pi} s^{2\alpha}\left[d_n^{(1)}(0) - B_n(0)\right] -$$

$$-a_1\frac{2}{n\pi} s^{\alpha+1}\left[\overline{\varphi_1}(s) - (-1)^n\overline{\varphi_2}(s)\right] + a_1\frac{2}{n\pi} s^{\alpha}\left[d_n^{(1)}(0) - B_n(0)\right] -$$

$$-a_2\frac{2}{n\pi} s^{2\alpha}\left[\overline{\varphi_1}(s) - (-1)^n\overline{\varphi_2}(s)\right] + a_2\frac{2}{n\pi} s^{2\alpha-1}\left[d_n^{(1)}(0) - B_n(0)\right] -$$

$$-a_3\frac{2}{n\pi} s\left[\overline{\varphi_1}(s) - (-1)^n\overline{\varphi_2}(s)\right] + a_3\frac{2}{n\pi}\left[d_n^{(1)}(0) - B_n(0)\right] -$$

$$-a_4\frac{2}{n\pi} s^{\alpha}\left[\overline{\varphi_1}(s) - (-1)^n\overline{\varphi_2}(s)\right] + a_4\frac{2}{n\pi} s^{\alpha-1}\left[d_n^{(1)}(0) - B_n(0)\right] - a_5\frac{2}{n\pi} e^{-s\tau}\left[\overline{\varphi_1}(s) - (-1)^n\overline{\varphi_2}(s)\right] +$$

$$+a_5\frac{2}{n\pi} e^{-s\tau}\left[\int_{-\tau}^{0} e^{-sp}\left[\varphi_1(p) - (-1)^n\varphi_2(p)\right] dp\right] + F_n(s).$$

By assumption $H(S) = \int_{-\tau}^{0} e^{-sp}\left[\varphi_1(p) - (-1)^n\varphi_2(p)\right] dp$, $G(s) = \int_{-\tau}^{0} e^{-sp} B_n(p)\, dp$ and $k_n = \frac{n\pi}{L}$, so we can write

$$\bar{B}_n(s) = \frac{B_n(0)\left[a_0 s^{2\alpha} + a_1 s^{\alpha} + a_2 s^{2\alpha-1} + a_3 + a_4 s^{\alpha-1} - b_1 {k_n}^2 s^{\beta-1}\right]}{a_0 s^{2\alpha+1} + a_1 s^{\alpha+1} + a_2 s^{2\alpha} + a_3 s + a_4 s^{\alpha} - a_5 e^{-s\tau} - b_1 {k_n}^2 s^{\beta} - b_2 {k_n}^2} +$$
$$+\frac{-\frac{2}{k_n L}\left[\overline{\varphi_1}(s) - (-1)^n\overline{\varphi_2}(s)\right]\left[a_0 s^{2\alpha+1} + a_1 s^{\alpha+1} + a_2 s^{2\alpha} + a_3 s + a_4 s^{\alpha} + a_5 e^{-s\tau}\right]}{a_0 s^{2\alpha+1} + a_1 s^{\alpha+1} + a_2 s^{2\alpha} + a_3 s + a_4 s^{\alpha} - a_5 e^{-s\tau} - b_1 {k_n}^2 s^{\beta} - b_2 {k_n}^2} +$$
$$+\frac{\frac{2}{k_n L}\left[d_n^{(1)}(0) - B_n(0)\right]\left\{a_0 s^{2\alpha} + a_1 s^{\alpha} + a_2 s^{2\alpha-1} + a_3 + a_4 s^{\alpha-1}\right\}}{a_0 s^{2\alpha+1} + a_1 s^{\alpha+1} + a_2 s^{2\alpha} + a_3 s + a_4 s^{\alpha} - a_5 e^{-s\tau} - b_1 {k_n}^2 s^{\beta} - b_2 {k_n}^2} +$$
$$+\frac{-a_5 G(s) e^{-s\tau} + a_5\frac{2}{k_n L} e^{-s\tau} H(S) + F_n(s)}{a_0 s^{2\alpha+1} + a_1 s^{\alpha+1} + a_2 s^{2\alpha} + a_3 s + a_4 s^{\alpha} - a_5 e^{-s\tau} - b_1 {k_n}^2 s^{\beta} - b_2 {k_n}^2}. \quad (14)$$

Using Eq.(14) we rewrite Eq.(13) as

$$\bar{B}_n(s) = e^{sm\tau}\sum_{m=0}^{\infty}\sum_{k,i,j,l,n,q\ge 0}^{k+i+j+l+n+q=m}\frac{(-1)^m}{{a_0}^{m+1}}\frac{m!\left(-{k_n}^2\right)^{n+q}}{k!i!j!l!q!}{a_1}^k{a_2}^i{a_3}^j{a_4}^l{b_1}^n{b_2}^q$$

$$\{B_n(0)[a_0\frac{s^{\alpha(k+2i+l+2)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_1\frac{s^{\alpha(k+2i+l+1)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+$$

$$+a_2\frac{s^{\alpha(k+2i+l+2)+k+\beta n-1}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_3\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+$$

$$+a_4\frac{s^{\alpha(k+2i+l+1)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_5e^{-s\tau}\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}]-$$

$$-\frac{2}{k_nL}\left[\overline{\varphi_1}(s)-(-1)^n\overline{\varphi_2}(s)\right][a_0\frac{s^{\alpha(k+2i+l+2)+k+\beta n+1}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_1\frac{s^{\alpha(k+2i+l+1)+k+\beta n+1}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+$$

$$+a_2\frac{s^{\alpha(k+2i+l+2)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_3\frac{s^{\alpha(k+2i+l)+k+\beta n+1}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+$$

$$+a_4\frac{s^{\alpha(k+2i+l+1)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_5e^{-s\tau}\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}]+$$

$$+\frac{2}{k_nL}\left[d_n^{(1)}(0)-B_n(0)\right][a_0\frac{s^{\alpha(k+2i+l+2)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_1\frac{s^{\alpha(k+2i+l+1)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+$$

$$+a_2\frac{s^{\alpha(k+2i+l+2)+k+\beta n-1}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_3\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_4\frac{s^{\alpha(k+2i+l+1)+k+\beta n-1}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}]-$$

$$-a_5G(s)e^{-s\tau}\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+a_5\frac{2}{k_nL}e^{-s\tau}H(S)\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}+$$

$$+F_n(s)\frac{s^{\alpha(k+2i+l)+k+\beta n}e^{-sm\tau}}{\left(s^{2\alpha+1}-\frac{a_5}{a_0}e^{-s\tau}\right)^{m+1}}\}.$$

Applying the discrete inverse Laplace transform to the preceding equation, we obtain

$$B_n(t)=\sum_{m=0}^{\infty}\sum_{k,i,j,l,n,q\geq 0}^{k+i+j+l+n+q=m}\frac{(-1)^m}{{a_0}^{m+1}}\frac{m!\left(-{k_n}^2\right)^{n+q}}{k!i!j!l!q!}{a_1}^k{a_2}^i{a_3}^j{a_4}^l{b_1}^n{b_2}^q$$

$$\{B_n(0)H(t-m\tau)[a_0G_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+1}^{\left(\frac{a_5}{a_0}\right),\tau,m}(t-m\tau)+a_1G_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+1}^{\left(\frac{a_5}{a_0}\right),\tau,m}(t-m\tau)+$$

$$+a_2G_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+2}^{\left(\frac{a_5}{a_0}\right),\tau,m}(t-m\tau)+a_3G_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}^{\left(\frac{a_5}{a_0}\right),\tau,m}(t-m\tau)+$$

$$+a_4G_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+2}^{\left(\frac{a_5}{a_0}\right),\tau,m}(t-m\tau)-b_1{k_n}^2G_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+2}^{\left(\frac{a_5}{a_0}\right),\tau,m}(t-m\tau)]-$$

$$-\frac{2}{k_nL}[\int_0^t\left[\varphi_1(t-u)-(-1)^n\varphi_2(t-u)\right]H(u-m\tau)(a_0G_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n}^{\left(\frac{a_5}{a_0}\right),\tau,m}(u-m\tau)+$$

$$+a_1G_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n}^{\left(\frac{a_5}{a_0}\right),\tau,m}(u-m\tau)+a_2G_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+1}^{\left(\frac{a_5}{a_0}\right),\tau,m}(u-m\tau)+$$

$$+a_3G_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n}^{\left(\frac{a_5}{a_0}\right),\tau,m}(u-m\tau)+a_4G_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+1}^{\left(\frac{a_5}{a_0}\right),\tau,m}(u-m\tau))du]-$$

$$-a_5\frac{2}{k_nL}\int_0^t\left[\varphi_1(t-u)-(-1)^n\varphi_2(t-u)\right]H(u-\tau(m+1))$$

$$G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(u-\tau(m+1))\,du+$$

$$+\frac{2}{k_nL}\left[d_n^{(1)}(0)-B_n(0)\right]\left[a_0G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+1}(t-m\tau)+\right.$$

$$+a_1G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+1}(t-m\tau)+a_2G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+2}(t-m\tau)+$$

$$\left.+a_3G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(t-m\tau)+a_4G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+2}(t-m\tau)\right]-$$

$$-a_5\int_0^t g(t-u)\,H(u-\tau(m+1))\,G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(u-\tau(m+1))\,du-$$

$$-a_5\frac{2}{k_nL}\int_0^t h(t-u)\,H(u-\tau(m+1))\,G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(u-\tau(m+1))\,du+$$

$$+\int_0^t f_n(t-u)\,H(u-m\tau)\,G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(u-\tau(m+1))\,du.$$

Once the $B_n(t)$ are known, so are the $W(y,t)$, and thus $u(y,t)$ as desired.

3.2 ($\frac{1}{2}\leq\alpha,\beta\leq 1$)

In the same way in the subsection 3.1 we could have

$$B_n(t)=\sum_{m=0}^{\infty}\sum_{k,i,j,l,n,q\geq 0}^{k+i+j+l+n+q=m}\frac{(-1)^m}{a_0{}^{m+1}}\frac{m!\left(-k_n{}^2\right)^{n+q}}{k!i!j!l!q!}a_1{}^k a_2{}^i a_3{}^j a_4{}^l b_1{}^n b_2{}^q$$

$$\left\{B_n(0)\,H(t-m\tau)\left[a_0G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+1}(t-m\tau)+a_1G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+1}(t-m\tau)+\right.\right.$$

$$+a_2G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+2}(t-m\tau)+a_3G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}(t-m\tau)+$$

$$\left.+a_4G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+2}(t-m\tau)-b_1k_n{}^2G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+2}(t-m\tau)\right]+$$

$$+B'_n(0)\,H(t-m\tau)\left[a_0G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+2}(t-m\tau)+a_1G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+2}(t-m\tau)+\right.$$

$$\left.+a_2G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+3}(t-m\tau)+a_4G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+3}(t-m\tau)\right]+$$

$$+B''_n(0)\,a_0G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+3}(t-m\tau)-$$

$$-\frac{2}{k_nL}\left[\int_0^t\left[\varphi_1(t-u)-(-1)^n\varphi_2(t-u)\right]H(u-m\tau)\left(a_0G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n}(u-m\tau)+\right.\right.$$

$$+a_1G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n}(u-m\tau)+a_2G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+1}(u-m\tau)+$$

$$\left.\left.+a_3G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n}(u-m\tau)+a_4G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+1}(u-m\tau)\right)du\right]-$$

$$-a_5\frac{2}{k_nL}\int\limits_0^t\left[\varphi_1\left(t-u\right)-\left(-1\right)^n\varphi_2\left(t-u\right)\right]H\left(u-\tau\left(m+1\right)\right)G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}\left(u-\tau\left(m+1\right)\right)du+$$

$$+\frac{2}{k_nL}\left[d_n^{(1)}\left(0\right)-B_n\left(0\right)\right]\left[a_0G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+2}\left(t-m\tau\right)+\right.$$

$$\left.+a_1G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+2}\left(t-m\tau\right)+a_2G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+3}\left(t-m\tau\right)\right]+$$

$$+\frac{2}{k_nL}\left[d_n^{(2)}\left(0\right)-B'_n\left(0\right)\right]\left[a_0G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+2}\left(t-m\tau\right)+\right.$$

$$\left.+a_1G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-1)-k-\beta n+2}\left(t-m\tau\right)+a_2G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+3}\left(t-m\tau\right)\right]+$$

$$+\frac{2}{k_nL}\left[d_n^{(3)}\left(0\right)-B''_n\left(0\right)\right]a_0G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l)-k-\beta n+3}\left(t-m\tau\right)-$$

$$-a_5\int\limits_0^t g\left(t-u\right)H\left(u-\tau\left(m+1\right)\right)G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}\left(u-\tau\left(m+1\right)\right)du-$$

$$-a_5\frac{2}{k_nL}\int\limits_0^t h\left(t-u\right)H\left(u-\tau\left(m+1\right)\right)G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}\left(u-\tau\left(m+1\right)\right)du+$$

$$+\int\limits_0^t f_n\left(t-u\right)H\left(u-m\tau\right)G^{\left(\frac{a_5}{a_0}\right),\tau,m}_{2\alpha+1,-\alpha(k+2i+l-2)-k-\beta n+1}\left(u-\tau\left(m+1\right)\right)du\}.$$

Examples

We consider the flow of an Oldroyd-B fluid when the body force and the pressure gradient are omitted and the plate is accelerating. We present the analytical solution in the different initial conditions

Example 1. In this example the plate is moving at speed ct, where c is constant. The corresponding initial problem is then given as

$$\frac{\partial u\left(y,t\right)}{\partial t}+\lambda_\alpha {D_t}^\alpha u\left(y,t\right)+\theta {D_t}^{2\alpha}u\left(y,t\right)=\nu\frac{\partial^2u\left(y,t\right)}{\partial y^2}+\nu\lambda_\beta D_t^\beta\frac{\partial^2u\left(y,t\right)}{\partial y^2}-Mu\left(y,t-\tau\right)$$

$$\begin{aligned}&u\left(y,t\right)=c_1,\quad u\left(0,t\right)=ct,\quad -\tau\le t\le 0\ ,\quad y>0,\\&u_t\left(y,t\right)=c_2,\quad u\left(L,t\right)=0,\quad \frac{1}{2}<\alpha,\beta<1,\\&u_{tt}\left(y,t\right)=0.\end{aligned}$$

Separating variables and use of the Laplace transformation yields,

$$\bar{B}_n\left(s\right)=e^{sm\tau}\sum_{m=0}^{\infty}\sum_{k,i,j\ge0}^{k+i+j=m}\frac{(-1)^m}{(\theta)^{m+1}}\frac{m!\left(-{k_n}^2\nu\right)^{j+l}{\lambda_\beta}^l{\lambda_\alpha}^i}{k!i!j!l!}$$

$$\{B_n\left(0\right)\left[\frac{s^{k+\alpha i+\beta l}e^{-sm\tau}}{\left(s^{2\alpha}-\frac{M}{\theta}e^{-s\tau}\right)^{m+1}}+\lambda_\alpha\frac{s^{k+\alpha(i+1)+\beta l-1}e^{-sm\tau}}{\left(s^{2\alpha}-\frac{M}{\theta}e^{-s\tau}\right)^{m+1}}+\theta\frac{s^{k+\alpha(i+2)+\beta l-1}e^{-sm\tau}}{\left(s^{2\alpha}-\frac{M}{\theta}e^{-s\tau}\right)^{m+1}}\right.$$

$$\left.-\nu\lambda_\beta {k_n}^2\frac{s^{k+\alpha i+\beta(1+l)-1}e^{-sm\tau}}{\left(s^{2\alpha}-\frac{M}{\theta}e^{-s\tau}\right)^{m+1}}\right]+B'_n\left(0\right)\left[\lambda_\alpha\frac{s^{k+\alpha(1+i)+\beta l-2}e^{-sm\tau}}{\left(s^{2\alpha}-\frac{M}{\theta}e^{-s\tau}\right)^{m+1}}+\theta\frac{s^{k+\alpha(i+2)+\beta l-2}e^{-sm\tau}}{\left(s^{2\alpha}-\frac{M}{\theta}e^{-s\tau}\right)^{m+1}}\right.$$

$$\left.+\nu\lambda_\beta {k_n}^2\frac{s^{k+\alpha i+\beta(1+l)-2}e^{-sm\tau}}{\left(s^{2\alpha}-\frac{M}{\theta}e^{-s\tau}\right)^{m+1}}\right]-Me^{-s\tau}G\left(s\right)\frac{s^{k+\alpha i+\beta l}e^{-sm\tau}}{\left(s^{2\alpha}-\frac{M}{\theta}e^{-s\tau}\right)^{m+1}}$$

$$+\frac{2c}{k_nL}\left[\frac{s^{k+\alpha i+\beta l-1}e^{-sm\tau}}{\left(s^{2\alpha}-\frac{M}{\theta}e^{-s\tau}\right)^{m+1}}-Me^{-s\tau}\frac{s^{k+\alpha i+\beta l-2}e^{-sm\tau}}{\left(s^{2\alpha}-\frac{M}{\theta}e^{-s\tau}\right)^{m+1}}+Me^{-s\tau}H\left(s\right)\frac{s^{k+\alpha i+\beta l}e^{-sm\tau}}{\left(s^{2\alpha}-\frac{M}{\theta}e^{-s\tau}\right)^{m+1}}-\lambda_\alpha\frac{s^{k+\alpha(1+i)+\beta l-2}e^{-sm\tau}}{\left(s^{2\alpha}-\frac{M}{\theta}e^{-s\tau}\right)^{m+1}}\right.$$

$$\left.-\theta\Gamma\left(-\alpha+2\right)\frac{s^{k+\alpha(2+i)+\beta l-2}e^{-sm\tau}}{\left(s^{2\alpha}-\frac{M}{\theta}e^{-s\tau}\right)^{m+1}}\right]\}.$$

Taking inverse Laplace transform gives us

$$B_n(t)=\sum_{m=0}^{\infty}\sum_{k,i,j\geq 0}^{k+i+j=m}\frac{(-1)^m}{(\theta)^{m+1}}\frac{m!\left(-k_n{}^2\nu\right)^{j+l}\lambda_\beta{}^l\lambda_\alpha{}^i}{k!i!j!l!}$$

$$\{B_n(0)\,H(t-m\tau)\,[G^{\left(\frac{M}{\theta}\right),\tau,m}_{2\alpha,-k-\alpha(i-2)-\beta l}(t-m\tau)+\lambda_\alpha G^{\left(\frac{M}{\theta}\right),\tau,m}_{2\alpha,-k-\alpha(i-1)-\beta l+1}(t-m\tau)+\theta G^{\left(\frac{M}{\theta}\right),\tau,m}_{2\alpha,-k-\alpha i-\beta l+1}(t-m\tau)$$

$$-\nu\lambda_\beta k_n{}^2G^{\left(\frac{M}{\theta}\right),\tau,m}_{2\alpha,-k-\alpha(i-2)-\beta(1+l)+1}(t-m\tau)]$$

$$+B'_n(0)\,[\lambda_\alpha G^{\left(\frac{M}{\theta}\right),\tau,m}_{2\alpha,-k-\alpha(i-1)-\beta l+2}(t-m\tau)+\theta G^{\left(\frac{M}{\theta}\right),\tau,m}_{2\alpha,-k-\alpha i-\beta l+2}(t-m\tau)$$

$$+\nu\lambda_\beta k_n{}^2G^{\left(\frac{M}{\theta}\right),\tau,m}_{2\alpha,-k-\alpha(i-2)-\beta(l+1)+2}(t-m\tau)]$$

$$-M\int_0^t g(t-u)\,H(u-\tau(m+1))\,G^{\left(\frac{M}{\theta}\right),\tau,m}_{2\alpha,-k-\alpha(i-2)-\beta l}(u-\tau(m+1))\,du$$

$$+\tfrac{2c}{k_nL}H(t-m\tau)\,[G^{\left(\frac{M}{\theta}\right),\tau,m}_{2\alpha,-k-\alpha(i-2)-\beta l+1}(t-m\tau)-\lambda_\alpha G^{\left(\frac{M}{\theta}\right),\tau,m}_{2\alpha,-k-\alpha(i-1)-\beta l+2}(t-m\tau)$$

$$-\theta\Gamma(-\alpha+2)\,G^{\left(\frac{M}{\theta}\right),\tau,m}_{2\alpha,-k-\alpha i-\beta l+2}(t-m\tau)]$$

$$+\tfrac{2cM}{k_nL}H(t-\tau(m+1))\,G^{\left(\frac{M}{\theta}\right),\tau,m}_{2\alpha,-k-\alpha(i-2)-\beta l+2}(t-\tau(m+1))$$

$$+\tfrac{2c}{k_nL}\int_0^t h(t-u)\,H(u-\tau(m+1))\,G^{\left(\frac{M}{\theta}\right),\tau,m}_{2\alpha,-k-\alpha(i-2)-\beta l}(u-\tau(m+1))\,du\}.$$

Example 2. We consider the flow of an Oldroyd-B fluid with the initial conditions $\psi_1(y)=c$, $\psi_2(y)=0$ and boundary conditions, $\varphi_1(t)=ct,\varphi_2(t)=0$ where c is constant. The problem now becomes,

$$\frac{\partial u(y,t)}{\partial t}+\lambda_\alpha D_t{}^\alpha u(y,t)=\nu\frac{\partial^2u(y,t)}{\partial y^2}+\nu\lambda_\beta D_t^\beta\frac{\partial^2u(y,t)}{\partial y^2}-Mu(y,t-\tau)$$

$$\begin{array}{lll} u(y,t)=c, & u(0,t)=ct, & -\tau\leq t\leq 0,\ y>0,\\ u_t(y,t)=0, & u(L,t)=0, & 0<\alpha,\beta<\tfrac{1}{2}.\end{array}$$

Using the preceding method we obtain,

$$B_n(t)=\sum_{m=0}^{\infty}\sum_{k,i,j,q\geq 0}^{k+i+j+q=m}\frac{(-1)^m}{(M\lambda_\alpha)^{m+1}}\frac{m!\left(-k_n{}^2\nu\right)^{i+j}\lambda_\beta{}^j}{k!i!j!}$$

$$\{B_n(0)\,H(t-m\tau)\,[G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,\alpha-k-\beta j}(t-m\tau)+\lambda_\alpha G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,-k-\beta q+1}(t-m\tau)$$

$$-\nu\lambda_\beta k_n{}^2G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,\alpha-k-\beta(j+1)+1}(t-m\tau)]\;-M\int_0^t g(t-u)\,H(u-\tau(m+1))\,G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,\alpha-k-\beta j}(u-\tau(m+1))\,du$$

$$+\tfrac{2c}{k_nL}H(t-m\tau)\left[G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,\alpha-k-\beta j+1}(t-m\tau)-\lambda_\alpha G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,-k-\beta j+2}(t-m\tau)\right]$$

$$+\tfrac{2cM}{k_nL}H(t-\tau(m+1))\,G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,\alpha-k-\beta j+2}(t-\tau(m+1)$$

$$+\tfrac{2c}{k_nL}\int_0^t h(t-u)\,H(u-\tau(m+1))\,G^{\left(\frac{M}{\lambda_\alpha}\right),\tau,m}_{\alpha,\alpha-k-\beta j}(t-m\tau)\,du\},$$ after which $W(y,t)$ and so $u(y,t)$ may be found.

Conclusion

In this paper we used a variant of the method of separation of variables to simplify the governing fractional-order partial differential equations of a generalized viscoelastic Oldroyd-B fluid with constant delay in time to a set of fractional-order ordinary differential equations with homogeneous boundary condition. The Laplace transformation (followed by its inverse) was then employed to obtain the exact solutions of the linear fractional ordinary differential equation. The solutions are given in terms of multivariate Green functions. We found exact solutions for three specific situations illustrated by examples.

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Тұтқыр сығынды сұйықтықтар теориясының бөлшек дифференциалдық теңдеуінің аналитикалық шешімі

Мақаланың мақсаты – Олдройд-Б сығылмайтын жалпылама сұйықтығын Капуто түріндегі бөлшек туындыларымен кешіктіру арқылы бөлшек дифференциалдық теңдеулердің аналитикалық шешімдерін ұсыну. Айнымалыларды бөлу әдісінің модификациясын қолдана отырып, біртекті емес шекаралық шарттары бар негізгі теңдеу біртекті шекаралық шарттары бар теңдеуге айналады, содан кейін алынған шешімдер Лаплас түрлендірулерінің көмегімен Грин функциялары арқылы көрінеді. Бұл нәтижелер екі жағдайда ұсынылған: бірінші қадамда $0 \leq \alpha, \beta \leq \frac{1}{2}$, ал екінші қадамда $\frac{1}{2} \leq \alpha, \beta \leq 1$, әр қадам үшін 1, 2 Олдройд-Б жалпыланған сұйықтығының стационарлық емес ағымдары үшін, оның ішінде жылжымалы плитасы бар ағын мысалдармен қарастырылды.

Кілт сөздер: Олдройд-Б сұйықтығы, бөлшек ретті жартылай туындылардағы теңдеулер, аналитикалық шешімдер, кешіктірілген дифференциалдық теңдеу, айнымалыларды бөлудің модификацияланған әдісі, Капутоның бөлшек туындылары.

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Аналитическое решение дробного дифференциального уравнения теории вязкоупругих жидкостей

Цель данной статьи – представить аналитические решения дробных дифференциальных уравнений с запаздыванием несжимаемой обобщенной жидкости Олдройда-Б с дробными производными типа Капуто. Используя модификацию метода разделения переменных, основное уравнение с неоднородными граничными условиями преобразуется в уравнение с однородными граничными условиями, а полученные решения затем выражаются через функции Грина с помощью преобразований Лапласа. Эти результаты представлены в двух условиях: на первом шаге, когда $0 \leq \alpha, \beta \leq \frac{1}{2}$, а на втором — при $\frac{1}{2} \leq \alpha, \beta \leq 1$. Для каждого шага 1, 2 для нестационарных течений обобщенной жидкости Олдройда-Б, включая поток с движущейся пластиной, приведены примеры.

Ключевые слова: жидкость Олдройда-Б, уравнения в частных производных дробного порядка, аналитические решения, дифференциальное уравнение с запаздыванием, модифицированный метод разделения переменных, дробные производные Капуто.

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On Discrete Solutions for Elliptic Pseudo-Differential Equations

We consider discrete analogue for simplest boundary value problem for elliptic pseudo-differential equation in a half-space with Dirichlet boundary condition in Sobolev–Slobodetskii spaces. Based on the theory of discrete boundary value problems for elliptic pseudo-differential equations we give a comparison between discrete and continuous solutions for certain model boundary value problem.

Keywords: Digital pseudo-differential operator, Discrete solution, Discrete boundary value problem, Rate of approximation.

Introduction

As soon as boundary value problems for partial differential equations were formulated, then at the same time the necessity of solving methods has appeared. Since finding exact solution for these problems is a very seldom phenomenon, numerical and approximate methods are extensively used. According to development of computer technologies, a preference is given to such methods which can be easily realized by computers.

There are a lot of approximate methods for solving boundary value problems in mathematical literature (see, for example, classical books [1–4] and many others) All authors consider a priori given boundary value problem and construct for it certain approximate structures. As a rule this way leads to final system of linear algebraic equations and the solution of the latter system us declared as an approximate solution for the starting problem.

In our opinion there is a reason to study discrete objects initially and then to apply their properties for studying approximation of starting continuous objects. This approach was started from papers [5–10] and further it was developed in [11–15]. We based on Eskin's approach for elliptic model pseudo-differential equations in a half-space [5] and have developed appropriate discrete theory. This report is devoted to a special case how we can approximate the infinite discrete objects by finite ones.

Digital Operators and Discrete Equations

We will use the following notations. Let $\mathbf{T}^m$ be m-dimensional cube $[-\pi,\pi]^m, h > 0, \hbar = h^{-1}$. We will consider all functions defined in the cube as periodic functions in $\mathbf{R}^m$ with the same cube of periods.

If $u_d(\tilde{x}), \tilde{x} \in h\mathbf{Z}^m$ is a function of a discrete variable, then we call it "discrete function". For such discrete functions one can define the discrete Fourier transform

$$(F_d u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x}\in h\mathbf{Z}^m} e^{-i\tilde{x}\cdot\xi} u_d(\tilde{x}) h^m, \quad \xi \in \hbar\mathbf{T}^m,$$

if the latter series converges, and the function $\tilde{u}_d(\xi)$ is a periodic function on $\mathbf{R}^m$ with the basic cube of periods $\hbar\mathbf{T}^m$. This discrete Fourier transform preserves basic properties of the integral Fourier transform, particularly the inverse discrete Fourier transform is given by the formula

$$(F_d^{-1}\tilde{u}_d)(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbf{T}^m} e^{i\tilde{x}\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in h\mathbf{Z}^m.$$

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Let $\mathbf{T}^m = [-\pi, \pi]^m, h > 0, A_d(\xi), \xi \in \mathbf{R}^m$ be a periodic function with basic cube of periods $\hbar\mathbf{T}^m, D \subset \mathbf{R}^m$ be a domain. We introduce a digital pseudo-differential operator

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbf{Z}^m} \int_{\hbar\mathbf{T}^m} A_d(\xi) e^{i(\tilde{y}-\tilde{x})\cdot\xi} u_d(\tilde{y}) d\xi h^m, \quad \tilde{x} \in D_d \equiv D \cap h\mathbf{Z}^m,$$

which is defined for functions of a discrete variable $\tilde{x} \in h\mathbf{Z}^m$.

We study operator equations

$$A_d u_d = v_d, \tag{1}$$

its solvability and approximate properties for small h.

Let us denote $\zeta^2 = h^{-2} \sum_{k=1}^{m} (e^{-ih\cdot\xi_k} - 1)^2$, $S(h\mathbf{Z}^m)$ is a discrete analogue of the Schwartz space $S(\mathbf{R}^m)$ [7] and introduce the following:

Definition 1. The space $H^s(h\mathbf{Z}^m)$ is a closure of the space $S(h\mathbf{Z}^m)$ with respect to the norm

$$||u_d||_s = \left(\int_{\hbar\mathbf{T}^m} (1 + |\zeta^2|)^s |\tilde{u}_d(\xi)|^2 d\xi \right)^{1/2}.$$

Further, let $D \subset \mathbf{R}^m$ be a domain, and $D_d = D \cap h\mathbf{Z}^m$ be a discrete domain.

Definition 2. The space $H^s(D_d)$ consists of discrete functions from $H^s(h\mathbf{Z}^m)$ which supports belong to $\overline{D_d}$. A norm in the space $H^s(D_d)$ is induced by a norm of the space $H^s(h\mathbf{Z}^m)$. The space $H_0^s(D_d)$ consists of discrete functions u_d with a support in D_d, and these discrete functions should admit a continuation into the whole $H^s(h\mathbf{Z}^m)$. A norm in the $H_0^s(D_d)$ is given by the formula

$$||u_d||_s^+ = \inf ||\ell u_d||_s,$$

where infimum is taken over all continuations ℓ.

Of course, all such norms are equivalent to the L_2-norm but this equivalence depends on h. Let us note that all constants below in our considerations do not depend on h.

To study the equation (1) in a discrete half-space ($D = \mathbf{R}_+^m \equiv \{x \in \mathbf{R}^m : x - (x', x_m), x_m > 0\}$) we use a special factorization for the symbol $A_d(\xi)$

$$A_d(\xi) = A_{d,+}(\xi) \cdot A_{d,-}(\xi)$$

where the factors $\tilde{A}_\pm(\xi)$ admit a holomorphic continuation into half-strips $\hbar\Pi_\pm$,

$$\Pi_\pm = \{z \in \mathbf{C} : z = \xi_m + i\tau, \xi_m \in [-h^{-1}\pi, h^{-1}\pi], \ \pm\tau > 0\}.$$

with respect to the last variable ξ_m under fixed $(\xi_1, \cdots, \xi_{m-1}) \in \hbar\mathbf{T}^{m-1}$ and satisfy some estimates [1–3].

Discrete Equations

We consider the class E_α, which includes symbols satisfying the following condition

$$c_1(1 + |\zeta^2|)^{\alpha/2} \leq |A_d(\xi)| \leq c_2(1 + |\zeta^2|)^{\alpha/2}$$

with universal positive constants c_1, c_2 non-depending on h and the symbol $A_d(\xi)$.

Definition 3. Periodic factorization of an elliptic symbol $A_d(\xi) \in E_\alpha$ is called its representation in the form

$$A_d(\xi) = A_{d,+}(\xi) A_{d,-}(\xi),$$

where the factors $A_{d,\pm}(\xi)$ admit an analytical continuation into half-strips $\hbar\Pi_\pm$ on the last variable ξ_m for almost all fixed $\xi' \in \hbar\mathbf{T}^{m-1}$ and satisfy the estimates

$$|A_{d,+}^{\pm 1}(\xi)| \leq c_1(1 + |\hat{\zeta}^2|)^{\pm\frac{æ}{2}}, \quad |A_{d,-}^{\pm 1}(\xi)| \leq c_2(1 + |\hat{\zeta}^2|)^{\pm\frac{\alpha-æ}{2}},$$

with constants c_1, c_2 non-depending on h,

$$\hat{\zeta}^2 \equiv \hbar^2 \left(\sum_{k=1}^{m-1} (e^{-ih\xi_k} - 1)^2 + (e^{-ih(\xi_m + i\tau)} - 1)^2 \right), \quad \xi_m + i\tau \in \hbar\Pi_{\pm}.$$

The number æ $\in \mathbf{R}$ is called an index of periodic factorization.

Such a representation can be constructed effectively and it fully determines a solvability picture for the equation (1).

Conditions for a Unique Solvability

Some auxiliaries Firstly, for an elliptic symbol $A_d(\xi)$ such periodic factorization exists always [5, 8].

Secondly, the index æ of periodic factorization determines how much additional conditions for the solution u_d or for the right hand side v_d we need [7, 9].

Thirdly, the equation (1) is uniquely solvable in the discrete half-space $H^s(D_d)$ for arbitrary right hand side $v_d \in H_0^{s-\alpha}(D_d)$ only under the condition

$$|æ - s| < 1/2, \tag{2}$$

Kernel of elliptic digital operator in a discrete half-space

In this paper we consider more complicated case when the condition (2) does not hold. There are two possibilities in this situation, and we consider one case which leads to typical boundary value problems. We use the following result from [7] in a simplest form.

Theorem 1. Let æ $- s = n + \delta, n \in \mathbf{N}, |\delta| < 1/2$. Then the Fourier image for a kernel of the operator A_d consists of the following functions

$$\tilde{u}_d(\xi) = \tilde{A}_{d,+}^{-1}(\xi) \sum_{k=0}^{n-1} c_k(\xi') \hat{\zeta}_m^k,$$

where $c_k(\xi'),\ k = 0, 1, \cdots, n-1$, are arbitrary functions from $H^{s_k}(h\mathbf{T}^{m-1}), s_k = s -$ æ $+ k - 1/2$.

The a priori estimate

$$||u_d||_s \leq a \sum_{k=0}^{n-1} [c_k]_{s_k}$$

holds, where $[\cdot]_{s_k}$ denotes a norm in the space $H^{s_k}(h\mathbf{T}^{m-1})$, and the constant a does not depend on h.

Discrete Structures as Approximating Objects.

Initial Observations for $D = \mathbf{R}^m$. Here and below we consider model pseudo-differential operators with symbols $A(\xi)$ satisfying the condition

$$c_1(1 + |\xi|)^\alpha \leq |A(\xi)| \leq c_2(1 + |\xi|)^\alpha.$$

Further, the symbol $A_d(\xi)$ will be defined in the following way. We take a restriction of $A(\xi)$ on the cube $\hbar\mathbf{T}^m$ and periodically extend it onto a whole $\mathbf{R}^m$. We consider such operator as an approximate operator for A. For arbitrary function u the notation $Q_h u$ will denote the same construction. So, to find an approximate discrete solution for the equation

$$(Au)(x) = v(x), \quad x \in D,$$

for $D = \mathbf{R}^m$ we can use the following discrete equation

$$A_d u_d = Q_h v.$$

Its solution is given by the formula

$$u_d(\tilde{x}) = \frac{1}{(2\pi)^m} \int_{\hbar\mathbf{T}^m} e^{i\tilde{x}\cdot\xi} A^{-1}(\xi) \tilde{v}(\xi) d\xi, \quad \tilde{x} \in h\mathbf{Z}^m,$$

so that we do not need to find an approximate solution for an infinite system of linear algebraic equations. For our case we need to apply any kind of cubature formulas for calculating the latter integral and a cubature formula for calculating the Fourier transform $\tilde{v}(\xi)$. For $v \in S(\mathbf{R}^m)$ the discrete solution $u_d(\tilde{x})$ tends to $u(\tilde{x})$ very fast under $h \to 0$ [12].

Rate of Approximation.

Infinite Discrete Half-Space Case. Here we consider the case $æ - s = 1+\delta, |\delta| < 1/2$. According to Theorem 1, the kernel of the operator A_d includes only one arbitrary function so that we need only one additional condition.

The continuous analogue of the discrete boundary value problem

$$(A_d u_d)(\tilde{x}) = 0, \quad \tilde{x} \in D_d, \tag{3}$$

$$u_d(\tilde{x}', 0) = g_d(x'), \quad \tilde{x}' \in h\mathbf{Z}^{m-1}, \tag{4}$$

is the following

$$(Au)(x) = 0, \quad x \in \mathbf{R}^m_+, \tag{5}$$

$$u(x', 0) = g(x'), \quad x' \in \mathbf{R}^{m-1}, \tag{6}$$

where A is a pseudo-differential operator with symbol $A(\xi)$. To obtain some comparison between discrete and continuous solutions we will remind how the continuous solution looks. If the index of factorization equals to $æ$ and $æ - s = 1 + \delta, |\delta| < 1/2$ then the unique solution for the problem (5),(6) is constructed by the similar formula

$$\tilde{u}(\xi) = b^{-1}(\xi')\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m),$$

where $A_\pm(\xi', \xi_m)$ are elements of factorization of the symbol $A(\xi)$ [5],

$$b(\xi') = \int\limits_{-\infty}^{+\infty} A_+^{-1}(\xi', \xi_m)d\xi_m,$$

assuming that $b(\xi') \neq 0, \forall \xi' \in \mathbf{R}^{m-1}$. Let us note that this is simplest variant of Shapiro–Lopatinskii condition [5].

We have the following discrete solution [8]

$$\tilde{u}_d(\xi) = b_d^{-1}(\xi')\tilde{g}_d(\xi')A_{d,+}^{-1}(\xi', \xi_m),$$

$$b_d(\xi') = \int\limits_{-\hbar\pi}^{+\hbar\pi} A_{d,+}^{-1}(\xi', \xi_m)d\xi_m,$$

in which we choose special approximations. We take $g_d = Q_h g$ and $A_{d,\pm}(\xi', \xi_m)$ we take as restrictions of $A_\pm(\xi', \xi_m)$ on $\hbar\mathbf{T}^m$. Then the periodic symbol

$$A_d(\xi) = A_{d,+}(\xi', \xi_m)A_{d,-}(\xi', \xi_m)$$

satisfies all conditions of periodic factorization with the same index $æ$. Moreover, $\tilde{g}_d(\xi')$ and $A_{d,+}(\xi', \xi_m)$ coincide with $\tilde{g}(\xi')$ and $A_+(\xi', \xi_m)$ respectively on $\hbar\mathbf{T}^m$.

Theorem 2. Let $æ > 1, s > m/2, g \in H^{s-1/2}(\mathbf{R}^{m-1})$. A comparison between solutions of problems (3), (4) and (5), (6) is given in the following way

$$|u(\tilde{x}) - u_d(\tilde{x})| \leq Ch^{æ-1}, \quad \tilde{x} \in h\mathbf{Z}^m.$$

Proof. We need to compare two integrals:

$$u(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\mathbf{R}^m} e^{i\tilde{x}\cdot\xi} b^{-1}(\xi')\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m)d\xi$$

and

$$u_d(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbf{T}^m} e^{i\tilde{x}\cdot\xi} b_d^{-1}(\xi')\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m)d\xi, \tag{7}$$

for $\tilde{x} \in h\mathbf{Z}^m$.

Thus, we have

$$u(\tilde{x}) - u_d(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\hbar \mathbf{T}^m} e^{i\tilde{x}\cdot\xi}(b^{-1}(\xi) - b_d^{-1}(\xi'))\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m)d\xi +$$

$$\frac{1}{(2\pi)^m} \int\limits_{\mathbf{R}^m \setminus \hbar \mathbf{T}^m} e^{i\tilde{x}\cdot\xi} b^{-1}(\xi')\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m)dxii,$$

because the functions $\tilde{g}, \tilde{g}_d$ and $A_+, A_{d,+}$ coincide in $\hbar \mathbf{T}^m$.

Now we estimate the second integral.

$$\left| \int\limits_{\mathbf{R}^m \setminus \hbar \mathbf{T}^m} e^{i\tilde{x}\cdot\xi} b^{-1}(\xi')\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m)d\xi \right| \leq \; const \int\limits_{\mathbf{R}^m \setminus \hbar \mathbf{T}^m} |\tilde{g}(\xi')||A_+^{-1}(\xi', \xi_m)|d\xi \leq$$

$$const \int\limits_{\mathbf{R}^{m-1} \setminus \hbar \mathbf{T}^{m-1}} |\tilde{g}(\xi')| \left(\int\limits_{-\infty}^{-\hbar\pi} + \int\limits_{\hbar\pi}^{+\infty} \right) |A_+^{-1}(\xi', \xi_m)|d\xi_m d\xi'.$$

Further, we estimate

$$\left(\int\limits_{-\infty}^{-\hbar\pi} + \int\limits_{\hbar\pi}^{+\infty} \right) |A_+^{-1}(\xi', \xi_m)|d\xi_m \leq const \int\limits_{\hbar\pi}^{+\infty} (1 + |\xi'| + |\xi_m|)^{-æ} d\xi_m =$$

$$\frac{const}{æ - 1}(1 + |\xi'| + \hbar\pi)^{1-æ} \leq c_6 h^{æ-1}.$$

Now by Cauchy–Schwartz inequality we have

$$\int\limits_{\mathbf{R}^{m-1} \setminus \hbar \mathbf{T}^{m-1}} |\tilde{g}(\xi')|d\xi' \leq$$

$$\left(\int\limits_{\mathbf{R}^{m-1} \setminus \hbar \mathbf{T}^{m-1}} |\tilde{g}(\xi')|^2 (1 + |\xi'|)^{2s-1} d\xi' \right)^{1/2} \left(\int\limits_{\mathbf{R}^{m-1} \setminus \hbar \mathbf{T}^{m-1}} (1 + |\xi'|)^{-2s+1} d\xi' \right)^{1/2}.$$

Since $g \in H^{s-1/2}(\mathbf{R}^{m-1})$ [5] the first factor is less than $[g]_{s-1/2}$ and the second one tends to zero if $s > m/2$.
For the first integral we use the estimate

$$|b^{-1}(\xi') - b_d^{-1}(\xi')| \leq const \cdot h^{æ-1}$$

(see [15]).

Finally,

$$\left| \frac{1}{(2\pi)^m} \int\limits_{\hbar \mathbf{T}^m} e^{i\tilde{x}\cdot\xi}(b^{-1}(\xi) - b_d^{-1}(\xi'))\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m)d\xi \right| \leq$$

$$const \cdot h^{æ-1} \int\limits_{\hbar \mathbf{T}^m} |\tilde{g}(\xi')||A_+^{-1}(\xi', \xi_m)|d\xi \leq const \cdot h^{æ-1} \int\limits_{\hbar \mathbf{T}^{m-1}} \frac{|\tilde{g}(\xi')|}{(1 + |\xi'|)^{æ-1}} d\xi'$$

and further as above using Cauchy–Schwartz inequality.

Finite Truncation. To obtain finite object for calculation we can apply an arbitrary cubature formula for the integral (7) and to approximately find its value in nodal points.

Conclusion

Here only model operators in a half-space were considered. We hope that these ideas and technique will be useful for more complicated situations in which both an operator depends on a spatial variable or a domain is not a half-space.

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Эллиптикалық псевдодифференциалды теңдеулердің дискретті шешімдері туралы

Соболев-Слободецкий кеңістігіндегі Дирихле шекаралық жағдайы бар жартылай кеңістіктегі эллиптикалық псевдодифференциалды теңдеудің қарапайым шекаралық есебінің дискретті аналогы қарастырылған. Эллиптикалық псевдодифференциалды теңдеулер үшін дискретті жиек есептері теориясына сүйене отырып, бір модельдік шекаралық есеп үшін дискретті және үздіксіз шешімдер арасындағы салыстыру берілген.

Кілт сөздер: дискретті псевдодифференциалды оператор, дискретті шешім, дискретті шекаралық есеп, жуықтау реті.

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О дискретных решениях эллиптических псевдодифференциальных уравнений

Рассмотрен дискретный аналог простейшей краевой задачи для эллиптического псевдодифференциального уравнения в полупространстве с граничным условием Дирихле в пространстве Соболева–Слободецкого. Основываясь на теории дискретных краевых задач для эллиптических псевдодифференциальных уравнений, дано сравнение между дискретными и непрерывными решениями для одной модельной краевой задачи.

Ключевые слова: дискретный псевдодифференциальный оператор, дискретное решение, дискретная краевая задача, порядок аппроксимации.

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On atomic and algebraically prime models obtained by closure of definable sets

This article discusses the properties of atomic and prime models obtained with the some closure operator given on definable subsets of the semantic model some fixed Jonsson theory. The main result is to obtain the equivalence of the thus defined atomic and prime models, and this coincidence follows the assumption that there is some model with nice-defined properties.

Keywords: Jonsson theory, semantic model, prime model, atomic model, algebraically prime model, pregeometry, definable subset.

The paper considered the syntactic and semantic characteristics of prime and atomic models [1]. A. Robinson defined a natural generalization of a prime model, and he called such a model an algebraically prime model. In work [2] the corresponding notions of atomicity and their connection with an algebraically prime model were systematically studied. We propose several new types of atomic models and refine these concepts for algebraically prime models within the framework of these types of atomic. We have previously obtained some results in connection with these new concepts in works [3–6].

With these concepts of types of atomic and primary models we can work in fixed classes of Jonsson theories, depending on the conditions of the problem under consideration. In work [7] generalizations of the concept of isomorphic embedding were considered and within the framework of this definition results were obtained connecting the concepts of atomic and algebraically prime within the framework of this generalization. Thus, this work is a synthesis of new results obtained using ideas and concepts of works [3–6] and [7]. In [8–13] some new directions related to the study of Jonsson theories and their companions were considered and studied. The results of this work can be useful for studying the properties of countable models related to the above topics from the list of papers [3–6], [8–13].

Remind some concepts from [7].

Let $\alpha \leq \omega$, $\mathfrak{A}, \mathfrak{B}$ are models first order of L. Then the mapping $f : \mathfrak{A} \to \mathfrak{B}$ is called α an embedding if for any formula $\varphi(\bar{x}) \in \Pi_\alpha$ and any tuple $\bar{a} \in A$ from the fact that $\mathfrak{A} \models \varphi(\bar{a})$, it follows $\mathfrak{B} \models \varphi(f(\bar{a}))$. A model $\mathfrak{A}$ of the theory T is called α-algebraically prime if $\mathfrak{A}$ α-embeddable into any model of the theory T.

From the above definitions it is easy to see that the concepts of an algebraically prime model and a prime model are obtained from the concept of an α-algebraically prime models for $\alpha = 0$ and $\alpha = \omega$ respectively. If Γ is a set of formulas, then we put $\Gamma^* \rightleftharpoons \{\neg\varphi/\varphi \in \Gamma\}$. If $\bar{a} =< a_0...a_n >$, $\mathfrak{A}$ is a model, then $\bar{a} \in \mathfrak{A}$ means that $a_i \in A, i < n$. A type p is called a Γ-type if $p \subseteq \Gamma$. Further, $t^{\mathfrak{A}}_{\Gamma}(\bar{a}) \rightleftharpoons \{\varphi(\bar{x})/\varphi(\bar{x}) \in L,\ \mathfrak{A} \models \varphi(\bar{a})\}$ is called a Γ- type $\bar{a}$ in $\mathfrak{A}$. Γ_2-type p is called a Γ_1-the main type if there is a Γ_1 is formula $\varphi(\bar{x})$ such that $T \models \forall(\bar{x})(\varphi(\bar{x}) \to \psi(\bar{x}))$ forall $\psi(\bar{x}) \in p$. In this case $\varphi(\bar{x})$ is said to generate p.

It is easy to see the following fact. Let $\mathfrak{A}$ is a model of the theory T, then $\mathfrak{A}$ is (Γ_1, Γ_2)-atomic model T if and only if for any $\bar{a} \in A$ there is such a formula $\varphi(\bar{x}) \in \Gamma_1$, which is true:

a) $\mathfrak{A} \models \varphi(\bar{x})$;

b) $\varphi(\bar{x})$ generates $t^{\mathfrak{A}}_{\Gamma \cup \Gamma^*}(\bar{a})$.

Similarly, if $\mathfrak{A} \models T$, then $\mathfrak{A}$ is weakly (Γ_1, Γ_2) atomic model of T if and only if for any $\bar{a} \in A$ there is a formula $\varphi(\bar{x}) \in \Gamma_1$ that is true:

a) $\mathfrak{A} \models \varphi(\bar{x})$;

b) $\varphi(\bar{x})$ generates $t^{\mathfrak{A}}_{\Gamma_2}(\bar{a})$.

In papers [3–6] the properties of atomic models were considered with the help of the closure operator specifying some pregeometry on subsets of the semantic model of a fixed Jonsson theory.

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Let cl is some closure operator defining a pregeometry over C (for example cl = acl or cl = dcl). It is clear that such operator is a special case of the closure operator and its example is a closure operator defined on any linear space as a linear shell. Further, the concepts under consideration are produced within the framework of a perfect Jonsson theory and if the contrary is not specified then the considered Jonsson theories are assumed to be complete for existential sentences.

Let us give definitions related to the atomic and prime model considered in this theory.

Definition 1. A set A will be called the (Γ_1, Γ_2)-*cl* atomic in T if:

1) $\forall \bar{a} \in A$, $\exists \varphi(\bar{x}) \in \Gamma_1$ such that $\mathfrak{A} \models \varphi(\bar{a})$;

2) $\varphi(\bar{x})$ generates $t^{\mathfrak{A}}_{\Gamma \cup \Gamma^*}(\bar{a})$;

3) $cl(A) = M, M \in E_T$, where E_T class of existentially closed models of the theory T;
and obtained model M is said to be the (Γ_1, Γ_2)-*cl* atomic model of the theory T.

Definition 2. A set A is said to be weakly the (Γ_1, Γ_2)-*cl* atomic in T, if $\forall \bar{a} \in A$ $\exists \varphi(\bar{x}) \in \Gamma_1$ such that:

1) $\varphi(\bar{x}) \cup T$ is consistent;

2) $\varphi(\bar{x})$ generates $t^{\mathfrak{A}}_{\Gamma_2 \cup \Gamma_2^*}(\bar{a})$;

3) $cl(A) = M, M \in E_T$, where E_T class of existentially closed models of the theory T; And obtained model M is said to be weakly (Γ_1, Γ_2)-*cl* atomic model of the theory T.

Definition 3. A set A is said to be almost-weakly (Γ_1, Γ_2)-*cl* atomic in T if for any $\bar{a} \in A$ there exists a formula $\varphi(\bar{x}) \in \Gamma_1$ such that:

1) $\varphi(\bar{x}) \cup T$ is consistent;

2) $\varphi(\bar{x})$ generates $t^{\mathfrak{A}}_{\Gamma_2}(\bar{a})$;

3) $cl(A) = M, M \in E_T$, where E_T is the class of existentially closed models of theory T; And obtained model M is said to be almost-weakly (Γ_1, Γ_2)-*cl* atomic model of the theory T.

Definition 4. A set A is said to be the (Γ_1, Γ_2)-*cl* algebraically prime of the theory T, if $cl(A) = M$, M is (Γ_1, Γ_2)-*cl* atomic model of the theory T, $M \in E_T \cap AP_T$, where $AP_T \bigcap E_T \neq \varnothing$ and obtained model M is said to be (Γ_1, Γ_2)-*cl* algebraically prime of the theory T.

Definition 5. A set A is said to be almost (Γ_1, Γ_2)-*cl* algebraically prime of the theory T, if $cl(A) = M$, M is been almost (Γ_1, Γ_2)-*cl* atomic model of the theory T, $M \in E_T \cap AP_T$, where $AP_T \bigcap E_T \neq 0$ and obtained model M is said to be almost the (Γ_1, Γ_2)-*cl* algebraically prime of the theory T.

Definition 6. A set A is said to be almost-weakly (Γ_1, Γ_2)-*cl* algebraically prime of theory T, if $cl(A) = M$, M is been almost-weakly (Γ_1, Γ_2)-*cl* atomic model of the theory T, $M \in E_T \cap AP_T$, where $AP_T \bigcap E_T \neq 0$ and obtained model M is said to be almost-weakly (Γ_1, Γ_2)-*cl* algebraically prime of the theory T.

For the convenience of expression

"$\mathfrak{A}$ is (Γ_1, Γ_2)-*cl* atomic model of the theory T";

"$\mathfrak{A}$ is weakly (Γ_1, Γ_2)-*cl* atomic model of theory T";

"$\mathfrak{A}$ is an almost (Γ_1, Γ_2)-*cl* atomic model of theory T";

"$\mathfrak{A}$ is an almost-weakly (Γ_1, Γ_2)-*cl* atomic model of theory T";

and denote by (1), (2), (3), (4), respectively.

Lemma 1.

1. If $(\Gamma_2 = \Gamma_2^*)$, then (1) $\Leftrightarrow$ (2), (3) $\Leftrightarrow$ (4).
2. If $(\Gamma_1^* \subset \Gamma_2)$, then (1) $\Leftrightarrow$ (3), (2) $\Leftrightarrow$ (4).
3. If $(\Gamma_2 \cup \Gamma_2^*) \subset \Gamma_3$, then if

a) $\mathfrak{A}$ is weakly (Γ_1, Γ_2)-*cl* atomic model of the theory T, then it is true (1);

b) $\mathfrak{A}$ is an almost-weakly (Γ_1, Γ_2)-*cl* atomic model of the theory T, then it is true (3).

4. If $(\Gamma_1^* \subset \Gamma_2 \subset \Gamma_2^*)$, then (1) $\Leftrightarrow$ (2)$\Leftrightarrow$ (3) $\Leftrightarrow$ (4).
5. If $(\Gamma_1 = \Gamma_2 = \Gamma_2^*)$, then (1) $\Leftrightarrow$ (2)$\Leftrightarrow$ (3) $\Leftrightarrow$ (4).
6. If $(\Gamma_1 \subset \Gamma_1')$, $(\Gamma_2 \subset \Gamma_2')$, then $\tau - (\Gamma_1, \Gamma_2')$-*cl* atomic model of the theory $T \Rightarrow \tau - (\Gamma_1', \Gamma_2)$-*cl* atomic model of the theory T, where $\tau \in \{\emptyset$, weakly, almost, almost-weakly$\}$.

Proof. The proof follows easily from the definition.

Lemma 2. If T is complete for $\exists\Gamma_2$ (i.e., if $\psi(\bar{x}) \cup T$ consistent and $\psi(\bar{x}) \in \Gamma_2$, then it is true that $T \models \exists \bar{x} \psi(\bar{x})$) and $(\Gamma_1 \cup \Gamma_1^*) \subset \Gamma_2$ then it is true (1) $\Leftrightarrow$ (2)$\Leftrightarrow$(3) $\Leftrightarrow$ (4).

Proof. Since $\Gamma_1^* \subset \Gamma_2$, then, by part 2) of Lemma 1, it suffices to show (2) $\Leftrightarrow$ (1). Let (2) $\bar{a} \in A$, $\mathfrak{A} \models \psi(\bar{a})$, $\psi(\bar{x}) \in \Gamma_1$, $\psi(\bar{x})$ generates $t^{\mathfrak{A}}_{\Gamma_2}(\bar{a})$.

Let $\neg\psi(\bar{x}) \in \Gamma_2^*$ and $\mathfrak{A} \models \neg\psi(\bar{a})$. Let us show that $T \models \psi(\bar{x}) \rightarrow \neg\varphi(\bar{x})$.

Suppose the opposite: $T \cup \{\psi(\bar{x}) \wedge \varphi(\bar{x})\}$ consistent. Since T is complete for $\exists\Gamma_2$, then $T \models \exists\bar{x}(\psi(\bar{x}) \wedge \varphi(\bar{x}))$. So there is $\bar{b} \in A$ such that $\mathfrak{A} \models \psi(\bar{b}) \wedge \varphi(\bar{b})$. Let $\theta(\bar{x}) \in \Gamma_1$, $\mathfrak{A} \models \theta(\bar{b})$ and $\theta(\bar{x})$ generates $t^{\mathfrak{A}}_{\Gamma_2}(\bar{b})$ by (2). Note that $T \models \theta(\bar{x}) \wedge \varphi(\bar{x})$ (1) as well as $T \vdash \psi(\bar{x}) \wedge \neg\theta(\bar{x})$ (2).

Since $\neg\theta(\bar{x}) \in \Gamma_1^* \subset \Gamma_2$ it follows from (2) that $\neg\theta(\bar{x}) \notin t^{\mathfrak{A}}_{\Gamma_2}(\bar{a})$, i.e., $\mathfrak{A} \models \theta(\bar{a})$. According to (1), in this case, $\mathfrak{A} \models \varphi(\bar{a})$ must be true. Contradiction. Recall that $\triangle_\kappa \rightleftharpoons \Sigma_\kappa \cap \Pi_\kappa$.

Corollary 3.

1) If $\Gamma_1 = \Gamma_2 = \Sigma_\omega$, then (1) $\Leftrightarrow$ (2)$\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

2) For any $\alpha \leq \beta \leq \omega$, if $\Gamma_1 = \triangle$, $\Gamma_2 = \Sigma$, T is complete for Σ, then it is true (1) $\Leftrightarrow$ (2)$\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

Proof. 1) follows from part 5) of Lemma 1;

2) from Lemma 2.

If $\Gamma \in \{\Sigma; \Pi\}$, then $\Gamma(\mathfrak{A}, \bar{a})_{\bar{a}\in A}$ denotes the set of sentences of the form Γ in the language L that are true on $(\mathfrak{A}, \bar{a})_{\bar{a}\in A}$.

Lemma 4. If $\mathfrak{A}$ is a model of T, then following conditions are equivalent:

1) $\mathfrak{A}$ is (Γ_1, Γ_2)-*cl* algebraically prime of the theory T.

2) Every model T can be enriched to the model $T \cup \Pi(\mathfrak{A}, \bar{a})_{\bar{a}\in A}$.

3) Every model T can be enriched to the model $T \cup \Sigma(\mathfrak{A}, \bar{a})_{\bar{a}\in A}$.

Proof 3)$\Leftrightarrow$ 2) $\Leftrightarrow$ 1) obviously.

Let's show 1)$\Leftrightarrow$ 3).

Let $f : \mathfrak{A} \to \mathfrak{B}$ be isomorphic embedding $\bar{a} \in A$, $\varphi(\bar{x}) \in \Sigma_1$: $\mathfrak{A} \models \varphi(\bar{a})$, $\varphi(\bar{x}) = \exists\bar{y}\psi(\bar{y}, \bar{x})$, $\psi(\bar{y}, \bar{x}) \in \Pi$, $\bar{a_1} \in A$, $\mathfrak{A} \models \psi(\bar{a_1}, \bar{a})$. Then $\mathfrak{B} \models \psi(f(\bar{a_1}), f(\bar{a}))$ due to the fact f is an isomorphic embedding.

Further, we have $\mathfrak{B} \models \exists y\psi(\bar{y}, f(\bar{a}))$ i.e., $\mathfrak{B} \models \varphi(f(\bar{a}))$. Hence $(\mathfrak{B}, f(\bar{a}))_{\bar{a}\in A}$ are the model of $T \cup \Sigma_1(\mathfrak{A}, \bar{a})_{\bar{a}\in A}$

Definition 7. Let $\Phi(x_1...x_\kappa)$ be some set of formulas of the language L from variables $x_1...x_\kappa$. We say that Γ_1 locally omitted Φ if for any formula consistent with T formulas $\varphi(x_1...x_\kappa) \in \Gamma_1$ there is such a formula $\theta(x_1...x_\kappa) \in \Phi$ such that $\varphi \wedge \neg\theta$ consistent with T.

Theorem 5. Let T be Π_2-axiomatizable consistent theory of a countable language L and for any $n < \omega$ let $\Phi(x_1...x_{m_n})$ be the set of the Π_1 are formulas of m_n variables. If T Σ_1 locally omitted every $\Phi^n, n < \omega$, then T has a countable model which omitted every set $\Phi^n, n < \omega$.

The proof can be taken from [15].

Theorem 6. Let T be a perfect Jonsson theory complete for Π_2 sentences. Then every (Σ, Σ)-*cl* algebraically prime model of theory T is an almost-weakly (Σ, Σ)-*cl* atomic model of the theory T.

Proof. Let $\mathfrak{A}$ be the (Σ, Σ)-*cl* algebraically prime model of theory T. Suppose there is a $\bar{a} \in A$, such that $t^{\mathfrak{A}}_{\Sigma}(\bar{a})$ is not be the Σ_2 is principle type. Since $\Sigma_1 \subset \Pi_2$ then by Theorem 5, there exists a model $\mathfrak{B}$ of the theory T, which omits $t^{\mathfrak{A}}_{\Sigma_1}(\bar{a})$. Let $f : \mathfrak{A} \to \mathfrak{B}$ be an embedding. Then by Lemma 4 we have $t^{\mathfrak{A}}_{\Sigma_1}(\bar{a}) \subseteq t^{\mathfrak{B}}_{\Sigma_1}(f(\bar{a}))$. It follows that $f(\bar{a})$ implements $t^{\mathfrak{A}}_{\Sigma_1}(\bar{a})$ to $\mathfrak{B}$. This contradicts Theorem 5.

Definition 8. Let t_1 be the Γ_1-type, t_2 be the Γ_2-type, then they say that t_1 and t_2 T-equivalent if $T \cup t_1 \vdash t_2$ & $T \cup t_2 \vdash t_1$. In this case, write $t_1 \sim_T t_2$. The following is known next lemma.

Lemma 7. Let T be perfect Jonsson theory complete for Π_2 sentences and $\mathfrak{A} \models T$, then there is a model $\mathfrak{B}$, such that:

1) $\mathfrak{B} \models T$;

2) $\mathfrak{A}$ is isomorphically embeddable in $\mathfrak{B}$;

3) for any $\bar{b} \in B$ $t^{\mathfrak{B}}_{\Sigma_1}(\bar{b}) \sim_T t^{\mathfrak{B}}_{\Sigma_2}(\bar{b})$.

Proof. The proof follows from [14] and the above definitions.

Theorem 8. Let T be the perfect Jonsson theory complete for Π_2 sentences. Then every (Σ, Σ)-*cl* algebraically prime model of the theory T is an almost-weakly (Σ, Σ)-*cl* atomic model of the theory T.

Proof. Firstly, we prove the following fact (F). If $\varphi(\bar{x}) \in \Sigma_1$ and $\varphi(\bar{x}) \cup T$ is consistent, then there is a formula $\psi(\bar{x}) \in \Sigma_1$ such that $T \cup \psi(\bar{x})$ is consistent and $T \models \psi(\bar{x}) \to \varphi(\bar{x})$. Indeed, let $\varphi(\bar{x}) \in \Sigma_1$ and $\varphi(\bar{x}) \cup T$ are consistent. Since that T is complete for Π_2 sentences we have $T \vdash \exists\bar{x}\varphi(\bar{x})$. Since T Π_2 is axiomatizable, then by Lemma 7 there exists a model $\mathfrak{B} \models T$, such that for any $\bar{b} \in B$ is holds

$$t^{\mathfrak{B}}_{\Sigma_1}(\bar{b}) \sim_T t^{\mathfrak{B}}_{\Sigma_2}(\bar{b}) \qquad (*)$$

Let $\bar{b} \in B$ such that $\mathfrak{B} \models \varphi(\bar{b})$. Due to (*) and the closedness concerning the conjunction of the type $t^{\mathfrak{B}}_{\Sigma_1}(\bar{b})$ there is a formula $\psi(\bar{x}) \in t^{\mathfrak{B}}_{\Sigma_1}(\bar{b})$, such that $T \vdash \psi(\bar{x}) \to \varphi(\bar{x})$. Fact (F) is proved.

Further $\mathfrak{A}$ be (Σ, Σ)-*cl* algebraically prime model of theory T, $\bar{a} \in A$, $t = t^{\mathfrak{A}}_{\Sigma}(\bar{a})$. By Theorem 6 $\mathfrak{A}$-almost-weakly (Σ, Σ)-*cl* atomic model of theory T. Therefore, there is a formula $\varphi(\bar{x}) \in \Sigma$ consistent with T, which generates $t^{\mathfrak{A}}_{\Sigma}(\bar{a})$. According to (F), there exists a formula $\psi(\bar{x}) \in \Sigma$ consistent with T, for which the following holds: $T \vdash \psi(\bar{x}) \to \varphi(\bar{x})$. Obviously $\psi(\bar{x})$ generates $t_{\Sigma}^{\mathfrak{A}}(\bar{a})$. Due to the arbitrariness $\bar{a} \in A$ a model $\mathfrak{A}$ is almost-weakly (Σ, Σ)-cl atomic model of the theory T.

Remark. Let $\alpha, \beta \leq \omega$, $\bar{x}^{\alpha} \rightleftharpoons < x_i : 1 \leq i \leq 1 + \alpha >, \bar{a}^{\alpha} = < a_i : 1 \leq i \leq 1 + \alpha >$.

Definition 9. 1) α-type is called any set of formulas consistent with T, the free variables of which are found in $\bar{x}$;

2) ω-type ρ is called Γ-ω-type, if $\rho \subseteq \Gamma$;

3) Γ-ω-type ρ is called Γ_1-principle type if there exists such a sequence $\langle \psi_n(\bar{x}^n) : 1 \leq n < \omega \rangle$ Γ_1-formulas, such that:

a) $T \cup \psi_n(\bar{x}^n)$ is consistent, $1 \leq n < \omega$;

b) $\psi_n(\bar{x}^n)$ generates $\rho \upharpoonright \bar{x}^n$, where $\rho \upharpoonright \bar{x}^n$ is set of formulas from ρ, the free variables of which are among $(x_1, ..., x_n)$, $1 \leq n < \omega$;

c) $T \vdash \psi_n(\bar{x}^n) \leftrightarrow \exists \bar{x}_{n+1} \psi_{n+1}(\bar{x}^{n+1})$, $1 \leq n < \omega$.

Definition 10. A model $\mathfrak{A}$ of the theory T is said to be the fine almost-weakly (Γ_1, Γ_2)-cl atomic model of T if each tuple of ω elements $\mathfrak{A}$ implements Γ_1-principle type Γ_2 ω-type.

Lemma 9. Let $\mathfrak{A}$ be a countable model of the perfect Jonsson theory T, $A = \bar{a}^{\omega} = < a_1 ..., a_n, ... >$ implements (Σ, Σ)-ω-type. $\mathfrak{B} \models T$, $\mathfrak{B}$ is the isomorphically embeddable in $\mathfrak{A}$. Then $\mathfrak{B}$ is a fine almost-weakly (Σ, Σ)-cl atomic model of T.

Proof. Let $\bar{b}^{\omega} = < b_1, ..., b_n, ... >$ be an arbitrary tuple of ω-elements $\mathfrak{B}$. Such that $\mathfrak{B}$ is the isomorphically embeddable in $\mathfrak{A}$, then $b_k = a_{i_k}$ for some $1 \leq k < \omega$.

Let $n_k \rightleftharpoons ij : 1 \leq j \leq k$,

$Z_k = 1, 2, 3, .., n_k \setminus ij : 1 \leq j \leq k; \bar{y}^k = < y_1, ..., y_k >$.

Such that $\bar{a}^{\omega}$ implements the Σ-principal type Σ-ω-type, then there exists a sequence of Σ-formulas $\langle \psi_n(\bar{x}^n) : 1 \leq n < \omega \rangle$, for which the following is true:

1)$\psi_n(\bar{x}^n) \cup T$ is consistent, $1 \leq n < \omega$;

2)$\psi_n(\bar{x}^n)$ generates $t_{\Sigma}^{\mathfrak{A}}(\bar{a}^n)$ $1 \leq n < \omega$;

3)$T \vdash \psi_n(\bar{x}^n) \leftrightarrow \exists \bar{x}^n \psi_{n+1}(\bar{x}^{n+1})$ $1 \leq n < \omega$.

Let us denote by what

$$S_k(\bar{y}^k) \rightleftharpoons \begin{cases} \psi_{n_k}(\bar{x}^{n_k}) \begin{pmatrix} x_{i1}, ..., x_{ik} \\ y_1, ..., y_k \end{pmatrix}, \text{ if } Z_k = \emptyset \\ \exists ... x_S ... \psi_{n_k}(\bar{x}^{n_k})_{S \in Z_k} \begin{pmatrix} x_{i1}, ..., x_{ik} \\ y_1, ..., y_k \end{pmatrix}, \text{ if } Z_k \neq \emptyset. \end{cases}$$

Then it is clear that:

a) $S_k(\bar{y}^k) \in \Sigma_1$ $1 \leq k < \omega$;

b)$S(\bar{y}^k)$ consistent with T, $1 \leq k < \omega$;

c)$S(\bar{y}^k)$ generates $t_{\Sigma_1}^{\mathfrak{A}}(\bar{b}^k)$, $1 \leq k < \omega$;

d)$T \vdash S_k(\bar{y}^k) \leftrightarrow \exists y_{k+1} S_{k+1}(\bar{y}^{k+1}), 1 \leq k < \omega$.

Further such that $\mathfrak{B}$ is isomorphic embedding $\mathfrak{A}$, then $t_{\Sigma}^{\mathfrak{B}}(\bar{b}^k) \subseteq t_{\Sigma}^{\mathfrak{A}}(\bar{a}^k)$. Hence $S(\bar{y}^k)$ generates $t_{\Sigma}^{\mathfrak{B}}(\bar{b}^k)$ $1 \leq k < \omega$. Thus, since $\bar{b}^{\omega}$ is arbitrary, the model $\mathfrak{B}$ is fine almost-weakly (Σ, Σ)-cl atomic model T.

Corollary 10. Let $\mathfrak{A} \models T$, $\bar{a}^{\omega} = A$. Then:

1) if $\bar{a}^{\omega}$ implements Σ is principle type Σ-ω-type, then any infinite a^{ω} implements some Σ-principle type Σ-ω-type;

2) if $\mathfrak{A}$ is the fine an almost-weekly (Σ, Σ)-cl atomic model T, then $\mathfrak{A}$ is almost-weekly (Σ, Σ)-cl atomic model T.

Proof. Follows from Lemma 9.

Lemma 11. Let T has a fine almost-weakly (Σ, Σ)-cl atomic model, then each (Σ, Σ)-cl algebraically prime model of theory T is a fine almost-weekly (Σ, Σ)-cl atomic model of the theory T.

Proof. Let $\mathfrak{B}$ be an arbitrary (Σ, Σ)-cl algebraically prime model of theory T, $\mathfrak{A}$ is fine an almost-weekly (Σ, Σ)-cl atomic model of the theory T, then there is an embedding $f : \mathfrak{A} \to \mathfrak{B}$. Let $\mathfrak{A}' = f[\mathfrak{B}]$. Obviously $\mathfrak{A}'$ embedded in $\mathfrak{A}$, and by Lemma 9 $\mathfrak{A}'$, therefore $\mathfrak{B}$ is also fine almost-weakly (Σ, Σ)-cl-atomic models of the theory T.

Lemma 12. Let T perfect Jonsson theory complete for Π_1-sentences. Then every fine almost-weakly (Σ, Σ)-cl-atomic models of the theory T is a (Σ, Σ)-cl algebraically prime model of T.

Proof. Let $\bar{a}^{\omega} =< a_1, ..., a_n, ... >$ are elements from A. Since $\bar{a}^{\omega}$ implements Σ_1-principal $\Sigma_1 - \omega$-type, there exists $\langle \psi_n(\bar{x}^n) : 1 \leq n < \omega \rangle$- is a sequence of Σ_1-formulas for which the condition of item 3 of Definition 7 is true. Such that T is complete for Π_1-sentences, then $\mathfrak{B} \models \exists \bar{x}^n \psi_n(\bar{x}^n), 1 \leq n < \omega$, where $\mathfrak{B} \models T$. Further, since $T \vdash \psi_n(\bar{x}^n) \leftrightarrow \exists \bar{x}^{n+1} \psi_{n+1}(\bar{x}^{n+1})$ for each $1 \leq n < \omega$, then it is possible (step by step) to gradually find such $b_1, ..., b_n$ from B, such that $\mathfrak{B} \models \psi_n(\bar{x}^n), 1 \leq n < \omega$, where $\bar{b}^n = \langle b_1, ..., b_n \rangle$. But $\psi_n(\bar{x}^n)$ generates $t^{\mathfrak{A}}_{\Sigma_1}(\bar{a}^n)$, so $t^{\mathfrak{A}}_{\Sigma_1}(\bar{a}^n) \subseteq t^{\mathfrak{B}}_{\Sigma_1}(\bar{b}^n), 1 \leq n < \omega$.

Therefore, the mapping $f : \mathfrak{A} \to \mathfrak{B}$, where $f(a_n) = b_n, 1 \leq n < \omega$, is an isomorphic embedding.

Theorem 13. Let T be the perfect Jonsson theory complete for Π_1-sentences and has fine almost-weakly (Σ, Σ)-*cl* atomic model. Then the following conditions are equivalent:

1) $\mathfrak{A}$ is the (Σ, Σ)-*cl* algebraically prime model of theory T.

2) $\mathfrak{A}$ is the fine almost-weakly (Σ, Σ)-*cl* atomic model of the theory T.

Proof. 1) $\Rightarrow$ 2) follows from Lemma 11. 2) $\Rightarrow$ 1) from Lemma 12.

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Анықталған жиынның тұйықтау операторы көмегімен алынған атомдық және алгебралық жай модельдер туралы

Мақалада қандай да бір бекітілген йонсондық теорияның семантикалық моделінің анықталған ішкі жиынында берілген қандайда бір тұйықтау операторының көмегімен алынған атомдық және жай модельдердің қасиеттері қарастырылған. Негізгі нәтиже ретінде атомдық және жай модельдерде анықталған эквиваленттілікті табу болып табылады, яғни бұл сәйкестік жақсы қасиеттерімен берілген қандай да бір модель бар деген шығады.

Кілт сөздер: йонсондық теориясы, семантикалық модель, жай модель, атомдық модель, алгебралық жай модель, предгеометрия, анықталған ішкі жиын.

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Об атомных и алгебраически простых моделях, полученных замыканием определимых множеств

В статье рассмотрены свойства атомных и простых моделей, полученных с помощью некоторого оператора замыкания, заданного на определимых подмножествах семантической модели некоторой фиксированной йонсоновской теории. Основным результатом явилось получение эквивалентности определенными таким образом атомной и простой моделей, причем это совпадение следует при предположении, что существует некоторая модель с хорошо заданными свойствами.

Ключевые слова: йонсоновская теория, семантическая модель, простая модель, атомная модель, алгебраически простая модель, предгеометрия, определимое подмножество.

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On the boundedness of the partial sums operator for the Fourier series in the function classes families associated with harmonic intervals

The article is devoted to the study of some data from the theory of functions approximation by trigonometric polynomials with a spectrum from special sets called harmonic intervals. Due to the limited perception range of devices, the perception range of the senses of the person himself, when studying a mathematical model it is often enough to find an approximation of the object so that the error (noise, interference, distortion) is outside the interval of perception. Harmonic intervals model problems of this kind to some extent. In the article the main components of the approximation theory of functions by trigonometric polynomials with a spectrum from harmonic intervals are presented, the theorem on estimating the best approximation of a function by trigonometric polynomials through the best approximations of a function by trigonometric polynomials with a spectrum from harmonic intervals is proved. Theorems on the boundedness of the partial sums operator for the Fourier series in the function classes families associated with harmonic intervals are considered; such a theorem for the Lorentz space is generalized and proved. The article is mainly aimed at scientific researchers dealing with practical applications of the approximation theory of functions by trigonometric polynomials with a spectrum from special sets.

Keywords: harmonic interval, trigonometric polynomials with a spectrum from harmonic intervals, best approximation of a function by trigonometric polynomials, partial sums operator of the Fourier series for a given function, interpolation theorem.

Introduction

In approximation theory one of the most relevant problems is the approximation of periodic functions by polynomials with a spectrum from special families of sets. Here we note the works of K.I. Babenko, S.A. Telyakovsky, V.N. Temlyakov [1] and others in the case when the spectrum is a hyperbolic cross; the works of V.I. Yudin, M.I. Dyachenko [2] in the case when the spectrum is a ball, etc.

In the study of many applied problems the question of approximating the mathematical model of the object under study naturally arises. However, due to the limited range of perception («window of perception») of devices, the range of perception of the human senses, when studying a mathematical model it is often enough to find an approximation of the object so that the error (noise, interference, distortion) is outside the interval («window») of perception.

In this paper we consider approximations of functions by trigonometric polynomials with a spectrum from harmonic intervals, which to some extent model problems of this kind.

Note that harmonic intervals are some fractal self-similar sets, the concept of which was introduced by E.D. Nursultanov in [3–5] and, as it turned out, harmonic intervals have an important role in harmonic analysis. Thus, in the works of N.T. Tleukhanova, K.S. Saydakhmetov, D.S. Karimov, such objects as harmonic segments and harmonic intervals were essentially used.

In the studying the problem of the boundedness of the partial sums operator for the Fourier series in the function classes families associated with the best approximations over harmonic intervals the method of real interpolation is used. Among the works devoted to the properties of interpolation spaces, as well as to the methods of interpolation, one should note the works of Y. Berg and J. Lefstrom [6], S.G. Krein, Yu.I. Petunin, E.M. Semenov, Yu.A. Brudny [7], [8], H. Tribel [9], [10].

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Definitions and auxiliary results

Let $k, \nu, N \in \mathbb{N}$, $k < N$. A set of the form

$$I_k^N = \bigcup_{\nu=-\infty}^{\infty} ([-k,k] + 2\nu N) = \bigcup_{\nu=-\infty}^{\infty} (m + 2\nu N : m \in [-k,k])$$

is called a harmonic interval in $\mathbb{Z}$.

We denote by T_k^N the set of trigonometric polynomials of the form

$$T_k^N = \left\{ \sum_{\nu=-s}^{s} a_\nu \cdot e^{i\nu x} : a_\nu = 0 \;\; if \;\; \nu \notin I_k^N, s \in \mathbb{N} \right\}.$$

The value

$$E_k^N(f)_p = \inf_{t \in T_k^N} \|f - t\|_p$$

is called the best approximation over the harmonic interval I_k^N of the function $f \in L_p[0, 2\pi)$, $1 \le p \le \infty$, by trigonometric polynomials from T_k^N of order less than or equal to k.

Let $f \in L_p[0, 2\pi)$, $1 \le p \le \infty$. The partial sum of the Fourier series for the function f over the harmonic interval I_k^N is called the function

$$S_k^N(f) = \sum_{\nu \in I_k^N} a_\nu \cdot e^{i\nu x}.$$

Theorem 1. [11] Let $f \in L_p[0, 2\pi)$, $1 < p < \infty$, $m \in \mathbb{N}$. $S_m^N(f)$ and $E_m^N(f)$ are the partial sum of the Fourier series and the best approximation of the function f over the harmonic interval I_m^N respectively, then we have the following relation

$$E_m^N(f)_p \sim \|f - S_m^N(f)\|_p.$$

Lemma 1. [11] Let $n \in \mathbb{N}$, $1 \le p < q \le \infty$, $1 \le r \le \infty$, then

$$\|T_n\|_{L_{q,r}} \le C n^{\frac{1}{p} - \frac{1}{q}} \|T_n\|_{L_p}. \tag{1}$$

Let $1 \le p$, $q \le \infty$, $r > 0$, $f \in L_p[0, 2\pi)$. The family of function classes $\left\{B_{p,q.N}^r\right\}_N$ is defined by the equality

$$B_{p,q.N}^r = \left\{ f : \|f\|_{B_{p,q.N}^r} < \infty \right\}, \quad N \in \mathbb{N},$$

where

$$\|f\|_{B_{p,q.N}^r} = \left(\sum_{k=1}^{N} k^{rq-1} \left(E_{k-1}^N(f)_p\right)^q \right)^{\frac{1}{q}}.$$

Let two families of function classes $\left\{A^N\right\}_N$ and $\left\{B^N\right\}_N$, $N \in \mathbb{N}$, be given. We assume that the ratio

$$\|f\|_{A^N} \sim \|f\|_{B^N}$$

holds if there are parameters C_1, C_2 such that for any $f \in A^N$ the following inequality

$$C_1 \|f\|_{B^N} \le \|f\|_{A^N} \le C_2 \|f\|_{B^N}$$

is valid, moreover, the parameters C_1, C_2 do not depend on f and N.

Theorem 2. [12] Let $f \in B_{p,q.2^m}^r$, $m \in \mathbb{N}$. Then for $1 \le p$, $q \le \infty$, $r > 0$ we have

$$\|f\|_{B_{p,q.2^m}^r} \sim \left(\sum_{k=1}^{m} 2^{rqk} \left(E_{2^k-1}^{2^m}(f)_p\right)^q \right)^{\frac{1}{q}}.$$

Theorem 3. [12] Let $m \in \mathbb{N}$, $1 \le p, p_0, p_1 \le \infty$, $0 < \theta < 1$, $r_0 > 0$, $r_1 > 0$, $r_0 \ne r_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $r = (1-\theta) \cdot r_0 + \theta \cdot r_1$, then

$$\left(B_{p_0,p_0,2^m}^{r_0}; B_{p_1,p_1,2^m}^{r_1}\right)_{\theta,p} = B_{p,p,2^m}^r.$$

Estimation by the best approximations over harmonic intervals

Theorem 4. Let $f \in L_p[0,2\pi)$, $1 < p < \infty$, $n \in \mathbb{N}$. $\sum\limits_{\nu\in\mathbb{Z}} a_\nu \cdot e^{i\nu x}$ is the trigonometric Fourier series of the function f, then the following inequality holds

$$E_n(f)_p \leq \sum_{j=1}^{\infty} E^{2^j\cdot n}_{(2^j-1)\cdot n}(f)_p.$$

Proof. By Lemma 9.3 [13] we have

$$E_n(f)_p \sim \|f - S_n(f)\|_p,$$

when $1 < p < \infty$ or

$$E_n(f)_p \sim \left\| \sum_{\nu\in Z\setminus[-n,n]} a_\nu \cdot e^{i\nu x} \right\|. \tag{2}$$

By entering the notation of harmonic intervals in $\mathbb{Z}$

$$V_j = \bigcup_{m=-\infty}^{\infty} \left\{ \left[\left(2^j-1\right)n; \left(2^j+1\right)n \right] + 2^{j+1}mn \right\} =$$

$$= \bigcup_{m=-\infty}^{\infty} \left\{ [-n;n] + 2^j n\,(2m+1) \right\}, \quad j = 1,2,...,$$

we obtain

$$\mathbb{Z} \setminus [-n;n] = \bigcup_{j=1}^{\infty} V_j.$$

Then from (2) we get the relation in this form

$$E_n(f)_p \sim \left\| \sum_{j=1}^{\infty} \sum_{\nu\in V_j} a_\nu \cdot e^{i\nu x} \right\|_p = \left\| \sum_{j=1}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{s=n[2^j(2m+1)-1]}^{n[2^j(2m+1)+1]} a_s \cdot e^{isx} \right\|_p. \tag{3}$$

We denote by W_j, $j = 1,2,...$, the following sets

$$W_j = \mathbb{Z} \setminus V_j, \tag{4}$$

where

$$W_j = \bigcup_{m=-\infty}^{\infty} \left\{ \left[-\left(2^j-1\right)n; \left(2^j-1\right)n \right] + 2^{j+1}mn \right\}$$

or

$$W_j = I^{2^j n}_{(2^j-1)n}.$$

We note that the sets W_j, $j = 1,2,...$ are also harmonic intervals in $\mathbb{Z}$ as complements of the harmonic intervals V_j, $j = 1,2,...$ in $\mathbb{Z}$. Then, according to Theorem 1, using (4), from (3) we obtain the required inequality

$$E_n(f)_p \sim \left\| \sum_{j=1}^{\infty} \sum_{\nu\in\mathbb{Z}\setminus W_j} a_\nu \cdot e^{i\nu x} \right\|_p = \left\| \sum_{j=1}^{\infty} \left(f - S_{W_j}(f)\right) \right\|_p \leq$$

$$\leq \sum_{j=1}^{\infty} \left\| \left(f - S_{W_j}(f)\right) \right\|_p \sim \sum_{j=1}^{\infty} E_{W_j}(f)_p,$$

$$E_n(f)_p \leq \sum_{j=1}^{\infty} E^{2^j\cdot n}_{(2^j-1)\cdot n}(f)_p.$$

The theorem is proved.

Theorems on the boundedness of the partial sums operator for the Fourier series of a function f in the function classes families $\left\{B^r_{p,q.N}\right\}_N$

Theorem 5. [11] Let $N \in \mathbb{N}$, $1 \le p < q \le \infty$, $1 \le r \le \infty$, $\beta > 0$, $\alpha - \beta = \frac{1}{p} - \frac{1}{q}$. $B^{\beta}_{q,r}$ is the Besov space [14], then the partial sums operator for the trigonometric Fourier series of the function f

$$S_N\left(f(x)\right) = \sum_{k=-N}^{N} \widehat{f}(k) e^{ikx}$$

such that

$$S_N \,:\, B^{\alpha}_{p,r,N} \,\to\, B^{\beta}_{q,r}$$

is bounded, that is, there is the inequality

$$\left\| S_N(f) \right\|_{B^{\beta}_{q,r}} \le\; C \left\| f \right\|_{B^{\alpha}_{p,r,N}},$$

where the parameter C do not depend on f and N.

Corollary 1. [11] Let $N \in \mathbb{N}$, $1 \le p < q \le \infty$, $1 \le r \le \infty$, $\beta > 0$, $\alpha - \beta = \frac{1}{p} - \frac{1}{q}$, then the partial sums operator for the trigonometric Fourier series of the function f

$$S_N \,:\, B^{\alpha}_{p,r,N} \,\to\, B^{\beta}_{q,r,N}$$

is bounded, that is, the following inequality

$$\left\| S_N(f) \right\|_{B^{\beta}_{q,r,N}} \le\; C \left\| f \right\|_{B^{\alpha}_{p,r,N}},$$

where the parameter C do not depend on f and N.

Theorem 6. [11] Let $m \in \mathbb{N}$, $1 \le p < q \le \infty$, $\alpha = \frac{1}{p} - \frac{1}{q}$, then the partial sums operator for the trigonometric Fourier series of the function f

$$S_{2^m} \,:\, B^{\alpha}_{p,q,2^m} \,\to\, L_q$$

is bounded, that is, we have the following inequality

$$\left\| S_{2^m}(f) \right\|_{L_q} \le\; C \left\| f \right\|_{B^{\alpha}_{p,q,2^m}},$$

where the parameter C do not depend on f and m.

Remark 1. Theorem 6 can be formulated in a more general form.

Let $N \in \mathbb{N}$, $1 \le p < q \le \infty$, $\alpha = \frac{1}{p} - \frac{1}{q}$, then the partial sums operator for the trigonometric Fourier series of the function f

$$S_N \,:\, B^{\alpha}_{p,q,N} \,\to\, L_q$$

is bounded, that is, there is the inequality of the form

$$\left\| S_N(f) \right\|_{L_q} \le\; C \left\| f \right\|_{B^{\alpha}_{p,q,N}},$$

where the parameter C do not depend on f and N.

We generalize Theorem 6 to Lorentz spaces.

Theorem 7. Let $N \in \mathbb{N}$, $1 \le p < q \le \infty$, $1 \le r \le \infty$, $\alpha = \frac{1}{p} - \frac{1}{q}$, then the partial sums operator for the trigonometric Fourier series of the function f

$$S_N \,:\, B^{\alpha}_{p,r,N} \,\to\, L_{q,r}$$

is bounded, that is, this inequality holds

$$\left\| S_N(f) \right\|_{L_{q,r}} \le\; C \left\| f \right\|_{B^{\alpha}_{p,r,N}},$$

where the parameter C do not depend on f and N.

Proof. We estimate the norm of the partial sum operator in the Lorentz space

$$\|S_N(f)\|_{L_{q,r}} \le \sum_{k=1}^{[\log_2 N]} \left\| \sum_{n=-2^{k-1}}^{2^k-1} a_n e^{inx} \right\|_{L_{q,r}} = \sum_{k=1}^{[\log_2 N]} \|\triangle_k \left(S_N(f)\right)\|_{L_{q,r}}. \tag{5}$$

Applying the inequality of different metrics (1), we transform the relation (5) as follows

$$\|S_N(f)\|_{L_{q,r}} \le C \sum_{k=1}^{[\log_2 N]} 2^{k\left(\frac{1}{p}-\frac{1}{q}\right)} \|\triangle_k \left(S_N(f)\right)\|_{L_p} = C \sum_{k=1}^{[\log_2 N]} 2^{\alpha k} \|\triangle_k \left(S_N(f)\right)\|_{L_p}. \tag{6}$$

Taking into account that $\triangle_k \left(S_N(f)\right)$ is a partial sum of the function $\sum\limits_{n\in\mathbb{Z}\setminus I^N_{2^{k-1}-1}} a_n e^{inx}$ and using the M. Riesz theorem [15], Theorem 1 and Theorem 2, we reduce relation (6) to the form

$$\|S_N(f)\|_{L_{q,r}} \le C \sum_{k=1}^{[\log_2 N]} 2^{\alpha k} \|\triangle_k \left(S_N(f)\right)\|_{L_p} \le$$

$$\le C \sum_{k=1}^{[\log_2 N]} 2^{\alpha k} \left\| \sum_{n\in\mathbb{Z}\setminus I^N_{2^{k-1}-1}} a_n e^{inx} \right\|_{L_p} = C \sum_{k=1}^{[\log_2 N]} 2^{\alpha k} \left\| f - S^N_{2^{k-1}-1}(f) \right\|_{L_p} \le$$

$$\le C \sum_{k=1}^{[\log_2 N]} 2^{\alpha k} E^N_{2^{k-1}-1}(f)_p = C \cdot 2^{\alpha} \sum_{k=1}^{[\log_2 N]} 2^{\alpha(k-1)} E^N_{2^{k-1}-1}(f)_p \le$$

$$\le C \sum_{k=1}^{[\log_2 N]} 2^{\alpha k} E^N_{2^k-1}(f)_p \sim C\|f\|_{B^{\alpha}_{p,1,N}}.$$

$$\Rightarrow \ \|S_N(f)\|_{L_{q,r}} \le C\|f\|_{B^{\alpha}_{p,1,N}}. \tag{7}$$

We take pairs (α_0, α_1), (q_0, q_1), (r_0, r_1) that satisfy the following conditions

$$\alpha_0 < \alpha < \alpha_1, \ \ q_0 < q < q_1, \ \ r_0 < r < r_1,$$

$$\alpha_0 = \frac{1}{p} - \frac{1}{q_0}, \ \ \alpha_1 = \frac{1}{p} - \frac{1}{q_1}.$$

Taking into account the relation (7), we obtain the following

$$S_N \ : \ B^{\alpha_0}_{p,1,N} \to L_{q_0,r_0},$$

$$S_N \ : \ B^{\alpha_1}_{p,1,N} \to L_{q_1,r_1}$$

then, by the interpolation theorem [6], we have

$$S_N \ : \ \left(B^{\alpha_0}_{p,1,N}; B^{\alpha_1}_{p,1,N}\right)_{\theta,r} \to \left(L_{q_0,r_0}; L_{q_1,r_1}\right)_{\theta,r}. \tag{8}$$

Using Theorem 3, we receive that this relation holds

$$\left(B^{\alpha_0}_{p,1,N}; B^{\alpha_1}_{p,1,N}\right)_{\theta,r} = B^{\alpha_\theta}_{p,r,N},$$

where

$$\alpha_\theta = (1-\theta)\cdot\alpha_0 + \theta\cdot\alpha_1, \ \ \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \ \ 0 < \theta < 1.$$

It follows from the theorem on the interpolation of Lorentz spaces [6] that

$$\left(L_{q_0,r_0}; L_{q_1,r_1}\right)_{\theta,r} = L_{q_\theta,r},$$

where
$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \ \ \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

Since there is a dependency
$$\alpha_\theta = (1-\theta)\cdot\alpha_0 + \theta\cdot\alpha_1 = (1-\theta)\left(\frac{1}{p} - \frac{1}{q_0}\right) + \theta\left(\frac{1}{p} - \frac{1}{q_1}\right) = \frac{1}{p} - \frac{1}{q_\theta},$$
then there is $\theta \in (0;1)$ such that
$$\alpha_\theta = \alpha, \ \ q_\theta = q.$$

As a result, from (8) we obtain the required relation
$$S_N \, : \, B^{\alpha}_{p,r,N} \, \rightarrow \, L_{q,r},$$
and
$$\|S_N(f)\|_{L_{q,r}} \leq \, C\, \|f\|_{B^{\alpha}_{p,r,N}} \, .$$
where the parameter C do not depend on f and N.

The theorem is proved.

Remark 2. In Theorems 5, 7 and Remark 1, the operator $S_N(f)$ can be replaced by the operator $S_n(f)$, where $0 \leq n \leq N$. Indeed, from M. Riesz's theorem we have
$$\|S_n(f)\|_{L_p} \leq C\, \|S_N(f)\|_{L_p} \, ,$$
where the parameter C do not depend on f and N.

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Гармоникалық интервалдармен байланысты функциялар кластары үйіріндегі Фурье қатарының дербес қосындылары операторының шенелгендігі туралы

Мақала гармоникалық интервалдар деп аталатын арнайы жиынтықтар спектрі бар тригонометрикалық полиномдар функцияларын жуықтау теориясының кейбір деректерін зерттеуге арналған. Математикалық модельді зерттеу кезінде құрылғылардың қабылдау ауқымы, адамның сезім мүшелерінің қабылдау ауқымы шектеулі болғандықтан қателік (шу, кедергі, бұрмалау) қабылдау интервалынан тыс болатындай етіп қажетті объектінің жуықтамасын табу көбінесе жеткілікті болады. Гармоникалық интервалдар осындай типтегі мәселелерді белгілі бір деңгейде модельдейді. Мақалада гармоникалық интервалдар деп аталатын арнайы жиынтықтар спектрі бар тригонометрикалық полиномдар функцияларын жуықтау теориясының негізгі компоненттері келтірілген, гармоникалық интервалдар деп аталатын арнайы жиынтықтар спектрі бар тригонометрикалық полиномдар функциясын ең жақсы жуықтау арқылы тригонометрикалық полиномдар функциясын ең жақсы жуықтауды бағалау туралы теоремасы дәлелденді. Гармоникалық интервалдармен байланысты функциялар кластары үйіріндегі Фурье қатарының дербес қосындылары операторының шенелгендігі туралы теоремалар келтірілген, мұндай теорема Лоренц кеңістігі үшін жалпыландырылған және дәлелденген. Негізінен мақала арнайы жиынтықтар спектрі бар тригонометрикалық полиномдар функцияларын жуықтау теориясының практикалық қолдануымен айналысатын ғылыми зерттеушілерге арналған.

Кілт сөздер: гармоникалық интервал, гармоникалық интервалдар спектрі бар тригонометрикалық полиномдар, тригонометрикалық полиномдар функциясын ең жақсы жуықтау, белгіленген функция үшін Фурье қатарының дербес қосындылары операторы, интерполяциялық теорема.

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Об ограниченности оператора частичных сумм ряда Фурье в семействах классов функций, связанных с гармоническими интервалами

Статья посвящена исследованию некоторых данных теории приближения функций тригонометрическими полиномами со спектром из специальных множеств, называемых гармоническими интервалами. В силу ограниченности диапазона восприятия приборов, диапазона восприятия органов чувств самого человека при исследовании математической модели часто достаточно найти приближение искомого объекта так, чтобы погрешность (шумы, помехи, искажения) оказалась вне интервала восприятия. Гармонические интервалы в некоторой степени моделируют задачи такого рода. В статье представлены основные компоненты теории приближения функций тригонометрическими полиномами со спектром из гармонических интервалов, доказана теорема об оценке наилучшего приближения функции тригонометрическими полиномами через наилучшие приближения функции тригонометрическими полиномами со спектром из гармонических интервалов. Приведены теоремы об ограниченности оператора частичных сумм ряда Фурье в семействах классов функций, связанных с гармоническими интервалами, обобщена и доказана такая теорема для пространства Лоренца. Статья ориентирована, в основном, на научных исследователей, занимающихся практическими приложениями теории приближений функций тригонометрическими полиномами со спектром из специальных множеств.

Ключевые слова: гармонический интервал, тригонометрические полиномы со спектром из гармонических интервалов, наилучшее приближение функции тригонометрическими полиномами, оператор частичных сумм ряда Фурье для заданной функции, интерполяционная теорема.

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On a Hilfer Type Fractional Differential Equation with Nonlinear Right-Hand Side

In this article we consider the questions of one-valued solvability and numerical realization of initial value problem for a nonlinear Hilfer type fractional differential equation with maxima. By the aid of uncomplicated integral transformation based on Dirichlet formula, this initial value problem is reduced to the nonlinear Volterra type fractional integral equation. The theorem of existence and uniqueness of the solution of given initial value problem in the segment under consideration is proved. For numerical realization of solution the generalized Jacobi–Galerkin method is applied. Illustrative examples are provided.

Keywords: Ordinary differential equation, equation with maxima, Hilfer operator, one-valued solvability, generalized Jacobi–Galerkin method.

Introduction

Let $(t_0;\, b) \subset \mathbb{R}^+ \equiv [0;\, \infty)$ be a finite interval on the set of positive real numbers, and let $\alpha > 0$. The Riemann–Liouville α-order fractional integral of a function $\eta(t)$ is defined as follows:

$$I_{t_0+}^{\alpha}\eta(t) = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-s)^{\alpha-1}\eta(s)\, d\, s, \quad \alpha > 0, \quad t \in (t_0;\, b),$$

where $\Gamma(\alpha)$ is the Gamma function [1; 112].

Let $n-1 < \alpha \le n,\ n \in \mathbb{N}$. The Riemann–Liouville α-order fractional derivative of a function $\eta(t)$ is defined as follows [2, Vol. 1, p. 27]:

$$D_{t_0+}^{\alpha}\eta(t) = \frac{d^n}{dt^n} I_{t_0+}^{n-\alpha}\eta(t), \quad t \in \big(t_0;\, b\big).$$

The Caputo α-order fractional derivative of a function $\eta(t)$ is defined [2, Vol. 1; 34] by

$${}_*D_{t_0+}^{\alpha}\eta(t) = I_{t_0+}^{n-\alpha}\eta^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_{t_0}^{t} \frac{\eta^{(n)}(s)\, d\, s}{(t-s)^{\alpha-n+1}}, \ \ t \in (t_0;\, b).$$

Both the derivatives are reduced to the n-th order derivatives for $\alpha = n \in \mathbb{N}$ [2, Vol. 1; 27–34]:

$$D_{t_0+}^{n}\eta(t) =_* D_{t_0+}^{n}\eta(t) = \frac{d^n}{dt^n}\eta(t), \quad t \in (t_0;\, b).$$

The so-called generalized Riemann–Liouville fractional derivative (referred to as the Hilfer fractional derivative) of order α, $n-1 < \alpha \le n$, $n \in \mathbb{N}$ and type β, $0 \le \beta \le 1$ is defined by the following composition of three operators: [1; 113]:

$$D_{t_0+}^{\alpha,\,\beta}\eta(t) = I_{t_0+}^{\beta(n-\alpha)} \frac{d^n}{dt^n} I_{t_0+}^{(1-\beta)(n-\alpha)}\eta(t), \quad t \in (t_0;\, b).$$

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For $\beta = 0$ this operator is reduced to the Riemann–Liouville fractional derivative $D^{\alpha,0}_{t_0+} = D^{\alpha}_{t_0+}$ and the case $\beta = 1$ corresponds to the Caputo fractional derivative $D^{\alpha,1}_{t_0+} =_* D^{\alpha}_{t_0+}$.

Let $\gamma = \alpha + \beta n - \alpha\beta$. It is easy to see that $\alpha \le \gamma \le n$. Then it is convenient to use another designation for the operator $D^{\alpha,\beta}_{t_0+}\eta(t)$:

$$D^{\alpha,\gamma}\eta(t) = D^{\alpha,\beta}_{t_0+}\eta(t).$$

The generalized Riemann–Liouville operator was introduced in [1] by R. Hilfer on the basis of fractional time evolutions that arise during the transition from the microscopic scale to the macroscopic time scale. Using the integral transforms, he investigated the Cauchy problem for the generalized diffusion equation, the solution of which is presented in the form of the Fox H-function. Note [3, 4], the generalized Riemann–Liouville operator was used in studying dielectric relaxation in glass-forming liquids with different chemical compositions. In [5] the properties of the generalized Riemann–Liouville operator were investigated in a special functional space, and an operational method was developed for solving fractional differential equations with this operator. Based on the results of the work [5], the authors of [6] have developed an operational method for solving fractional differential equations containing a finite linear combination of the generalized Riemann–Liouville operators with various parameters.

Fractional calculus plays an important role in the mathematical modelling of many scientific and engineering disciplines (see more detailed information in [7]). In [8] problems of continuum and statistical mechanics are considered. In [9] the mathematical problems of Ebola epidemic model are studied. In [10] and [11] the fractional model for the dynamics of tuberculosis infection and novel coronavirus (COViD-2019), respectively are studied. The construction of various models of theoretical physics by the aid of fractional calculus is described in [2, Vol. 4, 5], [12, 13]. A specific interpretation of the Hilfer fractional derivative, describing the random motion of a particle moving on the real line at Poisson paced times with finite velocity is given in [14]. A detailed review of the application of fractional calculus in solving problems of applied sciences is given in [2, Vol. 6–8], [15]. More detailed information related to the theory of fractional integro-differentiation, including the Hilfer fractional derivative one can find in the monograph [16]. In [17] the unique solvability of boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator is studied by analytical method. In [18] the solvability of nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator is studied. In [19] it is considered an inverse problem for a mixed type integro-differential equation with fractional order Caputo operators (see also [20–22]).

In the modern scientific world information technologies are widely used in various fields of science and engineering [23, 24]. In application of differential equations the numerical methods play an important role. Different methods are used for the numerical solution of differential, integral and integro-differential equations [25–34]. In particular, the book [28] is devoted to Chebyshev and Fourier spectral methods and [30] tells us about polynomial approximations of solving differential equations. The work [35] is devoted to study of nonlinear Volterra integral equations with weakly singular kernels by generalized Jacobi Spectral–Galerkin method.

In the present paper we consider the questions of one-valued solvability and numerical realization for a Hilfer type fractional differential equation with nonlinear right-hand side and maxima. This equation we solve under initial value condition. Differential equations with maxima play an important role in solving control problems of the sale of goods and investment of manufacturing companies in a market economy [36]. In [37] it is justified that the theoretical study of differential equations with maxima is relevant.

We consider the Hilfer type fractional differential equation on a interval $(t_0; T)$:

$$D^{\alpha,\gamma}x(t) + \omega x(t) = f\Big(t, \, x(t), \, \max\{x(\theta) \,|\, \theta \in [q_1 t; \, q_2 t]\}\Big) \tag{1}$$

under initial value condition

$$\lim_{t\to t_0} J^{1-\gamma}_{t_0+}x(t) = x_0, \quad x(t) = \varphi(t), \; t \notin (t_0, T), \tag{2}$$

where $f(t, u, \vartheta) \in C(\Omega)$, $\varphi(t) \in C\left([0; t_0] \cup [T; \infty]\right)$, $0 < \omega$ is real parameter, $x_0 = \text{const}$, $\Omega \equiv [t_0; T] \times \mathbb{X} \times \mathbb{X}$, $0 \le t_0$, $\mathbb{X} \subset \mathbb{R} \equiv (-\infty; \infty)$, $\mathbb{X}$ is closed set. Here

$$D^{\alpha,\gamma} = J^{\gamma-\alpha}_{t_0+}\frac{d}{dt}J^{1-\gamma}_{t_0+}, \quad 0 < \alpha \le \gamma \le 1$$

is Hilfer operator and J^{α}_{0+} is the Riemann–Liouville integral operator, which is defined by the formula

$$J^{\alpha}_{t_0+}\eta(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\eta(s)\,d\,s}{(t-s)^{1-\alpha}}, \quad \alpha > 0.$$

We set $0 < q_1 < q_2 < \infty$ and understand that there are possible cases: 1) $0 < q_1 < q_2 < 1$; 2) $0 < q_1 < 1$, $1 < q_2 < \infty$; 3) $1 < q_1 < q_2 < \infty$.

Fractional integral equation

Lemma. The solution of the differential equation (1) with initial value condition (2) is represented as follows

$$x\,(t) = \Im\,(t;\,x) \equiv x_{\,0}\,(t-t_0)^{\gamma-1} E_{\alpha,\,\gamma}\,(-\omega\,(t-t_0)^{\alpha}) +$$

$$+\int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\,\alpha}\,(-\omega\,(t-s)^{\alpha})\, f\,(s,\,x\,(s),\,\max\{x\,(\theta)\,|\,\theta\in[q_1 s;\,q_2 s]\})\,d\,s, \tag{3}$$

where $E_{\alpha,\,\gamma}(z)$ is Mittag–Leffler function and has the form [2, vol. 1, 269–295]

$$E_{\alpha,\,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^{\,k}}{\Gamma(\alpha\,k+\gamma)}, \quad z,\,\alpha,\gamma\in\mathbb{R}>0.$$

Proof. We rewrite the differential equation (1) in the form

$$J^{\gamma-\alpha}_{t_0+} D^{\gamma}_{t_0+}\,x\,(t) = -\omega\,x(t) + f\,(t,\,\cdot),$$

where $f\,(t,\,\cdot) = f\,(t,\,x\,(t),\,\max\{x\,(\theta)\,|\,\theta\in[q_1 t;\,q_2 t]\})$.

Applying the operator $J^{\alpha}_{t_0+}$ to both sides of this equation, taking into account the linearity of this operator and the formula [6]

$$J^{\gamma}_{t_0+} D^{\gamma}_{t_0+} x(t) = x(t) - \frac{1}{\Gamma\,(\gamma)}\, J^{1-\gamma}_{t_0+} x(t)|_{\,t=t_0+} (t-t_0)^{\,\gamma-1}$$

we obtain

$$x(t) = \frac{x_{\,0}}{\Gamma\,(\gamma)}\,(t-t_0)^{\,\gamma-1} + J^{\alpha}_{t_0+}\,f\,(t,\cdot) - \omega\,J^{\alpha}_{t_0+}\,x(t). \tag{4}$$

Using the lemma from [38], we represent the solution of equation (4) in the form

$$x(t) = \frac{x_{\,0}}{\Gamma\,(\gamma)}\,(t-t_0)^{\,\gamma-1} + J^{\alpha}_{t_0+}\,f\,(t,\cdot) -$$

$$-\omega \int_{t_0}^{t} (t-s)^{\,\alpha-1}\, E_{\alpha,\,\alpha}\,(-\omega\,(t-s)^{\,\alpha}) \left[\frac{x_{\,0}}{\Gamma\,(\gamma)}\,(s-t_0)^{\,\gamma-1} + J^{\alpha}_{t_0+}\,f\,(s,\cdot)\right] d\,s. \tag{5}$$

We rewrite the representation (5) as the sum of two expressions:

$$I_{\,1}\,(t) = x_{\,0} \left[\frac{(t-t_0)^{\,\gamma-1}}{\Gamma\,(\gamma)} - \frac{\omega}{\Gamma\,(\gamma)} \int_{t_0}^{t} (t-s)^{\,\alpha-1}\, E_{\alpha,\,\alpha}\,(-\omega\,(t-s)^{\,\alpha})\,(s-t_0)^{\,\gamma-1}\,d\,s\right], \tag{6}$$

$$I_{\,2}\,(t) = J^{\alpha}_{t_0+}\,f\,(t,\cdot) - \omega \int_{t_0}^{t} (t-s)^{\,\alpha-1}\, E_{\alpha,\,\alpha}\,(-\omega\,(t-s)^{\,\alpha})\, J^{\alpha}_{t_0+}\,f\,(s,\cdot)\,d\,s. \tag{7}$$

We apply the following representations [2, vol. 1, 269–295]

$$E_{\,\alpha,\,\gamma}\,(z) = \frac{1}{\Gamma\,(\gamma)} + z\,E_{\,\alpha,\,\gamma+\alpha}\,(z), \quad \alpha>0, \quad \gamma>0, \tag{8}$$

$$\frac{1}{\Gamma(k)}\int\limits_{t_0}^{z}(z-t)^{k-1}E_{\alpha,\gamma}\left(-\omega t^{\alpha}\right)t^{\gamma-1}dt=z^{\gamma+k-1}E_{\alpha,\gamma+k}\left(-\omega z^{\alpha}\right),\quad k>0,\quad \gamma>0. \tag{9}$$

Then for the integral (6) we obtain the representation

$$I_1(t)=x_0\,(t-t_0)^{\gamma-1}E_{\alpha,\,\gamma}\left(-\omega\,(t-t_0)^{\alpha}\right). \tag{10}$$

The integral in (7) is easily transformed to the form

$$\int\limits_{t_0}^{t}(t-\xi)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\,(t-\xi)^{\alpha}\right)\,J^{\alpha}_{t_0+}f\,(\xi,\,\cdot)\,d\xi=$$

$$=\frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^{t}(t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\,(t-\xi)^{\alpha}\right)\,d\xi\int\limits_{t_0}^{\xi}(\xi-s)^{\alpha-1}f\,(s,\,\cdot)\,ds=$$

$$=\frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^{t}f\,(s,\cdot)\,ds\int\limits_{s}^{t}(t-\xi)^{\alpha-1}(\xi-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\,(t-\xi)^{\alpha}\right)\,d\xi. \tag{11}$$

Taking (9) into account the second integral in the last equality of (11) can be written as

$$\int\limits_{s}^{t}(t-\xi)^{\alpha-1}(\xi-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\,(t-\xi)^{\alpha}\right)\,d\xi=\Gamma(\alpha)\,(t-\xi)^{2\alpha-1}E_{\alpha,\,2\alpha}\left(-\omega\,(t-\xi)^{\alpha}\right).$$

Then, taking into account (8), we represent (7) in the following form

$$I_2(t)=\int\limits_{t_0}^{t}(t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\,(t-\xi)^{\alpha}\right)\,f\,(\xi,\,\cdot)\,d\xi. \tag{12}$$

Substituting (10) and (12) into the sum $x(t)=I_1(t)+I_2(t)$, we obtain (3). The lemma is proved.

Existence and uniqueness of solution

Theorem. Let the following two conditions be satisfied:
1) $\max\limits_{t_0\le t\le T}|\,f\,(t,\,x,\,y)\,|\le M=\mathrm{const}<\infty$;
2) $\left|\,f\left(t,\,x_1,\,y_1\right)-f\left(t,\,x_2,\,y_2\right)\right|\le L\,\left(|\,x_1-x_2\,|+|\,y_1-y_2\,|\right),\ \ 0<L=\mathrm{const}<\infty$.
Then there exists a unique solution of the initial value problem (1), (2) in the space of continuous functions $C\ (t_0;\,T)$, which can be found by the method of successive approximations:

$$\begin{cases} x_0\,(t)=G(t), \\ x_{k+1}\,(t)=\Im\,(t;\,x_k),\quad k=0\,,\,1\,,\,2\,,\,\ldots, \end{cases} \tag{13}$$

where $G(t)=x_0\big(t-t_0\big)^{\gamma-1}E_{\alpha,\gamma}\left(-\omega\,\big(t-t_0\big)^{\alpha}\right)$.

Proof. Mittag–Leffler function $E_{\alpha,\gamma}(z)$ has the following property [39]: we assume that $0<\alpha<2$, γ is real constant and $\arg z=\pi$. Then there holds

$$|E_{\alpha,\,\gamma}(z)|\le\frac{A}{1+|z|},$$

where A is positive constant and does not dependent on z. Then it is not difficult to see that from the approximations (13) we obtain that there following estimate holds

$$\left|\,\big(t-t_0\big)^{1-\gamma}x_0(t)\,\right|\le|\,x_0|\cdot\left|\,E_{\alpha,\,\gamma}\left(-\omega\,\big(t-t_0\big)^{\alpha}\right)\right|\le|\,x_0\,|\cdot C_0, \tag{14}$$

where $|\,E_{\alpha,\,\alpha}\,(-\omega\,(t-s)^{\alpha})\,|\le C_0$.

By virtue of first condition of the theorem and estimate (14), from approximations (13) we obtain

$$|\,x_1(t)-x_0(t)\,|\leq$$

$$\leq\int\limits_{t_0}^{t}\left|(t-s)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\,(t-s)^{\alpha}\right)f\left(s,\,x_0(s),\,\max\left\{x_0(\theta)\,|\,\theta\in[q_1s;\,q_2s]\right\}\right)\right|d\,s\leq$$

$$\leq M\cdot C_0|x_0|\int\limits_{t_0}^{t}(t-s)^{\alpha-1}d\,s\leq\frac{|x_0|}{\alpha}M\cdot C_0\cdot(t-t_0)^{\alpha}. \tag{15}$$

We continue the Picard iteration process for the integral equation (3) according to the approximations (13). Then, by virtue of conditions of the theorem and taking the estimate (15) into account, we derive

$$|\,x_2(t)-x_1(t)\,|\leq\int\limits_{t_0}^{t}\Big|\,(t-s)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\,(t-s)^{\alpha}\right)\left[f\left(s,\,x_1(s),\,\max\left\{x_1(\theta)\,|\,\theta\in[q_1s;\,q_2s]\right\}\right)-\right.$$

$$\left.-f\left(s,\,x_0(s),\,\max\left\{x_0(\theta)\,|\,\theta\in[q_1s;\,q_2s]\right\}\right)\right]|\;d\,s\leq L\int\limits_{t_0}^{t}\left|\,(t-s)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\,(t-s)^{\alpha}\right)\right|\left[|x_1(s)-x_0(s)|+\right.$$

$$\left.+\Big|\max\left\{x_1(\theta)\,|\,\theta\in[q_1s;\,q_2s]\right\}-\max\left\{x_0(\theta)\,|\,\theta\in[q_1s;\,q_2s]\right\}\Big|\right]\;d\,s\leq$$

$$\leq 2C_0L\int\limits_{t_0}^{t}(t-s)^{\alpha-1}|x_1(s)-x_0(s)|d\,s\leq\frac{2|x_0|}{\alpha}M\cdot C_0^2L\int\limits_{t_0}^{t}(t-s)^{\alpha-1}(s-t_0)^{\alpha}d\,s.$$

By the changing the argument as $s=t_0+\big(t-t_0\big)\tau$, from the last estimate we obtain

$$|\,x_2(t)-x_1(t)\,|\leq\frac{2|x_0|}{\alpha}M\cdot C_0^2L\int\limits_{t_0}^{t}\big(t-t_0\big)^{\alpha-1}(1-\tau)^{\alpha-1}\big(t-t_0\big)^{\alpha}\tau^{\alpha}\big(t-t_0\big)d\tau\leq$$

$$\leq\frac{2\Gamma^2(\alpha)}{\Gamma(2\alpha+1)}\big|x_0\big|M\cdot L\cdot\left[C_0\cdot(t-t_0)^{\alpha}\right]^2. \tag{16}$$

Analogously, taking the estimate (16) into account, for the next difference we derive

$$|\,x_3(t)-x_2(t)\,|\leq L\int\limits_{t_0}^{t}\left|\,(t-s)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\,(t-s)^{\alpha}\right)\right|\Big[\,|\,x_2(s)-x_1(s)\,|+$$

$$+\,|\max\left\{x_2(\theta)\,|\,\theta\in[q_1s;\,q_2s]\right\}-\max\left\{x_1(\theta)\,|\,\theta\in[q_1s;\,q_2s]\right\}|\,\Big]\,d\,s\leq$$

$$\leq 2C_0L\int\limits_{t_0}^{t}(t-s)^{\alpha-1}\big|\,x_2(s)-x_1(s)\,\big|d\,s\leq$$

$$\leq\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha+1)}\big|\,x_0\,\big|M\cdot(2L)^2\cdot C_0^3\int\limits_{t_0}^{t}(t-s)^{\alpha-1}\big(s-t_0\big)^{2\alpha}d\,s\leq$$

$$\leq\frac{\Gamma^3(\alpha)}{\Gamma(3\alpha+1)}\cdot|x_0|\cdot M\cdot(2L)^2\cdot\left[C_0\cdot(t-t_0)^{\alpha}\right]^3. \tag{17}$$

Continuing the estimation processes (14)–(17) for arbitrary difference we obtain

$$|\,x_n(t)-x_{n-1}(t)\,|\leq\frac{\Gamma^n(\alpha)}{\Gamma(n\alpha+1)}\cdot\big|x_0\big|\cdot M\cdot(2L)^{n-1}\left[C_0\cdot\big(t-t_0\big)^{\alpha}\right]^n. \tag{18}$$

For the absolute value of difference $|\,x_n(t)-x_{n-1}(t)\,|$ we show that $\sum\limits_{n=1}^{\infty}|\,x_n(t)-x_{n-1}(t)\,|<\infty$ in the space $C\ (t_0;\,T)$. So, we denote the right-hand side of (18) as

$$a_n=\frac{\Gamma^n(\alpha)}{\Gamma(n\,\alpha+1)}\cdot(2L)^{n-1}\left[C_0\cdot\big(t-t_0\big)^{\alpha}\right]^{\,n}$$

and we put

$$a_{n+1}=\frac{\Gamma^{n+1}(\alpha)}{\Gamma((n+1)\,\alpha+1)}\cdot(2L)^{n}\left[C_0\cdot\big(t-t_0\big)^{\alpha}\right]^{\,n+1}.$$

Then we consider the following limit

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=2L\cdot\Gamma(\alpha)\cdot C_0\cdot\big(t-t_0\big)^{\alpha}\lim_{n\to\infty}\frac{\Gamma(n\,\alpha+1)}{\Gamma((n+1)\,\alpha+1)}. \tag{19}$$

Taking known formula [40]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}=z^{a-b}\left[1+\frac{(a-b)(a-b-1)}{2z}+O(z^{-2})\right]$$

into account, we obtain

$$\lim_{n\to\infty}\frac{\Gamma(n\,\alpha+1)}{\Gamma((n\,\alpha+\alpha+1)}=\lim_{n\to\infty}(n\,\alpha)^{1-\alpha-1}\left[1+\frac{(1-\alpha-1)(1-\alpha-1-1)}{2n\,\alpha}+O(n\,\alpha)^{-2}\right]=$$

$$=\frac{1}{\alpha^{\alpha}}\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left[1+\frac{\alpha\,(1+\alpha)}{2n\,\alpha}+O(n\,\alpha)^{-2}\right]=0.$$

Consequently, for (19) we have

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=2L\cdot\Gamma(\alpha)\cdot C_0\cdot\big(t-t_0\big)^{\alpha}\cdot\lim_{n\to\infty}\frac{\Gamma(n\,\alpha+1)}{\Gamma((n+1)\,\alpha+1)}=$$

$$=2\Gamma(\alpha)\cdot L\cdot C_0\cdot\big(t-t_0\big)^{\alpha}\cdot\frac{1}{\alpha^{\alpha}}\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left[1+\frac{\alpha\big(1+\alpha\big)}{2n\,\alpha}+O(n\,\alpha)^{-2}\right]=0.$$

Hence, according to d'Alembert's convergence criterion of series, we have

$$\sum_{n=1}^{\infty}|\,x_n(t)-x_{n-1}(t)\,|\le\sum_{n=1}^{\infty}\frac{\Gamma^n(\alpha)}{\Gamma(n\,\alpha+1)}\cdot C_0^n\cdot(2L)^{n-1}\big(t-t_0\big)^{n\,\alpha}<\infty \tag{20}$$

for all $t\ge t_0$. Since we consider the solution of the integral equation (3) in the space of continuous functions $C\ (t_0;\,T)$, it follows from (20) that the sequence of functions $\{x_k(t)\}_{k=1}^{\infty}$ converges absolutely and uniformly to solution of the integral equation (3) with respect to argument t. Hence implies the existence of a solution of the problem (1), (2) on the interval $(t_0;\,T)$. Now we show the uniqueness of this solution. Assuming that the integral equation (3) has two different solutions $x(t)$ and $y(t)$ on the interval $(t_0;\,T)$, we obtain the following integral inequality

$$|\,x(t)-y(t)\,|\le 2L\int\limits_{t_0}^{t}\big|\,(t-s)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\,(t-s)^{\alpha}\right)\big|\cdot|\,x(s)-y(s)\,|\;d\,s. \tag{21}$$

Applying Gronwall–Bellman inequality to estimate (21), we obtain that $|\,x\,(t)-y\,(t)\,|\equiv 0$ for all $t\in(t_0;\,T)$. Therefore, the Cauchy problem (1), (2) has a unique solution on the interval $(t_0;\,T)$. The theorem is proved.

The generalized Jakobi–Galerkin method

Now, to the problem (1), (2) we apply the generalized Jacobi–Galerkin method as a numerical realization of solution (3). This solution (3) is nonlinear Volterra type fractional integral equation. On the interval $(-1;\,1)$ for the given numbers $\beta_1,\ \beta_2 > -1$ we consider standard Jacobi polynomial $J_n^{(\beta_1,\,\beta_2)}(\xi)$ of degree n with weight function $\Lambda^{(\beta_1,\,\beta_2)}(\xi) = (1-\xi)^{\beta_1}(1+\xi)^{\beta_2}$. For the standard Jacobi polynomial the following relation is true

$$\int\limits_{-1}^{1} J_n^{(\beta_1,\,\beta_2)}(\xi)\, J_m^{(\beta_1,\,\beta_2)}(\xi)\, \Lambda^{(\beta_1,\,\beta_2)}(\xi)\, d\,\xi = \gamma_m^{(\beta_1,\,\beta_2)} \delta_{m,\,n}, \tag{22}$$

where $\delta_{m,\,n}$ is the Kronecker function and

$$\gamma_m^{(\beta_1,\,\beta_2)}(\xi) = \begin{cases} \frac{2^{\beta_1+\beta_2+1}\,\Gamma(\beta_1+1)\,\Gamma(\beta_2+1)}{\Gamma(\beta_1+\beta_2+2)}, & m=0, \\ \frac{2^{\beta_1+\beta_2+1}\,\Gamma(m+\beta_1+1)\,\Gamma(m+\beta_2+1)}{(2m+\beta_1+\beta_2+1)\,m!\,\Gamma(m+\beta_1+\beta_2+2)}, & m \geq 1. \end{cases}$$

From (22) we note that the set of standard Jacobi polynomial $J_n^{(\beta_1,\,\beta_2)}(\xi)$ is a complete orthogonal system in the space $L^2_{\Lambda^{(\beta_1,\,\beta_2)}}(-1;\,1)$ with weight function $\Lambda^{(\beta_1,\,\beta_2)}(\xi)$. In particular, $J_0^{(\beta_1,\,\beta_2)}(\xi) = 1$.

The shifted Jacobi polynomial of variable t and degree n is defined by the following formula

$$\tilde{J}_n^{(\beta_1,\,\beta_2)}(t) = J_n^{(\beta_1,\,\beta_2)}\left(\frac{2\,(t-t_0)}{T-t_0} - 1\right), \quad t \in \big(t_0;\,T\big). \tag{23}$$

We note that the set of shifted Jacobi polynomial $\tilde{J}_n^{(\beta_1,\,\beta_2)}(t)$ is a complete orthogonal system with weight function $\Lambda_T^{(\beta_1,\,\beta_2)}(t) = \big(T-t+t_0\big)^{\beta_1}\big(t-t_0\big)^{\beta_2}$ in the space $L^2_{\Lambda^{(\beta_1,\,\beta_2)}}\big(t_0;\,T\big)$ and by the aid of (23) we have the analogue of the (22)

$$\int\limits_{t_0}^{T} \tilde{J}_n^{(\beta_1,\,\beta_2)}(t)\, \tilde{J}_m^{(\beta_1,\,\beta_2)}(t)\, \Lambda_T^{(\beta_1,\,\beta_2)}(t)\, d\,t = \left(\frac{T+t_0}{2}\right)^{\beta_1+\beta_2+1} \gamma_m^{(\beta_1,\,\beta_2)}(t)\, \delta_{m,\,n}. \tag{24}$$

For any integer $N \geq 0$ we denote by $\left\{\xi_j^{(\beta_1,\,\beta_2)},\ \eta_j^{(\beta_1,\,\beta_2)}\right\}_{j=0}^{N}$ the nodes and the corresponding Christoffel numbers of the standard Jacobi–Gauss interpolation on the interval $(-1;\,1)$. By the $\tilde{P}_N(t_0;\,T)$ we denote the set of polynomials of degree at most N on the interval $\big(t_0;\,T\big)$ and by the $t_j^{(\beta_1,\,\beta_2)}$ we denote the shifted Jacobi-Gauss quadrature nodes on the interval $(t_0;\,T)$

$$t_j^{(\beta_1,\,\beta_2)} = \frac{T-t_0}{2}\,\left(\xi_j^{(\beta_1,\,\beta_2)} + 1\right) + t_0, \quad 0 \leq j \leq N.$$

By virtue of the property of the standard Jacobi–Gauss quadrature it's implied that for any $\phi\,(t) \in \tilde{P}_{2\,N+1}(t_0;\,T)$ we have

$$\int\limits_{t_0}^{T} \phi\,(t)\, \Lambda_T^{(\beta_1,\,\beta_2)}(t)\, d\,t = \left(\frac{T+t_0}{2}\right)^{\beta_1+\beta_2+1} \sum_{j=0}^{N} \phi\left(t_j^{(\beta_1,\,\beta_2)}\right)\, \eta_j^{(\beta_1,\,\beta_2)}. \tag{25}$$

By virtue of (25) from (24), we have for any $0 \leq m+n \leq 2N+1$,

$$\sum_{j=0}^{N} \tilde{J}_m^{(\beta_1,\,\beta_2)}\left(t_j^{(\beta_1,\,\beta_2)}\right)\, \tilde{J}_n^{(\beta_1,\,\beta_2)}\left(t_j^{(\beta_1,\,\beta_2)}\right)\, \eta_j^{(\beta_1,\,\beta_2)} = \gamma_m^{(\beta_1,\,\beta_2)} \delta_{m,\,n}.$$

By the aid of shifted Jacobi polynomial $\tilde{J}_n^{(\beta_1,\,\beta_2)}(t)$ we define the shifted generalized Jacobi function of degree n as (see [41])

$$P_n^{(\beta_1,\,\beta_2)}(t) = t^{\beta_2}\, \tilde{J}_n^{(\beta_1,\,\beta_2)}(t), \quad \beta_1,\ \beta_2 > -1, \quad t \in \big(t_0;\,T\big). \tag{26}$$

By virtue of (24) and (26), we see that

$$\int_{t_0}^{T} P_n^{(\beta_1,\,\beta_2)}(t)\,P_m^{(\beta_1,\,\beta_2)}(t)\,\Lambda_T^{(\beta_1,\,-\beta_2)}(t)\,d\,t=\left(\frac{T+t_0}{2}\right)^{\beta_1+\beta_2+1}\gamma_m^{(\beta_1,\,\beta_2)}\delta_{m,\,n}.$$

By virtue of (25), for any $\varphi\,(t)=t^{2\,\beta_2}\phi\,(t)$ we have

$$\int_{t_0}^{T}\varphi\,(t)\,\Lambda_T^{(\beta_1,\,-\beta_2)}(t)\,d\,t=\left(\frac{T+t_0}{2}\right)^{\beta_1+\beta_2+1}\sum_{j=0}^{N}\left(t_j^{(\beta_1,\,\beta_2)}\right)^{-2\beta_2}\varphi\left(t_j^{(\beta_1,\,\beta_2)}\right)\eta_j^{(\beta_1,\,\beta_2)}. \tag{27}$$

By the aid of (27) we introduce the inner product in $L^2_{\Lambda_T^{(\beta_1,\,-\beta_2)}}(0;\,T)$ as

$$\langle f,\,g\rangle_{\Lambda_T^{(\beta_1,\,-\beta_2)}}=\left(\frac{T+t_0}{2}\right)^{\beta_1+\beta_2+1}\sum_{j=0}^{N}\left(t_j^{(\beta_1,\,\beta_2)}\right)^{-2\beta_2}f\left(t_j^{(\beta_1,\,\beta_2)}\right)g\left(t_j^{(\beta_1,\,\beta_2)}\right)\eta_j^{(\beta_1,\,\beta_2)}.$$

We need also to introduce finite N-dimensional fractional polynomial space [41]

$$\tilde F_N^{(\beta_2)}(t_0;\,T)=\left\{t^{\beta_2}\,\psi\,(t):\ \psi\,(t)\in\tilde P_N^{(\beta_1,\,\beta_2)}\big(t_0;\,T\big)\right\}=\mathrm{span}\left\{P_n^{(\beta_1,\,\beta_2)}(t):\ 0\le n\le N\right\}.$$

Then we note that for any $\phi,\,\psi\in\tilde F_N^{(\beta_2)}\big(t_0;\,T\big)$ hold the equalities

$$(\phi,\,\psi)_{\Lambda_T^{(\beta_1,\,-\beta_2)}}=\langle\phi,\,\psi\rangle_{\Lambda_T^{(\beta_1,\,-\beta_2)}}\,.$$

Now in integral equation (3) we make variable transformation $s=\frac{t\,\tau}{T},\ \tau\in\big(t_0;\,T\big)$. Then the we describe integral equation (3) as

$$x\,(t)=\Im\,(t;\,x)\equiv G\,(t)+V\,x\,(t)=G\,(t)+\left(\frac{t}{T}\right)^{\alpha}\int_{t_0}^{t}(T-\tau)^{\alpha-1}\,E_{\alpha,\,\alpha}\left(-\omega\,\left(\frac{t}{T}\right)^{\alpha}(T-\tau)^{\alpha}\right)\times$$
$$\times f\left(\frac{t\,\tau}{T},\,x\,\left(\frac{t\,\tau}{T}\right),\,\max\left\{x\,(\theta)\,|\,\theta\in\left[\frac{q_1t\,\tau}{T};\,\frac{q_2t\,\tau}{T}\right]\right\}\right)\,d\,\tau. \tag{28}$$

For the Hilfer fractional operator's order $0<\alpha<1$ we denote $\alpha-1=-\mu$, where $0<\mu=\mathrm{const}$ Then for $U,\,\varphi\in\tilde F_N^{(1-\mu)}(t_0;\,T)$ we apply the generalized Jacobi–Galerkin method to equation (28):

$$(U,\,\varphi)_{\Lambda_T^{(-\mu,\,\mu-1)}}=(G,\,\varphi)_{\Lambda_T^{(-\mu,\,\mu-1)}}+(V\,U,\,\varphi)_{\Lambda_T^{(-\mu,\,\mu-1)}}\,. \tag{29}$$

We set

$$U\,(t)=\sum_{m=0}^{N}x_{\,m}(t)\,P_m^{(-\mu,\,1-\mu)}(t),\quad \varphi\,(t)=P_n^{(-\mu,\,1-\mu)}(t),\quad 0\le m,\ n\le N.$$

Then for (29) we have

$$\sum_{m=0}^{N}x_{\,m}(t)\,\left(P_m^{(-\mu,\,1-\mu)}(t),\,P_n^{(-\mu,\,1-\mu)}(t)\right)_{\Lambda_T^{(-\mu,\,\mu-1)}}=$$
$$=\left(G\,(t),\,P_n^{(-\mu,\,1-\mu)}(t)\right)_{\Lambda_T^{(-\mu,\,\mu-1)}}+\left(V\,U\,(t),\,P_n^{(-\mu,\,1-\mu)}(t)\right)_{\Lambda_T^{(-\mu,\,\mu-1)}}\,.$$

Hence, we come to nonlinear system

$$\bar B\,\bar x=\bar G+\bar\vartheta\,(\bar x), \tag{30}$$

after introducing designations:

$$\bar x=(x_0,\,x_1,\,\ldots,\,\bar x_N)^T\,,\quad B=(b_{\,n\,m})_{0\le n\,m\le N}\,,$$
$$b_{\,n\,m}=\left(P_m^{(-\mu,\,1-\mu)}(t),\,P_n^{(-\mu,\,1-\mu)}(t)\right)_{\Lambda_T^{(-\mu,\,\mu-1)}}=\left(\tfrac{T+t_0}{2}\right)^{2-2\,\mu}\gamma_m^{(-\,\mu,\,1-\,\mu)}\delta_{m,\,n},$$
$$\bar G=(G_{\,0},\,G_1,\,\ldots,\,G_N)^T\,,\ \ G_{\,n}(t)=\left(G\,(t),\,P_n^{(-\mu,\,1-\mu)}(t)\right)_{\Lambda_T^{(-\mu,\,\mu-1)}}\,,$$
$$\bar\vartheta\,(\bar x)=(\vartheta_{\,0},\,\vartheta_{\,1},\,\ldots,\,\vartheta_N)^T\,,\ \ \vartheta_{\,n}(x)=\left(V\,U\,(t),\,P_n^{(-\mu,\,1-\mu)}(t)\right)_{\Lambda_T^{(-\mu,\,\mu-1)}}\,,$$

where by $(u_0,\,u_1,\,\ldots,\,u_N)^T$ we denoted the transposition of the matrix $(u_0,\,u_1,\,\ldots,\,u_N)$.

We use the quadrature formula

$$\langle f,\, g\rangle_{\Lambda_T^{(\beta_1,\,-\beta_2)}} = \left(\frac{T+t_0}{2}\right)^{\beta_1+\beta_2+1} \sum_{j=0}^{N} \left(t_j^{(\beta_1,\,\beta_2)}\right)^{-2\beta_2} f\left(t_j^{(\beta_1,\,\beta_2)}\right) g\left(t_j^{(\beta_1,\,\beta_2)}\right) \eta_j^{(\beta_1,\,\beta_2)}$$

to obtain approximate formulas:

$$G_n(t) \approx \left\langle G(t),\, P_n^{(-\mu,\,1-\mu)}(t)\right\rangle_{\Lambda_T^{(-\mu,\,\mu-1)}} =$$

$$= \left(\frac{T+t_0}{2}\right)^{2-2\mu} \sum_{j=0}^{N} \left(t_j^{(-\mu,\,1-\mu)}\right)^{2\mu-2} G\left(t_j^{(-\mu,\,1-\mu)}\right) P_n^{(-\mu,\,1-\mu)}\left(t_j^{(-\mu,\,1-\mu)}\right) \eta_j^{(-\mu,\,1-\mu)}, \tag{31}$$

$$\bar{\vartheta}(\bar{x}) \approx \frac{(T+t_0)^{2-2\mu}}{2^{2-2\mu}} \sum_{i,\,j=0}^{N} \left(\frac{t_i^{(-\mu,\,1-\mu)}}{T}\right)^{1-\mu} (T-\tau)^{-\mu} E_{1-\mu,\,1-\mu}\left(-\omega\left(\frac{t_i^{(-\mu,\,1-\mu)}}{T}\right)^{1-\mu} (T-\tau)^{1-\mu}\right) \times$$

$$\times f\left(t_{ij},\, U(t_{ij}),\, \max\{U(\theta)\,|\,\theta\in[q_1\cdot t_{ij};\, q_2\cdot t_{ij}]\}\right) \times$$

$$\times P_n^{(-\mu,\,1-\mu)}\left(t_i^{(-\mu,\,1-\mu)}\right) \eta_i^{(-\mu,\,1-\mu)} \eta_j^{(-\mu,\,0)}, \tag{32}$$

where $t_{ij} = \frac{t_i^{(-\mu,\,1-\mu)} t_j^{(-\mu,\,0)}}{T}$.

In approximately solving the system (30) one can use the Newton iterative method.

Illustrarive examples

As an example, we consider the simple equation of the form

$$D_{0+}^{\alpha,\,\beta} u(t) = \lambda\, u(t) + f(t), \;\; t\in(0;\,T)$$

with initial value condition

$$\lim_{t\to+0} J_{0+}^{1-\gamma} u(t) = u_0.$$

The solution of this problem has the form

$$u(t) = u_0\, t^{\gamma-1} E_{\alpha,\,\gamma}\left(\lambda t^{\alpha}\right) + \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\,\alpha}\left(\lambda(t-s)^{\alpha}\right) f(s) ds, \tag{33}$$

where $\gamma = \alpha+\beta-\alpha\,\beta$.

Example 1. We consider cases $\alpha=\beta=\frac{1}{2}$, $f(t)=t^{\sigma}$, $\sigma>-1$. Since $\gamma=\frac{1}{2}+\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2}=\frac{3}{4}$, from (33) we have

$$u(t) = u_0\, t^{-\frac{1}{4}} E_{\frac{1}{2},\,\frac{3}{4}}\left(\lambda t^{\frac{1}{2}}\right) + \int\limits_0^t (t-s)^{-\frac{1}{2}} E_{\frac{1}{2},\,\frac{1}{2}}\left(\lambda(t-s)^{\frac{1}{2}}\right) s^{\sigma} ds. \tag{34}$$

Taking into account

$$\frac{1}{\Gamma(\nu)} \int\limits_0^z (z-t)^{\nu-1} E_{\alpha,\,\beta}\left(\lambda t^{\alpha}\right) t^{\beta-1} dt = z^{\beta+\nu-1} E_{\alpha,\,\beta+\nu}\left(\lambda z^{\alpha}\right), \;\; \nu>0, \;\; \beta>0,$$

we calculate the integral in (34):

$$\int\limits_0^t (t-s)^{-\frac{1}{2}} E_{\frac{1}{2},\,\frac{1}{2}}\left(\lambda(t-s)^{\frac{1}{2}}\right) s^{\sigma} ds = \Gamma(\sigma+1) t^{\frac{1}{2}+\sigma} E_{\frac{1}{2},\,\frac{3}{2}+\sigma}\left(\lambda t^{\frac{1}{2}}\right). \tag{35}$$

Substituting (35) into (34), we obtain

$$u(t) = \frac{u_0}{\sqrt[4]{t}} E_{\frac{1}{2},\,\frac{3}{4}}\left(\lambda\sqrt{t}\right) + \Gamma(\sigma+1)\, t^{\sigma}\sqrt{t} E_{\frac{1}{2},\,\frac{3}{2}+\sigma}\left(\lambda\sqrt{t}\right). \tag{36}$$

In particular case, when $\sigma = 0$, from (36) yields

$$u(t) = \frac{u_0}{\sqrt[4]{t}} E_{\frac{1}{2},\,\frac{3}{4}}\left(\lambda\sqrt{t}\right) + \sqrt{t} E_{\frac{1}{2},\,\frac{3}{2}}\left(\lambda\sqrt{t}\right). \tag{37}$$

Taking into account

$$E_{\alpha,\,\mu}(z) = \frac{1}{\Gamma(\mu)} + z\, E_{\alpha,\,\alpha+\mu}(z), \quad \alpha > 0, \quad \mu > 0,$$

we obtain

$$\sqrt{t} E_{\frac{1}{2},\,\frac{3}{2}}\left(\lambda\sqrt{t}\right) = \frac{1}{\lambda} E_{\frac{1}{2},\,1}\left(\lambda\sqrt{t}\right) - \frac{1}{\lambda}.$$

Therefore (37) takes form

$$u(t) = \frac{u_0}{\sqrt[4]{t}} E_{\frac{1}{2},\,\frac{3}{4}}\left(\lambda\sqrt{t}\right) + \frac{1}{\lambda} E_{\frac{1}{2},\,1}\left(\lambda\sqrt{t}\right) - \frac{1}{\lambda}.$$

Since $E_{\frac{1}{2},\,1}(z) = \cosh\sqrt{z}$, we present the solution as

$$u(t) = \frac{u_0}{\sqrt[4]{t}} E_{\frac{1}{2},\,\frac{3}{4}}\left(\lambda\sqrt{t}\right) + \frac{1}{\lambda}\left[\cosh\left(\sqrt{\lambda\sqrt{t}}\right) - 1\right].$$

Example 2. The case of Caputo operator: $\alpha = \frac{1}{2}$, $\beta = 1$, $f(t) = t^{\sigma}$, $\sigma > -1$.
Since $\gamma = \frac{1}{2} + 1 - \frac{1}{2}\cdot 1 = 1$, from (33) we have

$$u\,(t) = u_0 E_{\frac{1}{2},\,1}\left(\lambda t^{\frac{1}{2}}\right) + \int\limits_0^t (t-s)^{-\frac{1}{2}} E_{\frac{1}{2},\,\frac{1}{2}}\left(\lambda (t-s)^{\frac{1}{2}}\right) s^{\sigma} ds. \tag{38}$$

Taking (35) into account, from (38) we obtain

$$u\,(t) = u_0 E_{\frac{1}{2},\,1}\left(\lambda\sqrt{t}\right) + \Gamma(\sigma+1) t^{\sigma}\sqrt{t} E_{\frac{1}{2},\,\frac{3}{2}+\sigma}\left(\lambda\sqrt{t}\right). \tag{39}$$

We are looking for real solutions. Since $E_{\frac{1}{2},\,1}(z) = \cosh\sqrt{z}$, then for $\lambda \geq 0$ we present the solution (39) as

$$u\,(t) = u_0 \cosh\left(\sqrt{\lambda\sqrt{t}}\right) + \Gamma(\sigma+1)\, t^{\sigma}\sqrt{t} E_{\frac{1}{2},\,\frac{3}{2}+\sigma}\left(\lambda\sqrt{t}\right).$$

For the cases $\sigma = 0$ and $\lambda > 0$ we have

$$u\,(t) = u_0\, \cosh\left(\sqrt{\lambda\sqrt{t}}\right) + \sqrt{t} E_{\frac{1}{2},\,\frac{3}{2}}\left(\lambda\sqrt{t}\right). \tag{40}$$

Taking

$$E_{\alpha,\,\mu}(z) = \frac{1}{\Gamma(\mu)} + z\, E_{\alpha,\,\alpha+\mu}(z), \quad \alpha > 0, \quad \mu > 0$$

into account, the last summand easily presents as

$$\sqrt{t} E_{\frac{1}{2},\,\frac{3}{2}}\left(\lambda\sqrt{t}\right) = \frac{1}{\lambda} E_{\frac{1}{2},\,1}\left(\lambda\sqrt{t}\right) - \frac{1}{\lambda}.$$

So, taking $E_{\frac{1}{2},\,1}(z) = \cosh\sqrt{z}$ into account, from representation (40) we obtain the simple form of solution

$$u\,(t) = \frac{1}{\lambda}\left[(\lambda\, u_0 + 1)\, \cosh\left(\sqrt{\lambda\sqrt{t}}\right) - 1\right].$$

Now we consider an example of a nonlinear differential equation.

Example 3. The equation

$$ {}_C D_{0t}^{\alpha} y(t) = \frac{1}{5}\Gamma(\alpha+1)t^{-2\alpha}\left(y^2(t) + 4\cdot\max\left\{y^2(\theta)\,|\,\theta\in\left[\frac{1}{3}\,t;\,t\right]\right\}\right),\;\; \alpha > \frac{1}{2} \tag{41} $$

on the interval (0; 1) has a solution

$$ y(t) = t^{\alpha}. \tag{42} $$

Indeed,

$$ \frac{1}{5}\Gamma(\alpha+1)t^{-2\alpha}\left(y^2(t) + 4\cdot\max\left\{y^2(\theta)\,|\,\theta\in\left[\frac{1}{3}\,t;\,t\right]\right\}\right) = \frac{1}{5}\,\Gamma(\alpha+1)t^{-2\alpha}\left(5\,t^{2\alpha}\right) = \Gamma(\alpha+1) \tag{43} $$

and

$$ {}_C D_{0t}^{\alpha} y(t) = {}_C\, D_{0t}^{\alpha}\left(t^{\alpha}\right) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\alpha)}t^{\alpha-\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\alpha)} = \Gamma(\alpha+1). \tag{44} $$

From (43) and (44) we come to the conclusion that function (42) is a solution of the Caputo fractional differential equation (41) on the interval (0; 1).

Remark. The function (42) is not a solution of fractional differential equation (41) on the semiaxis $(1;\infty)$. If we consider the solvability of the differential equation (41) on the entire positive semiaxis $\mathbb{R}^+ \equiv (0;\infty)$, then this equation suffers a discontinuity of the first kind at the point $t = 1$.

Conclusion

In this paper we consider the questions of unique solvability of initial value problem for a nonlinear fractional differential equation (1) with maxima on the given segment $(t_0; T)$. We reduce this initial value problem to the fractional order nonlinear integral equation of Volterra type. Then we used the method of successive approximation and proved the theorem on existence and uniqueness of solution of the problem under consideration. We apply the generalized Jacobi–Galerkin method as a numerical realization of solution of the fractional order nonlinear integral equation (3). We make a variable transformation in integral equation (3): $s = \frac{t\,\tau}{T}$, $\quad \tau \in \left(t_0;\,T\right)$. Applying the generalized Jacobi–Galerkin method to equation (28), we come to the system (30). By using the quadrature formula we obtain the necessary approximation formulas (31) and (32).

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Сызықтық емес оң жағы бар Хильфер типтес бөлшек дифференциалдық теңдеу туралы

Мақалада максималды сызықтық емес бөлшек дифференциалдық теңдеу үшін бастапқы есепті біркелкі шешу және сандық іске асыру мәселелері қарастырылды. Дирихле формуласына негізделген қарапайым интегралдық түрлендіруді қолдана отырып, қарастырылып отырған бастапқы міндет Вольтерр типіндегі сызықты емес бөлшек интегралдық теңдеуге дейін азаяды. Қарастырылған сегментте берілген бастапқы есепті шешудің бар болуы мен бірегейлігі теоремасы дәлелденді. Шешімді сандық түрде жүзеге асыру үшін Галеркин Якобидің жалпыланған спектрлік әдісі қолданылған. Көрнекі мысалдар келтірілген.

Кілт сөздер: қарапайым дифференциалдық теңдеу, максимумдармен теңдеу, Хильфер операторы, бір мәнді шешімділік, Галеркин Якобидің жалпыланған спектрлік әдісі.

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Об одном дробном дифференциальном уравнении типа Хильфера с нелинейной правой частью

В статье рассмотрены вопросы однозначной разрешимости и численной реализации начальной задачи для нелинейного дробного дифференциального уравнения типа Хильфера с максимумами. С помощью несложного интегрального преобразования, основанного на формуле Дирихле, рассматриваемая начальная задача сведена к нелинейному дробно-интегральному уравнению типа Вольтерра. Доказана теорема существования и единственности решения заданной начальной задачи на рассматриваемом отрезке. Для численной реализации решения применен обобщенный спектральный метод Галеркина-Якоби. Приведены наглядные примеры.

Ключевые слова: обыкновенное дифференциальное уравнение, уравнение с максимумами, оператор Хильфера, однозначная разрешимость, обобщенный спектральный метод Галеркина-Якоби.

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