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DEAR READER!

We present you the 100th anniversary edition of the "Bulletin of the Karaganda University". "Bulletin of the Karaganda University" is the scientific periodical is aimed at publishing in the open press the results of research in various fields of science by scientists from Kazakhstan and other countries. The purpose of the journal is to create an effective environment for the exchange of important scientific and educational information, to acquaint the international scientific community with new methods and ideas. The journal is included in the list of publications recommended by the Committee for Control in Science and Education of the Ministry of Education and Science of the Republic of Kazakhstan for publication of the main results of scientific activity.

The journal was first published in 1996 in two series ("Humanities" and "Natural Sciences"), since 2004 "Mathematics"; "Physics"; "Chemistry"; “Biology. Medicine. Geography.”; "Economics"; "Pedagogy"; "Philology"; "History. Philosophy. Law" published materials in eight series, and in 2010 the ninth series was added.

Since 2015, the series "Chemistry", "Physics", "Mathematics" of the journal "Bulletin of the Karaganda University" are included in the platform "Emerging Sources Citation Index (ESCI)" of the international database Web of Science Core Collection. Currently, the Bulletin of the Karaganda University is a prestigious publication that publishes 9 series of research papers in the CIS and Germany, Poland, China, Egypt, Turkey, India and Pakistan, in addition to research on topical issues by domestic scientists. The journal's personal website in 3 languages, complying with international standards, contains the policy of the editorial board, the requirements for online submission of articles and online peer review. All articles published in the journal are assigned a digital object ID. The journal cooperates with leading Kazakhstan and foreign library systems and databases, which in turn provides quick and open access to published materials.

We have a clear signature in the development of science and education of our independent country. I believe that such a rise to the heights of prestige is the result of many years of hard work, constant search and tireless progress. I am convinced that the publication, which has made the solution of the most pressing problems facing humankind its eternal and noble goal, will continue to be the herald of scientific discoveries. We would like to express our gratitude to all the authors and researchers who have contributed to the growth of the scientific potential of the journal, and sincerely congratulate you on the publication of the 100th anniversary edition!

Chairman of the Editorial Board
Corresponding member of NAS RK,
Doctor of Law,
Professor ***N.O. Dulatbekov***

CONTENTS

MATHEMATICS

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Method of functional parametrization for solving a semi-periodic initial problem for fourth-order partial differential equations

A semi-periodic initial boundary-value problem for a fourth-order system of partial differential equations is considered. Using the method of functional parametrization, an additional parameter is carried out and the studied problem is reduced to the equivalent semi-periodic problem for a system of integro-differential equations of hyperbolic type second order with functional parameters and integral relations. An interrelation between the semi-periodic problem for the system of integro-differential equations of hyperbolic type and a family of Cauchy problems for a system of ordinary differential equations is established. Algorithms for finding of solutions to an equivalent problem are constructed and their convergence is proved. Sufficient conditions of a unique solvability to the semi-periodic initial boundary value problem for the fourth-order system of partial differential equations are obtained.

Keywords: semi-periodic initial boundary-value problem, fourth-order system of partial differential equations, the method of functional parametrization, semi-periodic problem, system of integro-differential equations of hyperbolic type second order, family of Cauchy problems, algorithm, unique solvability.

Introduction

In the present paper, on the domain $\Omega = [0,T] \times [0,\omega]$ we consider the following semi-periodic initial boundary value problem for a fourth order system of partial differential equations

$$\frac{\partial^4 u}{\partial t^3 \partial x} = A_1(t,x)\frac{\partial^3 u}{\partial t^2 \partial x} + A_2(t,x)\frac{\partial^3 u}{\partial t^3} + A_3(t,x)\frac{\partial^2 u}{\partial t^2} + A_4(t,x)\frac{\partial^2 u}{\partial t \partial x} +$$

$$+A_5(t,x)\frac{\partial u}{\partial t} + A_6(t,x)\frac{\partial u}{\partial x} + A_7(t,x)u + f(t,x), \tag{1}$$

$$u(0,x) = \varphi_1(x), \qquad x \in [0,\omega], \tag{2}$$

$$\frac{\partial u(t,x)}{\partial t}|_{t=0} = \varphi_2(x), \qquad x \in [0,\omega], \tag{3}$$

$$\frac{\partial^2 u(t,x)}{\partial t^2}|_{t=0} = \frac{\partial^2 u(t,x)}{\partial t^2}|_{t=T}, \qquad x \in [0,\omega], \tag{4}$$

$$u(t,0) = \psi(t), \qquad t \in [0,T], \tag{5}$$

where $u(t,x) = col(u_1(t,x),...,u_n(t,x))$ is unknown function, the $n \times n$ matrices $A_i(t,x), (i = \overline{1,7})$, and n vector–function $f(t,x)$ are continuous on Ω; n vector–function $\psi(t)$ are continuously three times differentiable on $[0,T]$; the n vector–functions $\varphi_1(x)$ and $\varphi_2(x)$ are continuously differentiable on $[0,\omega]$.

Let $C(\Omega, \mathbb{R}^n)$ be a space of continuous on Ω vector functions $u(t,x)$ with the norm
$||u||_0 = \max\limits_{(t,x)\in\Omega} ||u(t,x)||$, $||u(t,x)|| = \max\limits_{i=\overline{1,n}} |u_i(t,x)|$.

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A function $u(t,x) \in C(\Omega, \mathbb{R}^n)$ having partial derivatives

$$\frac{\partial u(t,x)}{\partial t} \in C(\Omega, \mathbb{R}^n), \frac{\partial u(t,x)}{\partial x} \in C(\Omega, \mathbb{R}^n), \frac{\partial^2 u(t,x)}{\partial t \partial x} \in C(\Omega, \mathbb{R}^n), \frac{\partial^2 u(t,x)}{\partial t^2} \in C(\Omega, \mathbb{R}^n),$$

$$\frac{\partial^3 u(t,x)}{\partial t^2 \partial x} \in C(\Omega, \mathbb{R}^n), \frac{\partial^3 u(t,x)}{\partial t^3} \in C(\Omega, \mathbb{R}^n), \frac{\partial^4 u(t,x)}{\partial t^3 \partial x} \in C(\Omega, \mathbb{R}^n),$$

is called a classical solution to problem (1)–(5) if it satisfies system (1) for all $(t,x) \in \Omega$, and the initial and the boundary conditions (2)–(5).

Mathematical modeling of various processes in physics, ecology, chemistry, biology and others are leaded to initial - boundary value problems for a higher-order partial differential equations with variable coefficients and boundary functions [1, 2]. Despite the presence of numerous works, general statements of initial-boundary value problems for the higher-order system of partial differential equations remain poorly studied up to now. Therefore, the problems of solvability of initial-boundary value problems for the fourth-order system of partial differential equations are important in applied problems [1-8]. Some classes of initial-boundary value problems for systems of fourth-order hyperbolic equations are studied in [4-8].

Aim of the paper is to study issues for an existence and uniqueness of classical solutions to the semi-periodic initial boundary value problem for the fourth-order system of partial differential equations (1)–(5). We will establish coefficient criteria for its unique solvability and construct algorithms for finding its approximate solutions. For reaching this goal, we use method of functional parametrization [9-19] for solving the problem (1)–(5).

First, we introduce a new unknown function $w(t,x) = \dfrac{\partial^2 u(t,x)}{\partial t^2}$ and rewrite problem (1)–(5) in the following from

$$\frac{\partial^2 w}{\partial t \partial x} = A_1(t,x)\frac{\partial w}{\partial x} + A_2(t,x)\frac{\partial w}{\partial t} + A_3(t,x)w + f(t,x) +$$

$$+ A_4(t,x)\frac{\partial^2 u}{\partial t \partial x} + A_5(t,x)\frac{\partial u}{\partial t} + A_6(t,x)\frac{\partial u}{\partial x} + A_7(t,x)u, \tag{6}$$

$$w(0,x) = w(T,x), \qquad x \in [0,\omega], \tag{7}$$

$$w(t,0) = \ddot{\psi}(t), \qquad t \in [0,T], \tag{8}$$

$$\frac{\partial u}{\partial t} = \varphi_2(x) + \int_0^t w(\tau,x)\,d\tau, \quad u(t,x) = \varphi_1(x) + t \cdot \varphi_2(x) + \int_0^t \int_0^\tau w(\tau_1,x)\,d\tau_1 d\tau, \tag{9}$$

$$\frac{\partial^2 u}{\partial t \partial x} = \dot{\varphi}_2(x) + \int_0^t \frac{\partial w(\tau,x)}{\partial x}\,d\tau, \quad \frac{\partial u}{\partial x} = \dot{\varphi}_1(x) + t \cdot \dot{\varphi}_2(x) + \int_0^t \int_0^\tau \frac{\partial w(\tau_1,x)}{\partial x}\,d\tau_1 d\tau. \tag{10}$$

A solution of problem (6)–(10) is a function $w(t,x) \in C(\Omega, \mathbb{R}^n)$ having partial derivatives $\frac{\partial w(t,x)}{\partial t} \in$ $\in C(\Omega, \mathbb{R}^n)$, $\frac{\partial w(t,x)}{\partial x} \in C(\Omega, \mathbb{R}^n)$, $\frac{\partial^2 w(t,x)}{\partial t \partial x} \in C(\Omega, \mathbb{R}^n)$, where the function $u(t,x)$ and its partial derivatives $\frac{\partial u(t,x)}{\partial t}$, $\frac{\partial u(t,x)}{\partial x}$ and $\frac{\partial^2 u(t,x)}{\partial t \partial x}$ are determined from integral relations (9), (10).

The method of functional parametrization is based on the introduction of additional parameters as the value of the desired solution on the line $t = 0$ of the domain Ω. The semi-periodic boundary-value problem for system of hyperbolic equations with integral conditions (6)–(10) is reduced to an equivalent semi-periodic problem for the system of integro-differential equations of hyperbolic type with functional parameter depending on x. The properties of solution and its partial derivatives pass into the properties of functional parameter. Using this method, we obtained coefficient conditions for the unique solvability of semi-periodic initial boundary value problem for the fourth-order system of partial differential equations (1)–(5).

Different types of initial-boundary value problems for some classes of fourth-order system of partial differential equations are studied in [20-22] by introducing additional new functions.

Scheme of the method functional parametrization without partitioning of the domain

We denote by $\lambda(x) = w(0, x)$ and in problem (6) – (10) make the change $\tilde{w}(t, x) = w(t, x) - \lambda(x)$. Then, the integral relations (9) and (10) have the following form

$$\frac{\partial u(t,x)}{\partial t} = \varphi_2(x) + t \cdot \lambda(x) + \int_0^t \tilde{w}(\tau, x)\, d\tau, \tag{11}$$

$$u(t,x) = \varphi_1(x) + t \cdot \varphi_2(x) + \frac{t^2}{2} \cdot \lambda(x) + \int_0^t \int_0^\tau \tilde{w}(\tau_1, x)\, d\tau_1 d\tau, \tag{12}$$

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} = \dot{\varphi}_2(x) + t \cdot \dot{\lambda}(x) + \int_0^t \frac{\partial \tilde{w}(\tau, x)}{\partial x}\, d\tau, \tag{13}$$

$$\frac{\partial u(t,x)}{\partial x} = \dot{\varphi}_1(x) + t \cdot \dot{\varphi}_2(x) + \frac{t^2}{2} \cdot \dot{\lambda}(x) + \int_0^t \int_0^\tau \frac{\partial \tilde{w}(\tau_1, x)}{\partial x}\, d\tau_1 d\tau. \tag{14}$$

Further, in system (6) instead of functions $\frac{\partial u(t,x)}{\partial t}$, $u(t, x)$ $\frac{\partial^2 u(t,x)}{\partial t \partial x}$ and $\frac{\partial u(t,x)}{\partial x}$ we substitute their representations from (11)-(14), respectively. We get the following equivalent nonlocal problem for system of integro-differential equations of hyperbolic type with an unknown function $\lambda(x)$:

$$\frac{\partial^2 \tilde{w}}{\partial t \partial x} = A_1(t,x)\frac{\partial \tilde{w}}{\partial x} + A_2(t,x)\frac{\partial \tilde{w}}{\partial t} + A_3(t,x)\tilde{w} + A_4(t,x)\int_0^t \frac{\partial \tilde{w}(\tau,x)}{\partial x}\, d\tau +$$

$$+A_5(t,x)\int_0^t \tilde{w}(\tau,x)\, d\tau + A_6(t,x)\int_0^t\int_0^\tau \frac{\partial \tilde{w}(\tau_1,x)}{\partial x}\, d\tau_1 d\tau + A_7(t,x)\int_0^t\int_0^\tau \tilde{w}(\tau_1,x)\, d\tau_1 d\tau +$$

$$+\Big[A_1(t,x) + A_4(t,x)t + A_6(t,x)\frac{t^2}{2}\Big]\dot{\lambda}(x) + \Big[A_3(t,x) + A_5(t,x)t + A_7(t,x)\frac{t^2}{2}\Big]\lambda(x) +$$

$$+f(t,x) + g_1(t,x) + g_2(t,x), \tag{15}$$

$$\tilde{w}(0,x) = 0, \qquad x \in [0,\omega], \tag{16}$$

$$\tilde{w}(t,0) = \ddot{\psi}(t) - \ddot{\psi}(0), \qquad t \in [0,T], \tag{17}$$

$$\tilde{w}(T,x) = 0, \qquad x \in [0,\omega], \tag{18}$$

where $g_1(t,x) = A_4(t,x)\dot{\varphi}_2(x) + A_5(t,x)\varphi_2(x)$,
$g_2(t,x) = A_6(t,x)[\dot{\varphi}_1(x) + t \cdot \dot{\varphi}_2(x)] + A_7(t,x)[\varphi_1(x) + t \cdot \varphi_2(x)]$.

The compatibility condition is valid:

$$\lambda(0) = \ddot{\psi}(0). \tag{19}$$

Problems (6)–(10) and (15)–(18) are equivalent in the sense that if the function $w(t, x)$ is a solution of problem (6)–(10), then the pair $\{\lambda(x) = w(0, x), \tilde{w}(t, x) = w(t, x) - w(0, x)\}$ will be a solution of problem (15)–(18), and vice versa, if a pair $\{\lambda(x), \tilde{w}(t, x)\}$ is a solution to problem (15)–(18), then the function $\{\lambda(x) + \tilde{w}(t, x)\}$ will be the solution to problem (6)–(10).

For fixed $\lambda(x)$, $\dot{\lambda}(x)$ the function $\tilde{w}(t, x)$ is a solution to the Goursat problem on Ω with conditions (16), (17). From (16), (17) we obtain $\frac{\partial \tilde{w}(0,x)}{\partial x} = 0$, $\frac{\partial \tilde{w}(t,0)}{\partial t} = \dddot{\psi}(t)$ and reduce the Goursat problem to an equivalent system of three integral equations

$$\frac{\partial \tilde{w}(t,x)}{\partial x} = \int_0^t \Big[A_1(\tau,x)\frac{\partial \tilde{w}(\tau,x)}{\partial x} + A_2(\tau,x)\frac{\partial \tilde{w}(\tau,x)}{\partial \tau} + A_3(\tau,x)\tilde{w}(\tau,x)\Big] d\tau +$$

$$+\int_0^t \Big[A_4(\tau,x)\int_0^\tau \frac{\partial \tilde{w}(\tau_1,x)}{\partial x}\, d\tau_1 + A_5(\tau,x)\int_0^\tau \tilde{w}(\tau_1,x)\, d\tau_1\Big] d\tau +$$

$$+\int_0^t \Big[A_6(\tau,x)\int_0^\tau\int_0^{\tau_1}\frac{\partial\tilde{w}(\tau_2,x)}{\partial x}\,d\tau_2 d\tau_1 + A_7(\tau,x)\int_0^\tau\int_0^{\tau_1}\tilde{w}(\tau_2,x)\,d\tau_2 d\tau_1\Big]d\tau+$$

$$+\int_0^t \Big[A_1(\tau,x)+A_4(\tau,x)\tau+A_6(\tau,x)\frac{\tau^2}{2}\Big]d\tau\dot{\lambda}(x)+\int_0^t \Big[A_3(\tau,x)+A_5(\tau,x)\tau+A_7(\tau,x)\frac{\tau^2}{2}\Big]d\tau\lambda(x)+$$

$$+\int_0^t [f(\tau,x)+g_1(\tau,x)+g_2(\tau,x)]d\tau, \tag{20}$$

$$\frac{\partial\tilde{w}(t,x)}{\partial t}=\dddot{\psi}(t)+\int_0^x \Big[A_1(t,\xi)\frac{\partial\tilde{w}(t,\xi)}{\partial\xi}+A_2(t,\xi)\frac{\partial\tilde{w}(t,\xi)}{\partial t}+A_3(t,\xi)\tilde{w}(t,\xi)\Big]d\xi+$$

$$+\int_0^x \Big[A_4(t,\xi)\int_0^t\frac{\partial\tilde{w}(\tau,\xi)}{\partial\xi}\,d\tau+A_5(t,\xi)\int_0^t\tilde{w}(\tau,\xi)\,d\tau\Big]d\xi+$$

$$+\int_0^x \Big[A_6(t,\xi)\int_0^t\int_0^\tau\frac{\partial\tilde{w}(\tau_1,\xi)}{\partial\xi}\,d\tau_1 d\tau+A_7(t,\xi)\int_0^t\int_0^\tau\tilde{w}(\tau_1,\xi)\,d\tau_1 d\tau\Big]d\xi+$$

$$+\int_0^x \Big[A_1(t,\xi)+A_4(t,\xi)t+A_6(t,\xi)\frac{t^2}{2}\Big]\dot{\lambda}(\xi)d\xi+\int_0^x \Big[A_3(t,\xi)+A_5(t,\xi)t+A_7(t,\xi)\frac{t^2}{2}\Big]\lambda(\xi)d\xi+$$

$$+\int_0^x [f(t,\xi)+g_1(t,\xi)+g_2(t,\xi)]d\xi, \tag{21}$$

$$\tilde{w}(t,x)=\dddot{\psi}(t)-\ddot{\psi}(0)+\int_0^x\frac{\partial\tilde{w}(\tau,\xi)}{\partial\xi}d\xi. \tag{22}$$

Instead of $\frac{\partial\tilde{w}(\tau,x)}{\partial x}$, $\frac{\partial\tilde{w}(\tau_1,x)}{\partial x}$, $\frac{\partial\tilde{w}(\tau_2,x)}{\partial x}$ we substitute the corresponding right-hand side of (20) and, repeating this procedure $m(m=1,2,3,...)$ times, we obtain

$$\frac{\partial\tilde{w}}{\partial x}=D_m(t,x)\cdot\dot{\lambda}(x)+E_m(t,x)\cdot\lambda(x)+G_m\Big(t,x,\frac{\partial\tilde{w}}{\partial x}\Big)+H_m\Big(t,x,\frac{\partial\tilde{w}}{\partial t},\tilde{w}\Big)+F_m(t,x), \tag{23}$$

where

$$D_m(t,x)=D_m^{(1)}(t,x)+D_m^{(2)}(t,x)+D_m^{(3)}(t,x),$$

$$E_m(t,x)=E_m^{(1)}(t,x)+E_m^{(2)}(t,x)+E_m^{(3)}(t,x),$$

$$G_m\Big(t,x,\frac{\partial\tilde{w}}{\partial x}\Big)=G_m^{(1)}\Big(t,x,\frac{\partial\tilde{w}}{\partial x}\Big)+G_m^{(2)}\Big(t,x,\frac{\partial\tilde{w}}{\partial x}\Big)+G_m^{(3)}\Big(t,x,\frac{\partial\tilde{w}}{\partial x}\Big),$$

$$H_m\Big(t,x,\frac{\partial\tilde{w}}{\partial t},\tilde{w}\Big)=H_m^{(1)}\Big(t,x,\frac{\partial\tilde{w}}{\partial t},\tilde{w}\Big)+H_m^{(2)}(t,x,\tilde{w})+H_m^{(3)}(t,x,\tilde{w}),$$

$$F_m(t,x)=F_m^{(1)}(t,x)+F_m^{(2)}(t,x)+F_m^{(3)}(t,x),$$

$$D_m^{(1)}(t,x)=\int_0^t A_1(\tau_1,x)d\tau_1+\int_0^t A_1(\tau_1,x)\int_0^{\tau_1}A_1(\tau_2,x)d\tau_2 d\tau_1+$$

$$+...+\int_0^t A_1(\tau_1,x)...\int_0^{\tau_{m-1}}A_1(\tau_m,x)d\tau_m...d\tau_1,$$

$$E_m^{(1)}(t,x)=\int_0^t A_3(\tau_1,x)d\tau_1+...+\int_0^t A_1(\tau_1,x)...\int_0^{\tau_{m-2}}A_1(\tau_{m-1},x)\int_0^{\tau_{m-1}}A_3(\tau_m,x)d\tau_m...d\tau_1,$$

$$G_m^{(1)}\Big(t,x,\frac{\partial\tilde{w}}{\partial x}\Big)=\int_0^t A_1(\tau_1,x)...\int_0^{\tau_{m-2}}A_1(\tau_{m-1},x)\int_0^{\tau_{m-1}}A_1(t_m,x)\frac{\partial\tilde{w}(\tau_m,x)}{\partial x}d\tau_m...d\tau_1,$$

$$H_m^{(1)}\Big(t,x,\tilde{w},\frac{\partial\tilde{w}}{\partial t}\Big)=\int_0^t[A_2(\tau_1,x)\frac{\partial\tilde{w}}{\partial\tau_1}+A_3(\tau_1,x)\tilde{w}]d\tau_1+...+$$

$$+\int_0^t A_1(\tau_1,x)...\int_0^{\tau_{m-2}} A_1(\tau_{m-1},x)\int_0^{\tau_{m-1}}[A_2(\tau_m,x)\frac{\partial\tilde{w}}{\partial\tau_m}+A_3(\tau_m,x)\tilde{w}]d\tau_m...d\tau_1,$$

$$F_m^{(1)}(t,x)=\int_0^t f(\tau_1,x)d\tau_1+...+\int_0^t A_1(\tau_1,x)...\int_0^{\tau_{m-2}} A_1(\tau_{m-1},x)\int_0^{\tau_{m-1}} f(\tau_m,x)d\tau_m...d\tau_1,$$

$$D_m^{(2)}(t,x)=\int_0^t A_4(\tau_1,x)\cdot\tau_1 d\tau_1+\int_0^t A_4(\tau_1,x)\int_0^{\tau_1}\int_0^{\tau_2} A_4(\tau_3,x)\cdot\tau_3 d\tau_3 d\tau_2 d\tau_1+$$

$$+...+\int_0^t A_4(\tau_1,x)\int_0^{\tau_1}\int_0^{\tau_2} A_4(\tau_3,x)...\int_0^{\tau_{2m-3}}\int_0^{\tau_{2m-2}} A_4(\tau_{2m-1},x)\tau_{2m-1}d\tau_{2m-1}d\tau_{2m-2}...d\tau_1,$$

$$E_m^{(2)}(t,x)=\int_0^t A_5(\tau_1,x)\tau_1 d\tau_1+...+$$

$$+\int_0^t A_4(\tau_1,x)...\int_0^{\tau_{2m-5}}\int_0^{\tau_{2m-4}} A_4(\tau_{2m-3},x)\int_0^{\tau_{2m-3}}\int_0^{\tau_{2m-2}} A_5(\tau_{2m-1},x)\tau_{2m-1}d\tau_{2m-1}...d\tau_1,$$

$$G_m^{(2)}\Big(t,x,\frac{\partial\tilde{w}}{\partial x}\Big)=\int_0^t A_4(\tau_1,x)...\int_0^{\tau_{2m-3}}\int_0^{\tau_{2m-2}} A_4(\tau_{2m-1},x)\int_0^{\tau_{2m-1}}\frac{\partial\tilde{w}(\tau_{2m},x)}{\partial x}d\tau_{2m}...d\tau_1,$$

$$H_m^{(2)}(t,x,\tilde{w})=\int_0^t A_5(\tau_1,x)\int_0^{\tau_1}\tilde{w}(\tau_2,x)d\tau_2 d\tau_1+...+\int_0^t A_4(\tau_1,x)...$$

$$...\int_0^{\tau_{2m-5}}\int_0^{\tau_{2m-4}} A_4(\tau_{2m-3},x)\int_0^{\tau_{2m-3}}\int_0^{\tau_{2m-2}} A_5(\tau_{2m-1},x)\int_0^{2m-1}\tilde{w}(\tau_{2m},x)d\tau_{2m}...d\tau_1,$$

$$F_m^{(2)}(t,x)=\int_0^t g_1(\tau_1,x)d\tau_1+...+$$

$$+\int_0^t A_4(\tau_1,x)...\int_0^{\tau_{2m-5}}\int_0^{\tau_{2m-4}} A_4(\tau_{2m-3},x)\int_0^{\tau_{2m-3}}\int_0^{\tau_{2m-2}} g_1(\tau_{2m-1},x)d\tau_{2m-1}...d\tau_1,$$

$$D_m^{(3)}(t,x)=\int_0^t A_6(\tau_1,x)\cdot\frac{\tau_1^2}{2}d\tau_1+$$

$$+\int_0^t A_6(\tau_1,x)\int_0^{\tau_1}\int_0^{\tau_2}\int_0^{\tau_3} A_6(\tau_4,x)\cdot\frac{\tau_4^2}{2}d\tau_4 d\tau_3 d\tau_2 d\tau_1+...+\int_0^t A_6(\tau_1,x)...$$

$$...\int_0^{\tau_{3m-8}}\int_0^{\tau_{3m-7}}\int_0^{\tau_{3m-6}} A_6(\tau_{3m-5},x)\int_0^{\tau_{3m-5}}\int_0^{\tau_{3m-4}}\int_0^{\tau_{3m-3}} A_6(\tau_{3m-2},x)\frac{\tau_{3m-2}^2}{2}d\tau_{3m-2}...d\tau_1,$$

$$E_m^{(3)}(t,x)=\int_0^t A_7(\tau_1,x)\frac{\tau_1^2}{2}d\tau_1+...+\int_0^t A_6(\tau_1,x)...$$

$$...\int_0^{\tau_{3m-8}}\int_0^{\tau_{3m-7}}\int_0^{\tau_{3m-6}} A_6(\tau_{3m-5},x)\int_0^{\tau_{3m-5}}\int_0^{\tau_{3m-4}}\int_0^{\tau_{3m-3}} A_7(\tau_{3m-2},x)\frac{\tau_{3m-2}^2}{2}d\tau_{3m-2}...d\tau_1,$$

$$G_m^{(3)}\Big(t,x,\frac{\partial\tilde{w}}{\partial x}\Big)=$$

$$=\int_0^t A_6(\tau_1,x)...\int_0^{\tau_{3m-5}}\int_0^{\tau_{3m-4}}\int_0^{\tau_{3m-3}} A_6(\tau_{3m-2},x)\int_0^{\tau_{3m-2}}\int_0^{\tau_{3m-1}}\frac{\partial\tilde{w}(\tau_{3m},x)}{\partial x}d\tau_{3m}...d\tau_1,$$

$$H_m^{(3)}(t,x,\tilde{w})=\int_0^t A_7(\tau_1,x)\int_0^{\tau_1}\int_0^{\tau_2}\tilde{w}(\tau_3,x)d\tau_3 d\tau_2 d\tau_1+...+\int_0^t A_6(\tau_1,x)...\int_0^{\tau_{3m-8}}\int_0^{\tau_{3m-7}}$$

$$\int_0^{\tau_{3m-6}} A_6(\tau_{3m-5},x)\int_0^{\tau_{3m-5}}\int_0^{\tau_{3m-4}}\int_0^{\tau_{3m-3}} A_7(\tau_{3m-2},x)\int_0^{\tau_{3m-2}}\int_0^{\tau_{3m-1}}\tilde{w}(\tau_{3m},x)d\tau_{3m}...d\tau_1,$$

$$F_m^{(3)}(t,x) = \int_0^t g_2(\tau_1,x)d\tau_1 + ... +$$

$$+ \int_0^t A_6(\tau_1,x)... \int_0^{\tau_{3m-8}} \int_0^{\tau_{3m-7}} \int_0^{\tau_{3m-6}} A_6(\tau_{3m-5},x) \int_0^{\tau_{3m-5}} \int_0^{\tau_{3m-4}} \int_0^{\tau_{3m-3}} g_2(\tau_{3m-2},x)d\tau_{3m-2}...d\tau_1.$$

Assumptions regarding the data of problem (6)–(10) allow us to differentiate relation (18) with respect to x:

$$\frac{\partial \tilde{w}(T,x)}{\partial x} = 0. \tag{24}$$

Relation (24) will be equivalent to relation (18) if the compatibility condition (19) is satisfied.

From the right-hand side of (23), finding the value of $\tilde{w}(t,x)$ for $t = T$ and substituting it in (24), we obtain a system of n ordinary first-order differential equations that are not resolved with respect to the derivatives:

$$D_m(T,x) \cdot \dot{\lambda}(x) = -E_m(T,x) \cdot \lambda(x) - G_m\Big(T, x, \frac{\partial \tilde{w}}{\partial x}\Big) - H_m\Big(T, x, \frac{\partial \tilde{w}}{\partial t}, \tilde{w}\Big) - F_m(T,x). \tag{25}$$

For fixed $\frac{\partial \tilde{w}}{\partial x}$, $\frac{\partial \tilde{w}}{\partial t}$, $\tilde{w}$ system of differential equations (25) with initial condition (19) is the Cauchy problem with respect to $\lambda(x)$ for all $x \in [0,\omega]$. We solve the Cauchy problem (25), (19) using the fundamental matrix.

Let the matrix $D_m(T,x)$ be invertible for all $x \in [0,\omega]$ and $\Phi(x)$ the fundamental matrix to system of differential equations

$$\frac{d\lambda(x)}{dx} = -[D_m(T,x)]^{-1} E_m(T,x) \cdot \lambda(x). \tag{26}$$

We re-write system (25) in the following form

$$\dot{\lambda}(x) = -[D_m(T,x)]^{-1} E_m(T,x) \cdot \lambda(x) - \tilde{F}\Big(T, x, \frac{\partial \tilde{w}}{\partial x}, \frac{\partial \tilde{w}}{\partial t}, \tilde{w}\Big), \tag{27}$$

where

$$\tilde{F}\Big(T, x, \frac{\partial \tilde{w}}{\partial x}, \frac{\partial \tilde{w}}{\partial t}, \tilde{w}\Big) = -[D_m(T,x)]^{-1}\Big\{G_m\Big(T, x, \frac{\partial \tilde{w}}{\partial x}\Big) + H_m\Big(T, x, \frac{\partial \tilde{w}}{\partial t}, \tilde{w}\Big) + F_m(T,x)\Big\}.$$

A solution to the Cauchy problem (27), (19) is written as

$$\lambda(x) = \Phi(x)\ddot{\psi}(0) + \Phi(x) \int_0^x \Phi^{-1}(\xi) \tilde{F}\Big(T, \xi, \frac{\partial \tilde{w}}{\partial \xi}, \frac{\partial \tilde{w}}{\partial t}, \tilde{w}\Big) d\xi, \qquad x \in [0,\omega].$$

Thus, the invertibility of the matrix $D_m(T,x)$ for all $x \in [0,\omega]$ allows us to find a solution to the original problem (1)–(5) by using the fundamental matrix of a system of ordinary differential equations (26) and constructing solutions to the Goursat problem (15)–(17).

Note that a similar technique was applied to the semi-periodic boundary value problem for systems of quasi-linear and semi-linear hyperbolic equations of second-order in [23-24]. These problems were reduced to an equivalent problems, consisting of a family of periodic boundary-value problems for quasi-linear and semi-linear ordinary differential equations, respectively, and functional relations. To solve a families of periodic boundary-value problems for ordinary differential the parametrization method were used. Algorithms for finding periodic boundary-value problem's solution for systems of the quasi-linear and semi-linear system of hyperbolic equations are offered. To construct the algorithms were used a solutions to families of Cauchy problems for systems of ordinary differential equations and systems of functional equations with respect to the introduced parameters. This approach allowed to establish sufficient conditions for the existence of an solution to considered problems.

Algorithm for finding solution to problem (6)–(10)

As well-known, the fundamental matrix can be constructed for a narrow class of differential equations. Therefore, we propose an algorithm for finding an approximate solution to problem (6)–(10) without using the fundamental matrix.

So, the method of functional parametrization divides the process of finding unknown functions into two stages:

1) finding the introduced functional parameter $\lambda(x) \quad (\dot{\lambda}(x))$ from system (25) with condition (19).

2) finding unknown functions $\frac{\partial \tilde{w}(t,x)}{\partial x}$, $\frac{\partial \tilde{w}(t,x)}{\partial t}$, $\tilde{w}(t,x)$ from the system of integral equations (20)–(22).

If the functions $\dot{\lambda}(x), \quad \lambda(x)$ are known, then we will find the functions $\frac{\partial \tilde{w}(t,x)}{\partial x}$, $\frac{\partial \tilde{w}(t,x)}{\partial t}$, $\tilde{w}(t,x)$ to solve the system of integral equations (20)–(22), and the function $\lambda(x)+\tilde{w}(t,x)$ will be the solution to problem (6)–(10). If the functions $\frac{\partial \tilde{w}(t,x)}{\partial x}$, $\frac{\partial \tilde{w}(t,x)}{\partial t}$, $\tilde{w}(t,x)$ are known, then solving system of differential equations (25) with condition (19), we find $\dot{\lambda}(x), \quad \lambda(x)$ and again determining the sum of the functions $\lambda(x)+\tilde{w}(t,x)$ we find a solution to problem (6)–(10).

Here unknown are both the functions $\dot{\lambda}(x), \quad \lambda(x)$ and the functions $\frac{\partial \tilde{w}(t,x)}{\partial x}$, $\frac{\partial \tilde{w}(t,x)}{\partial t}$, $\tilde{w}(t,x)$. Therefore, we use an iterative method and the solution to system of integral equations (20)–(22) and the Cauchy problem (25), (19) is found as the limits of the sequences $\{\dot{\lambda}(x), \lambda(x), \frac{\partial \tilde{w}(t,x)}{\partial x}, \frac{\partial \tilde{w}(t,x)}{\partial t}, \tilde{w}(t,x)\}$, determined by the following algorithm:

Step 0. Assuming on the right-hand side of (25) $\lambda(x) = \ddot{\psi}(0)$, $\frac{\partial \tilde{w}(t,x)}{\partial x} = 0$, $\frac{\partial \tilde{w}(t,x)}{\partial t} = \dddot{\psi}(t)$, $\tilde{w}(t,x) = \ddot{\psi}(t) - \ddot{\psi}(0)$, and taking into account the invertibility of the matrix $D_m(T,x)$ for all $x \in [0,\omega]$, we find $\dot{\lambda}^{(0)}(x)$ from equation (25). Using conditions (19) we find the function $\lambda^{(0)}(x)$: $\lambda^{(0)}(x) = \ddot{\psi}(0) + \int\limits_0^x \dot{\lambda}^{(0)}(\xi)d\xi, \quad x \in [0,\omega]$. From the system of integral equations (20)–(22), where $\lambda(x) = \lambda^{(0)}(x), \dot{\lambda}(x) = \dot{\lambda}^{(0)}(x)$, we define the functions $\frac{\partial \tilde{w}^{(0)}(t,x)}{\partial x}, \frac{\partial \tilde{w}^{(0)}(t,x)}{\partial t}, \tilde{w}^{(0)}(t,x)$ for all $(t,x) \in \Omega$.

Step 1. From equation (25), where on the right-hand side of $\lambda(x) = \lambda^{(0)}(x)$, $\frac{\partial \tilde{w}(t,x)}{\partial x} = \frac{\partial \tilde{w}^{(0)}(t,x)}{\partial x}$, $\frac{\partial \tilde{w}(t,x)}{\partial t} = \frac{\partial \tilde{w}^{(0)}(t,x)}{\partial t}$, $\tilde{w}(t,x) = \tilde{w}^{(0)}(t,x)$, by virtue of the invertibility of $D_m(T,x)$ for all $x \in [0,\omega]$, we find $\dot{\lambda}^{(1)}(x)$.

Using conditions (19) again, we find the function $\lambda^{(1)}(x) = \ddot{\psi}(0) + \int_0^x \dot{\lambda}^{(1)}(\xi)d\xi, \quad x \in [0,\omega]$. From the system of integral equations (20)–(22), where $\lambda(x) = \lambda^{(1)}(x)$, $\dot{\lambda}(x) = \dot{\lambda}^{(1)}(x)$, we define the functions $\frac{\partial \tilde{w}^{(1)}(t,x)}{\partial x}$, $\frac{\partial \tilde{w}^{(1)}(t,x)}{\partial t}$, $\tilde{w}^{(1)}(t,x)$ for all $(t,x) \in \Omega$.

And so on.

Step k. From equation (25), where on the right-hand side of $\lambda(x) = \lambda^{(k-1)}(x)$, $\frac{\partial \tilde{w}(t,x)}{\partial x} = \frac{\partial \tilde{w}^{(k-1)}(t,x)}{\partial x}$, $\frac{\partial \tilde{w}(t,x)}{\partial t} = \frac{\partial \tilde{w}^{(k-1)}(t,x)}{\partial t}$, $\tilde{w}(t,x) = \tilde{w}^{(k-1)}(t,x)$, by virtue of the reversibility of $D_m(T,x)$ for all $x \in [0,\omega]$ we find $\dot{\lambda}^{(k)}(x)$.

Using conditions (19), we find the function $\lambda^{(k)}(x) = \ddot{\psi}(0) + \int_0^x \dot{\lambda}^{(k)}(\xi)d\xi, \quad x \in [0,\omega]$. From the system of integral equations (20)–(22), where $\lambda(x) = \lambda^{(k)}(x)$, $\dot{\lambda}(x) = \dot{\lambda}^{(k)}(x)$, we define the functions $\frac{\partial \tilde{w}^{(k)}(t,x)}{\partial x}$, $\frac{\partial \tilde{w}^{(k)}(t,x)}{\partial t}$, $\tilde{w}^{(k)}(t,x)$ for all $(t,x) \in \Omega$.

Here $k = 1, 2, 3, \ldots$.

The following statement gives conditions for the convergence of the proposed algorithm and the unique solvability of problem (6)–(10) in terms of the initial data.

Theorem 1. Suppose that for some $m, m = 1,2,3,...$, the $n \times n$ -matrix $D_m(T,x)$ is invertible for all $x \in [0,\omega]$ and the following inequalities hold:

a) $||[D_m(T,x)]^{-1}|| \le \gamma_m(T,x)$, and $\gamma_m(T,x)$ is a positive continuous function for all $x \in [0,\omega]$;

b) $q_m(T,x) = \gamma_m(T,x) \cdot \left\{ e^{\alpha(x)T} - 1 - \alpha(x)T - ... - \frac{1}{m!}[\alpha(x)T]^m \right\} \le \chi < 1,$
where $\alpha(x) = \max\limits_{t\in[0,T]} (||A_1(t,x)||, ||A_4(t,x)||, ||A_6(t,x)||)$, χ *is constant.*

Then there is a unique solution $w^*(t,x)$ *to problem (6)–(10), determining by equality*

$$w^*(t,x) = \lambda^*(x) + \tilde{w}^*(t,x)$$

with

$$\frac{\partial u^*(t,x)}{\partial t} = \varphi_2(x) + t\cdot\lambda^*(x) + \int_0^t \tilde{w}^*(\tau,x)\,d\tau,$$

$$u^*(t,x) = \varphi_1(x) + t\cdot\varphi_2(x) + \frac{t^2}{2}\cdot\lambda^*(x) + \int_0^t\int_0^\tau \tilde{w}^*(\tau_1,x)\,d\tau_1 d\tau,$$

$$\frac{\partial^2 u^*(t,x)}{\partial t\partial x} = \dot{\varphi}_2(x) + t\cdot\dot{\lambda}^*(x) + \int_0^t \frac{\partial\tilde{w}^*(\tau,x)}{\partial x}\,d\tau,$$

$$\frac{\partial u^*(t,x)}{\partial x} = \dot{\varphi}_1(x) + t\cdot\dot{\varphi}_2(x) + \frac{t^2}{2}\cdot\dot{\lambda}^*(x) + \int_0^t\int_0^\tau \frac{\partial\tilde{w}^*(\tau_1,x)}{\partial x}\,d\tau_1 d\tau,$$

where $\lambda^*(x) = \lim\limits_{k\to\infty}\lambda^{(k)}(x)$, $\dot{\lambda}^*(x) = \lim\limits_{k\to\infty}\dot{\lambda}^{(k)}(x)$ *for all* $x\in[0,\omega]$,

$\tilde{w}^*(t,x) = \lim\limits_{k\to\infty}\tilde{w}^{(k)}(t,x)$, $\frac{\partial\tilde{w}^*(t,x)}{\partial x} = \lim\limits_{k\to\infty}\frac{\partial\tilde{w}^{(k)}(\tau,x)}{\partial x}$ *for all* $(t,x)\in\Omega$.

Proof of the Theorem 1 is provided according to proposed algorithm above.

Therefore, from the equivalence of problems (6)–(10) and (1)–(5) it follows

Theorem 2. Suppose that for some $m, m = 1,2,3,...,$ *the* $n\times n$ *-matrix* $D_m(T,x)$ *is invertible for all* $x\in[0,\omega]$ *and the inequalities a), b) of Theorem 1 are fulfilled.*

Then there is a unique classical solution $u^*(t,x)$ *to problem (1)–(5), defining from the following integral representation*

$$u^*(t,x) = \varphi_1(x) + t\cdot\varphi_2(x) + \int_0^t\int_0^\tau w^*(\tau_1,x)\,d\tau_1 d\tau, \qquad (t,x)\in\Omega.$$

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Төртінші ретті дербес туындылы дифференциалдық теңдеулер үшін жартылайпериодты бастапқы есепті шешудің функционалдық параметрлеу әдісі

Төртінші ретті дербес туындылы дифференциалдық теңдеулер жүйесі үшін жартылайпериодты бастапқы шеттік есеп қарастырылды. Функционалдық параметрлеу әдісі көмегімен авторлар қосымша параметрін енгізіп, зерттеліп отырған есеп екінші ретті гиперболалық тектес интегралдық-дифференциалдық теңдеулер жүйесі үшін функционалдық параметрлері мен интегралдық қатынастары бар пара-пар жартылайпериодты есепке келтірді. Гиперболалық тектес интегралдық-дифференциалдық теңдеулер жүйесі үшін жартылайпериодты есеп пен жай дифференциалдық теңдеулер жүйесі үшін Коши есептері әулетінің өзара байланысы тағайындалған. Пара-пар есептің шешімін табу алгоритмдері құрылған және олардың жинақтылығы дәлелденген. Төртінші ретті дербес туындылы дифференциалдық теңдеулер үшін жартылайпериодты бастапқы шеттік есептің бірмәнді шешілімділігінің жеткілікті шарттары алынған.

Кілт сөздер: жартылайпериодты бастапқы шеттік есеп, төртінші ретті дербес туындылы дифференциалдық теңдеулер жүйесі, функционалдық параметрлеу әдісі, жартылайпериодты есеп, екінші ретті гиперболалық тектес интегралдық-дифференциалдық теңдеулер жүйесі, Коши есептерінің әулеті, алгоритм, бірмәнді шешілімділік.

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Метод функциональной параметризации решения полупериодической начальной задачи для дифференциальных уравнений в частных производных четвертого порядка

Рассмотрена полупериодическая начальная краевая задача для системы дифференциальных уравнений в частных производных четвертого порядка. Авторами с помощью метода функциональной параметризации введен дополнительный параметр, и исследуемая задача сведена к эквивалентной полупериодической задаче для системы интегро-дифференциальных уравнений гиперболического типа второго порядка с функциональными параметрами и интегральными соотношениями. Установлена взаимосвязь полупериодической задачи для системы интегро-дифференциальных уравнений гиперболического типа и семейства задач Коши для системы обыкновенных дифференциальных уравнений. Построены алгоритмы нахождения решений эквивалентной задачи и доказана их сходимость. Получены достаточные условия однозначной разрешимости полупериодической начальной краевой задачи для системы дифференциальных уравнений в частных производных четвертого порядка.

Ключевые слова: полупериодическая начальная краевая задача, система дифференциальных уравнений в частных производных четвертого порядка, метод функциональной параметризации, полупериодическая задача, система интегро-дифференциальных уравнений гиперболического типа второго порядка, семейство задач Коши, алгоритм, однозначная разрешимость.

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The problem of trigonometric Fourier series multipliers of classes in $\lambda_{p,q}$ spaces

In this article, we consider weighted spaces of numerical sequences $\lambda_{p,q}$, which are defined as sets of sequences $a = \{a_k\}_{k=1}^{\infty}$, for which the norm

$$\|a\|_{\lambda_{p,q}} := \left(\sum_{k=1}^{\infty} |a_k|^q k^{\frac{q}{p}-1} \right)^{\frac{1}{q}} < \infty$$

is finite. In the case of non-increasing sequences, the norm of the space $\lambda_{p,q}$ coincides with the norm of the classical Lorentz space $l_{p,q}$. Necessary and sufficient conditions are obtained for embeddings of the space $\lambda_{p,q}$ into the space λ_{p_1,q_1}. The interpolation properties of these spaces with respect to the real interpolation method are studied. It is shown that the scale of spaces $\lambda_{p,q}$ is closed in the relative real interpolation method, as well as in relative to the complex interpolation method. A description of the dual space to the weighted space $\lambda_{p,q}$ is obtained. Specifically, it is shown that the space is reflective, where p', q' are conjugate to the parameters p and q. The paper also studies the properties of the convolution operator in these spaces. The main result of this work is an O'Neil type inequality. The resulting inequality generalizes the classical Young-O'Neil inequality. The research methods are based on the interpolation theorems proved in this paper for the spaces $\lambda_{p,q}$.

Keywords: trigonometric Fourier coefficients, O'Neil inequality, convolution operator, $M_{p_0,q_0}^{p_1,q_1}$ class.

Introduction

Let $1 \le p \le \infty$, $L_p \equiv L_p(\mathbb{R})$ and let the convolution operator be given by

$$(Af)(x) = (K * f)(x) = \int_{\mathbb{R}} K(x-y) f(y) dy.$$

The Young convolution inequality

$$\|A\|_{L_p \to L_q} \le \|K\|_{L_r}, \qquad 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}, \qquad 1 \le p \le q \le \infty,$$

has a very important role both in Harmonic Analysis and PDE (see, e.g., [1, Ch. 4, § 2, 4], [2]). $K(x) = |x|^{-\gamma}$, $\gamma > 0$. Young's estimates were generalized by O'Neil [3] who showed that for $1 < p < q < \infty$, $0 < t, s_1, s_2, \le \infty$, $1/r = 1 - 1/p + 1/q$ and $1/t = 1/s_1 + 1/s_2$

$$\|A\|_{L_{p,s_1} \to L_{q,s_2}} \le C \|K\|_{L_{r,t}},$$

and in particular

$$\|A\|_{L_p \to L_q} \le C \|K\|_{L_{r,\infty}}, \tag{1.1}$$

where $L_{p,s}$ is Lorentz space.

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Note that inequality (2.2) unlike (2.1) gives the Hardy-Littlewood fractional integration theorem, which corresponds to the model case in which $K(x) = |x|^{-1/r}$.

When $1 \le p \le q \le \infty$, were considered in [4], [5]. The estimate (1.1) was improved in [6], [7].

There are several generalizations of both Young and O'Neil's inequalities for various function spaces (weighted L_p, classical and weighted Lorentz spaces, weighted Besov and Hardy spaces, Wiener amalgam spaces, Orlicz spaces; see, e.g., [8], [9], [10], [11], [12], [13], [14], [15], [16] and references there in). We also remark that the sharp Young convolution inequality was obtained in [17] and [18].

Note that the norm estimates for convolution operators in various spaces are closely related to the problem of multipliers of Fourier transforms and Fourier series [19], [20], [21], [22], [23].

Let $0 < p < \infty,\ 0 < q \le \infty$. It is said that the sequence $a = \{a_k\}_{k=-\infty}^{\infty}$ belongs to the class $\lambda_{p,q}$, if

$$\|a\|_{\lambda_{p,q}} = \left(\sum_{k=-\infty}^{\infty} |a_k|^q \bar{k}^{\frac{q}{p}-1}\right)^{\frac{1}{q}} < \infty,$$

where $\bar{k} = \max(|k|, 1)$, and

$$\left\|a\right\|_{\lambda_{p,\infty}} = \sup_k |a_k| \bar{k}^{\frac{1}{p}} < \infty,$$

if $q = \infty$.

Our aim is to study the convolution inequalities in weighted spaces $\lambda_{p,q}$.

Throughout this paper, $F \lesssim G$ means that $F \le CG$; by C we denote positive constants that may be different on different occasions. Moreover, $F \asymp G$ means that $F \lesssim G \lesssim F$.

2. Properties of the spaces $\lambda_{p,q}$

Lemma 2.1 For embedding

$$\lambda_{p_0,q_0} \hookrightarrow \lambda_{p_1,q_1} \tag{2.1}$$

to hold it is necessary and sufficient that
for $q_0 \le q_1$, $\frac{1}{q_0} - \frac{1}{q_1} \le \frac{1}{p_0} - \frac{1}{p_1}$,
for $q_1 < q_0$, $\frac{1}{p_0} - \frac{1}{p_1} > 0$.

Proof. We consider the case $q_0 \le q_1$. Let $\frac{1}{q_0} - \frac{1}{q_1} \le \frac{1}{p_0} - \frac{1}{p_1}$. Then, using the inequality $(a+b)^\alpha \le$ $\le a^\alpha + b^\alpha$ for $\alpha < 1$, we get:

$$\left\|a\right\|_{\lambda_{p_1,q_1}} = \left(\sum_{k=-\infty}^{\infty} |a_k|^{q_1} \bar{k}^{(\frac{q_1}{p_1}-1)}\right)^{\frac{1}{q_1}} = \left(\sum_{k=-\infty}^{\infty} \left(|a_k|^{q_1} \bar{k}^{(\frac{1}{p_1}-\frac{1}{q_1})}\right)^{q_1}\right)^{\frac{1}{q_1}}$$

$$\le \left(\sum_{k=-\infty}^{\infty} \left(|a_k| \bar{k}^{(\frac{1}{p_0}-\frac{1}{q_0})}\right)^{q_1}\right)^{\frac{1}{q_1}} \le \left(\sum_{k=-\infty}^{\infty} \left(|a_k| \bar{k}^{(\frac{1}{p_0}-\frac{1}{q_0})}\right)^{q_0}\right)^{\frac{1}{q_0}} = \left\|a\right\|_{\lambda_{p_0,q_0}}.$$

Thus,

$$\lambda_{p_0,q_0} \hookrightarrow \lambda_{p_1,q_1}.$$

On the other hand, let $m \in \mathbb{N}$ and we consider the sequence $a = \{a_k\}_{k=-\infty}^{\infty}$:

$$a_k = \begin{cases} 1, & k = m \\ 0, & \text{in otherwise.} \end{cases}$$

Then according to embedding (2.1),

$$\bar{m}^{\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} = \left\|a\right\|_{\lambda_{p_1,q_1}} \le c \left\|a\right\|_{\lambda_{p_0,q_0}} = c\bar{m}^{\left(\frac{1}{p_0}-\frac{1}{q_0}\right)}.$$

Since m is arbitrary, which is possible if only if $\frac{1}{q_0} - \frac{1}{q_1} \le \frac{1}{p_0} - \frac{1}{p_1}$.

Let us pass to a case $q_1 < q_0$. Let $\frac{1}{p_0} - \frac{1}{p_1} > 0$. Denote $\varepsilon = \frac{1}{p_0} - \frac{1}{p_1} - \frac{1}{q_0} + \frac{1}{q_1}$. Further applying the Hölder inequality with the following parameters r_1 and r_2 such that $\frac{1}{r_1} = \frac{1}{q_0}$, $\frac{1}{r_2} = \frac{1}{q_1} - \frac{1}{q_0}$, $\left(\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{q_1}\right)$, we get

$$\|a\|_{\lambda_{p_1,q_1}} = \left(\sum_{k=-\infty}^{\infty} \left(|a_k| \bar{k}^{\frac{1}{p_0} - \frac{1}{q_0} - \varepsilon}\right)^{q_1}\right)^{\frac{1}{q_1}} = \left(\sum_{\bar{k}=-\infty}^{\infty} \left(|a_k| \bar{k}^{\left(\frac{1}{p_0} - \frac{1}{q_0}\right)} \bar{k}^{-\varepsilon}\right)^{q_1}\right)^{\frac{1}{q_1}}$$

$$\leq \|a\|_{\lambda_{p_0,q_0}} \left(\sum_{k=-\infty}^{\infty} \bar{k}^{-\varepsilon r_2}\right)^{\frac{1}{r_2}}.$$

Moreover $\varepsilon r_2 = \left[\left(\frac{1}{p_0} - \frac{1}{p_1}\right) + \left(\frac{1}{q_1} - \frac{1}{q_0}\right)\right] \left(\frac{1}{q_1} - \frac{1}{q_0}\right)^{-1} > 1$. We have $\sum\limits_{k=-\infty}^{\infty} \bar{k}^{-\varepsilon r_2} < \infty$.

To prove the necessity, we suppose that the embedding $\lambda_{p_0,q_0} \hookrightarrow \lambda_{p_1,q_1}$ and $m \in \mathbb{N}$ holds. We consider the sequence $a = \{a_k\}_{k=-\infty}^{\infty}$ such that

$$a_k = \begin{cases} k^{\alpha}, & 0 \leq k \leq m \\ 0, & \text{in otherwise.} \end{cases}$$

.

We have $\left\|a\right\|_{\lambda_{p_1,q_1}} = \left(\sum\limits_{i=1}^{m} \bar{i}^{\left(\frac{q_1}{p_1} + \alpha q_1 - 1\right)}\right)^{\frac{1}{q_1}} = c_1 \bar{m}^{\left(\frac{1}{p_1} + \alpha\right)}$,

$\left\|a\right\|_{\lambda_{p_0,q_0}} = \left(\sum\limits_{i=1}^{m} \bar{i}^{\left(\frac{q_0}{p_0} + \alpha q_0 - 1\right)}\right)^{\frac{1}{q_0}} = c_2 \bar{m}^{\left(\frac{1}{p_0} + \alpha\right)}$.

Therefore, since $\lambda_{p_0,q_0} \hookrightarrow \lambda_{p_1,q_1}$, and m is arbitrary we have that $\frac{1}{p_0} \geq \frac{1}{p_1}$.

In the case $\frac{1}{p_0} = \frac{1}{p_1}$. We consider the sequence $a = \{a_k\}_{k=-\infty}^{\infty}$: when

$$a_k = \begin{cases} \bar{k}^{\frac{1}{p_0}} \ln^{-\frac{1}{q_0 - \varepsilon}} \bar{k}, & 2 \leq k \\ 0, & \text{in otherwise,} \end{cases}$$

where ε is chosen so that $q_1 < q_0 - \varepsilon$. Then since $\frac{q_0}{q_0 - \varepsilon} > 1$, we get

$$\left\|a\right\|_{\lambda_{p_0,q_0}} = \left(\sum_{k=2}^{\infty} \left(\bar{k}^{-\frac{1}{p_0}} \ln^{-\frac{1}{q_0-\varepsilon}} \bar{k}\right)^{q_0} \bar{k}^{\left(\frac{q_0}{p_0} - 1\right)}\right)^{\frac{1}{q_0}} = \left(\sum_{k=2}^{\infty} \frac{\ln^{-\frac{q_0}{q_0-\varepsilon}} \bar{k}}{\bar{k}}\right)^{\frac{1}{q_0}} < \infty.$$

On the other hand since $\frac{q_1}{q_0 - \varepsilon} < 1$, we have

$$\left\|a\right\|_{\lambda_{p_1,q_1}} = \left(\sum_{k=2}^{\infty} \frac{1}{\bar{k} \ln^{\frac{q_1}{q_0-\varepsilon}} \bar{k}}\right)^{\frac{1}{q_1}} = \infty.$$

Therefore, the condition $\frac{1}{p_0} > \frac{1}{p_1}$ is necessary.

Lemma 2.2 Let $0 < p_0, q_0, p_1, q_1 < \infty$, $\quad 0 < \theta < 1$, then

$$\left(\lambda_{p_0,q_0}, \lambda_{p_1,q_1}\right)_{\theta,q} = \lambda_{p,q},$$

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Proof. By the well-known theorem of powers (see [24, Th. 3.11.6]), we have

$$\left((\lambda_{p_0,q_0})^{q_0}, (\lambda_{p_1,q_1})^{q_1}\right)_{\eta,1} = \Big((\lambda_{p_0,q_0}, \lambda_{p_1,q_1})_{\theta,q}\Big)^q,$$

where $\eta = \frac{\theta q}{q_1}$.

The norm of the element x in the space $\left((\lambda_{p_0,q_0})^{q_0}, (\lambda_{p_1,q_1})^{q_1}\right)_{\eta,1}$ is equal to

$$\int_0^\infty t^{-\eta} \inf_{x=x^0+x^1} \left(\sum_{k=-\infty}^{\infty} |x_k^0|^{q_0} \bar{k}^{\left(\frac{q_0}{p_o}-1\right)} + t \sum_{k=-\infty}^{\infty} |x_k^1|^{q_1} \bar{k}^{\left(\frac{q_1}{p_1}-1\right)} \right) \frac{dt}{t}$$

$$= \sum_k \int_0^\infty t^{-\eta} |x_k|^{q_0} \bar{k}^{\frac{q_0}{p_0}-1} \inf_{x_k=x_k^0+x_k^1} \left(\left|\frac{x_k{}^0}{x_k}\right|^{q_0} + t \left|\frac{x_k{}^1}{x_k}\right|^{q_1} |x_k|^{q_1-q_0} \bar{k}^{\frac{q_1}{p_1}-\frac{q_0}{p_0}} \right) \frac{dt}{t}$$

$$= \sum_{k=-\infty}^{\infty} \int_0^\infty \left(s^{-1} |x_k|^{q_1-q_0} \bar{k}^{\frac{q_1}{p_1}-\frac{q_0}{p_0}} \right)^\eta |x_k|^{q_0} \bar{k}^{\frac{q_0}{p_0}-1} \inf_{1=y_k^0+y_k^1} \left(|y_k^0|^{q_0} + s|y_k^1|^{q_1} \right) \frac{ds}{s}.$$

Considering that $\inf\limits_{1=y_k^0+y_k^1} \left(|y_k^0|^{q_0} + s|y_k^1|^{q_1} \right) \sim \min(1,s)$ and $\eta = \frac{\theta q}{q_1}$ the last expression is equal to $c \sum\limits_{k=-\infty}^{\infty} |x_k|^q \bar{k}^{\left(\frac{q}{p}-1\right)}$, whence the statement of the lemma follows.

Lemma 2.3 Let $0 < q < \infty,\ 0 < s \le \infty,\ \ 0 < \theta < 1$, then

$$[\lambda_{q,s}, \lambda_{q,\infty}]_\theta = \lambda_{q,t},$$

where $\frac{1}{t} = \frac{1-\theta}{s}$.

Proof. The interpolation theorem (see [25], p. 142) concerning to the complex interpolation method is known

$$[l_{p_0}(A_k), l_\infty(B_k)]_\theta = l_p([A_k, B_k]_\theta)$$

here $1 \le p_0 < \infty,\ \ 0 < \theta < 1,\ \ \frac{1}{p} = \frac{1-\theta}{p_0}$. In our case, the spaces $\lambda_{q,s}, \lambda_{q,\infty}$ can be represented as follows

$$\lambda_{q,s} = l_s(A_k), \qquad \lambda_{q,\infty} = l_\infty(B_k),$$

where $\|\cdot\|_{A_k} = |\cdot|\bar{k}^{\frac{1}{q}-\frac{1}{s}}, \qquad \|\cdot\|_{B_k} = |\cdot|\bar{k}^{\frac{1}{q}}$.

Therefore, we have

$$[\lambda_{q,s}, \lambda_{q,\infty}]_\theta = [l_s(A_k), l_\infty(B_k)]_\theta = l_t(C_k),$$

here $\frac{1}{t} = \frac{1-\theta}{s}$,

$$\|\cdot\|_{C_k} = |\cdot|\bar{k}^{(1-\theta)(\frac{1}{q}-\frac{1}{s})+\frac{\theta}{q}} = |\cdot|\bar{k}^{\frac{1}{q}-\frac{1-\theta}{s}} = |\cdot|\bar{k}^{\frac{1}{q}-\frac{1}{t}}$$

i.e. $l_t(C_k) = \lambda_{q,t}$

Let X be a linear normed space of numerical sequences. We define the dual space X' as a set of sequences $a = \{a_k\}_{k\in\mathbb{Z}}$ for which

$$\|a\|_{X'} := \sup_{\|b\|_X=1} \sum_{k\in\mathbb{Z}} a_k b_k.$$

Lemma 2.4 Let $1 < p < \infty,\ 1 \le q \le \infty,\ \ \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, then

$$(\lambda_{p,q})' = \lambda_{p',q'}.$$

Proof. The statement of Lemma follows from equality

$$\|a\|_{\lambda_{p',q'}} = \sup_{\|b\|_{\lambda_{p,q}=1}} \sum_{k\in\mathbb{Z}} a_k b_k \tag{2.2}$$

which could be proved using the Hölder inequality.

3. Convolution in the spaces $\lambda_{p,q}$

Let $a = \{a_k\}_{k=-\infty}^{\infty}$, $b = \{b_k\}_{k=-\infty}^{\infty}$ be such that

$$\sum_{k=-\infty}^{\infty} a_k b_{k-m} < \infty \,, \quad m \in \mathbb{Z}.$$

The sequence

$$\left\{ \sum_{k=-\infty}^{\infty} a_k b_{k-m} \right\}_{m=-\infty}^{\infty}$$

will be called a convolution and denoted by $a * b$.

Lemma 3.1 Let $1 < r, p, q < \infty$ and $\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p}$.

Then

$$\|a * b\|_{\lambda_{q,\infty}} \leq c\|a\|_{\lambda_{r,\infty}} \|b\|_{\lambda_{p,\infty}}.$$

Proof. By the definition of nonconforming transformations, we have

$$\left\|a * b\right\|_{\lambda_{q,\infty}} = \sup_k \left| \left(\sum_{m=-\infty}^{\infty} a_m b_{m-k} \right) \right| \bar{k}^{\frac{1}{q}} \leq \sup_m |a_m| \bar{m}^{\frac{1}{r}} \sup_k \sum_{m=-\infty}^{\infty} |b_{k-m}| \bar{m}^{-\frac{1}{r}} \bar{k}^{\frac{1}{q}}$$

$$= \left\|a\right\|_{\lambda_{r,\infty}} \sup_k \bar{k}^{\frac{1}{q}} \sum_{m=-\infty}^{\infty} |b_m| (\overline{m-k})^{-\frac{1}{r}} \leq \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,\infty}} \sup_k \bar{k}^{\frac{1}{q}} \sum_{m=-\infty}^{\infty} (\overline{m-k})^{-\frac{1}{r}} \bar{m}^{-\frac{1}{p}}.$$

Note that for $\bar{k} \neq \bar{0}$

$$\sum_{m=-\infty}^{\infty} (\overline{m-k})^{-\frac{1}{r}} \bar{m}^{-\frac{1}{p}} \asymp \int_{\mathbb{R}} \frac{dx}{|x-k|^{\frac{1}{r}} |x|^{\frac{1}{p}}} = k^{1-1/p-1/r} \int_{\mathbb{R}} \frac{dx}{|x-1|^{\frac{1}{r}} |x|^{\frac{1}{p}}} = ck^{-\frac{1}{q}}.$$

Therefore, we have

$$\left\|a * b\right\|_{\lambda_{q,\infty}} \leq c \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,\infty}}.$$

Lemma 3.2 Let one of the following conditions be fulfilled:
either $0 < s \leq 1$, $0 < s < p, r < q < \infty$, $\frac{1}{s} + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$
or $1 < s \leq \infty$, $1 < p, r < q < \infty$, $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$.
Then the following inequalities hold:

$$\left\|a * b\right\|_{\lambda_{q,s}} \leq c \left\|a\right\|_{\lambda_{r,s}} \left\|b\right\|_{\lambda_{p,\infty}},$$

$$\left\|a * b\right\|_{\lambda_{q,s}} \leq c \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,s}}.$$

Proof. Let $0 < s \leq 1$. According to the Jensen's inequality we have

$$\|a * b\|_{\lambda_{q,s}} = \left(\sum_{k=-\infty}^{\infty} \left| \sum_{m=-\infty}^{\infty} a_m b_{k-m} \right|^s \bar{k}^{\left(\frac{s}{q}-1\right)} \right)^{\frac{1}{s}} \leq \left(\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |a_m b_{m-k}|^s \, \bar{k}^{\left(\frac{s}{q}-1\right)} \right)^{\frac{1}{s}}$$

$$= \left(\sum_{m=-\infty}^{\infty} |a_m|^s \sum_{k=-\infty}^{\infty} |b_{k-m}|^s \bar{k}^{\left(\frac{s}{q}-1\right)} \right)^{\frac{1}{s}} \leq \|b\|_{\lambda_{r,\infty}} \left(\sum_{m=-\infty}^{\infty} |a_m|^s \sum_{k=-\infty}^{\infty} (\overline{k-m})^{-\frac{s}{p}} \bar{k}^{\left(\frac{s}{q}-1\right)} \right)^{\frac{1}{s}}.$$

Considering that

$$\sum_{k=-\infty}^{\infty} (\overline{k-m})^{-\frac{s}{p}} \bar{k}^{\left(\frac{s}{q}-1\right)} \asymp \bar{m}^{-\frac{s}{p}+\frac{s}{q}} = \bar{m}^{\frac{s}{r}-1},$$

we have $0 < s \leq 1$, $\frac{1}{s} + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$

$$\left\|a * b\right\|_{\lambda_{q,s}} \leq c \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,s}}$$

and in particular

$$\left\|a * b\right\|_{\lambda_{q,1}} \leq c \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,1}} .$$

Using Lemma 3.1 we have

$$\left\|a * b\right\|_{\lambda_{q,\infty}} \leq c \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,\infty}} .$$

Applying the bilinear interpolation theorem [2; Theorem 4.4.1] we obtain

$$\left\|a * b\right\|_{[\lambda_{q,1},\lambda_{q,\infty}]_\theta} \leq c\|a\|_{[\lambda_{r,\infty},\lambda_{r,\infty}]_\theta} \left\|b\right\|_{[\lambda_{p,1},\lambda_{p,\infty}]_\theta} .$$

Moreover, by Lemma 3.1 we obtain

$$\left\|a * b\right\|_{\lambda_{q,s}} \leq c \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,s}} ,$$

where $\frac{1}{s} = 1 - \theta$, $\theta \in [0, 1]$, $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$.

The second inequality is proved symmetrically.

For the proof of further results we need next statement also of independent interest.

Theorem 3.1 Let $0 < s, t_1, t_2 \leq \infty$, $\frac{1}{s} = \frac{1}{t_1} + \frac{1}{t_2}$. Let one of the following conditions be fulfilled: either $0 < s \leq 1$, $0 < s < p, r < q < \infty$, $\frac{1}{s} + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ or $1 < s \leq \infty$, $1 < p, r < q < \infty$, $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$.
Then

$$\left\|a * b\right\|_{\lambda_{q,s}} \leq c \left\|a\right\|_{\lambda_{r,t_1}} \left\|b\right\|_{\lambda_{p,t_2}} .$$

Proof. According to the Lemma 3.2 the following basic inequalities are known

$$\left\|a * b\right\|_{\lambda_{q,s}} \leq c \left\|a\right\|_{\lambda_{r,s}} \left\|b\right\|_{\lambda_{p,\infty}} .$$

$$\left\|a * b\right\|_{\lambda_{q,s}} \leq c \left\|a\right\|_{\lambda_{r,\infty}} \left\|b\right\|_{\lambda_{p,s}} .$$

Applying the bilinear interpolation theorem and using Lemma 3.1, we obtain the desired statement.

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А. Бахыт, Н.Т. Тлеуханова

Тригонометриялық Фурье қатарының көбейткіштері класының $\lambda_{p,q}$ кеңістігіндегі есебі

Мақалада $a = \{a_k\}_{k=1}^{\infty}$ тізбектің жиыны ретінде анықталатын $\lambda_{p,q}$ сандық тізбектерінің салмақты кеңістігі қарастырылды, олар үшін норма

$$\|a\|_{\lambda_{p,q}} := \left(\sum_{k=1}^{\infty} |a_k|^q k^{\frac{q}{p}-1}\right)^{\frac{1}{q}} < \infty$$

шектеулі. Өспейтін тізбектер болған жағдайда $\lambda_{p,q}$ кеңістігінің нормасы классикалық $l_{p,q}$ Лоренц кеңістігінің нормасына сәйкес келеді. $\lambda_{p,q}$ кеңістігінің λ_{p_1,q_1} кеңістігіне енгізу үшін қажетті және жеткілікті шарттары алынды. Нақты интерполяция әдісіне қатысты осы кеңістіктердің интерполяциялық қасиеттері зерттелген. $\lambda_{p,q}$ кеңістіктерінің шкаласы нақты интерполяция әдісіне қатысты, сондай-ақ біріктірілген интерполяция әдісіне қатысты тұйық екендігі көрсетілген. Қосарланған кеңістіктің $\lambda_{p,q}$ салмақты кеңістігіне сипаттама алынды. Атап айтқанда, кеңістік рефлексивті, мұндағы p', q' параметрлері p және q параметрлеріне түйіндес болып келеді. Сонымен қатар осы кеңістіктерде үйірткі операторларының қасиеттері зерттелді. Бұл жұмыстың негізгі нәтижесі О'Нейл типті теңсіздігі болып табылады. Алынған теңсіздік классикалық Юнг-О'Нейл теңсіздігін жалпылайды. Зерттеу әдісі $\lambda_{p,q}$ кеңістіктері үшін дәлелденген интерполяциялық теоремаларға негізделген.

Кілт сөздер: тригонометриялық Фурье коэффициенттері, О'Нейл теңсіздігі, үйірткі операторы, $M_{p_0,q_0}^{p_1,q_1}$ класы.

А. Бахыт, Н.Т. Тлеуханова

Задача классов множителей тригонометрических рядов Фурье в пространствах $\lambda_{p,q}$

В статье рассмотрены весовые пространства числовых последовательностей $\lambda_{p,q}$, которые определяются как множества последовательностей $a = \{a_k\}_{k=1}^{\infty}$, для которых конечна норма

$$\|a\|_{\lambda_{p,q}} := \left(\sum_{k=1}^{\infty} |a_k|^q k^{\frac{q}{p}-1}\right)^{\frac{1}{q}} < \infty.$$

В случае невозрастающих последовательностей норма пространства $\lambda_{p,q}$ совпадает с нормой классического пространтва Лоренца $l_{p,q}$. Получены необходимые и достаточные условия для вложений пространства $\lambda_{p,q}$ в пространство λ_{p_1,q_1}. Исследованы интерполяционные свойства этих пространств относительно вещественного интерполяционного метода. Показано, что шкала пространств $\lambda_{p,q}$ замкнута относительно вещественного интерполяционного метода, а также относительно комплексного интерполяционного метода. Получено описание двойственного пространства к весовому пространству $\lambda_{p,q}$, а именно: пространство рефлексивно, где p', q' сопряжены к параметрам p и q. Кроме того в статье изучены свойства оператора свертки в данных пространствах. Основным результатом данной работы является неравенство типа О'Нейла. Полученное неравенство обобщает классическое неравенство Юнга-О'Нейла. Метод исследования опирается на доказанные в этой работе интерполяционные теоремы для пространств $\lambda_{p,q}$.

Ключевые слова: тригонометрические коэффициенты Фурье, неравенство О'Нейла, оператор свертки, $M_{p_0,q_0}^{p_1,q_1}$ класс.

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On boundedness of the Hilbert transform on Marcinkiewicz spaces

We study boundedness properties of the classical (singular) Hilbert transform

$$(\mathcal{H}f)(t) = p.v.\frac{1}{\pi}\int\limits_{\mathbb{R}} \frac{f(s)}{t-s}ds$$

acting on Marcinkiewicz spaces. The Hilbert transform is a linear operator which arises from the study of boundary values of the real and imaginary parts of analytic functions. Questions involving the $\mathcal{H}$ arise therefore from the utilization of complex methods in Fourier analysis, for example. In particular, the $\mathcal{H}$ plays the crucial role in questions of norm-convergence of Fourier series and Fourier integrals. We consider the problem of what is the least rearrangement-invariant Banach function space $F(\mathbb{R})$ such that $\mathcal{H} : M_\phi(\mathbb{R}) \to F(\mathbb{R})$ is bounded for a fixed Marcinkiewicz space $M_\phi(\mathbb{R})$. We also show the existence of optimal rearrangement-invariant Banach function range on Marcinkiewicz spaces. We shall be referring to the space $F(\mathbb{R})$ as the optimal range space for the operator $\mathcal{H}$ restricted to the domain $M_\phi(\mathbb{R}) \subseteq \Lambda_{\varphi_0}(\mathbb{R})$. Similar constructions have been studied by J.Soria and P.Tradacete for the Hardy and Hardy type operators [1]. We use their ideas to obtain analogues of their some results for the $\mathcal{H}$ on Marcinkiewicz spaces.

Keywords: rearrangement-invariant Banach function space, Hilbert transform, Calderón operator, Marcinkiewicz space.

Introduction

The classical Hilbert transform $\mathcal{H}$(for measurable functions on $\mathbb{R}$) is given by the formula

$$(\mathcal{H}f)(t) = p.v.\frac{1}{\pi}\int\limits_{\mathbb{R}} \frac{f(s)}{t-s}ds \tag{1}$$

There is an uncountable number of papers devoted to research of the Hilbert transform defined by the formula (1). In 20th century, David Hilbert finally showed that the function $\sin(\omega t)$ is the Hilbert transform of $\cos(\omega t)$. After that, the Hilbert transform has been studied by many authors in different research areas of science. One of the important applications of the Hilbert transform in Interpolation theory and rearrangement invariant Banach function spaces has received a lot of attention since Boyd's pioneer work in 1966 [2] (see also [3], [4]), which is related to the main objective of this paper. The boundedness properties of some classical operators were studied in [5-8]. Also, the boundedness properties of the Hilbert transform were studied by many authors. For instance, [9], [10], [11] and recent papers [12-16], and references therein.

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Preliminaries

Let (I, m), where $I = \mathbb{R}_+ = (0, \infty)$(resp. $I = \mathbb{R}$) denote the measure space equipped with Lebesgue measure m. Let $L(I, m)$ be the space of all measurable real-valued functions on I equipped with Lebesgue measure m i.e. functions which coinside almost everywhere are considered identical. Define $L_0(I, m)$ to be the subset of $L(I, m)$ which consists of all functions f such that $m(\{t : |f(t)| > s\}) < \infty$ for some $s > 0$. For $f \in L_0(I)$ we denote $f^*(t)$ the decreasing rearrangement of the function $|f|$. That is,

$$f^*(t) = \inf\{s \geq 0 : m(|f| > s) \leq t\}, \ t > 0$$

Definition 1. [11; 49] A function φ defined on the semiaxis $[0, \infty)$ is said to be quasiconcave if
(i) $\varphi(t) = 0 \Leftrightarrow t = 0$.
(ii) $\varphi(t)$ is positive and increasing on $\mathbb{R}_+$.
(iii) $\frac{\varphi(t)}{t}$ is decreasing on $\mathbb{R}_+$.

Observe that every nonnegative function on $[0, \infty)$ that vanishes at origin is quasiconcave. The reverse, however, is not always true. However, we may replace, if necessary, a quasiconcave function φ by its least concave majorant $\widetilde{\varphi}$ such that

$$\frac{1}{2}\widetilde{\varphi} \leq \varphi \leq \widetilde{\varphi}$$

(see [10; 71]).

Definition 2. [10; 59] A Banach function space E is called rearrangement-invariant if, whenever f belongs to E and g is equimeasurable with f, then g also belongs to E and $\|f\|_E = \|g\|_E$.

Next we define the Köthe dual (or associate) space of rearrangement invariant Banach function spaces. Given rearrangement invariant Banach function space E on I, equipped with Lebesgue measure m the Köthe dual space $E^\times$ on I is defined by

$$E(I)^\times = \left\{ g \in L_0(I) : \int_I |f(t)g(t)|dt < \infty, \ \forall f \in E(I) \right\}.$$

$E^\times$ is a Banach space with the norm

$$\|g\|_{E(I)^\times} := \sup \left\{ \int_I |f(t)g(t)|dt : f \in E(I), \ \|f\|_{E(I)} \leq 1 \right\}.$$

If $E(I)$ is a rearrangement invariant Banach function space, then $(E^\times(I), \|\cdot\|_{E^\times(I)})$ is also rearrangement invariant Banach function space (cf. [9; Section 2.4]). For more details we refer to [10], [17].

Let Ω denote the set of increasing concave functions $\varphi : [0, \infty) \to [0, \infty)$ for which $\lim\limits_{t\to 0+} \varphi(t) = 0$(or simply $\varphi(+0) = 0$). For the function φ in Ω, the Lorentz space $\Lambda_\varphi(\mathbb{R}_+)$ is defined by setting

$$\Lambda_\varphi(\mathbb{R}_+) := \left\{ f \in L_0(I) : \int\limits_{\mathbb{R}_+} f^*(s)d\varphi(s) < \infty \right\}$$

and equipped with the norm

$$\|f\|_{\Lambda_\varphi(\mathbb{R}_+)} := \int\limits_{\mathbb{R}_+} f^*(s)d\varphi(s).$$

Let ψ be a quasiconcave function on $[0, \infty)$. Define the Marcinkiewicz space $M_\psi(I)$ by setting

$$M_\psi(I) := \Big\{ f \in L_0(I) : \|f\|_{M_\psi(\mathbb{R}_+)} < \infty \Big\}$$

equipped with the norm

$$\|f\|_{M_\psi(\mathbb{R}_+)} := \sup_{t>0} \frac{t}{\psi(t)} \int\limits_0^t f^*(s)ds.$$

These spaces are examples of rearrangement invariant Banach function space. For more information we refer [10], [11]. For more information we refer [10], [11]. The space $(L_1 + L_\infty)(\mathbb{R}_+) = L_1(\mathbb{R}_+) + L_\infty(\mathbb{R}_+)$ consists of functions which are sums of bounded measurable and summable functions $f \in L_0(\mathbb{R}_+)$ equipped with the norm given by

$$\|f\|_{(L_1+L_\infty)(\mathbb{R}_+)} = \inf \left\{ \|f_1\|_{L_1(\mathbb{R}_+)} + \|f_2\|_{L_\infty(\mathbb{R}_+)} : f = f_1 + f_2, f_1 \in L_1(\mathbb{R}_+), f_2 \in L_\infty(\mathbb{R}_+) \right\}$$

Define

$$\varphi_0(t) = \begin{cases} t\log(\frac{e^2}{t}), & 0 < t \leq 1 \\ 2\log(et), & 1 \leq t < \infty. \end{cases} \tag{2}$$

It is easy to show that φ_0 is a quasi-concave function on $[0, \infty)$. It was proved in [13; 5] that $\Lambda_{\varphi_0}(\mathbb{R}_+)$ is the maximal rearrangement invariant Banach function space such that $S : \Lambda_{\varphi_0}(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$ is bounded. For a function $f \in \Lambda_{\varphi_0}(\mathbb{R}_+)$ define the Calderón operator $S : \Lambda_{\varphi_0}(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$ as follows

$$(Sf)(t) := \frac{1}{t} \int\limits_0^t f(s)ds + \int\limits_t^\infty \frac{f(s)}{s} ds, \ t > 0.$$

Similarly, for a function $f \in \Lambda_{\varphi_0}(\mathbb{R})$, define the Hilbert transform as follows

$$(\mathcal{H}f)(t) = p.v. \frac{1}{\pi} \int\limits_{\mathbb{R}} \frac{f(s)}{t-s} ds.$$

For more details on these operators refer to [10], [11].

Main results

More general results for the Hilbert transform in quasi-Banach rearrangement invariant spaces were obtained in [12], [13]. In this work, we study the boundedness of the Hilbert transform on Marcinkiewicz spaces. The following is the main result of this paper.

Theorem 1. Let ϕ be an increasing concave function on $[0, \infty)$ such that $\phi(0) = 0$ and

$$\lim_{s\to\infty} \frac{\phi(s)}{s} = 0, \quad \lim_{s\to 0} \phi(s) \log(\frac{e}{s}) = 0. \tag{3}$$

Then

$$S : M_\phi(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$$

is bounded if and only if

$$tS(\frac{\phi(t)}{t}) < \infty, \ \forall t > 0.$$

Proof. Let

$$S : M_\phi(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+).$$

Since S is positive, it follows from Proposition 1.3.5 [18; 27] that $S : M_\phi(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$ is bounded. Then by duality and since $S = S^*$ (see Lemma 6 [14]), it follows that

$$S : (L_1 + L_\infty)^\times(\mathbb{R}_+) \to M_\phi^\times(\mathbb{R}_+)$$

is bounded. However, $(L_1 + L_\infty)^\times(\mathbb{R}_+) = (L_1 \cap L_\infty)(\mathbb{R}_+)$ and $M_\phi^\times(\mathbb{R}_+) = \Lambda_\phi(\mathbb{R}_+)$ (see [10], [11]). Hence,

$$S : (L_1 \cap L_\infty)(\mathbb{R}_+) \to \Lambda_\phi(\mathbb{R}_+) \tag{4}$$

is bounded.

Take $f = \chi_{(0,t)}$ and it is easy to show that $\chi_{(0,t)} \in (L_1 \cap L_\infty)(\mathbb{R}_+)$ for any $t > 0$. Therefore, it follows from (4) that $S\chi_{(0,t)} \in \Lambda_\phi(\mathbb{R}_+)$, that is,

$$\|S\chi_{(0,t)}\|_{\Lambda_\phi(\mathbb{R}_+)} < \infty. \tag{5}$$

But, the latter condition (5) is equivalent to $tS(\frac{\phi(t)}{t}) < \infty$ for any $t > 0$.

Indeed,

$$\begin{aligned}\|S\chi_{(0,t)}\|_{\Lambda_\phi(\mathbb{R}_+)} &= \int\limits_0^\infty S\chi_{(0,t)}(s)d\phi(s) = \int\limits_0^t (1 + \log(\frac{t}{s}))d\phi(s) + \int\limits_t^\infty \frac{t}{s}d\phi(s) \\ &= \left(\phi(s)\log(e\frac{t}{s})\right)\Big|_0^t + \int\limits_0^t \frac{\phi(s)}{s}ds + t\frac{\phi(s)}{s}\Big|_t^\infty + t\int\limits_t^\infty \frac{\phi(s)}{s^2}ds \\ &= t \cdot S(\frac{\phi(t)}{t}) + t\lim_{s\to\infty}\frac{\phi(s)}{s} - \lim_{s\to 0}\log\left(e\frac{t}{s}\right).\end{aligned}$$

Taking the assumptions (3) into account, we obtain the desired result.

Conversely, if $tS\left(\frac{\phi(t)}{t}\right) < \infty$, then, as we proved above, we have $\|S\chi_{(0,t)}\|_{\Lambda_\phi(\mathbb{R}_+)} < \infty$.

Let us show that $S : M_\phi(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$ is bounded.

Take $f \in M_\phi(\mathbb{R}_+)$. Then by formula (6.8) in [10; 76] and Hölder inequality (see [10; 9])

$$\begin{aligned}\|Sf\|_{L_1+L_\infty(\mathbb{R}_+)} &= \int\limits_0^1 (Sf)^*(s)ds \le \int\limits_0^1 (Sf^*)(s)ds = \int\limits_0^1 (Sf^*)(s)\chi_{(0,1)}(s)ds \\ &= \int\limits_0^\infty f^*(s)S\chi_{(0,1)}(s)ds \le \|f\|_{M_\phi(\mathbb{R}_+)}\|S\chi_{(0,1)}\|_{M_\phi^\times(\mathbb{R}_+)} \\ &= \|f\|_{M_\phi(\mathbb{R}_+)}\|S\chi_{(0,1)}\|_{\Lambda_\phi(\mathbb{R}_+)}.\end{aligned}$$

Since f is arbitrary and $\|S\chi_{(0,1)}\|_{\Lambda_\phi(\mathbb{R}_+)} < \infty$, the assertion follows.

Corollary 1. Let the assumptions of Theorem 1 hold. If $M_\phi(\mathbb{R}_+) \subset \Lambda_{\varphi_0}(\mathbb{R}_+)$, where φ_0 defined by (2), then there is a minimal rearrangement invariant Banach function space $F(\mathbb{R}_+)$ such that

$$S : M_\phi(\mathbb{R}_+) \to F(\mathbb{R}_+)$$

is bounded.

Proof. By Theorem 1,

$$S : M_\phi(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$$

is bounded if and only if $tS\left(\frac{\phi(t)}{t}\right) < \infty$ for any $t > 0$. As it was proved in Theorem 1, the latter condition is equivalent to $S\chi_{(0,t)} \in \Lambda_\phi(\mathbb{R}_+)$. Since $S = S^*$ and $\Lambda_\phi^\times(\mathbb{R}_+) = M_\phi(\mathbb{R}_+)$, it follows from

Proposition 3.9 [1; 876] that there is a minimal rearrangement invariant Banach function space $F(\mathbb{R}_+)$ such that

$$S : M_\phi(\mathbb{R}_+) \to F(\mathbb{R}_+)$$

is bounded.

Corollary 2. Let the assumptions of Corollary 1 hold. Then there is a minimal rearrangement invariant Banach function space $F(\mathbb{R})$ such that the Hilbert transform

$$\mathcal{H} : M_\phi(\mathbb{R}) \to F(\mathbb{R})$$

is bounded.

Proof. By Corollary 1, there is a rearrangement invariant Banach function space $F(\mathbb{R}_+)$ such that $S : M_\phi(\mathbb{R}_+) \to F(\mathbb{R}_+)$ is bounded. Hence, by Theorem 4.8 [10; 138], $\mathcal{H} : M_\phi(\mathbb{R}) \to F(\mathbb{R})$ is bounded. By assumming that $G(\mathbb{R})$ is another rearrangement invariant Banach function space such that $\mathcal{H} : M_\phi(\mathbb{R}) \to G(\mathbb{R})$ is bounded. Take $f \in M_\phi(\mathbb{R})$. Then $f^* \in M_\phi(\mathbb{R}_+)$. By [13, Lemma 5] there is a function $g \in M_\phi(\mathbb{R})$ with $f^* = g^*$ such that $Sf^*(t) \leq c_{abs}(\mathcal{H}g)^*(t),\ t > 0$, which shows $Sf^* \in G(\mathbb{R}_+)$. Since $f \in M_\phi(\mathbb{R})$ is arbitrary, it follows from the Corrollary 1 that $M_\phi(\mathbb{R}_+) \subset G(\mathbb{R}_+)$, i.e. $M_\phi(\mathbb{R}) \subset G(\mathbb{R})$.

This completes the proof.

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Н.Т. Бекбаев, Қ.С. Төленов

Марцинкевич кеңістіктеріндегі Гильберт түрлендіруінің шенелгендігі туралы

Мақалада Марцинкевич кеңістіктеріндегі функцияларға әсер ететін

$$(\mathcal{H}f)(t) = p.v. \frac{1}{\pi} \int\limits_{\mathbb{R}} \frac{f(s)}{t-s} ds$$

классикалық (сингулярлық) Гильберт түрлендіруінің шенелгендік қасиеттері зерттелді. Гильберт түрлендіруі аналитикалық функциялардың нақты және жорамал бөліктерінің шекаралық мәндерін зерттеу кезінде туындайтын сызықтық оператор болып табылады. Сондықтан $\mathcal{H}$ түрлендіруіне қатысты сұрақтар Фурье талдауының күрделі әдістерін қолданғанда пайда болады. Дербес жағдайда $\mathcal{H}$ түрлендіруі Фурье қатарлары мен Фурье интегралдарының норма бойынша жинақталуының мәселесінде шешуші рөл атқарады. Мақала авторлары бекітілген $M_\phi(\mathbb{R})$ Марцинкевич кеңістігі үшін $\mathcal{H} : M_\phi(\mathbb{R}) \to F(\mathbb{R})$ түрлендіруі шенелген болатындай ең кіші $F(\mathbb{R})$ алмастыру-инвариантты банах функциялар кеңістігі қандай болатыны туралы есепті қарастырды. Сонымен қатар, Марцинкевич кеңістіктерінде тиімді алмастыру-инвариантты банах функциялар жиыны бар болатынын көрсетті және $F(\mathbb{R})$ кеңістігі $M_\phi(\mathbb{R}) \subseteq \Lambda_{\varphi_0}(\mathbb{R})$ кеңістігіндегі Гильберт түрлендіруінің тиімді мәндер кеңістігі болатынын ескерді. Ұқсас есептер Дж.Сориа мен П.Традацет жұмысында Харди және Харди типтес операторлар үшін зерттелген [1]. Авторлар сол ғалымдардың кейбір нәтижелерінің аналогын Марцинкевич кеңістіктеріндегі Гильберт түрлендіруі үшін қолданған.

Кілт сөздер: алмастыру-инвариантты банах кеңістігі, Гильберт түрлендіруі, Кальдерон операторы, Марцинкевич кеңістігі.

Н.Т. Бекбаев, К.С. Туленов

Об ограниченности преобразования Гильберта в пространствах Марцинкевича

В статье исследованы свойства ограниченности классического (сингулярного) преобразования Гильберта

$$(\mathcal{H}f)(t) = p.v. \frac{1}{\pi} \int\limits_{\mathbb{R}} \frac{f(s)}{t-s} ds,$$

действующего на пространствах Марцинкевича. Преобразование Гильберта является линейным оператором, возникающим при изучении граничных значений вещественной и мнимой частей аналитических функций. Поэтому вопросы, связанные с $\mathcal{H}$, возникают из-за использования сложных методов в анализе Фурье. В частности, $\mathcal{H}$ играет решающую роль в вопросах сходимости по норме рядов и интегралов Фурье. Авторами статьи рассмотрен вопрос о том, каково будет минимальное перестановочно-инвариантное пространство банаховых функций $F(\mathbb{R})$ для того, чтобы $\mathcal{H} : M_\phi(\mathbb{R}) \to F(\mathbb{R})$ был ограниченным в фиксированном пространстве Марцинкевича $M_\phi(\mathbb{R})$. Кроме того, показаны существование перестановочно-инвариантной области значений функций в пространствах Марцинкевича, ссылка на пространство $F(\mathbb{R})$ как оптимальное пространство для ограниченного оператора $\mathcal{H}$ в области определения $M_\phi(\mathbb{R}) \subseteq \Lambda_{\varphi_0}(\mathbb{R})$. Подобные конструкции были изучены Дж. Сориа и П.Традацетом для операторов Харди и типа Харди [1]. Авторами использованы их идеи для получения аналогов некоторых их результатов для оператора $\mathcal{H}$ в пространствах Марцинкевича.

Ключевые слова: перестановочно-инвариантное банахово пространство, преобразование Гильберта, оператор Кальдерона, пространство Марцинкевича.

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Interpolation theorem for Nikol'skii-Besov type spaceswith mixed metric

In this paper we study the interpolation properties of Nikol'skii-Besov spaces with a dominant mixed derivative and mixed metric with respect to anisotropic and complex interpolation methods. An interpolation theorem is proved for a weighted discrete space of vector-valued sequences $l_{\mathbf{q}}^{\alpha}(A)$. It is shown that the Nikol'skii-Besov space under study is a retract of the space $l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}})$. Based on the above results, interpolation theorems were obtained for Nikol'skii-Besov spaces with the dominant mixed derivative and mixed metric.

Keywords: Nikol'skii-Besov type spaces, method of anisotropic interpolation, complex interpolation.

Introduction

The embedding theorems for spaces of differentiable functions play an important role in the study of boundary value problems for equations of mathematical physics and approximation theory. At the same time, interpolation of smooth function spaces is of great interest.

The interpolation of the Sobolev and Besov spaces was first studied by J. Petre [1], J.-L. Lions and J. Petre [2]. Further results on interpolation of spaces of smooth functions with respect to the classical real and complex methods can be found in the monographs of J. Berg and J. Lofstrom [3], H. Tribel [4]. In V.L. Krepkogorskii's papers [5], [6], I. Asecritova's and others [7] interpolation properties of Besov and Lizorkin-Triebel spaces were studied with respect to Sparr's method. E.D. Nursultanov and K.A. Bekmaganbetov considered interpolation properties of Besov spaces with respect to a method of multiparametric interpolation (see [8]). They also considered interpolation properties of classical Besov and Lizorkin-Triebel spaces with respect to an anisotropic interpolation method (see [9], [10]). In works [11]– [13] E.D. Nursultanov, K.A. Bekmaganbetov and Ye. Toleugazy considered interpolation properties of Besov spaces with dominant mixed derivative with anisotropic and mixed metric. The use of interpolation theorems for receiving embedding theorems and their further applications in approximation theory is shown in works [14]– [16].

In this paper we study the interpolation properties of Nikol'skii-Besov spaces with a dominant mixed derivative and mixed metric with respect to anisotropic and complex interpolation methods.

An interpolation theorem is proved for a weighted discrete space of vector-valued sequences $l_{\mathbf{q}}^{\alpha}(A)$. It is shown that the Nikol'skii-Besov space under study is a retract of the space $l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}})$. Based on the above results, interpolation theorems were obtained for Nikol'skii-Besov spaces with the dominant mixed derivative and mixed metric.

Preliminaries and auxiliary results

Let $E = \{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) : \varepsilon_i = 0 \text{ or } \varepsilon_i = 1, i = 1, \ldots, n\}$ be the set of vertices of an n-dimensional unit cube in $\mathbb{R}^n$, $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon \in E}$ is a set of Banach spaces that are subspaces of some linear Hausdorff

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space, which is called a compatible set of Banach spaces [3]. For element a from space $\sum_{\varepsilon\in E} A_\varepsilon$ we define

$$K(\mathbf{t}, a; \mathbf{A}) = \inf_{a=\sum_{\varepsilon\in E} a_\varepsilon} \sum_{\varepsilon\in E} \mathbf{t}^\varepsilon \|a_\varepsilon\|_{A_\varepsilon},$$

where $\mathbf{t}^\varepsilon = t_1^{\varepsilon_1} \cdot \ldots \cdot t_n^{\varepsilon_n}$.

Let $\mathbf{0} < \theta = (\theta_1, \ldots, \theta_n) < \mathbf{1}$, $\mathbf{0} < \mathbf{r} = (r_1, \ldots, r_n) \leq \infty$. By $\mathbf{A}_{\theta\mathbf{r}} = (A_\varepsilon; \varepsilon \in E)_{\theta\mathbf{r}}$ we denote the linear subspace of $\sum_{\varepsilon\in E} A_\varepsilon$ such that for its elements the following condition holds:

$$\|a\|_{\mathbf{A}_{\theta\mathbf{r}}} = \left(\int_{\mathbb{R}_+^n} \left(\mathbf{t}^{-\theta} K(\mathbf{t}, a; \mathbf{A})\right)^{\mathbf{r}} \frac{\mathbf{dt}}{\mathbf{t}}\right)^{\mathbf{1/r}} =$$

$$= \left(\int_0^\infty \left(t_n^{-\theta_n} \ldots \left(\int_0^\infty \left(t_1^{-\theta_1} K(\mathbf{t}, a; \mathbf{A})\right)^{r_1} \frac{dt_1}{t_1}\right)^{r_2/r_1} \ldots\right)^{r_n/r_{n-1}} \frac{dt_n}{t_n}\right)^{1/r_n} < \infty.$$

Lemma 1 ([9]). Let $\mathbf{0} < \theta < \mathbf{1}$, $\mathbf{0} < \mathbf{r} \leq \infty$ and let $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon\in E}$ and $\mathbf{B} = \{B_\varepsilon\}_{\varepsilon\in E}$ be two compatible sets of Banach spaces. Suppose that for a linear operator $T : \mathbf{A}_\varepsilon \to \mathbf{B}_\varepsilon$ there are two vectors $\mathbf{M_0} = (M_1^0, \ldots, M_n^0), \mathbf{M_1} = (M_1^1, \ldots, M_n^1)$ with positive components such that $\|T\|_{\mathbf{A}_\varepsilon\to\mathbf{B}_\varepsilon} \leq$ $\leq C_\varepsilon \prod_{i=1}^n M_i^{\varepsilon_i}$ for any $\varepsilon \in E$, then

$$T : \mathbf{A}_{\theta\mathbf{r}} \to \mathbf{B}_{\theta\mathbf{r}}$$

with estimate $\|T\|_{\mathbf{A}_{\theta\mathbf{r}}\to\mathbf{B}_{\theta\mathbf{r}}} \leq \max\limits_{\varepsilon\in E} C_\varepsilon \prod\limits_{i=1}^n \left(M_i^0\right)^{1-\theta_i} \left(M_i^1\right)^{\theta_i}$.

Lemma 2 ([4]). Let $\alpha_1 < \alpha < \alpha_0$ and $1 \leq q \leq \infty$. For a sequence of non-negative numbers $\{a_k\}_{k\in\mathbb{Z}}$ define transformations

$$I_0(a; j) = \sum_{k=-\infty}^{j} 2^{\alpha_0(k-j)} a_k,$$

$$I_1(a; j) = \sum_{k=j+1}^{\infty} 2^{\alpha_1(k-j)} a_k.$$

Then the following inequalities hold:

$$\left(\sum_{j=-\infty}^{\infty} \left(2^{\alpha j} I_0(a; j)\right)^q\right)^{1/q} \leq C_1 \left(\sum_{j=-\infty}^{\infty} \left(2^{\alpha j} a_j\right)^q\right)^{1/q}, \tag{1}$$

$$\left(\sum_{j=-\infty}^{\infty} \left(2^{\alpha j} I_1(a; j)\right)^q\right)^{1/q} \leq C_2 \left(\sum_{j=-\infty}^{\infty} \left(2^{\alpha j} a_j\right)^q\right)^{1/q}. \tag{2}$$

For multi-indices $\mathbf{b}_0 = (b_1^0, \ldots, b_n^0), \mathbf{b}_1 = (b_1^1, \ldots, b_n^1)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E$ we introduce the notation $\mathbf{b}_\varepsilon = (b_1^{\varepsilon_1}, \ldots, b_n^{\varepsilon_n})$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $\mathbf{1} \leq \mathbf{q} = (q_1, \ldots, q_n) \leq \infty$ and A be a Banach space. By $l_{\mathbf{q}}^\alpha(A)$ we denote the set of multi-sequences $\{a_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^n}$ with values in A for which the norm

$$\|a\|_{l_{\mathbf{q}}^\alpha(A)} = \left(\sum_{\mathbf{k}\in\mathbb{Z}^n} \left(2^{(\alpha,\mathbf{k})} \|a_{\mathbf{k}}\|_A\right)^{\mathbf{q}}\right)^{\mathbf{1/q}} =$$

$$= \left(\sum_{k_n=-\infty}^{+\infty} \left(\dots \left(\sum_{k_1=-\infty}^{+\infty} \left(2^{\sum_{i=1}^{n} \alpha_i k_i} \left\| a_{k_1,\dots,k_n} \right\|_A \right)^{q_1} \right)^{q_2/q_1} \dots \right)^{q_n/q_{n-1}} \right)^{1/q_n}$$

is finite, here $(\alpha, \mathbf{k}) = \sum_{i=1}^{n} \alpha_i k_i$ is the inner product.

Lemma 3. Let $\alpha_0 = (\alpha_1^0, \ \dots, \ \alpha_n^0) \neq \alpha_1 = (\alpha_1^1, \ \dots, \ \alpha_n^1)$, $\mathbf{1} \leq \mathbf{q_0} = (q_1^0, \ \dots, \ q_n^0)$, $\mathbf{q_1} = (q_1^1, \ \dots, \ q_n^1) \leq \infty$. Then for $\mathbf{0} < \theta = (\theta_1, \dots, \theta_n) < \mathbf{1}$, $\mathbf{1} \leq \mathbf{q} = (q_1, \dots, q_n)\infty$ the equality

$$\left(l_{\mathbf{q}_\varepsilon}^{\alpha_\varepsilon}(A); \varepsilon \in E \right)_{\theta \mathbf{q}} = l_{\mathbf{q}}^{\alpha}(A)$$

holds, here $\alpha = (\mathbf{1} - \theta)\alpha_{\mathbf{0}} + \theta \alpha_{\mathbf{1}}$.

Proof. Without loss of generality, we can assume that $\alpha_0 = (\alpha_1^0, \dots, \alpha_n^0) > \alpha_1 = (\alpha_1^1, \dots, \alpha_n^1)$. Due to the embeddings

$$l_{\mathbf{1}}^{\alpha}(A) \hookrightarrow l_{\mathbf{q}}^{\alpha}(A) \hookrightarrow l_{\infty}^{\alpha}(A)$$

it is enough to prove the embeddings

$$(l_{\infty}^{\alpha_\varepsilon}(A); \varepsilon \in E)_{\theta \mathbf{q}} \hookrightarrow l_{\mathbf{q}}^{\alpha}(A) \tag{3}$$

and

$$l_{\mathbf{q}}^{\alpha}(A) \hookrightarrow (l_{\mathbf{1}}^{\alpha_\varepsilon}(A); \varepsilon \in E)_{\theta \mathbf{q}}, \tag{4}$$

where $\alpha = (\mathbf{1} - \theta)\alpha_{\mathbf{0}} + \theta \alpha_{\mathbf{1}}$.

First we prove the embedding (3). If $a = \{a_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^n} \in l_{\infty}^{\alpha_{\mathbf{1}}}(A)$, then

$$K(\mathbf{t}, a; l_{\infty}^{\alpha_\varepsilon}(A); \varepsilon \in E) = \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sum_{\varepsilon \in E} \mathbf{t}^{\varepsilon} \left\| a_\varepsilon \right\|_{l_{\infty}^{\alpha_\varepsilon}(A)} =$$

$$= \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sum_{\varepsilon \in E} \mathbf{t}^{\varepsilon} \sup_{\mathbf{k} \in \mathbb{Z}^n} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \left\| a_{\mathbf{k}}^{(\varepsilon)} \right\|_A = \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sum_{\varepsilon \in E} \sup_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \left\| a_{\mathbf{k}}^{(\varepsilon)} \right\|_A \geq$$

$$\geq \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sum_{\varepsilon \in E} \sup_{\mathbf{k} \in \mathbb{Z}^n} \min_{\varepsilon \in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \right) \left\| a_{\mathbf{k}}^{(\varepsilon)} \right\|_A =$$

$$= \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sup_{\mathbf{k} \in \mathbb{Z}^n} \min_{\varepsilon \in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \right) \sum_{\varepsilon \in E} \left\| a_{\mathbf{k}}^{(\varepsilon)} \right\|_A \geq$$

$$\geq \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sup_{\mathbf{k} \in \mathbb{Z}^n} \min_{\varepsilon \in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \right) \left\| \sum_{\varepsilon \in E} a_{\mathbf{k}}^{(\varepsilon)} \right\|_A =$$

$$= \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sup_{\mathbf{k} \in \mathbb{Z}^n} \min_{\varepsilon \in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \right) \left\| a_{\mathbf{k}} \right\|_A = \sup_{\mathbf{k} \in \mathbb{Z}^n} \min_{\varepsilon \in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \right) \left\| a_{\mathbf{k}} \right\|_A.$$

Since $\alpha_0 > \alpha_1$, then $\mathbb{R}_+^n$ can be divided into parallelepipeds of the form $\left[2^{(\alpha_0 - \alpha_1)(\mathbf{j} - \mathbf{1})}; 2^{(\alpha_0 - \alpha_1)\mathbf{j}} \right)$, $\mathbf{j} \in \mathbb{Z}^n$. Then

$$\|a\|_{(l_{\infty}^{\alpha_\varepsilon}(A); \varepsilon \in E)_{\theta \mathbf{q}}} = \left(\int_{\mathbb{R}_+^n} \left(\mathbf{t}^{-\theta} K(\mathbf{t}, a; l_{\infty}^{\alpha_\varepsilon}(A); \varepsilon \in E) \right)^{\mathbf{q}} \frac{\mathbf{dt}}{\mathbf{t}} \right)^{\mathbf{1}/\mathbf{q}} \geq$$

$$\geq \left(\int_{\mathbb{R}_+^n} \left(\mathbf{t}^{-\theta} \sup_{\mathbf{k} \in \mathbb{Z}^n} \min_{\varepsilon \in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \frac{\mathbf{dt}}{\mathbf{t}} \right)^{\mathbf{1}/\mathbf{q}} =$$

$$= \left(\sum_{\mathbf{j}\in\mathbb{Z}^n} \int_{2^{(\alpha_0-\alpha_1)(\mathbf{j}-\mathbf{1})}}^{2^{(\alpha_0-\alpha_1)\mathbf{j}}} \left(\mathbf{t}^{-\theta} \sup_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k}\rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \frac{\mathbf{dt}}{\mathbf{t}} \right)^{\mathbf{1/q}} \geq$$

$$\geq C_1 \left(\sum_{\mathbf{j}\in\mathbb{Z}^n} \left(2^{-\langle \theta(\alpha_0-\alpha_1), \mathbf{j}\rangle} \sup_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(2^{\langle \varepsilon(\alpha_0-\alpha_1), \mathbf{j}\rangle + \langle \alpha_\varepsilon, \mathbf{k}\rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1/q}} =$$

$$= C_1 \left(\sum_{\mathbf{j}\in\mathbb{Z}^n} \left(\sup_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(2^{\langle (\varepsilon-\theta)(\alpha_0-\alpha_1), \mathbf{j}\rangle + \langle \alpha_\varepsilon, \mathbf{k}\rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1/q}} =$$

$$= C_1 \left(\sum_{\mathbf{j}\in\mathbb{Z}^n} \left(\sup_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(2^{\langle (\varepsilon-\theta)(\alpha_0-\alpha_1)+\alpha_\varepsilon, \mathbf{j}\rangle + \langle \alpha_\varepsilon, (\mathbf{k}-\mathbf{j})\rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1/q}}.$$

Since for any $\varepsilon \in E$ the equality

$$(\varepsilon-\theta)(\alpha_0-\alpha_1)+\alpha_\varepsilon = (\varepsilon-\theta)(\alpha_0-\alpha_1)+(\mathbf{1}-\varepsilon)\alpha_0+\varepsilon\alpha_1 =$$

$$= (\mathbf{1}-\theta)\alpha_0+\theta\alpha_1 = \alpha$$

holds, then we get

$$\|a\|_{(l_\infty^{\alpha_\varepsilon}(A);\varepsilon\in E)_{\theta\mathbf{q}}} \geq C_1 \left(\sum_{\mathbf{j}\in\mathbb{Z}^n} \left(\sup_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(2^{\langle \alpha, \mathbf{j}\rangle + \langle \alpha_\varepsilon, (\mathbf{k}-\mathbf{j})\rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1/q}} =$$

$$= C_1 \left(\sum_{\mathbf{j}\in\mathbb{Z}^n} \left(2^{\langle \alpha, \mathbf{j}\rangle} \sup_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(2^{\langle \alpha_\varepsilon, (\mathbf{k}-\mathbf{j})\rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1/q}} \geq$$

$$\geq C_1 \left(\sum_{\mathbf{j}\in\mathbb{Z}^n} \left(2^{\langle \alpha, \mathbf{j}\rangle} \|a_{\mathbf{j}}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1/q}} = C_1 \|a\|_{l_{\mathbf{q}}^{\alpha}(A)}.$$

The last inequality means the embedding (3).

Next, we prove the embedding (4). Let $a = \{a_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^n} \in l_{\mathbf{q}}^{\alpha}(A)$. We have

$$K(\mathbf{t}, a; l_{\mathbf{1}}^{\alpha_\varepsilon}(A); \varepsilon\in E) = \inf_{a=\sum_{\varepsilon\in E} a_\varepsilon} \sum_{\varepsilon\in E} \mathbf{t}^{\varepsilon} \|a_\varepsilon\|_{l_{\mathbf{1}}^{\alpha_\varepsilon}(A)} =$$

$$= \inf_{a=\sum_{\varepsilon\in E} a_\varepsilon} \sum_{\varepsilon\in E} \mathbf{t}^{\varepsilon} \sum_{\mathbf{k}\in\mathbb{Z}^n} 2^{\langle \alpha_\varepsilon, \mathbf{k}\rangle} \left\| a_{\mathbf{k}}^{(\varepsilon)} \right\|_A = \inf_{a=\sum_{\varepsilon\in E} a_\varepsilon} \sum_{\varepsilon\in E} \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k}\rangle} \left\| a_{\mathbf{k}}^{(\varepsilon)} \right\|_A \leq$$

$$\leq \sum_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k}\rangle} \right) \|a_{\mathbf{k}}\|_A,$$

here we put $a_{\mathbf{k}}^{(\varepsilon)} = a_{\mathbf{k}}$ for ε that corresponds to $\min_{\varepsilon\in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k}\rangle} \right)$.

As in the proof of (3), we obtain

$$\|a\|_{(l_{\mathbf{1}}^{\alpha_\varepsilon};\varepsilon\in E)_{\theta\mathbf{q}}} = \left(\int_{\mathbb{R}_+^n} \left(\mathbf{t}^{-\theta} K(\mathbf{t}, a; l_{\mathbf{1}}^{\alpha_\varepsilon}(A); \varepsilon\in E) \right)^{\mathbf{q}} \frac{\mathbf{dt}}{\mathbf{t}} \right)^{\mathbf{1/q}} \leq$$

$$\left(\int_{\mathbb{R}^n_+}\left(\mathbf{t}^{-\theta}\sum_{\mathbf{k}\in\mathbb{Z}^n}\min_{\varepsilon\in E}\left(\mathbf{t}^{\varepsilon}2^{\langle\alpha_\varepsilon,\mathbf{k}\rangle}\right)\|a_{\mathbf{k}}\|_A\right)^{\mathbf{q}}\frac{\mathbf{dt}}{\mathbf{t}}\right)^{\mathbf{1/q}}=$$

$$\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\int_{2^{(\alpha_0-\alpha_1)(\mathbf{j-1})}}^{2^{(\alpha_0-\alpha_1)\mathbf{j}}}\left(\mathbf{t}^{-\theta}\sum_{\mathbf{k}\in\mathbb{Z}^n}\min_{\varepsilon\in E}\left(\mathbf{t}^{\varepsilon}2^{\langle\alpha_\varepsilon,\mathbf{k}\rangle}\right)\|a_{\mathbf{k}}\|_A\right)^{\mathbf{q}}\frac{\mathbf{dt}}{\mathbf{t}}\right)^{\mathbf{1/q}}\le$$

$$C_2\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{-\langle\theta(\alpha_0-\alpha_1),\mathbf{j}\rangle}\sum_{\mathbf{k}\in\mathbb{Z}^n}\min_{\varepsilon\in E}\left(2^{\langle\varepsilon(\alpha_0-\alpha_1),\mathbf{j}\rangle+\langle\alpha_\varepsilon,\mathbf{k}\rangle}\right)\|a_{\mathbf{k}}\|_A\right)^{\mathbf{q}}\right)^{\mathbf{1/q}}=$$

$$C_2\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{\langle\alpha-\alpha_0,\mathbf{j}\rangle}\sum_{\mathbf{k}\in\mathbb{Z}^n}\min_{\varepsilon\in E}\left(2^{\langle\varepsilon(\alpha_0-\alpha_1),\mathbf{j}\rangle+\langle\alpha_\varepsilon,\mathbf{k}\rangle}\right)\|a_{\mathbf{k}}\|_A\right)^{\mathbf{q}}\right)^{\mathbf{1/q}}=$$

$$C_2\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{\langle\alpha,\mathbf{j}\rangle}\sum_{\mathbf{k}\in\mathbb{Z}^n}\min_{\varepsilon\in E}\left(2^{\langle\alpha_\varepsilon,\mathbf{k-j}\rangle}\right)\|a_{\mathbf{k}}\|_A\right)^{\mathbf{q}}\right)^{\mathbf{1/q}}=$$

$$C_2\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{\langle\alpha,\mathbf{j}\rangle}\sum_{\varepsilon\in E}I_\varepsilon(\|a\|_A;\mathbf{j})\right)^{\mathbf{q}}\right)^{\mathbf{1/q}}\le C_2\sum_{\varepsilon\in E}\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{\langle\alpha,\mathbf{j}\rangle}I_\varepsilon(\|a\|_A;\mathbf{j})\right)^{\mathbf{q}}\right)^{\mathbf{1/q}},$$

where $I_\varepsilon(\|a\|_A;\mathbf{j})=I_{\varepsilon_n}(\ldots I_{\varepsilon_1}(\|a\|_A;\mathbf{j}))$ is a composition of transformations from Lemma 2.

Further, using the Minkowski inequalities, (1) and (2) we obtain

$$\|a\|_{\left(l_{\mathbf{1}}^{\alpha_\varepsilon};\varepsilon\in E\right)_{\theta\mathbf{q}}}\le C_2\sum_{\varepsilon\in E}\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{\langle\alpha,\mathbf{j}\rangle}I_\varepsilon(\|a\|_A;\mathbf{j})\right)^{\mathbf{q}}\right)^{\mathbf{1/q}}=$$

$$=C_2\sum_{\varepsilon\in E}\left(\sum_{j_n=-\infty}^{\infty}\left(2^{\alpha_n j_n}\ldots\left(\sum_{j_1=-\infty}^{\infty}\left(2^{\alpha_1 j_1}I_{\varepsilon_n}(..I_{\varepsilon_1}(\|a\|_A;\mathbf{j}))\right)^{q_1}\right)^{q_2/q_1}\ldots\right)^{q_n/q_{n-1}}\right)^{1/q_n}\le$$

$$\le C_2\sum_{\varepsilon\in E}\left(\sum_{j_n=-\infty}^{\infty}\left(2^{\alpha_n j_n}I_{\varepsilon_n}\left(\ldots\left(\sum_{j_1=-\infty}^{\infty}\left(2^{\alpha_1 j_1}I_{\varepsilon_1}(\|a\|_A;\mathbf{j})\right)^{q_1}\right)^{q_2/q_1}\ldots\right)\right)^{q_n}\right)^{1/q_n}\le$$

$$\le C_3\sum_{\varepsilon\in E}\left(\sum_{j_n=-\infty}^{\infty}\left(2^{\alpha_n j_n}\ldots\left(\sum_{j_1=-\infty}^{\infty}\left(2^{\alpha_1 j_1}\|a_{j_1,\ldots,j_n}\|_A\right)^{q_1}\right)^{q_2/q_1}\ldots\right)^{q_n/q_{n-1}}\right)^{1/q_n}=$$

$$=C_4\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{\langle\alpha,\mathbf{j}\rangle}\|a_{\mathbf{j}}\|_A\right)^{\mathbf{q}}\right)^{\mathbf{1/q}}=C_4\|a\|_{l_{\mathbf{q}}^{\alpha}(A)}.$$

The last inequality means the embedding (4).

The lemma is completely proved. □

Nikol'skii-Besov type spaces and their interpolation

Let multi-index $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, $\mathbb{T}^{\mathbf{d}} = \{\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) : \mathbf{x}_i = (x_1^i, \ldots, x_{d_i}^i) \in [0, 2\pi)^{d_i}$, $i = 1, \ldots, n\}$. Let $f(\mathbf{x}) = f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be measurable function on $\mathbb{T}^{\mathbf{d}}$.

Further, let $\mathbf{1} \leq \mathbf{p} = (p_1, \ldots, p_n) \leq \infty$. The Lebesgue space $L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})$ with a mixed metric is a set of functions for which the following expression is finite

$$\|f\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} = \left(\int_{\mathbb{T}^{d_n}} \left(\ldots \left(\int_{\mathbb{T}^{d_1}} |f(\mathbf{x}_1, \ldots, \mathbf{x}_n)|^{p_1} d\mathbf{x}_1 \right)^{p_2/p_1} \ldots \right)^{p_n/p_{n-1}} d\mathbf{x}_n \right)^{1/p_n}.$$

Here, the expression $\left(\int_{\mathbb{T}^{d_i}} |f(\mathbf{x})|^{p_i} d\mathbf{x}_i\right)^{1/p_i}$ for $p_i{=}\infty$ we undestand as $\sup_{x \in \mathbb{T}^{d_i}} |f(\mathbf{x})|$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $\mathbf{1} \leq \mathbf{q} = (q_1, \ldots, q_n) \leq \infty$, $\mathbf{1} \leq \mathbf{p} = (p_1, \ldots, p_n) < \infty$.

For trigonometric series $f \sim \sum_{\mathbf{k} \in \mathbb{Z}^{\mathbf{d}}} a_{\mathbf{k}} e^{i\langle \mathbf{k}, \mathbf{x} \rangle}$ denote by

$$\Delta_{\mathbf{s}}(f, \mathbf{x}) = \sum_{\mathbf{k} \in \rho(\mathbf{s})} a_{\mathbf{k}} e^{i\langle \mathbf{k}, \mathbf{x} \rangle},$$

where $\langle \mathbf{k}, \mathbf{x} \rangle = \sum_{i=1}^{n} \sum_{j=1}^{d_i} k_j^i x_j^i$ is the inner product, $\rho(\mathbf{s}) = \{\mathbf{k} = (\mathbf{k}_1, \ldots, \mathbf{k}_n) \in \mathbb{Z}^{\mathbf{d}} : \left[2^{s_i - 1}\right] \leq$ $\leq \max_{j=1,\ldots,d_i} |k_j^i| < 2^{s_i}, i = 1, \ldots, n\}$.

The Nikol'skii-Besov space $B_{\mathbf{p}}^{\alpha \mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ with a mixed metric is the set of trigonometric series $f \sim \sum_{\mathbf{k} \in \mathbb{Z}^{\mathbf{d}}} a_{\mathbf{k}} e^{i\langle \mathbf{k}, \mathbf{x} \rangle}$, for which the norm

$$\|f\|_{B_{\mathbf{p}}^{\alpha \mathbf{q}}(\mathbb{T}^{\mathbf{d}})} = \left\| \left\{ \mathbf{2}^{\langle \alpha, \mathbf{s} \rangle} \|\Delta_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \right\}_{\mathbf{s} \in \mathbb{Z}_+^{\mathbf{n}}} \right\|_{l_{\mathbf{q}}}$$

is finite, where $\| \cdot \|_{l_{\mathbf{q}}}$ is the norm of a discrete Lebesgue space $l_{\mathbf{q}}$ with a mixed metric.

Definition 1. Let A and B be Banach spaces. An operator $R \in L(A, B)$ is called a retraction if there exists an operator $S \in L(B, A)$ such

$$RS = E \quad \text{(identity operator from } L(B, B)) \, .$$

Moreover, the operator S is called the coretraction (corresponding to R).

Theorem 1. Let $-\infty < \alpha = (\alpha_1, \ldots, \alpha_n) < \infty$, $\mathbf{1} < \mathbf{p} = (p_1, \ldots, p_n) < \infty$ and $\mathbf{1} \leq \mathbf{q} =$ $= (q_1, \ldots, q_n) \leq \infty$. Then the space $B_{\mathbf{p}}^{\alpha \mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ is a retraction of the space $l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))$.

Proof. We prove first the S-property. For the function $f \in B_{\mathbf{p}}^{\alpha \mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ we define the operator S as follows

$$Sf = \{\Delta_{\mathbf{s}}(f, \mathbf{x})\}_{\mathbf{s} \in \mathbb{Z}_+^n} = \{(\Delta_{\mathbf{s}} * f)(\mathbf{x})\}_{\mathbf{s} \in \mathbb{Z}_+^n},$$

here $\Delta_{\mathbf{s}}(\mathbf{x})$ is the Dirichlet kernel corresponding to the block $\rho(\mathbf{s})$.

Then, by definition, we have

$$\|Sf\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))} = \|\{\Delta_{\mathbf{s}}(f, \cdot)\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))} = \|f\|_{B_{\mathbf{p}}^{\alpha \mathbf{q}}(\mathbb{T}^{\mathbf{d}})},$$

which means the fulfillment of the S-property.

Next, we check the fulfillment of the R-property. For the sequence $f = \{f_{\mathbf{s}}(\mathbf{x})\}_{\mathbf{s} \in \mathbb{Z}_+^n}$ we define the operator

$$Rf = \sum_{\mathbf{s} \in \mathbb{Z}_+^n} (\Delta_{\mathbf{s}} * f_{\mathbf{s}})(\mathbf{x}).$$

Then, according to the inequality of M. Riesz about the boundedness of parallelepipedal partial sums, we obtain

$$\|\Delta_{\mathbf{m}} * f\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \leq C \|f\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})},$$

where C is the absolute constant, and further

$$\|Rf\|_{B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})} = \|\{(\Delta_{\mathbf{s}} * f_{\mathbf{s}}) * \Delta_{\mathbf{s}}\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))} =$$

$$= \|\{\Delta_{\mathbf{s}} * f_{\mathbf{s}}\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))} \leq C \|\{f_{\mathbf{s}}\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))} = C \|f\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))}.$$

The last inequality means the fulfillment of the R-property.

It remains to show that $RS = E$. Indeed,

$$RSf(\mathbf{x}) = R\left(\{\Delta_{\mathbf{s}}(f, \mathbf{x})\}\right) = \sum_{\mathbf{s}\in\mathbb{Z}_+^n} \left(\Delta_{\mathbf{s}}(f, \mathbf{x}) * \Delta_{\mathbf{s}}(\mathbf{x})\right) =$$

$$= \sum_{\mathbf{s}\in\mathbb{Z}_+^n} \left((S_{2^{\mathbf{s}}} - S_{2^{\mathbf{s}-\mathbf{1}}}) * f\right)(\mathbf{x}) = f(\mathbf{x}).$$

The theorem is completely proved. □

Theorem 2. Let $\mathbf{1} < \mathbf{p} = (p_1, \ldots, p_n) < \infty$, $\alpha_{\mathbf{0}} = (\alpha_1^0, \ldots, \alpha_n^0) \neq \alpha_{\mathbf{1}} = (\alpha_1^1, \ldots, \alpha_n^1)$, $\mathbf{1} \leq \mathbf{q_0} = (q_1^0, \ldots, q_n^0), \mathbf{q_1} = (q_1^1, \ldots, q_n^1) \leq \infty$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E$. Then for $\mathbf{0} < \theta = (\theta_1, \ldots, \theta_n) < \mathbf{1}$ and $\mathbf{1} \leq \mathbf{q} = (q_1, \ldots, q_n) \leq \infty$ the equality

$$\left(B_{\mathbf{p}}^{\alpha_\varepsilon \mathbf{q}_\varepsilon}(\mathbb{T}^{\mathbf{d}}); \varepsilon \in E\right)_{\theta\mathbf{q}} = B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$$

holds, where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

Proof. follows from Theorem 1 and the Lemma 3. □

Remark 1. In the case $\mathbf{d} = (1, \cdots, 1)$, the result of Theorem 2 was announced by E.D. Nursultanov in [9].

Theorem 3. Let $\mathbf{1} < \mathbf{p_0} = (p_1^0, \ldots, p_n^0), \mathbf{p_1} = (p_1^1, \ldots, p_n^1) < +\infty$, $\alpha_{\mathbf{0}} = (\alpha_1^0, \ldots, \alpha_n^0) \neq \alpha_{\mathbf{1}} = (\alpha_1^1, \ldots, \alpha_n^1)$, $\mathbf{1} \leq \mathbf{q_0} = (q_1^0, \ldots, q_n^0), \mathbf{q_1} = (q_1^1, \ldots, q_n^1) \leq \infty$. Then for $0 < \theta < 1$ the equality

$$\left(B_{\mathbf{p_0}}^{\alpha_{\mathbf{0}}\mathbf{q_0}}(\mathbb{T}^{\mathbf{d}}); B_{\mathbf{p_1}}^{\alpha_{\mathbf{1}}\mathbf{q_1}}(\mathbb{T}^{\mathbf{d}})\right)_{[\theta]} = B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$$

holds, where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\mathbf{1}/\mathbf{p} = (\mathbf{1}-\theta)/\mathbf{p_0} + \theta/\mathbf{p_1}$ and $\mathbf{1}/\mathbf{q} = (\mathbf{1}-\theta)/\mathbf{q_0} + \theta/\mathbf{q_1}$. Here $(\cdot,\cdot)_{[\theta]}$ is a complex interpolation functor (see [3]).

Proof. follows from Theorem 1, Theorems 5.1.2 and 5.6.3 from [3]. □

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Аралас метрикалы Никольский-Бесов типтес кеңістіктер үшін интерполяциялық теорема

Мақалада аралас туындылы және аралас метрикалы Никольский-Бесов кеңістігінің интерполяциялық қасиеттері анизотропты және кешенді интерполяция әдістері бойынша зерттелді. $l_{\mathbf{q}}^{\alpha}(A)$ векторлы мәнді салмақтық дискретті кеңістік үшін интерполяциялық теорема дәлелденді. Зерттелген Никольский-Бесов кеңістіктері $l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}})$ кеңістігінің ретракты болатындығы көрсетілді. Жоғарыда келтірілген нәтижелерге сүйене отырып, үстемдік ететін аралас туындылы және аралас метрикалы Никольский-Бесов кеңістіктері үшін интерполяциялық теоремалар алынды.

Кілт сөздер: Никольский-Бесов типтес кеңістіктер, анизотропты интерполяция әдісі, кешенді интерполяция.

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Интерполяционная теорема для пространств типа Никольского-Бесова со смешанной метрикой

В статье изучены интерполяционные свойства пространств Никольского-Бесова с доминирующей смешанной производной и смешанной метрикой относительно анизотропного и комплексного методов интерполяции. Доказана интерполяционная теорема для весового дискретного пространства векторнозначных последовательностей $l_{\mathbf{q}}^{\alpha}(A)$. Показано, что изучаемые пространства Никольского-Бесова являются ретрактом пространства $l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}})$. На основании перечисленных выше результатов получены интерполяционные теоремы для пространств Никольского-Бесова с доминирующей смешанной производной и смешанной метрикой.

Ключевые слова: пространства типа Никольского-Бесова, метод анизотропной интерполяции, комплексная интерполяция.

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Decomposition formulas for some quadruple hypergeometric series

Abstract: In the present work, the authors obtained operator identities and decomposition formulas for second order Gauss hypergeometric series of four variables into products containing simpler hypergeometric functions. A Choi–Hasanov method based on the inverse pairs of symbolic operators is used. The obtained expansion formulas for the hypergeometric functions of four variables will allow us to study the properties of these functions. These decompositions are used to study the solvability of boundary value problems for degenerate multidimensional partial differential equations.

Keywords: Appell hypergeometric function, Lauricella function, Saran function, Quadruple hypergeometric series, Decomposition formulas, Operator identities, Inverse symbolic operators.

Introduction

A variety of tasks related to almost all of the most significant sections of mathematical physics and answering urgent technical questions are associated with the special functions applying, such as the Bessel, Hermite, Gaussian hypergeometric functions, etc. Thus, for example, Bessel functions are actively used in solving hydrodynamics, radiophysics, acoustics problems of atomic and nuclear physics. There are applications of Bessel functions in problems of elasticity and thermal conductivity theories (determination of stress concentration near faults, plate oscillation) [1]. Many functions used in astronomy are arranged in series of hypergeometric functions [2]. Also, the hypergeometric functions of many complex variables are applicable to the research of analytic continuation problems of Mellin – Barnes type integrals [3], in the superstring theory [4], and in theoretical aspects of algebraic geometry [5].

Generalized hypergeometric functions are used in solving boundary value problems for shell theory equations whose applications are used in mechanical engineering. A.D. Kovalenko developed the application of the theory of generalized hypergeometric functions to determine the stress state in disks, circular plates of alternant thickness and conical shells of rotation according to the equilibrium linear theory [6]. Multiple hypergeometric series are used in research and development of aerospace systems [7]. At the same time, hypergeometric functions of many variables arise in quantum field theory as a solution of Knizhnik-Zamolodchikov equations [5]. In [8–10], the connection of special functions of the hypergeometric type with the actual problems of the theory of representations of Lie algebras and quantum groups is shown, as well as the application of hypergeometric functions and series to applied problems of various fields.

It should be noted that the Riemann functions and the fundamental solutions of degenerate partial differential equations are expressed in terms of multiple hypergeometric functions. Thus, hypergeometric functions are used in solving boundary value problems for degenerate differential equations [11]. In particular, hypergeometric functions are used in [12] to find the fundamental solutions of a four-dimensional degenerate equation of elliptic type, which can be used in solving known boundary value

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problems. Also in [13], Appel hypergeometric functions are used to construct a double layer potential theory.

Second order hypergeometric functions of four variables were introduced in [14, 15]. For one class of hypergeometric functions of four variables, various properties, such as decomposition formulas, integral representations were obtained in [16,17]. However, it should be noted that decompositions into products of simpler hypergeometric functions can be obtained not for all the introduced second order hypergeometric functions of four variables.

In this paper, we obtain decomposition formulas using operator identities for the following quadruple hypergeometric functions:

$$F_1^{(4)}\left(a_1, a_2, b; c_1, c_2, c_3, c_4; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_q(b)_{m+n+p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \quad (1)$$

$$F_3^{(4)}\left(a_1, a_2, a_3, b; c_1, c_2, c_3, c_4; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_p(a_3)_q(b)_{m+n+p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \quad (2)$$

$$F_4^{(4)}\left(a, b, c; c_1, c_2, c_3, c_4; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b)_{m+n+q}(c)_p}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \quad (3)$$

$$F_5^{(4)}\left(a_1, a_2, b_1, b_2; c_1, c_2, c_3, c_4; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_{p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \quad (4)$$

$$F_6^{(4)}\left(a_1,a_2,b_1,b_2,b_3; c_1,c_2,c_3,c_4; x,y,z,t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+q}(b_2)_n(b_3)_p}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \quad (5)$$

$$F_8^{(4)}\left(a_1, a_2, b_1, b_2; c_1, c_2, c_3, c_4; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{p+q}(b_1)_{m+p}(b_2)_n(b_3)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \quad (6)$$

$$F_{11}^{(4)}\left(a_1, a_2, b; c_1, c_2, c_3; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_p(a_3)_q(b)_{m+n+p+q}}{(c_1)_{m+p}(c_2)_n(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \quad (7)$$

$$F_{13}^{(4)}\left(a_1, a_2, a_3, a_4, b; c_1, c_2, c_3; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_m(a_2)_n(a_3)_p(a_4)_q(b)_{m+n+p+q}}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \quad (8)$$

where $(a)_n = \Gamma\left(a+m\right)/\Gamma\left(a\right)$ is a Pochhammer symbol.

Operator identities

By means of Burchnall–Chaundy pair of mutually inverse symbolic operators $\nabla_{x,y}\left(h\right)$ and $\Delta_{x,y}\left(h\right)$ [18–20], decomposition formulas were obtained for the Appel's hypergeometric functions of two variables by the products of hypergeometric functions of one variable [21].

To decompose multiple hypergeometric functions, a multivariable analogue of the above pair of mutually inverse symbolic operators

$$\tilde{\nabla}_{x_1;x_2,...,x_r}(h) = \frac{\Gamma(h)\Gamma(h+\delta_1+\delta_2+...+\delta_r)}{\Gamma(h+\delta_1)\Gamma(h+\delta_2+...+\delta_r)} = \sum_{k_2,k_3,...,k_r=0}^{\infty} \frac{(-\delta_1)_{k_2+...+k_r}(-\delta_2)_{k_2}\cdots(-\delta_r)_{k_r}}{(h)_{k_2+...+k_r}k_2!k_3!\cdots k_r!}$$

and

$$\tilde{\Delta}_{x_1;x_2,...,x_r}(h) = \frac{\Gamma(h+\delta_1)\Gamma(h+\delta_2+...+\delta_r)}{\Gamma(h)\Gamma(h+\delta_1+\delta_2+...+\delta_r)} =$$

$$= \sum_{k_2,k_3,...,k_r=0}^{\infty} \frac{(-\delta_1)_{k_2+...+k_r}(-\delta_2)_{k_2}\cdots(-\delta_r)_{k_r}}{(1-h-\delta_1-...-\delta_r)_{k_2+...+k_r}k_2!k_3!\cdots k_r!}$$

$\left(\delta_{x_j} = x_j\frac{\partial}{\partial x_j};\ j=1,2,...,r\right)$ was introduced in [22].

To study the various properties of another class of generalized multidimensional hypergeometric functions, J. Choi and A. Hasanov [23] introduced the following reciprocal operators:

$$H_{x_1,...x_r}(\alpha,\beta) = \frac{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)}{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)} = \sum_{k_1,\cdots,k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(\beta)_{k_1+\cdots+k_r}k_1!\cdots k_r!},$$

$$\overline{H}_{x_1,...x_r}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)}{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)} = \sum_{k_1,...k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(1-\alpha-\delta_1-\cdots-\delta_r)_{i+j}i!j!}$$

$$\left(\delta_{x_j} = x_j\frac{\partial}{\partial x_j}\left(j=1,...,r, r\in\mathbb{N}=\{1,2,...\}\right)\right).$$

Theorem 1. For the second order hypergeometric functions of four variables (1)–(8), the following operator identities are valid:

$$F_1^{(4)}(a_1,a_2,b;c_1,c_2,c_3,c_4;x,y,z,t) = H_t(a_2,c_4)(1-t)^{-b}F_C^{(3)}\left(a_1,b;c_1,c_2,c_3;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t}\right), \tag{9}$$

$$(1-t)^{-b}F_C^{(3)}\left(a_1,b;c_1,c_2,c_3;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t}\right) = \overline{H}_t(a_2,c_4)F_1^{(4)}(a_1,a_2,b;c_1,c_2,c_3,c_4;x,y,z,t), \tag{10}$$

$$F_3^{(4)}(a_1,a_2,a_3,b;c_1,c_2,c_3,c_4;x,y,z,t)$$
$$= H_z(a_2,c_3)H_t(a_3,c_4)(1-t-z)^{-b}F_4\left(a_1,b;c_1,c_2;\frac{x}{1-t-z},\frac{y}{1-t-z}\right), \tag{11}$$

$$(1-t-z)^{-b}F_4\left(a_1,b;c_1,c_2;\frac{x}{1-t-z},\frac{y}{1-t-z}\right)$$
$$= \overline{H}_z(a_2,c_3)\overline{H}_t(a_3,c_4)F_3^{(4)}(a_1,a_2,a_3,b;c_1,c_2,c_3,c_4;x,y,z,t), \tag{12}$$

$$F_4^{(4)}(a,b,c;c_1,c_2,c_3,c_4;x,y,z,t) =$$

$$= H_z\left(c, c_3\right)(1-z)^{-a} F_C^{(3)}\left(a, b; c_1, c_2, c_3; \frac{x}{1-z}, \frac{y}{1-z}, \frac{t}{1-z}\right), \tag{13}$$

$$(1-z)^{-a} F_C^{(3)}\left(a, b; c_1, c_2, c_3; \frac{x}{1-z}, \frac{y}{1-z}, \frac{t}{1-z}\right)$$
$$= \overline{H}_z\left(c, c_3\right) F_4^{(4)}\left(a, b, c; c_1, c_2, c_3, c_4; x, y, z, t\right), \tag{14}$$

$$F_5^{(4)}\left(a_1, a_2, b_1, b_2; c_1, c_2, c_3, c_4; x, y, z, t\right) = H_t\left(a_2, c_4\right)(1-t)^{-b_2} F_E\left(a_1; b_1, b_2; c_1, c_2, c_3; x, y, \frac{z}{1-t}\right), \tag{15}$$

$$(1-t)^{-b_2} F_E\left(a_1; b_1, b_2; c_1, c_2, c_3; x, y, \frac{z}{1-t}\right) = \overline{H}_t\left(a_2, c_4\right) F_5^{(4)}\left(a_1, a_2, b_1, b_2; c_1, c_2, c_3, c_4; x, y, z, t\right), \tag{16}$$

$$F_6^{(4)}\left(a_1, a_2, b_1, b_2, b_3; c_1, c_2, c_3, c_4; x, y, z, t\right)$$
$$= H_y\left(b_2, c_2\right) H_z\left(b_3, c_3\right)(1-y-z)^{-a_1} F_2\left(b_1; a_1, a_2; c_1, c_4; \frac{x}{1-y-z}, t\right), \tag{17}$$

$$(1-y-z)^{-a_1} F_2\left(b_1; a_1, a_2; c_1, c_4; \frac{x}{1-y-z}, t\right)$$
$$= \overline{H}_y\left(b_2, c_2\right) \overline{H}_z\left(b_3, c_3\right) F_6^{(4)}\left(a_1, a_2, b_1, b_2, b_3; c_1, c_2, c_3, c_4; x, y, z, t\right), \tag{18}$$

$$F_8^{(4)}\left(a_1, a_2, b_1, b_2; c_1, c_2, c_3, c_4; x, y, z, t\right)$$
$$= H_y\left(b_2, c_2\right) H_t\left(b_3, c_4\right)(1-y)^{-a_1}(1-t)^{-a_2} F_2\left(b_1; a_1, a_2; c_1, c_3; \frac{x}{1-y}, \frac{x}{1-y}\right), \tag{19}$$

$$(1-y)^{-a_1}(1-t)^{-a_2} F_2\left(b_1; a_1, a_2; c_1, c_3; \frac{x}{1-y}, \frac{x}{1-y}\right)$$
$$= \overline{H}_y\left(b_2, c_2\right) \overline{H}_t\left(b_3, c_4\right) F_8^{(4)}\left(a_1, a_2, b_1, b_2; c_1, c_2, c_3, c_4; x, y, z, t\right), \tag{20}$$

$$F_{11}^{(4)}\left(a_1, a_2, b; c_1, c_2, c_3; x, y, z, t\right) = H_t\left(a_3, c_3\right)(1-t)^{-b} F_F\left(b; a_1, a_2; c_2, c_1; \frac{y}{1-t}, \frac{z}{1-t}, \frac{x}{1-t}\right), \tag{21}$$

$$(1-t)^{-b} F_F\left(b; a_1, a_2; c_2, c_1; \frac{y}{1-t}, \frac{z}{1-t}, \frac{x}{1-t}\right) = \overline{H}_t\left(a_3, c_3\right) F_{11}^{(4)}\left(a_1, a_2, b; c_1, c_2, c_3; x, y, z, t\right), \tag{22}$$

$$F_{13}^{(4)}\left(a_1, a_2, a_3, a_4, b; c_1, c_2, c_3; x, y, z, t\right)$$
$$= H_z\left(a_3, c_2\right) H_t\left(a_4, c_3\right)(1-z-t)^{-b} F_1\left(b; a_1, a_2; c_1; \frac{x}{1-z-t}, \frac{y}{1-z-t}\right), \tag{23}$$

$$(1-z-t)^{-b}F_1\left(b;a_1,a_2;c_1;\frac{x}{1-z-t},\frac{y}{1-z-t}\right)$$
$$=\overline{H}_z\left(a_3,c_2\right)\overline{H}_t\left(a_4,c_3\right)F_{13}^{(4)}\left(a_1,a_2,a_3,a_4,b;c_1,c_2,c_3;x,y,z,t\right), \quad (24)$$

where $F_1,\ F_2,\ F_4$ are Appel hypergeometric functions [21], $F_C^{(3)}$ is Lauricella function [24], and $F_E,\ F_F$ are Saran functions [25]:

$$F_E\left(\alpha;\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{p+m+n}(\beta_1)_{m+n}(\beta_2)_p}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p m!n!p!}x^m y^n z^p,$$

$$F_F\left(\alpha;\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_{n+p} m!n!p!}x^m y^n z^p.$$

Proof. Theorem 1 is proved by dint of Mellin's transformations [26].

Decomposition formulas

Theorem 2. For second order hypergeometric functions (1)–(8) the following decomposition formulas are valid:

$$F_1^{(4)}\left(a_1,a_2,b;c_1,c_2,c_3,c_4;x,y,z,t\right)$$
$$=(1-t)^{-b}\sum_{i=0}^{\infty}\frac{(-1)^i(b)_i(c_4-a_2)_i}{(c_4)_i i!}\left(\frac{t}{1-t}\right)^i F_C^{(3)}\left(a_1,b+i;c_1,c_2,c_3;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t}\right), \quad (25)$$

$$(1-t)^{-b}F_C^{(3)}\left(a_1,b;c_1,c_2,c_3;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t}\right)$$
$$=\sum_{i=0}^{\infty}\frac{(c_4-a_2)_i(b)_i}{(c_4)_i i!}t^i F_1^{(4)}\left(a_1,a_2,b+i;c_1,c_2,c_3,c_4+i;x,y,z,t\right), \quad (26)$$

$$F_3^{(4)}\left(a_1,a_2,a_3,b;c_1,c_2,c_3,c_4;x,y,z,t\right)$$
$$=(1-t-z)^{-b}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(b)_{i+j}(c_3-a_2)_i(c_4-a_3)_j}{(c_3)_i(c_4)_j i!j!}\left(\frac{z}{1-t-z}\right)^i$$
$$\times\left(\frac{t}{1-t-z}\right)^j F_4\left(a_1,b+i+j;c_1,c_2;\frac{x}{1-t-z},\frac{y}{1-t-z}\right), \quad (27)$$

$$(1-t-z)^{-b}F_4\left(a_1,b;c_1,c_2;\frac{x}{1-t-z},\frac{y}{1-t-z}\right)$$
$$=\sum_{i,j=0}^{\infty}\frac{(c_3-a_2)_i(c_4-a_3)_j(b)_{i+j}}{(c_3)_i(c_4)_j i!j!}z^i t^j F_3^{(4)}\left(a_1,a_2,a_3,b+i+j;c_1,c_2,c_3+i,c_4+j;x,y,z,t\right), \quad (28)$$

$$F_4^{(4)}\left(a,b,c;c_1,c_2,c_3,c_4;x,y,z,t\right)$$
$$=(1-z)^{-a}\sum_{i=0}^{\infty}\frac{(-1)^i(a)_i(c_3-c)_i}{(c_3)_i i!}\left(\frac{z}{1-z}\right)^i F_C^{(3)}\left(a+i,b;c_1,c_2,c_4;\frac{x}{1-z},\frac{y}{1-z},\frac{t}{1-z}\right), \quad (29)$$

$$(1-z)^{-a}F_C^{(3)}\left(a,b;c_1,c_2,c_3;\frac{x}{1-z},\frac{y}{1-z},\frac{t}{1-z}\right)$$
$$=\sum_{i=0}^{\infty}\frac{(a)_i(c_3-c)_i}{(c_3)_i i!}z^iF_4^{(4)}\left(a+i,b,c;c_1,c_2,c_3+i,c_4;x,y,z,t\right), \quad (30)$$

$$F_5^{(4)}\left(a_1,a_2,b_1,b_2;c_1,c_2,c_3,c_4;x,y,z,t\right)$$
$$=(1-t)^{-b_2}\sum_{i=0}^{\infty}\frac{(-1)^i(b_2)_i(c_4-a_2)_i}{(c_4)_i i!}\left(\frac{t}{1-t}\right)^iF_E\left(a_1;b_1,b_2+i;\gamma_1,\gamma_2,\gamma_3;x,y,\frac{z}{1-t}\right), \quad (31)$$

$$(1-t)^{-b_2}F_E\left(a_1;b_1,b_2;c_1,c_2,c_3;x,y,\frac{z}{1-t}\right)$$
$$=\sum_{i=0}^{\infty}\frac{(b_2)_i(c_4-a_2)_i}{(c_4)_i i!}t^iF_5^{(4)}\left(a_1,a_2,b_1,b_2+i;c_1,c_2,c_3,c_4+i;x,y,z,t\right), \quad (32)$$

$$F_6^{(4)}\left(a_1,a_2,b_1,b_2,b_3;c_1,c_2,c_3,c_4;x,y,z,t\right)$$
$$=(1-y-z)^{-a_1}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(a_1)_{i+j}(c_2-b_2)_i(c_3-b_3)_j}{(c_2)_i(c_3)_j i!j!}\left(\frac{y}{1-y-z}\right)^i$$
$$\times\left(\frac{z}{1-y-z}\right)^jF_2\left(b_1;a_1+i+j,a_2;c_1,c_4;\frac{x}{1-y-z},t\right), \quad (33)$$

$$(1-y-z)^{-a_1}F_2\left(b_1;a_1,a_2;c_1,c_4;\frac{x}{1-y-z},t\right)=\sum_{i,j=0}^{\infty}\frac{(a_1)_{i+j}(c_2-b_2)_i(c_3-b_3)_j}{(c_2)_i(c_3)_j i!j!}y^iz^j$$
$$\times F_6^{(4)}\left(a_1+i+j,a_2,b_1,b_2,b_3;c_1,c_2+i,c_3+j,c_4;x,y,z,t\right), \quad (34)$$

$$F_8^{(4)}\left(a_1,a_2,b_1,b_2;c_1,c_2,c_3,c_4;x,y,z,t\right)=(1-y)^{-a_1}(1-t)^{-a_2}$$
$$\times\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(a_1)_i(a_2)_j(c_2-b_2)_i(c_4-b_3)_j}{(c_2)_i(c_4)_j i!j!}\left(\frac{y}{1-y}\right)^i\left(\frac{t}{1-t}\right)^j$$
$$\times F_2\left(b_1;a_1+i,a_2+j;c_1,c_3;\frac{x}{1-y},\frac{z}{1-y}\right), \quad (35)$$

$$(1-y)^{-a_1}(1-t)^{-a_2}F_2\left(b_1;a_1,a_2;c_1,c_3;\frac{x}{1-y},\frac{x}{1-y}\right)=\sum_{i,j=0}^{\infty}\frac{(a_1)_i(a_2)_j(c_2-b_2)_i(c_4-b_3)_j}{(c_2)_i(c_4)_j i!j!}y^it^j$$
$$\times F_8^{(4)}\left(a_1+i,a_2+j,b_1,b_2;c_1,c_2+i,c_3,c_4+j;x,y,z,t\right), \quad (36)$$

$$F_{11}^{(4)}\left(a_1,a_2,b;c_1,c_2,c_3;x,y,z,t\right)$$
$$=(1-t)^{-b}\sum_{i=0}^{\infty}\frac{(-1)^i(b)_i(c_3-a_3)_i}{(c_3)_i i!}\left(\frac{t}{1-t}\right)^iF_F\left(b+i;a_1,a_2;c_2,c_1;\frac{y}{1-t},\frac{z}{1-t},\frac{x}{1-t}\right), \quad (37)$$

$$(1-t)^{-b} F_F\left(b; a_1, a_2; c_2, c_1; \frac{y}{1-t}, \frac{z}{1-t}, \frac{x}{1-t}\right)$$
$$= \sum_{i=0}^{\infty} \frac{(b)_i (c_3-a_3)_i}{(c_3)_i i!} t^i F_{11}^{(4)}\left(a_1, a_2, b+i; c_1, c_2, c_3+i; x, y, z, t\right), \quad (38)$$

$$F_{13}^{(4)}\left(a_1, a_2, a_3, a_4, b; c_1, c_2, c_3; x, y, z, t\right)$$
$$= (1-z-t)^{-b} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} (b)_{i+j} (c_2-a_3)_i (c_3-a_4)_j}{(c_2)_i (c_3)_j i! j!} \left(\frac{z}{1-z-t}\right)^i \left(\frac{t}{1-z-t}\right)^j$$
$$\times F_1\left(b+i+j; a_1, a_2; c_1; \frac{x}{1-z-t}, \frac{y}{1-z-t}\right), \quad (39)$$

$$(1-z-t)^{-b} F_1\left(b; a_1, a_2; c_1; \frac{x}{1-z-t}, \frac{y}{1-z-t}\right)$$
$$= \sum_{i=0}^{\infty} \frac{(b)_{i+j} (c_2-a_3)_i (c_3-a_4)_j}{(c_2)_i (c_3)_j i! j!} z^i t^j F_{13}^{(4)}\left(a_1, a_2, a_3, a_4, b+i+j; c_1, c_2+i, c_3+j; x, y, z, t\right). \quad (40)$$

Proof. The proof of Theorem 2 is realized utilizing operator identities (9)–(24), some properties of hypergeometric functions of many variables and the following operator identities [27, p. 93]:

$$(\delta+\alpha)_n \{f(\xi)\} = \xi^{1-\alpha} \frac{d^n}{d\xi^n} \left\{\xi^{\alpha+n-1} f(\xi)\right\},$$

$$(-\delta)_n \{f(\xi)\} = (-\xi)^n \frac{d^n}{d\xi^n} \{f(\xi)\}, \quad (41)$$

$\delta = \xi \frac{d}{d\xi}$; $\alpha \in \mathbb{C}$; $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{N} = \{1, 2, 3, ...\}$, where $f(\xi)$ is analytical function.

As an example, we give a brief proof of the decomposition (25).

The following equality holds:

$$(1-t)^{-b} F_C^{(3)}\left(a_1, b; c_1, c_2, c_3; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p} (b)_{m+n+p+q}}{(c_1)_m (c_2)_n (c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{t^q}{q!}. \quad (42)$$

Considering operator definition $H_t(a_2, c_4)$ and identity (42), from (9) we have

$$F_1^{(4)}\left(a_1, a_2, b; c_1, c_2, c_3, c_4; x, y, z, t\right)$$
$$= \sum_{j=0}^{\infty} \frac{(c_4-a_2)_j (-\delta_t)_j}{(c_4)_j j!} \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p} (b)_{m+n+p+q}}{(c_1)_m (c_2)_n (c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{t^q}{q!}.$$

Using formula (41), we obtain

$$F_1^{(4)}\left(a_1, a_2, b; c_1, c_2, c_3, c_4; x, y, z, t\right)$$
$$= \sum_{i=0}^{\infty} \frac{(-1)^i (b)_i (c_4-a_2)_i}{(c_4)_i i!} t^i \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p} (b+i)_{m+n+p+q}}{(c_1)_m (c_2)_n (c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{t^q}{q!}.$$

By virtue of the validity of the identity $(\lambda)_{m+n} = (\lambda)_m(\lambda+m)_n$ we get

$$F_1^{(4)}\left(a_1,a_2,b;c_1,c_2,c_3,c_4;x,y,z,t\right) = (1-t)^{-b}\sum_{i=0}^{\infty}\frac{(-1)^i(b)_i(c_4-a_2)_i}{(c_4)_i i!}\left(\frac{t}{1-t}\right)^i$$
$$\times\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}(b+i)_{m+n+p}}{(c_1)_m(c_2)_n(c_3)_p}\frac{\left(\frac{x}{1-t}\right)^m}{m!}\frac{\left(\frac{y}{1-t}\right)^n}{n!}\frac{\frac{z}{1-t}^p}{p!}. \quad (43)$$

In view of the $F_C^{(3)}$ Lauricella hypergeometric function definition, from expression (43) we obtain decomposition (25).

Thus, the decomposition formula (25) is proved.

Similarly, we can prove each of the decomposition formulas (26)–(40).

Remark 1. The decomposition formulas $(25)-(40)$ can also be proved by comparing the coefficients before the factor $x^m y^n z^p t^q$ in both sides of the equality.

Conclusion

In conclusion, we proved the operator identities written via the mutually inverse operators H and $\overline{H}$ for the hypergeometric functions of four variables $F_1^{(4)}$, $F_3^{(4)}-F_6^{(4)}$, $F_8^{(4)}$, $F_{11}^{(4)}$, $F_{13}^{(4)}$, the validity of the former is proved using the Mellin transforms. By applying the obtained operator identities, differentiation formulas for hypergeometric functions, and properties of hypergeometric functions, we have proved decompositions for the functions $F_1^{(4)}$, $F_3^{(4)}-F_6^{(4)}$, $F_8^{(4)}$, $F_{11}^{(4)}$, $F_{13}^{(4)}$ by products of such known hypergeometric functions as the Appell's functions F_1, F_2, F_4; Lauricella's function $F_C^{(3)}$; the Saran functions F_E, F_F. Similarly, the decomposition formulas for hypergeometric functions of four variables $F_{17}^{(4)}$, $F_{18}^{(4)}$, $F_{19}^{(4)}$, $F_{20}^{(4)}$, $F_{21}^{(4)}$, etc. can be obtained.

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А.С. Бердышев, А. Хасанов, А.Р. Рысқан

Кейбір төрт айнымалы гипергеометриялық қатарлар үшін жіктеу формулалары

Мақалада төрт айнымалы гипергеометриялық Гаусс қатарлары үшін операторлық тепе-теңдік пен қарапайым функцияларға жіктеу формулалары алынды. Символдық операторлардың кері жұптарына негізделген Чои-Хасанов әдісі қолданылды. Алынған төрт айнымалы гипергеометриялық қатарлары үшін жіктеу формулалары осы функциялардың қасиеттерін зерттеп білуге мүмкіндік береді. Алынған жіктеулер көпөлшемді азғындалған дербес туындылы дифференциалдық теңдеулер үшін шеттік есептердің шешілімділік мәселелерін зерттеуде пайданылады.

Кілт сөздер: Аппель гипергеометриялық функциясы, Лауричелл функциясы, Саран функциясы, төрт айнымалы гипергеометриялық қатар, жіктеу формулалары, операторлық тепе-теңдіктер, кері символдық операторлар.

А.С. Бердышев, А. Хасанов, А.Р. Рыскан

Формулы разложения для некоторых гипергеометрических рядов четырех переменных

В статье получены операторные тождества и формулы разложения для гипергеометрических рядов Гаусса второго порядка четырех переменных по произведениям, содержащим более простые гипергеометрические функции. Авторами использован метод Чои-Хасанова, основанный на обратных парах символических операторов. Полученные формулы разложения для гипергеометрических функций четырех переменных позволят изучить свойства этих функций. Данные разложения применяются при исследовании вопросов разрешимости краевых задач для вырождающихся многомерных дифференциальных уравнений в частных производных.

Ключевые слова: гипергеометрическая функция Аппеля, функция Лауричелла, функция Сарана, гипергеометрический ряд четырех переменных, формулы разложения, операторные тождества, обратные символические операторы.

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Stabilization of a solution for two-dimensional loaded parabolic equation

In this paper we consider the stabilization problem of the solution of a boundary value problem for the heat equation with a loaded two-dimensional Laplace operator. The loaded terms represent the values of the required function and traces of the first-order partial derivatives of the required function at fixed points. An algorithm for constructing boundary control functions is proposed.

Keywords: problem of boundary stabilization, loaded heat equation, loaded Laplace operator, biorthogonal system, stabilization, algorithm.

Introduction

Along with the direct heat conduction problem - finding the temperature field by solving an equation with known boundary conditions, it is often necessary to solve inverse problems, where the corresponding boundary conditions from a given temperature distribution in space and time need to be determined. Such inverse problems have great practical applications in physics, technology, mechanics, and medicine.

Boundary value problems for loaded heat conduction equations, by themselves, have a large amount of applications; they also constitute a special class of equations with their own specific problems. Such problems arise when studying the unique solvability of semi-periodic (periodic in a time variable) problems in a bounded domain in problems of optimal agroecosystem management, for example, the problem of long-term forecasting and regulation of the level of groundwater and soil moisture.

Recently, among specialists in control problems, interest has significantly increased in the stabilization of solutions to boundary value problems [1]-[4]. First of all, this is due not only to their importance in theoretical terms, but also to the fact that one has to deal with them in many applied problems.

The problem considered in this paper on the stabilizability from the boundary $\partial\Omega$ of a solution of a parabolic equation given in a bounded domain $\Omega \in \mathbb{R}$ consists in choosing a boundary control such that the solution of the boundary value problem tends at $t \to \infty$ to a given stationary solution at a given rate $exp\,(-\sigma_0 t)$. In this case, it is required that the control be with feedback, i.e. so that it responds to unforeseen fluctuations of the system, suppressing the results of their influence on the stabilized solution.

In [5], the stabilization problem for a parabolic equation is reduced to solving an auxiliary boundary value problem in an extended domain of independent variables. This idea was further developed in [6]-[8]. Note that in [5]-[8], stabilization problems for differential equations without load were considered. At the same time, loaded differential equations [9]-[14] are actively used in control problems for nonlocal dynamical systems. In [15]-[20], stabilization problems were studied for a loaded one- and two-dimensional heat equation. In this paper, we consider the stabilization problem on the boundary (forming a parallelepiped) of the solution of the boundary value problem for the heat equation with a loaded two-dimensional Laplace operator, where the loaded terms are the values of the required function and traces of the derivatives of the required function at fixed points.

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1. Problem Formulation

Let $\Omega = \{(x,y) : -\frac{\pi}{2} < x, y < \frac{\pi}{2}\}$ be a domain with boundary $\partial\Omega$. In a parallelepiped $Q = \Omega \times \{t > 0\}$ with a lateral surface $\Sigma = \partial\Omega \times \{t > 0\}$ we consider boundary value problem for a loaded heat equation:

$$\frac{\partial u}{\partial t} - \triangle\, u + \alpha_1\, u(x,y,t)|_{x=0} + \alpha_2\, u(x,y,t)|_{y=0} + \alpha_3\, \left.\frac{\partial u(x,y,t)}{\partial x}\right|_{x=0} + \alpha_4\, \left.\frac{\partial u(x,y,t)}{\partial y}\right|_{y=0} = 0, \quad (1)$$

$$u(x,y,t)|_{t=0} = u_0(x,y),\ \{x,y\} \in \Omega, \quad (2)$$

$$u(x,y,t)|_{\Sigma} = p(x,y,t) = \\ = \left\{ u_1\left(\frac{\pi}{2}, y, t\right); u_2\left(x, \frac{\pi}{2}, t\right); u_3\left(-\frac{\pi}{2}, y, t\right); u_4\left(x, -\frac{\pi}{2}, t\right) \Big| \{x,y,t\} \in \Sigma \right\}, \quad (3)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$, $u_0(x,y)$ is a known function. Equation (1) is loaded [9], [10]. It is necessary to find such boundary functions $u_1\left(\frac{\pi}{2}, y, t\right)$; $u_2\left(x, \frac{\pi}{2}, t\right)$; $u_3\left(-\frac{\pi}{2}, y, t\right)$; $u_4\left(x, -\frac{\pi}{2}, t\right)$, so that the solution of the boundary value problem (1)-(3) satisfies the inequality:

$$\|u(x,y,t)\|_{L_2(\Omega)} \le C_0 e^{-\sigma t},\ \sigma > 0,\ t > 0, \quad (4)$$

where σ is a given constant, $C_0 \ge \|u_0(x,y)\|_{L_2(\Omega)}$ is an arbitrary bounded constant.

2. Auxiliary boundary value problem

Let $\Omega_1 = \{(x,y) : -\pi < x, y < \pi\}$ and $Q_1 = \Omega_1 \times \{t > 0\}$.

$$\frac{\partial z}{\partial t} - \triangle\, z + \alpha_1\, z(x,y,t)|_{x=0} + \alpha_2\, z(x,y,t)|_{y=0} + \alpha_3\, \left.\frac{\partial z(x,y,t)}{\partial x}\right|_{x=0} + \alpha_4\, \left.\frac{\partial z(x,y,t)}{\partial y}\right|_{y=0} = 0, \quad (5)$$

$$z(x,y,t)|_{t=0} = z_0(x,y),\ \{x,y\} \in \Omega, \quad (6)$$

$$\left.\begin{aligned} z(-\pi,y,t) = z(\pi,y,t);\ \frac{\partial z(-\pi,y,t)}{\partial x} = \frac{\partial z(\pi,y,t)}{\partial x},\ \{y,t\} \in (-\pi;\pi) \times \{t>0\} \\ z(x,-\pi,t) = z(x,\pi,t);\ \frac{\partial z(x,-\pi,t)}{\partial y} = \frac{\partial z(x,\pi,t)}{\partial y},\ \{x,t\} \in (-\pi;\pi) \times \{t>0\} \end{aligned}\right\} \quad (7)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$, $z_0(x,y)$ is a known function. It is necessary to find the function z, so that the solution of the auxiliary boundary value problem (5)-(7) satisfies the inequality:

$$\|z(x,y,t)\|_{L_2(\Omega_1)} \le C_1 e^{-\sigma t},\ \sigma > 0,\ t > 0, \quad (8)$$

where σ is a given constant, $C_1 \ge C_0$ is an arbitrary bounded constant.

3. Spectral problem for the loaded two-dimensional Laplace operator

Let's find the solution of the problem (5)-(7) by the method of separation of variables

$$z(x,y,t) = \sum_{k,n\in\mathbb{Z}} Z_{k,n}(t) \cdot \varphi_{k,n}(x,y)$$

$$\frac{\partial z}{\partial t} = \sum_{k,n\in\mathbb{Z}} Z'_{k,n}(t) \cdot \varphi_{k,n}(x,y);$$

$$\triangle\, z=\sum_{k,n\in\mathbb{Z}} Z_{k,n}\left(t\right)\left(\frac{\partial^2\varphi_{k,n}\left(x,y\right)}{\partial x^2}+\frac{\partial^2\varphi_{k,n}\left(x,y\right)}{\partial y^2}\right)=\sum_{k,n\in\mathbb{Z}} Z_{k,n}\left(t\right)\cdot\triangle\,\varphi_{k,n};$$

$$\alpha_1\, z\left(x,y,t\right)\big|_{x=0}+\alpha_2\, z\left(x,y,t\right)\big|_{y=0}=\sum_{k,n\in\mathbb{Z}}\left(\alpha_1 Z_{k,n}\left(t\right)\varphi_{k,n}\left(0,y\right)+\alpha_2 Z_{k,n}\left(t\right)\varphi_{k,n}\left(x,0\right)\right);$$

$$\alpha_3\,\frac{\partial z\left(x,y,t\right)}{\partial x}\bigg|_{x=0}+\alpha_4\,\frac{\partial z\left(x,y,t\right)}{\partial y}\bigg|_{y=0}=$$

$$=\sum_{k,n\in\mathbb{Z}}\left(\alpha_3 Z_{k,n}\left(t\right)\,\frac{\partial\varphi_{k,n}\left(x,y\right)}{\partial x}\bigg|_{x=0}+\alpha_4 Z_{k,n}\left(t\right)\,\frac{\partial\varphi_{k,n}\left(x,y\right)}{\partial y}\bigg|_{y=0}\right).$$

Now we substitute the obtained expressions into (5):

$$\sum_{k,n\in\mathbb{Z}}\Big(Z'_{k,n}\left(t\right)\cdot\varphi_{k,n}\left(x,y\right)-Z_{k,n}\left(t\right)\cdot\triangle\,\varphi_{k,n}+$$

$$+Z_{k,n}\left(t\right)\left(\alpha_1\varphi_{k,n}\left(0,y\right)+\alpha_2\varphi_{k,n}\left(x,0\right)+\alpha_3\,\frac{\partial\varphi_{k,n}\left(x,y\right)}{\partial x}\bigg|_{x=0}+\alpha_4\,\frac{\partial\varphi_{k,n}\left(x,y\right)}{\partial y}\bigg|_{y=0}\right)\Bigg)=0.$$

Hence, we get:

$$Z'_{k,n}\left(t\right)\cdot\varphi_{k,n}\left(x,y\right)-Z_{k,n}\left(t\right)\cdot\triangle\,\varphi_{k,n}+$$

$$+\,Z_{k,n}\left(t\right)\left(\alpha_1\varphi_{k,n}\left(0,y\right)+\alpha_2\varphi_{k,n}\left(x,0\right)+\alpha_3\,\frac{\partial\varphi_{k,n}\left(x,y\right)}{\partial x}\bigg|_{x=0}+\alpha_4\,\frac{\partial\varphi_{k,n}\left(x,y\right)}{\partial y}\bigg|_{y=0}\right)=0.$$

Dividing both sides of the equality by $Z_{k,n}\left(t\right)\cdot\varphi_{k,n}\left(x,y\right)$, we obtain:

$$\frac{Z'_{k,n}\left(t\right)}{Z_{k,n}\left(t\right)}=\frac{\triangle\,\varphi_{k,n}-\alpha_1\varphi_{k,n}\left(0,y\right)-\alpha_2\varphi_{k,n}\left(x,0\right)-\alpha_3\,\frac{\partial\varphi_{k,n}(x,y)}{\partial x}\Big|_{x=0}-\alpha_4\,\frac{\partial\varphi_{k,n}(x,y)}{\partial y}\Big|_{y=0}}{\varphi_{k,n}\left(x,y\right)}=-\lambda_{k,n}.$$

In order to find $\varphi_{k,n}\left(x,y\right)$, we consider the following spectral problem.

$$\left.\begin{aligned}\frac{\triangle\,\varphi_{k,n}-\alpha_1\varphi_{k,n}\left(0,y\right)-\alpha_2\varphi_{k,n}\left(x,0\right)-\alpha_3\,\frac{\partial\varphi_{k,n}(x,y)}{\partial x}\Big|_{x=0}-\alpha_4\,\frac{\partial\varphi_{k,n}(x,y)}{\partial y}\Big|_{y=0}}{\varphi_{k,n}\left(x,y\right)}=-\lambda_{k,n},\\ \varphi_{k,n}\left(-\pi,y\right)=\varphi_{k,n}\left(\pi,y\right);\ \ \frac{\partial\varphi_{k,n}\left(-\pi,y\right)}{\partial x}=\frac{\partial\varphi_{k,n}\left(\pi,y\right)}{\partial x},\\ \varphi_{k,n}\left(x,-\pi\right)=\varphi_{k,n}\left(x,\pi\right);\ \ \frac{\partial\varphi_{k,n}\left(x,-\pi\right)}{\partial y}=\frac{\partial\varphi_{k,n}\left(x,\pi\right)}{\partial y}.\end{aligned}\right\}\qquad(9)$$

Let's use the method of separation of variables again. Let $\varphi_{k,n}\left(x,y\right)=X_k\left(x\right)\cdot Y_n\left(y\right)$. Then the problem (9) can be written as follows:

$$\left.\begin{aligned}\frac{X''_k\left(x\right)-\alpha_1 X_k\left(0\right)-\alpha_3 X'_k\left(0\right)}{X_k\left(x\right)}=-\lambda_{k,n}-\frac{Y''_n\left(y\right)-\alpha_2 Y_n\left(0\right)-\alpha_4 Y'_n\left(0\right)}{Y_n\left(y\right)}=-\mu_k,\\ X_k\left(-\pi\right)=X_k\left(\pi\right);\ \ X'_k\left(-\pi\right)=X'_k\left(\pi\right),\\ Y_n\left(-\pi\right)=Y_n\left(\pi\right);\ \ Y'_n\left(-\pi\right)=Y'_n\left(\pi\right).\end{aligned}\right\}$$

This problem was reduced to finding solutions of the following two differential equations with periodic conditions:

$$\left.\begin{aligned}X''_k\left(x\right)+\mu_k X_k\left(x\right)-\left(\alpha_1 X_k\left(0\right)+\alpha_3 X'_k\left(0\right)\right)=0,\\ X_k\left(-\pi\right)=X_k\left(\pi\right);\ \ X'_k\left(-\pi\right)=X'_k\left(\pi\right).\end{aligned}\right\}\qquad(10)$$

$$\left.\begin{aligned} Y_n''(y) - (\mu_k - \lambda_{k,n}) Y_n(y) - \big(\alpha_2 Y_n(0) + \alpha_4 Y_n'(0)\big) = 0, \\ Y_n(-\pi) = Y_n(\pi); \;\; Y_n'(-\pi) = Y_n'(\pi). \end{aligned}\right\} \tag{11}$$

Note that the general solution of loaded ordinary differential equations (10) and (11) is represented as a linear combination of the complete system of periodic functions $\left\{\Phi_m = e^{imt},\;\; m \in \mathbb{Z}\right\}$. Therefore, we will find a solution of problem (10) in the form $X_k(x) = A_k e^{ikx} + C_k$ $(k \in \mathbb{Z})$. Let's consider several cases.

Case (a). Let $\alpha_1 \neq k^2$ $\forall k \in \mathbb{Z} \setminus \{0\}$. Then the solution to the problem (10) will be as follows:

$$X_k(x) = A_k e^{ikx} + C_k \;\Rightarrow\; X_k(0) = A_k + C_k,$$

$$X_k'(x) = ikA_k e^{ikx} \;\Rightarrow\; X_k'(0) = ikA_k,$$

$$X_k''(x) = -k^2 A_k e^{ikx},$$

$$-k^2 A_k e^{ikx} + \mu_k A_k e^{ikx} + \mu_k C_k - (\alpha_1 A_k + \alpha_1 C_k + i\alpha_3 k A_k) = 0.$$

$$\begin{cases} -k^2 A_k + \mu_k A_k = 0, \\ \mu_k C_k - (\alpha_1 A_k + \alpha_1 C_k + i\alpha_3 k A_k) = 0 \end{cases} \Rightarrow \begin{cases} A_k\left(-k^2 + \mu_k\right) = 0, \\ \mu_k C_k - (\alpha_1 A_k + \alpha_1 C_k + i\alpha_3 k A_k) = 0. \end{cases}$$

If $A_k = 0$, then the equation (10) has no nontrivial solutions. Therefore, it follows from the first equation of the last system that A_k can take any nonzero value. For simplicity, we will assume that $A_k = 1$. It should also be noted that from the equality $\mu_k = k^2$ it follows that $k \neq 0$, since for $\mu_k = 0$ the equation (10) also has no nontrivial solutions. Thus:

$$\begin{cases} A_k = 1, \\ \mu_k = k^2,\, k \neq 0 \\ k^2 C_k - \alpha_1 - \alpha_1 C_k - i\alpha_3 k = 0 \end{cases} \Rightarrow \begin{cases} A_k = 1, \\ \mu_k = k^2,\, k \neq 0 \\ C_k = \dfrac{\alpha_1}{k^2 - \alpha_1} + \dfrac{i\alpha_3 k}{k^2 - \alpha_1} \end{cases}$$

Therefore, we obtain a system of eigenfunctions:

$$X_k(x) = e^{ikx} + \frac{\alpha_1}{k^2 - \alpha_1} + \frac{i\alpha_3 k}{k^2 - \alpha_1},\;\; \alpha_1 \neq k^2 \;\; \forall k \in \mathbb{Z} \setminus \{0\},$$

which correspond to the eigenvalues $\mu_k = k^2$ $\forall k \in \mathbb{Z} \setminus \{0\}$.

Let $\alpha_1 \neq k^2$, $k = 0$. Then we have

$$\left.\begin{aligned} X_0''(x) + \mu_0 X_0(x) - \big(\alpha_1 X_0(0) + \alpha_3 X_0'(0)\big) = 0, \\ X_0(-\pi) = X_0(\pi); \;\; X_0'(-\pi) = X_0'(\pi). \end{aligned}\right\}$$

$$X_0(x) = A_0 + C_0 \;\Rightarrow\; X_0(0) = A_0 + C_0$$

$$X_0'(x) = X_0''(x) = 0$$

$$\mu_0(A_0 + C_0) - \alpha_1(A_0 + C_0) = 0 \quad\Rightarrow\quad (A_0 + C_0)(\mu_0 - \alpha_1) = 0 \quad\Rightarrow\quad \begin{cases} A_0 + C_0 = 1, \\ \mu_0 = \alpha_1. \end{cases}$$

We get the eigenfunction $X_0(x) = 1$, which corresponds to the eigenvalue $\mu_0 = \alpha_1$.

Hence, the system of eigenfunctions and eigenvalues of the problem (10) for the case (a) has the form:

$$X_k(x) = \begin{cases} e^{ikx} + \dfrac{\alpha_1}{k^2 - \alpha_1} + \dfrac{i\alpha_3 k}{k^2 - \alpha_1},\;\; \mu_k = k^2,\;\; \forall k \in \mathbb{Z} \setminus \{0\} \\ 1,\;\; \mu_0 = \alpha_1,\;\; k = 0. \end{cases}$$

Case (b). Let $\exists a \in \mathbb{Z}: \ \alpha_1 = a^2 \ \forall k \in \mathbb{Z}\backslash\{\pm a\}$. Reasoning similarly to the previous case, we obtain a system of eigenfunctions $X_k(x) = e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}$, which correspond to the eigenvalues $\mu_k = k^2, \ \forall k \in \mathbb{Z}\backslash\{0;\pm a\}$, and the eigenfunction $X_0(x) = 1$, which corresponds to the eigenvalue $\mu_0 = \alpha_1 = a^2$.

Let $\exists a \in \mathbb{Z}: \ \alpha_1 = a^2, \ k = \pm a$. Then we have

$$\left.\begin{aligned} X''_{\pm a}(x) + \mu_{\pm a}X_{\pm a}(x) - \left(\alpha_1 X_{\pm a}(0) + \alpha_3 X'_{\pm a}(0)\right) = 0, \\ X_{\pm a}(-\pi) = X_{\pm a}(\pi); \ X'_{\pm a}(-\pi) = X'_{\pm a}(\pi). \end{aligned}\right\}$$

$$X_{\pm a}(x) = A_{\pm a}e^{\pm iax} + C_{\pm a} \Rightarrow X_{\pm a}(0) = A_{\pm a} + C_{\pm a},$$

$$X'_{\pm a}(x) =_{\pm} iaA_{\pm a}e^{\pm ax} \Rightarrow X'_{\pm a}(0) = \pm iaA_{\pm a},$$

$$X''_{\pm a}(x) = -a^2 A_{\pm a}e^{\pm iax},$$

$$-a^2A_{\pm a}e^{\pm iax} + \mu_{\pm a}A_{\pm a}e^{\pm iax} + \mu_{\pm a}C_{\pm a} - \left(a^2A_{\pm a} + a^2C_{\pm a} \pm i\alpha_3 aA_{\pm a}\right) = 0.$$

$$\left\{\begin{aligned} -a^2A_{\pm a} + \mu_{\pm a}A_{\pm a} = 0, \\ \mu_{\pm a}C_{\pm a} - \left(a^2A_{\pm a} + a^2C_{\pm a} \pm i\alpha_3 aA_{\pm a}\right) = 0 \end{aligned}\right. \Rightarrow \left\{\begin{aligned} A_{\pm a} = 1, \\ \mu_{\pm a} = a^2, \\ a^2C_{\pm a} - a^2 - a^2C_{\pm a} \mp i\alpha_3 a = 0 \end{aligned}\right. \Rightarrow$$

$$\Rightarrow \left\{\begin{aligned} A_{\pm a} = 1, \\ \mu_{\pm a} = a^2, \\ -a^2 \mp i\alpha_3 a = 0 \end{aligned}\right. \Rightarrow \left\{\begin{aligned} A_{\pm a} = 1, \\ \mu_{\pm a} = a^2, \\ a(a \pm i\alpha_3) = 0 \end{aligned}\right. \Rightarrow \left\{\begin{aligned} A_{\pm a} = 1, \\ \mu_{\pm a} = a^2, \\ a = 0 \text{ или } \alpha_3 = \pm ia \end{aligned}\right.$$

Therefore, we get the eigenfunctions $X_{\pm a}(x) = e^{\pm iax}$, which correspond to the eigenvalues $\mu_{\pm a} = \alpha_1 = a^2$.

Hence, the system of eigenfunctions and eigenvalues of the problem (10) for the case (b) has the form:

$$X_k(x) = \left\{\begin{aligned} e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}, \ \mu_k = k^2, \ \forall k \in \mathbb{Z}\backslash\{0;\pm a\} \\ 1, \ \mu_0 = \alpha_1 = a^2, \ k = 0 \\ e^{\pm iax}, \ \mu_{\pm a} = \alpha_1 = a^2, \ k = \pm a \end{aligned}\right.$$

The solution of the problem (11) is defined similarly.

Case (c). Let $\alpha_2 \neq n \ \forall n \in \mathbb{Z}$. Then the system of eigenfunctions and eigenvalues of the problem (11) has the form:

$$Y_n(y) = \left\{\begin{aligned} e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2}, \ \lambda_{k,n} = k^2 + n^2, \ \forall k \in \mathbb{Z}, \ \forall n \in \mathbb{Z}\backslash\{0\} \\ 1, \ \lambda_{k,0} = k^2 + \alpha_2, \ \forall k \in \mathbb{Z}, \ n = 0. \end{aligned}\right.$$

Case (d). Let $\exists b \in \mathbb{Z}: \ \alpha_2 = b^2 \ \forall n \in \mathbb{Z}$. Then the system of eigenfunctions and eigenvalues of the problem (11) has the form:

$$Y_n(y) = \left\{\begin{aligned} e^{iny} + \frac{b^2}{n^2-b^2} + \frac{i\alpha_4 n}{n^2-b^2}, \ \lambda_{k,n} = k^2 + n^2, \ \forall k \in \mathbb{Z}, \forall n \in \mathbb{Z}\backslash\{0;\pm b\} \\ 1, \ \lambda_{k,0} = k^2 + \alpha_2 = k^2 + b^2, \ \forall k \in \mathbb{Z}, n = 0 \\ e^{\pm iby}, \ \lambda_{k;\pm b} = k^2 + \alpha_2 = k^2 + b^2, \ \forall k \in \mathbb{Z}, n = \pm b \end{aligned}\right.$$

Let's write the systems of eigenfunctions and eigenvalues of the problem (9).

Case 1. Let $\alpha_1 \neq k^2$ $\forall k \in \mathbb{Z}$ and $\alpha_2 \neq n^2$ $\forall n \in \mathbb{Z}$. Then the system of eigenfunctions and eigenvalues of the problem (9) has the form:

$$\begin{gathered}
\Big\{\varphi_{k,n}(x,y) = \left(e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}\right)\left(e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2}\right),\\
\lambda_{k,n} = k^2+n^2,\ \forall k,n \in \mathbb{Z}\setminus\{0\};\\
\varphi_{k,0}(x,y) = e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1},\ \ \lambda_{k,0} = k^2+\alpha_2,\ \forall k \in \mathbb{Z}\setminus\{0\};\\
\varphi_{0,n}(x,y) = e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2},\ \ \lambda_{0,n} = \alpha_1+n^2,\ \forall n \in \mathbb{Z}\setminus\{0\};\\
\varphi_{0,0}(x,y) = 1,\ \ \lambda_{0,0} = \alpha_1+\alpha_2\Big\}.
\end{gathered} \tag{12}$$

Case 2. Let $\alpha_1 \neq k^2$ $\forall k \in \mathbb{Z}$ and $\exists b \in \mathbb{Z}:\ \alpha_2 = b^2$ $\forall n \in \mathbb{Z}$. Then the system of eigenfunctions and eigenvalues of the problem (9) has the form:

$$\begin{gathered}
\Big\{\varphi_{k,n}(x,y) = \left(e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}\right)\left(e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2}\right),\\
\lambda_{k,n} = k^2+n^2,\ \forall k \in \mathbb{Z}\setminus\{0\}, n \in \mathbb{Z}\setminus\{0;\pm b\};\\
\varphi_{k,0}(x,y) = e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1},\ \ \lambda_{k,0} = k^2+\alpha_2,\ \forall k \in \mathbb{Z}\setminus\{0\};\\
\varphi_{k,\pm b}(x,y) = \left(e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}\right)e^{\pm iby},\ \ \lambda_{k,\pm b} = k^2+\alpha_2,\ \forall k \in \mathbb{Z}\setminus\{0\};\\
\varphi_{0,n}(x,y) = e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2},\ \ \lambda_{0,n} = \alpha_1+n^2,\ \forall n \in \mathbb{Z}\setminus\{0;\pm b\};\\
\varphi_{0,\pm b}(x,y) = e^{\pm iby},\ \ \lambda_{0,\pm b} = \alpha_1+\alpha_2;\\
\varphi_{0,0}(x,y) = 1,\ \ \lambda_{0,0} = \alpha_1+\alpha_2\Big\}.
\end{gathered} \tag{13}$$

Case 3. Let $\exists a \in \mathbb{Z}:\ \alpha_1 = a^2$ $\forall k \in \mathbb{Z}$ and $\alpha_2 \neq n^2$ $\forall n \in \mathbb{Z}$. Then the system of eigenfunctions and eigenvalues of the problem (9) has the form:

$$\begin{gathered}
\Big\{\varphi_{k,n}(x,y) = \left(e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}\right)\left(e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2}\right),\\
\lambda_{k,n} = k^2+n^2,\ \forall k \in \mathbb{Z}\setminus\{0;\pm a\}, n \in \mathbb{Z}\setminus\{0\};\\
\varphi_{k,0}(x,y) = e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1},\ \ \lambda_{k,0} = k^2+\alpha_2,\ \forall k \in \mathbb{Z}\setminus\{0;\pm a\};\\
\varphi_{0,n}(x,y) = e^{iny} + \frac{\alpha_2}{n^2-\alpha_2} + \frac{i\alpha_4 n}{n^2-\alpha_2},\ \ \lambda_{0,n} = \alpha_1+n^2,\ \forall n \in \mathbb{Z}\setminus\{0\};\\
\varphi_{\pm a,n}(x,y) = e^{\pm iax}\left(e^{ikx} + \frac{\alpha_1}{k^2-\alpha_1} + \frac{i\alpha_3 k}{k^2-\alpha_1}\right),\ \ \lambda_{\pm a,n} = \alpha_1+n^2,\ \forall n \in \mathbb{Z}\setminus\{0\};\\
\varphi_{\pm a,0}(x,y) = e^{\pm iax},\ \ \lambda_{\pm a,0} = \alpha_1+\alpha_2;\\
\varphi_{0,0}(x,y) = 1,\ \ \lambda_{0,0} = \alpha_1+\alpha_2\Big\}.
\end{gathered} \tag{14}$$

Case 4. Let $\exists a\in\mathbb{Z}:\ \ \alpha_1=a^2\ \forall k\in\mathbb{Z}$ and $\exists b\in\mathbb{Z}:\ \ \alpha_2=b^2\ \forall n\in\mathbb{Z}$. Then the system of eigenfunctions and eigenvalues of the problem (9) has the form:

$$\Big\{\varphi_{k,n}(x,y)=\left(e^{ikx}+\frac{\alpha_1}{k^2-\alpha_1}+\frac{i\alpha_3k}{k^2-\alpha_1}\right)\left(e^{iny}+\frac{\alpha_2}{n^2-\alpha_2}+\frac{i\alpha_4n}{n^2-\alpha_2}\right),$$

$$\lambda_{k,n}=k^2+n^2,\ \forall k\in\mathbb{Z}\backslash\{0;\pm a\},n\in\mathbb{Z}\backslash\{0;\pm b\};$$

$$\varphi_{k,0}(x,y)=e^{ikx}+\frac{\alpha_1}{k^2-\alpha_1}+\frac{i\alpha_3k}{k^2-\alpha_1},\ \ \lambda_{k,0}=k^2+\alpha_2,\ \forall k\in\mathbb{Z}\backslash\{0;\pm a\};\tag{15}$$

$$\varphi_{0,n}(x,y)=e^{iny}+\frac{\alpha_2}{n^2-\alpha_2}+\frac{i\alpha_4n}{n^2-\alpha_2},\ \ \lambda_{0,n}=\alpha_1+n^2,\ \forall n\in\mathbb{Z}\backslash\{0;\pm b\};$$

$$\varphi_{\pm a,n}(x,y)=e^{\pm iax}\left(e^{ikx}+\frac{\alpha_1}{k^2-\alpha_1}+\frac{i\alpha_3k}{k^2-\alpha_1}\right),\ \ \lambda_{\pm a,n}=\alpha_1+n^2,\ \forall n\in\mathbb{Z}\backslash\{0;\pm b\};$$

$$\varphi_{k,\pm b}(x,y)=\left(e^{ikx}+\frac{\alpha_1}{k^2-\alpha_1}+\frac{i\alpha_3k}{k^2-\alpha_1}\right)e^{\pm iby},\ \ \lambda_{k,\pm b}=k^2+\alpha_2,\ \forall k\in\mathbb{Z}\backslash\{0;\pm a\};$$

$$\varphi_{\pm a,\pm b}(x,y)=e^{\pm i(ax+by)},\ \ \lambda_{\pm a,\pm b}=\alpha_1+\alpha_2;$$

$$\varphi_{\pm a,0}(x,y)=e^{\pm iax},\ \ \lambda_{\pm a,0}=\alpha_1+\alpha_2;$$

$$\varphi_{0,\pm b}(x,y)=e^{\pm iby},\ \ \lambda_{0,\pm b}=\alpha_1+\alpha_2;$$

$$\varphi_{0,0}(x,y)=1,\ \ \lambda_{0,0}=\alpha_1+\alpha_2\Big\}.$$

Solution of the equation $\frac{Z'_{k,n}(t)}{Z_{k,n}(t)}=-\lambda_{k,n}$ has the form.

$$Z_{k,n}(t)=C_{k,n}\cdot e^{-\lambda_{k,n}t},\tag{16}$$

where $C_{k,n}=z_{0kn}$ are the expansion coefficients of the function $z_0(x,y)$ by system $\{\varphi_{k,n}(x,y),\ k,n\in\mathbb{Z}\}$.

Note that the obtained systems of eigenfunctions (12)-(15) are complete in the space $L_2(\Omega_1)$, forme a basis but is not orthogonal (the completeness of the systems of eigenfunctions (12)-(15) follows from the Paley-Wiener theorem [21], [22]). Therefore, the solution to problem (5)-(7) will be sought in the form

$$z(x,y,t)=\sum_{k,n}Z_{k,n}(t)\,\psi_{k,n}(x,y),\tag{17}$$

where $\{\psi_{k,n}(x,y),\ k,n\in\mathbb{Z}\}$ is a biorthogonal basis [23] of the space $L_2(\Omega_1)$ and $\mathbb{Z}=\{0;\pm1;\pm2;...\}$ to the system $\{\varphi_{k,n}(x,y),\ k,n\in\mathbb{Z}\}$.

3. Construction of biorthogonal systems of functions $\{\psi_{k,n}(x,y),\ k,n\in\mathbb{Z}\}$

The biorthogonal systems of functions in $L_2(\Omega_1)$ for (12)-(15) will be constructed as follows:

$$\{\psi_{k,n}(x,y),\ k,n\in\mathbb{Z}\}=\left\{\frac{1}{4\pi^2}e^{i(kx+ny)};\ \frac{1}{2\pi}g_0(y)\,e^{ikx};\ \frac{1}{2\pi}f_0(x)\,e^{iny};\ k,n\in\mathbb{Z}\backslash\{0\};\ f_0(x)\cdot g_0(y)\right\}$$

where $f_0(x)$, $g_0(y)$ are unknown functions. For the (12) $f_0(x)$ we will search in the form:

$$f_0(x)=C_0+\sum_{m\in\mathbb{Z}\backslash\{0\}}C_m\left(e^{imx}+\frac{i\alpha_1}{m}\right).$$

Coefficients C_0 and C_m we determine from the biorthogonality conditions:

$$C_0 = -\frac{1}{2\pi} - \sum_{m \in \mathbb{Z} \backslash \{0\}} C_m B_m^{\alpha_1,\alpha_3}, \ \ C_m = -\frac{1}{2\pi} B_m^{\alpha_1,\alpha_3}, \ \text{ where } \ B_m^{\alpha_1,\alpha_3} = \frac{\alpha_1}{m^2-\alpha_1} + \frac{i\alpha_3 m}{m^2-\alpha_1}.$$

Further, applying the found values of C_0 and C_m, we find the required function $f_0\left(x\right)$:

$$f_0\left(x\right) = -\frac{1}{2\pi} \sum_{m \in \mathbb{Z}} B_m^{\alpha_1,\alpha_3} e^{imx}, \ \text{ where } \ B_m^{\alpha_1,\alpha_3} = \frac{\alpha_1}{m^2-\alpha_1} + \frac{i\alpha_3 m}{m^2-\alpha_1}.$$

Function $g_0\left(y\right)$ for the (12) is defined similarly:

$$g_0\left(y\right) = -\frac{1}{2\pi} \sum_{m \in \mathbb{Z}} B_m^{\alpha_2,\alpha_4} e^{imy}, \ \text{ where } \ B_m^{\alpha_2,\alpha_4} = \frac{\alpha_2}{m^2-\alpha_2} + \frac{i\alpha_4 m}{m^2-\alpha_2}.$$

Thus, biorthogonal system for (12) is:

$$\left\{\psi_{k,n}\left(x,y\right), \ k,n \in \mathbb{Z}\right\} = \left\{ \frac{1}{4\pi^2} e^{i(kx+ny)}; \ -\frac{1}{4\pi^2} \sum_{l \in \mathbb{Z}} B_l^{\alpha_2,\alpha_4} e^{i(ly+kx)}; \ -\frac{1}{4\pi^2} \sum_{m \in \mathbb{Z}} B_m^{\alpha_1,\alpha_3} e^{i(mx+ny)}; \right.$$
$$\left. k,n \in \mathbb{Z} \backslash \left\{0\right\}; \ -\frac{1}{4\pi^2} \sum_{m,l \in \mathbb{Z}} B_m^{\alpha_1,\alpha_3} B_l^{\alpha_2,\alpha_4} e^{i(mx+ly)} \right\}. \quad (18)$$

The biorthogonal systems of functions in $L_2\left(\Omega_1\right)$ for (13)-(15) are defined similarly.
Biorthogonal system for (13) is:

$$\left\{\psi_{k,n}\left(x,y\right), \ k,n \in \mathbb{Z}\right\} = \left\{ \frac{1}{4\pi^2} e^{i(kx+ny)}; \ -\frac{1}{4\pi^2} \sum_{l \in \mathbb{Z} \backslash \{\pm b\}} B_l^{\alpha_2,\alpha_4} e^{i(ly+kx)}; \ -\frac{1}{4\pi^2} \sum_{m \in \mathbb{Z}} B_m^{\alpha_1,\alpha_3} e^{i(mx+ny)}; \right.$$
$$\left. k,n \in \mathbb{Z} \backslash \left\{0\right\}; \ -\frac{1}{4\pi^2} \sum_{\substack{m \in \mathbb{Z} \\ l \in \mathbb{Z} \backslash \{\pm b\}}} B_m^{\alpha_1,\alpha_3} B_l^{\alpha_2,\alpha_4} e^{i(mx+ly)} \right\}. \quad (19)$$

Biorthogonal system for (14) is:

$$\left\{\psi_{k,n}\left(x,y\right), \ k,n \in \mathbb{Z}\right\} = \left\{ \frac{1}{4\pi^2} e^{i(kx+ny)}; \ -\frac{1}{4\pi^2} \sum_{l \in \mathbb{Z}} B_l^{\alpha_2,\alpha_4} e^{i(ly+kx)}; \ -\frac{1}{4\pi^2} \sum_{m \in \mathbb{Z} \backslash \{\pm a\}} B_m^{\alpha_1,\alpha_3} e^{i(mx+ny)}; \right.$$
$$\left. k,n \in \mathbb{Z} \backslash \left\{0\right\}; \ -\frac{1}{4\pi^2} \sum_{\substack{l \in \mathbb{Z} \\ m \in \mathbb{Z} \backslash \{\pm a\}}} B_m^{\alpha_1,\alpha_3} B_l^{\alpha_2,\alpha_4} e^{i(mx+ly)} \right\}. \quad (20)$$

Biorthogonal system for (15) is:

$$\left\{\psi_{k,n}\left(x,y\right), \ k,n \in \mathbb{Z}\right\} =$$

$$= \left\{ \frac{1}{4\pi^2} e^{i(kx+ny)};\ -\frac{1}{4\pi^2} \sum_{l\in\mathbb{Z}\backslash\{\pm b\}} B_l^{\alpha_2,\alpha_4} e^{i(ly+kx)};\ -\frac{1}{4\pi^2} \sum_{m\in\mathbb{Z}\backslash\{\pm a\}} B_m^{\alpha_1,\alpha_3} e^{i(mx+ny)}; \right.$$

$$\left. k,n\in\mathbb{Z}\backslash\{0\};\ -\frac{1}{4\pi^2} \sum_{\substack{l\in\mathbb{Z}\backslash\{\pm b\}\\ m\in\mathbb{Z}\backslash\{\pm a\}}} B_m^{\alpha_1,\alpha_3} B_l^{\alpha_2,\alpha_4} e^{i(mx+ly)} \right\}. \quad (21)$$

The constructed biorthogonal systems define biorthogonal basises in $L_2(\Omega_1)$.
Hereinafter, we will assume that in the space $L_2(\Omega_1)$ we have:

- basises $\{\varphi_{k,n}(x,y),\ k,n\in\mathbb{Z}\}$, composed of the systems (12)-(15) of eigenfunctions and eigenvalues;
- the corresponding biorthogonal basises $\{\psi_{k,n}(x,y),\ k,n\in\mathbb{Z}\}$, defined by the relations (18)-(21).

Then solution (16) of the auxiliary boundary value problem (5)-(7) can be written as:
for the Case 1:

$$z(x,y,t) = \sum_{k,n\in\mathbb{Z}\backslash\{0\}} z_{0kn} e^{-(k^2+n^2)t} \psi_{kn}(x,y) + \sum_{k\in\mathbb{Z}\backslash\{0\}} z_{0k0} e^{-(k^2+\alpha_2)t} \psi_{k0}(x,y) +$$

$$+ \sum_{n\in\mathbb{Z}\backslash\{0\}} z_{00n} e^{-(\alpha_1+n^2)t} \psi_{0n}(x,y) + z_{000} e^{-(\alpha_1+\alpha_2)t} \psi_{00}(x,y); \quad (22)$$

for the Case 2:

$$z(x,y,t) = \sum_{\substack{k\in\mathbb{Z}\backslash\{0\}\\ n\in\mathbb{Z}\backslash\{0;\pm b\}}} z_{0kn} e^{-(k^2+n^2)t} \psi_{kn}(x,y) + \sum_{k\in\mathbb{Z}\backslash\{0\}} z_{0k0} e^{-(k^2+\alpha_2)t} \psi_{k0}(x,y) +$$

$$+ \sum_{k\in\mathbb{Z}\backslash\{0\}} z_{0k\pm b} e^{-(k^2+\alpha_2)t} \psi_{k\pm b}(x,y) + \sum_{n\in\mathbb{Z}\backslash\{0;\pm b\}} z_{00n} e^{-(\alpha_1+n^2)t} \psi_{0n}(x,y) +$$

$$+ z_{00\pm b} e^{-(\alpha_1+\alpha_2)t} \psi_{0\pm b}(x,y) + z_{000} e^{-(\alpha_1+\alpha_2)t} \psi_{00}(x,y); \quad (23)$$

for the Case 3:

$$z(x,y,t) = \sum_{\substack{k\in\mathbb{Z}\backslash\{0;\pm a\}\\ n\in\mathbb{Z}\backslash\{0\}}} z_{0kn} e^{-(k^2+n^2)t} \psi_{kn}(x,y) + \sum_{k\in\mathbb{Z}\backslash\{0;\pm a\}} z_{0k0} e^{-(k^2+\alpha_2)t} \psi_{k0}(x,y) +$$

$$+ \sum_{n\in\mathbb{Z}\backslash\{0\}} z_{00n} e^{-(\alpha_1+n^2)t} \psi_{0n}(x,y) + \sum_{n\in\mathbb{Z}\backslash\{0\}} z_{0\pm an} e^{-(\alpha_1+n^2)t} \psi_{\pm an}(x,y) +$$

$$+ z_{0\pm a0} e^{-(\alpha_1+\alpha_2)t} \psi_{\pm a0}(x,y) + z_{000} e^{-(\alpha_1+\alpha_2)t} \psi_{00}(x,y); \quad (24)$$

for the Case 4:

$$z(x,y,t) = \sum_{\substack{k\in\mathbb{Z}\backslash\{0;\pm a\}\\ n\in\mathbb{Z}\backslash\{0;\pm b\}}} z_{0kn} e^{-(k^2+n^2)t} \psi_{kn}(x,y) + \sum_{k\in\mathbb{Z}\backslash\{0;\pm a\}} z_{0k0} e^{-(k^2+\alpha_2)t} \psi_{k0}(x,y) +$$

$$+ \sum_{n\in\mathbb{Z}\backslash\{0;\pm b\}} z_{00n} e^{-(\alpha_1+n^2)t} \psi_{0n}(x,y) + \sum_{n\in\mathbb{Z}\backslash\{0;\pm b\}} z_{0\pm an} e^{-(\alpha_1+n^2)t} \psi_{\pm an}(x,y) +$$

$$+ \sum_{k\in\mathbb{Z}\backslash\{0;\pm a\}} z_{0k\pm b} e^{-(k^2+\alpha_2)t} \psi_{k\pm b}(x,y) + z_{0\pm a\pm b} e^{-(\alpha_1+\alpha_2)t} \psi_{\pm a\pm b}(x,y) +$$

$$+ z_{0\pm a0} e^{-(\alpha_1+\alpha_2)t} \psi_{\pm a0}(x,y) + z_{00\pm b} e^{-(\alpha_1+\alpha_2)t} \psi_{0\pm b}(x,y) + z_{000} e^{-(\alpha_1+\alpha_2)t} \psi_{00}(x,y); \quad (25)$$

where

$$z_{0kn}=\int\limits_{\Omega_1}\overline{\varphi_{kn}\left(x,y\right)}z_0\left(x,y\right)dxdy,\quad k,n\in\mathbb{Z}$$

are the Fourier coefficients of the function $z_0\left(x,y\right)$; and systems $\{\psi_{k,n}\left(x,y\right),\ k,n\in\mathbb{Z}\}$ are defined by (18)-(21).

From (16) and (22)-(25) it follows immediately that if

for the Case 1:

$$\begin{gathered}z_{0kn}=0\ \text{ for }\ k^2+n^2<\sigma,\\ z_{0k0}=0\ \text{ for }\ k^2+Re\left(\alpha_2\right)<\sigma,\\ z_{00n}=0\ \text{ for }\ Re\left(\alpha_1\right)+n^2<\sigma,\\ z_{00\pm b}\neq 0\,\text{and}\,z_{000}\neq 0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)\geq\sigma,\\ z_{00\pm b}=0\,\text{and}\,z_{000}=0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)<\sigma,\end{gathered}$$

then solution (22) of the problem (5)-(7) will satisfy the inequality (8);

for the Case 2:

$$\begin{gathered}z_{0kn}=0\ \text{ for }\ k^2+n^2<\sigma,\\ z_{0k0}=0\,\text{and}\,z_{0k\pm b}=0\ \text{ при }\ k^2+Re\left(\alpha_2\right)<\sigma,\\ z_{00n}=0\ \text{ for }\ Re\left(\alpha_1\right)+n^2<\sigma,\\ z_{00\pm b}\neq 0\,\text{and}\,z_{000}\neq 0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)\geq\sigma,\\ z_{00\pm b}=0\,\text{and}\,z_{000}=0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)<\sigma,\end{gathered}$$

then solution (23) of the problem (5)-(7) will satisfy the inequality (8);

for the Case 3:

$$\begin{gathered}z_{0kn}=0\ \text{ for }\ k^2+n^2<\sigma,\\ z_{0k0}=0\ \text{ for }\ k^2+Re\left(\alpha_2\right)<\sigma,\\ z_{00n}=0\,\text{and}\,z_{0\pm an}=0\ \text{ for }\ Re\left(\alpha_1\right)+n^2<\sigma,\\ z_{00\pm b}\neq 0\,\text{and}\,z_{000}\neq 0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)\geq\sigma,\\ z_{00\pm b}=0\,\text{and}\,z_{000}=0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)<\sigma,\end{gathered}$$

then solution (24) of the problem (5)-(7) will satisfy the inequality (8);

for the Case 4:

$$\begin{gathered}z_{0kn}=0\ \text{ for }\ k^2+n^2<\sigma,\\ z_{0k0}=0\,\text{and}\,z_{0k\pm b}=0\ \text{ for }\ k^2+Re\left(\alpha_2\right)<\sigma,\\ z_{00n}=0\,\text{and}\,z_{0\pm an}=0\ \text{ for }\ Re\left(\alpha_1\right)+n^2<\sigma,\\ z_{000}\neq 0,\ z_{00\pm b}\neq 0,\ z_{0\pm a0}\neq 0\,\text{and}\,z_{0\pm a\pm b}\neq 0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)\geq\sigma,\\ z_{000}=0,\ z_{00\pm b}=0,\ z_{0\pm a0}=0\,\text{and}\,z_{0\pm a\pm b}=0\ \text{ for }\ Re\left(\alpha_1\right)+Re\left(\alpha_2\right)<\sigma,\end{gathered}$$

then solution (25) of the problem (5)-(7) will satisfy the inequality (8);

We introduce the following notation for the sets of pairs of indices $(k,n)\quad k,n\in\mathbb{Z}$:

$$I_1=\left\{(k,n)\,|k^2+n^2\geq\sigma\right\},\ \ \overline{I_1}=\left\{(k,n)\,|k^2+n^2<\sigma\right\}$$

$$I_2=\left\{(k,0)\,,(k,\pm b)\,|k^2+Re\left(\alpha_2\right)\geq\sigma\right\},\ \ \overline{I_2}=\left\{(k,0)\,,(k,\pm b)\,|k^2+Re\left(\alpha_2\right)<\sigma\right\}$$

$$I_3=\left\{(0,n)\,,(\pm a,n)\,|Re\left(\alpha_1\right)+n^2\geq\sigma\right\},\ \ \overline{I_3}=\left\{(0,n)\,,(\pm a,n)\,|\left(\alpha_1\right)+n^2<\sigma\right\}$$

$$I_4=\{(0,0)\,,(0,\pm b)\,,(\pm a,0)\,,(\pm a,\pm b)\,|Re\left(\alpha_1\right)+Re\left(\alpha_2\right)\geq\sigma\}$$

$$\overline{I_4} = \{(0,0), (0,\pm b), (\pm a, 0), (\pm a, \pm b) \,|Re(\alpha_1) + Re(\alpha_2) < \sigma\}$$
$$\overline{I} = \overline{I_1} \cup \overline{I_2} \cup \overline{I_3} \cup \overline{I_4}.$$

Let condition

$$z_{0kn} = 0 \text{ при } (k,n) \in \overline{I}$$

satisfies for (22)-(25), then stabilized solution $z_{stab}(x,y,t)$ of the problem (5)-(7), satisfying the inequality (8), can be written as:

for the Case 1:

$$z(x,y,t) = \sum_{(k,n)\in I_1} z_{0kn} e^{-\left(k^2+n^2\right)t}\psi_{kn}(x,y) + \sum_{(k,0)\in I_2} z_{0k0} e^{-\left(k^2+\alpha_2\right)t}\psi_{k0}(x,y) +$$
$$+ \sum_{(0,n)\in I_3} z_{00n} e^{-\left(\alpha_1+n^2\right)t}\psi_{0n}(x,y) + A(\alpha_1,\alpha_2) e^{-(\alpha_1+\alpha_2)t}\psi_{00}(x,y),$$

where

$$A(\alpha_1,\alpha_2) = \begin{cases} z_{000}, & I_4 \neq \oslash \\ 0, & I_4 = \oslash \end{cases}$$

for the Case 2:

$$z(x,y,t) = \sum_{(k,n)\in I_1} z_{0kn} e^{-\left(k^2+n^2\right)t}\psi_{kn}(x,y) + \sum_{(k,0)\in I_2} z_{0k0} e^{-\left(k^2+\alpha_2\right)t}\psi_{k0}(x,y) +$$
$$+ \sum_{(k,\pm b)\in I_2} z_{0k\pm b} e^{-\left(k^2+\alpha_2\right)t}\psi_{k\pm b}(x,y) + \sum_{(0,n)\in I_3} z_{00n} e^{-\left(\alpha_1+n^2\right)t}\psi_{0n}(x,y) +$$
$$+ A_1(\alpha_1,\alpha_2) e^{-(\alpha_1+\alpha_2)t}\psi_{0\pm b}(x,y) + A_2(\alpha_1,\alpha_2) e^{-(\alpha_1+\alpha_2)t}\psi_{00}(x,y),$$

where

$$A_1(\alpha_1,\alpha_2) = \begin{cases} z_{00\pm b}, & I_4 \neq \oslash \\ 0, & I_4 = \oslash \end{cases}; \quad A_2(\alpha_1,\alpha_2) = \begin{cases} z_{000}, & I_4 \neq \oslash \\ 0, & I_4 = \oslash \end{cases}$$

for the Case 3:

$$z(x,y,t) = \sum_{(k,n)\in I_1} z_{0kn} e^{-\left(k^2+n^2\right)t}\psi_{kn}(x,y) + \sum_{(k,0)\in I_2} z_{0k0} e^{-\left(k^2+\alpha_2\right)t}\psi_{k0}(x,y) +$$
$$+ \sum_{(0,n)\in I_3} z_{00n} e^{-\left(\alpha_1+n^2\right)t}\psi_{0n}(x,y) + \sum_{(\pm a,n)\in I_3} z_{0\pm an} e^{-\left(\alpha_1+n^2\right)t}\psi_{\pm an}(x,y) +$$
$$+ A_1(\alpha_1,\alpha_2) e^{-(\alpha_1+\alpha_2)t}\psi_{\pm a0}(x,y) + A_2(\alpha_1,\alpha_2) e^{-(\alpha_1+\alpha_2)t}\psi_{00}(x,y),$$

where

$$A_1(\alpha_1,\alpha_2) = \begin{cases} z_{0\pm a0}, & I_4 \neq \oslash \\ 0, & I_4 = \oslash \end{cases}; \quad A_2(\alpha_1,\alpha_2) = \begin{cases} z_{000}, & I_4 \neq \oslash \\ 0, & I_4 = \oslash \end{cases}$$

for the Case 4:

$$z(x,y,t) = \sum_{(k,n)\in I_1} z_{0kn} e^{-\left(k^2+n^2\right)t}\psi_{kn}(x,y) + \sum_{(k,0)\in I_2} z_{0k0} e^{-\left(k^2+\alpha_2\right)t}\psi_{k0}(x,y) +$$
$$+ \sum_{(0,n)\in I_3} z_{00n} e^{-\left(\alpha_1+n^2\right)t}\psi_{0n}(x,y) + \sum_{(\pm a,n)\in I_3} z_{0\pm an} e^{-\left(\alpha_1+n^2\right)t}\psi_{\pm an}(x,y) +$$
$$+ \sum_{(k,\pm b)\in I_2} z_{0k\pm b} e^{-\left(k^2+\alpha_2\right)t}\psi_{k\pm b}(x,y) + A_1(\alpha_1,\alpha_2) e^{-(\alpha_1+\alpha_2)t}\psi_{\pm a\pm b}(x,y) +$$

$$+A_2(\alpha_1,\alpha_2)\,e^{-(\alpha_1+\alpha_2)t}\psi_{\pm a0}(x,y)+$$
$$+A_3(\alpha_1,\alpha_2)\,e^{-(\alpha_1+\alpha_2)t}\psi_{0\pm b}(x,y)+A_4(\alpha_1,\alpha_2)\,e_{-(\alpha_1+\alpha_2)t}\psi_{00}(x,y),$$

where

$$A_1(\alpha_1,\alpha_2)=\begin{cases} z_{0\pm a\pm b}, & I_4\neq\oslash \\ 0, & I_4=\oslash \end{cases};\quad A_2(\alpha_1,\alpha_2)=\begin{cases} z_{0\pm a0}, & I_4\neq\oslash \\ 0, & I_4=\oslash \end{cases}$$

$$A_3(\alpha_1,\alpha_2)=\begin{cases} z_{00\pm b}, & I_4\neq\oslash \\ 0, & I_4=\oslash \end{cases};\quad A_4(\alpha_1,\alpha_2)=\begin{cases} z_{000}, & I_4\neq\oslash \\ 0, & I_4=\oslash \end{cases}$$

Algorithm for solving the stabilization problem

The results of the previous sections make it possible to implement the following algorithm for the approximate construction of boundary control functions (and even in the form of synthesis that work out random perturbations) that provide a monotonic (no slower than a given exponent) decrease on time according to the formula (4) of the $L_2(\Omega)$-norm solution.

Step 1. To the original boundary value problem (1)–(3) on a parallelepiped the base of which is a square with side π, with the nonhomogeneous Dirichlet boundary conditions and an initial condition on the square Ω determined by the given function $u_0(x,y)$ is posed an auxiliary boundary value problem (5)–(7) on an extended parallelepiped, the base of which is a square with side 2π, with periodicity conditions (instead of the Dirichlet conditions) and an initial function $z_0(x,y)$ on the bottom base of the extended parallelepiped Ω_1. The function $z_0(x,y)$ will be defined as a continuation of the given function $u_0(x,y)$.

Thus, in the auxiliary boundary value problem (5)–(7) it is necessary to redefine the function $z_0(x,y)$ on the square Ω_1, so that for the solution $z(x,y,t)$ of the problem (5)–(7) the requirement (8) will be satisfied. In this case, the condition (4) will be also satisfied for its restriction $u(x,y,t)$ and the required boundary control $p(x,y,t)\quad \{x,y,t\}\in\Sigma$ will be defined as the trace of the function $z(x,y,t)$ при $\{x,y,t\}\in\Sigma$.

Step 2. Construction of complete biorthogonal systems of functions on the square Ω_1 by solving the corresponding spectral problems.

Step 3. Find the expansion coefficients of the required function $z_0(x,y)$ on the square Ω_1 according to the complete biorthogonal system constructed in the previous step, so that condition (8) is satisfied. Note that condition (8) ensures requirement (4) for the solution of the boundary value problem (1)–(3).

Step 4. Using the found solution $z(x,y,t)$ of the auxiliary boundary value problem (5)–(7), as its restriction on the parallelepiped Q we find the solution $u(x,y,t)$ of the original boundary value problem (1)–(3), satisfying the required condition (4). Boundary control $p(x,y,t)\quad \{x,y,t\}\in\Sigma$ we find as a trace of the solution $z_{stab}(x,y,t)$, i.e.

$$p(x,y,t)=z_{stab}(x,y,t)|_{\{x,y,t\}\in\Sigma}.$$

Conclusion

The paper proposes a problem formulation of boundary stabilization (forming a parallelepiped) of the solution of the boundary value problem for the heat equation with a loaded two-dimensional Laplace operator, where the loaded terms are the values of the required function and traces of the derivatives of the required function at fixed points, and an algorithm for the approximate construction of boundary controls.

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М.Т. Дженалиев, М.И. Рамазанов, А.Х. Аттаев, Н.К. Гульманов

Екіөлшемді жүктелген параболалық теңдеуінің шешімін тұрақтандыру

Мақалада екіөлшемді жүктелген Лаплас операторымен жылуөткізгіштік теңдеуі үшін шекаралық есептің шешімін тұрақтандыру есебі қарастырылды. Жүктелген қосылғыштар белгіленген нүктелердегі ізделінді функцияның мәндері мен оның бірінші ретті дербес туындыларының іздері болып табылады. Сондай-ақ есепте шекаралық басқару функцияларын құру алгоритмі ұсынылған.

Кілт сөздер: шекара бойынша тұрақтандыру есебі, жылуөткізгіштіктің жүктелген теңдеуі, жүктелген Лаплас операторы, биортогоналды жүйе, тұрақтандыру, алгоритм.

М.Т. Дженалиев, М.И. Рамазанов, А.Х. Аттаев, Н.К. Гульманов

Стабилизация решения для двумерного нагруженного параболического уравнения

В статье рассмотрена задача стабилизации решения граничной задачи для уравнения теплопроводности с нагруженным двумерным оператором Лапласа. Нагруженные слагаемые представляют собой значения искомой функции и следы ее частных производных первого порядка в фиксированных точках. Предложен алгоритм построения граничных управляющих функций.

Ключевые слова: задача стабилизации по границе, нагруженное уравнение теплопроводности, нагруженный оператор Лапласа, биортогональная система, стабилизация, алгоритм.

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Problem of describing the function of a GPR source

In this paper, we consider the problem of determining the source $h(t)\delta(x)$ of electromagnetic waves from GPR data. The task of electromagnetic sensing is to find the pulse characteristic of the medium $r(t)$ and consists in calculating the response of the medium to the pulse source of excitation $\delta(t)$ (Dirac Delta function). To determine the analytical expression of the impulse response of a homogeneous medium $r(t)$, we use the method proposed in [1-2]. To determine $h(t)$, the inverse problem is reduced to a system of Volterra integral equations. The source function $h(\tau)$, is defined as the solution of the Volterra integral equation of the first kind, $f(t) = \int_0^t r(t-\tau)h(\tau)d\tau$ in which $f(t)$ is the data obtained by the GPR at the observation points. The problem of calculating the function of the GPR source $h(\tau)$ consists in numerically solving the inverse problem, in which the function of the source $h(\tau)$ is unknown, and the electromagnetic parameters of the medium are known: the permittivity ε; the conductivity σ; the magnetic permeability μ and the response of the medium to a given excitation $h(\tau)$.

Keywords: radargram processing, source recovery, mathematical simulation, calculation results.

1 Introduction

Ground-penetrating radars have builtin software, the output of which is a radarogram, i.e. time scans of the reflected signal taken along the route. To interpret radarograms, engineering techniques are used, and it also depends on the geophysicist's experience and skills in reading radarograms. On the other hand, there is a different direction of interpretation of radarograms, based on mathematical and computer modeling of the propagation and reflection of electromagnetic waves in the medium. The radarogram is a function of the run time to inhomogeneity. In practice, geophysicists are interested in the physical characteristics of inhomogeneities that depend on spatial coordinates. For the numerical solution of the inverse coefficient problem, it is necessary to have a table value of the source of the disturbance, as well as table values of the reflected signals (medium responses) at the measurement points. To solve these problems, we have developed an algorithm for restoring the source, and as a result, determining the response of media corresponding to real GPR data at observation points. Here is a brief overview of the work related to these problems. Questions of uniqueness of the solution of inverse coefficient problems are studied in [3]. Numerical algorithms for solving such a class of inverse problems are described in [4], which also studies the convergence of iterative methods for determining coefficients for hyperbolic equations. The problem of restoring the source of a tsunami is considered in [5].

In [6], we consider the inverse problem of identifying a source that depends on the spatial variable $F(x)$ in the one-dimensional wave equation $u_{tt} = c^2 u_{xx} + F(x)H(t - \frac{x}{c})$, $(x,t) \in \{(x,t)\,|x > 0, -\infty \le t \le T\}$. The measured data is taken as $g(t) := u(0,t)$. The relationship between this task and the GPR data interpretation task is shown. An iterative algorithm for restoring an unknown source $F(x)$ is developed. The algorithm is based on the decomposition of $F(x)$ functions

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into a Fourier series and representation of the solution of a direct problem using the $F(x)$ function. Next, we solve the minimization problem for the discrete form of the Tikhonov functional, which is reduced to a linear algebraic system and solved numerically. Calculations show that the proposed algorithm allows reconstructing the x-dependent $F(x)$ source with sufficient accuracy for clean and noisy data.

In [7], it was assumed that a function h(t) of a special type was defined for each carrier frequency. Assuming that the coefficients of dielectric, magnetic permeability, and conductivity are smooth functions, a fundamental solution for the system of the Maxwell equation is constructed in [8]. The original problem is reduced to an auxiliary problem for vector and scalar potentials. In [9], we derive a formula for solving the Cauchy problem of a multidimensional Telegraph equation, which allows us to reduce the problem to quadrature forms and obtain exact solutions explicitly. Later, using these formulas, we can obtain formulas for calculating the impulse response of an arbitrary sufficiently smooth medium.

The need to solve inverse coefficient problems for hyperbolic equations follows from practical applications that arise in problems of seismology, electrical exploration, tomography, rock mechanics, archeology, and many problems of natural science. A class of questions of existence, uniqueness of solutions, regularization and stability are considered in a series of works by scientists from near and far abroad (see, for example, [10]-[24]). Algorithms for numerical solutions of coefficient inverse problems for hyperbolic equations are covered in monographs [15]-[17].

Note that the development of interpretation methods is still in demand in geophysical research. As noted above, in practice, the inverse problems that arise in georadar methods are solved by various approximate methods, the most commonly used ones are described in [25] and in the review [26]. To study the horizontally-layered media are used for economical methods of solution of direct problems of radar. This method is based on the method of layer-by-layer recalculation, which was proposed in [27], and was further improved in [28]-[29]. This algorithm was used for electrical exploration and elasticity problems in [30]-[33].

2 Description of the method

The problem of accurately describing the GPR source function occurs in all known GPR series produced. An approximate value of the source function leads to an error in interpreting the GPR data.

One of the reasons that leads to an inaccurate description of the GPR source is the effect of a pulse of the order of 10 nanoseconds. It is almost impossible to measure the amplitude of the pulse carrier in the specified time interval.

Note also that knowledge of the source function is necessary to solve the inverse problem, since effective algorithms are constructed not for the function $f(t)$ that is the response of the medium, radar data from the source $h(t)$, but for the $r(t)$ pulse characteristic of the medium perturbed by the Delta function $\delta(t)$ of the source.

The proposed method for determining the functions of the GPR source $h(t)$ is based on the numerical solution of the inverse problem, in which the function of the source $h(t)$ is unknown, and the electromagnetic parameters of the medium are known: the permittivity ε, magnetic permeability μ, conductivity σ, and the response of the medium to a given excitation $h(t)$.

We give to the description of the mathematical model. Let us consider the problem statement formulated and studied in monographs [3,4] for the geoelectric equation:

$$\mu\varepsilon\omega_{tt} = \omega_{zz} + \omega_{xx} - \mu\sigma\omega_t + \delta\left(t\right)\delta(z)\eta\left(x\right) \tag{1}$$

$$\omega|_{t<0} = 0, \omega_t|_{t<0} = 0 \tag{2}$$

$$\omega(0, x, t) = r(x, t) \tag{3}$$

Here: ε is permittivity, μ is magnetic permittivity,

σ is medium conductivity, $\delta(t)$ is Dirac Delta function,
$r(x,t)$ is an impulse response of the medium,
We introduce the equation of geoelectrics, in which electromagnetic waves are excited by the source of the GPR $h\,(t)$:

$$\mu\varepsilon\omega u_{tt} = u_{zz} + u_{xx} - \mu\sigma u_t + \delta\,(z)\,\eta\,(x)\,h\,(t) \tag{4}$$

$$u|_{t<0} = 0 u_t\,|_{t<0} = 0 \tag{5}$$

$$u(0,x^*,t) = f(x^*,t) \tag{6}$$

Here: $h(t)$ is a function describing the GPR source as a function of time, $f(x^*,t)$ is a the response of the medium at the observation point x^* (the radarogram trace).
To determine the impulse response of the medium, we consider the Volterra equation of the first kind:

$$f(x,t) = \int_0^t r(x,t-\tau)h(\tau)d\tau \tag{7}$$

In this equation, the left side is known, i.e. the radar data at the observation point. The $h(t)$ is a function describes the radar source. The relation (7) shows the relationship between the response of the medium, which is a trace of the solution of the problem (4)-(6) and the impulse response (3). Obviously, it is advisable to determine the impulse response $r(x,t)$ analytically. For this purpose, in the future we use the method of solving the direct problem (1)-(3), with constant coefficients, given in [1]. To analyze the numerical algorithm, we conduct experimental studies in a homogeneous medium with known geoelectric properties.

Analytical method for determining the impulse response

Following [34], we denote:

$$a_0^2 = \frac{1}{\mu\varepsilon}, a_1 = -\frac{\sigma}{\varepsilon}, \tag{8}$$

Then, taking into account the notation (8), we write problem (1)-(3) in the form:

$$\omega_{tt} = a_0^2(\omega_{zz} + \omega_{xx}) + a_1\omega_t - a_0\delta\,(t)\,\delta(z)\eta\,(x) \tag{9}$$

$$\omega|_{t<0} =, \omega_t|_{t<0} = 0 \tag{10}$$

$$\omega(0,x,t) = r(x,t) \tag{11}$$

Let's introduce a new function ϑ instead of ω using the formula

$$\omega = e^{\alpha t}\vartheta$$

Assuming $\alpha = \frac{1}{2}a_1$,$c^2 = -\alpha^2 + 2a_1$ from the relations (9)-(10), we get :

$$\vartheta_{tt} = a_0^2(\vartheta_{zz} + \vartheta_{xx}) + c^2\vartheta + a_0\eta(x)\delta(t)$$

$$\vartheta|_{t<0} = 0,\, \vartheta_t|_{t<0} = 0$$

Next, to get an explicit analytical expression for the impulse response of the medium, we use the method of work [1].
We decompose the following functions into a Fourier series in the system of function $\left\{e^{ijx}\right\}$.

$$\vartheta\,(x,z,t) = \sum_y \vartheta^j\,(z,t)\,e^{ijx}$$

$$\eta\left(x\right)=\sum_{j}\eta_{j}e^{ijx}$$

Then be

$$(\vartheta_{tt}^{j}-a_{0}^{2}\vartheta_{zz}+(\lambda^{j})^{2}\vartheta^{j}-c^{2}\vartheta^{j})=\eta_{j}(z)\delta(t)$$

Finally, after performing a series of calculations of relations (9)-(10), we write it differently:

$$\vartheta_{tt}^{j}-a_{0}^{2}\vartheta_{zz}^{j}+(\lambda^{j})^{2}\vartheta^{j}=0 \quad (12)$$

$$\vartheta^{j}\big|_{t<0}=0,\,\vartheta_{t}^{j}\Big|_{t<0}=0 \quad (13)$$

$$[\vartheta_{z}]_{z=0}=a_{0}\eta^{j}\delta(t) \quad (14)$$

By analogy, after applying the Fourier transform, condition (11) has the form

$$\vartheta(0,x,t)=r(x,t)=\sum_{j}r_{j}\left(t\right)e^{ijx}$$

$$\vartheta^{j}(0,t)=r^{j}(t)$$

Solution of problem (12)-(14) have the form

$$\vartheta^{j}(z,t)=\frac{1}{2}-a_{0}\eta^{i}(0)J_{0}\left(\lambda^{j}\sqrt{t-z^{2}}\right) \quad (15)$$

Assuming $z=0$,in expression (15), we obtain an explicit analytical expression for the impulse response of the medium:

$$r^{j}(t)=-J_{0}(\lambda^{j}t)\lambda^{j}\eta_{0}^{j}j=1,N$$

3 A description of the method in the General case

In [9], a formula for solving the Cauchy problem for a linear Telegraph equation in three-dimensional space is derived and the Kirchhoff formula for a linear wave equation that passes into it at zero conductivity. Reducing the problem of the field of a given is derived external current source in an infinite homogeneous isotropic conductor to a generalized Cauchy problem for a three-dimensional Telegraph equation is considered, which allows us to reduce this problem to quadratures, and in some cases to obtain accurate three-dimensional solutions with a propagating front, which are of great applied value for testing methods for the numerical solution of Maxwell's equations. As an example, an exact solution to the problem of the field of the Hertz electric dipole with an arbitrary dependence of the current on time in an infinite homogeneous isotropic conductor is constructed is constructed.

Let us present formulas for solving the Cauchy problem for the telegraph equation described in [9]. The Cauchy problem for the spatially three-dimensional linear telegraph equation is considered, the formulation of which is completely similar to that for the wave equation

$$LE_{T}=\frac{\partial^{2}}{\partial t^{2}}E_{T}+\lambda_{\delta}\frac{\partial}{\partial t}E_{T}-c^{2}\Delta E_{T}=\delta(x,t),\;\;x\in R^{3},\;t>0 \quad (16)$$

$$E_{T}(x,t)\;|_{t=0}=E_{T}^{0}(x),\;\;\frac{\partial}{\partial t}E_{T}(x,t)\;|_{t=0}=(E_{T})_{t}^{0}(x),\;\;x\in R^{3} \quad (17)$$

Here: $\lambda_\delta > 0$, $c > 0$ are set constants, $\delta(x,t)$ is the Delta function, $E_T(x,t)$, $(E_T)_t^0(x)$ are set functions. Then the exact solution of problem (16)-(17) for a spatially three-dimensional linear Telegraph equation has the form:

$$E_T(x,t) = e^{-\frac{1}{2}\lambda_\delta t}(\frac{\theta(t)}{4\pi c^2 t}\delta_{S_{ct}}(x) + \frac{\omega\theta(t)\theta\,(ct-|x|)}{4\pi c^2\sqrt{(ct)^2-|x|^2}}J_1\,(\frac{\omega}{c}\sqrt{(ct)^2-|x|^2})) \tag{18}$$

Here: $\theta(t)$ is a theta function, $\delta_{S_{ct}}(x)$ is a simple layer on the sphere $S_{ct} = \{x : |x| = ct\}$ with density 1, $\lambda_\delta = \frac{\delta}{EE_0}$.

In [8], a fundamental solution for the system of the Maxwell equation is constructed.

$$\begin{array}{l} rot\,H = \varepsilon\left(\frac{\partial}{\partial t}E + \sigma E\right) + j^0\delta\left(x-x^0\right)\,\delta\left(t\right), \; x^0 \in R^3 \\ rot\,H = -\mu\frac{\partial}{\partial t}H, \; (x,\,t) \in R^4 \end{array} \tag{19}$$

on the construction of its generalized solution satisfying the conditions

$$H\,|_{t<0} = E\,|_{t<0} = 0. \tag{20}$$

Assuming that ε, μ, $\sigma$are smooth functions of the point $x \in R^3$,$\varepsilon > 0$, $\mu > 0$,

$$j = j^0\delta\left(x-x^0\right)\,\delta\left(t\right), \; x^0 \in R^3, \tag{21}$$

j^0 is some numeric vector, δ is the Dirac Delta function. We consider the vector potential

$$H = \frac{1}{\mu}\;rot\,A, \;\; E = -\frac{\partial}{\partial t}\;A - \nabla\varphi \tag{22}$$

Here the Lorentz gauge condition is

$$div\,A + \varepsilon\mu\left(\varphi_t + \sigma\varphi\right) = 0, \quad \varphi\,|_{t<0} = 0. \tag{23}$$

The scalar potential is found through the vector by the formula

$$\varphi\left(x,t,x^0\right) = -\frac{1}{\varepsilon\left(x\right)\mu\left(x\right)}\int_0^t e^{\sigma(x)(z-t)}div\;A\left(x,z,x^0\right)\,dz, \tag{24}$$

Problem (19)—(21) is reduced to some auxiliary problem, for vector (22) and scalar potentials AA, φ : (see (23)-(24))

For a vector potential, the Cauchy problem is studied:

$$\begin{array}{l} LA \equiv \frac{\partial^2}{\partial t^2}A + \sigma\frac{\partial A}{\partial t} - \frac{1}{\varepsilon\mu}\left|\Delta A + \frac{1}{\varepsilon}\;\nabla\frac{1}{\mu}\times rot\,A - \right. \\ -\nabla\left(\frac{1}{\varepsilon\mu}\right)\;div\;A + \frac{1}{\varepsilon\mu}\nabla\sigma\int_0^t e^{\sigma(x)(z-t)}div\;A\;\left(x,z,x^0\right)dz = \frac{1}{\varepsilon}j, \end{array}$$

$$A\,|_{t<0} = 0.$$

Conclusions

When numerically modeling the solution of the inverse coefficient problem, the question arises about the table value of the source of the disturbance, as well as the table value of the reflected signals (medium responses), at the measurement points. To solve these issues, we have developed an algorithm for restoring the source. Next, it is necessary to carry out measurements using ground-penetrating radar in a homogeneous environment, for example, a sand pit with known geoelectric properties. The response of the medium obtained by georadar from a test environment is used to calculate the table values definition source excited by the GPR. Then the obtained source value is used in algorithms for determining the geoelectric section of the object under study. The authors of this article have developed a series of algorithms for the numerical solution of inverse and ill-posed problems, and they can be found in published monographs and scientific articles [3, 4, 6, 20].

In General, using exact formulas (18) for solving the Cauchy problem for a spatially three-dimensional linear Telegraph equation, one can obtain formulas for calculating the impulse response of an arbitrary sufficiently smooth medium.

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С.И. Кабанихин, Қ.Т. Искаков, Д.Қ. Тоқсеит, М.А. Шишленин, А. Тойбеков

Георадар дереккөзінің функциясын түсіндіру мәселесі

Мақалада георадар мәліметтерінен электромагниттік толқындардың $h(t)\delta(x)$ дереккөзін анықтау мәселесі қарастырылды. Электромагниттік зондтаудың міндеттері $r(t)$ ортасының импульстік реакциясын табу болып табылады және ортаның импульстің қозу дереккөзіне реакциясын есептеп шығару $\delta(t)$(Dirac delta функциясы). Авторлар біртекті $r(t)$ ортаның импульс реакциясының аналитикалық өрнегін анықтау үшін [1, 2] ұсынылған әдісті қолданды. $h(t)$ анықтау үшін кері есеп вольтердің интегралдық теңдеулер жүйесіне келтірілді. Функция бірінші типтегі Вольтерраның интегралдық теңдеуінің шешімі ретінде анықталды, $f(t) = \int_0^t r(t-\tau)h(\tau)d\tau$-де $h(\tau)$ — бұл GPR бақылау нүктелерінде алынған мәліметтер. GPR дереккөзі $h(\tau)$ функциясын есептеу есебі кері есепті сандық шешуден тарады, онда $h(\tau)$ дереккөзі функциясы белгісіз, ал ортаның электромагниттік параметрлері белгілі: өткізгіштік ε; өткізгіштік conductivity σ; магнит өткізгіштігі permeability μ және ортаның берілген қоздыруға реакциясы $h(\tau)$.

Кілт сөздер: радарограмманы өңдеу, дереккөзді қалпына келтіру, математикалық модельдеу, есептеу нәтижелері.

С.И. Кабанихин, К.Т. Искаков, Д.К. Токсеит, М.А. Шишленин, А. Тойбеков

Задача описания функции источника георадара

В статье рассмотрена задача определения источника $h(t)\delta(x)$ электромагнитных волн по данным георадара. Задача электромагнитного зондирования заключается в нахождении импульсной характеристики среды $r(t)$ и состоит в вычислении отклика среды на импульсный источник возбуждения $\delta(t)$ (дельта-функция Дирака). Для определения аналитического выражения импульсной характеристики однородной среды $r(t)$ авторами использован метод, предложенный в [1, 2]. Для определения $h(t)$ рассматриваемая обратная задача сводится к системе вольтеровских интегральных уравнений. Функция источника $h(\tau)$ определяется как решение интегрального уравнения Вольтерра первого рода $f(t) = \int_0^t r(t-\tau)h(\tau)d\tau$, в котором $f(t)$ — данные, полученные георадаром в точках наблюдения. Задача вычисления функции источника георадара $h(\tau)$ состоит в численном решении обратной задачи, в которой неизвестной является функция источника $h(\tau)$, а известными предстают электромагнитные параметры среды: диэлектрическая проницаемость ε; проводимость σ; магнитная проницаемость μ и отклик среды на заданное возбуждение $h(\tau)$.

Ключевые слова: обработка радарограммы, восстановление источника, математическое моделирование, результаты расчетов.

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Numerical implementation of solving a control problem for loaded differential equations with multi-point condition

A linear boundary value problem with a parameter for loaded differential equations with multi-point condition is considered. The method of parameterization is used for solving the considered problem. We offer an algorithm for solving a control problem for the system of loaded differential equations with multi-point condition. The linear boundary value problem with a parameter for loaded differential equations with multi-point condition by introducing additional parameters at the partition points is reduced to equivalent boundary value problem with parameters. The equivalent boundary value problem with parameters consists of the Cauchy problem for the system of ordinary differential equations with parameters, multi-point condition, and continuity conditions. The solution of the Cauchy problem for the system of ordinary differential equations with parameters is constructed using the fundamental matrix of differential equation. The system of linear algebraic equations concerning the parameters is composed by substituting the values of the corresponding points in the built solutions to the multi-point condition and continuity conditions. The numerical method for finding the solution of the problem is suggested, which based on the solving the constructed system and solving Cauchy problem on the subintervals by Adams method and Bulirsch-Stoer method. The proposed numerical implementation is illustrated by example.

Keywords: problem with parameter, loaded differential equation, multi-point condition, numerical method, algorithm.

Introduction

The problem of constructing effective models finds its solution in many areas of science and technology. Therefore, a modern approach in the theory of control and identification of parameters should be directed to the development of new constructive methods and modifications of known methods for solving boundary value problems with parameters for ordinary and loaded differential equations with multi-point condition [1-7].

In recent years, an intensive study of loaded differential equations associated with various applications of problems has been observed. The problems of the applications described by these equations include the problems of long-term forecasting and regulation of the level of groundwater and soil resources, simulation of processes of transported particles, and some optimal control problems [8]. The theory of boundary value problems for the loaded differential equations with parameters is rapidly developing and is used in various fields of applied mathematics, biophysics, biomedicine, chemistry, etc. [8-13]. Despite this, the questions of finding the effective criterions of unique solvability and constructing the numerical algorithms for finding the solutions of boundary value problems for the system of loaded differential equations with parameters remain open. One of the constructive methods for investigating and solving the boundary value problems with parameters for the system of ordinary differential equations is the parameterization method [14].

The parameterization method was developed for the investigating and solving the boundary value problems for the system of ordinary differential equations. Later, this method was developed for the two-point boundary value problems for the Fredholm integro-differential equations [15-19]. The algorithms

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for finding the numerical solutions of the problems are considered. This approach are applied to two-point boundary value problems for system of ordinary and loaded differential equations with parameter [20, 21].

In the present paper, we offer numerical algorithm of parametrization method for solving the control problem for the loaded differential equation with multi-point condition.

So, we consider a linear boundary value problem with a parameter for loaded differential equation with multi-point condition

$$\frac{dx}{dt} = A(t)x + \sum_{j=1}^{N} K_j(t)x(\theta_j) + A_0(t)\mu + f(t), \quad x \in R^n, \ \mu \in R^m, \ t \in (0,T), \tag{1}$$

$$\sum_{i=0}^{N+1} C_i x(\theta_i) + B_0\mu = d, \quad d \in R^{n+m}. \tag{2}$$

Here the $(n \times n)$ matrices $A(t)$, $K_j(t)$ are continuous on $[0,T]$, $j = \overline{1,N}$; the $(n \times m)$ matrix $A_0(t)$ is continuous on $[0,T]$; the n vector $f(t)$ is continuous on $[0,T]$; the $((n+m) \times m)$ matrix B_0 and the $((n+m)\times n)$ matrices C_i, $i = \overline{0, N+1}$ are constants; $0 = \theta_0 < \theta_1 < \theta_2 < ... < \theta_{N-1} < \theta_N < \theta_{N+1} = T$; $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

$C([0,T], R^n)$ is the space of continuous functions $x : [0,T] \to R^n$ with the norm $||x||_1 = \max\limits_{t\in[0,T]} ||x(t)||$.

A pair $(x^*(t), \mu^*)$, with $x^*(t) \in C([0,T], R^n)$, $\mu^* \in R^m$, where n vector function $x^*(t)$ is continuously differentiable on $(0,T)$, is called a solution to problem (1), (2), if it satisfies the loaded differential equation (1) and condition (2) for $\mu = \mu^*$.

1. Scheme of the method

Points $0 = \theta_0 < \theta_1 < \theta_2 < ... < \theta_{N-1} < \theta_N < \theta_{N+1} = T$ are given and the interval $[0,T)$ is divided into N subintervals: $[0,T) = \bigcup\limits_{r=1}^{N+1} [\theta_{r-1}, \theta_r)$.

$C([0,T], \theta_N, R^{n(N+1)})$ is the space of systems functions $x[t] = (x_1(t), x_2(t), ..., x_{N+1}(t))$, where x_r: $[\theta_{r-1}, \theta_r) \to R^n$ are continuous and have finite left-sided $\lim\limits_{t\to\theta_r-0} x_r(t)$ for all $r = \overline{1, N+1}$, with the norm $||x[\cdot]||_2 = \max\limits_{r=\overline{1,N+1}} \sup\limits_{t\in[\theta_{r-1},\theta_r)} ||x_r(t)||$.

Let $x_r(t)$ be the restriction of function $x(t)$ to the $r-$th interval $[\theta_{r-1}, \theta_r)$, i.e. $x_r(t) = x(t)$ for $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N+1}$. Then we reduce problem (1), (2) to the equivalent multipoint boundary value problem

$$\frac{dx_r}{dt} = A(t)x_r + \sum_{j=1}^{N} K_j(t)x_{j+1}(\theta_j) + A_0(t)\mu + f(t), \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, N+1}, \tag{3}$$

$$\sum_{i=0}^{N} C_i x_{i+1}(\theta_i) + C_{N+1} \lim_{t\to T-0} x_{N+1}(t) + B_0\mu = d, \tag{4}$$

$$\lim_{t\to\theta_p-0} x_p(t) = x_{p+1}(\theta_p), \qquad p = \overline{1,N}, \tag{5}$$

where (5) are conditions for matching the solution at the interior points of partition.

The solution of problem (3)-(5) is the pair $(x^*[t], \mu^*)$ with elements $x^*[t] = (x_1^*(t), x_2^*(t), ..., x_{N+1}^*(t)) \in$ $\in C([0,T], \theta_N, R^{n(N+1)})$, $\mu \in R^m$, where functions $x_r^*(t)$, $r = \overline{1, N+1}$, are continuously differentiable on $[\theta_{r-1}, \theta_r)$, which satisfies system of loaded differential equations (3) and condition (4) with $\mu = \mu*$ and continuity conditions (5).

We introduce the additional parameters λ_r as a values of required functions at the points of partition: $\lambda_r = x_r(\theta_{r-1})$, $r = \overline{1, N+1}$, the $(N+2)-$th component is assigned the original parameter μ, i.e. $\lambda_{N+2} = \mu$. Making the substitution $x_r(t) = u_r(t) + \lambda_r$ on every r-th interval $[\theta_{r-1}, \theta_r)$, $r = \overline{1, N+1}$, we obtain multipoint boundary value problem with parameters

$$\frac{du_r}{dt} = A(t)(u_r + \lambda_r) + \sum_{j=1}^{N} K_j(t)\lambda_{j+1} + A_0(t)\lambda_{N+2} + f(t), \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, N+1}, \quad (6)$$

$$u_r(\theta_{r-1}) = 0, \qquad r = \overline{1, N+1}, \quad (7)$$

$$\sum_{i=0}^{N} C_i\lambda_{i+1} + C_{N+1}\lambda_{N+1} + C_{N+1} \lim_{t\to T-0} u_{N+1}(t) + B_0\lambda_{N+2} = d, \quad (8)$$

$$\lambda_p + \lim_{t\to\theta_p-0} u_p(t) = \lambda_{p+1}, \qquad p = \overline{1, N}, \quad (9)$$

A solution to problem with parameters (6)–(9) is a pair of functions $(u^*[t], \lambda^*)$, where the function $u^*[t] = \big(u_1^*(t), u_2^*(t), \ldots, u_{N+1}^*(t)\big) \in C([0,T], \theta_N, R^{n(N+1)})$ with continuously differentiable on $[\theta_{r-1}, \theta_r)$ components $u_r^*(t)$, $r = \overline{1, N+1}$, and $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_{N+1}^*, \lambda_{N+2}^*) \in R^{n(N+1)+m}$, satisfies system of ordinary differential equations (6), initial conditions (7), and relations (8), (9) for $\lambda_j = \lambda_j^*$, $j = \overline{1, N+2}$.

If the pair $(x^*(t), \mu^*)$ is a solution to problem (1), (2), then the pair $(u^*[t], \lambda^*)$ with elements $u^*[t] = \big(u_1^*(t), u_2^*(t), \ldots, u_{N+1}^*(t)\big) \in C([0,T], \theta_N, R^{n(N+1)})$, $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_{N+1}^*, \lambda_{N+2}^*) \in R^{n(N+1)+m}$, where $u_r^*(t) = x^*(t) - x^*(\theta_{r-1})$, $t \in [\theta_{r-1}, \theta_r)$, $\lambda_r^* = x^*(\theta_{r-1})$, $r = \overline{1, N+1}$, $\lambda_{N+2}^* = \mu^*$ is a solution to problem (6)–(9). Conversely, if the pair $(\widetilde{u}[t], \widetilde{\lambda})$ with elements $\widetilde{u}[t] = \big(\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_{N+1}(t)\big) \in$ $\in C([0,T], \theta_N, R^{n(N+1)})$, $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_{N+1}, \widetilde{\lambda}_{N+2}) \in R^{n(N+1)+m}$, is a solution to problem (6)–(9), then the pair $(\widetilde{x}(t), \widetilde{\mu})$ defined by the equalities $\widetilde{x}(t) = \widetilde{u}_r(t) + \widetilde{\lambda}_r$, $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N+1}$, and $\widetilde{x}(T) = \lim_{t\to T-0} \widetilde{u}_{N+1}(t) + \widetilde{\lambda}_{N+1}$, and $\widetilde{\mu} = \widetilde{\lambda}_{N+2}$, is a solution to the origin problem with parameter (1), (2).

Let $X_r(t)$ be a fundamental matrix to the differential equation $\dfrac{dx}{dt} = A(t)x$ on $[\theta_{r-1}, \theta_r]$, $r = \overline{1, N+1}$. Then the unique solution to the Cauchy problem for the system of ordinary differential equations (6), (7) at the fixed values $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{N+1}, \lambda_{N+2})$, has the following form

$$u_r(t) = X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)A(\tau)d\tau\lambda_r + X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)A_0(\tau)d\tau\lambda_{N+2} +$$

$$+ X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\sum_{j=1}^{N} K_j(\tau)d\tau\lambda_{j+1} + X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)f(\tau)d\tau, \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, N+1}. \quad (10)$$

Substituting the corresponding right-hand sides of (10) into the conditions (8), (9), we obtain a system of linear algebraic equations with respect to the parameters λ_r, $\quad r = \overline{1, N+2}$:

$$\sum_{i=0}^{N} C_i\lambda_{i+1} + C_{N+1}\lambda_{N+1} + C_{N+1}\Big\{X_{N+1}(T)\int_{\theta_N}^{T} X_{N+1}^{-1}(\tau)A(\tau)d\tau\lambda_{N+1} +$$

$$+X_{N+1}(T)\int\limits_{\theta_N}^{T} X_{N+1}^{-1}(\tau)A_0(\tau)d\tau\lambda_{N+2}+X_{N+1}(T)\int\limits_{\theta_N}^{T} X_{N+1}^{-1}(\tau)\sum_{j=1}^{N}K_j(\tau)d\tau\lambda_{j+1}\Big\}+B_0\lambda_{N+2}=$$

$$=d-C_{N+1}X_{N+1}(T)\int\limits_{\theta_N}^{T} X_{N+1}^{-1}(\tau)f(\tau)d\tau, \tag{11}$$

$$\lambda_p+X_p(\theta_p)\int\limits_{\theta_{p-1}}^{\theta_p} X_p^{-1}(\tau)A(\tau)d\tau\lambda_p+X_p(\theta_p)\int\limits_{\theta_{p-1}}^{\theta_p} X_p^{-1}(\tau)A_0(\tau)d\tau\lambda_{N+2}+$$

$$+X_p(\theta_p)\int\limits_{\theta_{p-1}}^{\theta_p} X_p^{-1}(\tau)\sum_{j=1}^{N}K_j(\tau)d\tau\lambda_{j+1}-\lambda_{p+1}=-X_p(\theta_p)\int\limits_{\theta_{p-1}}^{\theta_p} X_p^{-1}(\tau)f(\tau)d\tau, \qquad p=\overline{1,N}. \tag{12}$$

Denoting by $Q_*(\theta_N)$ the matrix corresponding to the left-hand side of system (11), (12) which is consist of the coefficients at the parameters λ_r, $\quad r=\overline{1,N+2}$, and then introducing the vector

$$F_*(\theta_N)=\begin{pmatrix} d-C_{N+1}X_{N+1}(T)\int\limits_{\theta_N}^{T} X_{N+1}^{-1}(\tau)f(\tau)d\tau \\ -X_1(\theta_1)\int\limits_{0}^{\theta_1} X_1^{-1}(\tau)f(\tau)d\tau \\ \dots \\ -X_N(\theta_N)\int\limits_{\theta_{N-1}}^{\theta_N} X_N^{-1}(\tau)f(\tau)d\tau, \end{pmatrix}$$

we write the system (11), (12) as:

$$Q_*(\theta_N)\lambda=F_*(\theta_N), \quad \lambda\in R^{n(N+1)+m}. \tag{13}$$

It is not difficult to establish that the solvability of the boundary value problem (1), (2) is equivalent to the solvability of the system (13). The solution of the system (13) is a vector $\lambda^*=(\lambda_1^*,\lambda_2^*,...,\lambda_{N+1}^*,$ $\lambda_{N+2}^*)\in R^{n(N+1)+m}$, consists of the values of the solutions of the original problem (1), (2) in the initial points of subintervals, i.e. $\lambda_r^*=x^*(\theta_{r-1})$, $\quad r=\overline{1,N+1}$, $\lambda_{N+2}^*=\mu^*$.

Further we consider the Cauchy problems for ordinary differential equations on subintervals

$$\frac{dz}{dt}=A(t)z+P(t), \quad z(\theta_{r-1})=0, \quad t\in[\theta_{r-1},\theta_r], \quad r=\overline{1,N+1}, \tag{14}$$

where $P(t)$ is either $(n\times n)$ matrix, or n vector, both continuous on $[\theta_{r-1},\theta_r], r=\overline{1,N+1}$. Consequently, solution to problem (14) is a square matrix or a vector of dimension n. Denote by a(P,t) the solution to the Cauchy problem (14). Obviously,

$$a(P,t)=X_r(t)\int\limits_{\theta_{r-1}}^{t} X_r^{-1}(\tau)P(\tau)d\tau, \quad t\in[\theta_{r-1},\theta_r], \quad r=\overline{1,N+1},$$

where $X_r(t)$ is a fundamental matrix of differential equation (14) on the r-th interval.

3. Algorithm for finding of solution to problem (1), (2)

We offer the following numerical implementation of algorithm. This algorithm is based on the the Adams method and the Bulirsch-Stoer method to solve the Cauchy problems for ordinary differential equations.

1. Suppose we have a partition: $0 = \theta_0 < \theta_1 < \theta_2 < ... < \theta_{N-1} < \theta_N < \theta_{N+1} = T$. Divide each r-th interval $[\theta_{r-1}, \theta_r]$, $r = \overline{1, N+1}$, into N_r parts with step $h_r = (\theta_r - \theta_{r-1})/N_r$. Assume on each interval $[\theta_{r-1}, \theta_r]$, $r = \overline{1, N+1}$, the variable $\widehat{\theta}$ takes its discrete values: $\widehat{\theta} = \theta_{r-1}$, $\widehat{\theta} = \theta_{r-1} + h_r$, ..., $\widehat{\theta} = \theta_{r-1} + (N_r - 1)h_r$, $\widehat{\theta} = \theta_r$, and denote by $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1, N+1}$, the set of such points.

2. Solving the Cauchy problems for ordinary differential equations

$$\frac{dz}{dt} = A(t)z + A(t), \quad z(\theta_{r-1}) = 0, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N+1},$$

$$\frac{dz}{dt} = A(t)z + K_j(t), \quad z(\theta_{r-1}) = 0, \quad t \in [\theta_{r-1}, \theta_r], \quad j = \overline{1, N}, \quad r = \overline{1, N+1},$$

$$\frac{dz}{dt} = A(t)z + A_0(t), \quad z(\theta_{r-1}) = 0, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N+1},$$

$$\frac{dz}{dt} = A(t)z + f(t), \quad z(\theta_{r-1}) = 0, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N+1},$$

by using the Adams method or the Bulirsch-Stoer method, we find the values of $(n \times n)$ matrices $a_r(A, \widehat{\theta})$, $a_r(K_j, \widehat{\theta})$, $j = \overline{1, N}$, $(n \times m)$ matrices $a_r(A_0, \widehat{\theta})$ and n vector $a_r(f, \widehat{\theta})$ on $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1, N+1}$.

3. Construct the system of linear algebraic equations with respect to parameters

$$Q_*^{\widetilde{h}}(\theta_N)\lambda = F_*^{\widetilde{h}}(\theta_N), \quad \lambda \in R^{n(N+1)+m}. \tag{15}$$

Solving the system (15), we find $\lambda^{\widetilde{h}}$. As noted above, the elements of $\lambda^{\widetilde{h}} = (\lambda_1^{\widetilde{h}}, \lambda_2^{\widetilde{h}}, ..., \lambda_{N+1}^{\widetilde{h}}, \lambda_{N+2}^{\widetilde{h}})$ are the values of approximate solution to problem (1), (2) in the starting points of subintervals: $x^{\widetilde{h}_r}(\theta_{r-1}) = \lambda_r^{\widetilde{h}}$, $r = \overline{1, N+1}$, $\mu^* = \lambda_{N+2}^*$.

4. To define the values of approximate solution at the remaining points of set $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1, N+1}$, we solve the Cauchy problems

$$\frac{dx}{dt} = A(t)x + \sum_{j=1}^{N} K_j(t)\lambda_{j+1}^{\widetilde{h}} + A_0(t)\lambda_{N+2}^{\widetilde{h}} + f(t),$$

$$x(\theta_{r-1}) = \lambda_r^{\widetilde{h}}, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N+1}.$$

And the solutions to Cauchy problems are found by the Adams method or the Bulirsch-Stoer method. Thus, the algorithm allows us to find the numerical solution to the problem (1), (2). To illustrate the proposed approach for the numerical solving linear boundary value problem with a parameter for an loaded differential equation with multipoint condition (1), (2) on the basis of parameterization method, let us consider the following example.

4. Example

Consider a linear boundary value problem with a parameter for loaded differential equation with multipoint condition

$$\frac{dx}{dt} = A(t)x + \sum_{j=1}^{N} K_j(t)x(\theta_j) + A_0(t)\mu + f(t), \quad x \in R^2, \ \mu \in R^3, \ t \in (0, 1), \tag{16}$$

$$\sum_{i=0}^{N+1} C_i x(\theta_i) + B_0\mu = d, \quad d \in R^5. \tag{17}$$

where $A(t) = \begin{pmatrix} t & cos(t) \\ 1 & t^2-2 \end{pmatrix}$, $A_0(t) = \begin{pmatrix} t & e^t & t-3 \\ 0 & t^2 & t+7 \end{pmatrix}$, $B_0 = \begin{pmatrix} 1 & 4 & 6 \\ -4 & 6 & 2 \\ 1 & 7 & -2 \\ -4 & 3 & 11 \\ 1 & 0 & 4 \end{pmatrix}$.

Case 1. Let $N = 1$. $\theta_0 = 0$, $\theta_1 = \frac{1}{2}$, $\theta_2 = 1$, $K_1(t) = \begin{pmatrix} t & t-1 \\ 3t+1 & t \end{pmatrix}$,

$$f(t) = \begin{pmatrix} -t^4 + 7t^2 - \frac{21t}{8} + \frac{47}{8} - 19e^t - cos(\pi t)cos(t) - tcos(t) \\ -2t^3 - 19t^2 + \frac{17t}{8} - \frac{481}{8} - t^2cos(\pi t) - \pi sin(\pi t) + 2cos(\pi t) \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 2 & 5 \\ 0 & 2 \\ 6 & -4 \\ 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 6 \\ 4 & 5 \\ 3 & 2 \\ 0 & -4 \\ 3 & 5 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -2 & 5 \\ 0 & 4 \\ 6 & 4 \\ 8 & 0 \\ 3 & -1 \end{pmatrix}, \quad d = \begin{pmatrix} \frac{1097}{8} \\ 149 \\ \frac{667}{8} \\ 150 \\ \frac{159}{8} \end{pmatrix}.$$

We use the numerical implementation of algorithm. Accuracy of solution depends on the accuracy of solving the Cauchy problem on subintervals. We provide the results of the numerical implementation of algorithm based on the Adams method and the Bulirsch-Stoer method by partitioning the subintervals [0, 0.5], [0.5, 1] with step h = 0.05.

Solution to problem with parameter (16), (17) is pair $(x^*(t), \mu^*)$, where $x^*(t) = \begin{pmatrix} t^3 - 4t \\ t + cos(\pi t) \end{pmatrix}$, $\mu^* = \begin{pmatrix} -5 \\ 19 \\ 9 \end{pmatrix}$. Table 1 provides the numerical solution values $(\widetilde{x}(t), \widetilde{\mu})$.

The following estimates are true:
using the Adams method for solving the Cauchy problems for ordinary differential equations

$$\max \|\mu^* - \widetilde{\mu}\| < 0.00005, \qquad \max_{j=\overline{0,20}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.00008;$$

using the Bulirsch-Stoer method for solving the Cauchy problems for ordinary differential equations

$$\max \|\mu^* - \widetilde{\mu}\| < 0.00000002, \qquad \max_{j=\overline{0,20}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.00000002.$$

Case 2. Let $N = 3$. $\theta_0 = 0$, $\theta_1 = \frac{1}{4}$, $\theta_2 = \frac{1}{2}$, $\theta_3 = \frac{3}{4}$, $\theta_4 = 1$, $K_2(t) = \begin{pmatrix} t^2 & 1 \\ t & t^2-1 \end{pmatrix}$,

$$K_3(t) = \begin{pmatrix} 2 & 2t \\ \frac{t^2}{2} & 3 \end{pmatrix}, \quad C_3 = \begin{pmatrix} -1 & 6 \\ 4 & -5 \\ 7 & 2 \\ 0 & 4 \\ -3 & 5 \end{pmatrix} \quad C_4 = \begin{pmatrix} 3 & 5 \\ 1 & 2 \\ 0 & -4 \\ -1 & 0 \\ 3 & -1 \end{pmatrix} \quad d = \begin{pmatrix} \frac{4315}{32} \\ 5\sqrt{2} + \frac{545}{4} \\ \frac{311}{4} \\ 166 - 4\sqrt{2} \\ \frac{853}{32} \end{pmatrix},$$

$$f(t) = \begin{pmatrix} -t^4 + \frac{71t^2}{8} - \frac{305}{64} + \frac{\sqrt{2}t}{2} + \frac{\sqrt{2}}{2} + \frac{893}{32} - 19e^t - cos(\pi t)cos(t) - tcos(t) \\ -2t^3 - \frac{2331t^2}{128} + \frac{101t}{64} - \frac{\sqrt{2}t}{2} + \frac{3\sqrt{2}}{2} - \frac{4017}{64} - t^2cos(\pi t) - \pi sin(\pi t) + 2cos(\pi t) \end{pmatrix}.$$

Table 1

Results received by using MathCad15

t	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$
	Adams method		Bulirsch-Stoer method	
0	-0.0000768202	1.0000483623	0.0000000206	0.9999999887
0.05	-0.1999451311	1.0377205669	-0.1998749798	1.0376883286
0.1	-0.3990642033	1.0510744329	-0.3989999803	1.0510565038
0.15	-0.5966839083	1.0410116624	-0.5966249809	1.0410065115
0.2	-0.7920541444	1.0090107586	-0.7919999817	1.0090169819
0.25	-0.984424811	0.9570903823	-0.9843749827	0.9571067693
0.3	-1.173045817	0.8877597447	-1.1729999838	0.8877852415
0.35	-1.3571670797	0.8039568233	-1.3571249852	0.8039904905
0.4	-1.5360385196	0.7089759575	-1.5359999867	0.709016987
0.45	-1.7089100632	0.6063867668	-1.7088749883	0.6064344599
0.5	-1.875031638	0.4999462346	-1.87499999	0.4999999973
0.55	-2.0336244073	0.393563497	-2.0336249926	0.3935655316
0.6	-2.1839925346	0.2909724033	-2.1839999946	0.2909830051
0.65	-2.3253604761	0.1959893476	-2.3253749967	0.1960095024
0.7	-2.4569780171	0.112185956	-2.4569999988	0.1122147524
0.75	-2.5780953578	0.0428566274	-2.578125001	0.0428932257
0.8	-2.6879621167	-0.0090590279	-2.6880000032	-0.0090169855
0.85	-2.7858287117	-0.0410520677	-2.7858750057	-0.0410065138
0.9	-2.8709446048	-0.0511055772	-2.8710000083	-0.0510565047
0.95	-2.9425595558	-0.0377405131	-2.9426250112	-0.037688328
1	-2.9999235398	-0.0000548018	-3.0000000144	0.0000000132
	$\widetilde{\mu}_1 = -4.9999451449$		$\widetilde{\mu}_1 = -5.0000000147$	
	$\widetilde{\mu}_2 = 19.0000319553$		$\widetilde{\mu}_2 = 18.9999999821$	
	$\widetilde{\mu}_3 = 8.9999764358$		$\widetilde{\mu}_3 = 9.0000000058$	

In this case we provide the results of the numerical implementation of algorithm by partitioning the interval $[0,1]$ with step $h = 0.25$ and partitioning the subintervals $[0, 0.25]$, $[0.25, 0.5]$, $[0.5, 0.75]$, $[0.75, 1]$ with step $h_1 = 0.025$. For the second case the following estimates are true:
The errors of using the Adams method

$$\max \|\mu^* - \widetilde{\mu}\| < 0.00002, \qquad \max_{j=\overline{0,40}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.00002;$$

The errors of using the Bulirsch-Stoer method

$$\max \|\mu^* - \widetilde{\mu}\| < 0.000000001, \qquad \max_{j=\overline{0,40}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.000000003.$$

As we can see, the numerical algorithm based on the Bulirsch-Stoer method proposed is effective and allows us to obtain the numerical solution to the the problem with a parameter for loaded differential equation with multipoint condition of higher order accuracy.

Below in the Figure 1, we plot graphs of the exact and numerical solutions to the problem (16), (17) on the interval [0, 1].

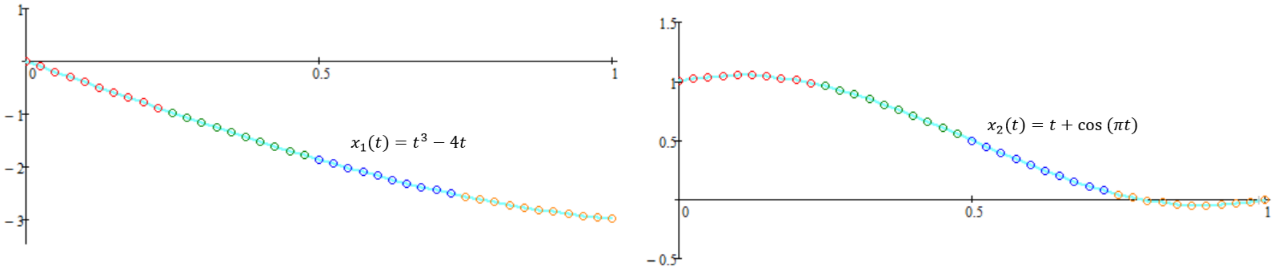

Figure 1. The exact solution values are indicated by the light blue solid line and the numerical solution values obtained by the Bulirsch-Stoer method are indicated by the symbol o

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Ж.М. Кадирбаева, А.Д. Джумабаев

Көпнүктелі шарты бар жүктелген дифференциалдық теңдеулер үшін басқару есебін шешудің сандық жүзеге асырылуы

Көпнүктелі шарты бар жүктелген дифференциалдық теңдеулер үшін параметрі бар сызықтық шеттік есеп зерттелді. Қарастырылып отырған есепті шешу үшін параметрлеу әдісі қолданылды. Көпнүктелі шарты бар жүктелген дифференциалдық теңдеулер үшін басқару есебін шешу алгоритмі ұсынылды. Көпнүктелі шарты бар жүктелген дифференциалдық теңдеулер үшін параметрі бар сызықтық шеттік есеп бөлу нүктелерінде қосымша параметрлер енгізу арқылы пара-пар параметрлері бар шеттік есепке келтірілді. Пара-пар параметрлері бар шеттік есеп жай дифференциалдық теңдеулер жүйесі үшін параметрлері бар Коши есебінен, көпнүктелі шартынан және үзіліссіздік шарттарынан тұрады. Параметрлері бар жай дифференциалдық теңдеулер жүйесі үшін Коши есебінің шешімі дифференциалдық теңдеудің фундаменталдық матрицасының көмегімен тұрғызылды. Тұрғызылған шешімнің сәйкес нүктелеріндегі мәндерін көпнүктелі шартқа және үзіліссіздік шарттарына қоя отырып, параметрлерге қатысты сызықты алгебралық теңдеулер жүйесі құрылды. Қарастырылып отырған есепті шешудің құрылған жүйені және ішкі аралықтардағы Коши есептерін Адамс және Булирш-Штёр әдістерін қолданып, шешуге негізделген сандық әдісі ұсынылды және ол жүзеге асырылу мысалмен көрнектелді.

Кілт сөздер: параметрі бар есеп, жүктелген дифференциалдық теңдеу, көпнүктелі шарт, сандық әдіс, алгоритм.

Ж.М.Кадирбаева, А.Д.Джумабаев

Численная реализация решения задачи управления для нагруженных дифференциальных уравнений с многоточечным условием

Исследована линейная краевая задача с параметром для нагруженных дифференциальных уравнений с многоточечным условием. Для решения рассматриваемой задачи применен метод параметризации. Предложен алгоритм решения задачи управления для системы нагруженных дифференциальных уравнений с многоточечным условием. Линейная краевая задача с параметром для нагруженных дифференциальных уравнений с многоточечным условием путем введения дополнительных параметров в точках разбиения сводится к эквивалентной краевой задаче с параметрами. Эквивалентная краевая задача с параметрами состоит из задачи Коши для системы обыкновенных дифференциальных уравнений с параметрами, многоточечного условия и условия склеивания. Решение задачи Коши для системы обыкновенных дифференциальных уравнений с параметрами строится с помощью фундаментальной матрицы дифференциального уравнения. Подставляя значения в соответствующих точках построенного решения в многоточечное условие и условия склеивания, составляется система линейных алгебраических уравнений относительно параметров. Предложен численный метод нахождения решения задачи, основанный на решении построенной системы и задачи Коши на подынтервалах по методам Адамса и Булирша-Штёра. Предлагаемая численная реализация проиллюстрирована примером.

Ключевые слова: задача с параметром, нагруженное дифференциальное уравнение, многоточечное условие, численный метод, алгоритм.

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Internal boundary layer in a singularly perturbed problem of fractional derivative

This paper is devoted to the study of internal boundary layer. Such motions are often associated with effect of boundary layer, i.e. low flow viscosity affects only in a narrow parietal layer of a streamlined body, and outside this zone the flow is as if there is no viscosity - the so-called ideal flow. Number of exponentials in the boundary layer is determined by the number of non-zero points of the limit operator spectrum. In the paper we consider the case when spectrum of the limit operator vanishes at the point To study the problem the Lomov regularization method is used. The original problem is regularized and the main term of asymptotics of the problem solution is constructed as the low viscosity tends to zero. Numerical results of solutions are obtained for different values of low viscosity.

Keywords: singular perturbation, small parameter, regularization, spectrum stability, asymptotic convergence.

Introduction

A mathematical model of motion of a viscous flow, where a non-uniform transition from one physical characteristic to another occurs, is described by various differential equations with large or small parameters, which are responsible for non-uniformity of the transition. If we consider the self-made flows, then the Navier-Stokes motion equations and the continuity equation are reduced to ordinary differential equations. In addition, if we introduce a small positive parameter then the motion equation will have a small parameter at the highest derivative. Such an equation is called singularly perturbed. Solution of singularly perturbed differential equations is fundamentally different from a solution of ordinary differential equations with a small parameter. Solution of such equations has an area of rapid change of the function, which is located, as a rule, in a neighborhood of one (or two) boundary points of the problem. Such an area of rapid change of function is called area of mathematical boundary layer. Location of the mathematical boundary layer coincides with hydrodynamic boundary layer. Thickness of the boundary layer depends on size of the small parameter, and as the small parameter decreases, the thickness of the boundary layer also decreases. The domain of integration is divided into the external (outside the boundary layer) and the internal (inside the boundary layer). A solution of the singularly perturbed equation is sought as a solution suitable for the external a domain which is then refined in neighborhood of a boundary point where the boundary layer is located [1]. A problem with an internal boundary layer does not belong to the number of standard problems in the singular perturbations theory. This is due to the fact that value of a small parameter is singular for a singularly perturbed equation (see, for example, the equation (2)). In these cases, it is habitually to talk about a "singular point". The singular point gives rise to a double dependence of the solution on singular and regular. We will illustrate this fact with the following specific example - the Cauchy problem for an ordinary differential equation of the second order [2]:

$$\begin{gathered}\varepsilon^2\ddot{y}(t,\varepsilon)+\varepsilon\left(\lambda_1(t)+\lambda_2(t)\right)\dot{y}(t,\varepsilon)+\lambda_1(t)\lambda_2(t)y(t,\varepsilon)=h(t),\\ y(0,\varepsilon)=y^0,\ \dot{y}(0,\varepsilon)=y^1,\end{gathered}\tag{1}$$

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where $\varepsilon > 0$ is a low viscosity, for mathematics it is a small dimensionless parameter; $y(t,\varepsilon)$ is a desired function; function $h(t)$ is a given known function, y^0, y^1 are known constants. It is necessary to study the problem as $\varepsilon \to +0$.

Let functions $\lambda_1(t), \lambda_2(t)$ satisfy the following spectrum stability conditions:

1) $\lambda_i(t) \neq 0,\ i = 1,2$;

2) $\lambda_1(t) \neq \lambda_2(t)\ \forall t \in [0,T]$.

In this case structure of a solution of the problem (1) will be as follows:

$$\begin{aligned} y(t,\varepsilon) &= e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_1(x)dx}[y_{10}(t) + \varepsilon y_{11}(t) + \ldots] + \\ &+ e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(x)dx}[y_{20}(t) + \varepsilon y_{21}(t) + \ldots] + \\ &+ [w_0(t) + \varepsilon w_1(t) + \ldots] \equiv \\ &\equiv y_1(t)e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_1(x)dx} + y_2(t)e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(x)dx} + w(t,\varepsilon). \end{aligned} \tag{2}$$

Let the spectrum stability conditions be violated only at one point:

$$\lambda_1(t) = (t-1)\lambda(t), \quad \lambda(t) \neq 0, \quad \forall t \in [0,T].$$

and the condition 2) hold as usual, then instead of the decomposition (2) the following decomposition of the problem (1) will take place:

$$\begin{aligned} y(t,\varepsilon) = v_\varepsilon(t) &+ e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_1(x)dx}\int\limits_0^t e^{-\frac{1}{\varepsilon}\int\limits_0^\tau \lambda_1(x)dx} d\tau\, [w_{10}(t) + \varepsilon w_{11}(t) + \ldots] + \\ &+ e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(x)dx}\int\limits_0^t \tau e^{-\frac{1}{\varepsilon}\int\limits_0^\tau \lambda_1(x)dx} d\tau\, [w_{20}(t) + \varepsilon w_{21}(t) + \ldots] + \\ &+ [w_{30}(t) + \varepsilon w_{31}(t) + \ldots] \equiv \varphi_1 y_1(t,\varepsilon) + \varphi_2 y_2(t,\varepsilon) + \\ &+ \sigma_1\left(t, \frac{1}{\varepsilon}\right) w_1(t,\varepsilon) + \sigma_2 w_2(t,\varepsilon) + w_3(t,\varepsilon), \end{aligned}$$

where the new type of singularity

$$\sigma\left(t, \frac{1}{\varepsilon}\right) = e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_1(x)dx}\left[\int\limits_0^t e^{-\frac{1}{\varepsilon}\int\limits_0^\tau \lambda_1(x)dx} d\tau + \int\limits_0^t \tau e^{-\frac{1}{\varepsilon}\int\limits_0^\tau \lambda_1(x)dx} d\tau\right]$$

gives the main contribution to describe the internal boundary layer

$$\sigma_1\left(t, \frac{1}{\varepsilon}\right) w_1(t,\varepsilon) + \sigma_2\left(t, \frac{1}{\varepsilon}\right) w_2(t,\varepsilon).$$

1. Statement of the problem

Internal boundary layers in singularly perturbed problems were considered in [3,4] from the standpoint of the regularization method [1,2], in [5-18] from the standpoint of the normal forms method. In this paper the internal boundary layers are investigated in a scalar singularly perturbed problem with fractional derivative:

$$L_\varepsilon y(t,\varepsilon) \equiv \varepsilon y^{(\alpha)} + ty = h(t), \quad y(0,\varepsilon) = y^0, \quad t \in [0,T], \tag{3}$$

where $\varepsilon > 0$ is a small parameter, $\alpha = 1/2$, $h(t) \in C^{\infty}[0, T]$ is a given known function, y^0 is a constant number. It is required to find an asymptotic solution of the problem (1) as $\varepsilon \to +0$.

Singularly perturbed problems with fractional derivatives were studied in [19-22] from the standpoint of the regularization method. In these problems, due to fulfillment of spectrum stability condition, internal boundary layers do not arise. Presence of a singular point at $t = 0$ generates an additional singularity in solution of the problem (3), which is not described in terms of limit operator spectrum of the problem (3). By definition of a fractional derivative [23], the derivative $y^{(1/2)}$ is denoted as $\sqrt{t}\frac{dy(t,\varepsilon)}{dt}$. Then problem (3) has the following form:

$$L_{\varepsilon}y(t,\varepsilon) \equiv \varepsilon\sqrt{t}\frac{dy}{dt} + ty = h(t), \qquad y(0,\varepsilon) = y^0. \tag{4}$$

2. Regularization of problem (4)

We introduce the following regularizing variable:

$$\tau = -\frac{1}{\varepsilon}\int_0^t \sqrt{t}dt = -\frac{2}{3\varepsilon}\sqrt{t^3} \equiv p(t,\varepsilon),$$

and the additional regularizing variable, which takes into account the essentially special singularity, induced by instability of the spectrum at the point $t = 0$.

According to the regularization method [1], we must move from the problem (4), the order of which is reduced when $\varepsilon = 0$, to some extended problem, which preserves its own order at $\varepsilon = 0$. Let us construct the extended problem. If we denote a solution of the extended problem by $\tilde{y}(t, \tau, \sigma, \varepsilon)$, and by $y(t, \varepsilon)$ a solution of the original problem (4), then the following identity holds

$$\tilde{y}(t,\tau,\sigma,\varepsilon)|_{\tau=p(t,\varepsilon),\sigma=q(t,\varepsilon)} \equiv y(t,\varepsilon).$$

This identity will be satisfied if the derivatives with respect to of the functions $\tilde{y}(t, \tau, \sigma, \varepsilon)$ and $y(t, \varepsilon)$ will coincide.

Then for the functions $\tilde{y}(t, \tau, \sigma, \varepsilon)$ the following "extended"task corresponds:

$$\begin{gathered}\tilde{L}_{\varepsilon}\tilde{y}(t,\tau,\sigma,\varepsilon) \equiv \varepsilon\sqrt{t}\tfrac{\partial\tilde{y}}{\partial t} - t\tfrac{\partial\tilde{y}}{\partial\tau} - t\sigma\tfrac{\partial\tilde{y}}{\partial\sigma} + \varepsilon\sqrt{t}\tfrac{\partial\tilde{y}}{\partial\sigma} + t\tilde{y} = h(t),\\ \tilde{y}(0,0,0,\varepsilon) = y^0.\end{gathered} \tag{5}$$

The main advantage of the problem (5) over the task (4) is that its solution $\tilde{y}(t, \tau, \sigma, \varepsilon)$ can be searched in the form of a regular classical series in powers of ε :

$$\tilde{y}(t,\tau,\sigma,\varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k y_k(t,\tau,\sigma), \tag{6}$$

where $y_k(t, \tau, \sigma) \in C^{\infty}[0, T]$, that could not be done for the original problem (4).

Substituting the series (6) into the "extended"problem (5) and equating the coefficients with the same powers of ε, we obtain the following iteration problems:

$$L_0y_0(t,\tau) \equiv -t\frac{\partial y_0}{\partial\tau} - t\sigma\frac{\partial y_0}{\partial\sigma} + ty_0 = h(t), \quad y_0(0,0,0) = y^0; \tag{7_0}$$

$$L_0y_1(t,\tau) = -\sqrt{t}\frac{\partial y_0}{\partial t} - \sqrt{t}\frac{\partial y_0}{\partial\sigma}, \qquad y_1(0,0,0) = 0; \tag{7_1}$$

$$\ldots$$

$$L_0y_k(t,\tau) = -\sqrt{t}\tfrac{\partial y_{k-1}}{\partial t} - \sqrt{t}\tfrac{\partial y_{k-1}}{\partial\sigma}, \quad y_k(0,0,0) = 0; \quad k \geq 2, \tag{7_k}$$

$$\ldots$$

3. Solvability of iterative problems

Solutions of the iteration problems (7_k) will be defined in the following space of functions:

$$U = \{y(t,\tau,\sigma): \ y(t,\tau,\sigma) = y_0(t) + y_1(t)e^{\tau} + y_2(t)\sigma, \quad y_j(t) \in C^{\infty}\left([0,T],^1\right), \ j = 0,1,2\right\}.$$

The problem (7_k) has a solution in the space U, which can be written in the form:

$$y_0(t,\tau,\sigma) = \alpha_1(t)e^{\tau} + \beta_1(t)\sigma + h_0(t), \tag{8}$$

where $\alpha_1(t), \beta_1(t) \in C^{\infty}[0,T]$ are still arbitrary scalar functions, $h_0(t) = h(t)/t$. Here expression of the type $h(t)/t$ at the point $t = 0$ is understood in the limit sense:

$$\left[\frac{h(t)}{t}\right]_{t=0} = \lim_{t\to 0}\left[\frac{h(t)}{t}\right].$$

To calculate the arbitrary functions $\alpha_1(t)$ and $\beta_1(t)$ we subject the right-hand side of the equation (7_1) to the orthogonality conditions (see, for example, [1]). We get the equations:

$$\sqrt{t}\alpha_1'(t) = 0, \qquad \sqrt{t}\beta_1'(t) = 0. \tag{9}$$

Subjecting (8) to the initial condition $y_0(0,0,0) = y^0$, we find that

$$\alpha_1(0) = y^0 - h_0(0),$$

therefore, from the equation (9) the function $\alpha_1(t)$ will be defined complitely:

$$\alpha_1(t) = y^0 - h_0(0).$$

Now let us calculate the function $\beta_1(t)$. From (9) it follows that $\beta_1(t) = const$,

$$h_0(0) + \beta_1(0) = 0.$$

Thus we uniquely find the function:

$$\beta_1(t) = -h_0(0),$$

hence, the solution (8) of the problem (7_0) will be found in the form

$$y_0(t,\tau,\sigma) = [y^0 - h_0(0)]e^{\tau} - h_0(0)\sigma + h_0(t).$$

Doing here constriction on functions $\tau = p(t,\varepsilon), \ \sigma = q(t,\varepsilon)$, we obtain the main term of the asymptotics:

$$y(t,p(t,\varepsilon),q(t,\varepsilon)) \equiv$$

$$\equiv y_{0\varepsilon}(t) = \left[y^0 - h_0(0)\right] e^{-\frac{2}{3\varepsilon}\sqrt{t^3}} -$$

$$-h_0(0)e^{-\frac{2}{3\varepsilon}\sqrt{t^3}} \int\limits_0^t e^{\frac{2}{3\varepsilon}\sqrt{s^3}} + \frac{h(t)}{t}.$$

a solution of the problem (4). The following approximations are calculated in the same way.

We formulate the corresponding result in the form of the following proposition.

Theorem 1. During consistent solution all iteration problems (7_k) are uniquely solvable in the space U.

4. Numerical results

Now we find a solution of the problem (4) by using the computer math systems Maple [24]:

```
>restart; with(plots); odu:=(epsilon)*sqrt(t)*diff(y(t),t)+t*y(t)=h(t);
```

$$odu := \varepsilon \sqrt{t} \left(\frac{d}{dt} y(t)\right) + t\, y(t) = h(t)$$

```
>ins:=y(0)=A;
```

$$ins := y(0) = A$$

```
>dsolve ([odu,ins]);
```

$$y(t) = \left(\int_0^t \frac{h(\_z1)\, e^{\left(\frac{2\,\_z1^{(3/2)}}{3\,\varepsilon}\right)}}{\varepsilon \sqrt{\_z1}}\, d\_z1 + A \right) e^{\left(-\frac{2\,t^{(3/2)}}{3\,\varepsilon}\right)}$$

Different values of these solutions depending on values of the small parameter ε and the constant y^0 are given in the following table:

	t=0	t=0,2	t=0,4	t=0,6	t=0,8	t=1
$\varepsilon = 0,010$	0	0,19557	0,39372	0,59230	0,79110	0,99005
$\varepsilon = 0,025$	0	0,18913	0,38450	0,58094	0,77795	0,97531
$\varepsilon = 0,050$	0	0,17888	0,36962	0,56252	0,75652	0,95125
$\varepsilon = 0,075$	0	0,16927	0,35537	0,54471	0,73573	0,92781
$\varepsilon = 0,100$	0	0,16027	0,34175	0,52754	0,71555	0,90500

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Б.Т. Калимбетов, А.Н. Темирбеков, Б.И. Ескараева

Бөлшек ретті туындылы сингулярлы ауытқыған есепте ішкі шекара қабаты

Мақала ішкі шекара қабатын зерттеуге арналған. Мұндай қозғалыстар көбінесе шекара қабатының әсерімен байланысты, яғни төмен ағынның тұтқырлығы ағынды дененің тар париеталды (қабырғалы) қабатына ғана әсер етеді, ал бұл аймақтың сыртында ағын тұтқырлық жоқ — идеалды ағын деп аталады. Шекаралық қабаттағы экспоненталардың саны шекті оператор спектрінің нөлдік емес нүктелерінің санымен анықталды. Мақала авторлары шекті оператор спектрінің бір нүктеде бұзылған жағдайын қарастырған. Есепті зерттеу үшін Ломовтың регуляриация әдісі қолданылған. Бастапқы есептің регуляризациясы жүргізілген және аз тұтқырлық нөлге ұмтылғандағы есептің шешімінің асимптотикасының бас мүшесі құрылған. Тұтқырлықтын әртүрлі мәндері үшін шешімнің сандық нәтижелері алынған.

Кілт сөздер: сингуляр ауытқу, кіші параметр, регуляризация, спектрдің тұрақтылығы, асимптотикалық жинақтылық.

Б.Т. Калимбетов, А.Н. Темирбеков, Б.И. Ескараева

Внутренний пограничный слой в сингулярно возмущенной задаче с производным дробного порядка

Статья посвящена изучению внутреннего пограничного слоя. Такие движения чаще всего связаны с воздействием пограничного слоя, то есть низкая вязкость потока влияет только в узком париетальном слое обтекаемого тела, а вне этой зоны поток, как если бы не было вязкости, — так называемый идеальный поток. Количество экспонент в пограничном слое определяется количеством ненулевых точек предельного операторного спектра. В статье рассмотрен случай необратимости спектра предельного оператора в одной точке. Для исследования задачи использован метод регуляризации Ломова. Произведена регуляризация исходной задачи, и построен главный член асимптотики решения задачи при стремлении малой вязкости к нулю. Для различных значений малой вязкости получены численные результаты решения.

Ключевые слова: сингулярное возмущение, малый параметр, регуляризация, стабильность спектра, асимптотическая сходимость.

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Characterizing the Ordered AG-Groupoids Through the Properties of Their Different Classes of Ideals

In this article, we have presented some important charcterizations of the ordered non-associative semigroups in relation to their ideals. We have initially characterized the ordered AG-groupoid through the properties of the their ideals, then we characterized the two important classes of these AG-groupoids, namely the regular and intragregular non-associative AG-groupoids. Our aim is also to encourage the research and the maturity of the associative algebraic structures by studying a class of non-associative and non-commutative algebraic structures called the ordered AG-groupoid.

Keywords: Ordered AG-groupoids, left (right, interior, quasi-, bi-, generalized bi-) ideals, regular (intra-regular) ordered AG-groupoids.

Introduction

In 1972, a generalization of commutative semigroups has been established by Kazim et. al [1]. In ternary commutative law: $abc = cba$, they introduced the braces on the left side of this law and explored a new pseudo associative law, that is $(ab)c = (cb)a$. They have called the left invertive law of this law. A groupoid S is said to be a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law : $(ab)c = (cb)a$. This structure is also known as Abel-Grassmann's groupoid (abbreviated as AG-groupoid) in [2]. An AG-groupoid is a midway structure between an abelian semigroup and a groupoid. Mushtaq et. al [3], investigated the concept of ideals in AG-groupoids.

In [4] (resp. [5]), a groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [1], an AG-groupoid is medial, but in general an AG-groupoid needs not to be paramedial. Every AG-groupoid with left identity is paramedial by Protic et. al [2] and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

In [6, 7], if $(S, \cdot, \leq)$ is an ordered semigroup and $\emptyset \neq A \subseteq S$, we define a subset of S as follows : $(A] = \{s \in S : s \leq a \text{ for some } a \in A\}$. A non-empty subset A of S is called a subsemigroup of S if $A^2 \subseteq A$.

A non-empty subset A of S is called a left (resp. right) ideal of S if following hold (1) $SA \subseteq A$ (resp. $AS \subseteq A$). (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$. Equivalent definition: A is called a left(resp. right) ideal of S if $(A] \subseteq A$ and $SA \subseteq A$ (resp. $AS \subseteq A$).

A non-empty subset A of S is called an interior (resp. quasi-) ideal of S if (1) $SAS \subseteq A$ (resp. $(AS] \cap (SA] \subseteq A$). (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$.

A subsemigroup (A non-empty subset) A of S is called a bi- (generalized bi-) ideal of S if (1) $ASA \subseteq A$. (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$. Every bi-ideal of S is a generalized bi-ideal of S.

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In [7, 8], an ordered semigroup is said to be regular if for every $a \in S$,there exists an element $x \in S$ such that $a \leq axa$. Equivalent definitions are as follows: (1) $A \subseteq (ASA]$ for every $A \subseteq S$. (2) $a \in (aSa]$ for every $a \in S$.

In [9,10], an ordered semigroup S is intra-regular if for every $a \in S$ there exist elements $x, y \in S$ such that $a \leq xa^2y$. Equivalent definitions are as follows: (1) $A \subseteq (SA^2S]$ for every $A \subseteq S$. (2) $a \in (Sa^2S]$ for every $a \in S$.

We will define left (right, interior, quasi-, bi-, generalized bi-) ideals in ordered AG-groupoids. We will establish a study by discussing the different properties of such ideals. We will also characterize regular (resp. intra-regular, both regular and intra-regular) ordered AG-groupoids by the properties of left (right, quasi-, bi-, generalized bi-) ideals.

Ideals in Ordered AG-groupoids

An ordered AG-groupoid S, is a partially ordered set, at the same time an AG-groupoid such that $a \leq b$, implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in S$. Two conditions are equivalent to the one condition $(ca)d \leq (cb)d$ for all $a, b, c, d \in S$.

Example 1. Consider a set $S = \{e, f, a, b, c\}$ with the following multiplication "$\cdot$" and order relation "$\leq$"

$\cdot$	e	f	a	b	c
e	e	f	a	b	c
f	f	f	f	b	c
a	a	f	c	b	c
b	c	c	c	f	b
c	b	b	b	c	f

$\leq= \{(e, e), (e, a), (e, b), (e, c), (f, f), (f, b), (f, c), (a, a), (a, c), (b, b), (b, c), (c, c)\}$.

Then $(S, \cdot, \leq)$ is an ordered AG-groupoid with left identity e.

For $\emptyset \neq A \subseteq S$, we define a subset $(A] = \{s \in S : s \leq a$ for some $a \in A\}$ of S and obviously $A \subseteq (A]$. For $\emptyset \neq A, B \subseteq S$, then $((A]] = (A], (A](B] \subseteq (AB]$, $((A](B]] = (AB]$, if $A \subseteq B$, then $(A] \subseteq (B]$, $(A \cap B] \neq (A] \cap (B]$, in general.

For $\emptyset \neq A \subseteq S$. Then A is called an ordered AG-subgroupoid of S if $A^2 \subseteq A$. A is called a left (resp. right) ideal of S if the following hold (1) $SA \subseteq A$ (resp. $AS \subseteq A$). (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$. A is called an ideal of S if A is both a left and a right ideal of S.

We denote by $L(a), R(a), I(a)$ the left ideal, the right ideal and the ideal of S, respectively, generated by a. we have $L(a) = \{s \in S : s \leq a$ or $s \leq xa$ for some $x \in S\} = (a \cup Sa]$, $R(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup (Sa)S]$.

A non-empty subset A of an ordered AG-groupoid S is called an interior (resp. quasi-) ideal of S if (1) $(SA)S \subseteq A$ (resp. $(AS] \cap (SA] \subseteq A$). (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$.

An AG-subgroupoid A of S is called a bi-ideal of S if (1) $(AS)A \subseteq A$. (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$. A non-empty subset A of S is called generalized bi-ideal of S if (1) $(AS)A \subseteq A$. (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$.

Now we give the imperative properties of such ideals of an ordered AG-groupoid S, which will be play a vital rule in the later sections. Specifically we show:

(1) Let S be an ordered AG-groupoid with left identity e. Then every right ideal of S is a ideal of S.

(2) Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then every quasi-ideal of S is a bi-ideal of S.

Lemma 1. Let S be an ordered AG-groupoid with left identity e. Then $SS = S$ and $eS = S = Se$.

Proof: Since $SS \subseteq S$ and $x = ex \in SS$, i.e., $S \subseteq SS$, thus $SS = S$. Obviously, $eS = S$ and $Se = (SS)\, e = (eS)\, S = SS = S$.

Lemma 2. Let S be an odered AG-groupoid with left identity e and $a \in S$. Then Sa is a smallest left ideal of S containing a.

Proof: Let $x \in Sa$ and $s \in S$, this implies that $x = s_1 a$, where $s_1 \in S$. Now

$$\begin{aligned} sx &= s(s_1 a) = (es)(s_1 a) = ((s_1 a)s)e = ((s_1 a)(es))e \\ &= ((s_1 e)(as))e = (e(as))(s_1 e) = (as)(s_1 e) = ((s_1 e)s)a \in Sa. \end{aligned}$$

Thus $sx \in Sa$ and $(Sa] \subseteq Sa$. Since $a = ea \in Sa$, hence Sa is a left ideal of S containing a. Let I be another left ideal of S containing a. Since $sa \in I$, because I is a left ideal of S. But $sa \in Sa$, this means that $Sa \subseteq I$. Therefore Sa is a smallest left ideal of S containing a.

Lemma 3. Let S be an odered AG-groupoid with left identity e and $a \in S$. Then aS is a left ideal of S.

Proof: Straight forward.

Proposition 1. Let S be an ordered AG-groupoid with left identity e and $a \in S$. Then $aS \cup Sa$ is a smallest right ideal of S containing a.

Proof: Let $x \in aS \cup Sa$. We have to show that $(aS \cup Sa)S \subseteq aS \cup Sa$. Now

$$\begin{aligned} (aS \cup Sa)S &= (aS)S \cup (Sa)S = (SS)a \cup (Sa)(eS) \\ &\subseteq Sa \cup (Se)(aS) = Sa \cup S(aS) \\ &= Sa \cup a(SS) \subseteq Sa \cup aS = aS \cup Sa. \end{aligned}$$

Thus $(aS \cup Sa)S \subseteq aS \cup Sa$ and $(aS \cup Sa] \subseteq aS \cup Sa$. Therefore $aS \cup Sa$ is a right ideal of S. Since $a \in Sa$, i.e., $a \in aS \cup Sa$. Let I be another right ideal of S containing a. Now $aS \in IS \subseteq I$ and $Sa = (SS)a = (aS)S \in (IS)S \subseteq IS \subseteq I$, i.e., $aS \cup Sa \subseteq I$. Hence $aS \cup Sa$ is a smallest right ideal of S containing a.

Lemma 4. Let S be an ordered AG-groupoid with left identity e. Then every right ideal of S is an ideal of S.

Proof: Let R be a right ideal of S and $r \in R, s \in S$. Now $sr = (es)r = (rs)e \in (RS)S \subseteq RS \subseteq R$. Thus $SR \subseteq R$ and $(R] \subseteq R$. Hence R is an ideal of S.

Lemma 5. Let S be an ordered AG-groupoid with left identity e such that $(xe)\,S = xS$ for all $x \in S$. Then $(AS)S \subseteq AS$ and $(AS]S \subseteq (AS]$.

Proof: Since

$$\begin{aligned} (AS)S &= (AS)(eS) = (Ae)(SS) \subseteq (Ae)S = AS. \\ \text{and } (AS]S &= (AS](S] \subseteq ((AS)S] \subseteq (AS]. \end{aligned}$$

Remark 1. Let S be an ordered AG-groupoid with left identity e such that $(xe)\,S = xS$ for all $x \in S$, then $(AS]$ is an ideal of S.

Let S be an ordered AG-groupoid with left identity e such that $(xe)\,S = xS$ for all $x \in S$ and $A, B \subseteq S$. Then $(AS)(BS) \subseteq (AB)S$ and $(AS](BS] \subseteq ((AB)S]$. Similarly $(SA)(SB) \subseteq S(AB)$ and $(SA](SB] \subseteq (S(AB)]$.

In general for $A_1, A_2, ..., A_n \subseteq S$, then $(A_1S)(A_2S)...(A_nS) \subseteq (A_1A_2, ..., A_n)S$ and $(A_1S](A_2S]...(A_nS] \subseteq ((A_1A_2, ..., A_n)S]$.

Similarly, $(SA_1)(SA_2)...(SA_n) \subseteq S(A_1A_2, ..., A_n)$ and $(SA_1](SA_2]....(SA_n] \subseteq (S(A_1A_2, ..., A_n)]$.

Lemma 6. Let S be an ordered AG-groupoid. A is a right ideal of S and B is a right ideal of A, then $(B] = B$.

Proof: Since $(B] = \{s \in S \mid s \leq b \text{ for some } b \in B\}$ and $s \in (B]$, this implies that there exists an element $s \in S$ such that $s \leq b$ for some $b \in B \subseteq A$. Thus $S \ni s \leq b \in A$. Now $A \ni s \leq b \in B$ and B is a right ideal of A, i.e., $s \in B$, so $(B] \subseteq B$. Since $B \subseteq (B]$, thus $(B] = B$.

Proposition 2. Let S be an ordered AG-groupoid with left identity e such that $(xe)S = xS$ for all $x \in S$. A is a right ideal of S and B is a right ideal of A such that $(B^2] = B$. Then B is an ideal of S.

Proof: We have to show that B is a right ideal of S. Now

$$\begin{aligned} BS &= (B^2]S = (B^2](S] \subseteq (B^2S] = ((BB)S] \\ &= ((SB)B] \subseteq ((SB)A] = ((SB)(eA)] \\ &= ((Se)(BA)] = (B((Se)A)] = (B((Ae)S)] \\ &= (B(AS)] \subseteq (BA] \subseteq (B] = B \text{ by the Lemma 6.} \end{aligned}$$

Thus $BS \subseteq B$ and $(B] \subseteq B$, i.e., B is a right ideal of S. Hence B is an ideal of S by the Lemma 4.

Lemma 7. Let S be an ordered AG-groupoid. A is a left ideal of S and B is a left ideal of A, then $(B] = B$.

Proof: Same as Lemma 6.

Proposition 3. Let S be an ordered AG-groupoid with left identity e. A is a left ideal of S and B is a left ideal of A such that $(B^2] = B$. Then B is left ideal of S.

Proof: We have to show that B is a left ideal of S. Now

$$\begin{aligned} SB &= S(B^2] = (S](B^2] \subseteq (SB^2] = (S(BB)] \\ &= ((Se)(BB)] = ((SB)(eB)] \\ &\subseteq ((SA)(eB)] \subseteq (AB] \subseteq (B] = B, \text{ by the Lemma 7.} \end{aligned}$$

Thus $SB \subseteq B$ and $(B] \subseteq B$. Hence B is a left ideal of S.

Lemma 8. Every two-sided ideal of S is an interior ideal of S.

Proof: Straight forward.

Proposition 4. Let S be an ordered AG-groupoid with left identity e. Then any non-empty subset I of S is an ideal of S if and only if I is an interior ideal of S.

Proof: Suppose that I is an interior ideal of S. Let $i \in I$ and $s \in S$. Now $is = (ei)s \in (SI)S \subseteq I$, this implies that $IS \subseteq I$ and $(I] \subseteq I$, i.e., I is a right ideal of S. Hence I is an ideal of S by the Lemma 4. Converse is true by the Lemma 8.

Lemma 9. Every right (two-sided) ideal of S is a bi-ideal of S.

Proof: Straight forward.

Lemma 10. Every bi-ideal of S is a generalized bi-ideal of S.

Proof: Obvious.

Lemma 11. Every left (right, two-sided) ideal of S is a quasi-ideal of S.

Proof: Let I be a right ideal of S. Now $(IS] \cap (SI] \subseteq (IS] \subseteq (I] \subseteq I$ and $(I] \subseteq I$. Thus I is a quasi-ideal of S.

Proposition 5. Every quasi-ideal of S is an ordered AG-subgroupoid of S.

Proof: Suppose that I is a quasi-ideal of S. Now $II \subseteq IS \subseteq (I](S] \subseteq (IS]$ and $II \subseteq SI \subseteq (S](I] \subseteq$ $\subseteq (SI]$, i.e., $I^2 = II \subseteq (IS] \cap (SI] \subseteq I$. Hence I is an AG-subgroupoid of S.

Proposition 6. Let R be a right ideal and L be a left ideal of an ordered AG-groupoid S, respectively. Then $R \cap L$ is a quasi-ideal of S.

Proof: Since $((R \cap L)S] \cap (S(R \cap L)] \subseteq (RS] \cap (SL] \subseteq (R] \cap (L] \subseteq R \cap L$ and $(R \cap L] = R \cap L$. Thus $R \cap L$ is a quasi-ideal of S.

Lemma 12. Let S be an ordered AG-groupoid with left identity e such that $(xe)\,S = xS$ for all $x \in S$. Then every quasi-ideal of S is a bi-ideal of S.

Proof: Let Q be a quasi-ideal of S. Now $(QS)Q \subseteq (SS)Q \subseteq SQ \subseteq (SQ]$ and $(QS)Q \subseteq (QS)S =$ $= (QS)(eS) = (Qe)(SS) = (Qe)S = QS \subseteq (QS]$, thus $(QS)Q \subseteq (QS] \cap (SQ] \subseteq Q$. Therefore $(QS)Q \subseteq Q$ and $(Q] \subseteq Q$. Hence Q is a bi-ideal of S.

Regular Ordered AG-groupoids

An ordered AG-groupoid S is called regular if for every $a \in S$, there exists an element $x \in S$ such that $a \leq (ax)a$. Equivalent definitions are as follows:

(1) $A \subseteq ((AS)A]$ for every $A \subseteq S$.

(2) $a \in ((aS)a]$ for every $a \in S$.

An ideal I of an ordered AG-groupoid S is called idempotent if $(I^2] = I$.

In this section, we characterize regular ordered AG-groupoids by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

Lemma 13. Every right ideal of a regular ordered AG-groupoid S

Proof: Let R be a right ideal of S. Let $r \in R$ and $a \in S$, this implies that there exists an element $x \in S$ such that $a \leq (ax)a$. Now $ar \leq ((ax)a)r = (ra)(ax) \in RS \subseteq R$, thus $SR \subseteq R$ and $(R] = R$. Hence R is an ideal of S.

Lemma 14. Every ideal of a regular ordered AG-groupoid S is an idempotent.

Proof: Suppose that I is an ideal of S and $(I^2] = (II] \subseteq (I] = I$. Let $a \in I$, this mean that there exists an element $x \in S$ such that $a \leq (ax)a$. Now $a \leq (ax)a \in (IS)I \subseteq II = I^2$, i.e., $I \subseteq (I^2]$. Therefore $(I^2] = I$.

Remark 2. Every right ideal of a regular ordered AG-groupoid S is an idempotent.

Proposition 7. Let S be a regular ordered AG-groupoid. Then any non-empty subset I of S is an ideal of S if and only if I is an interior ideal of S.

Proof: Assume that I is an interior ideal of S. Let $a \in I$ and $s \in S$, then there exists an element $x \in S$, such that $a \leq (ax)a$. Now $as \leq ((ax)a)s = (sa)(ax) \in (SI)S \subseteq I$. Thus $IS \subseteq I$ and $(I] = I$, i.e., I is a right ideal of S. Hence I is an ideal of S by the Lemma 4. Converse is true by the Lemma 13.

Proposition 8. Let S be a regular ordered AG-groupoid with left identity e. Then $(IS] \cap (SI] = I$, for every right ideal I of S.

Proof: Let I be an ideal of S. This implies that $(IS] \cap (SI] \subseteq I$, because every ideal of S is a quasi-ideal of S. Let $a \in I$, this means that there exists an element $x \in S$ such that $a \leq (ax)a$. Now $a \leq (ax)a \in (IS)I \subseteq II \subseteq IS$, i.e., $I \subseteq (IS]$. Now $a \leq (ax)a = (ax)(ea) = (ae)(xa) \in II \subseteq SI$, i.e., $I \subseteq (SI]$. Thus $I \subseteq (IS] \cap (SI]$. Hence $(IS] \cap (SI] = I$.

Lemma 15. Let S be a regular ordered AG-groupoid. Then $(RL] = R \cap L$, for every right ideal R and every left ideal L of S.

Proof: Since $(RL] \subseteq (RS] \subseteq (R] = R$ and $(RL] \subseteq (SL] \subseteq (L] = L$, i.e., $(RL] \subseteq R \cap L$. Let $a \in R \cap L$, this implies that there exists an element $x \in S$ such that $a \leq (ax)a$. Now $a \leq (ax)a \in (RS)L \subseteq RL$, i.e., $R \cap L \subseteq (RL]$. Therefore $(RL] = R \cap L$.

Theorem 1. Let S be an ordered AG-groupoid with left identity e such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.

(1) S is a regular.

(2) $R \cap L = (RL]$ for every right ideal R and every left ideal L of S.

(3) $Q = ((QS)Q]$ for every quasi-ideal Q of S.

Proof: Suppose that (1) holds. Let Q be a quasi-ideal of S and $a \in Q$, this implies that there exists an element $x \in S$ such that $a \leq (ax)a$. Now $a \leq (ax)a \in (QS)Q$, i.e., $Q \subseteq ((QS)Q] \subseteq (Q] = Q$, because every quasi-ideal of S is a bi-ideal of S. Hence $Q = ((QS)Q]$, i.e., (1) $\Rightarrow$ (3). Assume that (3) holds, let R be a right ideal and L be a left ideal of S. Then R and L be quasi-ideals of S by the Lemma 11, so $R \cap L$ be a quasi-ideal of S. Now $R \cap L = (((R \cap L)S)(R \cap L)] \subseteq ((RS)L] \subseteq (RL]$. Since $(RL] \subseteq R \cap L$, so $(RL] = R \cap L$, i.e., (3) $\Rightarrow$ (2). Suppose that (2) is true, let $a \in S$, then Sa is a left ideal of S containing a by the Lemma 2 and $aS \cup Sa$ is a right ideal of S containing a by the

Proposition 1. By (2),

$$\begin{aligned}(aS \cup Sa) \cap Sa &= ((aS \cup Sa)(Sa)] = ((aS)(Sa) \cup (Sa)(Sa)].\\ (Sa)(Sa) &= ((Se)a)(Sa) = ((ae)S)(Sa) = (aS)(Sa).\end{aligned}$$

Thus

$$\begin{aligned}(aS \cup Sa) \cap Sa &= ((aS)(Sa) \cup (Sa)(Sa)]\\ &= ((aS)(Sa) \cup (aS)(Sa)] = ((aS)(Sa)].\end{aligned}$$

Since $a \in (aS \cup Sa) \cap Sa$, Implies $a \in ((aS)(Sa)]$. Then $a \leq (ax)(ya) = ((ya)x)a = (((ey)a)x)a = = (((ay)e)x)a = ((xe)(ay))a = (a((xe)y))a \in (aS)a$ for any $x, y \in S$, i.e., $a \in ((aS)a]$. Hence a is regular, so S is a regular, i.e., $(2) \Rightarrow (1)$.

Theorem 2. Let S be an ordered AG-groupoid with left identity e such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.

(1) S is a regular.

(2) $Q = ((QS)Q]$ for every quasi-ideal Q of S.

(3) $B = ((BS)B]$ for every bi-ideal B of S.

(4) $G = ((GS)G]$ for every generalized bi-ideal G of S.

Proof: $(1) \Rightarrow (4)$, is obvious. $(4) \Rightarrow (3)$, since every bi-ideal of S is a generalized bi-ideal of S by the Lemma 10. $(3) \Rightarrow (2)$, since every quasi-ideal of S is bi-ideal of S by the Lemma 12. $(2) \Rightarrow (1)$, by the Theorem 1.

Theorem 3. Let S be an ordered AG-groupoid with left identity e such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.

(1) S is a regular.

(2) $Q \cap I = ((QI)Q]$ for every quasi-ideal Q and every ideal I of S.

(3) $B \cap I = ((BI)B]$ for every bi-ideal B and every ideal I of S.

(4) $G \cap I = ((GI)G]$ for every generalized bi-ideal G and every ideal I of S.

Proof: Suppose that (1) is true. Let G be a generalized bi-ideal and I be an ideal of S. Now $((GI)G] \subseteq ((SI)S] \subseteq (I] = I$ and $((GI)G] \subseteq ((GS)G] \subseteq (G] = G$, thus $((GI)G] \subseteq G \cap I$. Let $a \in G \cap I$, this means that there exists an element $x \in S$ such that $a \leq (ax)a$. Now $a \leq (ax)a = = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a \in (GI)G$, thus $G \cap I \subseteq ((GI)G]$. Hence $G \cap I = ((GI)G]$, i.e., $(1) \Rightarrow (4)$. $(4) \Rightarrow (3)$, since every bi-ideal of S is a generalized bi-ideal of S by the Lemma 10. $(3) \Rightarrow (2)$, since every quasi-ideal of S is a bi-ideal of S by the Lemma 12. Assume that (2) is true. Now $Q \cap S = ((QS)Q]$, i.e., $Q = ((QS)Q]$, where Q is a quasi-ideal of S. Hence S is a regular by the Theorem 1.

Theorem 4. Let S be an ordered AG-groupoid with left identity e such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.

(1) S is a regular.

(2) $R \cap Q \subseteq (RQ]$ for every quasi-ideal Q and every right ideal R of S.

(3) $R \cap B \subseteq (RB]$ for every bi-ideal B and every right ideal R of S.

(4) $R \cap G \subseteq (RG]$ for every generalized bi-ideal G and every right ideal R of S.

Proof: $(1) \Rightarrow (4)$, is obvious. $(4) \Rightarrow (3)$, since every bi-ideal of S is a generalized bi-ideal of S. $(3) \Rightarrow (2)$, since every quasi-ideal of S is a bi-ideal of S by the Lemma 12. . Suppose that (2) is true. Now $R \cap Q = Q \cap R \subseteq (RQ]$, where Q is a left ideal and R is right ideal of S, because every left ideal of S is a quasi-ideal of S. Since $(RQ] \subseteq R \cap Q$, thus $R \cap Q = (RQ]$. Hence S is a regular, by the Theorem 1.

Intra-regular Ordered AG-groupoids

An ordered AG-groupoid S is called intra-regular if for every $a \in S$, there exist elements $x, y \in S$ such that $a \leq (xa^2)y$. Equivalent definitions are as follows:

(1) $A \subseteq ((SA^2)S]$ for every $A \subseteq S$.

(2) $a \in ((Sa^2)S]$ for every $a \in S$.

In this section, we characterize intra-regular ordered AG-groupoids by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

Lemma 16. Every left (right) ideal of an intra-regular ordered AG-groupoid S is an ideal of S.

Proof: Let R be a right ideal of S. Let $r \in R$ and $a \in S$, this implies that there exist elements $x, y \in S$ such that $a \leq (xa^2)y$. Now $ar \leq ((xa^2)y)r = (ry)(xa^2) \in RS \subseteq R$. Thus $SR \subseteq R$ and $(R] \subseteq R$. Hence R is an ideal of S.

Lemma 17. Every ideal of an intra-regular ordered AG-groupoid S with left identity e, is an idempotent.

Proof: Suppose that I is an ideal of S and $(I^2] = (II] \subseteq (I] = I$. Let $a \in I$, this means that there exist elements $x, y \in S$ such that $a \leq (xa^2)y$. Now

$$\begin{aligned} a &\leq (xa^2)y = (x(aa))y = (a(xa))y \\ &= (a(xa))(ey) = (ae)((xa)y) = (xa)((ae)y) \in II. \end{aligned}$$

Thus $a \in (II] = (I^2]$. Therefore $(I^2] = I$.

Proposition 9. Let S be an intra-regular ordered AG-groupoid with left identity e. Then any non-empty subset I of S is an ideal of S if and only if I is an interior ideal of S.

Proof: Assume that I is an interior ideal of S. Let $i \in I$ and $a \in S$, then there exist elements $x, y \in S$ such that $x \leq (yx^2)z$. Now

$$\begin{aligned} ia &\leq i((xa^2)y) = i((x(aa))y) \\ &= i((a(xa))y) = i((a(xa))(ey)) \\ &= i((ae)((xa)y)) = i((xa)((ae)y)) \\ &= (xa)(i((ae)y)) = (xi)(a((ae)y)) \in (SI)S \subseteq I. \end{aligned}$$

Thus $IS \subseteq I$ and $(I] \subseteq I$, i.e., I is a right ideal of S. So I is an ideal of S by the Lemma 16. Converse is obvious.

Lemma 18. Let S be an intra-regular ordered AG-groupoid with left identity e. Then $L \cap R \subseteq (LR]$ for every left ideal L and every right ideal R of S.

Proof: Let $a \in L \cap R$, where L is a left ideal and R is a right ideal of S, respectively, this implies that there exist elements $x, y \in S$ such that $a \leq (xa^2)y$. Now

$$\begin{aligned} a &\leq (xa^2)y = (x(aa))y = (a(xa))y = (a(xa))(ey) \\ &= (ae)((xa)y) = (xa)((ae)y) \in LR. \\ &\Rightarrow L \cap R \subseteq (LR]. \end{aligned}$$

Theorem 5. Let S be an ordered AG-groupoid with left identity e such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.

(1) S is an intra-regular.

(2) $L \cap R \subseteq (LR]$ for every left ideal L and every right ideal R of S.

Proof: Since (1) $\Rightarrow$ (2) holds by the Lemma 18. Suppose that (2) holds and $a \in S$, then Sa is a left ideal of S containing a and $aS \cup Sa$ is a right ideal of S containing a. By our supposition

$$\begin{aligned} Sa \cap (aS \cup Sa) &\subseteq ((Sa)(aS \cup Sa)] = ((Sa)(aS) \cup (Sa)(Sa)]. \\ (Sa)(aS) &= (Sa)((ea)S) = (Sa)((Sa)e) = (Sa)((Sa)(ee)) \\ &= (Sa)((Se)(ae)) = (Sa)(S(ae)) = (Sa)(Sa). \end{aligned}$$

Thus

$$\begin{aligned}(aS\cup Sa)\cap Sa &\subseteq ((Sa)(aS)\cup(Sa)(Sa)]\\ &= ((Sa)(Sa)\cup(Sa)(Sa)]\\ &= ((Sa)(Sa)]=(S^2a^2]=(Sa^2]\\ &= (S(a^2e)]=((SS)(a^2e)]=((eS)(a^2S)]=(S(a^2S)]\\ &= (a^2(SS)]=((ea^2)(SS)]=((Sa^2)(Se)]=((Sa^2)S].\end{aligned}$$

Since $a\in(aS\cup Sa)\cap Sa$, implies $a\in((Sa^2)S]$, thus a is an intra regula. Hence S is an intra-regular, i.e., $(2)\Rightarrow(1)$.

Theorem 6. Let S be an ordered AG-groupoid with left identity e such that $(xe)S=xS$ for all $x\in S$. Then the following conditions are equivalent.

(1) S is an intra-regular.

(2) $Q\cap I=((QI)Q]$ for every quasi-ideal Q and every ideal I of S.

(3) $B\cap I=((BI)B]$ for every bi-ideal B and every ideal I of S.

(4) $G\cap I=((GI)G]$ for every generalized bi-ideal G and every ideal I of S.

Proof: Suppose that (1) holds. Let $a\in G\cap I$, where G is a generalized bi-ideal and I is an ideal of S, this implies that there exist elements $x,y\in S$ such that $a\le(xa^2)y$. Now

$$\begin{aligned}a &\le (xa^2)y=(x(aa))y=(a(xa))y=(y(xa))a.\\ y(xa) &\le y(x((xa^2)y))=y((xa^2)(xy))=(xa^2)(y(xy))\\ &= (xa^2)(xy^2)=(x(aa))m,\ \text{say } xy^2=m\\ &= (a(xa))m=(m(xa))a.\\ m(xa) &\le m(x((xa^2)y))=m((xa^2)(xy))=(xa^2)(m(xy))\\ &= (x(aa))n,\ \text{say } m(xy)=n\\ &= (a(xa))n=(n(xa))a\\ &= va,\ \text{say } n(xa)=v.\\ &\Rightarrow y(xa)=(m(xa))a=(va)a=(va)(ea)=(ve)(aa)=a((ve)a).\end{aligned}$$

Thus $a\le(xa^2)y=(y(xa))a=(a((ve)a))a\in(GI)G$. This means that $a\in((GI)G]$, i.e., $G\cap I\subseteq((GI)G]$. Now $((GI)G]\subseteq((SI)S]\subseteq(I]=I$ and $((GI)G]\subseteq((GS)G]\subseteq(G]=G$, thus $((GI)G]\subseteq G\cap I$. Hence $G\cap I=((GI)G]$, i.e., $(1)\Rightarrow(4)$. $(4)\Rightarrow(3)$, every bi-ideal of S is a generalized bi-ideal of S by the Lemma 10. $(3)\Rightarrow(2)$, every quasi-ideal of S is a bi-ideal of S by the Lemma 12. Assume that (2) is true and let R be a right ideal and I be a two-sided ideal of S. Now $I\cap R=((RI)R]\subseteq((SI)R]\subseteq(IR]$, since every right ideal of S is a quasi-ideal of S. Therefore S is an intra-regular by the Theorem 5, i.e., $(2)\Rightarrow(1)$.

Theorem 7. Let S be an ordered AG-groupoid with left identity e such that $(xe)S=xS$ for all $x\in S$. Then the following conditions are equivalent.

(1) S is an intra-regular.

(2) $L\cap Q\subseteq(LQ]$ for every quasi-ideal Q and every left ideal L of S.

(3) $L\cap B\subseteq(LB]$ for every bi-ideal B and every left ideal L of S.

(4) $L\cap G\subseteq(LG]$ for every generalized bi-ideal G and every left ideal L of S.

Proof: Suppose that (1) holds. Let $a\in L\cap G$, where L is a left ideal and G is a generalized bi-ideal of S, this means that there exist elements $x,y\in S$ such that $a\le(xa^2)y$. Now $a\le(xa^2)y=(x(aa))y=$ $=(a(xa))y=(y(xa))a\in LG$, i.e., $a\in(LG]$. Thus $L\cap G\subseteq(LG]$, i.e., $(1)\Rightarrow(4)$. $(4)\Rightarrow(3)$, every bi-ideal of S is a generalized bi-ideal of S. $(3)\Rightarrow(2)$, every quasi-ideal of S is a bi-ideal of S. Assume that (2) is true and let R be a right ideal of S and L be a left ideal of S. Now $L\cap R\subseteq(LR]$, where R is a quasi-ideal of S. Hence S is an intra-regular by the Theorem 5, i.e., $(2)\Rightarrow(1)$.

Theorem 8. Let S be an ordered AG-groupoid with left identity e such that $(xe)\,S = xS$ for all $x \in S$. Then the following conditions are equivalent.

(1) S is an intra-regular.

(2) $L \cap Q \cap R \subseteq ((LQ)R]$ for every quasi-ideal Q, every right ideal R and every left ideal L of S.

(3) $L \cap B \cap R \subseteq ((LB)R]$ for every bi-ideal B, every right ideal R and every left ideal L of S.

(4) $L \cap G \cap R \subseteq ((LG)R]$ for every generalized bi-ideal G, every right ideal R and every left ideal L of S.

Proof: Suppose that (1) holds. Let $a \in L \cap G \cap R$, where L is a left ideal, G is a generalized bi-ideal and R is a right ideal of S, this implies that there exist elements $x, y \in S$ such that $a \leq (xa^2)y$. Now

$$\begin{aligned} a &\leq (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a. \\ y(xa) &\leq y(x((xa^2)y)) = y((xa^2)(xy)) = (xa^2)(y(xy)) \\ &= (xa^2)(xy^2) = (x(aa))m, \text{ say } xy^2 = m \\ &= (a(xa))m = (m(xa))a. \end{aligned}$$

Thus $a \leq (xa^2)y = (y(xa))a = ((m(xa))a)a \in (LG)R$, i.e., $a \in ((LG)R]$. Hence $L \cap G \cap R \subseteq$ $\subseteq ((LG)R]$, i.e., $(1) \Rightarrow (4)$. $(4) \Rightarrow (3)$, every bi-ideal of S is a generalized bi-ideal of S. $(3) \Rightarrow (2)$, every quasi-ideal of S is a bi-ideal of S. Assume that (2) is true. Now

$$\begin{aligned} L \cap S \cap R &\subseteq ((LS)R] = (((eL)S)R] = (((SL)e)R] = (((SL)(ee))R] \\ &= (((Se)(Le))R] \subseteq ((S(Le))R] \subseteq ((SL)R] \subseteq (LR]. \\ &\Rightarrow L \cap R \subseteq (LR]. \end{aligned}$$

Hence S is an intra-regular by the Theorem 5, i.e., $(2) \Rightarrow (1)$.

Regular and Intra-regular Ordered AG-groupoids

In this section, we characterize both regular and intra-regular ordered AG-groupoids by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

Theorem 9. Let S be an ordered AG-groupoid with left identity e such that $(xe)\,S = xS$ for all $x \in S$. Then the following conditions are equivalent.

(1) S is a regular and an intra-regular.

(2) $(B^2] = B$ for every bi-ideal B of S.

(3) $B_1 \cap B_2 = (B_1B_2] \cap (B_2B_1]$ for all bi-ideals B_1, B_2 of S.

Proof: Suppose that (1) holds and B be a bi-ideal of S. Since $(B^2] = (BB] \subseteq (B] = B$. Let $a \in B$, this implies that there exists an element $x \in S$ such that $a \leq (ax)a$, also there exist elements $y, z \in S$ such that $a \leq (ya^2)z$. Now

$$\begin{aligned} a &\leq (ax)a \leq (ax)((ya^2)z) = (((ya^2)z)x)a. \\ ((ya^2)z)x &= (xz)(ya^2) = m(ya^2), \text{ say } m = xz \\ &= m(y(aa)) = m(a(ya)) = a(m(ya)) \\ &\leq ((ax)a)(m(ya)) = ((ax)m)(a(ya)) \\ &= ((mx)a)(a(ya)) = (na)(a(ya)), \text{ say } n = mx \\ &= ((en)a)(a(ya)) = ((an)e)(a(ya)) \\ &= ((an)a)(e(ya)) = ((an)a)(ya) = (sa)(ya), \text{ say } s = an \\ &= (aa)(ys) = (aa)t, \text{ say } t = ys \\ &\leq (((ax)a)a)t = ((aa)(ax))t = (t(ax))(aa) \\ &= (a(tx))(aa) = (aw)(aa), \text{ say } w = tx. \end{aligned}$$

Thus $a \le (((ya^2)z)x)a \le ((aw)(aa))a \in ((BS)B)B \subseteq B^2$, i.e., $a \in (B^2]$. So $B \subseteq (B^2]$, i.e., $(B^2] = B$. Hence (1) $\Rightarrow$ (3) . Assume that (2) is true. Let B_1, B_2 be bi-ideals of S, then $B_1 \cap B_2$ be also a bi-ideal of S. Now $B_1 \cap B_2 = ((B_1 \cap B_2)(B_1 \cap B_2)] \subseteq (B_1B_2]$ and $B_1 \cap B_2 = ((B_1 \cap B_2)(B_1 \cap B_2)] \subseteq$ $\subseteq (B_2B_1]$, thus $B_1 \cap B_2 \subseteq (B_1B_2] \cap (B_2B_1]$. First of all we have to show that $(B_1B_2]$ is a bi-ideal of S. It is enough to show that $((B_1B_2]S)(B_1B_2] \subseteq (B_1B_2]$. Now

$$\begin{aligned} ((B_1B_2]S)(B_1B_2] &= ((B_1B_2](S])(B_1B_2] \\ &\subseteq ((B_1B_2)S](B_1B_2] \\ &\subseteq (((B_1B_2)S)(B_1B_2)] \\ &= (((B_1B_2)(SS))(B_1B_2)] \\ &= (((B_1S)(B_2S))(B_1B_2)] \\ &= (((B_1S)B_1)((B_2S)B_2)] \subseteq (B_1B_2] \\ &\Rightarrow (((B_1B_2)S)(B_1B_2))] \subseteq (B_1B_2]. \end{aligned}$$

Thus $(B_1B_2]$ is a bi-ideal of S, similarly $(B_2B_1]$ is also a bi-ideal of S. Then $(B_1B_2] \cap (B_2B_1]$ is also a bi-ideal of S. Now

$$\begin{aligned} (B_1B_2] \cap (B_2B_1] &= (((B_1B_2] \cap (B_2B_1])((B_1B_2] \cap (B_2B_1])] \\ &\subseteq ((B_1B_2](B_2B_1]] \subseteq (((B_1B_2)(B_2B_1)]] \\ &= ((B_1B_2)(B_2B_1)] \subseteq ((B_1S)(SB_1)] \\ &= (((SB_1)S)B_1] = ((((Se)B_1)S)B_1] \\ &= ((((B_1e)S)S)B_1] = (((B_1S)S)B_1] \\ &= (((SS)B_1)B_1] = ((SB_1)B_1] = (((Se)B_1)B_1] \\ &= (((B_1e)S)B_1] = ((B_1S)B_1] \subseteq (B_1] \\ &\Rightarrow (B_1B_2] \cap (B_2B_1] \subseteq (B_1] = B_1. \end{aligned}$$

Similarly, we have $(B_1B_2] \cap (B_2B_1] \subseteq (B_2] = B_2$, thus $(B_1B_2] \cap (B_2B_1] \subseteq B_1 \cap B_2$. Therefore $B_1 \cap B_2 = (B_1B_2] \cap (B_2B_1]$, i.e., (2) $\Rightarrow$ (3) . Suppose that (3) holds, let R be right ideal of S and I be an ideal of S. Then R and I be bi-ideals of S, because every right ideal and two sided ideal of S is bi-ideal of S by the Lemma 9. Now $R \cap I = (RI] \cap (IR]$, this implies that $R \cap I \subseteq (RI] \cap (IR]$. Thus $R \cap I \subseteq (RI]$ and $R \cap I \subseteq (IR]$, where I is also a left ideal of S. Since $(RI] \subseteq R \cap I$, i.e., $(RI] = R \cap I$, thus S is a regular by the Theorem 1. Also, $R \cap I \subseteq (IR]$, thus S is an intra-regular by the Theorem 5. Hence (3) $\Rightarrow$ (1) .

Theorem 10. Let S be an ordered AG-groupoid with left identity e such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.

(1) S is regular and intra-regular.

(2) Every quasi-ideal of S is an idempotent.

Proof: Suppose that (1) holds. Let Q be a quasi-ideal of S and $(Q^2] = (QQ] \subseteq (Q] = Q$, i.e., $(Q^2] \subseteq Q$. Let $a \in Q$, this implies that there exists an element $x \in S$ such that $a \le (ax)a$, also there exist elements $y, z \in S$ such that $a \le (ya^2)z$. Now

$$\begin{array}{rcl} a & \leq & (ax)a \leq (ax)((ya^2)z) = (((ya^2)z)x)a. \\ ((ya^2)z)x & = & (xz)(ya^2) = m(ya^2), \text{ say } m = xz \\ & = & m(y(aa)) = m(a(ya)) = a(m(ya)) \\ & \leq & ((ax)a)(m(ya)) = ((ax)m)(a(ya)) \\ & = & ((mx)a)(a(ya)) = (qa)(a(ya)), \text{ say } q = mx \\ & = & ((eq)a)(a(ya)) = ((aq)e)(a(ya)) \\ & = & ((aq)a)(e(ya)) = ((aq)a)(ya) = (sa)(ya), \text{ say } s = aq \\ & = & (aa)(ys) = (aa)t, \text{ say } t = ys \\ & \leq & (((ax)a)a)t = ((aa)(ax))t = (t(ax))(aa) \\ & = & (a(tx))(aa) = (aw)(aa), \text{say } w = tx \end{array}$$

Thus $a \leq (((ya^2)z)x)a \leq ((aw)(aa))a \in ((QS)Q)Q \subseteq QQ \subseteq Q^2$, i.e., $a \in (Q^2]$, because every quasi-ideal of S is a bi-ideal of S by the Lemma 12. Thus $Q \subseteq (Q^2]$, i.e., $(Q^2] = Q$. Hence $(1) \Rightarrow (2)$. Assume that (2) is true. Let $a \in S$, then Sa is a left ideal of S containing a, so Sa is a quasi-ideal of S, because every left ideal of S is a quasi-ideal of S. Now $Sa = ((Sa)^2] = ((Sa)(Sa)]$, i.e., $a \in ((Sa)(Sa)]$. Thus S is an intra-regular by the Theorem 5. Now $Sa = ((Sa)(Sa)] = (((Se)a)(Sa)] = (((ae)S)(Sa)] = ((aS)(Sa)]$, i.e., $a \in ((aS)(Sa)]$. Thus S is a regular by the Theorem 1. Therefore $(2) \Rightarrow (1)$.

Theorem 11. Let S be an ordered AG-groupoid with left identity e such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.

(1) S is regular and intra-regular.

(2) Every quasi-ideal of S is an idempotent.

(3) Every bi-ideal of S is an idempotent.

Proof: $(1) \Rightarrow (3)$, by the Theorem 9. $(3) \Rightarrow (2)$, every quasi-ideal of S is a bi-ideal of S, by the Lemma 12. $(2) \Rightarrow (1)$, by the Theorem 10.

Theorem 12. Let S be an ordered AG-groupoid with left identity e such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.

(1) S is regular and intra-regular.

(2) $Q_1 \cap Q_2 \subseteq (Q_1Q_2]$ for all quasi-ideals Q_1, Q_2 of S.

(3) $Q \cap B \subseteq (QB]$ for every quasi-ideal Q and every bi-ideal B of S.

(4) $B \cap Q \subseteq (BQ]$ for every bi-ideal B and every quasi-ideal Q of S.

(5) $B_1 \cap B_2 \subseteq (B_1B_2]$ for all bi-ideals B_1, B_2 of S.

Proof: Suppose that (1) holds. Let B_1, B_2 be bi-ideals of S, then $B_1 \cap B_2$ be also a bi-ideal of S. Since every bi-ideal of S is an idempotent by the Theorem 9, then $B_1 \cap B_2 = ((B_1 \cap B_2)^2] = ((B_1 \cap B_2)(B_1 \cap B_2)] \subseteq (B_1B_2]$. Hence $(1) \Rightarrow (5)$. Since $(5) \Rightarrow (4) \Rightarrow (2)$ and $(5) \Rightarrow (3) \Rightarrow (2)$, because every quasi-ideal of S is a bi-ideal of S by the Lemma 12. Assume that (2) is true. Now $R \cap L \subseteq (RL]$, where R is a right ideal and L is a left ideal of S. Since $(RL] \subseteq R \cap L$, i.e., $R \cap L = (RL]$, thus S is regular. Again by (2) $L \cap R \subseteq (LR]$, thus S is an intra-regular. Therefore $(2) \Rightarrow (1)$.

Conclusion

In this article, we have characterized the non-associative ordered semigroups in terms of their one-sided ideals, ideals, interior ideals, bi-ideals and quais ideals. We have also characterized the intraregular and regular orderded AG-groupoids through the properties of their ideals.

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Реттелген AG-группоидтардың әртүрлі идеалды кластарының қасиеттері бойынша сипаттамасы

Мақалада ассоциативті емес жартылай группалардың идеалдарына қатысты кейбір маңызды сипаттамалар ұсынылған. Біріншіден, біз реттелген AG-группоидты оның идеалының қасиеттері тұрғысынан сипаттадық, содан кейін осы AG-группоидтардың екі маңызды класына, яғни регулярлық және ішкі регулярлық емес ассоциативті емес AG-группоидтарға сипаттама бердік. Біздің мақсатымыз – реттелген AG-группоид деп аталатын ассоциативті емес және коммутативті емес алгебралық құрылымдар класын зерттеу арқылы ассоциативті алгебралық құрылымдарды зерттеу мен дамытуды ынталандыру.

Кілт сөздер: реттелген AG-группоидтар, солға (оң, ішкі, квази-, би-, жалпыланған би-) идеалдар, регулярлық (ішкі регулярлық) реттелген AG-группоидтар.

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Характеризация упорядоченных AG-группоидов через свойства их различных классов идеалов

В статье представлены некоторые важные характеристики упорядоченных неассоциативных полугрупп относительно их идеалов. Сначала были охарактеризован упорядоченный AG-группоид через свойства его идеалов, затем два важных класса этих AG-группоидов, а именно, регулярные и внутрирегулярные неассоциативные AG-группоиды. Цель настоящей работы — стимулирование исследования и развитие ассоциативных алгебраических структур путем изучения класса неассоциативных и некоммутативных алгебраических структур, называемых упорядоченным AG-группоидом.

Ключевые слова: упорядоченные AG-группоиды, левые (правые, внутренние, квази-, би-, обобщенные би-) идеалы, регулярные (внутрирегулярные) упорядоченные AG-группоиды.

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Smoothness and approximative properties of solutions of the singular nonlinear Sturm-Liouville equation

It is known that the eigenvalues $\lambda_n (n = 1, 2, ...)$ numbered in decreasing order and taking the multiplicity of the self-adjoint Sturm-Liouville operator with a completely continuous inverse operator L^{-1} have the following property

(*) $\lambda_n \to 0$, when $n \to \infty$, moreover, than the faster convergence to zero so the operator L^{-1} is best approximated by finite rank operators.

The following question:

- Is it possible for a given nonlinear operator to indicate a decreasing numerical sequence characterized by the property (*)?

naturally arises for nonlinear operators.

In this paper, we study the above question for the nonlinear Sturm-Liouville operator. To solve the above problem the theorem on the maximum regularity of the solutions of the nonlinear Sturm-Liouville equation with greatly growing and rapidly oscillating potential in the space $L_2(R)\,(R = (-\infty, \infty))$ is proved. Two-sided estimates of the Kolmogorov widths of the sets associated with solutions of the nonlinear Sturm-Liouville equation are also obtained. As is known, the obtained estimates of Kolmogorov widths give the opportunity to choose approximation apparatus that guarantees the minimum possible error.

Keywords: maximum regularity; singular nonlinear equation; Sturm-Liouville equation; smoothness of solutions; approximative properties; approximate numbers; Kolmogorov widths; rapidly oscillating potential; greatly growing potential; two-sided estimates.

Introduction

In this paper we study the nonlinear Sturm-Liouville equation

$$Ly = -y'' + q(x, y)y = f(x) \in L_2(R),\ \ R = (-\infty, \infty).$$

The existence and the smoothness of nonlinear elliptic equations solutions in a bounded domain have been studied quite well. A very comprehensive bibliography is contained, for example, in [1-6] and the works cited there.

However, nonlinear equations in an unbounded domain with greatly increasing and rapidly oscillating coefficients arise in applications. For example, the nonlinear Sturm-Liouville equation, which is especially interesting for quantum mechanics.

Here we are interested in the question:

A) to find out the conditions on the potential function $q\,(x, y)$ which provide $y'' \in L_2\,(R)$, when $y\,(x)$ is a solution of the nonlinear equation $Ly = f \in L_2\,(R)$.

We note that the linear case is well studied and reviews are available in [7–12].

It is known that eigenvalues $\lambda_n\,(n = 1, 2, ...)$ of the self-adjoint positive completely continuous operator A in the Hilbert space H are numbered according to their decreasing magnitude and observing their multiplicities have the following approximative properties

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a) $\lambda_n = \min\limits_{k \in l_n} \|A - K\|$, where l_n is the set of all finite-dimensional operators with dimension no greater than n;
b) $\lambda_n \to 0$, when $n \to \infty$, wherein the faster convergence to zero, the operator A better approximated by finite rank operators.

It will be natural to explore a similar issues for a nonlinear Sturm-Liouville operator, i.e. to study the question

B) Is it possible for a given non-linear operator to specify a numerical sequence that characterizes properties a)–b)?

This paper is devoted to the study of the issues A) and B) for the nonlinear Sturm-Liouville equation.

Formulation of the main results. Example

We will make some notation and definitions for the statement of results.

The set of integrable functions with respect to the square of the module in each strictly internal subdomain $\Omega \subset R$ is denoted by $L_{2,loc}(R)$.

The set of functions from $L_{2,loc}(R)$ having generalized first-order derivatives (from $L_{2,loc}(R)$) will be denoted by $W^1_{2,loc}(R)$. We denote the subset of $W^1_{2,loc}(R)$ by $W^1_2(R)$, which elements together with the first generalized derivatives belong to $L_2(R)$. By $W^2_{2,loc}(R)$ we denote the set of all functions $u \in L_{2,loc}(R)$ which with their generalized derivatives up to and including the second order belong to $L_{2,loc}(R)$.

$\|\cdot\|_2$ is the norm of an element in $L_2(R)$, $\|\cdot\|_{2,1}$ is the norm of an element in $W^1_2(R)$, $\|\cdot\|_{2,loc}$ is the norm of an element in $L_{2,loc}(R)$.

Consider the nonlinear Sturm-Liouville equation

$$Ly = -y'' + q(x,y)\,y = f(x) \in L_2(R), \quad R = (-\infty, \infty). \tag{1}$$

Suppose that $q(x,y)$ satisfies the conditions:

i) $q(x,y)$ is a continuous mapping $R \times C$ in $[\delta, \infty)$, $\delta > 0$, C is a set of complex numbers;

ii) $\sup\limits_{|x-\eta|\le 1} \sup\limits_{|c_1-c_2|\le A} \frac{q(x,c_1)}{q(\eta,c_2)} \le \mu(A) < \infty$, where A is a finite value, $\mu(A)$ is a continuous function from A.

Definition 1. The function $y \in L_2(R)$ is called a solution of the equation (1) if there exist a sequence $\{y_n\}_{n=1}^{\infty} \subset W^1_2(R)$ such that $\{y_n\}_{n=1}^{\infty} \subset W^2_{2,loc}(R)$, $\|y_n - y\|_{L_{2,loc}} \to 0$ $\|Ly_n - f\|_{L_{2,loc}} \to 0$ as $n \to \infty$.

Definition 2. Following [13-15], we say that the solution $y(x) \in L_2(R)$ of equation (1) is called the maximal regular in $L_2(R)$ if $q(x,y)\,y \in L_2(R)$, $y'' \in L_2(R)$.

Theorem 1.1. Let the conditions $i) - ii)$ be fulfilled. Then there is the most regular solution to equation (1).

The condition $ii)$, imposed in Theorem 1.1 and in [16], limits the potential oscillations. This condition is removed in the following theorem. In order to formulate the theorem, we introduce the following condition:

$i_0)$ $\sup\limits_{x \in R} \sup\limits_{|c_1-c_2|} \frac{q(x,c_1)}{Q^2(x,c_2)} < \infty$, $Q(x,c_2)$ is a special averaging of the function $q(x,c_1)$ [11], i.e.

$$Q(x,c_2) = \inf_{d>0}\left(d^{-1} + \int_{x-\frac{d}{2}}^{x+\frac{d}{2}} q(t,c_2)\,dt\right),$$

where A is a finite value.

Theorem 1.2. Let the conditions $i) - i_0)$ be fulfilled. Then there exist the maximal regular solution to equation (1).

Example 1. Let $q(x,y) = e^{|x|} \cdot \sin^2 e^{|x|} + e^{|y|}$. Then it is not difficult to verify that all conditions of Theorem 1.2 are satisfied for the equation

$$Ly = -y'' + \left(e^{|x|} \cdot \sin^2 e^{|x|} + e^{|y|}\right) y = f(x).$$

Therefore, there exists a solution $y(x)$ for the equation such that $y''(x) \in L_2(R)$.

This shows that Theorem 1.2 holds for a very wide class of nonlinear equations, including equations with potentials that are rapidly oscillating at infinity.

Now, we consider consider the question B), i.e. finding such sequences of numbers that have approximative properties of the type a)-b). To do this, we study the behavior of the Kolmogorov k-widths of the set

$$M = \left\{u \in W_2^1(R) : \left\|-y'' + q(x,y)\,y\right\|_2^2 \le T\right\}.$$

By definition [17], the Kolmogorov k-width of the set M is called the quantity

$$d_k(M, L_2) = d_k = \inf_{\{\ell_k\}} \sup_{u \in M} \inf_{v \in \ell_k} \|u - v\|_2,$$

where ℓ_k is a subspace of dimension k.

Note that the Kolmogorov widths of a compact set have the following properties: 1) $d_0 \ge d_1 \ge d_2 \ge ... \ge d_k \ge ...$, 2) $d_k \to 0$ as $k \to \infty$.

By $L_2^2(R, q(x,0))$ we denote the space obtained by completing $C_0^\infty(R)$ with respect to the norm

$$\left\|y \cdot L_2^2(R, q(x,0))\right\|_2 = \left(\int_{-\infty}^{\infty} \left(\left|y''\right|^2 + q(x,0)\,|y|^2\right) dx\right)^{1/2}$$

Theorem 1.3. Let the conditions $i)$-$ii)$ be fulfilled. Then any bounded set is compact in $L_2^2(R, q(x,0))$ if and only if

$$\lim_{|x| \to \infty} q(x,0) = \infty. \qquad (*)$$

We introduce the following counting function $N(\lambda) = \sum_{d_k > \lambda} 1$ of those widths d_k greater than $\lambda > 0$.

Theorem 1.4. Let the conditions $i)$-$ii)$ be fulfilled. Then the estimate

$$c^{-1}\lambda^{-1/2} mes\left(x \in R : q(x,0) \le c^{-1}\lambda^{-1}\right) \le$$
$$\le N(\lambda) \le c\lambda^{-1/2} mes\left(x \in R : q(x,0) \le c\lambda^{-1}\right),$$

holds, $c > 0$ is a constant depending, generally speaking, on T.

Example 2. Let us take $q(x) = |x| + |y| + 1$. Then, by virtue of Theorem 1.4, the estimate $c^{-1}\lambda^{-3/2} \le N(\lambda) \le c\lambda^{-3/2}$ holds for the distribution function of the widths of the set $M = \left\{y \in W_2^1(R) : \ \|-y'' + q(x,y)\,y\|_2^2 \le 1\right\}$, where $c > 0$ is a constant. Since $N(\lambda)$ is a monotone function then we have $d_k \sim \frac{c}{k^{2/3}}, \ k = 1, 2, 3, ...$

On the existence of solutions of the nonlinear Sturm-Liouville equation

In this section we prove a lemma on the existence of solutions.

Lemma 2.1. Let the condition $i)$ be fulfilled. Then there exists a solution of the equation (1) in the space $W_2^1(R)$ for any $f \in L_2(R)$.

To prove this lemma we need some auxiliary assertions.

Consider the following problem

$$L_{n,v}y = -y''(x) + q(x,v)\,y = f \cdot \chi_n, \qquad (2)$$

$$y(-n) = y(n) = 0, \tag{3}$$

where χ_n is the characteristic function of the segment $[-n, n]$, $n = 1, 2, ..., v(x) \in C[-n, n]$, $C[-n, n]$ is a space of continuous functions.

Lemma 2.2. Let the condition *i*) be fulfilled and let $v \in C[-n, n]$. Then there exists a unique solution of problem (2)-(3) for any $f \cdot \chi_n \in L_2(-n, n)$ such that

$$\|y\|_{W_2^1[-n,n]} \le c_0 \|f\|_2, \tag{4}$$

$$\|y\|_{C[-n,n]} \le c \|f\|_2. \tag{5}$$

where $c_0 > 0$ and $c > 0$ are constant numbers.

Proof. Assume $q(x) = q(x, v)$. Then the problem (2)-(3) takes the form

$$L_{n,v} y = -y''(x) + q(x) y = f \cdot \chi_n, \tag{6}$$

$$y(-n) = y(n) = 0. \tag{7}$$

From the general theory of boundary value problems [7] it follows that the problem (6)-(7) has a unique solution $W_2^2(-n, n)$ such that

$$\|y\|_{W_2^1[-n,n]} \le c_0(\delta) \|f \chi_n\|_2 \le c_0(\delta) \|f\|_2. \tag{8}$$

It is known that $W_2^1(-n, n)$ is completely continuously embedded in the space $C[-n, n]$. Therefore, the following estimate

$$\|y\|_{C[-n,n]} \le c_1 \|y\|_{W_2^1(-n,n)}, \tag{9}$$

holds, where $c_1 > 0$ is the constant of the embedding theorem.

So problem (6)-(7) has a unique solution

$$y_{n,v} = L_{n,v}^{-1}(f \chi_n), \tag{10}$$

where $L_{n,v}^{-1}$ is the inverse operator of the operator $L_{n,v}$ corresponding to the problem (6)-(7). And

$$\left\|L_{n,v}^{-1}\right\|_{C[-n,n]} \le c, \tag{11}$$

where $c = c_1 \cdot c_0(\delta)$.

Lemma 2.3. Let the condition *i*) be fulfilled. Then the operator $L_{n,v}^{-1}$ maps the ball $\bar{s}$ into itself, where $\bar{s} = \left\{v \in C[-n, n] : \|v\|_{C[-n,n]} \le A\right\}$ is a ball in the space $C[-n, n]$ and A is an arbitrary positive number.

Proof. If the radius A of the ball $\bar{s}$ is equal to the right side of the inequality (5), i.e. $A = c\|f\|_2$, then Lemma 2.2 implies that the operator $L_{n,v}^{-1}$ maps the set $\bar{s}$ into itself. Lemma 2.3 is proved.

Let $K = \left\{y_{n,v} \in C[-n, n] : y_{n,v} = L_{n,v}^{-1}(f \chi_n), v \in \bar{s}, f \in L_2(R)\right\}$ is the image of the ball $\bar{s}$ under the mapping $L_{n,v}^{-1}$.

Lemma 2.4. Let the condition *i*) be fulfilled. Then the set K is compact in the space $C[-n, n]$.

Proof. Lemma 2.2 implies that the inequality

$$\|y_{n,v}\|_{W_2^1(-n,n)} \le c_0 \|f\|,$$

holds for any function $y_{n,v}(x)$ from K, where $c_0 > 0$ is a constant.

This and the embedding theorem imply that the set K is compact in $C[-n, n]$. Lemma 2.4 is proved.

Lemma 2.5. Let the condition *i*) be fulfilled. Then the operator $L_{n,v}^{-1}$ is continuous.

Proof. Let $f(x) \in L_2(R)$ and let the sequence $\{v_k\}_{k=1}^{\infty}$ converge to the element $v(x)$ of the ball $\bar{s}$ in the norm of the space $C[-n,n]$ and

$$L_{n,v_k} y_{n,v_k} = f(x) \cdot \chi_n, \tag{12}$$

$$L_{n,v} y_{n,v} = f(x) \cdot \chi_n, \tag{13}$$

From the equality (12)-(13) we find that

$$-\left(y_{n,v_k} - y_{n,v}\right)'' + q(x, v_k)\left(y_{n,v_k} - y_{n,v}\right) + \left(q(x, v_k) - q(x, v)\right) y_{n,v} = 0.$$

Hence

$$L_{n,v_k}\left(y_{n,v_k} - y_{n,v}\right) = \left(q(x, v) - q(x, v_k)\right) y_{n,v}. \tag{14}$$

It is easy to verify that the coefficients of the operator L_{n,v_k} satisfy the conditions of Lemma 2.2, therefore there exist an inverse operator L_{n,v_k}^{-1} and the equality

$$y_{n,v_k} - y_{n,v} = L_{n,v_k}^{-1}\left(q(x, v) - q(x, v_k)\right) y_{n,v}$$

holds.

From this and the inequalities (4)-(5) and (9)-(11) we obtain that

$$\left\|y_{n,v_k} - y_{n,v}\right\|_{C[-n,n]} = \left\|L_{n,v_k}^{-1}\left(q(x, v) - q(x, v_k)\right) y_{n,v}\right\|_{C[-n,n]} \le$$

$$\le \left\|L_{n,v_k}^{-1}\right\|_{C[-n,n]} \cdot \left\|\left(q(x, v) - q(x, v_k)\right) y_{n,v}\right\|_{C[-n,n]} \le$$

$$\le c \cdot \sup_{x\in[-n,n]} \left|q(x, v) - q(x, v_k)\right| \cdot \left\|y_{n,v}\right\|_{L_2(-n,n)}.$$

From this and from the inequality (4) we have

$$\begin{aligned}\left\|y_{n,v_k} - y_{n,v}\right\|_{C[-n,n]} &\le c \cdot \sup_{x\in[-n,n]} \left|q(x, v) - q(x, v_k)\right| \cdot A_0 \cdot \|f\|_2 = \\ &= c_1 \cdot \sup_{x\in[-n,n]} \left|q(x, v) - q(x, v_k)\right| \cdot \|f\|_2,\end{aligned} \tag{15}$$

where $c_1 = c \cdot c_0$.

Since $\|v_k - v\|_{C[-n,n]} \to 0$ for $k \to \infty$ then we obtain from (15) that

$$\lim_{k\to\infty} \left\|y_{n,v_k} - y_{n,v}\right\|_{C[-n,n]} \le c_0 \cdot \lim_{k\to\infty} \sup_{x\in[-n,n]} \left|q(x, v) - q(x, v_k)\right| \cdot \|f\|_2 \to 0$$

The last relation shows that the operator L_{n,v_k}^{-1} is continuous. Lemma 2.5 is proved.

Now, consider the following nonlinear problem

$$L_n y_n \equiv -y_n'' + q(x, y_n)\, y_n = f \cdot \chi_n \tag{16}$$

$$y_n(-n) = y_n(n) = 0. \tag{17}$$

Lemma 2.6. Let the condition *i*) be fulfilled. Then there exist a solution of the problem (16)-(17) for any $f \in L_2(R)$ such that

$$\|y_n\|_{C[-n,n]} + \|y_n\|_{W_2^1(-n,n)} \le c \cdot \|f\|_2, \tag{18}$$

where $c > 0$ is a constant.

Proof. The function $y_{n,v} = L_{n,v}^{-1}(f\chi_n)$ belongs to the domain $D(L_n)$ of the operator L_n for each function $v \in C[-n,n]$ corresponding to the problem (16)-(17). Therefore, the existence of a solution to problem (16)-(17) is equivalent to the existence of a fixed point of the operator $L_{n,v}^{-1}$ in the space $C[-n,n]$, i.e., to the existence of a function $y_n \in C[-n,n]$ such that $y_n = L_{n,y_n}^{-1} f \cdot x_n$. Thus $y_n \in D(L_n)$, since $L_{n,v}^{-1}(f\chi_n) \in D(L_n)$ for any $v(x)$ from $C[-n,n]$.

To find a fixed point, it remains to show that the operator $L_{n,v}^{-1}$ maps the convex set into itself and it is completely continuous. The proof of this assertion follows from Lemmas 2.2-2.5. Lemma 2.6 is proved.

Proof of Lemma 2.1. Each y_n is continued by zero outside $[-n,n]$ and the continuation is denoted by $\tilde{y}_n$. As you know, we get the elements $W_2^1(R)$ with such a continuation and (18) implies that their norm is bounded

$$\|\tilde{y}_n\|_{W_2^1(R)} \leq c \cdot \|f\|_{L_2(R)}. \tag{19}$$

Therefore, from the sequence $\{\tilde{y}_n\}$ one can select a subsequence $\tilde{y}_{n_k}$ such that

$$\tilde{y}_{n_k} \to y \text{ weakly in } W_2^1(R), \tag{20}$$

$$\tilde{y}_{n_k} \to y \text{ strongly in } L_{2,loc}, \tag{21}$$

and the estimate

$$\|y\|_{W_2^1(R)} \leq c \cdot \|f\|_2 \tag{22}$$

holds. The last estimate follows from (19) and (20).

Let $[-\alpha,\alpha]$ is an arbitrary fixed segment in R, where $\alpha > 0$ is any number. Then there exists a number N for any $\varepsilon > 0$ such that

$$\|L\tilde{y}_{n_k} - f\|_{L_2(-\alpha,\alpha)} \to 0 \text{ for } n_k \to \infty \tag{23}$$

for $n \geq N$ $[-\alpha,\alpha] \subset \operatorname{supp} \tilde{y}_{n_k}$ and by virtue of (16).

(21) and (23) imply that $y(x)$ is a solution to the equation (1). Lemma 2.1 is proved. □

On smoothness of solutions

Proof of Theorem 1.1. Let $|x-\eta| \leq 1$, then by Lemma 2.1 and from the inequality (22) we have

$$|y(x) - y(\eta)| = \left| \int_\eta^x y'(t)\,dt \right| \leq \sqrt{x-\eta} \cdot c\,\|f\|_2 \leq c\,\|f\|_2.$$

Now supposing $y(x) = c$, $y(\eta) = c_2$ $A = c\,\|f\|_2$ we obtain that

$$\sup_{|x-\eta|\leq 1} \frac{q(x,y(x))}{q(\eta,y(\eta))} \leq \sup_{|x-\eta|\leq 1} \sup_{|c_1-c_2|\leq A} \frac{q(x,c_1)}{q(\eta,c_2)} \leq \mu(A) < \infty.$$

Hence, according to Theorem 3 in [11] y'', $q(x,y)\,y$ belongs to $L_2(R)$. Theorem 1.1 is proved. □

Proof of Theorem 1.2. By Lemma 2.1, there exist a solution $y(x)$ for the equation (1) such that $y(x) \in W_2^1(R)$. Consequently, by the Sobolev embedding theorem $y(x) \in C(R)$. In the space $C(R)$ the norm is defined by the formula

$$\|y\|_{C(R)} = \sup_{x\in R} |y(x)|.$$

Then, according to the condition *i)* $q(x,y(x)) \in C_{loc}(R)$. Further, the inequality

$$|y(x) - y(\eta)| \leq |c_1 - c_2| \leq A$$

holds, where $y(x) = c_1$, $y(\eta) = c_2$.

Hence, we have:

$$\sup_{x\in R}\frac{q\left(x,y\left(x\right)\right)}{Q^{2}\left(x,y\left(x\right)\right)}\leq\sup_{x\in R}\sup_{|c_{1}|\leq A}\frac{q\left(x,c_{1}\right)}{Q_{A}^{2}\left(x,c_{1}\right)}\leq\sup_{x\in R}\sup_{|c_{1}-c_{2}|\leq A}\frac{q\left(x,c_{1}\right)}{Q^{2}\left(x,c_{2}\right)}$$

From the last inequality according to the condition $i)$ we find that

$$\sup_{x\in R}\frac{q\left(x,y\left(x\right)\right)}{Q^{2}\left(x,y\left(x\right)\right)}<\sup_{x\in R}\sup_{|c_{1}-c_{2}|\leq A}\frac{q\left(x,c_{1}\right)}{Q^{2}\left(x,c_{2}\right)}<\infty.$$

It follows that all the conditions of Theorem 4 of [11] are fulfilled. Consequently, $q\left(x,y\right)y\left(x\right)$, $y''\in L_{2}\left(R\right)$. Theorem 1.2 is proved. □

Two-sided estimates of the approximate numbers of solutions of the nonlinear Sturm-Liouville equation

As is known for a compact set, especially, when it contains solutions of a differential equation, the problem of the asymptotics of their widths is meaningful. The Kolmogorov widths estimation of the set M can be used to determine for the equation $Ly=f$ the convergence rate of approximate solutions to the exact one.

In order to prove Theorem 1.3, first we prove several lemmas.

Lemma 3.1 Let the conditions $i)-ii)$ be fulfilled. Then there exist a number $K\left(T\right)$ such that

$$\tilde{M}\subseteq M\subseteq\tilde{\tilde{M}},$$

where

$$\tilde{\tilde{M}}=\left\{y\in L_{2}\left(R\right):\left\|-y''\right\|_{2}^{2}+\left\|q\left(x,y\right)y\right\|_{2}^{2}\leq K\left(T\right)\right\},$$

$$\tilde{M}=\{y\in L_{2}\left(R\right):\left\|-y''\right\|_{2}^{2}+\left\|q\left(x,y\right)y\right\|_{2}^{2}\leq\frac{T}{2}\bigg\}.$$

Proof. Let $y\in\tilde{M}$. Then, using the triangle inequality, we get

$$\left\|-y''+q\left(x,y\right)y\right\|_{2}^{2}\leq2\left(\left\|-y''\right\|_{2}^{2}+\left\|q\left(x,y\right)y\right\|_{2}^{2}\right)\leq2\cdot\frac{T}{2}\leq T.$$

It follows that $y\in M$, i.e. $\tilde{M}\subseteq M$.

Let $y\in M$. Then, by virtue of Lemma 2.1 and the estimate (22) and the embedding theorem $W_{2}^{1}\left(R\right)$ in the space $C\left(R\right)$ we have

$$\left\|y\right\|_{C(R)}\leq c\left\|-y''+q\left(x,y\right)y\right\|_{2},$$

where c is independent of $y\left(x\right)$ и $q\left(x,y\right)$.

It follows that

$$\sup_{y\in M}\left\|y\left(x\right)\right\|_{C(R)}\leq c\cdot T^{1/2}\tag{24}$$

On the other hand, using the estimate (22), we have

$$\left|y\left(x\right)-y\left(\eta\right)\right|\leq c\left\|-y''+q\left(x,y\right)y\right\|\leq c\cdot T^{1/2}\tag{25}$$

for any $y\in M$, where $c>0$ is a constant independent of $y\left(x\right)$.

Now, supposing $y\left(x\right)=c_{1},\ y\left(\eta\right)=c_{2},\ A=c\cdot T^{1/2}$ from (25) we obtain that $\left|c_{1}-c_{2}\right|\leq A$.

Let $y_0(x) \in M$ and suppose $q_0(x) = q(x, y_0(x))$. Denote by L the closure in the norm of $L_2(R)$ of the operator defined on $C_0^\infty(R)$ by the equality

$$L_0 y = -y''(x) + q_0(x)\, y.$$

It is easy to verify that the operator L is self-adjoint, positive definite and $y_0(x) \in D(L)$, wherein the estimate

$$\left\|-y_0''\right\|_2 \le \mu(A) \left\|-y_0 + q(x, y_0)\, y\right\|_2 \tag{26}$$

holds. The estimate (26) follows from Theorem 1.1.

This shows that the inequality

$$\left\|-y''\right\|_2 \le \mu(A)\, T^{1/2}. \tag{27}$$

holds for all $y \in M$.

From the inequality (27) we have

$$\|q(x,y)\,y\|_2 = \left\|-y'' + q(x,y)\,y + y''\right\|_2 \le \left\|y''\right\|_{,2} + \left\|-y'' + q(x,y)\,y\right\|_2 \le$$

$$\le \mu(A) \cdot T^{1/2} + T^{1/2} \le 2\mu(A) \cdot T^{1/2} \tag{28}$$

for any $y \in M$. Here we take into account that the condition $ii)$ implies that $\mu(A) \ge 1$.

From the inequalities (27) and (28) we find

$$\left\|-y''\right\|_2^2 + \|q(x,y)\,y\|_2^2 \le \mu^2(A) \cdot T + 4\mu^2(A) \cdot T \le K(T) \tag{29}$$

for any $y \in M$, where $K(T) = 5\mu^2(A) \cdot T$. The estimate (29) proves Lemma 3.1.

Lemma 3.2. Let the conditions $i) - ii)$ be fulfilled. Then $\tilde{\tilde{M}} \subseteq \tilde{\tilde{B}}$, where

$$\tilde{\tilde{B}} = \left\{u \in L_2(R) : \left\|-y''\right\|_2^2 + \|q(x,0)\,y\|_2^2 \le K_1(T)\right\}.$$

Proof. By the embedding theorems, we have

$$\|y\|_{C(R)} \le c\left(\left\|-y''\right\|_2^2 + \|q(x,y)\,y\|_2^2\right)^{1/2} \le c \cdot K(T) \tag{30}$$

for any $y(x) \in \tilde{\tilde{M}}$, where $c > 0$ is the constant of the embedding theorem.

Hence, using the computations and arguments used in the proof of (29), we obtain that

$$y(x) = c_1,\ y(\eta) = c_2,\ |c_1 - c_2| \le A,\ A = 2c \cdot K^{1/2}(T). \tag{31}$$

Hence, using the conditions of $ii)$ for all $y(x) \in \tilde{\tilde{M}}$, we have

$$\mu^{-1}(A)\, q(x,0) \le q(x, y(x)) \le \mu(A)\, q(x,0), \tag{32}$$

where $A = 2c \cdot K^{1/2}(T),\ \mu(A) = \mu\left(2cK^{1/2}(T)\right)$.
From (32) we have

$$\left\|-y''\right\|_2^2 + \|q(x,0)\,y\|_2^2 \le \left\|-y''\right\|_2^2 + \mu^2(A)\,\|q(x,y)\,y\|_2^2 \le \mu^2(A)\left(\left\|-y''\right\|_{,2}^2 +\right.$$

$$\left. + \|q(x,y)\,y\|_2^2\right) \le \mu^2(A) \cdot K(T) \le K_1(T),\ K_1(T) = \mu^2\left(2cK^{1/2}\right) \cdot K(T)$$

for any $y(x) \in \tilde{\tilde{M}}$. This implies $\tilde{\tilde{M}} \subseteq \tilde{\tilde{B}}$.

Lemma 3.3. Let the conditions $i) - ii)$ be fulfilled. Then $\tilde{B} \subseteq \tilde{M}$, where

$$\tilde{B} = \left\{ u \in L_2(R) : \left\| -y'' \right\|_2^2 + \left\| q(x,0)\, y \right\|_2^2 \leq K_2(T) \right\},$$

$K_2(T)$ is a positive number depending on T, such that $K_2(T) \leq \frac{T}{2}$.

Proof. Let $u \in \tilde{B}$. Then, using the embedding theorem, we have

$$\|y\|_{C(R)} \leq c \cdot K_2(T) \leq c \cdot \frac{T}{2}, \tag{33}$$

$c > 0$ is the constant of the embedding theorem from $W_2^2(R)$ to $C(R)$.

Now, using the condition $ii)$, we obtain from (33) that for all $u \in \tilde{B}$

$$\mu^{-1}\left(c \cdot \frac{T}{2}\right) q(x,0) \leq q(x, y(x)) \leq \mu\left(c \cdot \frac{T}{2}\right) q(x,0). \tag{34}$$

Hence, we find

$$\left\| -y'' \right\|_2^2 + \| q(x,y)\, y \|_2^2 \leq \left\| -y'' \right\|_2^2 + \mu^2\left(c \cdot \frac{T}{2}\right) \cdot \| q(x,0)\, y \|_2^2 \leq$$
$$\leq \mu^2\left(c \cdot \frac{T}{2}\right)\left(\| -y \|_{,2}^2 + \| q(x,0)\, y \|_2^2 \right) \leq \mu^2\left(c \cdot \frac{T}{2}\right) K_2(T)$$

for any $y \in \tilde{B}$.

If we assume $K_2(T) = \frac{\frac{T}{2}}{\mu^2\left(c \cdot \frac{T}{2}\right)}$ then the inequality $\| -y'' \|_2^2 + \| q(x,y)\, y \|_2^2 \leq \frac{T}{2}$ holds for all $y \in \tilde{B}$. Therefore $\tilde{B} \subseteq \tilde{M}$. Lemma 3.3 is proved.

Lemma 3.4. Let the conditions $i) - ii)$ be fulfilled. Then the estimates

$$c^{-1} d_k \leq \tilde{d}_k \leq c d_k,\ \ k = 1, 2, ...$$

hold, where $c > 0$ depends only on T, $\tilde{d}_k, d_k$ k are the Kolmogorov widths of the sets M and B, respectively, where $B = \left\{ y \in L_2(R) : \| -y'' \|_2^2 + \| q(x,0)\, y \|_2^2 \leq 1 \right\}$.

This lemma is proved in the same way as Lemma 4.3 in [18].

Lemma 3.5. Let the conditions $i) - ii)$ be fulfilled. Then the estimates

$$N(c\lambda) \leq \tilde{N}(\lambda) \leq N\left(c^{-1}\lambda\right)$$

hold, where $N(\lambda) = \sum_{d_k > \lambda} 1$ is the counting function of those d_k greater than $\lambda > 0$, $\tilde{N}(\lambda) = \sum_{\tilde{d}_k > \lambda} 1$ is the counting function of those $\tilde{d}_k$ greater than $\lambda > 0$, $c > 0$ is a constant.

The proof of this lemma follows from Lemma 3.4.

Proof of Theorems 1.3-1.4. Repeating the computations and arguments used in the proof of Theorems 1.2–1.3 from [18] we obtain the proof of Theorem 1.3. □

Proof of Theorem 1.4. Using Lemmas 3.4–3.5, the proofs of Theorems 1.1—1.4 from [18] and the results from [19], we obtain the proof of Theorem 1.4. □

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Сингулярлы сызықты емес Штурм-Лиувилль теңдеуінің шешімінің тегістігі мен аппроксимативті қасиеттері туралы

Кему тәртібімен еселігі бойынша реттелген өз-өзіне түйіндес Штурм-Лиувилль операторымен L^{-1} жете үзіліссіз кері операторының меншікті сандарының келесі қасиеті бар екендігі белгілі: $^{(*)}$ $\lambda_n \to 0$, егер $n \to \infty$, сонымен қатар нөлге ұмтылу жылдам болған сайын, L^{-1} операторы ақырлы рангілі операторлар арқылы жақсырақ жуықталады. Сонымен қатар сызықты емес операторларға келесі сұрақ туындайды: «Берілген сызықты емес операторға (*) қасиетімен сипатталатын кемімелі сандық тізбегін көрсетуге бола ма?» Мақалада сызықты емес Штурм-Лиувиль операторына арналған жоғарыда келтірілген сұрақ зерттелді. Аталған мәселені шешу үшін $L_2(R)\,(R = (-\infty, \infty))$ кеңістігінде жылдам өспелі және жылдам тербелмелі потенциалы бар сызықты емес Штурм-Лиувилль теңдеуінің шешімдерінің максималды регулярлығы туралы теорема дәлелденді және сызықты емес Штурм-Лиувилль теңдеуінің шешімдерімен байланысты жиындардың Колмогоров енінің екіжақты бағалаулары алынды. Белгілі болғандай, Колмогоров енінің алынған бағалаулар ең аз қателікке кепілдік беретін жуықтау аппаратын таңдауға мүмкіндік береді.

Кілт сөздер: максималды регулярлығы, сингулярлы сызықты емес теңдеу, Штурм-Лиувилль теңдеуі, шешімдердің тегістігі, аппроксимативті қасиеттер, аппроксимативті сандар, Колмогоров ені, жылдам тербелмелі потенциал, қатты өспелі потенциал, екіжақты бағалау.

М.Б. Муратбеков, М.М. Муратбеков

О гладкости и аппроксимативных свойствах решений сингулярного нелинейного уравнения Штурма-Лиувилля

Известно, что собственные числа $\lambda_n(n = 1, 2, ...)$, пронумерованные в порядке убывания и с учетом кратности самосопряженного оператора Штурма-Лиувилля с вполне непрерывным обратным оператором L^{-1}, обладают следующим свойством: $^{(*)}$ $\lambda_n \to 0$, когда $n \to \infty$, причем, чем быстрее стремление к нулю, тем оператор L^{-1} лучше аппроксимируется с операторами конечного ранга. Естественным образом возникает следующий вопрос для нелинейных операторов: «Можно ли для заданного нелинейного оператора указать убывающую числовую последовательность, которая характеризуется свойством (*)?». В статье изучен указанный выше вопрос для нелинейного оператора Штурма-Лиувилля. Для решения задачи доказана теорема о максимальной регулярности решений нелинейного уравнения Штурма-Лиувилля с сильно растущим и быстро колеблющимся потенциалом в пространстве $L_2(R)\,(R = (-\infty, \infty))$, а также получены двусторонние оценки поперечников по Колмогорову множества, связанные с решениями нелинейного уравнения Штурма-Лиувилля. Как известно, полученные оценки поперечников по Колмогорову дают возможность выбирать аппарат приближения, который гарантирует минимально возможную погрешность.

Ключевые слова: максимальная регулярность, сингулярное нелинейное уравнение, уравнение Штурма-Лиувилля, гладкость решений, аппроксимативные свойства, аппроксимативные числа, поперечники по Колмогорову, быстро колеблющийся потенциал, сильно растущий потенциал, двусторонние оценки.

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Recovery problem for a singularly perturbed differential equation with an initial jump

The article investigates the asymptotic behavior of the solution to reconstructing the boundary conditions and the right-hand side for second-order differential equations with a small parameter at the highest derivative, which have an initial jump. Asymptotic estimates of the solution of the reconstruction problem are obtained for singularly perturbed second-order equations with an initial jump. The rules for the restoration of boundary conditions and the right parts of the original and degenerate problems are established. The asymptotic estimates of the solution of the perturbed problem are determined as well as the difference between the solution of the degenerate problem and the solution of the perturbed problem. A theorem on the existence, uniqueness, and representation of a solution to the reconstruction problem from the position of singularly perturbed equations is proved. The results obtained open up possibilities for the further development of the theory of singularly perturbed boundary value problems for ordinary differential equations.

Keywords: Perturbed problems, degenerate problems, small parameter, boundary value problem, initial jump, asymptotic behavior.

Introduction

At the end of the last century and over the past decade (ten years), approximate methods for solving differential equations, asymptotics construction, solution for singularly perturbed differential equations attracted the attention of many researches. This interest is caused by the needs of numerical methods for solving differential equations. Since in many cases the asymptotic approximation of solution of boundary value problem is useful to use as a first approximation in numerical calculations [1,2].

The difficulty of constructing an asymptotic approximation to the solution of initial and boundary value problems for differential and integro-differential equations is related to the perturbation feature. In this regard, the researchers proposed various asymptotic methods for constructing the asymptotic behavior of the solution of initial and boundary value problems. However, without a preliminary study of the asymptotic behavior of the solution of singularly perturbed initial problems with singular initial conditions and boundary value problems with boundary jumps phenomena, the greatest difficulty is the selection of a suitable asymptotic method for solving these problems [3-8].

The most general cases of the existence of the phenomenon of initial jumps were investigated in the works of Dauylbaev [9], Kasymov and Nurgabyl [10,11,12]. In the works of Nurgabyl [13,14], Mirzakulova [15] for the third-order equation with a small parameter at higher derivatives, the phenomena of boundary jumps were studied.

The constructions of an approximate solution of a boundary value problem defined with various additional conditions were studied by Hikosaka-Nobory [16], Kibenko and Perov [17], Dzhumabaev [18]. In these works, using the well-known structure of the differential equation and additional information, the problem of restoring the right side of the differential equation is solved.

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So in the work of A.V. Kibenko and A.M. Perov [17] problem of simultaneously finding the function $y(t)$ and the parameter λ from the relations $y' = \lambda \cdot f(t, y)$, $y(0, \varepsilon) = \alpha$, $y(t_1, \varepsilon) = \beta$, with given $\alpha, \beta, t_1 \in \mathbb{R}$ were considered.

On the other hand, in some singularly perturbed boundary value problems, it might turn out that the number of additional conditions exceeds the order of the equation, and the equation contains unknown parameters. Interest in such problems is caused by the problems of the optimal management.

So, for example, [19], the restoration problem for singularly perturbed differential equations was investigated in the work, in case when the right-hand side of the differential equation and boundary conditions linearly depend on an unknown parameter. A priori estimates were established to solve a parametrized singularly perturbed boundary value problem for a second-order equation, by Mustafa Kudu et al. [20].

The proposed work is devoted to the study of the solution of a singularly perturbed boundary value problem with initial a jump in case when the boundary conditions depend on an unknown parameter in a nonlinear way.

1. Set the problem's condition

Let $\mathbb{R} = (-\infty, +\infty)$, $J = [0, 1]$ and $\Lambda-$ be some bounded set from $\mathbb{R}$.

Consider the boundary value problem:

$$L_\varepsilon y \equiv \varepsilon y" + A(t)y' + B(t)y = \lambda h(t), \tag{1}$$

$$y(0, \varepsilon) = \alpha_0,\ y(1, \varepsilon) = \beta_0(\lambda), \quad y'(1, \varepsilon) = \beta_1(\lambda), \tag{2}$$

where $\alpha_0, \beta_0, \beta_1 \in R, \lambda$ is an unknown parameter, $\varepsilon > 0$ is a small parameter.

The problem is to determine the pair $(y(t, \lambda(\varepsilon), \varepsilon), \lambda(\varepsilon))$, where the function $y(t, \lambda, \varepsilon)$ at $0 \leq t \leq 1$ satisfies equation (1) and the boundary conditions (2), to establish an asymptotic estimate for the solution to problem (1), (2), to formulate a degenerate problem, to define condition for the occurrence of a jump.

A pair (λ, y) is called a solution to problem (1), (2) if, for each fixed value of $\lambda = \overline{\lambda} \in \Lambda$, the function $y(t, \varepsilon)$ is a solution to problem (1) (2).

In this article, using the results of the research[11], an analytical representation of the problem solution (1), (2) will be constructed and on its basis the existence and uniqueness of a solution are proved, a degenerate problem is formulated, an proximity of a solutions of the original and degenerate problem are proved at $\varepsilon \to 0$, the nature of the derivative growth of the solution of the problem (1), (2), the condition for the appearance of a jump, and asymptotic estimates of the solution of problem will be established (1), (2).

Let be:

1^0. $A(t), B(t), h(t) \in C^1(J)$;

2^0. $A(t) \geq \nu > 0, t \in J$;

3^0. The equation $R_0(\lambda_0) = -\beta_0 \dfrac{B(1)}{A(1)} + \lambda_0 \dfrac{F(1)}{A(1)} - \beta_1(\lambda_0) = 0$ has a unique solution $\lambda_0 = \bar{\lambda}_0$, at that $R_0'(\bar{\lambda}_0) = \dfrac{F(1)}{A(1)} - \beta_1'(\bar{\lambda}_0) \neq 0$.

2. Construction of the initial function

We consider the homogeneous equation

$$L_\varepsilon y\,(t, \varepsilon) \equiv \varepsilon y''\,(t, \varepsilon) + A\,(t)\,y'\,(t, \varepsilon) + B\,(t)\,y\,(t, \varepsilon) = 0, \tag{3}$$

which corresponds to the inhomogeneous equation (1).

The following lemma holds [4].

Lemma 1. Let conditions 1^0 - 2^0 be satisfied. Then for the fundamental system of solutions $y_1(t,\varepsilon)$, $y_2(t,\varepsilon)$ homogeneous equation (3) the following asymptotic representations are valid at $\varepsilon \to 0$:

$$y_1^{(j)}(t,\varepsilon) = y_0^{(j)}(t) + O(\varepsilon), y_2^{(j)}(t,\varepsilon) = \frac{1}{\varepsilon^j}\exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu(x)\,dx\right) u_0(t)\,\mu^j(t)\,[1+O(\varepsilon)],\ j=0,1 \quad (4)$$

where $y_0(t) = \exp\left(-\int\limits_0^t \frac{B(s)}{A(s)}ds\right)$, $\mu(t) = -A(t)$, $u_0(t) = \frac{A(0)}{A(t)}\cdot\exp\left(-\int\limits_0^t \frac{B(x)}{A(x)}dx\right)$.
We introduce the initial function

$$K(t,s,\varepsilon) = \frac{W(t,s,\varepsilon)}{W(s,\varepsilon)}, \quad (5)$$

where $W(t,\varepsilon)$ is the Wronskian of the fundamental system of solutions $y_1(t,\varepsilon)$, $y_2(t,\varepsilon)$ of equation (3), $W(t,s,\varepsilon)$ is a second-order determinant obtained from $W(s,\varepsilon)$ by replacing the second row with a row with elements$y_1(t,\varepsilon)$, $y_2(t,\varepsilon)$.
Obviously, the function $K(t,s,\varepsilon)$ satisfies by at the variable t the homogeneous equation (3) and the initial conditions $K(t,t,\varepsilon)=0$, $\quad K_t'(t,t,\varepsilon)=1$, and does not depend on the choice of the fundamental system of equation solutions (4). Therefore, the initial function $K(t,s,\varepsilon)$ for equation (4) is uniquely determined.
From (5) taking into account (4) we obtain:

$$W(s,\varepsilon) = \frac{1}{\varepsilon}\exp\left(\frac{1}{\varepsilon}\int\limits_0^s \mu(x)\,dx\right) u_0(s)\,y_0(s)\,\mu(s)\,[1+O(\varepsilon)] \neq 0; \quad (6)$$

$$W^{(q)}(t,s,\varepsilon) = u_0(s)e^{\frac{1}{\varepsilon}\int\limits_0^s \mu(x)dx}\left[y_0^{(q)}(t) + \frac{1}{\varepsilon^q}e^{\frac{1}{\varepsilon}\int\limits_s^t \mu(x)dx}\frac{u_0(t)\mu^q(t)}{u_0(s)} + O\left(\varepsilon + \frac{\varepsilon}{\varepsilon^q}e^{\frac{1}{\varepsilon}\int\limits_s^t \mu(x)dx}\right)\right]. \quad (7)$$

Now using these estimates, it is easy to verify the validity of the following lemma.

Lemma 2. If conditions 1), 2) are satisfied, then the initial function $K(t,s,\varepsilon)$ for $0 \le s \le t \le 1$ and sufficiently small $\varepsilon > 0$ is representable in the form

$$K(t,s,\varepsilon) = \frac{\varepsilon}{\mu(s)}\left(\frac{y_0(t)}{y_0(s)} + \frac{u_0(t)}{u_0(s)}\exp\left(\frac{1}{\varepsilon}\int\limits_s^t \mu(x)dx\right) + O(\varepsilon)\right), \quad (8)$$

$$K_t'(t,s,\varepsilon) = \frac{\varepsilon}{\mu(s)}\left(\frac{y_0'(t)}{y_0(s)} + \frac{1}{\varepsilon}\cdot\frac{u_0(t)\mu(t)}{u_0(s)}\exp\left(\frac{1}{\varepsilon}\int\limits_s^t \mu(x)dx\right) + O\left(\varepsilon + e^{\frac{1}{\varepsilon}\int\limits_s^t \mu(x)dx}\right)\right).$$

Proof. Estimating function (5) with regard for (6) and (7), we obtain estimate (8).

3. Solution representation of the auxiliary boundary value problem

Since the initial function $K(t,s,\varepsilon)$ satisfies the homogeneous equation (3) and the initial conditions $K(t,t,\varepsilon)=0$, $\quad K_t'(t,t,\varepsilon)=1$,so at fixed value of the parameterλ, the general solution of the inhomogeneous equation (1) can be represented in the form

$$y(t,\varepsilon) = \tilde{c}_1 y_1(t,\varepsilon) + \tilde{c}_2 y_2(t,\varepsilon) + \frac{\lambda}{\varepsilon}\int\limits_0^t K(t,s,\varepsilon)F(s)ds, \quad (9)$$

where $\tilde{c}_1,\ \ \tilde{c}_2$ are arbitrary constants. Then the boundary value problem (1), (2) is solvable, and can be represented in the form (9), if and only if, when the coefficients $\tilde{c}_1,\ \ \tilde{c}_2$ can be chosen so that (9) satisfies the boundary conditions (2). Thus, substituting (9) in (2), we find

$$\begin{gathered} \tilde{c}_1 y_1(0,\varepsilon)+\tilde{c}_2 y_2(0,\varepsilon)=\alpha_0, \\ \tilde{c}_1 y_1(1,\varepsilon)+\tilde{c}_2 y_2(1,\varepsilon)+\frac{\lambda}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds=\beta_0(\lambda), \\ \tilde{c}_1 y_1'(1,\varepsilon)+\tilde{c}_2 y_2'(1,\varepsilon)+\frac{\lambda}{\varepsilon}\int\limits_0^1 K_t'(1,s,\varepsilon)F(s)ds=\beta_1(\lambda). \end{gathered} \tag{10}$$

From the first two equations of system (10) we have:

$$J(\varepsilon)=\begin{vmatrix} y_1(0,\varepsilon) & y_2(0,\varepsilon) \\ y_1(1,\varepsilon) & y_2(1,\varepsilon) \end{vmatrix}=-y_0(1)+O(\varepsilon)\neq 0, \tag{11}$$

$$\tilde{c}_1=\frac{1}{J(\varepsilon)}\begin{vmatrix} \alpha_0 & y_2(0,\varepsilon) \\ \beta_0-\sigma_0 & y_2(1,\varepsilon) \end{vmatrix},\quad \tilde{c}_2=\frac{1}{J(\varepsilon)}\begin{vmatrix} y_1(0,\varepsilon) & \alpha_0 \\ y_1(1,\varepsilon) & \beta_0-\sigma_0 \end{vmatrix}. \tag{12}$$

Substituting the found values $\tilde{c}_1$, $\tilde{c}_2$ from (12) into (9), we find that for each fixed value of the parameter λ, the solution of the auxiliary boundary value problem is representable in the form

$$y=\alpha_0\Phi_1(t,\varepsilon)+\beta_0\Phi_2(t,\varepsilon)-\Phi_2(t,\varepsilon)\frac{\lambda}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds+\frac{\lambda}{\varepsilon}\int\limits_0^t K(t,s,\varepsilon)F(s)ds, \tag{13}$$

where the functions

$$\Phi_1(t,\varepsilon)=\frac{J_1(t,\varepsilon)}{J(\varepsilon)},\ \Phi_2(t,\varepsilon)=\frac{J_2(t,\varepsilon)}{J(\varepsilon)} \tag{14}$$

are called boundary functions of the boundary value problem (1), (2), which satisfy the homogeneous equation (3) and the boundary conditions:

$$\Phi_1(0,\varepsilon)=1,\quad \Phi_1(1,\varepsilon)=0,\quad \Phi_2(0,\varepsilon)=0,\quad \Phi_2(1,\varepsilon)=1, \tag{15}$$

where the determinant $J_1(t,\varepsilon)$ is obtained from the determinant $J(\varepsilon)$ by replacing the first row with the row $y_1(t,\varepsilon)$, $y_2(t,\varepsilon)$, and $J_2(t,\varepsilon)$ is obtained from $J(\varepsilon)$ by replacing the second row with the row $y_1(t,\varepsilon)$, $y_2(t,\varepsilon)$.

Based on (14), it can be proved that the boundary functions $\Phi_k(t,\varepsilon),\ \ k=1,2$ satisfy the boundary conditions (15) do not depend on the fundamental system of solutions $y_1(t,\varepsilon),\ y_2(t,\varepsilon)$ of equation (3). Then, for sufficiently small $\varepsilon>0$, the boundary functions $\Phi_k(t,\varepsilon),\ \ k=1,2$ on the interval [0,1] exist are unique, and are expressed by formula (14).

Lemma 3. If conditions 1^0-3^0, are satisfied, then at sufficiently small$\varepsilon>0$, for the boundary functions $\Phi_i^{(q)}(t,\varepsilon)$ on the interval $0\le t\le 1$ the following estimates are true:

$$\Phi_1(t,\varepsilon)=u_0(t)\,e^{\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx}+O(\varepsilon),\Phi_2(t,\varepsilon)=\frac{y_0(t)}{y_0(1)}+u_0(t)\,e^{\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx}+O(\varepsilon),$$

$$\Phi_1'(t,\varepsilon)=\frac{1}{\varepsilon}u_0(t)\mu(t)\,e^{\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx}+O\left(\varepsilon+e^{\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx}\right), \tag{16}$$

$$\Phi_2'(t,\varepsilon)=\frac{y_0'(t)}{y_0(1)}+u_0(t)\mu(t)\,e^{\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx}+O\left(\varepsilon+e^{\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx}\right).$$

The proof of the lemma directly follows from (14), taking into account estimates (11) and (4).

4. On the unique solvability of the solution of the recovery problem

Now, substituting (13) in the third equation of system (10), we obtain

$$R(\lambda,\varepsilon)\equiv\alpha_0\Phi_1'(1,\varepsilon)+\beta_0\Phi_2'(1,\varepsilon)-\Phi_2'(1,\varepsilon)\frac{\lambda}{\varepsilon}\int\limits_0^1K(1,s,\varepsilon)F(s)ds+ \tag{17}$$

$$+\frac{\lambda}{\varepsilon}\int\limits_0^1K_t'(1,s,\varepsilon)F(s)ds-\beta_1(\lambda)=0.$$

Thus, the solution of the boundary value problem (1), (2) is uniquely solvable if and only if the equation

$$R(\lambda,\varepsilon)=0 \tag{18}$$

regarding a parameter λ has a unique solution.
Let us prove that equation (17) is solvable with respect to λ. To this end, from (17) we find $R'(\lambda,\varepsilon)$ and we study the asymptotic behavior of the functions $R(\lambda,\varepsilon)$ and

$$R'(\lambda,\varepsilon)\equiv-\frac{\Phi_2'(1,\varepsilon)}{\varepsilon}\int\limits_0^1K(1,s,\varepsilon)F(s)ds+\frac{1}{\varepsilon}\int\limits_0^1K_t'(1,s,\varepsilon)F(s)ds-\beta_1'(\lambda). \tag{19}$$

at $\varepsilon\to0$.

Estimating the expressions from (17) and (18) at sufficiently small $\varepsilon>0$, we obtain:

$$\Phi_1'(1,\varepsilon)=o(\varepsilon),\Phi_2'(1,\varepsilon)=\frac{y_0'(1)}{y_0(1)}+O(\varepsilon)$$

$$K(1,s,\varepsilon)=\frac{\varepsilon}{\mu(s)}\left(\frac{y_0(1)}{y_0(s)}+\frac{u_0(1)}{u_0(s)}\exp\left(\frac{1}{\varepsilon}\int\limits_s^1\mu(x)dx\right)+O\left(\varepsilon\right)\right),$$

$$K_t'(1,s,\varepsilon)=\frac{\varepsilon}{\mu(s)}\left(\frac{y_0'(1)}{y_0(s)}+\frac{1}{\varepsilon}\frac{u_0(1)\mu(1)}{u_0(s)}\exp\left(\frac{1}{\varepsilon}\int\limits_s^1\mu(x)dx\right)+O\left(\varepsilon+e^{\frac{1}{\varepsilon}\int\limits_s^{\lambda 1}\mu(x)dx}\right)\right),$$

$$\frac{\lambda}{\varepsilon}\int\limits_0^1K(1,s,\varepsilon)F(s)ds=\lambda\int\limits_0^1\frac{y_0(1)}{y_0(s)\mu(s)}F(s)ds+O(\varepsilon), \tag{20}$$

$$\frac{\lambda}{\varepsilon}\int\limits_0^1K_t'(1,s,\varepsilon)F(s)ds=\lambda\int\limits_0^1\frac{y_0'(1)F(s)}{y_0(s)\mu(s)}ds-\lambda\frac{F(1)}{\mu(1)}+O(\varepsilon).$$

$$\frac{\lambda}{\varepsilon}\int\limits_0^tK(t,s,\varepsilon)F(s)ds=\lambda\int\limits_0^t\frac{y_0(t)}{y_0(s)\mu(s)}F(s)ds+O(\varepsilon).$$

$$\frac{\lambda}{\varepsilon}\int\limits_0^t K'(t,s,\varepsilon)F(s)ds = \lambda\int\limits_0^t \frac{y_0'(t)F(s)}{y_0(s)\mu(s)}ds - \lambda\frac{F(t)}{\mu(t)} + \lambda\frac{u_0(t)\mu(t)}{u_0(0)\mu^2(0)}e^{\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx} + O(\varepsilon)$$

Taking into account estimates (20) for the functions $R(\lambda,\varepsilon)$ and $R'(\lambda,\varepsilon)$ at sufficiently small $\varepsilon > 0$, the following representations are valid:

$$R(\lambda,\varepsilon) = -\beta_0\frac{B(1)}{A(1)} + \lambda\frac{F(1)}{A(1)} - \beta_1(\lambda) + O(\varepsilon) = R_0(\lambda) + O(\varepsilon).$$

$$R'(\lambda,\varepsilon) = \frac{F(1)}{A(1)} - \beta'(\lambda) + O(\varepsilon) = R_0'(\lambda) + O(\varepsilon)$$

Hence, by virtue of condition 3^0, we conclude that at the point λ_0 for sufficiently small $\varepsilon > 0$ the following asymptotic representations are valid:

$$R(\lambda_0,\varepsilon) = O(\varepsilon);$$

$$R'(\lambda_0,\varepsilon) = R_0'(\lambda_0) + O(\varepsilon) \neq 0.$$

Consequently, in a sufficiently small neighborhood of the point λ_0 there is a unique point $\tilde{\lambda}(\varepsilon)$ such that will be fulfilled equality

$$R(\tilde{\lambda}(\varepsilon),\varepsilon) = 0,$$

at that

$$\left|\tilde{\lambda}(\varepsilon) - \lambda_0\right| \leq K\varepsilon.$$

Thus, we proved that there a unique solution $(y(t,\varepsilon),\, \tilde{\lambda}(\varepsilon))$ of the boundary value problem (1), (2) exists. Thus, the following theorem holds.

Theorem 1. If conditions 1^0 – 3^0 are satisfied, then the boundary-value problem (1), (2) has a unique solution and this solution can be represented in the form

$$y(t,\tilde{\lambda}(\varepsilon)) = \alpha_0\Phi_1(t,\varepsilon) + \beta_0\Phi_2(t,\varepsilon) - \Phi_2(t,\varepsilon)\frac{\tilde{\lambda}(\varepsilon)}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds +$$

$$+\frac{\tilde{\lambda}(\varepsilon)}{\varepsilon}\int\limits_0^t K(t,s,\varepsilon)F(s)ds. \tag{21}$$

5. Limit Transition Theorem. The phenomena of the initial jump

The following estimate holds for solution (21):

$$y(t,\varepsilon) = \beta_0\exp\left(-\int\limits_1^t \frac{B(x)}{A(x)}dx\right) - \lambda_0\int\limits_0^1 \exp\left(-\int\limits_s^t \frac{B(x)}{A(x)}dx\right)\frac{F(s)}{\mu(s)}ds +$$

$$+\lambda_0\int\limits_0^t \exp\left(-\int\limits_s^t \frac{B(x)}{A(x)}dx\right)\frac{F(s)}{\mu(s)}ds + O\left(\varepsilon + \exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx\right)\right). \tag{22}$$

Now, we define a degenerate problem. Without any additional considerations, we cannot formulate the boundary conditions for the unperturbed (degenerate) equation

$$L_0\bar{y} \equiv A(t)\bar{y}' + B(t)\bar{y} = \lambda_0 F(t), \tag{23}$$

obtained from (1) at $\varepsilon = 0$.

Such an additional consideration we can obtain from estimate (22). It follows from (22) that the limit function $y(t,\varepsilon)$ does not contain α_0 and β_1 at $\varepsilon \to 0$. Therefore, the boundary conditions for the solution are defined in the form

$$\bar{y}(1) = \beta_0(\lambda_0). \tag{24}$$

Therefore, the solution to problem (23), (24) is representable in the form

$$\bar{y}(t) = \beta_0(\lambda_0)\exp\left(-\int\limits_1^t \frac{B(x)}{A(x)}dx\right) + \lambda_0\int\limits_1^t \exp\left(-\int\limits_s^t \frac{B(x)}{A(x)}dx\right)\frac{F(s)}{\mu(s)}ds, \tag{25}$$

Theorem 2. Let conditions 1^0-3^0 be satisfied. Then, for sufficiently small $\varepsilon > 0$, the following estimate holds:

$$\left| y\left(t, \widetilde{\lambda}(\varepsilon), \varepsilon\right) - \bar{y}\left(t, \lambda_0\right)\right| = O\left(\varepsilon + \exp\left(-\frac{\gamma t}{\varepsilon}\right)\right). \tag{26}$$

The proof follows from representations (22), (25).

Thus, it directly follows from Theorem 2 that the solution $\left(y(t,\widetilde{\lambda}(\varepsilon),\varepsilon),\ \widetilde{\lambda}(\varepsilon)\right)$ of the singularly perturbed problem (1), (2) at tends small parameter ε to zero tends to the solution of $\bar{y}(t,\lambda_0)$:

$$\lim_{\varepsilon\to 0} y(t,\widetilde{\lambda}(\varepsilon)\varepsilon) = \bar{y}(t,\lambda_0), \quad 0 < t \le 1. \tag{27}$$

Hence, we conclude that

$$\lim_{\varepsilon\to 0} y(0,\tilde{\lambda}(\varepsilon),\varepsilon) - \bar{y}(0,\lambda_0) = \Delta, \tag{28}$$

where Δ is a some magnitude. We define magnitude of the jump Δ. Using formulas (23), (26), (27) and the condition $y(0,\varepsilon) = \alpha_0$, we determine the magnitude of the initial jump:

$$\Delta = \alpha_0 - \beta_0(\lambda_0)\exp\left(\int\limits_0^1 \frac{B(x)}{A(x)}\right) + \lambda_0\int_0^1 \exp\left(-\int\limits_s^0 \frac{B(x)}{A(x)}dx\right)\frac{F(s)}{A(s)}ds.$$

Conclusions

The proposed algorithm serves as the basis for constructing asymptotic solutions of some linear and nonlinear singularly perturbed boundary value problems with parameters for higher order equations with more complex additional conditions such $U_i(y) = 0,\ i = \overline{1,.n}$ where $U_i(y)$ the linear form of $y^{(j)}(0,\varepsilon), y^{(j)}(1,\varepsilon), j = \overline{0.n-1}$.

In this work, the asymptotic behavior of the solution to the problem of reconstructing the boundary conditions and the right-hand side for second-order differential equations with a small parameter at the highest derivative were studied. At that the following new results were obtained:

- Asymptotic estimates are obtained for the solution of the reconstruction problem for singularly perturbed second-order equations with an initial jump;
- rules for the restoration of boundary conditions and the right side of the original and degenerate problems were established;
- Asymptotic estimates are obtained for the solution of the perturbed problem and the difference between the solution of the degenerate problem and the solution of the perturbed problem.

The results obtained open up possibilities for the further development of the theory of singularly perturbed boundary value problems for ordinary differential equations.

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Д.Н. Нургабыл, С.С. Нажим

Бастапқы секірумен сингулярлы ауытқыған дифференциалдық теңдеуді қалпына келтіру есебі

Мақалада бастапқы секіру құбылысына ие жоғары туынды кезінде кіші параметрі бар екінші ретті дифференциалдық теңдеулер үшін оң және шеттік жағдайларды қалпына келтіру есептерін шешудің асимптотикалық шешімі зерттелген. Бастапқы секірумен екінші ретті сингулярлы ауытқыған теңдеулер үшін қалпына келтіру есебін шешудің асимптотикалық бағалары алынды. Шеттік жағдайларды қалпына келтіру ережелері, бастапқы және қалыптасқан міндеттердің оң бөліктері белгіленген. Ауытқыған есептің шешімін асимптотикалық бағалау, сонымен қатар ауытқыған және азғындалған есептердің шешімдері арасындағы айырмашылық анықталды. Сингулярлы ауытқыған теңдеулер позициясындағы қалпына келтіру есебінің бар болуы, біртұтастығы және шешімін ұсыну туралы теорема дәлелденген. Алынған нәтижелер жай дифференциалдық теңдеулер үшін сингулярлы ауытқыған шеттік есептер теориясының одан әрі дамуына мүмкіндік береді.

Кілт сөздер: ауытқыған есептер, кіші параметр, шеттік есептер, бастапқы секіріс, асимптотикалық қасиет.

Д.Н. Нургабыл, С.С. Нажим

Задача восстановления сингулярно возмущенного дифференциального уравнения с начальным скачком

В статье исследовано асимптотическое поведение решения задачи восстановления краевых условий и правой части для дифференциальных уравнений второго порядка с малым параметром при старшей производной, обладающих явлением начального скачка. Получены асимптотические оценки решения задачи восстановления для сингулярно возмущенных уравнений второго порядка с начальным скачком. Установлены правила восстановления краевых условий и правые части исходной и вырожденной задач. Определены асимптотические оценки решения возмущенной задачи и разности между решением вырожденной задачи и решением возмущенной задачи. Доказана теорема о существовании, единственности и представлении решения задачи восстановления с позиции сингулярно возмущенных уравнений. Полученные результаты открывают возможности для дальнейшего развития теории сингулярно возмущенных краевых задач для обыкновенных дифференциальных уравнений.

Ключевые слова: возмущенные задачи, вырожденные задачи, малый параметр, краевая задача, начальный скачок, асимптотическое поведение.

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On a bottom layer in a group

We consider the problem of recognizing a group by its bottom layer. This problem is solved in the class of layer-finite groups. A group is layer-finite if it has a finite number of elements of every order. This concept was first introduced by S. N. Chernikov. It appeared in connection with the study of infinite locally finite p-groups in the case when the center of the group has a finite index. S. N. Chernikov described the structure of an arbitrary group in which there are only finite elements of each order and introduced the concept of layer-finite groups in 1948. Bottom layer of the group G is a set of its elements of prime order. If have information about the bottom layer of a group we can receive results about its recognizability by bottom layer. The paper presents the examples of groups that are recognizable, almost recognizable and unrecognizable by its bottom layer under additional conditions.

Keywords: group, layer-finiteness, bottom layer, thin layer-finite group, spectrum, periodic group, Sylow subgroup, Abelian group, quasi-cyclic group, complete group.

Introduction

Every direct product of finite groups, having for each prime number p only a finite number of factors with orders divisible by p, is a periodic group with a finite set of elements of each order. The direct product of quasi-cyclic p-groups containing only a finite number of factors of this type for each p has the same property. Each direct product of a group of the first kind and a group of the second kind has the same property. However, the last products do not exhaust all periodic groups possessing a finite set of elements of each order. Groups with a finite set of elements of each order are called layer-finite, and a layer is called a set of elements of the same order.

This concept was first introduced by S. N. Chernikov. It appeared in connection with the study of infinite locally finite p-groups in the case when the center of the group has a finite index in it. S. N. Chernikov in 1948 described the structure of an arbitrary group in which there are an finite number of elements of each order, and in this work the term layer-finite groups appeared. The main result describing the structure of layer-finite groups was obtained by S. N. Chernikov also in 1948. It says that a group is then and only then layer-finite when it can be represented as the product of two elementwise permutation subgroups, of which the first is a layer-finite complete Abelian group, and the second is a layer-finite group with finite Sylow subgroups. The bottom layer of the group G is the set of its elements of prime orders. In this work we will recognize the group by its bottom layer under additional conditions. It will be convenient for us to do this in the class of layer-finite groups.

For the convenience of reading the article, the last section contains well-known results that we referred to in the proof of the theorems as proposition with the corresponding number.

Main part

A group G is called *recognizable by the bottom layer* under additional conditions if it is uniquely reconstructed by the bottom layer under these conditions. A group G is said to be *almost recognizable by*

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its bottom layer under additional conditions if there are finitely many pairwise non-isomorphic groups with the same bottom layer the same as in the group G under these conditions. A group G is called *unrecognizable by the bottom layer* under additional conditions if there is an infinite number of pairwise non-isomorphic groups with the same bottom layer as in the group G under these conditions.

Recall that the set of elements of prime order in a group is called its *bottom layer*.

Among the results on recognizability by the bottom layer, we can name those that describe the entire structure of the group by its bottom layer. For example: if the bottom layer of an infinite group consists of elements of order 2 and the group does not have non-identity elements of other orders, then G is an infinite elementary Abelian 2-group. That is, the group under such conditions is recognizable by the bottom layer.

V. P. Shunkov proved that if the bottom layer in an infinite layer-finite group consists of one element of order 2, then the group is either quasi-cyclic or an infinite generalized group of quaternions [1]. In this example, groups are almost recognizable by the bottom layer.

The following series of groups gives an example of unrecognizability by the bottom layer: in groups $C_{p^\infty} \times C_q$, $C_{p^\infty} \times C_{q^2}$, $C_{p^\infty} \times C_{q^3}, \ldots$ the same bottom layer consisting of the $p-1$ element of order p and the $q-1$ element of order q.

If the set of orders of elements of the bottom layer of an infinite group is small in terms of the number of its constituent numbers, but not in magnitude, then such examples of groups are quite rare. According to the figurative expression of Yu. I. Merzlyakov, they are comparable with "samples of lunar soil". Such examples include monsters of A. Yu. Olshansky [2]. Olshansky groups, as well as direct products of cyclic groups of prime order, are examples of groups without a single element coinciding with their bottom layer.

N. D. Gupta and V. D. Mazurov proved that for the group G, which, without a unit element, coincides with its bottom layer consisting of elements of orders 3, 5, one of the statements is true: 1) $G = FT$; where F is a normal 5-subgroup of nilpotent class at most two and $|T| = 3$; 2) G contains a normal 3-subgroups T of nilpotent class at most three such that G/T is a 5-group [3]. In the same work, it was shown that a group that, without a unit element, coincides with its bottom layer consisting of elements of orders 2, 5, either contains an elementary Abelian 5-subgroup of index 2, or an elementary Abelian normal Sylow 2-subgroup [3].

Sometimes you can restore a group by the bottom layer of a group, sometimes you can say something about the properties of such a group. Among the results of the first type, we can name those that describe the entire structure of the group by its bottom layer, for example: if the bottom layer consists of elements of orders 2, 3, 5 and the group does not have non-identity elements of other orders, then A. S. Kondratiev and V. D. Mazurov proved that this is a group of even permutations on five elements [4]. The results of the second type include the establishment by V. D. Mazurov of the local finiteness of a group with a bottom layer consisting of elements of orders 2, 3, 5, in which all other non-unit elements are of order 4 [5].

In the theory of finite groups, a similar concept of spectrum recognition of a group is considered.

The spectrum of a finite group is the set of orders of its elements. The spectrum of a group G is denoted by $\omega(G)$. A finite group G is called spectrum recognizable if any finite group whose spectrum coincides with the spectrum of the group G is isomorphic to G. A group G is said to be almost recognizable by spectrum if there exist finitely many pairwise nonisomorphic groups with the same spectrum as the group G. A group G is said to be unrecognizable by spectrum if there is an infinite number of pairwise nonisomorphic groups with the same spectrum as the group G.

It was proved in [6] that the symmetric groups S_n are recognizable by spectrum for $n \notin \{2, 3, 4, 5, 6, 8, 10, 15, 16, 18, 21, 27, 33, 35, 39, 45\}$.

In 1994, W. Shi and R. Brandl proved spectrum recognizability of an infinite series of simple linear groups $L_2(q)$, $q \neq 9$ [7, 8].

Let G be a finite group and $\omega(G) = \omega(S_6(2))$. Then G is isomorphic to $S_6(2)$ or O_+^8. In particular, the group $S_6(2)$ is almost recognizable by spectrum [9].

An example of a group not recognized by the spectrum is the group A_6 with the spectrum $1, 2, 3, 5, 4, 8, 9$ (there are infinitely many groups, one of which is the group A_6) [10]. Also, the group $L_3(3)$ with the spectrum, $1, 2, 3, 4, 8, 9, 13, 16, 27$, is unrecognizable by the spectrum [10].

By the theorem of A. V. Vasiliev (Proposition 1), a finite simple non-Abelian group $U_4(5)$ is not recognizable by spectrum. In this regard, we prove the result of recognizability of this group simultaneously by the spectrum and by the bottom layer.

Theorem 1. Let G be a finite simple group $U_4(5)$ and H be a finite group with the property $\omega(H) = \omega(G)$ and the bottom layer same as the group $U_4(5)$. Then $H \cong G$. That is, the group $U_4(5)$ is an unique finite group with such a spectrum and a bottom layer.

Proof. Indeed, let G be a finite simple group $U_4(5)$ and H be a finite group with the property $\omega(H) = \omega(G)$.

By the theorem of A. V. Vasiliev (Proposition 1), in addition to the group $U_4(5)$, there is only one such group $H \cong G(\gamma)$, where γ is a field automorphism of the group G of order 2. The groups $U_4(5)$ and H have the same spectrum, while these groups have different bottom layers, which differ at least by an element of order 2. Thus, the group $U_4(5)$ is an unique finite group with such a spectrum and a bottom layer. The theorem is proved.

Definition. Recall that if the orders of all elements of a group are finite, then the group is called *periodic*.

Theorem 2. If G is a complete group in which $Z(G)$ is layer-finite and $G/Z(G)$ is a periodic group containing for each prime p only a finite number of p-elements, then the group G is recognizable by the bottom layer among groups with such properties.

Proof. Indeed, let the group G satisfy the indicated conditions. Since $Z(G)$ is layer-finite and $G/Z(G)$ is a periodic group containing for each prime p only a finite number of p-elements, by proposition 2, the group G is layer-finite. Since, by Proposition 3, each complete subgroup of a layer-finite group G is contained in the center of the group G, then since G is complete, then it is Abelian.

By Proposition 4, the complete Abelian group G decomposes into a direct sum of subgroups isomorphic to the additive group of rational numbers or to quasicyclic groups, possibly for different prime numbers. There can be no rational groups in this extension, since G is a layer-finite group and, therefore, there are no elements of infinite order in it. Since the direct product of quasi-cyclic groups is obviously restored from the bottom layer, the group G is recognizable by the bottom layer among groups with the properties as in the theorem. The theorem is proved.

Definition. Layer-finite group is called a *thin layer-finite group* if all of its Sylow subgroups are finite.

Theorem 3. Let G be a group in which the center contains a complete layer-finite subgroup R such that the factor group G/R is a thin layer-finite group. Then the group G is recognizable by the bottom layer among groups with such properties.

Proof. Suppose that the group G satisfies the indicated conditions. Since G is a group in which the center contains such a complete layer-finite subgroup R such that the factor group G/R is a thin layer-finite group, by Proposition 5 the group G is layer-finite .

Since by Proposition 3 each complete subgroup of a layer-finite group G is contained in the center of the group G, the group G, being complete, is Abelian. Then, by Proposition 4, the group G decomposes into a direct sum of subgroups isomorphic to the additive group of rational numbers or to quasi-cyclic groups, possibly for different prime numbers.

Among the direct components there can only be quasi-cyclic groups, since the group of rational numbers has elements of infinite order, and G is a layer-finite group and therefore cannot contain elements of infinite order. So G decomposes into a direct sum of quasi-cyclic primary groups, and such a group is recognizable by the bottom layer among groups with the properties as in the theorem. The theorem is proved.

Theorem 4. Let G be a complete nilpotent p-group with finite bottom layer. Then the group G is recognizable by the bottom layer among groups with such properties.

Proof. Indeed, since G is a complete nilpotent p-group with a finite bottom layer, by Proposition 6 the group G is layer-finite. Given that G is a complete group and repeating the final part of the previous proof, we see that the group G is recognizable by the bottom layer among groups with the properties as in the theorem. The theorem is proved.

Theorem 5. Let G be a complete periodic group in which for each prime p there is only a finite number of Sylow p-subgroups and for every prime p there is at least, one Sylow p-subgroup in G, which is a layer-finite group. Then the group is recognizable by the bottom layer among groups with such properties.

Proof. Suppose that the group G satisfies the given conditions. Because G is the group in which the conditions are satisfied: G is the periodic group; for each prime number p there is only a finite number of Sylow p-subgroups; for every prime number p there is at least one Sylow p-subgroup in G, which is a layer-finite group, then by Proposition 7 the group G is layer-finite. Based on the fact that the group is layer-finite, complete, and applying Propositions 3 and 4, we have that the group G is complete Abelian and decomposes into a direct sum of subgroups isomorphic to the additive rational group or quasi-cyclic groups, may be according to different prime numbers.

Among the direct components there can be only quasi-cyclic groups, since the group of rational numbers has elements of infinite order, and G is a layer-finite group and therefore cannot contain elements of infinite order. Therefore, G decomposes into a direct sum of quasi-cyclic primary groups, and such a group is recognizable by the bottom layer among groups with the properties as in the theorem. The theorem is proved.

In proving the results of the paper, we used the following theorems, which were referred to as proposition with the corresponding number.

Proposition 1 (A.V. Vasiliev [10]). Let G be a finite simple group $U_4(5)$ and H be a finite group with the property $\omega(H) = \omega(G)$. Then $H \cong G$ or $H \cong G(\gamma)$, where γ is a field automorphism of the group G of order 2. In particular, $h(G) = 2$.

By $h(G)$ we denote the number of non-isomorphic groups with the same spectrum.

Proposition 2 (R. Baer [11]). The following properties are equivalent;

a) G is a layer-finite group;

b) $Z(G)$ is layer-finite and $G/Z(G)$ is a periodic group containing for each prime p only a finite number of p-elements;

c) there is a subgroup S in the center of G such that S and G/S are layer-finite groups.

Proposition 3 (S.N. Chernikov, Lemma 3.1. from [12]). Each complete subgroup of a locally normal (in particular, layer-finite) group is contained in the center of the group.

Proposition 4 (Theorem 9.1.6 from [13]). A nonzero complete Abelian group can be decomposed into a direct sum of subgroups isomorphic to the additive rational group or quasi-cyclic groups, may be for different prime numbers.

Proposition 5 (S. N. Chernikov, Theorem 1 from [14]). A group G if and only if is layer-finite if its center contains such a complete layer-finite subgroup R such that the factor group G/R is a thin layer-finite group.

Proposition 6 (S. N. Chernikov [15]). If a nilpotent p-group of G contains only a finite set of elements of some non-unit order, then it is layer-finite.

Proposition 7. (R. Baer [11]). A group G is layer-finite if and only if the following conditions are satisfied:

a) G is a periodic group;

b) for each prime number p there is only a finite number of Sylow p-subgroups;

c) for every prime number p there is at least one Sylow p-subgroup in G, which is a layer-finite group.

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В.И. Сенашов, И.А. Паращук

Группадағы төменгі қабат туралы

Мақалада группадағы төменгі қабат бойынша танылу мәселесі қарастырылды. Бұл мәселе қабатты шекті группалардың класында шешіледі. Группа қабатты шекті деп аталады, егер группаның әрбір ретінде шекті санды элемент бар болса. Бұл ұғымды С.Н. Черников енгізген. Ол шексіз локальды-шекті p-группаларын группаның центрінде шекті индекс болған жағдайда зерттеуге байланысты пайда болды. С.Н. Черников 1948 жылы әрбір реттегі элементтер жиыны шексіз кез келген группаның құрылымын сипаттады және осы жұмыста қабатты шекті группалар термині пайда болды.

G группасының төменгі қабаты деп оның жай ретті элементтер жиынын айтамыз. Төменгі қабат туралы мәліметтер бойынша группаның төменгі қабаты бойынша танылуы туралы нәтижелерді аламыз. Жұмыста төменгі қабаты бойынша танылатын, танылатын дерлік, танылмайтын группалардың мысалдары келтірілген.

Кілт сөздер: группа, қабатты шектілік, төменгі қабат, жұқа қабатты шекті группа, спектр, периодты группа, силов ішкі группасы, абельдік группа, квазициклдік группа, толық группа.

В.И. Сенашов, И.А. Паращук

О нижнем слое в группе

В статье рассмотрен вопрос о распознавании группы по её нижнему слою. Этот вопрос решается в классе слойно конечных групп. Группа называется слойно конечной, если она имеет конечное число элементов каждого порядка. Это понятие впервые было введено С.Н. Черниковым. Оно появилось в связи с изучением бесконечных локально конечных p-групп в случае, когда центр группы имеет конечный индекс в ней. В 1948 г. С.Н. Черников описал строение произвольной группы, в которой бесконечно множество элементов каждого порядка, и ввел понятие слойно конечных групп. Нижним слоем группы G называется множество её элементов простых порядков. По информации о нижнем слое авторами статьи получены результаты о распознаваемости группы по нижнему слою. Приведены примеры групп, распознаваемых по нижнему слою, почти рапознаваемых и нераспознаваемых групп при дополнительных ограничениях.

Ключевые слова: группа, слойная конечность, нижний слой, тонкая слойно конечная группа, спектр, периодическая группа, силовская подгруппа, абелева группа, квазициклическая группа, полная группа.

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A source inverse problem for the pseudo–parabolic equation with the fractional Sturm–Liouville operator

A class of inverse problems for restoring the right-hand side of the pseudo-parabolic equation with one fractional Sturm–Liouville operator is considered. In this paper, we prove the existence and uniqueness results of the solutions using by the variable separation method that is to say the Fourier method. We are especially interested in proving the existence and uniqueness of the solutions in the abstract setting of Hilbert spaces. The mentioned results are presented as well as for the Caputo time fractional pseudo-parabolic equation. There are many cases in which practical needs lead to problems determining the coefficients or the right side of a differential equation from some available decision data. These are called inverse problems of mathematical physics. Inverse problems arise in various areas of human activity, such as seismology, mineral exploration, biology, medicine, industrial quality control goods, and so on. All these circumstances put the inverse problems among the important problems of modern mathematics.

Keywords: Pseudo–parabolic equation, inverse problem, fractional Sturm–Liouville operator, Caputo fractional derivative.

Introduction

In this paper we consider pseudo–parabolic equation generated by fractional Sturm–Liouville operator with Caputo time-fractional derivative. We investigate the equation

$$\mathcal{D}_t^\alpha[u(t,x) + \partial_{+a,x}^\alpha D_{b-,x}^\alpha u(t,x)] + \partial_{+a,x}^\alpha D_{b-,x}^\alpha u(t,x) = f(x), \tag{1}$$

for $(t,x) \in \Omega = \{(t,x)|\ 0 < t \leq T < \infty,\ a \leq x \leq b\}$, where $\mathcal{D}_t^\alpha$ is the Caputo derivative and $\partial_{+a,x}^\alpha D_{b-,x}^\alpha$ is the fractional Sturm–Liouville operator which is defined in the next section.

In many physical problems, it is required to determine the coefficients or the right-hand side (the original term, in the case of the diffusion equation) in the differential equation from some available information; These problems are known as inverse problems. Similar problems are poorly formulated in the sense of Hadamard. A number of papers consider the problem of solvability of inverse problems for the equations of diffusion and anomalous diffusion (see [1–9] and references therein).

1 Definitions of fractional operators

We begin this paper with a brief introduction of several concepts that are important for the further studies.

Definition 1 [10]. The Riemann-Liouville fractional integral I^α of order $\alpha > 0$ for an integrable function is defined by

$$I^\alpha[f](t) = \frac{1}{\Gamma(\alpha)}\int_c^t (t-s)^{\alpha-1} f(s)ds, t \in [c,d],$$

where Γ denotes the Euler gamma function.

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Definition 2 [10]. The Riemann-Liouville fractional derivative D^{α} of order $\alpha \in (0,1)$ of a continuous function is defined by

$$D^{\alpha}[f](t) = \frac{d}{dt} I^{\alpha}[f](t), t \in [c,d].$$

Definition 3 [10]. The Caputo fractional derivative of order $0 < \alpha < 1$ of a differentiable function is defined by

$$\mathcal{D}_t^{\alpha}[f](t) = D^{\alpha}[f'(t)], t \in [c,d].$$

Definition 4 [10]. Let $f \in L^1[a,b]$, $-\infty \leq a < t < b \leq +\infty$ and $f * K_{m-\alpha}(t) \in W^{m,1}[a,b]$, $m = [\alpha]$, $\alpha > 0$. The Caputo fractional derivative ∂_{+a}^{α} of order $\alpha \in \mathbb{R}$ $(m-1 < \alpha < m,\, m \in \mathbb{N})$ is defined as

$$\partial_{+a}^{\alpha} f(t) = D_{+a}^{\alpha} \left[f(t) - f(a) - f'(a) \frac{(t-a)}{1!} - \dots - f^{(m-1)}(a) \frac{(t-a)^{m-1}}{(m-1)!} \right].$$

If $f \in C^m[a,b]$ then, the Caputo fractional derivative ∂_{+a}^{α} of order $\alpha \in \mathbb{R}$ $(m-1 < \alpha < m,\, m \in \mathbb{N})$ is defined as

$$\partial_{+a}^{\alpha}[f](t) = I_{+a}^{m-\alpha} f^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} f^{(m)}(s) ds.$$

2 Fractional Sturm–Liouville operator

We study the operator generated by the integro-differential expression

$$\mathcal{L}(u) = \partial_{+a}^{\alpha} D_{b-}^{\alpha} u,\ a < x < b, \tag{2}$$

and the conditions

$$I_{b-}^{1-\alpha} u(a) = 0,\ I_{b-}^{1-\alpha} u(b) = 0, \tag{3}$$

where ∂_{+a}^{α} is the left Caputo derivative of order $\alpha \in (0,1]$ of u,

$$D_{b-}^{\alpha} u(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (\xi - x)^{-\alpha} u(\xi) d\xi$$

is the right Riemann-Liouville derivative of order $\alpha \in (0,1]$ of u, and

$$I_{b-}^{\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\xi - x)^{\alpha-1} u(\xi) d\xi$$

is the right Riemann-Liouville integral of order $\alpha \in (0,1]$ of u, [10]. The fractional Sturm–Liouville operator (2)–(3) is self-adjoint and positive in $L^2(a,b)$ (see [11–14]). The spectrum of the fractional Sturm–Liouville operator generated by the equations (2)–(3) is discrete, positive and real valued, and the system of eigenfunctions is a complete orthogonal basis in $L^2(a,b)$.

So we can denote eigenvalues and eigenfunctions accordingly by λ_{ξ} and $e_{\xi}(x)$. That say us for $e_{\xi}(x) \in L^2(a,b)$ following identity is hold:

$$\mathcal{L} e_{\xi}(x) = \lambda_{\xi} e_{\xi}(x),\ \lambda_{\xi} \in \mathbb{R}_{+}. \tag{4}$$

Where $\mathcal{I}$ is a countable set and $\forall \xi \in \mathcal{I}$.

3 Formulation of the problem

We aim to find a couple of functions $(u(t,x), f(x))$ satisfying the equation (1) with an initial condition

$$u(0,x) = \varphi(x), \ x \in [a,b] \tag{5}$$

and with an additional information

$$u(T,x) = \psi(x), \ x \in [a,b]. \tag{6}$$

By using $\mathcal{L}$–Fourier analysis we obtain existence and uniqueness results for this problem.

We say a solution of the problem (1), (5), (6) is a pair of functions $(u(t,x), f(x))$ such that they satisfy equation (1) and conditions (5), (6) where $u(t,x) \in C^{\alpha}([0,T], \mathcal{H}^1)$, $0 < \alpha \leq 1$ and $f(x) \in L^2(a,b)$.

Now, to investigate our problem, we need to define the Hilbert space $\mathcal{H}^1$.

Definition 5. The Hilbert space $\mathcal{H}^1$ is defined by

$$\mathcal{H}^1 := \{u \in L^2(a,b) : \mathcal{L}u \in L^2(a,b)\}.$$

4 Main results

For problem (1), (5), (6) the following theorem holds.

Theorem. Let $\varphi, \psi \in \mathcal{H}^1$.Then a solution $u(t,x) \in C^{\alpha}\left([0,T], \mathcal{H}^1\right)$, $0 < \alpha \leq 1$, $f(x) \in L^2(a,b)$ of problem (1), (5), (6) exists, is unique, and can be written in the form

$$u(x,t) = \varphi(x) + \sum_{\xi \in \mathcal{I}} \frac{\left[\left(\partial^{\alpha}_{+a,x} D^{\alpha}_{b-,x}\psi, e_{\xi}\right)_{L^2(a,b)} - \left(\partial^{\alpha}_{+a,x} D^{\alpha}_{b-,x}\varphi, e_{\xi}\right)_{L^2(a,b)}\right]\left(1 - E_{\alpha,1}\left(-\frac{\lambda_{\xi}}{1+\lambda_{\xi}} t^{\alpha}\right)\right) e_{\xi}(x)}{\lambda_{\xi}\left(1 - E_{\alpha,1}\left(-\frac{\lambda_{\xi}}{1+\lambda_{\xi}} T^{\alpha}\right)\right)},$$

$$f(x) = \partial^{\alpha}_{+a,x} D^{\alpha}_{b-,x}\varphi(x) + \sum_{\xi \in \mathcal{I}} \frac{\left[\left(\partial^{\alpha}_{+a,x} D^{\alpha}_{b-,x}\psi, e_{\xi}\right)_{L^2(a,b)} - \left(\partial^{\alpha}_{+a,x} D^{\alpha}_{b-,x}\varphi, e_{\xi}\right)_{L^2(a,b)}\right] e_{\xi}(x)}{1 - E_{\alpha,1}\left(-\frac{\lambda_{\xi}}{1+\lambda_{\xi}} T^{\alpha}\right)}.$$

Where $E_{\alpha,\beta}$ is the Mittag-Leffler type function [15]:

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}.$$

Proof. First of all, we start by proving an existence result. Let us look for functions $u(t,x)$ and $f(x)$ in the forms:

$$u(t,x) = \sum_{\xi \in \mathcal{I}} u_{\xi}(t) e_{\xi}(x), \tag{7}$$

and

$$f(x) = \sum_{\xi \in \mathcal{I}} f_{\xi} e_{\xi}(x), \tag{8}$$

where $u_{\xi}(t)$ and f_{ξ} are unknown. Substituting (7) and (8) into problem (1), (5), (6) and using relationship (4) we obtain the following problem for the functions $u_{\xi}(t)$ and for the constants f_{ξ}, $\xi \in \mathcal{I}$:

$$\mathcal{D}^{\alpha} u_{\xi}(t) + \frac{\lambda_{\xi}}{1+\lambda_{\xi}} u_{\xi}(t) = \frac{f_{\xi}}{1+\lambda_{\xi}}, \tag{9}$$

$$u_{\xi}(0) = \varphi_{\xi}, \tag{10}$$

$$u_{\xi}(T) = \psi_{\xi}, \tag{11}$$

where φ_ξ, ψ_ξ are $\mathcal{L}$-Fourier coefficients of $\varphi(x)$ and $\psi(x)$:

$$\varphi_\xi = (\varphi, e_\xi)_{L^2(a,b)},$$

$$\psi_\xi = (\psi, e_\xi)_{L^2(a,b)}.$$

General solution of the equation (9):

$$u_\xi(t) = \frac{f_\xi}{\lambda_\xi} + C_\xi E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}t^\alpha\right),$$

where the constants C_ξ, f_ξ are unknown. By using conditions (10) and (11), we can find they. We first find C_ξ:

$$u_\xi(0) = \frac{f_\xi}{\lambda_\xi} + C_\xi = \varphi_\xi,$$

$$u_\xi(T) = \frac{f_\xi}{\lambda_\xi} + C_\xi E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}T^\alpha\right) = \psi_\xi,$$

$$\varphi_\xi - C_\xi + C_\xi E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}T^\alpha\right) = \psi_\xi.$$

Then

$$C_\xi = \frac{\varphi_\xi - \psi_\xi}{1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}T^\alpha\right)}.$$

f_ξ is represented as

$$f_\xi = \lambda_\xi\varphi_\xi - \lambda_\xi C_\xi.$$

Substituting $f_\xi, u_\xi(t)$ into formula (7), we find

$$u(x,t) = \varphi(x) + \sum_{\xi\in\mathcal{I}} C_\xi\left(E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}t^\alpha\right) - 1\right)e_\xi(x). \quad (12)$$

Using self-adjoint property of operator $\mathcal{L}$

$$(\mathcal{L}\varphi, e_\xi)_{L^2(a,b)} = (\varphi, \mathcal{L}e_\xi)_{L^2(a,b)}$$

and in respect that (4) we obtain

$$(\varphi, e_\xi)_{L^2(a,b)} = \frac{(\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}}{\lambda_\xi},$$

and for $\psi(x)$ we can write analogously. Substituting these equality into formula of C_ξ we can get that

$$C_\xi = \frac{(\mathcal{L}\varphi, e_\xi)_{L^2(a,b)} - (\mathcal{L}\psi, e_\xi)_{L^2(a,b)}}{\lambda_\xi\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}T^\alpha\right)\right)}.$$

Putting this into the formula (12), we have

$$u(t,x) = \varphi(x) + \sum_{\xi\in\mathcal{I}} \frac{\left[(\mathcal{L}\psi, e_\xi)_{L^2(a,b)} - (\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right]\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}t^\alpha\right)\right)e_\xi(x)}{\lambda_\xi\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}T^\alpha\right)\right)}, \quad (13)$$

As the same way as (13), we obtain

$$f(x) = \mathcal{L}\varphi(x) + \sum_{\xi\in\mathcal{I}} \frac{\left[(\mathcal{L}\psi, e_\xi)_{L^2(a,b)} - (\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right] e_\xi(x)}{1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}T^\alpha\right)}. \quad (14)$$

The following Mittag-Leffler function's estimate is known by [16]:

$$|E_{\alpha,\beta}(z)| \leq \frac{M}{1+|z|}, arg(z) = \pi, |z| \to \infty. \quad (15)$$

Now, we show that $u(t,x) \in C^\alpha([0,T],\mathcal{H}^1)$, $f(x) \in L^2(a,b)$, that is

$$\|u\|_{C^\alpha([0,T],\mathcal{H}^1)} = \max_{t\in[0,T]} \|u(t,\cdot)\|_{\mathcal{H}^1} + \max_{t\in[0,T]} \|\mathcal{D}_t^\alpha u(t,\cdot)\|_{\mathcal{H}^1} < \infty,$$

and

$$\|f\|_{L^2(a,b)} < \infty.$$

Where

$$\|u(t,\cdot)\|_{\mathcal{H}^1} = \|u(t,\cdot)\|_{L^2(a,b)} + \|\mathcal{L}u(t,\cdot)\|_{L^2(a,b)}$$

and

$$\|\mathcal{D}_t^\alpha u(t,\cdot)\|_{\mathcal{H}^1} = \|\mathcal{D}_t^\alpha u(t,\cdot)\|_{L^2(a,b)} + \|\mathcal{D}_t^\alpha \mathcal{L}u(t,\cdot)\|_{L^2(a,b)}.$$

Using by the estimate (15) we get following estimates for $u(t,x), \mathcal{L}u(t,x)$ and $\mathcal{D}_t^\alpha u(t,x)$:

$$\|u(t,x)\|^2_{C([0,T],L^2(a,b))} = \|\varphi(x)$$

$$+ \sum_{\xi\in\mathcal{I}} \frac{\left[(\mathcal{L}\psi, e_\xi)_{L^2(a,b)} - (\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right]\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}t^\alpha\right)\right) e_\xi(x)}{\lambda_\xi\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}T^\alpha\right)\right)}\|^2_{C([0,T],L^2(a,b))}$$

$$\lesssim \|\varphi\|^2_{L^2(a,b)} + \sum_{\xi\in\mathcal{I}} \max_{t\in[0,T]} \left| \frac{\left[(\mathcal{L}\psi, e_\xi)_{L^2(a,b)} - (\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right]\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}t^\alpha\right)\right)}{\lambda_\xi\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}T^\alpha\right)\right)} \right|^2 \|e_\xi\|^2_{L^2(a,b)}$$

$$\lesssim \|\varphi\|^2_{L^2(a,b)} + \sum_{\xi\in\mathcal{I}} \left[\frac{\left|(\mathcal{L}\psi, e_\xi)_{L^2(a,b)}\right|^2 + \left|(\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right|^2}{\lambda_\xi^2} \right] < \infty,$$

$$\|\mathcal{L}u(x,t)\|^2_{C([0,T],L^2(a,b))} = \|\mathcal{L}\varphi(x)$$

$$+ \sum_{\xi\in\mathcal{I}} \frac{\left[(\mathcal{L}\psi, e_\xi)_{L^2(a,b)} - (\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right]\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}t^\alpha\right)\right) \mathcal{L}e_\xi(x)}{\lambda_\xi\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}T^\alpha\right)\right)}\|^2_{C([0,T],L^2(a,b))}$$

$$\lesssim \|\mathcal{L}\varphi\|^2_{L^2(a,b)} + \sum_{\xi\in\mathcal{I}} \max_{t\in[0,T]} \left| \frac{\left[(\mathcal{L}\psi, e_\xi)_{L^2(a,b)} - (\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right]\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}t^\alpha\right)\right)}{\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}T^\alpha\right)\right)} \right|^2 \|e_\xi\|^2_{L^2(a,b)}$$

$$\lesssim \|\mathcal{L}\varphi\|^2_{L^2(a,b)} + \sum_{\xi\in\mathcal{I}} \left[\left|(\mathcal{L}\psi, e_\xi)_{L^2(a,b)}\right|^2 + \left|(\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right|^2 \right] < \infty,$$

$$\|\mathcal{D}_t^\alpha u(x,t)\|^2_{C([0,T],L^2(a,b))}$$

$$= \| \sum_{\xi\in\mathcal{I}} \frac{\left[(\mathcal{L}\psi, e_\xi)_{L^2(a,b)} - (\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right] D_t^\alpha \left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} t^\alpha\right)\right) e_\xi(x)}{\lambda_\xi \left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)\right)} \|^2_{C([0,T],L^2(a,b))}$$

$$\leq \sum_{\xi\in\mathcal{I}} \max_{t\in[0,T]} \left| \frac{\left[(\mathcal{L}\psi, e_\xi)_{L^2(a,b)} - (\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right] E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} t^\alpha\right)}{(1+\lambda_\xi)\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)\right)} \right|^2 \|e_\xi\|^2_{L^2(a,b)}$$

$$\lesssim \sum_{\xi\in\mathcal{I}} \left[\frac{\left|(\mathcal{L}\psi, e_\xi)_{L^2(a,b)}\right|^2 + \left|(\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right|^2}{(1+\lambda_\xi)^2} \right] < \infty,$$

and

$$\|\mathcal{D}_t^\alpha \mathcal{L} u(x,t)\|^2_{C([0,T],L^2(a,b))}$$

$$= \| \sum_{\xi\in\mathcal{I}} \frac{\left[(\mathcal{L}\psi, e_\xi)_{L^2(a,b)} - (\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right] D_t^\alpha \left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} t^\alpha\right)\right) \mathcal{L} e_\xi(x)}{\lambda_\xi \left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)\right)} \|^2_{C([0,T],L^2(a,b))}$$

$$\leq \sum_{\xi\in\mathcal{I}} \max_{t\in[0,T]} \left| \frac{\lambda_\xi \left[(\mathcal{L}\psi, e_\xi)_{L^2(a,b)} - (\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right] E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} t^\alpha\right)}{(1+\lambda_\xi)\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)\right)} \right|^2 \|e_\xi\|^2_{L^2(a,b)}$$

$$\lesssim \sum_{\xi\in\mathcal{I}} \left[\left|(\mathcal{L}\psi, e_\xi)_{L^2(a,b)}\right|^2 + \left|(\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right|^2 \right]$$

$$+ \sum_{\xi\in\mathcal{I}} \left[\frac{\left|(\mathcal{L}\psi, e_\xi)_{L^2(a,b)}\right|^2 + \left|(\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right|^2}{(1+\lambda_\xi)^2} \right] < \infty.$$

Similarly for $f(x)$ we have the estimate

$$\|f\|^2_{L^2(a,b)} = \|\mathcal{L}\varphi(x) + \sum_{\xi\in\mathcal{I}} \frac{\left[(\mathcal{L}\psi, e_\xi)_{L^2(a,b)} - (\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right] e_\xi(x)}{1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)} \|^2_{L^2(a,b)}$$

$$\lesssim \|\mathcal{L}\varphi\|^2_{L^2(a,b)} + \sum_{\xi\in\mathcal{I}} \left| \frac{(\mathcal{L}\psi, e_\xi)_{L^2(a,b)} - (\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}}{1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)} \right|^2 \|e_\xi\|^2_{L^2(a,b)}$$

$$\lesssim \|\mathcal{L}\varphi\|^2_{L^2(a,b)} + \sum_{\xi\in\mathcal{I}} \left[\left|(\mathcal{L}\psi, e_\xi)_{L^2(a,b)}\right|^2 + \left|(\mathcal{L}\varphi, e_\xi)_{L^2(a,b)}\right|^2 \right] < \infty.$$

Where, $L \lesssim Q$ denotes $L \leq CQ$ for some positive constant C independent of L and Q. Existence of the solution of problem (1), (5), (6) is proved.

Now, we start proving the uniqueness of the solution.

Let us suppose that $\{u_1(x,t), f_1(x)\}$ and $\{u_2(x,t), f_2(x)\}$ are solution of problem (1), (5), (6). Then $u(x,t) = u_1(x,t) - u_2(x,t)$ and $f(x) = f_1(x) - f_2(x)$ are solution of following problem:

$$\mathcal{D}_t^\alpha [u(x,t) + \partial^\alpha_{+a,x} D^\alpha_{b-,x} u(x,t)] + \partial^\alpha_{+a,x} D^\alpha_{b-,x} u(x,t) = f(x), \tag{16}$$

$$u(x,0) = 0, \tag{17}$$

$$u(x,T) = 0. \tag{18}$$

By using (13) and (14) for (16)–(18) we easily see $u(x,t) \equiv 0, f(x) \equiv 0$. Uniqness of the solution of problem (1), (5), (6) is proved.

Discussion on further generalisations. Note that the results are derived here can be generalised by using the non–harmonic analysis developed in the papers [17, 18] with the general setting settled in [19, 20]. Moreover, the reader is referred to [21–25] for interesting applications of the non–harmonic analysis to the different branches of partial differential equations.

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Штурм-Лиувилль бөлшек туынды операторлы псевдопараболалық теңдеуі үшін қайнар көзді анықтаудың кері есебі

Мақалада Штурм-Лиувилль бөлшек туынды операторлы псевдопараболалық теңдеудің оң жағын қалпына келтіру кері есептер класы қарастырылды. Авторлар айнымалыларды ажырату әдісін, яғни Фурье әдісін қолдана отырып, шешімнің бар және жалғыздығын дәлелдеді. Сонымен қатар абстракты Гильберт кеңістігінде шешімнің бар және жалғыздығы туралы нәтижелерді алды. Көрсетілген нәтижелер уақыт бойынша Капуто бөлшек туындылы псевдопараболалық теңдеу үшін алынды. Дифференциалдық теңдеудің шешімдеріне қатысты кейбір қосымша ақпараттар арқылы теңдеудің оң жағын анықтау немесе теңдеудің коэффициенттерін анықтау есептері практикалық жұмыстардан туындап отыр. Бұл математикалық физиканың кері есептері. Олар адам қызметінің әртүрлі салаларында пайда болады, мысалы, сейсмология, минералды барлау, биология, медицина, өнеркәсіптік сапаны бақылау өнімдері және т.б. Осы жағдайлардың барлығы қазіргі математиканың маңызды мәселелерінің қатарына кері есептер саласын енгізіп отыр.

Кілт сөздер: псевдопараболалық теңдеу, кері есеп, бөлшек туынды Штурм-Лиувилль операторы, Капуто бөлшек туындысы.

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Обратная задача определения источника для псевдопараболического уравнения с дробным оператором Штурма-Лиувилля

В статье рассмотрен класс обратных задач восстановления правой части псевдопараболического уравнения с дробным оператором Штурма-Лиувилля. Авторами доказаны результаты существования и единственности решений, с использованием метода разделения переменных, то есть методом Фурье. Кроме того, особенная заинтересованность наблюдается в доказательстве существования и единственности решений в абстрактной постановке гильбертовых пространств. Указанные результаты представлены для дробного псевдо-параболического уравнения Капуто по времени. Есть много случаев, в которых практические потребности приводят к задачам определения коэффициентов или правой части дифференциального уравнения по некоторым доступным данным решения. Это так называемые обратные задачи математической физики. Они возникают в различных областях человеческой деятельности, таких как сейсмология, разведка полезных ископаемых, биология, медицина, промышленные товары контроля качества и т.д. Все эти обстоятельства ставят обратные задачи в число важных проблем современной математики.

Ключевые слова: псевдопараболическое уравнение, обратная задача, дробный оператор Штурма-Лиувилля, производная Капуто.

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Method of the rheostat for studying properties of fragments of theoretical sets

In this article discusses the model-theoretical properties of fragments of theoretical sets and the rheostat method. These two concepts: theoretical set and rheostat are new. The study of this topic in the framework of the study of Jonsson theories, the Jonsson spectrum, classes of existentially closed models of such fragments is a new promising class of problems and their solution is closely related to many problems that once defined the classical problems of model theory. The purpose of this article is to determine the rheostat of the transition from complete theory to Jonsson theory, which will be consistent with the corresponding concepts for any α and any α-Jonsson theory. For this we define a theoretical set. On the basis of research by the author formulated a model-theoretical definition of the concept of a rheostat in the transition from complete theories to $\varphi(x)$-theoretically convex Jonsson sets. Also was formulated an application of h-syntactic similarity to α-Jonsson theories.

Keywords: Jonsson theory, Jonsson spectrum, Jonsson set, theoretical set, fragment, rheostat.

This article is devoted to the study of model-theoretical properties of fragments [1-3] of theoretical sets. The concept of a theoretical set is defined as a special case of a Jonsson set [1]. In order to define a theoretical set, we take a fixed Jonsson set [4] and then we apply the universal quantifier for all free variables from existential formula which defined this set. And received universal-existential sentence should be satisfy for demand of Jonsson theory, i.e. to be Jonsson theory. It is clear that we can define in such way just finitely axiomatizable Jonsson theories. We will also consider the rheostat method and give a model-theoretical definition of the concept of a rheostat. Since both concepts: theoretical set and rheostat are new, we consider that the study of this topic in the framework of the study of Jonsson theories [2, 5, 6, 7], the Jonsson spectrum [5], classes of existentially closed models [5] of such fragments represents from itself a new promising class of problems and their solution is closely related to many problems that defined in their time the classical problems of Model Theory [8, 9].

Until now, the study of Jonsson theories and their classes of models [6, 10] was a complex of model-theoretic problems, the formulation of which was due to adaptations of the conceptual apparatus and related content from the arsenal of the Model Theory course for the study of complete theories. It is clearly seen from the definition of Jonsson theory that these theories are, generally speaking, incomplete. Therefore, the direct transfer of results on complete theories to the field of study of Jonsson theories is not a simple and easy thing. Until now, we have used the so-called semantic method in solving problems related to the study of Jonsson theories. The essence of this method is to translate the elementary properties of the first order to the theory itself, while adapting the model-theoretical properties inherent for the center of this theory.

Let us consider in more detail on the need to study the above two new concepts: a theoretical set and a rheostat. Research of the definable subsets of the semantic model of a fixed Jonsson theory allows one to transfer many of the central ideas of modern methods of studying complete theories and their classes of models to the field of studying Jonsson theories and their classes of existentially closed models. These methods include, first of all, the methods of the so-called geometric stability [11, 12]. At the same time, it should be noted the outstanding contribution to the development of this direction

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of modern Model Theory by the results of E. Hrushovski [13, 14], as well as the works of B. Zilber [15, 16]. The concept of a Jonsson set allows us to consider issues related to geometric stability in the framework of the study of Jonsson theories and their classes of existentially closed models.

In this work, we want to focus our attention on the adaptation of problems from work [8, 9] in the framework of studying the concepts of Jonsson rheostat and theoretical set. The concept of a rheostat arose in physics as a technical means for the study of such properties as current strength, voltage and resistance. Drawing an analogy with this technical means, we consider the classical model-theoretical concepts from the technical and conceptual apparatus for the study of complete theories, we will adapt and investigate the corresponding model-theoretical concepts for the study of Jonsson theories. It is clear that a concept from the arsenal of complete theories is transformed in a certain way when translated into the Jonsson theory. And at the same time, we want to interpret the difference in the transformation of the concept by some mathematical property. Conditionally, a certain imaginary rheostat is responsible for the transformation, and therefore a necessary mathematical property that will characterize the transition from a concept in a complete theory to a concept in a Jonsson theory, and, accordingly, describe this transition in a syntactic or semantic way, and in what follows we will call this mathematical property the corresponding rheostat.

To formulate the main results, we will need some definitions and model-theoretical properties of these concepts. Those results that will not be determined can be extracted from the following works [6, 17, 18, 19, 20, 21].

By Π_n we denote the set of all formulas in the language L of the form $\forall\exists...\varphi$ ((i.e., formulas with n variables quantifiers starting with $\forall$), $\Sigma_n = \{\varphi : \neg\varphi \in \Pi_n\}$, $\nabla_n = \Pi_n \cup \Sigma_n$. Then

$$\Pi_\omega = \Sigma_\omega = \nabla_\omega = \Pi_{\omega+1} = \Sigma_{\omega+1} = \Delta_{\omega+1} = ... = \bigcup_{n<\omega} \nabla_n.$$

Definition 1 [10]. Let $\Gamma \subset L$. Then:

1) $T \in \Gamma C_\Delta$ means that $T \cap \Gamma \vdash \varphi$ for all $\varphi \in T$;

2) if $B \subseteq\mid \mathfrak{A} \mid$, then $Th_\Gamma(\mathfrak{A}, B)$ denotes the set of all Γ-sentences in L_B, that are true in $\mathfrak{A}$;

3) the mapping $f : \mathfrak{A} \to \mathfrak{B}$ is called a Γ-embedding if for any $\overline{a} \in \mathfrak{A}$ and $\varphi(\overline{x}) \in \Gamma$ in $\mathfrak{A} \models \varphi(\overline{a})$ follows $\mathfrak{B} \models \varphi(f(\overline{a}))$;

4) if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \subseteq_\Gamma \mathfrak{B}$ means that $Th_\Gamma(\mathfrak{A}, |\mathfrak{A}|) \subseteq Th_\Gamma(\mathfrak{B}, |\mathfrak{A}|)$;

5) a sequence of models $\mathfrak{A}_i$, $i < \beta$, is called a Γ-chain if $\mathfrak{A}_i \subseteq_\Gamma \mathfrak{A}_j$ for $i < j < \beta$.

Lemma 1 [10]. The mapping $f : \mathfrak{A} \to \mathfrak{B}$ is a Π_α-embedding if and only if it is a $\Sigma_{\alpha+1}$-embedding.

Definition 2 [10].

1. The theory T is stable under the union of Π_α-chains (or α-inductive), if the union of any Π_α-chain of models of T is again a model of T.

2. The theory T has the property of α-joint embedding (α-JEP),if for any $\mathfrak{A}, \mathfrak{B} \models T$ there are $\mathfrak{M} \models T$ and Π_α-embeddings $f : \mathfrak{A} \to \mathfrak{M}$ and $g : \mathfrak{B} \to \mathfrak{M}$.

3. The theory T has the property of α-amalgamation (α-AP), if for any $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \models T$ and Π_α-embeddings $f_1 : \mathfrak{A} \to \mathfrak{B}_1$ and $f_2 : \mathfrak{A} \to \mathfrak{B}_2$ there are $\mathfrak{M} \models T$ и Π_α-embeddings $g_1 : \mathfrak{B}_1 \to \mathfrak{M}$ and $g_2 : \mathfrak{B}_2 \to \mathfrak{M}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Definition 3 [10]. A theory T is called α-Jonsson theory, if

1) T has an infinite models;

2) T is α-inductive;

3) T has the α-JEP;

4) T has the α-AP.

Lemma 2 [10]. 1. The theory T is complete if and only if T is ω-Jonsson.

2. The theory T is Jonsson in the sense if and only if T is 0-Jonsson.

Proposition 3 [10]. The following conditions are equivalent:

1) T has the α-JEP;

2) T has the α-JEP for countable models;

3) If $\overline{x} \cap \overline{y} = \emptyset$, $p(\overline{x})$ and $q(\overline{y})$ are an arbitrary sets $\Sigma_{\alpha+1}$-formulas such that $T \cup p(\overline{x})$ and $T \cup q(\overline{y})$ are consistent, then $T \cup p(\overline{x}) \cup q(\overline{y})$ is consistent.

Proposition 4 [10]. The following conditions are equivalent:

1) T has the α-AP;

2) T has the α-AP for countable models;

3) If $p(\overline{x})$ and $q(\overline{x})$ are sets of $\Sigma_{\alpha+1}$- formulas such that

$$T \cup p(\overline{x}),\ T \cup q(\overline{x}),$$

$$T \cup \{\neg\varphi(\overline{x}) : \varphi(\overline{x}) \in \Sigma_{\alpha+1}, \varphi(\overline{x}) \notin p(\overline{x}) \cap q(\overline{x})\}$$

are consistent sets, then the set $T \cup p(\overline{x}) \cup q(\overline{x})$ is consistent;

4) for any $\mathfrak{A} \models T$ and $\overline{a} \in \mathfrak{A}$ the set $Th_{\Sigma_{\alpha+1}}(\mathfrak{A}, \overline{a})$ is contained in the only maximal consistent with T set of $\Sigma_{\alpha+1}$- sentences of the language $L(\overline{a})$.

Proposition 5 [10]. The following conditions are equivalent:

1) $T \in \Pi_{\alpha+2}C_{\alpha}$;

2) the theory T is α-inductive.

Proposition 6 [10]. The property of a theory to be or not to be α-Jonsson is absolute, i.e. does not depend on complementary to ZF axioms of set theory.

Proposition 7 [10]. The following conditions are equivalent:

1) $T \in \Pi_{\alpha+1}C_{\Delta}$;

2) If $\mathfrak{B} \models T$ and $\mathfrak{A} \subseteq_{\Pi_\alpha} \mathfrak{B}$, then $\mathfrak{A} \models T$.

Proposition 8 [10]. The following conditions are equivalent:

1) $T \in \Sigma_{\alpha+1}C_{\Delta}$;

2) If $\mathfrak{A} \models T$ and $\mathfrak{A} \subseteq_{\Pi_\alpha} \mathfrak{B}$, then $\mathfrak{B} \models T$.

Proposition 9 [10]. The following conditions are equivalent:

1) $T \in \nabla_{\alpha+1}C_{\Delta}$;

2) If $\mathfrak{A}, \mathfrak{B} \models T$ and $\mathfrak{A} \subseteq_{\Pi_\alpha} \mathfrak{M} \subseteq_{\Pi_\alpha} \mathfrak{B}$, then $\mathfrak{M} \models T$.

Definition 4 [10]. Let $\alpha \leq \omega$.

1. An alternative chain over $\mathfrak{A} \subseteq \mathfrak{B}$ is a sequence of models $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq ... \subseteq \mathfrak{M}_\beta \subseteq ..., \beta < \alpha$, satisfying the relations

(a) $\mathfrak{A} \preceq \mathfrak{M}_0 \preceq \mathfrak{M}_2 \preceq ... \preceq \mathfrak{M}_{2i} \preceq ..., 2i < \alpha$;

(b) $\mathfrak{A} \preceq \mathfrak{M}_1 \preceq \mathfrak{M}_3 \preceq ... \preceq \mathfrak{M}_{2i+1} \preceq ..., 2i+1 < \alpha$;

2. A theory T is called α-alternative if for any $\mathfrak{A}, \mathfrak{B} \models T$ the equivalence $\mathfrak{A} \preceq \mathfrak{B} \Leftrightarrow$ ($\mathfrak{A} \subseteq \mathfrak{B}$ and there is an alternative α-chain over $\mathfrak{A} \subseteq \mathfrak{B}$).

Proposition 10 [10]. The following conditions are equivalent:

1) for any formula $\varphi(x) \in L$ there exists a $\Sigma_{\alpha+1}$-formula $\psi(x)$ such that $T \models \varphi(x) \leftrightarrow \psi(x)$;

2) for any formula $\varphi(x) \in L$ there exists a $\Pi_{\alpha+1}$-formula $\theta(x)$ such that $T \models \varphi(x) \leftrightarrow \theta(x)$;

3) the theory T is the α-model complete;

4) the theory T is the α-alternative;

5) for any $\mathfrak{A}, \mathfrak{B} \models T$ the relation holds

$$\mathfrak{A} \subseteq_{\Pi_\alpha} \mathfrak{B} \Leftrightarrow \mathfrak{A} \preceq \mathfrak{B} \Leftrightarrow \mathfrak{A} \subseteq_{\Sigma_{\alpha+1}} \mathfrak{B}$$

Proposition 11 [10]. The following conditions are equivalent:

1) in the T theory, any formula is equivalent to a Boolean combination of ∇_α-formulas;

2) for any $\mathfrak{A} \models T$ and $\overline{a} \in \mathfrak{A}$ theory $T \cup Th_{\nabla_\alpha}(\mathfrak{A}, \overline{a})$ is complete;

3) (for the complete theory T) if $\mathfrak{A}$ is a saturated model of T and $\overline{a} \in \mathfrak{A}$, then the theory $T \cup Th_{\nabla_\alpha}(\mathfrak{A}, \overline{a})$ is complete.

Definition 5 [10]. The α-Jonsson theory T is called a perfect if $\mathfrak{G}_T$ is a saturated model $Th(\mathfrak{G}_T)(= T^*)$.

Proposition 12 [10]. For the α-Jonsson theory T the following conditions are equivalent:

1) the theory T is perfect;

2) T^* is the α-model completion (that is, the $\mathcal{D}(T^*)$ is α-model completion) of the theory T.

All of the above facts and definitions are directly related to the transition from a complete theory to a Jonsson theory. Using these facts and definitions from [10], we can notice that in the language of generalized Jonsson theories, it is possible to formulate the results from [8, 9] in a unified language for the α -Jonsson theory, which was considered in [10].

The purpose of this article is to determine the rheostat of the transition from complete theory to Jonsson theory, which will be consistent with the corresponding concepts for any α and any α-Jonsson theory. For this we define a theoretical set.

Definition 6. Let T be some Jonsson theory, C is the semantic model of the theory T, $X \subseteq C$.

A set X is called theoretical set, if

1) X is Jonsson set, and let $\varphi(\overline{x})$ be the formula that defines the set X ;

2) $\varphi(\overline{x}) = \exists\overline{y}\psi(\overline{x}, \overline{y})$ and let θ be the universal closure of the formula $\varphi(\overline{x})$, i.e. θ is the sentence $\forall x\exists\overline{y}\psi(x, \overline{y})$ defines some Jonsson theory.

It is easy to see from Definition 6 that θ can only be a finitely axiomatizable theory and belongs to the Jonsson spectrum $JSp(M)$, where $M = cl(X)$ and $M \in E_T$. It can be seen that $\forall\Delta \in JSp(M)$, if Δ is a finitely axiomatizable theory, then we can consider sentences θ' such that θ' defines a Jonsson theory Δ. If we eliminate the universality quantifier in θ', then we get formula, which defines a theoretical set that will be a subset of M. Thus, all the finitely axiomatizable Jonsson theories from $JSp(M)$ will be define uniquely some theoretical subset in the model M.

Definition 7. We will say that an existentially closed model M is convex (strongly convex) if its Kaiser hull $(Th_{\forall\exists}(M))$ is a convex (strongly convex) theory.

Definition 8. We will say that an existentially closed model M is $\varphi(x)$-convex if it is convex and

1) $\varphi(x)$ defines a Jonsson set in the model M;

2) if for all substructures $\mathfrak{A}_i$ of the model M, $\bigcap\mathfrak{A}_i$ contains $\varphi(M)$, given that this intersection is not empty.

If this intersection is never empty and the model M is a strongly convex, then the model M is called the $\varphi(x)$-strongly convex.

If the semantic model C of a Jonsson theory T is $\varphi(x)$-convex (strongly convex), then the theory T itself is correspondingly the same. Note that the classical definition of convexity [3; 41; Def. 2] of the theory coincides with this definition if we omit the prefix $\varphi(x)$.

Definition 9. A model $M \in E_T$ is called $\varphi(x)$-theoretical convex (strongly convex) if this model $\varphi(x)$-convex (strongly convex) and $\varphi(x)$ defines a theoretical set.

The following definition provides a generalization of the concept of the syntactic similarity of Jonsson theories.

Definition 10. Let T_1 and T_2 are an arbitrary Jonsson theories. We say, that T_1 and T_2 are the h-syntactically similar, where h is map $h : E(T_1) \longrightarrow E(T_2)$ such that

1) restriction h to $E_n(T_1)$ is homomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;

2) $h(\exists v_{n+1}\varphi) = \exists v_{n+1}h(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;

3) $h(v_1 = v_2) = (v_1 = v_2)$.

If the kernel $Ker(h)$ of this homomorphism is trivial, then we obtain a definition of the syntactic similarity of two Jonsson theories [22; 167; def. 10].

We are now ready to give a model-theoretical definition of the concept of a rheostat in the transition from complete theories to $\varphi(x)$-theoretically convex Jonsson sets.

Definition 11. Let T be some Jonsson theory, C is the semantical model of the theory T, $X \subseteq C$, X is the theoretical set. $\varphi(C) = X$, $\varphi(x) \in L$.

If the universal closure $\varphi(x)$ will be the Jonsson theory and the Kaiser hull $M^0 = Th_{\forall\exists}(M)$, $M \in E_T$, where $M = cl(\varphi(C))$, then we will say that $\varphi(x)$ is a rheostat if exists h-syntactic similarity between theories T and $Th_{\forall\exists}(M)$.

The next result is an application of h-syntactic similarity to α-Jonsson theories.

Theorem. Let T_1, T_2 be $\varphi(x)$-convex (strongly convex) complete for existential sentences perfect α-Jonsson theories. Then the following conditions are equivalent:

1) T_1, T_2 are the h-syntactically similar and the kernel $Ker(h)$ is trivial;

2) T_1^*, T_2^* are the syntactically similar in the sense of [23] and $\alpha = \omega$;

3) $Th_{\forall\exists}(M_1) = T_1^*$, $Th_{\forall\exists}(M_2) = T_2^*$, where C_1 is a semantic model of the theory T_1, C_2 is a semantic model of the theory T_2, $cl(\varphi(C_1)) = M_1$, $cl(\varphi(C_2)) = M_2$.

Proof. The proof follows from Theorem 2.7.1 [6; 182] and from Definition 11.

Consequence. All Propositions 3-12 are true for the corresponding $\varphi(x)$-rheostat and the ordinal α, which defines the α-Jonsson theory. Moreover, there is an h-syntactic similarity between the α-Jonsson theory and the corresponding Kaiser hull of the fragment of the theoretical set defined by the formula $\varphi(x)$ and the corresponding α-Jonsson theory.

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А.Р. Ешкеев

Теоретикалық жиындардың фрагменттерінің қасиеттерін зерттеу үшін реостат әдісі

Мақалада теоретикалық жиындардың фрагменттерінің модельді-теоретикалық қасиеттері және реостат әдісі қарастырылған. Бұл екі ұғым — теоретикалық жиын және реостат — жаңа. Йонсондық теориялар, йонсондық спектрді, осындай фрагменттердің экзистенциалды тұйық модельдерінің кластарының аясында осы тақырыпты зерттеу — бұл проблемалардың жаңа перспективалы класы және оларды шешу модельдер теориясының классикалық мәселелерін анықтаған көптеген сұрақтармен тығыз байланысты. Бұл жұмыстың мақсаты — кез келген α және α-йонсондық теорияның ұғымдарына сәйкес келетін толық теориядан йонсондық теориясына өту реостатын анықтау. Ол үшін автор теоретикалық жиынды анықтады. Зерттеу негізінде ол толық теориялардан $\varphi(x)$-теоретикалық дөңес йонсондық жиындарға өту кезіндегі реостат ұғымының модельді-теоретикалық анықтамасын, сонымен қатар h-синтактикалық ұқсастығын α-йонсондық теорияларына қолдануды тұжырымдады.

Кілт сөздер: йонсондық теория, йонсондық спектр, йонсондық жиын, теоретикалық жиын, фрагмент, реостат.

А.Р. Ешкеев

Метод реостата для изучения свойств фрагментов теоретических множеств

В статье рассмотрены теоретико-модельные свойства фрагментов теоретических множеств и метод реостата. Эти два понятия — теоретическое множество и реостат — являются новыми. Изучение данной тематики в рамках йонсоновских теорий, йонсоновского спектра, классов экзистенциально замкнутых моделей представляет из себя новый перспективный класс задач, и решение их тесно связано со многими проблемами, которые определяли в свое время классические проблемы теории моделей. Цель настоящей работы — определить реостат перехода от полной теории к йонсоновской теории, который будет согласован соответствующими понятиями для любого α и любой α-йонсоновской теории. Для этого автором определено теоретическое множество. На основе проведенного исследования сформулированы теоретико-модельное определение понятия реостата при переходе от полных теорий к $\varphi(x)$-теоретически выпуклым йонсоновским множествам, а также приложение h-синтаксического подобия к α-йонсоновским теориям.

Ключевые слова: йонсоновская теория, йононовский спектр, йонсоновское множество, теоретическое множество, фрагмент, реостат.

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Small models of convex fragments of definable subsets

This article discusses the problems of that part of Model Theory that studies the properties of countable models of inductive theories with additional properties, or, in other words, Jonsson theories. The characteristic features are analyzed on the basis of a review of works devoted to research in the field of the study of Jonsson theories and enough examples are given to conclude that the vast area of Jonsson theories is relevant to almost all branches of algebra. This article also discusses some combinations of Jonsson theories, presents the concepts of Jonsson theory, elementary theory, core Jonsson theories, as well as their combinations that admit a core model in the class of existentially closed models of this theory. The concepts of convexity, perfectness of theory semantic model, existentially closed model, algebraic primeness of model of the considered theory, as well as the criterion of perfection and the concept of rheostat are considered in this article. On the basis of the research carried out, the authors formulated and proved a theorem about the $(\nabla_1, \nabla_2) - cl$ coreness of the model for some perfect, convex, complete for existential sentences, existentially prime Jonsson theory T.

Keywords: Jonsson theory, Jonsson set, convex theory, fragment, existentially closed model.

We have studied the special countable models of inductive theories [1] with additional properties. These properties are the amalgamation property and the joint embedding property. In other such theories are called Jonsson theory [1]. The class of theory, which is determined by the conditions of Jonssonness, is quite wide. These include classical examples of theories as group theory, abelian group theory, field theory of fixed characteristic, theories of various kinds of rings, modules theory, and finally theory of polygons. The last example is essential for all problems of classical Model Theory by the results from work [2]. This paper shows that any complete theory is similar in some formal sense to the theory of polygons. As we can see from the above the areas of Jonsson theories a very wide and relates to almost all fields of algebra. Also, it should be noted that Jonsson theories are not complete, and therefore many classical problems from Model Theory and universal algebra considered for elementary theories of classes of algebras that are not complete are directly related to the study of Jonsson theories. In the study of Jonsson theories within the framework of this problem, the notion of a Jonsson spectrum and the notion of cosemanticness were defined, respectively. In connection with these new concepts, which respectively generalize the study of the problem of elementary equivalence of fixed classes of algebras in the framework of the study of Jonsson theories, results related to abelian groups and modules were obtained [3, 4]. In connection with these new concepts, which respectively generalize the study of the problem of elementary equivalence of fixed classes. On the other hand, if we consider the class of models of an arbitrary Jonsson theory, then this class can be conditionally divided into two subclasses. The class of existentially closed models and the class of models that are not. It is well known that the elementarity of the subclass of existentially closed models is directly related to the perfection of the considered Jonsson theory [5]. Thus, the problems associated with the study of the behavior of a class of existentially closed models of an arbitrary or fixed Jonsson theory is an actual class of problems related to both the classical model theory and universal algebra.

Recently, quite a lot of works have been devoted to the study of Jonsson theories [6-9]. After the definition of Jonsson set [10], we noticed the usefulness of this concept in the sense that it made it

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possible to define the rheostat principle [11]. Among these works, one can single out the work related to definable subsets of the semantic model of the fixed Jonsson theory [11]. One of the new technical concepts is a fragment of a definable subset of the semantic model. As part of the study of this concept, the following works were considered [12-16].

Small models are usually understood as countable models of the considered theory. Since we will work with fragment models of some fixed Jonsson theory's models, generally speaking, it is not necessary that small fragment models coincide with small models of this theory. The concept of convexity of theory was introduced by A. Robinson [7; 41; Def. 2] and D. Kueker [17] studied the properties of core models for convex theories.

The main result of this article is Theorem 4. $\mathfrak{M}$ is a $(\nabla_1, \nabla_2)-cl$ core model for some perfect convex complete for existential sentences existentially prime Jonsson theory T if and only if it is $(\nabla_1, \nabla_2) - cl$ core model T^*, where T^* is the center of the theory T.

We give the following definitions and the related results, which we need for further work.

We begin with a classic definition of Jonsson theory and all the concepts needed to work with Jonsson theories.

Definition 1 [5]. A theory T is called a Jonsson theory if:

1) the theory T has infinite models;

2) the theory T is inductive;

3) the theory T has the joint embedding property (JEP);

4) the theory T has the property of amalgam (AP).

Definition 2 [5]. Let $\kappa \geq \omega$. Model M of theory T is called κ-universal for T, if each model T with the power strictly less κ isomorphically embedded in M; κ- homogeneous for T, if for any two models A and A_1 of theory T, which are submodels of M with the power strictly less then κ and for isomorphism $f : A \to A_1$ for each extension B of model A, which is a submodel of M and is model of T with the power strictly less then κ there exist the extension B_1 of model A_1, which is a submodel of M and an isomorphism $g : B \to B_1$ which extends f.

Definition 3 [5]. A model C of a Jonsson theory T is called semantic model, if it is ω^+-homogeneous-universal.

Definition 4 [5]. The center of a Jonsson theory T is an elementary theory T^* of the semantic model C of T, i.e. $T^* = Th(C)$.

Fact 1 [5]. Each Jonsson theory T has k^+-homogeneous-universal model of power 2^k. Conversely, if a theory T is inductive and has an infinite model and ω^+-homogeneous-universal model then the theory T is a Jonsson theory.

Fact 2 [5]. Let T is a Jonsson theory. Two k-homogeneous-universal models M and M_1 of T are elementary equivalents.

One of the main results obtained previously in the above definitions is the following result:

Theorem 1 (Criterion of perfectness) [5; 158]. Let T be a Jonsson theory. Then the following conditions are equivalent:

1) Theory T is perfect;

2) Theory T^* is a model companion of theory T.

Since there are much fewer perfect theories than imperfect ones, and only the perfectness of Jonsson theory guarantees the elementary class of existentially closed models of this theory, the study of properties of the class of existentially closed models is a very important task. Recall the classic definition of an existentially closed model for any theory.

Definition 5. A model A of theory T is called existentially closed if for any model B and any existential formula $\varphi(\overline{x})$ with constants of A we have $A \models \exists\overline{x}\varphi(\overline{x})$ provided that A is a submodel of B and $B \models \exists\overline{x}\varphi(\overline{x})$.

We denote by E_T the class of all existentially closed models of the theory T.

The existence of an existentially closed model is not necessary for any theory. But, as is well known from the following theorem [1], for any inductive theory T E_T is non-empty.

Theorem 2 [1; 97; Proposition 8.12]. If the theory T is inductive, then any model of the theory T is embedded in an existentially closed model of the theory T.

The next aspect related to the models of the theory is called the convexity of theory. The concept of convexity of theory was first introduced by the well-known specialist in the field of Model Theory A. Robinson.

Definition 6 [7]. A theory T is called convex if for any its model A and any family $\{B_i \mid i \in I\}$ of substructures of A, which are models of the theory T, the intersection $\bigcap_{i \in I} B_i$ is a model of T, provided it is non-empty. If besides such an intersection is never empty, then T is called strongly convex.

As a simple example of a convex but not strongly convex theory, we can give the following example: consider a theory T that defines an equivalence relation. It is clear that any substructure of the model T also satisfies the axioms of T, but the intersection of two substructures may well turn out to be empty. An example of a strongly convex theory is group theory. A simple example of a nonconvex theory is given by the following example: the theory of densely ordered sets with different end elements. On the other hand, if the language of a theory T contains at least one constant a and the theory T is convex, then it is also strongly convex since any model from $ModT$ contains an element that realizes this constant a.

The concept of an algebraically prime model was also introduced by A. Robinson. This concept is a generalization of the concept of a prime model, which says that the model is prime if and only if, it is elementary embeddable in any model of the considered theory. A model is called algebraically prime if and only if, it is isomorphically embedded in any model of the considered theory. Since in the definition of Jonsson theory we see only isomorphic embeddings, it is naturally important for us to know the behavior of an algebraically prime model. If in the case of a prime model we have a good criterion in the form of R. Vaught's theorem: a model is prime if and only if it is countable and atomic. In the case of algebraic primeness there is no such criterion. Therefore, the study of algebraically prime models is an important task in the study of Jonsson theory.

Definition 7. Model A of theory T is called core if it is isomorphically embedded in any model of a given theory and this isomorphism exactly one.

As part of the study of inductive theories, we define the core theory [11].

Definition 8. An inductive theory T is called a core theory if there exists a model $A \in E_T$ such that for any model $B \in E_T$ there exists a unique isomorphism from A to B.

Definition 9. The inductive theory T is called the existentially prime if:

1) it has an algebraically prime model, the class of its AP (algebraically prime models) denote by AP_T;

2) class E_T nontrivial intersects with class AP_T, i.e. $AP_T \bigcap E_T \neq 0$.

When studying Jonsson theories, we noticed that not all of them have an algebraically prime model. But there are also such Jonsson theories that have such a model. Therefore, it was natural to define the following subclasses of inductive theories.

The following definition makes sense in the case of imperfect Jonsson theories. In the perfect case, the concept of algebraic primeness is considered in the class of the considered theory's models. Since in the perfect case will be $ModT^* = E_T$.

Definition 10 [11]. Theory T is called existentially algebraically prime (EAP) if it has a model $A \in E_T$ such that for any $B \in E_T$, A is isomorphically embedded in B.

In the modern Model Theory, the definable subsets of the considered models play an important role. A set is called definable if there is a formula of the language, the solution of which is the given set. We distinguish such special definable subsets of the semantic model of the considered Jonsson theory through the following definition.

Definition 11. Let $X \subseteq C$. We will say that a set X is $\nabla - cl$-Jonsson subset of C if X satisfies the following conditions:

1) X is ∇-definable set (this means that there is a formula from ∇, the solution of which in the C is the set X, where $\nabla \subseteq L$, that is ∇ is a view of formula, for example $\exists, \forall, \forall\exists$ and so on.);

2) $cl(X) = M$, $M \in E_T$, where cl is some closure operator defining a pregeometry [16] over C (for example $cl = acl$ or $cl = dcl$).

In order to take advantage of the rheostat principle, which will be applied to a specific model, we must define a series of definable subsets, in which the foundations of a rheostat are laid in the form of a formula subset with additional conditions regarding primeness atomicity, coreness, and existentially closeness, when necessary.

Definition 12 [11]. A set A is said to be $(\nabla_1, \nabla_2) - cl$ atomic in the theory T, if

1) $\forall a \in A, \exists\varphi \in \nabla_1$ such that for any formula $\psi \in \nabla_2$ follows that φ is a complete formula for ψ and $C \models \varphi(a)$;

2) $cl(A) = M, M \in E_T$,

and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$ atomic model of theory T.

Definition 13 [11]. A set A is said to be weakly $(\nabla_1, \nabla_2) - cl$ is atomic in T, if

1) $\forall a \in A, \exists\varphi \in \nabla_1$ such that in $C \models \varphi(a)$ for any formula $\psi \in \nabla_2$ follow that $T \models (\varphi \to \psi)$ whenever $\psi(x)$ of ∇_2 and $C \models \psi(a)$;

2) $cl(A) = M, M \in E_T$,

and obtained model M is said to be weakly $(\nabla_1, \nabla_2) - cl$ atomic model of theory T.

Definition 14 [11]. A set A is said to be $(\nabla_1, \nabla_2) - cl$-algebraically prime in the theory T, if

1) If A is $(\nabla_1, \nabla_2) - cl$-atomic set in T;

2) $cl(A) = M, M \in AP_T$,

and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$ algebraically prime model of theory T.

Definition 15 [11]. The set A is said to be $(\nabla_1, \nabla_2) - cl$-core in the theory T, if

1) A is (∇_1, ∇_2) a cl - atomic set in the theory T;

2) $cl(A) = M$, where M is the core model of theory T

and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$ core model of theory T.

Theorem 3 [17; Th. 2.1] For any T the following conditions are equivalent:

1) $\mathfrak{C}$ is a core structure for T.

2) $\mathfrak{C}$ is a model of every universal sentence consistent with T, and there are existential formulas $\varphi_i(x)$ and $k_i \in \omega$, for $i \in I$, such that

$$\mathfrak{C},\ T \models \exists^{=k_i} x\varphi_i \text{ for all } i \in I,$$

and

$$\mathfrak{C} \models \forall x \bigvee_{i\in I} \varphi_i$$

The following two definitions (Def. 16 and Def. 17) will have a value for the formulation of further results of this article.

Definition 16. Let T_1 and T_2 are an arbitrary Jonsson theories. We say, that T_1 and T_2 are the h-syntactically similar, where h is map $h : E(T_1) \longrightarrow E(T_2)$ such that

1) restriction h to $E_n(T_1)$ is homomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;

2) $h(\exists v_{n+1}\varphi) = \exists v_{n+1} h(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;

3) $h(v_1 = v_2) = (v_1 = v_2)$.

The following definition belongs to the first author and is a measure of the change in the rheostat principle for Jonsson theories.

Definition 17. Let T be some Jonsson theory, C a semantic model of the theory T, $X \subseteq C$, X a theoretical set. $\varphi(C) = X,\ \varphi(x) \in L$.

If the universal closure $\varphi(x)$ is a Jonsson theory and the Kaiser hull $M^0 = Th_{\forall\exists}(M), M \in E_T$, where $M = cl(\varphi(C))$, then we will say that $\varphi(x)$ is a rheostat if there is an h-syntactic similarity between the theories T and $Th_{\forall\exists}(M)$.

The next result connects the convexity of the theory and its center in connection with the existence of the above form of the core of the model (Def. 15).

Symbols ∇_1, ∇_2, cl s in definitions 11-16.

Further, by the requirements for the content of Theorem 4: let $\nabla_1 = \{\varphi(x)\}$, $\nabla_2 = \{\varphi(x)\}$, $cl = acl, cl = dcl$, $\varphi(x)$ there is a rheostat such that $cl(X) = M$, $M^0 = Th_{\forall\exists}(M)$, $\theta = \forall x \exists \overline{y} \psi(x, \overline{y})$, θ there is Jonsson theory, $\varphi(x) = \exists \overline{y} \psi(x, \overline{y})$ and $h : E(\theta) \to M^0$, satisfying the definition 16.

Let $Ker\, h$ be trivial, i.e. consists only of identical congruence.

Theorem 4. $\mathfrak{M}$ is $(\nabla_1, \nabla_2) - cl$ a core model for some perfect, convex, complete for existential sentences, existentially prime Jonsson theory T if and only if it is a $(\nabla_1, \nabla_2) - cl$ core model T^*, where T^* is the center of T.

Proof. Let's prove the sufficiency. Suppose that T^* is strongly convex and $\mathfrak{M}$ is the $(\nabla_1, \nabla_2) - cl$ core model T^*. Let C be a semantic model of T. Let $\mathfrak{A}$ be any model of T and $\mathfrak{A}_0$ be a sufficiently saturated existentially closed extension of $\mathfrak{A}$. Then $\mathfrak{M} \exists \mathfrak{A}_o$, which means that any $\exists$-sentence true in $\mathfrak{M}$ implies being true in $\mathfrak{A}_o$. This is true since any model of the theory T can be isomorphically embedded in C, in particular, the models $\mathfrak{A}_o$, $\mathfrak{A}_o$ are also embedded in C since the theory T is perfect, the model $\mathfrak{A}_o$. And due to the strong convexity of the theory T^* the model $\mathfrak{M}$ is isomorphically embedded in the model $\mathfrak{A}_o$. Therefore, $\mathfrak{M} \cong \mathfrak{M}_1$, for some $\mathfrak{M}_1 \subseteq \mathfrak{A}_o$. Since no proper submodel of $\mathfrak{M}$ is a model of T^*, we must have

$$\mathfrak{M}_1 = \cap\{\mathfrak{B} : \mathfrak{B} \subseteq \mathfrak{A}_o \text{ and } \mathfrak{B} \models T\}$$

In particular $\mathfrak{M}_1 \subseteq \mathfrak{A}$. If $\mathfrak{M}$ is isomorphic to some other $\mathfrak{M}_2 \subseteq \mathfrak{A}$, then the same argument shows that $\mathfrak{M}_1 = \mathfrak{M}_2$. Therefore, $\mathfrak{M}$ is $(\nabla_1, \nabla_2) - cl$ core model for T.

Let's prove the necessity. Let $\mathfrak{M}$ - $(\nabla_1, \nabla_2) - cl$ core model for T, and let T^* be the center of the theory T, then it is obvious that the set of all existential and universal sentences is true in $\mathfrak{M}$, since $T^* = Th(C)$. Condition (2) of Theorem 2.1 [17; 157] holds for $\mathfrak{M}$ and T^* so that $\mathfrak{M}$ is $(\nabla_1, \nabla_2) - cl$ core model for T^*. Let $\mathfrak{A}$ any model T^*. $\mathfrak{M}$ is isomorphic to exactly one $\mathfrak{M}' \subseteq \mathfrak{A}$ and can also be embedded in any other model T^*, therefore,

$$\mathfrak{M}' = \cap\{\mathfrak{B} : \mathfrak{B} \subseteq \mathfrak{A} \text{ and } \mathfrak{B} \models T^*\}$$

It follows that T^* is strongly convex and that $\mathfrak{M}$ is $(\nabla_1, \nabla_2) - cl$ core model T^*, as clamed.

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А.Р. Ешкеев, Н.В. Попова

Анықталған ішкі жиындардың дөңес фрагменттерінің кішігірім модельдерінің типтері

Мақалада модельдер теориясының қосымша қасиеттері бар индуктивті теориялардың саналымды модельдерінің қасиеттерін зерттейтін бөлігінің, немесе, басқаша айтқанда, йонсондық теориялардың, мәселелері қарастырылды. Йонсондық теорияды зерттеу аясында анықтауға арналған жұмыстарға шолу негізінде сипаттамалық ерекшеліктер талданды және йонсондық теориялардың кең аумағы алгебраның барлық дерлік салаларына қатысты деген тұжырым жасауға жеткілікті мысалдар келтірілген. Сонымен қатар авторлар йонсондық теориялардың кейбір комбинацияларын талқылап, йонсондық теорияның, элементарлық теорияның, ядролық йонсондық теорияның ұғымдарын, сондай-ақ осы теорияның экзистенциалды тұйық модельдері класында ядролық модельді рұқсат ететін олардың комбинацияларын келтірген. Мақалада дөңестілік, теорияның кемелділігі, семантикалық модель, экзистенциалды тұйық модель, қарастырылатын теорияның алгебралық жай моделі, сондай-ақ кемелдік критерийі және реостат ұғымы қарастырылған. Зерттеу негізінде авторлар экзистенциалды сөйлемдер үшін толық, дөңес, экзистенциалды жай йонсондық теория T үшін модельдің $(\nabla_1, \nabla_2) - cl$ ядролылығы туралы теореманы тұжырымдады және дәлелдеді.

Кілт сөздер: йонсондық теория, йонсондық спектр, йонсондық жиын, теоретикалық жиын, фрагмент.

А.Р. Ешкеев, Н.В. Попова

Малые модели выпуклых фрагментов определимых подмножеств

В статье рассмотрены проблемы той части теории моделей, которая изучает свойства счетных моделей индуктивных теорий с дополнительными свойствами, или, иначе говоря, йонсоновские теории. Проанализированы характерные особенности, на основании обзора работ, посвященных исследованиям в области изучения йонсоновских теорий, и приведено достаточно примеров, позволяющих сделать вывод, что обширный ареал йонсоновских теорий имеет отношение практически ко всем разделам алгебры. Авторами обсуждены некоторые комбинации йонсоновских теорий, приведены понятия йонсоновской теории, элементарной теории, ядерной йонсоновской теории, а также их комбинаций, допускающих ядерную модель в классе экзистенциально замкнутых моделей этой теории. Кроме того, в статье понятия выпуклости, совершенства теории, семантической модели, экзистенциально замкнутой модели, алгебраической простоты модели рассматриваемой теории, а также критерий совершенности и понятие реостата всесторонне изучены. На основе проведенного исследования авторами сформулирована и доказана теорема о $(\nabla_1, \nabla_2) - cl$ ядерности модели для некоторой совершенной, выпуклой, полной для экзистенциальных предложений, экзистенциально простой йонсоновской теории T.

Ключевые слова: йонсоновская теория, йононовский спектр, йонсоновское множество, теоретическое множество, фрагмент.

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IN MEMORIAM OF SCIENTIST