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MATHEMATICS

Preface

This issue is a collection of 12 selected papers. These papers are presented at the Fifth International Conference on Analysis and Applied Mathematics (ICAAM 2020) organized by Near East University, Lefkosa (Nicosia), Mersin 10, Turkey.

The meeting was held on September 23–30, 2020 in North Cyprus, Turkey. The main organizer of the conference is Near East University, Nicosia (Lefkosa), Mersin 10, Turkey. The conference was also supported by Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan and Analysis & PDE Center, Ghent University, Belgium.

The conference is organized biannually. Previous conferences were held in Gumushane, Turkey in 2012; in Shymkent, Kazakhstan in 2014; in Almaty, Kazakhstan in 2016; in 2018 Lefkosa,Mersin 10, Turkey. The proceedings of ICAAM 2012, ICAAM 2014, ICAAM 2016, and ICAAM 2018 were published in AIP Conference Proceedings (American Institute of Physics) and in some rating scientific journals.

Near East University was pleased to host the fifth conference which was focused on various topics of analysis and its applications, applied mathematics and modeling. The main aim of the International Conferences on Analysis and Applied Mathematics (ICAAM) is to bring mathematicians working in the area of analysis and applied mathematics together to share new trends of applications of mathematics. In mathematics, the developments in the field of applied mathematics open new research areas in analysis and vice versa. That is why, we planned to find the conference series to provide a forum for researches and scientists to communicate their recent developments and to present their original results in various fields of analysis and applied mathematics. This issue presents papers by authors from different countries: Azerbaijan, Iraq, Russia, Turkey, Turkmenistan, USA, Kazakhstan. Especially we are pleased with the fact that many articles are written by co-authors who work in different countries. We are confident that such international integration provides an opportunity for a significant increase in the quality and quantity of scientific publications.

Finally, but not least, we would like to thank the Editorial board of the "Bulletin of the Karaganda University - Mathematics", who kindly provided an opportunity for the formation of this special issue.

July 2020

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Numerical solution to elliptic inverse problem with Neumann-type integral condition and overdetermination

In modeling various real processes, an important role is played by methods of solution source identification problem for partial differential equation. The current paper is devoted to approximate of elliptic over determined problem with integral condition for derivatives. In the beginning, inverse problem is reduced to some auxiliary nonlocal boundary value problem with integral boundary condition for derivatives. The parameter of equation is defined after solving that auxiliary nonlocal problem. The second order of accuracy difference scheme for approximately solving abstract elliptic overdetermined problem is proposed. By using operator approach existence of solution difference problem is proved. For solution of constructed difference scheme stability and coercive stability estimates are established. Later, obtained abstract results are applied to get stability estimates for solution Neumann-type overdetermined elliptic multidimensional difference problems with integral conditions. Finally, by using MATLAB program, we present numerical results for two dimensional and three dimensional test examples with short explanation on realization on computer.

Keywords: difference scheme, inverse elliptic problem, overdetermination, source identification problem, stability, coercive stability, estimate.

Introduction

Methods of solutions and theory nonlocal boundary value problems (BVPs) for differential equations have been studied by numerous authors (see [1–5, 7–12, 14–16, 18, 19] and references herein).

Let us I is identity operator and A is a selfadjoint and positive definite operator (SAPDO) in an arbitrary Hilbert space H. It is known that $A > \delta I$ for some positive number δ, and the operator $C = \frac{\tau}{2}A + \sqrt{A + \frac{\tau^2 A^2}{4}})$ is also SAPDO.

Assume that given function $f \in C^1\left([0,T], H\right)$, elements $\phi, \eta, \zeta \in H$, number $\lambda_0 \in [0,1]$. Denote by $[0,1]_\tau = \{t_i = i\tau,\ i = 1, \cdots, N,\ \tau N = T\}$ the uniform grid space with step size $\tau > 0$, where N is a fixed integer number. Let β be known scalar continuous function satisfying condition

$$\sum_{j=1}^{N} \left|\beta\left(t_{j-\frac{1}{2}}\right)\right| \tau < 1. \tag{1}$$

In the study [10] established well-possedness of elliptic inverse problem with Neumann-type overdetermination and integral condition for obtaining a function $u \in C^2([0,T], H) \cap C([0,T], D(A))$ and an element $p \in H$ such that

$$\left\{ \begin{array}{l} -u''(t) \ + \ Au(t) = \ f(t) \ + \ p, \quad t \in (0,T), \\ u'(0) = \phi, \ u'(T) = \int\limits_0^T \beta(\lambda) u'(\lambda) d\lambda + \eta, \ u(\lambda_0) = \zeta. \end{array} \right. \tag{2}$$

Moreover, in [10], the stability inequalities for solution of inverse problem (2) were applied to investigate the following source identificating problem (SIP) for multi dimensional elliptic partial differential equation

$$\left\{ \begin{array}{l} -u_{tt}(t,x) \ - \ \sum\limits_{r=1}^{n} (a_r(x) u_{x_r}(t,x))_{x_r} + \sigma u(t,x) \ = \ f(t,x) \ + \ p(x), \ (t,x) \in (0,T) \times \Omega, \\ u_t(0,x) = \phi(x), \ u_t(T,x) = \int\limits_0^T \beta(\gamma) u_\gamma(\gamma,x) d\gamma + \eta(x), \ u(\lambda_0, x) = \zeta(x), x \in \overline{\Omega}, \\ u(t,x) \ = \ 0, \quad (t,x) \ \in [0,T] \times S. \end{array} \right. \tag{3}$$

Here $\Omega = (0,T)^n$ is open cube in $\mathbb{R}^n$ with boundary S, $\overline{\Omega} = \Omega \cup S$; a_r, ζ, ϕ, η , f are given sufficiently smooth functions; $\forall x \in \Omega$, $a_r(x) \geq a_0 > 0$; $\sigma > 0, 0 < \lambda_0 < T$ are known numbers.

We denote by R, P, and D, the corresponding operators $R = (I + \tau C)^{-1}$, $P = (I - R^{2N})^{-1}$, $D = (I + \tau C)(2I + \tau C)^{-1} C^{-1}$.

Now, let us to give some lemmas that will be used in further.

Lemma 1. [8] The following estimates hold:

$$\| R^k \|_{H \to H} \leq M(\delta)(1 + \delta^{\frac{1}{2}} \tau)^{-k}, \ \| CR^k \|_{H \to H} \leq \frac{1}{k\tau} M(\delta), k \geq 1, \| P \|_{H \to H} \leq M(\delta), \ \delta > 0. \tag{4}$$

Lemma 2.

Suppose that inequality (1) is satisfied, then the operator

$$\begin{aligned} G_2 = & \left[-3(I - R^{2N}) + 4\left(R - R^{2N-1}\right) - \left(R^2 - R^{2N-2}\right)\right] \left[\left(3 - \tau\beta\left(t_{N-\frac{3}{2}}\right)\right)\left(I - R^{2N}\right) \right. \\ & + \left(-4 - \tau\beta\left(t_{N-\frac{5}{2}}\right)\right)\left(R - R^{2N-1}\right) + \left(1 - \tau\beta\left(t_{N-\frac{7}{2}}\right) + \tau\beta\left(t_{N-\frac{3}{2}}\right)\right)\left(R^2 - R^{2N-2}\right) \\ & \left. + \tau\beta\left(t_{\frac{3}{2}}\right)\left(R^{N-1} - R^{N+1}\right) + \sum_{i=2}^{N-3} \tau \left[\beta\left(t_{i+\frac{1}{2}}\right) - \beta\left(t_{i-\frac{3}{2}}\right)\right]\left(R^{N-i} - R^{N+i}\right)\right] \\ & - \left[R^{N-1} - R^{N+1} - R^{N-2} + R^{N+2}\right] \left[-\left(4 + \tau\beta\left(t_{N-\frac{5}{2}}\right)\right)\left(R^{N-1} - R^{N+1}\right) \right. \\ & + \left(1 - \tau\beta\left(t_{N-\frac{7}{2}}\right) + \tau\beta\left(t_{N-\frac{3}{2}}\right)\right)\left(R^{N-2} - R^{N+2}\right) + \sum_{i=2}^{N-3} \tau \left[\beta\left(t_{i+\frac{1}{2}}\right) - \beta\left(t_{i-\frac{3}{2}}\right)\right]\left(R^{i} - R^{2N-i}\right) \\ & \left. + \tau\beta\left(t_{\frac{3}{2}}\right)\left(R - R^{2N-1}\right) + \tau\beta\left(t_{\frac{1}{2}}\right)\left(I - R^{2N}\right)\right] \end{aligned} \tag{5}$$

has an inverse G_2^{-1} and its norm is bounded, i.e.

$$\| G_2^{-1} \|_{H \to H} \leq M(\delta). \tag{6}$$

In the paper [8] , for given v_0 and v_N,the solution of difference scheme

$$-\tau^{-2}(v_{i+1} - 2v_i + v_{i-1}) + Av_i = f_i, \quad 1 \leq i \leq N-1 \tag{7}$$

was represented by formula

$$\begin{aligned} v_i = & P\left[\left(R^i - R^{2N-i}\right)v_0 + \left(R^{N-i} - R^{N+i}\right)v_N\right] - P\left(R^{N-i} - R^{N+i}\right)D \\ & \times \sum_{j=1}^{N-1}\left(R^{N-j} - R^{N+j}\right)f_j\tau + D\sum_{j=1}^{N-1}\left(R^{|i-j|} - R^{i+j}\right)f_j\tau \ , \ 1 \leq i \leq N-1. \end{aligned} \tag{8}$$

Let $\alpha \in (0,1)$ is a given number. Introduce notations for $C_\tau(H), C_\tau^\alpha(H)$, and $C_\tau^{\alpha,\alpha}(H)$, the Banach spaces of H-valued grid functions $w_\tau = \{w_k\}_{k=1}^{N-1}$ with the corresponding norms,

$$\|w_\tau\|_{C_\tau(H)} = \max_{1\le k\le N-1} \|w_k\|_H\ ,\ \|w_\tau\|_{C_\tau^\alpha(H)} = \sup_{1\le k<k+n\le N-1} (n\tau)^{-\alpha}\,\|w_{k+n}-w_k\|_H + \|w_\tau\|_{C_\tau(H)}\,,$$
$$\|w_\tau\|_{C_\tau^{\alpha,\alpha}(H)} = \|w_\tau\|_{C_\tau(H)} + \sup_{1\le k<k+n\le N-1} (1-k\tau)^\alpha\,(n\tau)^{-\alpha}\,(k\tau+n\tau)^\alpha\|w_{k+n}-w_k\|_H.$$

In the current study, we construct the second order accuracy difference scheme (ADS) for approximately solution of inverse problem (2) and study well-posedness of difference problem. Then, we discuss the second order ADS for SIP (3).

The second order of ADS for SIP (3)

Now, we study second order of ADS

$$\begin{cases} -\tau^{-2}(u_{k+1}-2u_k+u_{k-1}) + Au_k = f_k + p,\ \ f_k = f(t_k), 1\le k\le N-1, \\ -3u_0+4u_1-u_2 = 2\tau\phi, 3u_N - 4u_{N-1} + u_{N-2} = \sum\limits_{i=1}^{N-1} \tau\beta\left(t_{i-\frac{1}{2}}\right)(u_{i+1}-u_{i-1}) + 2\tau\eta, \\ u_l + \mu(u_{l+1}-u_l) = \zeta\ \ \left(\mu = \frac{\lambda_0}{\tau} - l\right) \end{cases} \tag{9}$$

for approximate solution inverse problem (2).

Theorem 1. Let us $\phi, \eta, \zeta \in D(A)$, and $f_\tau \in C_\tau(H)$ and inequality (1) is satisfied. Then, solution $\left(\{u_k\}_{k=1}^{N-1}, p\right)$ of difference problem (9) exists in $C_\tau(H)\times H$ and the next stability estimates for solution

$$\left\|\{u_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} \le M(\delta)\left(\|\phi\|_H + \|\zeta\|_H + \|\eta\|_H + \|f_\tau\|_{C_\tau(H)}\right), \tag{10}$$
$$\left\|A^{-1}p\right\|_H \le M(\delta)\left(\|\phi\|_H + \|\zeta\|_H + \|\eta\|_H + \|f_\tau\|_{C_\tau(H)}\right) \tag{11}$$

are fulfilled.

Proof. Firstly, by using
$$u_k = v_k + A^{-1}p, \tag{12}$$
we get auxiliary difference problem for unknowns $\{v_k\}_{k=0}^N$:

$$\begin{cases} -\tau^{-2}(v_{k+1}-2v_k+v_{k-1}) + Av_k = f_k\ , \quad 1\le k\le N-1, \\ -3v_0+4v_1-v_2 = 2\tau\phi, \left(3-\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)v_N + \left(-4-\tau\beta\left(t_{N-\frac{5}{2}}\right)\right)v_{N-1} \\ +\left(1-\tau\beta\left(t_{N-\frac{7}{2}}\right) + \tau\beta\left(t_{N-\frac{3}{2}}\right)\right)v_{N-2} + \sum\limits_{i=2}^{N-3}\tau\left[\beta\left(t_{i+\frac{1}{2}}\right) - \beta\left(t_{i-\frac{3}{2}}\right)\right]v_i \\ +\tau\beta\left(t_{\frac{3}{2}}\right)v_1 + \tau\beta\left(t_{\frac{1}{2}}\right)v_0 = 2\tau\eta. \end{cases} \tag{13}$$

We seek solution of (13) by (8). By using (8), from first condition of difference problem (13), we get equation

$$\begin{array}{l} \left[-3(I-R^{2N}) + 4\left(R-R^{2N-1}\right) - \left(R^2-R^{2N-2}\right)\right]v_0 \\ +\left[4\left(R^{N-1}-R^{N+1}\right) - \left(R^{N-2}-R^{N+2}\right)\right]v_N = F_1, \end{array} \tag{14}$$

for unknowns v_0 and v_N, where

$$F_1 = 2\tau(I-R^{2N})\phi + 4\left(R^{N-1}-R^{N+1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau - 4(I-R^{2N})D$$
$$\times\sum_{j=1}^{N-1}\left(R^{|1-j|}-R^{1+j}\right)f_j\tau - \left(R^{N-2}-R^{N+2}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau$$
$$+(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|2-j|}-R^{2+j}\right)f_j\tau.$$

From integral condition follows the next equation

$$\left(3-\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)(I-R^{2N})v_N+\left(-4-\tau\beta\left(t_{N-\frac{5}{2}}\right)\right)\left[\left(R^{N-1}-R^{N+1}\right)v_0+\left(R-R^{2N-1}\right)v_N\right]$$
$$+\left(1-\tau\beta\left(t_{N-\frac{7}{2}}\right)+\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)\left[\left(R^{N-2}-R^{N+2}\right)v_0+\left(R^2-R^{2N-2}\right)v_N\right]$$
$$+\sum_{i=2}^{N-3}\tau\left[\beta\left(t_{i+\frac{1}{2}}\right)-\beta\left(t_{i-\frac{3}{2}}\right)\right]\left[\left(R^i-R^{2N-i}\right)v_0+\left(R^{N-i}-R^{N+i}\right)v_N\right]$$
$$+\tau\beta\left(t_{\frac{3}{2}}\right)\left[\left(R-R^{2N-1}\right)v_0+\left(R^{N-1}-R^{N+1}\right)v_N\right]+\tau\beta\left(t_{\frac{1}{2}}\right)(I-R^{2N})v_0=F_2 \tag{15}$$

for unknowns v_0 and v_N, where

$$F_2=\left(-4-\tau\beta\left(t_{N-\frac{5}{2}}\right)\right)\left[\left(R-R^{2N-1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\right.$$
$$\left.\times\sum_{j=1}^{N-1}\left(R^{|N-1-j|}-R^{N-1+j}\right)f_j\tau\right]+\left(1-\tau\beta\left(t_{N-\frac{7}{2}}\right)+\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)$$
$$\times\left[\left(R^2-R^{2N-2}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|N-2-j|}-R^{N-2+j}\right)f_j\tau\right]$$
$$-\sum_{i=2}^{N-3}\tau\left[\beta\left(t_{i+\frac{1}{2}}\right)-\beta\left(t_{i-\frac{3}{2}}\right)\right]\left[\left(R^{N-i}-R^{N+i}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau\right.$$
$$\left.-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|i-j|}-R^{i+j}\right)f_j\tau\right]-\tau\beta\left(t_{\frac{3}{2}}\right)\left[\left(R^{N-1}-R^{N+1}\right)D\right.$$
$$\left.\times\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|1-j|}-R^{1+j}\right)f_j\tau+2\tau(I-R^{2N})\eta\right].$$

Thus, determinant operator G_2 of linear system equatıon (14), (15) has bounded inverse G_2^{-1}. Therefore solution of linear system equatıon (14), (15) is defined by

$$v_0=G_2^{-1}\left\{\left[\left(3-\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)(I-R^{2N})+\left(-4-\tau\beta\left(t_{N-2}-\tfrac{\tau}{2}\right)\right)\left(R-R^{2N-1}\right)\right.\right.$$
$$+\left(1-\tau\beta\left(t_{N-\frac{7}{2}}\right)+\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)\left(R^2-R^{2N-2}\right)$$
$$\left.+\sum_{i=2}^{N-3}\tau\left[\beta\left(t_{i+\frac{1}{2}}\right)-\beta\left(t_{i-\frac{3}{2}}\right)\right]\left(R^{N-i}-R^{N+i}\right)+\tau\beta\left(t_{\frac{3}{2}}\right)\left(R^{N-1}-R^{N+1}\right)\right]$$
$$\times\left[2\tau(I-R^{2N})\phi+4\left(R^{N-1}-R^{N+1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-4(I-R^{2N})D\right.$$
$$\times\sum_{j=1}^{N-1}\left(R^{|1-j|}-R^{1+j}\right)f_j\tau-\left(R^{N-2}-R^{N+2}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau$$
$$\left.+(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|2-j|}-R^{2+j}\right)f_j\tau\right]-\left(R^{N-1}-R^{N+1}-R^{N-2}+R^{N+2}\right)$$
$$\times\left\{2\tau(I-R^{2N})\eta+\left(-4-\tau\beta\left(t_{N-\frac{5}{2}}\right)\right)\left[\left(R-R^{2N-1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau\right.\right.$$
$$\left.\times-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|N-1-j|}-R^{N-1+j}\right)f_j\tau\right]+\left(1-\tau\beta\left(t_{N-\frac{7}{2}}\right)+\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)$$
$$\times\left[\left(R^2-R^{2N-2}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|N-2-j|}-R^{N-2+j}\right)f_j\tau\right]$$

$$-\sum_{i=2}^{N-3}\tau\left[\alpha\left(t_{i+1}-\tfrac{\tau}{2}\right)-\alpha\left(t_{i-1}-\tfrac{\tau}{2}\right)\right]\left[\left(R^{N-i}-R^{N+i}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau\right.$$
$$\left.-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|i-j|}-R^{i+j}\right)f_j\tau\right]-\tau\alpha\left(t_2-\tfrac{\tau}{2}\right)\left[\left(R^{N-1}-R^{N+1}\right)D\right. \tag{16}$$
$$\left.\times\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|1-j|}-R^{1+j}\right)f_j\tau\right]\Bigg]\Bigg\}\Bigg\},$$

and

$$v_N=G_2^{-1}\left\{\left[-3(I-R^{2N})+4\left(R-R^{2N-1}\right)-\left(R^2-R^{2N-2}\right)\right]2\tau(I-R^{2N})\eta\right.$$
$$+\left(-4-\tau\beta\left(t_{N-2}-\tfrac{\tau}{2}\right)\right)\left[\left(R-R^{2N-1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau\right.$$
$$\left.-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|N-1-j|}-R^{N-1+j}\right)f_j\tau\right]+\left(1-\tau\beta\left(t_{N-3}-\tfrac{\tau}{2}\right)+\tau\beta\left(t_{N-1}-\tfrac{\tau}{2}\right)\right)$$
$$\times\left[\left(R^2-R^{2N-2}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|N-2-j|}-R^{N-2+j}\right)f_j\tau\right]$$
$$-\sum_{i=2}^{N-3}\tau\left[\beta\left(t_{i+\frac{1}{2}}\right)-\beta\left(t_{i-\frac{3}{2}}\right)\right]$$
$$\left[\left(R^{N-i}-R^{N+i}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|i-j|}-R^{i+j}\right)f_j\tau\right]$$
$$-\tau\beta\left(t_{\frac{3}{2}}\right)\left[\left(R^{N-1}-R^{N+1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|1-j|}-R^{1+j}\right)f_j\tau\right]$$
$$-\left[-\left(4+\tau\beta\left(t_{N-\frac{5}{2}}\right)\right)\left(R^{N-1}-R^{N+1}\right)\right.$$
$$+\left(1-\tau\beta\left(t_{N-\frac{7}{2}}\right)+\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)\left(R^{N-2}-R^{N+2}\right)+\tau\beta\left(t_{\frac{1}{2}}\right)\left(I-R^{2N}\right)$$
$$\left.+\sum_{i=2}^{N-3}\tau\left[\beta\left(t_{i+\frac{1}{2}}\right)-\beta\left(t_{i-\frac{3}{2}}\right)\right]\left(R^{i}-R^{2N-i}\right)+\tau\beta\left(t_{\frac{3}{2}}\right)\left(R-R^{2N-1}\right)\right]$$
$$\times\left[2\tau(I-R^{2N})\phi+4\left(R^{N-1}-R^{N+1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau\right.$$
$$-4(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|1-j|}-R^{1+j}\right)f_j\tau$$
$$\left.\left.-\left(R^{N-2}-R^{N+2}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau+(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|2-j|}-R^{2+j}\right)f_j\tau\right]\right\}. \tag{17}$$

Thus solution of difference problem (13) exists and it is defined by (8) with the corresponding v_0 and v_N via (16) and (17). From (8), (16),(17), estimates (4), (6), it follows that for solution of difference problem (13) stability estimates

$$\left\|\{v_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\leq M\left(\delta\right)\left(\|\phi\|_H+\|\zeta\|_H+\|\eta\|_H+\|f_\tau\|_{C_\tau(H)}\right), \tag{18}$$

$$\left\|\{Av_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}+\left\|\left\{\tfrac{v_{k+1}-2v_k+v_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}$$
$$\leq M\left(\delta\right)\left(\tfrac{1}{\alpha(1-\alpha)}\left\|f_\tau\right\|_{C_\tau^{\alpha,\alpha}(H)}+\|A\zeta\|_H+\|A\phi\|_H+\|A\eta\|_H\right). \tag{19}$$

are fulfilled. (12) and estimates (18) permit us to get estimates estimates (11) (10) and (19).

Theorem 2. Let us $f_\tau \in C_\tau^{\alpha,\alpha}(H)$, and $\phi, \zeta, \eta \in D(A)$ and inequality (1) is satisfied. Then, for solution $\left(\{u_k\}_{k=1}^{N-1}, p\right)$ of difference problem (9) the coercive stability inequality

$$\begin{gathered}\left\|\left\{\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)} + \left\|\{Au_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)} + \|p\|_H \\ \leq M(\delta)\left(\frac{1}{\alpha(1-\alpha)}\|f_\tau\|_{C_\tau^{\alpha,\alpha}(H)} + \|A\zeta\|_H + \|A\phi\|_H + \|A\eta\|_H\right)\end{gathered} \tag{20}$$

is valid.

The proof of inequality (20) is based on formulas (8), (12), (16), (17), and (19).

Approximation of (3)

Denote by

$$\widetilde{\Omega}_h = \left\{x = (h_1 m_1, ..., h_n m_n); m = (m_1, ..., m_n), m_i = \overline{0, M_i},\ h_i M_i = 1, i = \overline{1,n}\ \right\},$$

$$\Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S$$

and by A_h^x difference operator

$$A_h^x u^h(x) = -\sum_{i=1}^{n}\left(a_i(x) u_{\overline{x}_i}^h(x)\right)_{x_i, j_i} + \sigma u^h(x)$$

acting in the space of grid functions $u^h(x)$, satisfying boundary condition $u^h(x) = 0$ for all $x \in S_h$.

In the beginning, by using approximation in variable x and later by approximation in variable t, one can get the following difference scheme for approximately solution of SIP (3):

$$\begin{gathered}-\tau^{-2}\left(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)\right) + Au_k^h(x) = f_k^h(x) + p^h(x), \quad 1 \leq k \leq N-1, x \in \Omega_h \\ -3u_0^h(x) + 4u_1^h(x) - u_2^h(x) = \tau\phi^h(x), u_l^h(x) + \mu(u_{l+1}^h(x) - u_l^h(x)) = \zeta^h(x) \\ 3u_N^h(x) - 4u_{N-1}^h(x) + u_{N-2}^h(x) = \sum_{i=1}^{N-1}\tau\alpha\left(t_i - \frac{\tau}{2}\right)\left(u_{i+1}^h(x) - u_i^h(x)\right) + 2\tau\eta^h(x), x \in \widetilde{\Omega}_h. \\ .\end{gathered} \tag{21}$$

Let $L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$, the Banach spaces of the grid functions $u^h(x) = \{u(h_1 m_1, \cdots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the corresponding norms

$$\begin{gathered}\left\|u^h\right\|_{L_{2h}} = (\textstyle\sum_{x\in\widetilde{\Omega}_h} |u^h(x)|^2 h_1 \cdots h_n)^{1/2}, \\ \left\|u^h\right\|_{W_{2h}^2} = \left\|u^h\right\|_{L_{2h}} + +(\textstyle\sum_{x\in\widetilde{\Omega}_h}\sum_{i=1}^n \left|(u^h(x))_{x_i\overline{x_i},\, m_i}\right|^2 h_1 \cdots h_n)^{1/2}.\end{gathered}$$

Theorem 3. Assume that (1) is valid, $f_\tau \in C_\tau^{\alpha,\alpha}(L_{2h})$, and $\phi^h, \eta^h, \zeta^h \in D(A_h^x) \cap L_{2h}$. Then, the solution of difference problem (21) exists and for solution the stability estimates hold:

$$\begin{gathered}\left\|\{u_k^h\}_1^{N-1}\right\|_{C_\tau(L_{2h})} \leq M(\delta)\,[\left(\left\|\phi^h\right\|_{L_{2h}} + \left\|\eta^h\right\|_{L_{2h}} + \left\|\zeta^h\right\|_{L_{2h}} + \|f_\tau\|_{C_\tau(L_{2h})}\right), \\ \left\|p^h\right\|_{L_{2h}} \leq M(\delta)\left(\left\|\zeta^h\right\|_{W_{2h}^2} + \left\|\eta^h\right\|_{W_{2h}^2} + \left\|\phi^h\right\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)}\|f_\tau\|_{C_\tau^{\alpha,\alpha}(L_{2h})}\right).\end{gathered}$$

Theorem 4. Assume that (1) is true, $f_\tau \in C_\tau^{\alpha,\alpha}(W_{2h}^2)$, and $\phi^h, \eta^h, \zeta^h \in D(A_h^x) \cap W_{2h}^2$. Then, for the solution of difference problem (21) the coercive stability estimate obeys

$$\begin{gathered}\left\|\left\{\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^k}{\tau^2})\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})} + \left\|\{u_k^h\}_1^{N-1}\right\|_{C_\tau(W_{2h}^2)} + \left\|p^h\right\|_{L_{2h}} \\ \leq M(\delta)\,[\left(\left\|\zeta^h\right\|_{W_{2h}^2} + \left\|\eta^h\right\|_{W_{2h}^2} + \left\|\phi^h\right\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)}\|f_\tau\|_{C_\tau^{\alpha,\alpha}(W_{2h}^2)}\right).\end{gathered}$$

The proofs of Theorems 3 and 4 are based on the symmetry property of the operator A_h^x in the Hilbert space L_{2h} and the corresponding theorem in [20] on the coercivity stability inequality for the solution of the elliptic difference problem in L_{2h} with first kind boundary condition.

Test examples

In the present section, we illustrate computed results for twodimensional and threedimentional examples of inverse elliptic problem with Neumann-type overdetermination and integral condition. All computed results are carried out by using MATLAB.

2D example

Notice that pair functions $(p(x), u(t,x)) = \left(\left(\pi^2+1\right)\ \sin(\pi x), \left(e^{-t}\ +t+1\right)\sin(\pi x)\right)$ is exact solution of the following 2D overdetermined elliptic problem with integral boundary condition:

$$\begin{cases} -u_{tt}(t,x) - u_{xx}\ (t,x) + u(t,x) = f(t,x) + p(x), \quad t, x \in (0,1)\,, \\ u_t(0,x) = 0,\ u(0.3,x) = \zeta(x), u_t(1,x) = \int\limits_0^1 e^{-\lambda} u_\lambda(\lambda) d\lambda\ + \eta(x),\ x \in [0,1]\,, \\ u(t,0) = 0, u(t,1) = 0,\ t \in [0,1]\,, \end{cases} \tag{22}$$

where

$$f(t,x) = \left[-e^{-t} + \left(\pi^2+1\right)\left(e^{-t}+t\right)\ \right]\sin(\pi x), \zeta(x) = \left(e^{-0.3}\ +1.3\right)\sin(\pi x)$$
$$\eta(x) = \left[\tfrac{1}{2} - \tfrac{1}{2}e^{-2}\ \right]\ \sin(\pi x).$$

The notation $[0,1]_\tau \times [0,1]_h$ means the set of grid points

$$[0,1]_\tau \times [0,1]_h = \{(t_i, x_n) : t_i = i\tau,\ i = \overline{0,N},\ x_n = nh,\ n = \overline{0,M}\},$$

which depends on the small parameters τ and h such that $N\tau = 1, Mh = 1$. Let us

$$l_0 = \left[0.3\tau^{-1}\right], \mu_0 = 0.3\tau^{-1} - l_0; \phi_n = 0, \eta_n = \eta\left(x_n\right), \zeta_n = \zeta\left(x_n\right),\ n = \overline{0,M};$$
$$f_n^k = f(t_k, x_n), k = \overline{0,N},\ n = \overline{0,M}.$$

To approximately soving (22), we use algorithm which contains three stages. Firstly, we find approximately solution of auxiliary NBVP

$$\begin{cases} \tau^{-2}\left(v_n^{k+1} - 2v_n^k + v_n^{k-1}\right) + h^{-2}\left(v_{n+1}^k - 2v_n^k + v_{n-1}^k\right) - v_n^k = -f(t_k, x_n), \\ k = \overline{1, N-1},\ n = \overline{1, M-1}, \\ v_0^k = v_M^k = 0,\ k = \overline{0,N},\ -3v_n^0 + 4v_n^1 - v_n^2 = 0, \\ 3v_n^N - 4v_n^{N-1} + v_n^{N-2} = \sum\limits_{j=1}^{N-1} \frac{\tau}{2} e^{-\left(t_j - \frac{\tau}{2}\right)}\ \left(v_n^{j+1} - v_n^{j-1} + v_n^j - v_n^{j-2}\right) + 2\tau\eta_n,\ n = \overline{0,M}. \end{cases} \tag{23}$$

Secondly, we find p_n. It is caried out by

$$p_n = -\tfrac{1}{h^2}[(\zeta_{n+1} - (\mu_0 v_{n+1}^{l_0+1} - (\mu_0 - 1)v_{n+1}^{l_0})) - 2(\zeta_n - (\mu_0 v_n^{l_0+1} - (\mu_0-1)v_n^{l_0}))$$
$$+ (\zeta_{n-1} - (\mu_0 v_{n-1}^{l_0+1} - (\mu_0 - 1)v_{n-1}^{l_0}))] + \zeta_n - (\mu_0 v_n^{l_0+1} - (\mu_0 - 1)v_n^{l_0}), \quad , n = \overline{1, M-1}.$$

Difference problem (23) can be rewritten in the matrix form

$$\begin{aligned} &A v_{n+1} + B v_n + C v_{n-1} = I g^{(n)},\ n = \overline{1, M-1}, \\ &v_0 = \overrightarrow{0}, v_M = \overrightarrow{0}. \end{aligned} \tag{24}$$

Here, A, B, C, I are $(N+1)\times(N+1)$ square matrices, and I is identity matrix, $v_s, s = n-1, n, n+1, g^{(n)}$ are column matrices with $(N+1)$ rows, $v_s = \left[\begin{array}{ccc} v_s^0 & \dots & v_s^N \end{array}\right]^t$. Denote by

$$a = \frac{1}{h^2}, c = \frac{1}{h^2},\ q = -\frac{2}{h^2} - \frac{2}{\tau^2} - 1, r = \frac{1}{\tau^2}.$$

Then,

$$A_n = diag(0, a, a, \dots, a, 0), C_n = A_n, g_k^{(n)} = -f(t_k, x_n),\ k = \overline{1, N-1},\ n = \overline{1, M-1},$$

$$b_{i,i} = q, b_{i-1,i} = r, b_{i,i-1} = r, i = \overline{2, N}, b_{1,1} = -3, b_{1,2} = 4, b_{1,3} = -1,$$
$$b_{N+1,N+1} = 2\tau\left(\frac{e^{t_{N-\frac{3}{2}}}}{4} + e^{-t_{N-\frac{3}{2}}}\right) - 3,\ b_{N+1,N} = 2\tau\left(\frac{e^{-t_{N-5}}}{4} + \frac{e^{-t_{N-\frac{3}{2}}}}{4} - e^{-t_{N-\frac{1}{2}}}\right) + 4,$$
$$b_{N+1,N-1} = \frac{\tau e^{-t_{N-\frac{7}{2}}}}{2} + \frac{\tau e^{-t_{N-\frac{5}{2}}}}{2} - \frac{\tau e^{-t_{N-\frac{3}{2}}}}{2} - 1,$$

$$b_{N+1,1} = 2\tau\left(-\frac{e^{-t_{\frac{3}{2}}}}{4} - e^{-t_{\frac{1}{2}}}\right),\ b_{N+1,2} = 2\tau\left(-\frac{e^{-t_{\frac{3}{2}}}}{4} - \frac{e^{-t_{\frac{5}{2}}}}{4} + e^{-t_{\frac{1}{2}}}\right),$$

$$b_{N+1,3} = \frac{\tau e^{-t_{\frac{3}{2}}}}{2} - \frac{\tau e^{-t_{\frac{5}{2}}}}{2} - \frac{\tau e^{-t_{\frac{7}{2}}}}{2},$$

$$b_{N+1,j} = \frac{\tau}{2}\left(e^{-t_{j-\frac{3}{2}}} + e^{-t_{j-\frac{1}{2}}} - e^{-t_{j+\frac{1}{2}}} - e^{-t_{j+\frac{3}{2}}}\right)\ , j = 4, ..., N-2;$$

$$b_{i,j} = 0,\ \text{for other } i \text{ and } j; g_n^0 = 2\tau\phi_n,\ g_n^N = 2\tau\eta_n,\ n = \overline{1, M-1}.$$

To solve (24), we use modified Gauss elimination method.

Thirdly, we define $\{u_n^k\}$ by $u_n^k = v_n^k + \zeta_n - \left(\mu_0 v_n^{l_0+1} - (\mu_0 - 1)\, v_n^{l_0}\right)$.

Errors are presented in Tables 1-3 for second order ADS in case N=M=10,20, 40, 80, 160 and 320. It can be seen from Tables 1-3 when N, M are increased two times that errors are decreased with approximately ratio $\frac{1}{4}$.

Table 1

Test example (22) - error v

DS \ (N, M)	$(10, 10)$	$(20, 20)$	$(40, 40)$	$(80, 80)$	$(160, 160)$	$(320, 320)$
2nd order of ADS	6.29×10^{-3}	1.57×10^{-3}	3.93×10^{-4}	9.84×10^{-5}	2.46×10^{-5}	6.15×10^{-6}

Table 2

Test example (22) - error u

DS \ (N, M)	$(10, 10)$	$(20, 20)$	$(40, 40)$	$(80, 80)$	$(160, 160)$	$(320, 320)$
2nd order of ADS	3.13×10^{-4}	7.95×10^{-5}	2.02×10^{-5}	5.10×10^{-6}	1.28×10^{-6}	3.22×10^{-7}

Table 3

Test example (22) - error p

Appr. \ (N, M)	$(10, 10)$	$(20, 20)$	$(40, 40)$	$(80, 80)$	$(160, 160)$	$(320, 320)$
2nd order	5.03×10^{-3}	1.28×10^{-3}	3.21×10^{-4}	8.06×10^{-5}	2.02×10^{-5}	5.05×10^{-6}

3D example

Now, consider the three dimensional inverse elliptic problem with integral condition

$$\begin{cases} -u_{tt}(t,x,y)-u_{xx}(t,x,y)-u_{yy}(t,x,y)+u(t,x,y)=f(t,x,y)+p(x,y),x,y,\ t\in(0,1)\,, \\ u(t,0,y)=u(t,1,y)=0,\ y,t\in[0,1]\,,u(t,x,0)=u(t,x,1)=0,\ x,t\in[0,1]\,, \\ u_t(0,x,y)=\phi(x,y),u(0.6,x,y)=\zeta(x,y), \\ u_t(1,x,y)-\int\limits_0^1 e^{-\lambda}u_\lambda(\lambda,x,y)d\lambda \quad =\eta(x,y),x,y\in[0,1]\,, \end{cases} \tag{25}$$

where

$$f(t,x,y)=2\pi^2e^{-t}q(x,y),\phi(x,y)=-q(x,y),\eta(x,y)=\left[-e^{-1}+\tfrac{1}{3}\left(e^{-0.6}+e^{-1.2}\right)\right]q(x,y),$$
$$\zeta(x,y)=\left(e^{-\frac{3}{5}}+1\right)q(x,y),q(x,y)=\sin(\pi x)\sin(\pi y)$$

It is clear that pair funcions $p(x,y)=\left(2\pi^2+1\right)q(x,y)$ and $u(t,x,y)=\left(e^{-t}+1\right)q(x,y)$ is exact solution of (25).

Denote by $[0,1]_\tau\times[0,1]_h\times[0,1]_h$ set of grid points depending on the small parameters τ and h

$$[0,1]_\tau\times[0,1]_h^2 \ =\{(t_i,x_n,y_m):t_i=i\tau,\ i=\overline{0,N},\ x_n=nh,\ n=\overline{0,M},$$
$$y_m=mh,\ m=\overline{0,M},\ \tau N=1,hM=1\}.$$

Let us

$$l_0=\left[0.3\tau^{-1}\right],\mu_0=0.3\tau^{-1}-l_0\ ,\phi_{m,n}=\phi\left(x_n,y_m\right),\eta_{m,n}=\eta\left(x_n,y_m\right),\zeta_{m,n}=\xi\left(x_n,y_m\right),$$
$$n=\overline{0,M},\ m=\overline{0,M};f_{m,n}^i=f(t_i,x_n,y_m),\ i=\overline{0,N},\ n=\overline{0,M},\ m=\overline{0,M}.$$

Firstly, difference scheme for approximate solution of NBVP can be written in the following form:

$$\begin{cases} -\tau^{-2}\left(v_{m,n}^{k+1}-2v_{m,n}^k+v_{m,n}^{k-1}\right)-h^{-2}\left(v_{m,n+1}^k-2v_{m,n}^k+v_{m,n-1}^k\right) \\ -h^{-2}\left(v_{m+1,n}^k-2v_{m,n}^k+v_{m-1,n}^k\right)+v_{m,n}^k=f_{m,n}^k, \\ k=\overline{1,N-1},\ n=\overline{1,M-1},\ m=\overline{1,M-1}, \\ v_{0,n}^k=v_{M,n}^k=v_{m,n}^k=v_{m,M}^k=0,\ k=0,\cdots,N,\ n=\overline{1,M-1},\ m=\overline{1,M-1}, \\ -3v_{m,n}^0+4v_{m,n}^1-v_{m,n}^2=2\tau\phi_{m,n},\quad 3v_{m,n}^N-4v_{m,n}^{N-1}+v_{m,n}^{N-2} \\ =\sum\limits_{j=1}^{N-1}\frac{\tau}{2}e^{-\left(t_j-\frac{\tau}{2}\right)}\ \left(v_{m,n}^{j+1}-v_{m,n}^{j-1}+v_{m,n}^j-v_{m,n}^{j-2}\right)+2\tau\eta_{m,n}, \\ n=\overline{1,M-1},n=\overline{1,M-1}. \end{cases} \tag{26}$$

Secondly, calculation of p_n $\left(n=\overline{1,M-1},m=\overline{1,M-1}\right)$ is caried out by

$$p_{m,n}=-\tfrac{1}{h^2}\left\{\left[\zeta_{m,n+1}-\left(\mu_0v_{m,n+1}^{l_0+1}-\left(\mu_0-1\right)v_{m,n+1}^{l_0}\right)\right]-2\left[\zeta_{m,n}-\left(\mu_0v_{m,n}^{l_0+1}-\left(\mu_0-1\right)v_{m,n}^{l_0}\right)\right]\right.$$
$$\left.+\left[\zeta_{m,n-1}-\left(\mu_0v_{m,n-1}^{l_0+1}-\left(\mu_0-1\right)v_{m,n-1}^{l_0}\right)\right]\right\}-\tfrac{1}{h^2}\left\{\left[\zeta_{m+1,n}-\left(\mu_0v_{m+1,n}^{l_0+1}-\left(\mu_0-1\right)v_{m+1,n}^{l_0}\right)\right]\right.$$
$$\left.-2\left[\zeta_{m,n}-\left(\mu_0v_{m,n}^{l_0+1}-\left(\mu_0-1\right)v_{m,n}^{l_0}\right)\right]+\left[\zeta_{m-1,n}-\left(\mu_0v_{m-1,n}^{l_0+1}-\left(\mu_0-1\right)v_{m-1,n}^{l_0}\right)\right]\right\}.$$

Thirdly, we calculate $\left\{u_n^k\right\}$ by

$$u_{m,n}^k=v_{m,n}^k+\zeta_{m,n}-\left(\mu_0v_{m,n}^{l_0+1}-\left(\mu_0-1\right)v_{m,n}^{l_0}\right).$$

Difference problem (26) can be rewritten in the matrix form (24). In this case, g_n is a column matrix with $(N+1)(M+1)$ elements, A, B, C,I are square matrices with $(N+1)(M+1)$ rows and columns, and I is the identity matrix, v_s is column matrix with $(N+1)(M+1)$ elements such that

$$v_s=\left[\begin{array}{ccccccccc} v_{0,s}^0 & \cdots & v_{0,s}^N & v_{1,s}^0 & \cdots & v_{1,s}^N & \cdots & v_{M,s}^0 & \cdots & v_{M,s}^N \end{array}\right]^t,s=n-1,n,n+1.$$

Denote by

$$a = \frac{1}{h^2}, q = 1 + \frac{2}{\tau^2} + \frac{4}{h^2}, r = \frac{1}{\tau^2}.$$

Then,

$$A = C = \begin{bmatrix} O & O & \cdots & O & O \\ O & E & \cdots & O & O \\ \ldots & \ldots & \ddots & \ldots & \ldots \\ O & O & \cdots & E & O \\ O & O & \cdots & O & O \end{bmatrix}, \quad B = \begin{bmatrix} Q & O & \cdots & O & O \\ O & D & \cdots & O & O \\ \ldots & \ldots & \ddots & \ldots & \ldots \\ O & O & \cdots & D & O \\ O & O & \cdots & O & Q \end{bmatrix},$$

$$E = diag(0, a, a, \ldots, a, 0), Q = I_{(N+1)\times(N+1)}, O = O_{(N+1)\times(N+1)},$$
$$g^k_{m,n} = -f(t_k, x_n, y_m), \ k = \overline{1, N-1}, \ n = \overline{1, M-1}, m = \overline{1, M-1},$$

$$d_{i,i} = q, d_{i-1,i} = r, d_{i,i-1} = r, i = \overline{2, N}, d_{1,1} = -3, d_{1,2} = 4, d_{1,3} = -1,$$
$$d_{N+1,N+1} = 2\tau\left(\frac{e^{-\left(t_{N-1}-\frac{\tau}{2}\right)}}{4} + e^{-\left(t_N-\frac{\tau}{2}\right)}\right) - 3, \ d_{N+1,N} = 2\tau\left(\frac{e^{-\left(t_{N-2}-\frac{\tau}{2}\right)}}{4} + \frac{e^{-\left(t_{N-1}-\frac{\tau}{2}\right)}}{4} - e^{-\left(t_N-\frac{\tau}{2}\right)}\right) + 4,$$
$$d_{N+1,N-1} = \frac{\tau e^{-\left(t_{N-3}-\frac{\tau}{2}\right)}}{2} + \frac{\tau e^{-\left(t_{N-2}-\frac{\tau}{2}\right)}}{2} - \frac{\tau e^{-\left(t_{N-1}-\frac{\tau}{2}\right)}}{2} - 1, \ d_{N+1,1} = 2\tau\left(-\frac{e^{-\left(t_2-\frac{\tau}{2}\right)}}{4} - e^{-\left(t_1-\frac{\tau}{2}\right)}\right),$$

$$d_{N+1,2} = 2\tau\left(-\frac{e^{-\left(t_2-\frac{\tau}{2}\right)}}{4} - \frac{e^{-\left(t_3-\frac{\tau}{2}\right)}}{4} + e^{-\left(t_1-\frac{\tau}{2}\right)}\right), \ d_{N+1,3} = \frac{\tau e^{-\left(t_2-\frac{\tau}{2}\right)}}{2} - \frac{\tau e^{-\left(t_3-\frac{\tau}{2}\right)}}{2} - \frac{\tau e^{-\left(t_4-\frac{\tau}{2}\right)}}{2},$$
$$d_{N+1,j} = \frac{\tau}{2}\left(e^{-\left(t_{j-1}-\frac{\tau}{2}\right)} + e^{-\left(t_j-\frac{\tau}{2}\right)} - e^{-\left(t_{j+1}-\frac{\tau}{2}\right)} - e^{-\left(t_{j+2}-\frac{\tau}{2}\right)}\right), j = 4, ..., N-2;$$
$$d_{i,j} = 0, \text{ for other } i \text{ and } j; g^0_{m,n} = 2\tau\phi_{m,n}, \ g^N_{m,n} = 2\tau\eta_{m,n}, \ n = \overline{1, M-1}, m = \overline{1, M-1}.$$

In Tables 4-6, errors approximations in case N=M=10, 20, 40 for u, v and p are displayed. It can be seen from Tables 4-6 when N, M are increased two times that errors are decreased with approximately ratio $\frac{1}{4}$.

Table 4

Test example (25) - error v

DS $\setminus (N, M)$	$(10, 10)$	$(20, 20)$	$(40, 40)$
2nd order of ADS	4.75×10^{-3}	1.18×10^{-3}	2.97×10^{-4}

Table 5

Test example (25) - error u

DS $\setminus (N, M)$	$(10, 10)$	$(20, 20)$	$(40, 40)$
2nd order of ADS	2.43×10^{-4}	6.23×10^{-5}	1.56×10^{-5}

Table 6

Test example (25) - error p

Approximation $\setminus (N, M)$	$(10, 10)$	$(20, 20)$	$(40, 40)$
2nd order	3.42×10^{-4}	1.15×10^{-4}	3.07×10^{-5}

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Интегралдық шарты бар және қайта анықталған Нейман типті эллипстік кері есептің сандық есептеуі

Әртүрлі нақты процестерді модельдеу кезінде дербес туындылы дифференциалдық теңдеу үшін дереккөздерді сәйкестендіру есептерін шешу әдістері маңызды рөл атқарады. Мақала интегралдық шарты бар туынды үшін белгілі бір есептің эллипстік аппроксимациясына арналған. Алғашқыда, кері есеп туынды үшін интегралдық шарттары бар бейлокаль қандай да бір көмекші шеттік есептерге әкеледі. Теңдеудің параметрі бейлокаль көмекші есепті шешкен соң анықталады. Абстрактілі анықталған эллипстік есепті жуықтап шешу үшін екінші дәлдіктің айырымдық схемасы ұсынылған. Оператор тәсілін қолдана отырып, айырымдық есептің шешімінің бар екендігі дәлелденді. Салынған айырмашылық сызбасын шешу үшін тұрақтылық пен мәжбүрлік тұрақтылық бағалануы орнатылған. Кейін алынған абстрактілі нәтижелер интегралды шарттары бар Нейман типіндегі эллипстік көп өлшемді айырымдық есептерінің шешімінің орнықтылық бағамын алу үшін қолданылады. Қорытындылай келе, MATLAB бағдарламасын қолдана отырып, екі өлшемді және үш өлшемді тестік мысалдарын қысқаша түсіндірмесімен және сандық нәтижесін ұсынамыз.

Кілт сөздер: айырымдық схема, эллипстік кері есеп, қайта анықталған, дереккөзді сәйкестендіру мәселесі, орнықтылық, мәжбүрлі тұрақтылық, бағамы.

Ч. Ашыралыев, А. Чай

Численное решение эллиптической обратной задачи с интегральным условием и переопределением типа Неймана

При моделировании различных реальных процессов важную роль играют методы решения задачи идентификации источника для уравнения в частных производных. Настоящая статья посвящена аппроксимации эллиптической переопределенной задачи с интегральным условием для производных. Вначале обратная задача сводится к некоторой вспомогательной нелокальной краевой задаче с интегральным граничным условием для производных. Параметр уравнения определяется после решения этой вспомогательной нелокальной задачи. Предложена разностная схема второго порядка точности для приближенного решения абстрактной переопределенной эллиптической задачи. С помощью операторного подхода доказано существование решения разностной задачи. Для решения построенной разностной схемы установлены оценки устойчивости и коэрцитивной устойчивости. Позднее полученные абстрактные результаты применяются для получения оценок устойчивости решения переопределенных эллиптических многомерных разностных задач типа Неймана с интегральными условиями. Кроме того, используя программу MATLAB, авторами представлены численные результаты для двух- и трехмерных тестовых примеров с кратким объяснением реализации на компьютере.

Ключевые слова: разностная схема, обратная эллиптическая задача, переопределение, проблема идентификации источника, устойчивость, коэрцитивная устойчивость, оценка.

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A comparison between the fourth order linear differential equation with its boundary value problem

In this paper, we study a fourth order linear differential equation. We found an upper bound for the solutions of this differential equation and also, we prove that all the solutions are in $L^4(0,\infty)$. By comparing these results we obtain that all the eigenfunction of the boundary value problem generated by this differential equation are bounded and in $L^4(0,\infty)$.

Keywords: linear differential equation, eigenvalue, eigenfunction, upper bound, linearly independent solution, $L^2(0,\infty)$, wrongskian, Gronwall inequality, Variation of parameters.

Introduction

The method of finding an upper bound for the solutions of a differential equation has been investigated by many authors. In papers [2,4] by authors were investigated the solutions of the second order linear differential equation. They obtained some important properties of this equation such that all solutions of the differential equation are bounded and in the space $L^2(0,\infty)$. Here $L^2(0,\infty)$ is the space of all functions f which are continuous and satisfy the conditions:

$$\int_0^\infty |f(x)|^2 dx < \infty.$$

The estimate of upper bounds for the eigenfunctions of a boundary value problem was investigated by many authors. In papers [2–6, 10] by authors were investigated a second order differential equation of the form

$$y'' + q(x)\,y = \lambda^2 \rho(x)\,y, x \in [0,a].$$

They found a normalized eigenfunctions for this problem and an upper bound for this solution under a certain condions.
Methods of finding of general solution of a fourth order differential equation were studied by many authors, see: [1, 7–9, 11].

This paper is specified to study some important properties of solutions of a fourth order linear differential equation of the form:

$$y^{(4)}(x) + \{q(x) + r(x)\}\,y(x) = 0, \qquad 0 \le x < \infty, \tag{1}$$

where $r(x)$ is a function satisfying the condition:

$$\int_0^\infty |r(x)|\,dx < \infty. \tag{2}$$

We investigate whether the solutions of (1) are related to any general properties such as boundedness of the solutions of the differential equation

$$y^{(4)}(x) + q(x)\,y(x) = 0, \qquad 0 \le x < \infty. \tag{3}$$

Let $L^4(0,\infty)$ is the space of all continuous functions f for which satisfy the condition

$$\int_0^\infty |f(x)|^4 dx < \infty.$$

In this paper we show that all solutions of (1) are in $L^4(0,\infty)$. It is based on the fact that the solutions of (3) are in $L^4(0,\infty)$ under the condition (2). Moreover, we show that eigenfunctions of the boundary value problem which is generated by the differential equation $y^{(4)}(x)+\{\lambda+r(x)\}y(x)=0$ are bounded under a certain condion.

Let $f(x)$ and $g(x)$ be real-valued, continuous, and nonnegative in $[a,b]$ and suppose that $f(x)\leq c+\int_a^x f(t)g(t)\,dt$, in $[a,b]$ where $c>0$ is a constant. Then,

$$f(x)\leq c\ e^{\int_a^x g(t)dt}. \tag{4}$$

This is known as Gronwall inequality [2].

Expression for the solutions

In this section we found a general solutions for (1) by using the method of variation of parameter. We need some properties of the differential equations (1) and (3) which are immediate consequence of the results of chapter two in [2].

Lemma 1. There are solutions $\phi_j(x)$, $\{j=1,2,3,4\}$ of (3) such that $W(\phi_1,\phi_3,\phi_2,\phi_4)=1$ in $[0,\infty)$.

Proof. Let $y_1(x)$, $y_2(x)$, $y_3(x)$ and $y_4(x)$be a fundamental system of solution of (3), then we obtain $W(\phi_1,\phi_3,\phi_2,\phi_4)=c\ in\ [0,\infty)$, where c is a non zero constant , we take $\phi_1(x)=y_1(x)$, $\phi_2(x)=y_2(x)$, $\phi_3(x)=y_3(x)$ and $\phi_4(x)=\frac{y_4(x)}{c}$, then we can easily establish that $W(\phi_1,\phi_3,\phi_2,\phi_4)=1$.

Lemma 2. If $\phi_j(x)$, $\{j=1,2,3,4\}$ are as in Lemma 1 and $\psi(x)$ is any solution of (1), then there are unique constants c_j for $j=1:4$ such that

$$\psi(x)=c_1\phi_1(x)+c_2\phi_2(x)+c_3\phi_3(x)+c_4\phi_4(x)+\psi_0(x), \tag{5}$$

where

$$\begin{aligned}
\psi_0(x) = \int_0^x &[\phi_2(t)\phi_3'(t)\phi_4''(t)\phi_1(x)+\phi_2''(t)\phi_3(t)\phi_4'(t)\phi_1(x)+\phi_2'(t)\phi_3''(t)\phi_4(t)\phi_1(x)\\
&-\phi_2'(t)\phi_3(t)\phi_4''(t)\phi_1(x)-\phi_2(t)\phi_3''(t)\phi_4'(t)\phi_1(x)-\phi_2''(t)\phi_3'(t)\phi_4(t)\phi_1(x)\\
&-\phi_1(t)\phi_3'(t)\phi_4''(t)\phi_2(x)-\phi_1''(t)\phi_3(t)\phi_4'(t)\phi_2(x)-\phi_1'(t)\phi_3''(t)\phi_4(t)\phi_2(x)\\
&+\phi_1'(t)\phi_3(t)\phi_4''(t)\phi_2(x)+\phi_1(t)\phi_3''(t)\phi_4'(t)\phi_2(x)+\phi_1''(t)\phi_3'(t)\phi_4(t)\phi_2(x)\\
&+\phi_1(t)\phi_2'(t)\phi_4''(t)\phi_3(x)+\phi_1''(t)\phi_2(t)\phi_4'(t)\phi_3(x)+\phi_1'(t)\phi_2''(t)\phi_4(t)\phi_3(x)\\
&-\phi_1'(t)\phi_2(t)\phi_4''(t)\phi_3(x)-\phi_1(t)\phi_2''(t)\phi_4'(t)\phi_3(x)-\phi_1''(t)\phi_2'(t)\phi_4(t)\phi_3(x)\\
&-\phi_1(t)\phi_2'(t)\phi_3''(t)\phi_4(x)-\phi_1''(t)\phi_2(t)\phi_3'(t)\phi_4(x)-\phi_1'(t)\phi_2''(t)\phi_3(t)\phi_4(x)\\
&+\phi_1'(t)\phi_2(t)\phi_3''(t)\phi_4(x)+\phi_1(t)\phi_2''(t)\phi_3'(t)\phi_4(x)+\phi_1''(t)\phi_2'(t)\phi_3(t)\phi_4(x)]\\
&\times r(t)\psi(t)\,dt.
\end{aligned}$$

Proof. If $\psi(x)$ is a solution of (1), then as we see in [2] by using variation of parameter there is unique constants $c_j$such that

$$\psi(x)=c_1\phi_1(x)+c_2\phi_2(x)+c_3\phi_3(x)+c_4\phi_4(x)+\psi_0(x), \tag{6}$$

where

$$\psi_0(x)=c_1(x)\phi_1(x)+c_2(x)\phi_2(x)+c_3(x)\phi_3(x)+c_4(x)\phi_4(x) \tag{7}$$

and

$$c_r(x) = \int_0^x \frac{W_r(\phi_1, \phi_3, \phi_2, \phi_4)(t)}{W(\phi_1, \phi_3, \phi_2, \phi_4)(t)} r(t)\,\psi(t)\,dt. \tag{8}$$

From Lemma 1 it follows that $W(\phi_1, \phi_3, \phi_2, \phi_4) = 1$.Therefore, (8) has the form

$$c_r(x) = \int_0^x W_r(\phi_1, \phi_3, \phi_2, \phi_4)(t) r(t)\,\psi(t)\,dt. \tag{9}$$

For $r = 1$, we have that

$$W_1(\phi_1, \phi_2, \phi_3, \phi_4)(t) = \begin{vmatrix} 0 & \phi_2(t) & \phi_3(t) & \phi_4(t) \\ 0 & \phi_2'(t) & \phi_3'(t) & \phi_4'(t) \\ 0 & \phi_2''(t) & \phi_3''(t) & \phi_4''(t) \\ 1 & \phi_2'''(t) & \phi_3'''(t) & \phi_4'''(t) \end{vmatrix}$$

$$\begin{aligned} = \;& \phi_2(t)\,\phi_3'(t)\,\phi_4''(t) + \phi_2''(t)\,\phi_3(t)\,\phi_4'(t) + \phi_2'(t)\,\phi_3''(t)\,\phi_4(t) - \phi_2'(t)\,\phi_3(t)\,\phi_4''(t) \\ & -\phi_2(t)\,\phi_3''(t)\,\phi_4'(t) - \phi_2''(t)\,\phi_3'(t)\,\phi_4(t). \end{aligned}$$

That is

$$\begin{aligned} W_1(\phi_1, \phi_2, \phi_3, \phi_4)(t) = \;& \phi_2(t)\,\phi_3'(t)\,\phi_4''(t) + \phi_2''(t)\,\phi_3(t)\,\phi_4'(t) + \phi_2'(t)\,\phi_3''(t)\,\phi_4(t) \\ & -\phi_2'(t)\,\phi_3(t)\,\phi_4''(t) - \phi_2(t)\,\phi_3''(t)\,\phi_4'(t) - \phi_2''(t)\,\phi_3'(t)\,\phi_4(t). \end{aligned}$$

For $r = 2, 3, 4$,applying the same way, we obtain

$$\begin{aligned} W_2(\phi_1, \phi_2, \phi_3, \phi_4)(t) = \;& -\phi_1(t)\,\phi_3'(t)\,\phi_4''(t) - \phi_1''(t)\,\phi_3(t)\,\phi_4'(t) - \phi_1'(t)\,\phi_3''(t)\,\phi_4(t) \\ & +\phi_1'(t)\,\phi_3(t)\,\phi_4''(t) + \phi_1(t)\,\phi_3''(t)\,\phi_4'(t) + \phi_1''(t)\,\phi_3'(t)\,\phi_4(t), \end{aligned}$$

$$\begin{aligned} W_3(\phi_1, \phi_2, \phi_3, \phi_4)(t) = \;& \phi_1(t)\,\phi_2'(t)\,\phi_4''(t) + \phi_1''(t)\,\phi_2(t)\,\phi_4'(t) + \phi_1'(t)\,\phi_2''(t)\,\phi_4(t) \\ & -\phi_1'(t)\,\phi_2(t)\,\phi_4''(t) - \phi_1(t)\,\phi_2''(t)\,\phi_4'(t) - \phi_1''(t)\,\phi_2'(t)\,\phi_4(t), \end{aligned}$$

$$\begin{aligned} W_4(\phi_1, \phi_2, \phi_3, \phi_4)(t) = \;& -\phi_1(t)\,\phi_2'(t)\,\phi_3''(t) - \phi_1''(t)\,\phi_2(t)\,\phi_3'(t) - \phi_1'(t)\,\phi_2''(t)\,\phi_3(t) \\ & +\phi_1'(t)\,\phi_2(t)\,\phi_3''(t) + \phi_1(t)\,\phi_2''(t)\,\phi_3'(t) + \phi_1''(t)\,\phi_2'(t)\,\phi_3(t). \end{aligned}$$

Substituting these values of W_r in (9) and then (9) in (7), we get the result.

Bounded solution

In this section we obtain that all solutions of (1) are bounded. It is based on boundednees of solutions of (3) and condition (2).

Theorem 1. Let that all solutions and their derivatives up to order three of (3) be bounded in $[0, \infty)$ and the condition (2) is hold, then all the solutions of (1) are bounded in $[0, \infty)$.

Proof. Let $\phi_1(x), \phi_2(x), \phi_3(x)$ and $\phi_4(x)$ be four linearly independent solutions of (3) such that $W(\phi_1, \phi_2, \phi_3, \phi_4) = 1$ and let $\psi(x)$ be any solution of (1), then by Lemma 2 there are constants c_1, c_2, c_3 and c_4 such that

$$\begin{aligned}\psi(x) &= c_1\phi_1(x)+c_2\phi_2(x)+c_3\phi_3(x)+c_4\phi_4(x)+\int_0^x [\phi_2(t)\phi_3^{'}(t)\phi_4^{''}(t)\phi_1(x)\\
&+\phi_2^{''}(t)\phi_3(t)\phi_4^{'}(t)\phi_1(x)+\phi_2^{'}(t)\phi_3^{''}(t)\phi_4(t)\phi_1(x)-\phi_2^{'}(t)\phi_3(t)\phi_4^{''}(t)\phi_1(x)\\
&-\phi_2(t)\phi_3^{''}(t)\phi_4^{'}(t)\phi_1(x)-\phi_2^{''}(t)\phi_3^{'}(t)\phi_4(t)\phi_1(x)-\phi_1(t)\phi_3^{'}(t)\phi_4^{''}(t)\phi_2(x)\\
&-\phi_1^{''}(t)\phi_3(t)\phi_4^{'}(t)\phi_2(x)-\phi_1^{'}(t)\phi_3^{''}(t)\phi_4(t)\phi_2(x)+\phi_1^{'}(t)\phi_3(t)\phi_4^{''}(t)\phi_2(x)\\
&+\phi_1(t)\phi_3^{''}(t)\phi_4^{'}(t)\phi_2(x)+\phi_1^{''}(t)\phi_3^{'}(t)\phi_4(t)\phi_2(x)+\phi_1(t)\phi_2^{'}(t)\phi_4^{''}(t)\phi_3(x)\\
&+\phi_1^{''}(t)\phi_2(t)\phi_4^{'}(t)\phi_3(x)+\phi_1^{'}(t)\phi_2^{''}(t)\phi_4(t)\phi_3(x)-\phi_1^{'}(t)\phi_2(t)\phi_4^{''}(t)\phi_3(x)\\
&-\phi_1(t)\phi_2^{''}(t)\phi_4^{'}(t)\phi_3(x)-\phi_1^{''}(t)\phi_2^{'}(t)\phi_4(t)\phi_3(x)-\phi_1(t)\phi_2^{'}(t)\phi_3^{''}(t)\phi_4(x)\\
&-\phi_1^{''}(t)\phi_2(t)\phi_3^{'}(t)\phi_4(x)-\phi_1^{'}(t)\phi_2^{''}(t)\phi_3(t)\phi_4(x)+\phi_1^{'}(t)\phi_2(t)\phi_3^{''}(t)\phi_4(x)\\
&+\phi_1(t)\phi_2^{''}(t)\phi_3^{'}(t)\phi_4(x)+\phi_1^{''}(t)\phi_2^{'}(t)\phi_3(t)\phi_4(x)]r(t)\psi(t)\,dt, \qquad (10)\end{aligned}$$

By our hypothesis, there are constants k_0, k_1, k_2 such that
$|\phi_j(x)| \le k_0$, $\left|\phi_j^{'}(x)\right| \le k_1$, $\left|\phi_j^{''}(x)\right| \le k_2$ in $[0,\infty]$.
Hence from 10 it follows that

$$|\psi(x)| \le (|c_1|+|c_2|+|c_3|+|c_4|)\,k_0 + 18k_0^2k_1k_2\int_0^x |r(t)|\,|\psi(t)|\,dt.$$

Then, using Gronwall's Inequality, we obtain

$$|\psi(x)| \le (|c_1|+|c_2|+|c_3|+|c_4|)\,k_0 e^{18k_0^2k_1k_2\int_0^x |r(t)|dt}.$$

Since by our hypothesis $\int_0^x |r(t)|\,dt$ is bounded in $[0,\infty)$, then $\psi(x)$ is bounded in $[0,\infty)$. which it completed the proof.

$L^4(0,\infty)$ property of the solution

In this section we obtain that all solutions of (1) are $L^4(0,\infty)$ when the solutions of (3) are in $L^4(0,\infty)$ and $r(x)$ satisfy the condition (2).

Theorem 2. Suppose that all solutions and their derivatives up to order three of (3) be in $L^4(0,\infty)$ and $r(x)$ is bounded in $[0,\infty)$. Then all the solutions of (1) are in $L^4(0,\infty)$.

Proof. Let $\phi_1(x), \phi_2(x), \phi_3(x)$ and $\phi_4(x)$ be four linearly independent solutions of (3) such that $W(\phi_1,\phi_2,\phi_3,\phi_4) = 1$ and let $\psi(x)$ be any solution of (1), then by Lemma 2 there are constants c_1, c_2, c_3 and c_4 such that

$$\begin{aligned}\psi(x) &= c_1\phi_1(x)+c_2\phi_2(x)+c_3\phi_3(x)+c_4\phi_4(x)+\int_0^x [\phi_2(t)\phi_3^{'}(t)\phi_4^{''}(t)\phi_1(x)\\
&+\phi_2^{''}(t)\phi_3(t)\phi_4^{'}(t)\phi_1(x)+\phi_2^{'}(t)\phi_3^{''}(t)\phi_4(t)\phi_1(x)-\phi_2^{'}(t)\phi_3(t)\phi_4^{''}(t)\phi_1(x)\\
&-\phi_2(t)\phi_3^{''}(t)\phi_4^{'}(t)\phi_1(x)-\phi_2^{''}(t)\phi_3^{'}(t)\phi_4(t)\phi_1(x)-\phi_1(t)\phi_3^{'}(t)\phi_4^{''}(t)\phi_2(x)\\
&-\phi_1^{''}(t)\phi_3(t)\phi_4^{'}(t)\phi_2(x)-\phi_1^{'}(t)\phi_3^{''}(t)\phi_4(t)\phi_2(x)+\phi_1^{'}(t)\phi_3(t)\phi_4^{''}(t)\phi_2(x)\\
&+\phi_1(t)\phi_3^{''}(t)\phi_4^{'}(t)\phi_2(x)+\phi_1^{''}(t)\phi_3^{'}(t)\phi_4(t)\phi_2(x)+\phi_1(t)\phi_2^{'}(t)\phi_4^{''}(t)\phi_3(x)\\
&+\phi_1^{''}(t)\phi_2(t)\phi_4^{'}(t)\phi_3(x)+\phi_1^{'}(t)\phi_2^{''}(t)\phi_4(t)\phi_3(x)-\phi_1^{'}(t)\phi_2(t)\phi_4^{''}(t)\phi_3(x)\\
&-\phi_1(t)\phi_2^{''}(t)\phi_4^{'}(t)\phi_3(x)-\phi_1^{''}(t)\phi_2^{'}(t)\phi_4(t)\phi_3(x)-\phi_1(t)\phi_2^{'}(t)\phi_3^{''}(t)\phi_4(x)\\
&-\phi_1^{''}(t)\phi_2(t)\phi_3^{'}(t)\phi_4(x)-\phi_1^{'}(t)\phi_2^{''}(t)\phi_3(t)\phi_4(x)+\phi_1^{'}(t)\phi_2(t)\phi_3^{''}(t)\phi_4(x)\\
&+\phi_1(t)\phi_2^{''}(t)\phi_3^{'}(t)\phi_4(x)+\phi_1^{''}(t)\phi_2^{'}(t)\phi_3(t)\phi_4(x)]r(t)\psi(t)\,dt. \qquad (11)\end{aligned}$$

Then by hypothesis there are constants C, k_0, k_1, k_2 such that $|r(x)| \le C$ in $[0,\infty)$, and $\int_0^\infty |\phi_j(x)|^4 dx \ \le k_0,\ \int_0^\infty \left|\phi_j'(x)\right|^4 dx \ \le k_1,\ \int_0^\infty \left|\phi_j''(x)\right|^4 dx \ \le k_2$ for $j = 1,2,3,4$.

Now, applying the Holder's inequality for integral, we get

$$\begin{aligned}
&|\int_0^x [\phi_2(t)\phi_3'(t)\phi_4''(t)\phi_1(x) + \phi_2''(t)\phi_3(t)\phi_4'(t)\phi_1(x) + \phi_2'(t)\phi_3''(t)\phi_4(t)\phi_1(x) \\
&-\phi_2'(t)\phi_3(t)\phi_4''(t)\phi_1(x) - \phi_2(t)\phi_3''(t)\phi_4'(t)\phi_1(x) - \phi_2''(t)\phi_3'(t)\phi_4(t)\phi_1(x) \\
&-\phi_1(t)\phi_3'(t)\phi_4''(t)\phi_2(x) - \phi_1''(t)\phi_3(t)\phi_4'(t)\phi_2(x) - \phi_1'(t)\phi_3''(t)\phi_4(t)\phi_2(x) \\
&+\phi_1'(t)\phi_3(t)\phi_4''(t)\phi_2(x) + \phi_1(t)\phi_3''(t)\phi_4'(t)\phi_2(x) + \phi_1''(t)\phi_3'(t)\phi_4(t)\phi_2(x) \\
&+\phi_1(t)\phi_2'(t)\phi_4''(t)\phi_3(x) + \phi_1''(t)\phi_2(t)\phi_4'(t)\phi_3(x) + \phi_1'(t)\phi_2''(t)\phi_4(t)\phi_3(x) \\
&-\phi_1'(t)\phi_2(t)\phi_4''(t)\phi_3(x) - \phi_1(t)\phi_2''(t)\phi_4'(t)\phi_3(x) - \phi_1''(t)\phi_2'(t)\phi_4(t)\phi_3(x) \\
&-\phi_1(t)\phi_2'(t)\phi_3''(t)\phi_4(x) - \phi_1''(t)\phi_2(t)\phi_3'(t)\phi_4(x) - \phi_1'(t)\phi_2''(t)\phi_3(t)\phi_4(x) \\
&+\phi_1'(t)\phi_2(t)\phi_3''(t)\phi_4(x) + \phi_1(t)\phi_2''(t)\phi_3'(t)\phi_4(x) + \phi_1''(t)\phi_2'(t)\phi_3(t)\phi_4(x)] \\
&\times r(t)\psi(t)\,dt|
\end{aligned}$$

$$\begin{aligned}
\le\ & 6C|\phi_1(x)|(k_0k_1k_2)^{\frac14}\left[\int_0^x |\psi(t)|^4 dt\right]^{\frac14} + 6C|\phi_2(x)|(k_0k_1k_2)^{\frac14}\left[\int_0^x |\psi(t)|^4 dt\right]^{\frac14} \\
& +6C|\phi_3(x)|(k_0k_1k_2)^{\frac14}\left[\int_0^x |\psi(t)|^4 dt\right]^{\frac14} + 6C|\phi_4(x)|(k_0k_1k_2)^{\frac14}\left[\int_0^x |\psi(t)|^4 dt\right]^{\frac14} \\
=\ & 6C(k_0k_1k_2)^{\frac14}(|\phi_1(x)| + |\phi_2(x)| + |\phi_3(x)| + |\phi_4(x)|)\,\Psi^{\frac14}(x).
\end{aligned}$$

Now, from the equation (11) it follows that

$$\begin{aligned}
&|\psi(x)| \le |c_1|\ |\phi_1(x)| + |c_2|\ |\phi_2(x)| + |c_3|\ |\phi_3(x)| + |c_4|\ |\phi_4(x)| \\
&+6C(k_0k_1k_2)^{\frac14}(|\phi_1(x)| + |\phi_2(x)| + |\phi_3(x)| + |\phi_4(x)|)\Psi^{\frac14}(x).
\end{aligned}$$

Then

$$\begin{aligned}
|\psi(x)|^4 \ \le\ & (|c_1|\ |\phi_1(x)| + |c_2|\ |\phi_2(x)| + |c_3|\ |\phi_3(x)| + |c_4|\ |\phi_4(x)| \\
& +6C(k_0k_1k_2)^{\frac14}(|\phi_1(x)| + |\phi_2(x)| + |\phi_3(x)| + |\phi_4(x)|)\,\Psi^{\frac14}(x))^4. \qquad (12)
\end{aligned}$$

Using the elementary inequality for any two real numbers x, y

$$(x+y)^4 \le 8\left(x^4 + y^4\right)$$

and equation (12), we get

$$\begin{aligned}
|\psi(x)|^4 \ \le\ & (|c_1|\ |\phi_1(x)| + |c_2|\ |\phi_2(x)| + |c_3|\ |\phi_3(x)| + |c_4|\ |\phi_4(x)|)^4 \\
& +1296C^4(k_0k_1k_2)(|\phi_1(x)| + |\phi_2(x)| + |\phi_3(x)| + |\phi_4(x)|)^4\Psi(x). \qquad (13)
\end{aligned}$$

Using the elementary inequality for any real numbers a, b, c, d

$$(a+b+c+d)^4 \le 64\left(a^4+b^4+c^4+d^4\right)$$

and equation (13), we obtain

$$\begin{aligned}
|\psi(x)|^4 \ \le\ & 64\left(|c_1|^4\ |\phi_1(x)|^4 + |c_2|^4\ |\phi_2(x)|^4 + |c_3|^4\ |\phi_3(x)|^4 + |c_4|^4\ |\phi_4(x)|^4\right) \\
& +82944C^4(k_0k_1k_2)\left(\ |\phi_1(x)|^4 + \ |\phi_2(x)|^4 + \ |\phi_3(x)|^4 + \ |\phi_4(x)|^4\right) \\
& \times\Psi(x). \qquad (14)
\end{aligned}$$

Integrating (14) over $[0, X]$, we can write

$$\begin{aligned}\int_0^X |\psi(x)|^4 dx &\le 64(|c_1|^4 \int_0^X |\phi_1(x)|^4 dx + |c_2|^4 \int_0^X |\phi_2(x)|^4 dx \\ &\quad + |c_3|^4 \int_0^X |\phi_3(x)|^4 dx \ + |c_4|^4 \int_0^X |\phi_4(x)|^4 dx) \\ &\quad + 82944 C^4 (k_0 k_1 k_2) \int_0^X (\ |\phi_1(x)|^4 + \ |\phi_2(x)|^4 \\ &\quad + |\phi_3(x)|^4 + \ |\phi_4(x)|^4) \Psi(x)\, dx \\ &\le 64 k_0 \left(|c_1|^4 \ + |c_2|^4 \ + |c_3|^4 \ + |c_4|^4 \ \right) \\ &\quad + 82944 C^4 (k_0 k_1 k_2) \int_0^X (\ |\phi_1(x)|^4 + \ |\phi_2(x)|^4 \\ &\quad + |\phi_3(x)|^4 + \ |\phi_4(x)|^4) \Psi(x)\, dx.\end{aligned}$$

That means

$$\begin{aligned}&\Psi(x) \le 64 k_0 \left(|c_1|^4 \ + |c_2|^4 \ + |c_3|^4 \ + |c_4|^4 \ \right) \\ &+ 82944 C^4 (k_0 k_1 k_2) \int_0^X \left(\ |\phi_1(x)|^4 + \ |\phi_2(x)|^4 + \ |\phi_3(x)|^4 + \ |\phi_4(x)|^4 \right) \Psi(x)\, dx.\end{aligned}$$

Then, using the Gronwall's Inequality, we obtain

$$\begin{aligned}\Psi(X) &\le 64 k_0 \left(|c_1|^4 \ + |c_2|^4 \ + |c_3|^4 \ + |c_4|^4 \ \right) \\ &\quad \times e^{82944 C^4 (k_0 k_1 k_2) \int_0^X \left(\ |\phi_1(x)|^4 + \ |\phi_2(x)|^4 + \ |\phi_3(x)|^4 + \ |\phi_4(x)|^4 \right) dx}.\end{aligned}$$

This means that $\Psi(x)$ is a bounded as $X \to \infty$. Thus we get $\psi(x) \in L^4(0, \infty)$.

Corollary 1. Let λ be a complex parameter and there be a value λ_0 such that all solution and their derivatives up to order three of the equation

$$y^{(4)} + \{\lambda - Q(x)\} y(x) = 0 \tag{15}$$

are in $L^4(0, \infty)$ when $\lambda = \lambda_0$. Then all solutions of the equation are in $L^4(0, \infty)$ for every λ.

Proof. We can write

$$\lambda - Q(x) = \lambda_0 + Q(x) + (\lambda - \lambda_0).$$

Then the differential equation has the following form

$$y^{(4)} + \{\lambda_0 - Q(x) + (\lambda - \lambda_0)\} y(x) = 0.$$

Comparing with (1) and (3), we obtain $q(x) = \lambda_0 - Q(x)$ and $r(x) = \lambda - \lambda_0$. This means that $r(x)$ is a constant function which is bounded in $[0, \infty)$. Then by using the Theorem 3 we obtain that all solutions of (15) are in $L^4(0, \infty)$ for every λ.

Conclusion

In the present paper, we study some properties of a general linear differential equation of fourth order in infinite interval of the form: $y^{(4)}(x) + \{q(x) + r(x)\} y(x) = 0, \qquad 0 \le x < \infty$, where $r(x)$ is a function which satisfies the condition: $\int_0^\infty |r(x)|\, dx < \infty$. A simple application of this result is provided.

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Карван Х.Ф. Жвемер, Рандо Р.К. Расул

Төртінші ретті сызықты дифференциалдық теңдеуді оның шеттік есебімен салыстыру

Мақалада төртінші ретті сызықты дифференциалдық теңдеу қарастырылған. Авторлар бұл дифференциалдық теңдеудің жоғарғы бағамын, сонымен қатар барлық шешімі $L^4(0, \infty)$ табылатындығын дәлелдеген. Алынған нәтижелерді салыстыра келе, осы дифференциалдық теңдеуден туындаған шеттік есептің барлық меншікті функциялары шектелген және $L^4(0, \infty)$ орналасқан болып табылады.

Кілт сөздер: сызықты дифференциалдық теңдеу, меншікті мән, меншікті функция, жоғарғы бағамы, сызықты тәуелсіз шешімі, $L^2(0, \ infty)$, вронскиан, Гронуолла теңсіздігі, тұрақтыны вариациялау.

Карван Х.Ф. Жвемер, Рандо Р.К. Расул

Сравнение линейного дифференциального уравнения четвертого порядка с его краевой задачей

В статье изучено линейное дифференциальное уравнение четвертого порядка. Авторами найдена верхняя оценка для решений этого дифференциального уравнения, а также доказано, что все решения находятся в $L^4(0,\infty)$. Сравнивая эти результаты, авторы пришли к выводу, что все собственные функции краевой задачи, порожденные этим дифференциальным уравнением, ограничены и находятся в $L^4(0,\infty)$.

Ключевые слова: линейное дифференциальное уравнение, собственное значение, собственная функция, верхняя оценка, линейно независимое решение, $L^2(0,\ infty)$, вронскиан, неравенство Гронуолла, вариация постоянных (параметров).

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Existence and uniqueness of solutions for the system of integro-differential equations with three-point and nonlinear integral boundary conditions

The paper examines a system of nonlinear integro-differential equations with three-point and nonlinear integral boundary conditions. The original problem demonstrated to be equivalent to integral equations by using Green function. Theorems on the existence and uniqueness of a solution to the boundary value problems for the first order nonlinear system of integro- differential equations with three-point and nonlinear integral boundary conditions are proved. A proof of uniqueness theorem of the solution is obtained by Banach fixed point principle, and the existence theorem then follows from Schaefer's theorem.

Keywords: three-point boundary conditions, nonlinear integral boundary value problems, existence and uniqueness of solutions, fixed point theorems.

Introduction

Multipoint boundary value problems for ordinary differential equations play a crucial role in various applications. It is epitomized the fact that, given a dynamical system with n degrees of freedom, there may exist exactly n states detected at n different times. A mathematical description of such a system results in an n-point boundary value problem. Another source of multipoint problems is the discretization of certain boundary value problems for partial differential equations over irregular domains with the method of lines. Multipoint problems for ordinary differential equations are a particular class of interface problems, and hence solvable with different techniques [1–4].

Integro-differential equations are encountered in many engineering and scientific disciplines, the problems can be represented as continuum phenomena and can be described approximately to partial differential equations. Many forms of these equations are possible. Some of the applications are unsteady aerodynamics and aeroelastic phenomena, viscoelasticity, viscoelastic panel in supersonic gas flow, fluid dynamics, electrodynamics of complex medium, many models of population growth, polymer rheology, neural network modeling, sandwich system identification, materials with fading memory, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, heat conduction in materials with memory, theory of lossless transmission lines, theory of population dynamics, compartmental systems, nuclear reactors, and mathematical modeling of a hereditary phenomenon. For details, see [5–7] and the references therein. Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermos-elasticity, underground water flow, population dynamics, and so forth. For a detailed description of the integral boundary conditions, we refer the reader to a recent paper [8]. For more details of nonlocal and integral boundary conditions, see [9–16] and references therein.

In the last few decades, the study of differential equations with nonlocal boundary conditions has been an interesting subject of mathematics, that has recently received the most significant attention of researchers; the reader is referred to [17–27]. It has been proposed by several authors that existence results for boundary value problems may be useful in real world problems. (see e.g., [28–30] and the references therein)

Problem statement and preliminaries

In this section, we set up problem statement and lemmas which are used throughout this paper. We denote by $C([0,T],R^n)$ the Banach space of all continuous functions from $[0,T]$ into R^n with the norm $\|x\| = \max\{|x(t)| : t \in [0,T]\}$, where $|\cdot|$ is the norm in space R^n.

We consider the existence, uniqueness of the system of nonlinear differential equations of the type

$$\dot{x}(t) = f(t, x(t), (\chi x)(t)), \quad t \in [0,T], \tag{1}$$

subject to three-point and nonlinear integral boundary conditions

$$Ax(0) + Bx(t_1) + Cx(T) = \int_0^T q(x(t))\, dt, \tag{2}$$

where A, B, C are constant square matrices of order n such that $\det N \neq 0$, $N = A + B + C$; $f : [0,T] \times R^n \times R^n \to R^n$ $q : R^n \to R^n$, $g : [0,T] \times [0,T] \times R^n \to R^n$ are given functions, t_1 satisfies the condition $0 < t_1 < T$ and $(\chi x)(t) = \int_0^t g(t,s,x(s))\, ds$.

For simplicity, the problem can be interpreted as solving the following problem:

Lemma 1. Suppose $\mu \in C([0,T],R^n)$ and $\det N \neq 0$. Then the unique solution of the following problem

$$\dot{x}(t) = \mu(t), \qquad t \in [0,T] \tag{3}$$

with three-point boundary conditions

$$Ax(0) + Bx(t_1) + Cx(T) = \int_0^T \eta(s)\, ds, \tag{4}$$

is given by

$$x(t) = d + \int_0^T G(t,\tau)\mu(\tau) d\tau, \tag{5}$$

where

$$G(t,\tau) = \begin{cases} G_1(t,\tau), & t \in [0,t_1], \\ G_2(t,\tau), & t \in (t_1,T], \end{cases}$$

such that

$$G_1(t,\tau) = \begin{cases} N^{-1}A, & 0 \le \tau \le t, \\ -N^{-1}(B+C), & t < \tau \le t_1, \\ -N^{-1}C, & t_1 < \tau \le T, \end{cases}$$

and

$$G_2(t,\tau) = \begin{cases} N^{-1}A, & 0 \le \tau \le t_1, \\ N^{-1}(A+B), & t_1 < \tau \le t, \\ -N^{-1}C, & t < \tau \le T, \end{cases}$$

$$d = N^{-1} \int_0^T \eta(s)\, ds.$$

Proof. If function $x = x(\cdot)$ is a solution of the differential equation (1), then for $t \in (0,T)$,

$$x(t) = x_0 + \int_0^t \mu(\tau) d\tau, \tag{6}$$

where x_0 is an arbitrary constant vector. Now we define x_0 so that, the function in equality (6) satisfies the condition (4)

$$x_0 = d - N^{-1}B\int\limits_0^{t_1}\mu(t)dt - N^{-1}C\int\limits_0^{T}\mu(t)dt. \tag{7}$$

Now we put the value x_0 determined from the equality (7) in (6) and obtain

$$x(t) = d - N^{-1}B\int\limits_0^{t_1}\mu(t)dt - N^{-1}C\int\limits_0^{T}\mu(t)dt + \int\limits_0^{t}\mu(\tau)d\tau. \tag{8}$$

Assume that, $t \in [0, t_1]$. Then we can write the equality (8) as follows:

$$x(t) = d - N^{-1}B\left(\int\limits_0^{t}\mu(\tau)d\tau + \int\limits_t^{t_1}\mu(\tau)d\tau\right) - N^{-1}C\left(\int\limits_0^{t}\mu(\tau)d\tau + \int\limits_t^{t_1}\mu(\tau)d\tau\right)$$

$$-N^{-1}C\int\limits_{t_1}^{T}\mu(t)dt + \int\limits_0^{t}\mu(\tau)d\tau. \tag{9}$$

We get (9) combining similar terms, and using the common technique for simplifying:

$$x(t) = d + \left(E - N^{-1}B - N^{-1}C\right)\int\limits_0^{t}\mu(\tau)d\tau - \left(N^{-1}B + N^{-1}C\right)\int\limits_t^{t_1}\mu(\tau)d\tau$$

$$-N^{-1}C\int\limits_{t_1}^{T}\mu(t)dt = d + N^{-1}A\int\limits_0^{t}\mu\left(\tau\right)d\tau$$

$$-N^{-1}\left(B + C\right)\int\limits_t^{t_1}\mu(\tau)d\tau - N^{-1}C\int\limits_{t_1}^{T}\mu(t)dt, \tag{10}$$

where E is an identity matrix.

Define new function as follows:

$$G_1(t,\tau) = \begin{cases} N^{-1}A, & 0 \le \tau \le t, \\ -N^{-1}(B+C), & t < \tau \le t_1, \\ -N^{-1}C, & t_1 < \tau \le T. \end{cases}$$

Equality (10) can be rewritten as integral equation below:

$$x(t) = d + \int_0^T G_1(t,\tau)\mu(\tau)d\tau.$$

Now assume that, $t \in (t_1, T]$. Then we can write the equality (8) as follows:

$$x(t) = d - N^{-1}B\int\limits_0^{t_1}\mu(t)dt - N^{-1}C\int\limits_0^{t_1}\mu(t)dt - N^{-1}C\left(\int\limits_{t_1}^{t}\mu(\tau)d\tau + \int\limits_t^{T}\mu(\tau)d\tau\right)$$

$$+\int_0^{t_1}\mu(t)dt+\int_{t_1}^{t}\mu(\tau)d\tau=d+\left(E-N^{-1}B-N^{-1}C\right)\int_0^{t_1}\mu(t)dt+\left(E-N^{-1}C\right)\int_{t_1}^{t}\mu(\tau)d\tau$$

$$-N^{-1}C\int_t^T\mu(\tau)d\tau=d+N^{-1}A\int_0^{t_1}\mu(t)dt+N^{-1}\left(A+B\right)\int_{t_1}^{t}\mu(\tau)d\tau-N^{-1}C\int_t^T\mu(\tau)d\tau.$$

We establish a new function as follows:

$$G_2(t,\tau)=\begin{cases} N^{-1}A, & 0\le\tau\le t_1,\\ N^{-1}(A+B), & t_1<\tau\le t\,,\\ -N^{-1}C, & t<\tau\le T.\end{cases}$$

Hence, if $t\in(t_1,T]$, then we can write the equality (8) as follows:

$$x(t)=d+\int_0^T G_2(t,\tau)\mu(\tau)d\tau.$$

Thus, the solution of the boundary value problem (3)-(4) can be shown as follows:

$$x(t)=d+\int_0^T G(t,\tau)\mu(\tau)d\tau.$$

We showed that the argument given above is valid (5). Proof is completed.

Lemma 2. Assume that $f:[0,T]\times R^n\times R^n\to R^n$, $q:R^n\to R^n$ are given functions. Then the function $x(t)$ is a solution of the boundary value problem (1)-(2) if and only if $x(t)$ is a solution of the integral equation

$$x(t)=D+\int_0^T G(t,\tau)f(\tau,x(\tau),(\chi x)(\tau))d\tau, \tag{11}$$

where

$$D=N^{-1}\int_0^T q(x(t))dt.$$

Proof. Let $x(t)$ be a solution of the boundary value problem (1)-(2).Proving statements similar to Lemmas 1 this lemma can be derived. By checking directly we identify the solution of integral equation (11) satisfies the boundary value problem (1)-(2). Lemma 2 is proved.

Existence results

Let P be an operator such that, $P:C([0,T],R^n)\to C([0,T],R^n)$ as

$$(Px)(t)=N^{-1}\int_0^T q(x(t))\,dt+\int_0^T G(t,\tau)f(\tau,x(\tau),(\chi x)(\tau))d\tau.$$

It is evident that, the problem (1)-(2) is equivalent to the fixed point problem $x=Px$. Thus, the problem (1)-(2) has a solution if and only if the operator P has a fixed point.

In Lemma 1, we use the most basic fixed point theorem named the contraction mapping principle and it uses the assumptions:

H1) There exist constants M_1, M_2 such that

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le M_1 |x_1 - y_1| + M_2 |x_2 - y_2|$$

for each $t \in [0,T]$ and all $x_1, x_2, y_1, y_2 \in R^n$.
H2) There exists a constant M_3 such that

$$|g(t,s,x) - g(t,s,y)| \le M_3 |x - y|$$

for each $t, s \in [0,T]$ and all $x, y \in R^n$.
H3) There exists a constant M_4 such that

$$|q(x) - q(y)| \le M_4 |x - y|$$

for all $x, y \in R^n$.
Theorem 1. Assume that, the assumptions H1)-H3) hold, and

$$L = \left[S \left(M_1 T + \frac{M_2 M_3 T^2}{2} \right) + M_4 T \left\| N^{-1} \right\| \right] < 1, \tag{12}$$

then the boundary-value problem (1)-(2) has a unique solution on $[0,T]$, where

$$S = \max_{[0,T]\times[0,T]} \|G(t,\tau)\| .$$

Proof. Setting $\max\limits_{[0,T]} |f(t,0,0)| = M_f$, $\max\limits_{[0,T]} |q(0)| = m_q$ and choosing $r \ge \frac{M_f TS + m_q T \|N^{-1}\|}{1-L}$ we show that $PB_r \subset B_r$ where

$$B_r = \{x \in C([0,T]R^n) : \|x\| \le r\} .$$

For $x \in B_r$, we have

$$\|(Px)(t)\| \le \left\| N^{-1} \int_0^T (|q(x(t)) - q(0)| + |q(0)|)\, dt \right\|$$

$$+ \int_0^T |G(t,\tau)| \left(|f(\tau, x(\tau), (\chi x)(\tau)) - f(\tau,0,0)| + |f(\tau,0,0)| \right) d\tau$$

$$\le M_4 T \left\| N^{-1} \right\| \|x\| + m_q T \left\| N^{-1} \right\| + S \left(M_1 T + \frac{M_2 M_3 T^2}{2} \right) \|x\| + STM_f$$

$$\le \left[M_4 T \left\| N^{-1} \right\| + S \left(M_1 T + \frac{M_2 M_3 T^2}{2} \right) \right] r + STM_f + m_q T \left\| N^{-1} \right\| \le r.$$

Now for any $x, y \in B_r$ we have

$$|(Px)(t) - (Py)(t)| \le \int_0^T \left| G(t,\tau) \left(f(\tau, x(\tau), \int_0^t g(t,s,x(s))\, ds) - f(\tau, y(\tau), \int_0^t g(t,s,y(s))\, ds \right) \right| d\tau$$
$$+ \left| N^{-1} \int_0^T (q(x(t)) - q(y(t)))\, dt \right|$$
$$\le S \int_0^T \left\{ M_1 |x(t) - y(t)| + M_2 \left| \int_0^t g(t,s,x(s))\, ds - \int_0^t g(t,s,y(s))\, ds \right| \right\} dt$$
$$+ M_4 \left\| N^{-1} \right\| \int_0^T |x(t) - y(t)|\, dt \le \left[S \left(M_1 T + \frac{M_2 M_3 T^2}{2} \right) + M_4 T \left\| N^{-1} \right\| \right] \|x - y\| ,$$

or

$$\|Px - Py\| \leq L\|x - y\|.$$

It is seen that, P is contraction by condition (12). So, the boundary-value problem (1)-(2) has a unique solution.

Theorem 2 (Schafer's fixed point theorem). Let X be a Banach space. Assume that, $G : X \to X$ is a completely continuous operator and the set $\rho = \{x \in X | x = \beta Gx,\ 0 < \beta < 1\}$ is bounded. Then G has a fixed point in X.

Now we apply Schafer's fixed point theorem and it uses the following assumption:

Theorem 3. Assume that the functions $f : [0,T] \times R^n \times R^n \to R^n$ and $q : R^n \to R^n$ are continuous and there exist functions $\rho, \lambda \in C([0,T], R^+)$ such that $|f(t, x(t), (\chi x)(t))| \leq \rho(t)$, $|q(x(t))| \leq \lambda(t)$, $\forall t \in [0,T]$, $x \in C([0,T], R^n)$ and with $\sup\limits_{t\in[0,T]} |\rho(t)| = \|\rho\|$, $\sup\limits_{t\in[0,T]} |\lambda(t)| = \|\lambda\|$. Then the boundary value problem (1)-(2) has at least one solution on $[0,T]$.

Proof. Let P be the operator defined in (12). We use Schaefer's fixed point theorem to prove that P has a fixed point. The proof will be given in several steps.

Step 1: Here we prove that P is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C([0,T]; R^n)$. Then, for each $t \in [0,T]$

$$|(Px)(t) - (Px_n)(t)|$$
$$= \left| N^{-1} \int\limits_0^T (q(x(t)) - q(x_n(t)))\,dt + \int\limits_0^T G(t,\tau)\left(f(\tau, x(\tau), (\chi x)(\tau)) - f(\tau, x_n(\tau), (\chi x_n)(\tau))\right) d\tau \right|$$

$$\leq \left[S\left(M_1 T + \frac{M_2 M_3 T^2}{2} \right) + M_4 T \left\| N^{-1} \right\| \right] |x(t) - x_n(t)| \leq L\|x - x_n\|.$$

From here we get $\|(Px)(t) - (Px_n)(t)\| \to 0$ as $n \to \infty$, which implies that the operator P is continuous.

Step 2: P maps bounded sets into bounded sets in $C([0,T]; R^n)$. Indeed, it is enough to show that for any $\eta > 0$ there exists a positive constant ω such that for each $x \in B_\eta = \{x \in C([0,T]; R^n) : \|x\| \leq \eta\}$ we have $\|P(x)\| \leq \omega$. We have for each $t \in [0,T]$

$$|(Px)(t)| \leq T\left\|N^{-1}\right\| \|\lambda\| + TS\|\rho\|.$$

This implies that

$$\|(Px)(t)\| \leq T\left\|N^{-1}\right\| \|\lambda\| + TS\|\rho\|.$$

*Step 3.*The operator P maps bounded sets into equicontinuous sets of $C([0,T], R^n)$. Let $\tau_1, \tau_2 \in [0,T]$, $\tau_1 < \tau_2$, B_η be a bounded set of $C([0,T]; R^n)$ as in Step 2, and let $x \in B_\eta$.

Case 1. Let be $\tau_1, \tau_2 \in [0, t_1]$. Then,

$$(Px)(\tau_2) - (Px)(\tau_1) = N^{-1} A \int\limits_0^{\tau_2} f(\tau, x(\tau), (\chi x)(\tau))\,d\tau$$

$$-N^{-1}(B + C) \int\limits_{\tau_2}^{t_1} f(\tau, x(\tau), (\chi x)(\tau))\,d\tau - N^{-1} A \int\limits_0^{\tau_1} f(\tau, x(\tau), (\chi x)(\tau))\,d\tau$$

$$+N^{-1}(B + C) \int\limits_{\tau_1}^{t_1} f(\tau, x(\tau), (\chi x)(\tau))\,d\tau = \int\limits_{\tau_1}^{\tau_2} f(\tau, x(\tau), (\chi x)(\tau))\,d\tau.$$

Case 2: Let be $\tau_1 \in [0, t_1]$ and $\tau_2 \in (t_1, T]$. Then

$$(Px)(\tau_2) - (Px)(\tau_1) = N^{-1}A\int_0^{t_1} f(\tau, x(\tau), (\chi x)(\tau))\,d\tau$$

$$+N^{-1}(A+B)\int_{t_1}^{\tau_2} f(\tau, x(\tau), (\chi x)(\tau))\,d\tau - N^{-1}C\int_{\tau_2}^{T} f(\tau, x(\tau), (\chi x)(\tau))\,d\tau$$

$$-N^{-1}A\int_{\tau_1}^{t_1} f(\tau, x(\tau), (\chi x)(\tau))\,d\tau + N^{-1}(B+C)\int_{\tau_1}^{t_1} f(\tau, x(\tau), (\chi x)(\tau))\,d\tau$$

$$+N^{-1}C\int_{t_1}^{T} f(\tau, x(\tau), (\chi x)(\tau))\,d\tau = \int_{\tau_1}^{\tau_2} f(\tau, x(\tau), (\chi x)(\tau))\,d\tau.$$

Apparently, in both cases

$$|(Px)(\tau_2) - (Px)(\tau_1)| \leq \int_{\tau_1}^{\tau_2} |f(\tau, x(\tau), (\chi x)(\tau))|\,d\tau.$$

As $\tau_2 \to \tau_1$, the right hand side of the preceding inequality tends to zero. Taking into account that the mapping P is continuous and equivalently continuous, we conclude that the mapping $P : C([0,T], R^n) \to C([0,T], R^n)$ is completely continuous by Arzela-Ascoli theorem.
Step 4. We show that a set $\Omega = \{x \in C([0,T], R^n) : x = \lambda P(x),\ \text{for some}\ 0 < \lambda < 1\}$ is bounded. Assume that, $x = \lambda P(x)$ for some $0 < \lambda < 1$. Then for each $t \in [0,T]$, we can write

$$x(t) = \lambda N^{-1}\int_0^T q(x(t))\,dt + \lambda\int_0^T G(t,\tau) f(\tau, x(\tau), (\chi x)(\tau))d\tau.$$

From here we get

$$\|x\| \leq T\left\|N^{-1}\right\|\|\lambda\| + TS\|\rho\|.$$

Therefore, the set Ω is bounded. The conclusion of Theorem 2 applies and the operator P has at least one fixed point. So, there exists at least one solution for the problem (1)-(2) on $[0,T]$.

Example

Consider the following system of integro-differential equation

$$\begin{cases} \dot{x}_1 = \sin\alpha\, x_2, \\ \dot{x}_2 = \cos\left(\beta\int_0^t \frac{\sin(\gamma x_1)}{1+t^2}dt\right), \end{cases} \tag{A}$$

subject to

$$\begin{cases} x_1(0) + x_2(0) - x_2(\frac{1}{2}) = 1, \\ -x_1(\frac{1}{2}) + x_1(1) + x_2(1) = \int_0^1 \cos\delta x_2(t)\,dt. \end{cases} \tag{B}$$

Evidently,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

$$A + B + C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $t \in \left[0, \frac{1}{2}\right]$, we obtain

$$G_1(t,\tau) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, & 0 \le \tau \le t, \\ \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, & t < \tau \le \frac{1}{2}, \\ \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, & \frac{1}{2} < \tau \le 1, \end{cases}$$

and for $t \in \left(\frac{1}{2}, 1\right]$

$$G_2(t,\tau) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, & 0 \le \tau \le \frac{1}{2}, \\ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, & \frac{1}{2} < \tau \le t, \\ \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, & t < \tau \le 1. \end{cases}$$

Obviously, $M_1 = |\alpha|$, $M_2 = |\beta|$, $M_3 = |\gamma|$, $M_4 = |\delta|$ and $\|S\| \le 2$. If $L = 2\left(|\alpha| + \frac{|\beta||\gamma|}{2}\right) + |\delta| < 1$, then boundary value problem (A)-(B) has a unique solution.

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Үш нүктелі және интегралдық шеттік шарттарымен берілген сызықты емес интегралды-дифференциалдық теңдеулер жүйесінің шешімінің бар болуы және жалғыздығы

Мақалада үш нүктелі және интегралдық шекаралық шарттарымен берілген сызықты емес интегралды-дифференциалдық теңдеулер жүйесі зерттелген. Бастапқыда, Грин функциясы арқылы эквиваленттті интегралдық теңдеуге алып келді. Кейіннен, қозғалмайтын нүктелер туралы теореманы қолдана отырып, шеттік есептің шешімінің бар болуы және жалғыздығының жеткілікті шарты алынды. Шешімінің жалғыздығы теоремасының дәлелдемесі қозғалмайтын нүкте туралы Банах принципі бойынша алынды, содан кейін бар болуы теоремасы Шефер теоремасынан шығады.

Кілт сөздер: үш нүктелі шекаралық шарттар, сызықты емес интегралдық шеттік есептер, шешімінің бар болуы және жалғыздығы, қозғалмайтын нүкте туралы теорема.

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Существование и единственность решений для систем интегро-дифференциальных уравнений с трёхточечными и нелинейными интегральными краевыми условиями

В статье исследована система нелинейных интегро-дифференциальных уравнений с трехточечными и интегральными граничными условиями. Сначала с помощью функции Грина она приведена к эквивалентному интегральному уравнению. Далее, с использованием теоремы о неподвижных точках, получены достаточные условия существования и единственности решения краевой задачи. Доказательство теоремы единственности решения получено по принципу Банаха о неподвижной точке, а затем теорема существования следует из теоремы Шефера.

Ключевые слова: трехточечные граничные условия, нелинейные интегральные краевые задачи, существование и единственность решений, теоремы о неподвижной точке.

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Nonlocal boundary value problem with Poisson's operator on a rectangle and its difference interpretation

In the present paper, differential and difference variants of nonlocal boundary value problem (NLBVP) for Poisson's equation in open rectangular domain are studied. The existence, uniqueness and a priori estimate of classical solution are established. The second order of accuracy difference scheme is presented. The applications with weighted integral condition are provided in differential and difference variants.

Keywords: Poisson's operator, nonlocal boundary value problem, rectangle, difference scheme.

Introduction

Firstly, NLBVP for Laplace's equation in a rectangular domain was considered by A.V. Bitsadze and A.A. Samarskii [1]. Later, the n-dimensional problem was studied by A. L. Skubachevskii [2].

V. A. Il'in and E. I. Moiseev [3] studied 2-d NLBVP with Poisson's operator on rectangle Π

$$\begin{cases} \Delta u = f(x,y),\ (x,y) \in \Pi = (0,1)\times(0,\pi), \\ u(x,0) = u(x,\pi) = u(0,y) = 0,\ u(1,y) = \sum\limits_{k=1}^{m} \alpha_k u(\xi_k, y),\ x\in[0,1],\ y\in[0,\pi],\ \xi_k \in (0,1) \end{cases}$$

and proved the existence and uniqness of classical solution when $\sum\limits_{k=1}^{m} \frac{1}{2}(\alpha_k + |\alpha_k|) \leq 1$, established a priori estimate $||u||_{W_2^2(\Pi)} \leq C||f||_{L_2(\Pi)}$ when $-\infty < \sum\limits_{k=1}^{m} \alpha_k \leq 1$ and if all $\alpha_k,\ k=\overline{1,m}$ have the same sign and given this condition offered the second order of accuracy difference scheme on a uniform grid.

In [4], E. A. Volkov demonstrated a simple proof of the existence and uniqueness of classical solution for Laplace's equation with the original Bitsadze-Samarskii nonlocal boundary value condition (NLBVC), proposed a finite-difference method on a square mesh that produces a uniform approximation by the second order of accuracy in the difference metric C, applied the method to Poisson's equation $\Delta u = g$ when $g \in C^{2,\lambda}$ for $0 < \lambda < 1$. In [5], he studied a solvability of the multilevel NLBVP for Poisson's operator on rectangular domain by applying the contraction mapping principle.

In [6], A. Ashyralyev established well-posedness of NLBVP in the open square $\Omega = (0,1)\times(0,1)$ by proving the coercive inequalities for solution of the differential problem

$$u_{tt}(t,x) + a(x)u_{xx}(t,x) - \delta u(t,x) = f(t,x) \text{ in } \Omega\ ,\ u(0,x) = u(t,0) = u(t,1) = 0,\ u(1,x) = u(\lambda,x) \text{ in } \overline{\Omega}\ ,$$

when smooth functions $a(x)$ and $f(t,x)$ satisfy the conditions

$$a(x) \geq 0\ ,\ f(0,x) = 0\ ,\ f(1,x) = f(\lambda,x)\ ,\ 0 \leq x \leq 1\ ,\ 0 \leq \lambda < 1\ ,$$

where $\delta > 0$ is sufficiently large number. In Ω, under the condition $\int\limits_0^1 |\rho(t)|dt < 1$, E. Ozturk [7] studied well-posedness of NLBVP for elliptic equation with integral type of NLBVC (in $\overline{\Omega}$) by reaching the coercive inequalities for solution of the problem

$$u_{tt}(t,x) + (a(x)u_x(t,x))_x = f(t,x),\ u(t,0) = u(t,1) = 0,\ u(0,x) = \varphi(x),$$

$$u(1,x)=\int\limits_0^1 \rho(t)u(t,x)dt+\psi(x)$$

and offered the first order of accuracy difference scheme against the term $\sum\limits_{j=1}^{N}|\rho(t_j)\tau|<1,\ \tau=1/N.$

By returning to Laplace's operator on rectangular domain we note, that various numerical methods on multilevel and integral type of NLBVPs were researched in [8–11] and other papers.

In the present paper, we generalize and prove the statements of the preliminary abstract [19] and, additionally, apply our results to NLBVP with integral conditions. We study the problem

$$\begin{cases} \Delta u(x,y)=f(x,y),\ (x,y)\in\Pi, \\ u(x,0)=u(x,\pi)=u(0,y)=0,\ u(1,y)=\sum\limits_{r=1}^{n}\alpha_r u(\zeta_r,y)-\sum\limits_{s=1}^{m}\beta_s u(\eta_s,y)=0,\ x\in[0,1],\ y\in[0,\pi], \end{cases}$$

where $f\in C(\overline{\Pi})$, $\alpha_r>0$, $\beta_s>0$, $0<\zeta_1<...<\zeta_n<1$ and $0<\eta_1<...<\eta_m<1$, $-\infty<\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s\le 1$ when $\zeta_n<\eta_1$; $\sum\limits_{r=1}^{n}\alpha_r\le 1$ when $\zeta_n\ge\eta_1$. We prove the existence, uniqueness and a priori estimate $||u||_{W_2^2(\Pi)}\le C||f||_{L_2(\Pi)}$ of the classical solution. Particularly, we consider the problem when $n=m$ and $\zeta_r<\eta_r$, $r=\overline{1,n}$ and for this special subcase we prove the existence, uniqueness and a priori estimate when $\sum\limits_{r=1}^{n}\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2}\le 1$. We offer the finite difference variants on a uniform grid and prove the second order of accuracy in terms of $h=\sqrt{h_1^2+h_2^2}$ for $h_1\le c_0h_2$, $h_2\to 0$ in respect of each difference metrics C and W_2^2.

As an application, we study NLBVP for Poisson's equation with weighted integral condtion (WIC)

$$\begin{cases} \Delta u(x,y)=f(x,y),\ (x,y)\in\Pi, \\ u(x,0)=u(x,\pi)=u(0,y)=0,\ u(1,y)=\int\limits_0^1\rho(x)u(x,y)dx=0,\ 0\le x\le 1,\ 0\le y\le\pi \end{cases}$$

respectively the behavior of $\rho(x)$, $\rho(x)\in C^0[\tau_0,\tau_1]$, i.e., $[\tau_0,\tau_1]\subset(0,1)$, $\rho(x)\equiv 0$ in $[0,1]\setminus[\tau_0,\tau_1]$. We prove the existence, uniqueness and a priori estimate under the conditions on $\rho(x)$ subject to whether or no the weight function changes the sign, whether or no the sign changing acts from plus to minus or vice verca, whether or no the number of sign changes is an even or odd. Particularly, when $\rho(x)$ does not change the sign and $-\infty<\int\limits_{\tau_0}^{\tau_1}\rho(x)dx\le 1$, we prove the existence, uniqueness, a priori estimate and offer the second order of accuracy difference sheme.

Differential problem

We consider NLBVP in the rectangle $\Pi=(0<x<1)\times(0<y<\pi)$

$$\begin{cases} \Delta u(x,y)=f(x,y),\ (x,y)\in\Pi, \\ u(x,0)=u(x,\pi)=0,\ 0\le x<1,\ \ u(0,y)=0,\ \ell[u](y)=0,\ 0\le y\le\pi, \end{cases} \tag{1}$$

where

$$\ell[u](y)\equiv u(1,y)-\sum_{r=1}^{n}\alpha_r u(\zeta_r,y)+\sum_{s=1}^{m}\beta_s u(\eta_s,y)\ , \tag{2}$$

$0<\zeta_1<...<\zeta_n<1,\ 0<\eta_1<...<\eta_m<1,\ \zeta_r\ne\eta_s,\ \alpha_r>0,\ \beta_s>0,\ r=\overline{1,n},\ s=\overline{1,m}$. We study the classical solution $u(x,y)\in C^2(\Pi)\cap C(\overline{\Pi})$ that satisfies the equation and all conditions of (1).

Further, on default, the symbol ***A1*** denotes the term: $-\infty<\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s\le 1$ holds when $\zeta_n<\eta_1$.

The symbol ***A2*** denotes: $\sum\limits_{r=1}^{n}\alpha_r\le 1$ holds when $\zeta_n\ge\eta_1$. The ***A*** denotes that ***A1*** holds or ***A2*** holds.

Theorem 1. Let $f(x,y) \in C(\overline{\Pi})$. If $\boldsymbol{A}$ holds, then classical solution of (1) exists and it is an unique.

Proof. Assume that classical solution of (1) exists. To prove the uniqueness it is sufficiently to show that $u(x,y) \equiv 0$ if $f(x,y) \equiv 0$. Put $f(x,y) \equiv 0$ in $\overline{\Pi}$. Then $u(x,y)$ is the solution of Laplace's equation, therefore, for each natural number $k \in N$ the function

$$X_k(x) = \sqrt{2/\pi} \int_0^\pi u(x,y)\sin(ky)dy \tag{3}$$

satisfies the equation $X_k''(x) - k^2 X_k(x) = 0, \ \ 0 < x < 1$. Moreover, since $u(0,y) = \ell[u](y) = 0$, then

$$X_k(0) = 0, \ \ X_k(1) = \sum_{r=1}^{n} \alpha_r X_k(\zeta_r) - \sum_{s=1}^{m} \beta_s X_k(\eta_s).$$

Hence, $X_k(x)$ is the solution of the multipoint problem

$$X_k''(x) - k^2 X_k(x) = 0, \ \ 0 < x < 1, \quad X_k(0) = 0, \ \ \ell[X_k] = 0, \tag{4}$$

where $\ell[X_k] = X_k(1) - \sum\limits_{r=1}^{n} \alpha_r X_k(\zeta_r) + \sum\limits_{s=1}^{m} \beta_s X_k(\eta_s)$. By virtue of mean value (MV) property [12, p. 1198-1199] (see also [13, 18, 20]) we get that solution of (4) satisfies the problem[1] [17, p. 92-93]

$$X_k''(x) - k^2 X_k(x) = 0, \ 0 < x < 1, \quad X_k(0) = 0, \ X_k(1) = \alpha X_k(\zeta_{[k]}) - \beta X_k(\eta_{[k]}) \ , \tag{5}$$

where [2] $\alpha = \sum\limits_{r=1}^{n} \alpha_r, \ \beta = \sum\limits_{s=1}^{m} \beta_s, \ \zeta_{[k]} \in [\zeta_1, \zeta_n], \ \eta_{[k]} \in [\eta_1, \eta_m]$ and $\zeta_{[k]} < \eta_{[k]}$ when $\zeta_n < \eta_1$. By virtue of [16, p. 1298-1299] we conclude that (5) has only trivial solution since $\boldsymbol{A}$ holds, i.e., $X_k(x) \equiv 0$ in the interval $[0,1]$. Hence, from (3), using the completeness of orthonormal system $\{\sqrt{2/\pi}\sin(ky), \ k \in N\}$ on the interval $0 \leq y \leq \pi$, we result $u(x,y) \equiv 0$ in $\overline{\Pi}$. Since the uniqueness is proved, then the existence follows from Fredholm's property [2] inherent (1). Theorem 1 is proved.

Theorem 2. Let $f \in C(\overline{\Pi})$. If $\boldsymbol{A}$ holds, then for classical solution of (1) a priori estimate holds

$$||u||_{W_2^2(\Pi)} \leq C||f||_{L_2(\Pi)}. \tag{6}$$

Proof. To prove (6) it is sufficiently to establish the estimates

$$||X_k||_{L_2[0,1]} \leq \frac{C_1}{k^2}||f_k||_{L_2[0,1]}, \ \ ||X_k'||_{L_2[0,1]} \leq \frac{C_2}{k}||f_k||_{L_2[0,1]}, \ \ ||X_k''||_{L_2[0,1]} \leq C_3||f_k||_{L_2[0,1]} \tag{7}$$

for $k \in N$, where

$$f_k(x) = \sqrt{2/\pi} \int_0^\pi f(x,y)\sin(ky)dy \ , \tag{8}$$

so that (7) [3, p. 142-143] results in

$$||u||_{W_2^2(\Pi)} \leq C_1||f||_{L_2(\Pi)}, \ \ ||u_{xx}||_{W_2^2(\Pi)} \leq C_2||f||_{L_2(\Pi)}, \ \ ||u_{xy}||_{W_2^2(\Pi)} \leq C_3||f||_{L_2(\Pi)}, \tag{9}$$

[1] Further in similar obstacles we will say, for example: the problem (4) is reducible to the problem (5), or the nonlocal condition (4) is reducible to the nonlocal condition (5), or we reduce (4) to (5).

[2] Further in this section the symbols α and β denote the sums $\alpha = \sum\limits_{r=1}^{n} \alpha_r$ and $\beta = \sum\limits_{s=1}^{m} \beta_s$.

and, after all, (9) results in (6). Hence, our target is to prove (7). Thereto, using (3) and (8) for equation $\Delta u(x,y)=f(x,y)$ and conditions $u(0,y)=0$, $u(1,y)=\sum\limits_{r=1}^{n}\alpha_r u(\zeta_r,y)-\sum\limits_{s=1}^{m}\beta_s u(\eta_s,y)$, we conclude that $X_k(x)$ satisfies the nonhomogeneous multipoint problem (this problem was studied in [16, 17])

$$X_k''(x)-k^2X_k(x)=f_k(x),\ 0<x<1,\ \ X_k(0)=0,\ X_k(1)=\sum_{r=1}^{n}\alpha_r X_k(\zeta_r)-\sum_{s=1}^{m}\beta_s X_k(\eta_s)\ . \tag{10}$$

Actually, the estimate

$$|\ X_k(1)\ |\ \le C\frac{\sqrt{2}}{k^{3/2}}\ ||f_k(x)||_{L_2[0,1]} \tag{11}$$

results in the estimates (7). Indeed, put $X_k(x)=\overline{X}_k(x)+\overline{\overline{X}}_k(x)$, where $\overline{X}_k(x)$ is the solution of

$$\overline{X}_k''(x)-k^2\overline{X}_k(x)=f_k(x),\ 0<x<1,\ \ \overline{X}_k(0)=\overline{X}_k(1)=0\ , \tag{12}$$

and $\overline{\overline{X}}_k(x)$ is the soluion of

$$\overline{\overline{X}}_k''(x)-k^2\overline{\overline{X}}_k(x)=0,\ 0<x<1,\ \ \overline{\overline{X}}_k(0)=0,\ \overline{\overline{X}}_k(1)=X_k(1)\ . \tag{13}$$

Thereby, it is sufficiently to show that the analog of (7) holds for each of the functions $\overline{X}_k(x)$ and $\overline{\overline{X}}_k(x)$. Thereto, we use the explicit solution of (13) to get

$$||\overline{\overline{X}}_k||_{L_2[0,1]}\le |\ X_k(1)\ |\ \Big(\frac{\int_0^1\sinh^2(kx)dx}{\sinh^2 k}\Big)^{1/2}\ , \tag{14}$$

$$||\overline{\overline{X}}_k'||_{L_2[0,1]}\le k\ |\ X_k(1)\ |\ \Big(\frac{\int_0^1\cosh^2(kx)dx}{\sinh^2 k}\Big)^{1/2}\ , \tag{15}$$

$$||\overline{\overline{X}}_k''||_{L_2[0,1]}\le k^2\ |\ X_k(1)\ |\ \Big(\frac{\int_0^1\sinh^2(kx)dx}{\sinh^2 k}\Big)^{1/2}\ , \tag{16}$$

and then, in view of $\frac{\int_0^1\sinh^2(kx)dx}{\sinh^2 k}\le\frac{1}{k}$ and $\frac{\int_0^1\cosh^2(kx)dx}{\sinh^2 k}\le\frac{5}{2k}$, from (14)-(16), we get

$$||\overline{\overline{X}}_k||_{L_2[0,1]}\le\frac{C\sqrt{2}}{k^2}||f_k||_{L_2[0,1]},\ \ ||\overline{\overline{X}}_k'||_{L_2[0,1]}\le\frac{C\sqrt{5}}{k}||f_k||_{L_2[0,1]},\ \ ||\overline{\overline{X}}_k''||_{L_2[0,1]}\le C\sqrt{2}||f_k||_{L_2[0,1]}. \tag{17}$$

It means that if (11) holds, then (7) holds for the function $\overline{\overline{X}}_k(x)$. Moreover, if (11) holds, then (7) holds for $\overline{X}_k(x)$ [3, p. 143-144]. Therefore, to establish (7) for $X_k(x)$ it is sufficiently to prove (11).

Let we prove (11). In view of [17, 92-93] the multipoint problem (10) is reducible to 3-point problem

$$X_k''(x)-k^2X_k(x)=f_k(x),\ 0<x<1,\ \ \ X_k(0)=0,\ \ X_k(1)=\alpha X_k(\zeta_{[k]})-\beta X_k(\eta_{[k]})\ , \tag{18}$$

where the points $\zeta_{[k]}\in[\zeta_1,\zeta_n]$, $\eta_{[k]}\in[\eta_1,\eta_m]$, so that $\zeta_{[k]}<\eta_{[k]}$ when $\zeta_n<\eta_1$. Therefore, it is sufficiently to obtain the estimate (11) for the solution of (18) when the term $\boldsymbol{A}$ holds.

Let ***A1*** holds, i.e., $-\infty<\alpha-\beta\le 1$ and $\zeta_n<\eta_1$. Put $sign\big(X_k(1)\ X_k(\eta_{[k]})\ X_k(\zeta_{[k]})\big)\ne 0$. We consider the alternate subcases: $sign\big(X_k(1)X_k(\eta_{[k]})\big)=-1$ and $sign\big(X_k(1)X_k(\eta_{[k]})\big)=1$. Note in advance, if $sign\big(X_k(1)\ X_k(\eta_{[k]})\ X_k(\zeta_{[k]})\big)=0$, then (11) results from the current proof.

Subcase 1.1 :

If $sign\big(X_k(1)X_k(\eta_{[k]})\big)=-1$, then in view of Bolzano theorem $X_k(\tau_k)=0$ for $\tau_k\in(\eta_{[k]},1)$. Then by virtue of [3, 143-144]

$$||\ X_k\ ||_{L_2[0,\tau_k]}\ \le\frac{1}{k^2}\ ||f_k||_{L_2[0,\tau_k]}\ ,\ \ ||\ X_k'\ ||_{L_2[0,\tau_k]}\ \le\frac{1}{k}\ ||f_k||_{L_2[0,\tau_k]}\ . \tag{19}$$

Since $X_k(0)=0$, then by virtue of Cauchy-Bunyakovskii inequality

$$X_k^2(\zeta_{[k]}) = |\int\limits_0^{\zeta_{[k]}} [X_k^2(x)]' dx| = 2\ |\int\limits_0^{\zeta_{[k]}} X_k(x)X_k'(x)dx| \le 2\ ||X_k||_{L_2[0,\zeta_{[k]}]}||X_k'||_{L_2[0,\zeta_{[k]}]}, \tag{20}$$

$$X_k^2(\eta_{[k]}) = |\int\limits_0^{\eta_{[k]}} [X_k^2(x)]' dx| = 2\ |\int\limits_0^{\eta_{[k]}} X_k(x)X_k'(x)dx| \le 2\ ||X_k||_{L_2[0,\eta_{[k]}]}||X_k'||_{L_2[0,\eta_{[k]}]}. \tag{21}$$

Using (19) in (20) and (21) we get

$$|\ X_k(\zeta_{[k]})\ |\ \le \frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ ,\ \ |\ X_k(\eta_{[k]})\ |\ \le \frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ . \tag{22}$$

Put $c_1 = \alpha+\beta$. From the 3-point condition (18), in view of (22), we obtain the desired estimate

$$|\ X_k(1)\ |\ \le c_1 \frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ . \tag{23}$$

Subcase 1.2 :
Let $sign\big(X_k(1)X_k(\eta_{[k]})\big) = 1$. Then $sign\big(X_k(1)X_k(\zeta_{[k]})\big) = 1$ in view of (18). By virtue of MV property [12, p. 1198-1199] we reduce the 3-point condition (18) to

$$X_k(0)=0,\ X_k(\xi_k) = \nu X_k(\zeta_{[k]}) \tag{24}$$

for $\xi_k \in [\eta_{[k]}, 1]$ and $\nu = \frac{\alpha}{1+\beta}$. Note, $0<\nu\le 1$ since $\alpha-\beta\le 1$, $\zeta_{[k]} < \xi_k$ since $\zeta_{[k]} < \eta_k$. By virtue of [12, p. 1199-1200] we specify an appropriate point $\tau_k \in [\zeta_{[k]}, \xi_k]$, so that the solution of (18) satisfies the classical boundary value condition

$$X_k(0)=0,\ \ X_k'(\tau_k) + h_k X_k(\tau_k) = 0 \tag{25}$$

for $h_k \ge 0$. Therefore, (19) holds [3, 143-144]. Since $\zeta_{[k]} \le \tau_k$, then (20) holds, and then the first estimate (22) holds. Since $X_k(1),\ X_k(\eta_{[k]}),\ X_k(\zeta_{[k]})$ have the same sign, then in view of (18)

$$(1+\beta)\min\{|X_k(1)|, |X_k(\eta_{[k]})|\} \le \alpha|\ X_k(\zeta_{[k]})\ |\ \le \alpha\frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ ,$$

$$\min\{|X_k(1)|, |X_k(\eta_{[k]})|\} \le \frac{\alpha}{1+\beta}\frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ . \tag{26}$$

Hence, the estimate (11) follows from (26) or, in view of (22), results from (18), i.e.:

$$|\ X_k(1)\ |\ \le c_2 \frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ , \tag{27}$$

$$c_2 = \begin{cases} \frac{\alpha}{1+\beta}, & \text{if}\ \ |X_k(1)| \le |X_k(\eta_{[k]})|\ , \\ \frac{\alpha\beta}{1+\beta} + \alpha, & \text{if}\ \ |X_k(1)| > |X_k(\eta_{[k]})|\ . \end{cases}$$

Let ***A2*** holds, i.e., $\alpha \le 1$ and $\zeta_n \not< \eta_1$. Put $\zeta_{[k]} \ne \eta_{[k]}$, because if this two points coincide, then NLBVC (18) transfoms to

$$X_k(0)=0,\ X_k(1) = (\alpha-\beta)X_k(\xi_k)\ \ \text{for}\ \ \xi_k = \zeta_{[k]} = \eta_{[k]}\ \ \text{while}\ \ -\infty < \alpha-\beta < 1\ ,$$

so that the estimate (11) holds in view of [3]. Moreover, we consider the layout $\zeta_{[k]} > \eta_{[k]}$ only, since for the alternate order when $\zeta_{[k]} < \eta_{[k]}$ (note that $-\infty < \alpha-\beta < 1$ since $\alpha \le 1$)

the estimate (16) is proved already in the above case under the term ***A1***. Additionally, we put $sign\big(X_k(1)\ X_k(\eta_{[k]})\ X_k(\zeta_{[k]})\big) \neq 0$. Note in advance, if $sign\big(X_k(1)\ X_k(\eta_{[k]})\ X_k(\zeta_{[k]})\big) = 0$, then the estimate (11) results from the current proof. In summary, we have to consider the alternate subcases when $sign\big(X_k(1)X_k(\zeta_{[k]})\big) = -1$ and $sign\big(X_k(1)X_k(\zeta_{[k]})\big) = 1$ for $\eta_{[k]} < \zeta_{[k]}$.

Subcase 2.1 :

If $sign\big(X_k(1)X_k(\zeta_{[k]})\big) = -1$ and $\eta_{[k]} < \zeta_{[k]}$, then by analogy with the subcase 1.1 we obtain all estimates (19)-(23).

Subcase 2.2 :

Put $sign\big(X_k(1)X_k(\zeta_{[k]})\big) = 1$ and $\eta_{[k]} < \zeta_{[k]}$. Then we have the alternate inequalities: $|X_k(\zeta_{[k]})| \geq$ $\geq |X_k(1)|$ and $|X_k(\zeta_{[k]})| < |X_k(1)|$.

If $X_k(\zeta_{[k]}) = X_k(1)$, then by virtue of Rolle's theorem $X_k'(\tau_{k1}) = 0$ for $\tau_{k1} \in [\zeta_{[k]}, 1]$.

If $|X_k(\zeta_{[k]})| > |X_k(1)|$, then $X_k(1) = \nu_k X_k(\zeta_{[k]})$ for an appropriate value ν_k, $0 < \nu_k < 1$. Hence, by virtue of [12, p. 1199-1200] we specify an appropriate point $\tau_{k2} \in [\zeta_{[k]}, 1]$, so that the classical boundary value condition holds for $h_k > 0$: $X_k(0) = 0$, $X_k'(\tau_{k2}) + h_k X_k(\tau_{k2}) = 0$. Thereby, if $|X_k(\zeta_{[k]})| \geq |X_k(1)|$, then for some $\tau_k \in [\zeta_{[k]}, 1]$ and $h_k \geq 0$

$$X_k(0) = 0,\ \ X_k'(\tau_k) + h_k X_k(\tau_k) = 0\ .$$

Since $\eta_{[k]} < \zeta_{[k]}$, then using the method of section 1.1 we succesively obtain the estimates (19)-(23).

If $|X_k(\zeta_{[k]})| < |X_k(1)|$, then $sign\big(X_k(\eta_{[k]})X_k(1)\big) = sign\big(X_k(\eta_{[k]})X_k(\zeta_{[k]})\big) = -1$ since $\alpha \leq 1$ and because $sign\big(X_k(1)X_k(\zeta_{[k]})\big) = 1$. By virtue of Bolzano theorem $X_k(\tau_k) = 0$ for $\tau_k \in [\eta_{[k]}, \zeta_{[k]}]$. Then, by analogy with subcase 1.1 we get (19), (21) and the second estimate in (22). Hence, if $\alpha < 1$, then in view of (18)

$$(1-\alpha)|X_k(\zeta_{[k]})| < \beta\frac{\sqrt{2}}{k^{3/2}}||f_k||_{L_2[0,1]}\ . \tag{28}$$

Put $c_3 = \frac{\alpha\beta}{1-\alpha} + \beta$. Using (18), in view of (22) and (28), we obtain the desired estimate

$$|\ X_k(1)\ |\ \leq c_3\frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ . \tag{29}$$

At least, if $|X_k(\zeta_{[k]})| < |X_k(1)|$ but $\alpha = 1$, then to estimate $X_k(1)$ we reduce NLBVP (18) to

$$L[X_k(x)] = f_k(x),\ 0 < x < 1, \quad X_k(0) = 0,\ X_k(1) = X_k(\zeta_{[k]}) - \gamma_k\ , \tag{30}$$

where $L[X_k(x)] = X_k''(x) - k^2 X_k(x)$ and $\gamma_k = \beta X_k(\eta_{[k]})$. In view of the second estimate in (22)

$$|\gamma_k| < \beta\frac{\sqrt{2}}{k^{3/2}}||f_k||_{L_2[0,1]}\ . \tag{31}$$

Put $X_k(x)$ is the sum $X_k(x) = V_k(x) + W_k(x)$, so that $V_k(x)$ is the solution of

$$L[V_k(x)] = f_k(x),\ 0 < x < 1, \quad V_k(0) = 0,\ V_k(1) - V_k(\zeta_{[k]}) = 0\ , \tag{32}$$

and $W_k(x)$ is the solution of

$$L[W_k(x)] = 0,\ 0 < x < 1, \quad W_k(0) = 0,\ W_k(1) - W_k(\zeta_{[k]}) = -\gamma_k\ . \tag{33}$$

The classical solution of (32) exists and is a unique [12, p. 1198-1200]. By virtue of Rolle theorem $V_k'(\tau_k) = 0$ for $\tau_k \in (\zeta_{[k]}, 1)$. Then similar subcase 1.1 the analogs of (19)-(20) and the first estimate (22) hold for $V_k(x)$. Hence, since $V_k(1) = V_k(\zeta_{[k]})$

$$|\ V_k(1)\ |\ \leq \frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ . \tag{34}$$

On the other hand, for $\mathcal{C}_k = -\gamma_k\Big(1 - \frac{\sinh k\zeta_{[k]}}{\sinh k}\Big)^{-1}$ the function $W_k(x) = \mathcal{C}_k \frac{\sinh kx}{\sinh k}$ is the solution of (33) since $1 - \frac{\sinh k\zeta_{[k]}}{\sinh k} > 0$ for $\zeta_{[k]} < 1$. Then, in view of 2-point condition (33),

$$|W_k(1)| \leq \Big(1 - \frac{\sinh k\zeta_{[k]}}{\sinh k}\Big)^{-1} \beta \frac{\sqrt{2}}{k^{3/2}} ||f_k||_{L_2[0,1]} \ . \tag{35}$$

Hence, for $M = \frac{\sinh \zeta_n}{\sinh 1}$

$$|W_k(1)| \leq \frac{1}{1-M} \beta \frac{\sqrt{2}}{k^{3/2}} ||f_k||_{L_2[0,1]} \ . \tag{36}$$

Then, in view of (34) and (36), $|X_k(1)| \leq c_4 \frac{\sqrt{2}}{k^{3/2}} ||f_k(x)||_{L_2[0,1]}$ for $c_4 = 1 + \beta \frac{1}{1-M}$.

Finally we resume, that for the classical solution of (10) the estimate (11) is proved for the constant $C = \max\{c_1, c_2, c_3, c_4\}$. Theorem 2 is proved.

Theorem 3. Let $f \in C(\overline{\Pi})$, $m = n$ and $\zeta_r < \eta_r$, $r = \overline{1,n}$. If $\sum\limits_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} \leq 1$, then classical solution of NLBVP (1) exists, it is an unique and a priori estimate (6) holds.

Proof. Suppose that classical solution exists. In view of Theorem 2, we rewrite (10) as

$$L[X_k(x)] = f_k(x), \ 0 < x < 1, \quad X_k(0) = 0, \ \ \ell[X_k] = 0, \tag{37}$$

where $L[X_k(x)] = X_k''(x) - k^2 X_k(x)$ and $\ell[X_k] \equiv X_k(1) - \sum\limits_{r=1}^{n} [\alpha_r X_k(\zeta_r) - \beta_r X_k(\eta_r)]$. To obtain the estimate (11) we put $X_k(x) = V_k(x) + W_k(x)$, so that $V_k(x)$ is the solution of problem

$$L[V_k(x)] = f_k(x), \ 0 < x < 1, \quad V_k(0) = 0, \ V_k(1) = 0 \ , \tag{38}$$

and $W_k(x)$ is the solution of problem

$$L[W_k(x)] = 0, \ 0 < x < 1, \quad W_k(0) = 0, \ \ \ell[W_k] = -\ell[V_k] \ . \tag{39}$$

For solution of (38) the analog of (7) holds (see Theorem 2). Hence, since $V_k(0) = 0$ and $\zeta_r \in (0,1)$, $\eta_r \in (0,1)$, $r = \overline{1,n}$, then

$$|V_k(\zeta_r)| \leq \frac{\sqrt{2}}{k^{3/2}} ||f_k||_{L_2[0,1]}, \quad |V_k(\eta_r)| \leq \frac{\sqrt{2}}{k^{3/2}} ||f_k||_{L_2[0,1]} \ .$$

Therefore,

$$|\ell[V_k]| \leq \Big(\sum_{r=1}^{n} (\alpha_r + \beta_r)\Big) \frac{\sqrt{2}}{k^{3/2}} ||f_k||_{L_2[0,1]} \ . \tag{40}$$

The problem (39) has the solution $W_k(x) = \mathcal{W}_k \frac{\sinh kx}{\sinh k}$, $\mathcal{W}_k = \frac{-\ell[V_k]}{1 - (\sinh k)^{-1} \sum\limits_{r=1}^{n} [\alpha_r \sinh k\zeta_r - \beta_r \sinh k\eta_r]}$, where the denominator of $\mathcal{W}_k$ is nozero when $\frac{1}{2} \sum\limits_{r=1}^{n} \Big[(\alpha_r - \beta_r) + |\alpha_r - \beta_r|\Big] < 1$. In view of (40),

$$|W_k(1)| \leq \frac{\sqrt{2} \sum\limits_{r=1}^{n} (\alpha_r + \beta_r)}{k^{3/2} \big[1 - \frac{1}{2} \sum\limits_{r=1}^{n} (\alpha_r - \beta_r + |\alpha_r - \beta_r|)\big]} ||f_k||_{L_2[0,1]} \ .$$

Hence, (11) holds since $V_k(1) = 0$, i.e., $|\ X_k(1)\ | \ \leq C \frac{\sqrt{2}}{k^{3/2}} \ ||f_k||_{L_2[0,1]}$.

At least, put $\frac{1}{2}\sum\limits_{r=1}^{n}\Big[(\alpha_r-\beta_r)+|\alpha_r-\beta_r|\Big]=1$, then similar (35), but in view of (40), we get

$$|W_k(1)|\le\Big(1-\frac{\sinh\zeta_p}{\sinh 1}\Big)^{-1}\Big[\sum_{r=1}^{n}(\alpha_r+\beta_r)\Big]\frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ ,$$

where p , $1\le p\le n$ is a natural number, so that

$$\frac{(\alpha_p-\beta_p)+|\alpha_p-\beta_p|}{2}>0\ ,\quad \text{but}\quad \frac{(\alpha_{p+i}-\beta_{p+i})+|\alpha_{p+i}-\beta_{p+i}|}{2}=0\quad \text{for all}\quad i,\ \ p<i\le n\ ,$$

and $p=n$ if i does not exists. Hence, (11) holds for $\frac{1}{2}\sum\limits_{r=1}^{n}\big[(\alpha_r-\beta_r)+|\alpha_r-\beta_r|\big]=1$ since $V_k(1)=0$. In summary, for the solution of (37) the estimate (11) holds when $\frac{1}{2}\sum\limits_{r=1}^{n}\big[(\alpha_r-\beta_r)+|\alpha_r-\beta_r|\big]\le 1$. Hence, in view of Theorem 2, a priori estimate (6) holds for NLIVP (1), thereto the solution of (1) is a unique and, therefore, in view of Theorem 1 the solution exists. Theorem 3 is proved.

Difference variant

We consider the difference variant of NLBVP (1)

$$\begin{cases} \Lambda Y=Y_{\bar{x}x}+Y_{\bar{y}y}=f(x_i,y_j),\ \ (x_i,y_j)\in\Pi,\\ Y|_{y=0}=Y|_{y=\pi}=0,\ x_i\in[0,1)\,,\quad Y|_{x=0}=0,\ \ y_j\in[0,\pi]\,,\\ \mathcal{L}Y=\sum\limits_{r=1}^{n}\alpha_r\Big\{Y_{i_{\zeta_r},j}\frac{[(i_{\zeta_r}+1)h_1-\zeta_r]}{h_1}+Y_{i_{\zeta_r}+1,j}\frac{[\zeta_r-i_{\zeta_r}h_1]}{h_1}\Big\}-\\ -\sum\limits_{s=1}^{m}\beta_s\Big\{Y_{i_{\eta_s},j}\frac{[(i_{\eta_s}+1)h_1-\eta_s]}{h_1}+Y_{i_{\eta_s}+1,j}\frac{[\eta_s-i_{\eta_s}h_1]}{h_1}\Big\}-Y_{N_1,j}=0,\ j=\overline{1,N_2-1}, \end{cases}\tag{41}$$

where $i_{\zeta_r}h_1\le\zeta_r<(i_{\zeta_r}+1)h_1$, $r=\overline{1,n}$, $i_{\eta_s}h_1\le\eta_s<(i_{\eta_s}+1)h_1$, $s=\overline{1,m}$ for $h_1=1/N_1$, $h_1<\frac{1}{2}\min\{\zeta_{r+1}-\zeta_r,\ r=\overline{0,n},\ \eta_{s+1}-\eta_s,\ s=\overline{0,m},\ |\zeta_r-\eta_s|,\ r=\overline{1,n},\ s=\overline{1,m}\}$, $\zeta_0=\eta_0=0$, $\zeta_{n+1}=\eta_{m+1}=1$, $h_1\le c_0h_2$, $h_2=\pi/N_2$.

Theorem 4. Let the term $\boldsymbol{A}$ holds and $u\in C^{(4)}(\overline{\Pi})$ is the solution of NLBVP (1). Then solution of the difference problem (41) approximates $u(x,y)$ by the second order of accuracy in terms of $h=\sqrt{h_1^2+h_2^2}$ when $h_2\to 0$ in respect of difference metrics C, W_2^2 .

Proof. We denote $z=Y-u$ and obtain the difference problem

$$\Lambda z=f-\Lambda u=F,\ \ (ih_1,jh_2)\in\Pi,\quad z|_{x=0}=z|_{y=0}=z|_{y=\pi}=0,\ \ \mathcal{L}z=-\mathcal{L}u.\tag{42}$$

For this problem $F=O(h^2)$, $\mathcal{L}u=O(h^2)$ [14, p. 81, 229]. Put $z=\tilde{z}+\hat{z}$, where $\tilde{z}$ is the solution of

$$\Lambda\tilde{z}=0,\ \ (ih_1,jh_2)\in\Pi,\quad \tilde{z}|_{x=0}=\tilde{z}|_{y=0}=\tilde{z}|_{y=\pi}=0,\ \ \mathcal{L}\tilde{z}=-\mathcal{L}u\ ,\tag{43}$$

and $\hat{z}$ is the solution of

$$\Lambda\hat{z}=F,\ (ih_1,jh_2)\in\Pi,\quad \hat{z}|_{x=0}=\hat{z}|_{y=0}=\hat{z}|_{y=\pi}=0,\ \ \mathcal{L}\hat{z}=0\ .\tag{44}$$

To estimate $\tilde{z}$ we use [14, p. 113] the orthogonal system of mesh functions $\{\sin(ky)\}_{k=1}^{k=N_2-1}$, so that

$$\tilde{z}=\sum_{k=1}^{N_2-1}\tilde{z}_k\sin(ky),\quad y=jh_2,\ \ \overline{j=0,N_2}$$

thereto $\tilde{z}_k$, $k=\overline{1,N_2-1}$ is the solution of difference problem

$$\Lambda_1\tilde{z}_k-\lambda_k\tilde{z}_k=0,\quad \tilde{z}_k|_{x=0}=0,\ \ \mathcal{L}\tilde{z}_k=-Q_k\ , \tag{45}$$

where $\Lambda_1\tilde{z}=\tilde{z}_{\bar{x}x}$, $\lambda_k=4h_2^{-2}\sin^2(kh_2)$, $Q_k=(\mathcal{L}u)_k$ so that, in view of [3, p. 142-143],

$$\tilde{z}_{k_i}=A_k\sinh(i\ln q_k),\ \ A_k=-Q_k/\mathcal{L}[\sinh(i\ln q_k)],\ \ i=\overline{0,N_1},\ \ q_k=1+\lambda_k h_1^2/2+\sqrt{\lambda_k h_1^2+\lambda_k^2 h_1^4/4}\ .$$

By acting $\mathcal{L}$ in the denominator of the fraction A_k, we get

$$-\mathcal{L}[\sinh(i\ln q_k)]\geq\sinh(N_1\ln q_k)-\sum_{r=1}^{n}\alpha_r\sinh((i_{\zeta_n}+1)\ln q_k)+\sum_{s=1}^{m}\beta_s\sinh(i_{\eta_1}\ln q_k)\ . \tag{46}$$

Hence,

$$-\mathcal{L}[\sinh(i\ln q_k)]\geq\sinh(N_1\ln q_k)-S\sinh((i_{\zeta_n}+1)\ln q_k) \tag{47}$$

for

$$S=\begin{cases}\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s\ ,\ \text{if}\ \ \zeta_n<\eta_1,\\ \sum\limits_{r=1}^{n}\alpha_r\ ,\quad \text{if}\quad \zeta_n>\eta_1.\end{cases}$$

Then

$$-\mathcal{L}[\sinh(i\ln q_k)]\geq C\sinh(N_1\ln q_k) \tag{48}$$

for $C>0$,

$$C=\begin{cases}1\ ,\quad \text{if}\quad -\infty<\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s\leq 0,\ \ \zeta_n<\eta_1;\\ 1-\big(\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s\big)\ ,\quad \text{if}\quad 0<\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s<1,\ \ \zeta_n<\eta_1;\\ 1-\sum\limits_{r=1}^{n}\alpha_r\ ,\quad \text{if}\quad \alpha_r<1,\ \ \zeta_n>\eta_1.\end{cases}$$

Let we show that when $S=1$ in (47), then the inequality (48) holds for $C=1-\frac{1}{(1+4/\pi)^\delta}$ subject to an appropriate δ, $0<\delta\leq 1$. Indeed, in view of (47)

$$-\mathcal{L}[\sinh(i\ln q_k)]\geq\sinh(N_1\ln q_k)\Big[1-\frac{\sinh((i_{\zeta_n}+1)\ln q_k)}{\sinh(N_1\ln q_k)}\Big]\geq 0\ .$$

Hence,

$$-\mathcal{L}[\sinh(i\ln q_k)]\geq\sinh(N_1\ln q_k)\Big[1-\frac{q_k^{i_{\zeta_n}+1}-q_k^{-(i_{\zeta_n}+1)}}{q_k^{N_1}-q_k^{-N_1}}\Big]\ . \tag{49}$$

Since $q_k\geq 1$, then

$$\frac{q_k^{i_{\zeta_n}+1}-q_k^{-(i_{\zeta_n}+1)}}{q_k^{N_1}-q_k^{-N_1}}\leq\frac{q_k^{i_{\zeta_n}+1}[1-q_k^{-2(i_{\zeta_n}+1)}]}{q_k^{N_1}[1-q_k^{-2N_1}]}\leq\frac{q_k^{i_{\zeta_n}+1}}{q_k^{N_1}}\ . \tag{50}$$

Since $h_1<\theta$ for $\theta=\frac{1}{2}\min\{\zeta_{r+1}-\zeta_r,\ r=\overline{0,n},\ \eta_{s+1}-\eta_s,\ s=\overline{0,m},\ |\zeta_r-\eta_s|,\ r=\overline{1,n},\ s=\overline{1,m}\}$, then for specified $\delta=1-\zeta_n-\theta$ the inequality $\zeta_n+h_1\leq 1-\delta$ holds. Hence, $i_{\zeta_n}+1\leq h_1^{-1}(1-\delta)$. Then from (60) it follows that

$$\frac{q_k^{i_{\zeta_n}+1}-q_k^{-(i_{\zeta_n}+1)}}{q_k^{N_1}-q_k^{-N_1}}\leq\frac{q_k^{N_1(1-\delta)}}{q_k^{N_1}}\leq\frac{1}{q_k^{N_1\delta}}\ .$$

Hence, in view of (49),

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq \Big(1-\frac{1}{q_k^{N_1\delta}}\Big)\sinh(N_1\ln q_k)\ . \tag{51}$$

Since $q_k^{N_1} \geq (1+\sqrt{\lambda_k}h_1)^{N_1} \geq (1+\sqrt{\lambda_1}h_1)^{N_1} \geq (1+\sqrt{\lambda_1}) \geq 1+\frac{4}{\pi}$, then from (51) we obtain

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq \Big[1-\frac{1}{(1+4/\pi)^{\delta}}\Big]\sinh(N_1\ln q_k)\ . \tag{52}$$

In summary, if the term $\boldsymbol{A}$ holds, then

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq C\sinh(N_1\ln q_k) > 0\ . \tag{53}$$

Finally, in view of (53), by virtue of [3, 150-151], we obtain the estimates

$$\max_{i,j}|\tilde{z}_{ij}| = O(h^2),\ \ ||\tilde{z}||_{W_2^2} = O(h^2),\ \ \max_{i,j}|\hat{z}_{ij}| = O(h^2),\ \ ||\hat{z}||_{W_2^2} = O(h^2).$$

Therefore, $\max_{i,j}|z_{ij}| = O(h^2)$, $||z||_{W_2^2} = O(h^2)$. Theorem 4 is proved.

Corollary 1. Let $n = m$, $\zeta_r < \eta_r$, $r = \overline{1,n}$. Let $u \in C^{(4)}(\overline{\Pi})$ is the solution of NLBVP (1). If $\sum\limits_{r=1}^{n}\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2} \leq 1$, then difference solution of (41) approximates $u(x,y)$ by the second order of accuracy in terms of $h=\sqrt{h_1^2+h_2^2}$ when $h_2 \to 0$ in respect of difference metrics C, W_2^2 .

Proof. By virtue of (42)-(46) we get the inequality for the denominator of the fraction A_k:

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq \sinh(N_1\ln q_k) - \sum_{r=1}^{n}\alpha_r\sinh((i_{\zeta_r}+1)\ln q_k) + \sum_{r=1}^{n}\beta_r\sinh(i_{\eta_r}\ln q_k)\ .$$

Since $i_{\zeta_r}+1 < i_{\eta_r}$, $r=\overline{1,n}$, then

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq \sinh(N_1\ln q_k) - \sum_{r=1}^{n}(\alpha_r-\beta_r)\ \sinh((i_{\zeta_r}+1)\ln q_k)\ .$$

Hence,

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq\ \Big[1-\sum_{r=1}^{n}(\alpha_r-\beta_r)\ \Big(\frac{q_k^{i_{\zeta_r}+1}-q_k^{-(i_{\zeta_r}+1)}}{q_k^{N_1}-q_k^{-N_1}}\Big)\Big]\ \sinh(N_1\ln q_k).$$

Then

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq \Big[1-\sum_{r=1}^{n}\Big(\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2}\Big)\Big(\frac{q_k^{i_{\zeta_r}+1}-q_k^{-(i_{\zeta_r}+1)}}{q_k^{N_1}-q_k^{-N_1}}\Big)\Big]\sinh(N_1\ln q_k). \tag{54}$$

Put p is a natural number, $1 \leq p \leq n$, so that

$$\frac{(\alpha_p-\beta_p)+|\alpha_p-\beta_p|}{2} > 0\ ,\ \ \text{but}\ \ \frac{(\alpha_{p+i}-\beta_{p+i})+|\alpha_{p+i}-\beta_{p+i}|}{2} = 0\ \ \text{for all}\ \ i,\ p<i\leq n$$

(if such p does not exists, or if such i does not exists, then put $p=n$). Hence, in view of (54),

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq \Big[1-S\ \frac{q_k^{i_{\zeta_p}+1}-q_k^{-(i_{\zeta_p}+1)}}{q_k^{N_1}-q_k^{-N_1}}\Big]\ \sinh(N_1\ln q_k) \tag{55}$$

for $S=\sum\limits_{r=1}^{n}\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2}$. By analogy with (50), for $q_k\geq 1$ and for $\delta=1-\zeta_p-\theta$ we get

$$\frac{q_k^{i_{\zeta_p}+1}-q_k^{-(i_{\zeta_p}+1)}}{q_k^{N_1}-q_k^{-N_1}}\leq\frac{1}{q_k^{N_1\delta}} \tag{56}$$

since the inequalities $\zeta_p+h_1\leq 1-\delta$ and $i_{\zeta_p}+1\leq h_1^{-1}(1-\delta)$ hold. Hence, the analog of (47) holds, then (51)-(53) hold, too. Thereby, in view of Theorem 4, the proof is finished. Corollary 1 is proved

NLBVP with integral condition

Here we apply the results of the previous sections to NLBVP with weighted integral condition (WIC). We consider the differential problem in the rectangular Π

$$\begin{cases}\Delta u(x,y)=f(x,y),\ (x,y)\in\Pi,\\ u(x,0)=u(x,\pi)=0,\ 0\leq x<1,\ \ u(0,y)=0,\ \mathcal{I}[u](y)=0,\ 0\leq y\leq\pi\ ,\end{cases} \tag{57}$$

$$\mathcal{I}[u](y)\equiv u(1,y)-\int\limits_{\tau_0}^{\tau_1}\rho(x)u(x,y)dx\ , \tag{58}$$

where $\rho(x)\in C[\tau_0,\tau_1]$, $[\tau_0,\tau_1]\subset(0,1)$, $\tau_0<\tau_1$ and $\rho(x)\not\equiv 0$ in $[\tau_0,\tau_1]$.

Theorem 5. Let the function $\rho(x)$ changes the sign[3] no more than once in the interval (τ_0,τ_1). Let :

$$-\infty<\int\limits_{\tau_0}^{\tau_1}\rho(x)dx\leq 1\ ,\ \ \text{if } \rho(x) \text{ does not change the sign, or changes it from plus to minus ;}$$

$$\int\limits_{\tau_0}^{\tau_1}\frac{\rho(x)+|\rho(x)|}{2}dx\leq 1\ ,\ \ \text{if } \rho(x) \text{ changes the sign from minus to plus .}$$

Then classical solution of (57) exists, it is an unique and a priori estimate (6) holds.

Proof. Assume that classical solution exits. Since

$$\int\limits_0^{\pi}u(1,y)\sin(ky)dy=\int\limits_0^{\pi}\int\limits_{\tau_0}^{\tau_1}\rho(x)u(x,y)dx\ \sin(ky)dy=\int\limits_{\tau_0}^{\tau_1}\rho(x)\left(\int\limits_0^{\pi}u(x,y)\sin(ky)dy\right)dx\ ,$$

then from (57)-(58), in view of (3) and by virtue of Theorem 1, we conclude that the function $X_k(x)$ satisfies the problem

$$X_k''(x)-k^2X_k(x)=0,\ 0<x<1,\quad X_k(0)=0,\ \ \mathcal{I}[X_k]=0\ , \tag{59}$$

where $\mathcal{I}[X_k]=X_k(1)-\int\limits_{\tau_0}^{\tau_1}\rho(x)X_k(x)dx$. By virtue of the integral type of mean value theorem, we reduce WIC problem (59) to the 3-point problem

$$X_k''(x)-k^2X_k(x)=0,\ 0<x<1,\quad X_k(0)=0,\ \ \ell[X_k]=0\ , \tag{60}$$

where

$$\ell[X_k]=X_k(1)-\left(\int\limits_{\tau_0}^{\tau_1}\frac{\rho(x)+|\rho(x)|}{2}dx\right)X_k(\zeta_k)+\left(\int\limits_{\tau_0}^{\tau_1}\frac{|\rho(x)|-\rho(x)}{2}dx\right)X_k(\eta_k) \tag{61}$$

[3]The sign changing number and order are regarded as argument x shifs towards τ_1.

for some $\zeta_k \in (\tau_0, \tau_1)$ and $\eta_k \in (\tau_0, \tau_1)$. Denote

$$\alpha = \int_{\tau_0}^{\tau_1} \frac{\rho(x) + |\rho(x)|}{2} dx \ , \quad \beta = \int_{\tau_0}^{\tau_1} \frac{|\rho(x)| - \rho(x)}{2} dx \ . \tag{62}$$

If $\rho(x)$ does not change the sign, then:

$$\ell[X_k] = X_k(1) - \alpha X_k(\zeta_k) \ \text{ and } \ 0 \le \alpha \le 1 \ , \ \text{if } \ \rho(x) \text{ is a nonnegative function} \ ,$$

$$\ell[X_k] = X_k(1) + \beta X_k(\eta_k) \ \text{ and } \ -\infty < -\beta \le 0 \ , \ \text{if } \ \rho(x) \text{ is a nonpositive function} \ .$$

If $\rho(x)$ changes the sign, then $\ell[X_k] = X(1) - \alpha X_k(\zeta_k) + \beta X_k(\eta_k)$, so that

$$-\infty < \alpha - \beta \le 1, \ \ \zeta_k < \eta_k \ \text{ if } \ \rho(x) \text{ changes the sign from plus to minus} \ ,$$

$$\alpha \le 1, \ \ \eta_k < \zeta_k \ \text{ if } \ \rho(x) \text{ changes the sign from minus to plus} \ .$$

Hence, in view of (61)-(62), for the 3-point NLBVP (60) the term $\boldsymbol{A}$ holds in extended form [16, p. 917], i.e., includes the option when $\alpha = 0$ or $\beta = 0$. Then, in view of Theorem 1, the problem (60) (and in turn the problem (59) of course) has only trivial solution $X_k(x) \equiv 0$, and, therefore, $u(x, y) \equiv 0$ in the rectangle Π. Since the uniqueness for the problem (57) is proved, then the existence follows from the Fredholm's property inherent such NLBVP with WIC [15, p. 68-70].

To prove a priori estimate (6) we follow Theorem 2 and, in view of (8), get WIC problem

$$X_k''(x) - k^2 X_k(x) = f_k(x), \ 0 < x < 1, \quad X_k(0) = 0, \ \ \mathcal{I}[X_k] = 0 \tag{63}$$

(this problem was studied in [17]) and, in view of (60), reduce it to the multipoint problem

$$X_k''(x) - k^2 X_k(x) = f_k(x) \ , \ 0 < x < 1, \quad X_k(0) = 0, \ \ \ell[X_k] = 0 \ . \tag{64}$$

In view of (61)-(62) and by virtue of Theorem 2, we ascertain that (11) holds for solution of (64) and, thereby, it holds for solution of (63). Further proof is similarly of Theorem 2. Theorem 5 is proved.

Corollary 2. Let the function $\rho(x)$ has an arbitrary order and a finite number of sign changings. If $\int_{\tau_0}^{\tau_1} \frac{\rho(x)+|\rho(x)|}{2} dx \le 1$, then classical solution of (57) exists, it is an unique and a priori estimate (6) holds.

Proof. The proof results from Theorem 1 and Theorem 2 by using Theorem 5. Corollary 2 is proved.

Corollary 3. Let starting from plus to minus the function $\rho(x)$ changes the sign $2n-1$ times in the interval (τ_0, τ_1) for specified natural number n and $\xi_1, ..., \xi_{2n-1}$ are the sign changing points. Put $\xi_0 = \tau_0$ and $\xi_{2n} = \tau_1$. If

$$\sum_{k=1}^{n} \frac{1}{2} \Big(\int_{\xi_{2(k-1)}}^{\xi_{2k}} \rho(x) dx \ + \ \Big| \int_{\xi_{2(k-1)}}^{\xi_{2k}} \rho(x) dx \Big| \Big) \le 1 \ ,$$

then classical solution of (57) exists, it is an unique and a priori estimate (6) holds

Proof. It results from Theorem 1-2 and by using of Theorem 3, Theorem 5. Corollary 3 is proved.

Difference application for WIC

We consider the difference problem

$$\begin{cases} \Lambda Y = Y_{\bar{x}x} + Y_{\bar{y}y} = f(x_i, y_j), \ \ (x_i, y_j) \in \Pi, \\ Y|_{y=0} = Y|_{y=\pi} = 0, \ x_i \in [0,1)\,, \ \ Y|_{x=0} = 0, \ \ y_j \in [0,\pi]\,, \\ \mathcal{T}Y = \sum\limits_{i=1}^{N_1} 2^{-1}(\rho_i Y_{i,j} + \rho_{i-1} Y_{i-1,j})h_1 - Y_{N_1,j} = 0, \ j = \overline{1, N_2 - 1}, \end{cases} \tag{65}$$

where $\rho(x)$ does not change the sign, $\rho(x) \in C[0,1]$ and $\rho(x) \equiv 0$ in $[0,\tau_0] \cup [\tau_1, 1]$, $\rho_i = \rho(x_i)$, $h_1 < \frac{1}{2}\min\{\tau_0, \ 1-\tau_1\}$, $h_1 \le c_0 h_2$, $h_1 = 1/N_1$, $h_2 = \pi/N_2$.

Corollary 4. Let $u \in C^{(4)}(\overline{\Pi})$ is solution of WIC NLBVP (57). If $-\infty < \int\limits_{\tau_0}^{\tau_1} \rho(x)dx < 1$, then the solution of (65) approximates $u(x,y)$ by the second order of accuracy in terms of $h = \sqrt{h_1^2 + h_2^2}$ when $h_2 \to 0$ in respect of difference metrics C, W_2^2.

Proof. Following Theorem 4, for $z = Y - u$ we obtain the difference problem

$$\Lambda z = f - \Lambda u = F \ , \ (ih_1, jh_2) \in \Pi, \ \ z|_{x=0} = z|_{y=0} = z|_{y=\pi} = 0 \ , \ \ \mathcal{T}z = -\mathcal{T}u \ , \tag{66}$$

thereto $F = O(h^2)$ and $\mathcal{T}u = O(h^2)$ as a neglect of the trapezoid method. Put $z = \tilde{z} + \hat{z}$, where $\tilde{z}$ is the solution of

$$\Lambda \tilde{z} = 0 \ , \ (ih_1, jh_2) \in \Pi \ , \ \ \tilde{z}|_{x=0} = \tilde{z}|_{y=0} = \tilde{z}|_{y=\pi} = 0, \ \ \mathcal{T}\tilde{z} = -\mathcal{T}u \ , \tag{67}$$

and $\hat{z}$ is the solution of

$$\Lambda \hat{z} = F \ , \ (ih_1, jh_2) \in \Pi \ , \ \ \hat{z}|_{x=0} = \hat{z}|_{y=0} = \hat{z}|_{y=\pi} = 0 \ , \ \ \mathcal{T}\hat{z} = 0 \tag{68}$$

By virtue of the orthogonal system [14, p. 113] of the mesh functions $\{\sin(ky)\}_{k=1}^{k=N_2-1}$

$$\tilde{z} = \sum_{k=1}^{N_2-1} \tilde{z}_k \sin(ky), \quad y = jh_2, \ \ \overline{j = 0, N_2} \ ,$$

thereto $\tilde{z}_k$, $k = \overline{1, N_2 - 1}$ is solution of the problem

$$\Lambda_1 \tilde{z}_k - \lambda_k \tilde{z}_k = 0 \ , \ \ \tilde{z}_k|_{x=0} = 0, \ \ \mathcal{T}\tilde{z}_k = -Q_k \tag{69}$$

for $\Lambda_1 \tilde{z} = \tilde{z}_{\bar{x}x}$, $\lambda_k = 4h_2^{-2}\sin^2(kh_2)$, $Q_k = (\mathcal{T}u)_k$ and, in view of [3, p. 142-143],

$$\tilde{z}_{k_i} = A_k \sinh(i \ln q_k), \ A_k = -Q_k/\mathcal{T}[\sinh(i \ln q_k)], \ \ i = \overline{0, N_1}, \ q_k = 1 + \lambda_k h_1^2/2 + \sqrt{\lambda_k h_1^2 + \lambda_k^2 h_1^4/4}.$$

Acting by $\mathcal{T}$ we get the inequality for the denominator of the fraction A_k:

$$-\mathcal{T}[\sinh(i \ln q_k)] \ge \sinh(N_1 \ln q_k) - \sum_{i=1}^{N_1} 2^{-1}\Big(\rho_i \sinh(i \ln q_k) + \rho_{i-1}\sinh([i-1]\ln q_k)\Big)h_1 \ . \tag{70}$$

If $\rho(x) \le 0$, then $-\mathcal{T}[\sinh(i \ln q_k)] \ge \sinh(N_1 \ln q_k)$. If $\rho(x) \ge 0$, then for $i_{\tau_0}h_1 \le \tau_0 < (i_{\tau_0}+1)h_1$ and $i_{\tau_1}h_1 \le \tau_1 < (i_{\tau_1}+1)h_1$

$$-\mathcal{T}[\sinh(i \ln q_k)] \ge \sinh(N_1 \ln q_k) - \sinh\left((i_{\tau_1}+1)\ln q_k\right) \sum_{i=i_{\tau_0}+1}^{i_{\tau_1}+1} 2^{-1}(\rho_i + \rho_{i-1})h_1 \ .$$

Denote $S_{h_1} = \sum\limits_{i=i_{\tau_0}+1}^{i_{\tau_1}+1} 2^{-1}(\rho_i + \rho_{i-1})h_1$, then

$$-\mathcal{T}[\sinh(i \ln q_k)] \geq (1 - S_{h_1}) \sinh(N_1 \ln q_k) \ .$$

Since $\int\limits_0^1 \rho(x)dx < \lambda$ for specified λ, $0 < \lambda < 1$, then $S_{h_1} < \lambda$ for sufficiently small h_1. Hence,

$$-\mathcal{T}[\sinh(i \ln q_k)] \geq (1 - \lambda) \sinh(N_1 \ln q_k) > 0 \ .$$

In summary,

$$-\mathcal{T}[\sinh(i \ln q_k)] \geq C \sinh(N_1 \ln q_k) \tag{71}$$

for

$$C = \begin{cases} 1 > 0, & \text{if} \quad \rho(x) \leq 0 \ , \\ 1 - \lambda > 0, & \text{if} \quad \rho(x) \geq 0 \ . \end{cases}$$

In view of (71) and by virtue of Theorem 4, the proof is finished. Corollary 4 is proved.

Conclusion

We considered NLBVP for the Poisson's operator on a rectangular domain and obtained new accurate conditions of the existence, uniqueness and a priori estimate of classical solution. We applied our results and researched NLBVPs with weighted integral condition. We offered the difference variants and proved the second order of accuracy on a uniform grid.

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Д.М. Довлетов

Тікбұрышта Пуассон операторымен берілген бейлокальді шеттік есебі және оның айырымдық интерпретациясы

Жұмыста ашық тікбұрышты облыста Пуассон теңдеуі үшін бейлокальді шеттік есебінің дифференциалдық және айырымдық нұсқалары қарастырылған. Классикалық шешімінің бар болуы, жалғыздығы және априорлық бағамы анықталған. Екінші ретті дәлдікпен айырымдық схемасы көрсетілген. Салмақты интегралдық шарттары бар қосымшалар дифференциалдық және айырымдық нұсқада ұсынылған.

Кілт сөздер: пуассон операторы, бейлокальді шеттік есебі, тікбұрыш, айырымдық схемасы.

Д.М. Довлетов

Нелокальная краевая задача с оператором Пуассона на прямоугольнике и ее разностная интерпретация

В статье изучены дифференциальные и разностные варианты нелокальной краевой задачи для уравнения Пуассона в открытой прямоугольной области. Установлены существование, единственность и априорная оценка классического решения. Представлена разностная схема второго порядка точности. Приложения с весовым интегральным условием даны в дифференциальном и разностном вариантах.

Ключевые слова: оператор Пуассона, нелокальная краевая задача, прямоугольник, разностная схема.

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Stability analysis of an eco-epidemiological model consisting of a prey and two competing predators with SI-disease in prey and toxicant

In the present paper, we study two eco-epidemiological models. The first one consists of a prey and two competing predators with SI-disease in prey species spreading by contacts between susceptible prey and infected prey. This model assumes linear functional response. The second model is the modification of the first one when the effect of toxicant is taken into account. In this paper, we examine the dynamical behavior of non-survival and free equilibrium points of our proposed model.

Keywords: Stability analysis, epidemiological model, prey, predator.

Introduction

In the nature, no species live alone. There are many hundreds or thousands of species in any given environment, in which two populations interact either by competition or mutualism or prey-predator. In the beginning of twentieth century, a number of attempts were made to predict the evolution and existence of species mathematically. Indeed, the first major attempt in this direction was due to the well known classical Lotka-Volterra model in 1927. Since then many complicated models for two or more interacting species have been proposed according to the Lotka-Volterra model by taking into account the effect of competition, time delay, functional response, etc. (see, e.g., [1, 2] and the references therein). On the other hand, over the last few decades, mathematics has been used to understand and predict the spread of diseases, relating important public-health questions to basic transmission parameters. The detailed history of mathematical epidemiology and basics for SIR epidemic models (or Kermack-McKendrick model) can be found in the classical books [1, 3]. However, recently Haque and Venturino [4] have discussed mathematical models of diseases spreading in symbiotic communities. During the last three decades, there has been growing interest in the study of infectious disease coupled with prey-predator interaction models. In many ecological studies of prey-predator systems with disease, it is reported that the predators take a disproportionately high number of parasite-infected prey. Some studies have even shown that parasites could change the external features or behavior of the prey so that infected preys are more vulnerable to predation (see [5,6] and the references therein). Later on, many authors have proposed and studied eco-epidemiological mathematical models incorporating ratio-dependent functional response, toxicant, external sources of disease, predator switching and infected prey refuge [1, 2, 7, 8].

In the present paper, we formulate two types of eco-epidemiological models, the first one consisting of a prey and two competing predators with SI-disease in prey species. The disease spreads by contact between susceptible prey and infected prey; the proposed model includes linear functional response. The second model is the modification of the first one by taking into account the effect of toxicant.

Model formulation

In this section, a prey-predator model consisting of a prey and two competing predators with SI-disease in prey species proposed and analyzed. The disease spreads by contact between susceptible prey and infected prey. The proposed model includes linear functional response and is given by

$$\frac{dS}{dT} = rS\left(1 - \frac{S}{K}\right) - \left(m + \frac{\lambda I}{1+I}\right)S, \tag{1}$$

$$\frac{dI}{dT} = \left(m + \frac{\lambda I}{1+I}\right)S - \mu_1 IY - \mu_2 IZ - d_1 I, \tag{2}$$

$$\frac{dY}{dT} = e_1 IY - \alpha_1 YZ - d_2 Y, \tag{3}$$

$$\frac{dZ}{dT} = e_2 IZ - \alpha_2 YZ - d_3 Z, \tag{4}$$

where $r, k, m, e_1, e_2, \alpha_1, \alpha_2, \mu_1, \mu_2, d_1, d_2, d_3$ are positive parameters. At time $T \geq 0$ prey population is divided into two classes, namely, susceptible $S(T) \geq 0$ and infected $I(T) \geq 0$ due to the existence of infectious disease, interacting with two competing predators species $Y(T) \geq 0$ and $Z(T) \geq 0$, which describe the population densities of the first and second predator, respectively.

The modified model is given by

$$\frac{dS}{dt} = rS\left(1 - \frac{S}{K}\right) - \left(m + \frac{\lambda I}{1+I}\right)S - \sigma_1 WS, \tag{5}$$

$$\frac{dI}{dt} = \left(m + \frac{\lambda I}{1+I}\right)S - \mu_1 IY - \mu_2 IZ - \sigma_1 IW - d_1 I, \tag{6}$$

$$\frac{dY}{dt} = e_1 IY - \alpha_1 YZ - d_2 Y, \tag{7}$$

$$\frac{dZ}{dt} = e_2 IZ - \alpha_2 YZ - d_3 Z, \tag{8}$$

$$\frac{dU}{dt} = \pi - \sigma_3 U(S+I) - d_4 U, \tag{9}$$

$$\frac{dW}{dt} = \sigma_3 U(S+I) - d_5 W, \tag{10}$$

where $W(t)$ is the toxicant concentration in the prey population at time t and $U(t)$ is the environment concentration of toxicant at time t. Here, the new parameters can be described as follows: π is the exogenous inputrate of the toxicant in the environment; d_4 is the natural depletion rate of the environmental toxicant; d_5 is the natural washout of the toxicant from organism; σ_1 and σ_2 are the rates at which susceptible and infected prey are decreeing due to the toxicant and σ_3 is the uptake rate of toxicant by organism.

The existence of the equilibrium points of system (2) can be guaranteed easily by using basic routine techniques and following Routh-Hurwitz criteria. It turns out that we have the trivial equilibrium point $J(0, 0, 0, 0, \frac{\pi}{d_4}, 0)$, which always exists, and the predators free equilibrium point $(S_0, I_0, 0, 0, U_0, W_0)$.

Boundedness

The following theorem ensures the boundedness of the system (2).

Theorem 1. All solutions of the system (2) that start in R^6_+ are uniformly bounded, that is

$$\mathrm{Sup}(S+I+Y+Z+W+U) \geq \frac{\pi+(r+d)K}{d} \tag{11}$$

Proof. The proof of theorem is similar to the case where the extra conditions are not included, it is omitted here as it is easy.

Analysis of non survival equilibrium point

The variation matrix of the non survival equilibrium point is

$$J(0,0,0,0,\frac{\pi}{d_4},0) = (b_{ij})_{6\times 6},$$

where $b_{11} = r-m$, $b_{21} = m$, $b_{22} = -d_1$, $b_{33} = -d_2$, $b_{44} = -d_3$, $b_{55} = -d_4$, $b_{66} = -d_5$ and all other entries are zeros. So the eigenvalues of $J(0,0,0,0,\frac{\pi}{d_4},0)$ are $r-m$, $-d_1$, $-d_2$, $-d_3$, $-d_4$ and $-d_5$. $(0,0,0,0,\frac{\pi}{d_4},0)$ is locally asymptotically stable if and only if $r<m$.

Analysis of the free predator equilibrium point

The free predator equilibrium point is $(S_0, I_0, 0, 0, U_0, W_0)$, where S_0, I_0, U_0 and W_0 are positive solutions of the following system

$$r\left(1-\frac{S}{K}\right) - (m+\frac{\lambda I}{1+I}) - \sigma_1 W = 0,$$

$$\left(m+\frac{\lambda I}{1+I}\right)S - \mu_1 IY - \mu_2 IZ - \sigma_1 IW - d_1 I = 0,$$

$$\pi = \sigma_3 U(S+I) + d_4 U,$$

$$\sigma_3 U(S+I) = d_5 W.$$

Theorem 2. The free predator equilibrium point of the system (2) is locally stable if the following conditions hold

$$\frac{\lambda S_0}{(1+I_0)^2} < \sigma_1 W_0 + d_1,$$

$$I_0 < \min\{\frac{d_2}{e_1}\frac{d_3}{e_2}\},$$

$$\frac{r}{K} < \frac{\lambda}{(1+I_0)^2} + \sigma_1,$$

$$\left|\frac{\lambda S_0}{(1+I_0)^2} - \sigma_1 W_0 - d_1\right| < m + \frac{\lambda I_0}{1+I_0} + \mu_1 I_0 + \mu_2 I_0 + \sigma_2 I_0,$$

$$\max\{d_4, d_5\} < 2\sigma_3 U_0.$$

Proof. Using Gerschgorin Theorem, one can easily prove the theorem.

Permanence of the population

In this section we give criteria for the persistence of the population in the system as shown in the following theorem.

Theorem 3. If the following conditions
(i) $(m + \theta\lambda + \sigma_1\theta) \leq 1$,
(ii) $\max\{\mu_1, \mu_2, \sigma_1\} \leq 1$,
(iii) $\max\left\{\frac{\alpha_1\theta+d_2}{e_1}, \frac{\alpha_2\theta+d_3}{e_2}\right\} < \frac{m\beta}{(d_1+\theta)}$
hold, then the population in the system (2) is persistent.

Proof. From the first equation of the system (2) and using (11), we obtain

$$\frac{dS}{dt} \geq rS\left(1 - \frac{S}{K}\right) - (m + \lambda\theta)\,S - \sigma_1\theta S = \ rS\left(1 - (m + \theta\lambda + \sigma_1\theta) - \frac{S}{K}\right),$$

where $\theta = \frac{\pi+(r+d)K}{d}$. It gives us

$$\lim_{t\to\infty} \inf\left(S\left(t\right)\right) \ \geq \left(1 - (m + \theta\lambda + \sigma_1\theta)\right) K = \beta. \tag{12}$$

Due to condition (i) we have

$$\lim_{t\to\infty} \inf\left(S\left(t\right)\right) \ > 0.$$

Using (12) and condition (ii), we get

$$\frac{dI}{dt} \geq m\beta - (d_1 + \theta)I,$$

that is

$$\lim_{t\to\infty} \inf\left(I(t)\right) \ \geq \frac{m\beta}{(d_1 + \theta)} > 0\ . \tag{13}$$

By using (13) and condition (iii), we get

$$\frac{dY}{dt} \geq \left(e_1\frac{m\beta}{(d_1 + \theta)} - \alpha_1\theta - d_2\right)Y,$$

so that

$$\lim_{t\to\infty} \inf\left(Y(t)\right) \ \geq Y_0 > 0\ ,$$

and

$$\frac{dZ}{dt} \geq (e_2\frac{m\beta}{(d_1 + \theta)} - \alpha_2\theta - d_3)Z,$$

hence

$$\lim_{t\to\infty} \inf\left(Z(t)\right) \ \geq Z_0 > 0\ ,$$

where Y_0 and Z_0 are initial values.

Numerical simulations

With the following parameter values

$$\begin{gathered} r = 0.999,\ K = 50,\ m = \mu_1 = \mu_2 = \lambda = \sigma_3 = 1, \\ \sigma_1 = d_1 = d_2 = d_3 = 0.5,\ \sigma_2 = 0.6,\ e_1 = e_2 = 0.9, \\ \alpha_1 = 2,\ \alpha_2 = 1.9,\ d_4 = d_5 = 0.1,\ \pi = 100 \end{gathered} \tag{14}$$

the system approaches the non survival equilibrium point as shown in Figure 1.

But if we neglect the affect of the toxicant then with the same parameter values (14) the system approaches the predator free equilibrium point $(0.0509, 0.1133, 0, 0)$. That is the prey population will survive as shown in Figure 2.

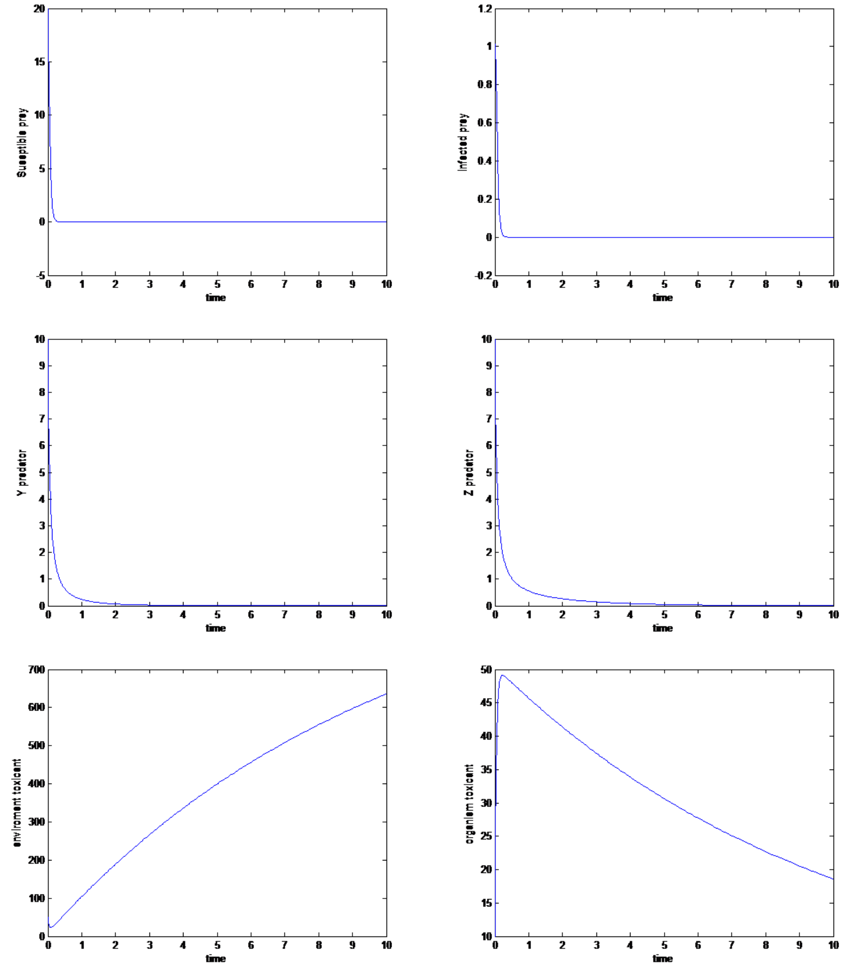

Figure 1.

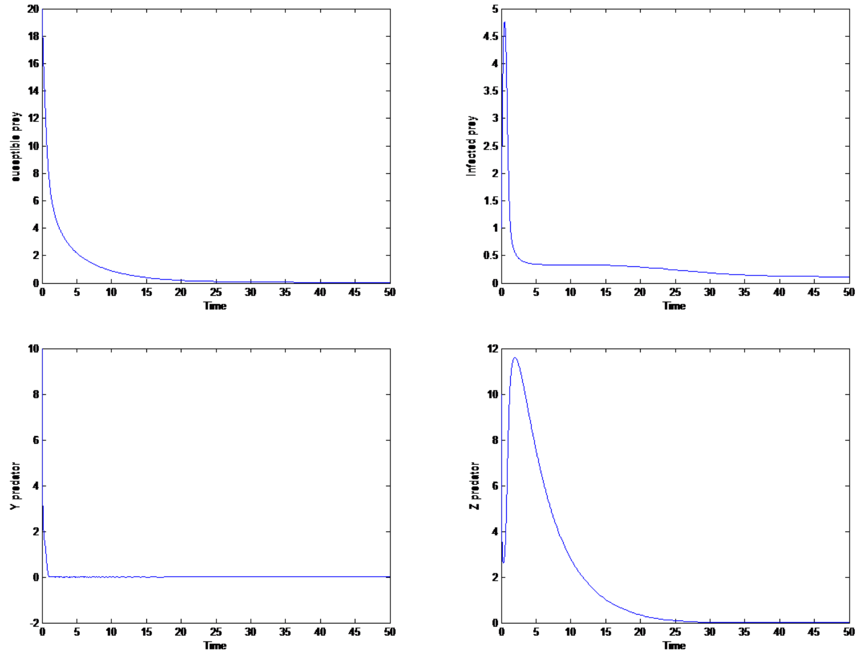

Figure 2.

Conclusions

In this paper, we study the effect of toxicant on dynamical behavior of proposed model (1). We give the sufficient conditions for permanence of the system and local asymptotic stability of non survival equilibrium point and predator free equilibrium point. We have discovered that decreasing the intrinsic grow rate of the susceptible prey below a contact rate value, as shown in (14), the system (2) approaches a locally asymptotically stable non survival point. However, if we neglect the effect of the toxicant, then for the same set of parameter values (14) system approaches the predator free equilibrium point $(0.0509, 0.1133, 0, 0)$. That is the prey population will survive.

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Э. Хинжал, Ш. Мохаммед, Б. Каймакамзаде

Токсиканттан және тарайтын SI ауруы бар екі бәсекелес жыртқыштардан және құрбаннан тұратын экоэпидемиологиялық моделдің тұрақтылығын талдау

Мақалада екі экоэпидемиологиялық модель зерттелген. Біріншісі сезімтал құрбан мен жұқтырған құрбанның байланысы арқылы таралатын, SI ауруы түрін шығаратын екі бәсекелес жыртқыштан тұрады. Бұл модель сызықты функционалды болжайды. Екінші модель токсиканттың әсерін есептегенде, біріншінің модификациясы болып табылады. Авторлар ұсынылған модель бойынша тіршілік етпейтін және еркін тепе-теңдік нүктелерінің динамикалык тәртібін қарастырған.

Кілт сөздер: орнықтылық талдауы, эпидемиологиялық моделі, олжа, жыртқыш.

Э. Хинжал, Ш. Мохаммед, Б. Каймакамзаде

Анализ устойчивости экоэпидемиологической модели, состоящей из жертвы и двух конкурирующих хищников с SI-болезнью в добыче и токсиканте

В статье исследованы две экоэпидемиологические модели. Первая состоит из добычи и двух конкурирующих хищников с SI-болезнью у видов добычи, распространяющихся путем контактов между восприимчивой жертвой и инфицированной жертвой. Эта модель предполагает линейный функциональный отклик. Вторая модель является модификацией первой, когда учитывается влияние токсиканта. Авторами рассмотрено динамическое поведение точек невыживания и свободного равновесия предложенной ими модели.

Ключевые слова: анализ устойчивости, эпидемиологическая модель, добыча, хищник.

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On the numerical schemes for Langevin-type equations

In this paper, a numerical approach is proposed based on the variation-of-constants formula for the numerical discretization Langevin-type equations. Linear and non-linear cases are treated separately. The proofs of convergence have been provided for the linear case, and the numerical implementation has been executed for the non-linear case. The order one convergence for the numerical scheme has been shown both theoretically and numerically. The stability of the numerical scheme has been shown numerically and depicted graphically.

Keywords: difference schemes, stochastic oscillators, Langevin equation, variation of constants

Introduction

In the beginning of the 20th century Paul Langevin discovered a very successful representation of the Brownian Motion [1]. This representation has been used as a fundamental building block, modified and generalized to analyze a large class of important stochastic processes. In simple terms, he applied the Newton's second law to a Brownian particle and obtained the differential equation that is known as the Langevin equation.

Due to its fundamental nature the generalized and modified versions of the Langevin equation has been used for modeling particle movements in so many different fields. [2] shows how it could be utilized in the statistical mechanics . Kubo introduces a generalized version of the equation for different applications [3, 4]. [5] introduces a structure of energetics into the stochastic system described by the Langevin equation and applies it in the thermodynamics context. [6] shows that how the Heisenberg-Langevin equation can be used to derive a Schrödinger equation for a Brownian particle interacting with a thermal environment. [7] used an approximate time-evolution equation of the Langevin type in modeling chemically reacting systems. [8] applies the Langevin equation in a stochastic control problem. [9] numerically investigates the Brownian motion of particles in a fluid with inhomogeneous temperature field.

In this study, a modified version of the Langevin equation has been studied from a numerical perspective. The convergence rate analysis of numerical schemes designed for these type of equations have been examined thoroughly in the literature. For a general treatment of numerical solutions of stochastic differential equations the reader is referred to [10].

[11] considers similar stochastic differential equations and analyzes the convergence rate of a numerical method where the approximation of the drift coefficient is done by the local linearization method and the diffusion coefficient by the Euler method. It is shown that order one convergence is obtained which is in line with the results obtained in this paper. The order of convergence of the Euler method for neutral stochastic functional differential equations has been studied in [12] where also similar order of convergence has been achieved. Convergence performance of different numerical integrators have been discussed in [13, 14] specifically for the Langevin-type equations, and weak convergence of order one has been obtained.

[15] considered the same Langevin-type equation

$$\ddot{X}_t = X_t - X_t^3 - \nu \dot{X}_t + \sigma \dot{W}_t \tag{1}$$

and approached to solve the equation by putting it into the form of

$$\ddot{X}_t + \nu \dot{X}_t = X_t - X_t^3 + \sigma \dot{W}_t \tag{2}$$

[15] obtained numerical schemes for the approximation of the solution (2). While discretizing the integral he used the trapezoidal rule. The numerical schemes are obtained by the variation-of-constants formula, however, no analysis of convergence of the numerical schemes has been given.

In this study, equation (1) has been considered under the form of

$$\ddot{X}_t + \nu \dot{X}_t - X_t = -X_t^3 + \sigma \dot{W}_t \tag{3}$$

Therefore, slightly different numerical schemes are obtained for the approximation of the solution of the equation (1). In addition to this, while discretizing the integral the left hand rule has been used as opposed to the trapezoidal rule. The results in the existing literature have been obtained but in an easier and more straight forward way. Furthermore, higher order of convergence rates have been established both for one step convergence and general n step convergence.

The organization of the paper is as follows. In section 2, an explicit numerical scheme has been derived for equation (1). The convergence analysis has been worked out in detail and order h convergence has been proved. In section 3, the theoretical results obtained in the previous section have been verified and a further stability analysis has been carried out. Finally, in section 4, the results are summarized and the paper is concluded.

Numerical schemes for Langevin-type equations

Now, let us consider the oscillator with cubic restoring force and additive noise from [15].

$$\ddot{X}_t = X_t - X_t^3 - \nu \dot{X}_t + \sigma \dot{W}_t. \tag{4}$$

Let us consider the Langevin-type Eq.(4) in the form

$$\ddot{X}_t + \nu \dot{X}_t - X_t = -X_t^3 + \sigma \dot{W}_t. \tag{5}$$

Let us write Eq.(5) as a system of first-order Ito stochastic differential equations

$$\begin{pmatrix} dX_t \\ dY_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -\nu \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ -X_t^3 + \sigma dW_t \end{pmatrix} \tag{6}$$

Let us find the unique solution of Eq.(6) using the method of variation of constants formula. Namely, first let us find the solution of homogeneous part. For this consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -\nu \end{pmatrix}.$$

The eigenvalues of the matrix A are $r = \frac{-\nu+\sqrt{\nu^2+4}}{2}$ and $-r-\nu$, with the corresponding eigenvectors $(1\ ,r)^T$ and $(1\ ,-r-\nu)^T$, respectively. Using these information, we can write the matrix A as a Jordan canonic form to write the exponential matrix e^{At} as

$$e^{At} = \begin{pmatrix} 1 & 1 \\ r & -r-\nu \end{pmatrix} \begin{pmatrix} e^{rt} & 0 \\ 0 & e^{(-r-\nu)t} \end{pmatrix} \frac{1}{-2r-\nu} \begin{pmatrix} -r-\nu & -1 \\ -r & 1 \end{pmatrix}.$$

From here the solution of homogeneous part is found as

$$X_t = \frac{1}{2r+\nu} \left(\alpha_{11}(t) X_0 + \alpha_{12}(t) Y_0 \right)$$

$$Y_t = \frac{1}{2r+\nu}\left(\alpha_{21}(t)X_0 + \alpha_{22}(t)Y_0\right),$$

where

$$\alpha_{11}(t) = (r+\nu)e^{rt} + re^{(-r-\nu)t},\ \alpha_{12}(t) = e^{rt} - e^{(-r-\nu)t},$$
$$\alpha_{21}(t) = r(r+\nu)e^{rt} - r(r+\nu)e^{(-r-\nu)t} \text{ and } \alpha_{22}(t) = re^{rt} + (r+\nu)e^{(-r-\nu)t}.$$

Therefore, by the variation of constants formula the solution of the non-homogeneous Eq.(6) is

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = e^{At}\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \int_0^t e^{A(t-s)}\begin{pmatrix} 0 \\ -X_s^3 + \sigma dW_s \end{pmatrix} ds.$$

Hence,

$$X_t = \frac{1}{2r+\nu}(\alpha_{11}(t)X_0 + \alpha_{12}(t)Y_0) + \frac{1}{2r+\nu}\int_0^t \alpha_{12}(t-s)(-X_s^3 + \sigma\dot{W}_s)ds,$$
$$Y_t = \frac{1}{2r+\nu}\left(\alpha_{21}(t)X_0 + \alpha_{22}(t)Y_0\right) + \frac{1}{2r+\nu}\int_0^t \alpha_{22}(t-s)(-X_s^3 + \sigma\dot{W}_s)ds.$$

Using the fact that $e^{At}e^{As} = e^{A(t+s)}$, discretizing the integrals with the left hand rule gives the following explicit numerical scheme

$$X_{n+1} = \frac{1}{2r+\nu}(\alpha_{11}(h)X_n + \alpha_{12}(h)Y_n) - \frac{h}{2r+\nu}\alpha_{12}(h)X_n^3 + \frac{\sigma}{2r+\nu}\alpha_{12}(h)\Delta W_n, \tag{7}$$

$$Y_{n+1} = \frac{1}{2r+\nu}(\alpha_{21}(h)X_n + \alpha_{22}(h)Y_n) - \frac{h}{2r+\nu}\alpha_{22}(h)X_n^3 + \frac{\sigma}{2r+\nu}\alpha_{22}(h)\Delta W_n. \tag{8}$$

It is clearly seen that the solution of linear part of non-homogeneous equation is

$$X_t = \frac{1}{2r+\nu}(\alpha_{11}(t)X_0 + \alpha_{12}(t)Y_0) - \frac{\sigma}{2r+\nu}\int_0^t \alpha_{12}(t-s)dW_s, \tag{9}$$

$$Y_t = \frac{1}{2r+\nu}\left(\alpha_{21}(t)X_0 + \alpha_{22}(t)Y_0\right) - \frac{\sigma}{2r+\nu}\int_0^t \alpha_{22}(t-s)dW_s, \tag{10}$$

and discretization of linear part is

$$X_{n+1} = \frac{1}{2r+\nu}(\alpha_{11}(h)X_n + \alpha_{12}(h)Y_n) + \frac{\sigma}{2r+\nu}\alpha_{12}(h)\Delta W_n, \tag{11}$$

$$Y_{n+1} = \frac{1}{2r+\nu}(\alpha_{21}(h)X_n + \alpha_{22}(h)Y_n) + \frac{\sigma}{2r+\nu}\alpha_{22}(h)\Delta W_n. \tag{12}$$

Lemma 1. For the numerical solution of linear first order system of differential equation

$$\begin{pmatrix} dX_t \\ dY_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \nu \end{pmatrix}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma dW_t \end{pmatrix} \tag{13}$$

consider numerical scheme (11) and (12). Then, the mean square errors after one step of the numerical schemes satisfy the following estimates:

$$(E[|X_1 - X_h|^2])^{1/2} \leq C_1(T)\sigma h^{3/2}, \tag{14}$$

$$(E[|Y_1 - Y_h|^2])^{1/2} \leq C_2(T)\sigma h^{3/2}, \tag{15}$$

where the constants $C_1(T)$ and $C_2(T)$ are independent of σ and h, but depend on T. Here, X_h, Y_h denote the exact solution after a time h and X_1, Y_1 denote the numerical solution after one step. That is the local errors are of order $3/2$ uniformly.

Proof. By definition,

$$E[|X_1 - X_h|^2] = (\frac{\sigma}{2r+\nu})^2 E(\int_0^h [(\alpha_{12}(h) - \alpha_{12}(h-s))dw_s])^2,$$

but using Itô isometry, we get

$$= (\frac{\sigma}{2r+\nu})^2 \int_0^h [\alpha_{12}(h) - \alpha_{12}(h-s)]^2 ds.$$

Then by the mean value theorem, we have

$$= (\frac{\sigma}{2r+\nu})^2 \int_0^h [\alpha'_{12}(\xi(s))(h-(h-s))]^2 ds$$

for some $h - s < \xi(s) < h$.
Since we have $|\alpha'_{12}(\xi(s))| = |re^{r\xi} + (r+\nu)e^{(-r-\nu)\xi}| \le |re^{r\xi} + (r+\nu)e^{r\xi}| \le |(2r+\nu)e^{rh}|$ then we get

$$(E[|X_1 - X_h|^2]) \le \sigma^2 e^{2rh} h^3/3 \le \sigma^2 e^{2rT} h^3/3.$$

Hence, we have

$$(E[|X_1 - X_h|^2])^{1/2} \le \sigma C_1(T) h^{3/2}$$

for some positive constant $C_1(T)$ does not depend on h and σ, but depends on T.

The mean square error after one step for numerical scheme for velocity is

$$E[|Y_1 - Y_h|^2] = (\frac{\sigma}{2r+\nu})^2 E(\int_0^h [(\alpha_{22}(h) - \alpha_{22}(h-s))dw_s])^2.$$

But using Itô isometry, we get

$$= (\frac{\sigma}{2r+\nu})^2 \int_0^h [\alpha_{22}(h) - \alpha_{22}(h-s)]^2 ds.$$

Then by the mean value theorem, we have

$$= (\frac{\sigma}{2r+\nu})^2 \int_0^h [\alpha'_{22}(\xi(s))(h-(h-s))]^2 ds$$

for some $h - s < \xi < h$.
Since $|\alpha'_{22}(\xi(s))| = |r^2 e^{r\xi(s)} - (r+\nu)^2 e^{(-r-\nu)\xi(s)}| \le |r^2 e^{rh} - (r+\nu)^2 e^{(-r-\nu)h}| \le r^2 e^{rh} \le (2r+\nu)e^{rh}$ and since $\alpha'_{22}(\xi(s))$ is an increasing function, then we have

$$(E[|Y_1 - Y_h|^2])^{1/2} \le \sigma e^{rh} h^{3/2}/\sqrt{3} \le \sigma C_2(T) h^{3/2},$$

for some positive constant $C_2(T)$ does not depend on h and σ, but depends on T.

Corollary 1. Let c_p be a solution of the equation $e^x x^p = 1$, $0 < p < 1.5$. If we take in Lemma 1 the step size h with $h < (c_p)^{1/p}/2r$ then we have the mean square errors after one step of the numerical schemes satisfy the following estimates

$$(E[|X_1 - X_h|^2])^{1/2} \le C_1 \sigma h^{(3-p)/2}, \tag{16}$$

$$(E[|Y_1 - Y_h|^2])^{1/2} \le C_2 \sigma h^{(3-p)/2} \tag{17}$$

where the constants C_1 and C_2 are independent of σ, h and T. If we take for example $p = 1.2$, then we get the case $c_{1.2} = 0.6043$. Hence, for any $h < 0.6572/2r$, the mean square errors after one step of the numerical schemes satisfy

$$(E[|X_1 - X_h|^2])^{1/2} \leq C_1 \sigma h^{0.9},$$

$$(E[|Y_1 - Y_h|^2])^{1/2} \leq C_2 \sigma h^{0.9}.$$

To show general mean square errors at time T, we need to obtain the following estimates.

Lemma 2. a) We have $E|d_n^X| = E|d_n^Y| = 0$.
b) We have $E[(d_n^X)^2] = O(h^3)$, $E[(d_n^Y)^2] = O(h^3)$ and $E[|d_n^X d_n^Y|] = O(h^3)$, where

$$d_n^X = \frac{\sigma}{2r+\nu}\left(\int_{t_n}^{t_{n+1}} \alpha_{12}(t_{n+1}-s)dw_s - \alpha_{12}(h)\Delta W_n\right)$$

and

$$d_n^Y = \frac{\sigma}{2r+\nu}\left(\int_{t_n}^{t_{n+1}} \alpha_{22}(t_{n+1}-s)dw_s - \alpha_{22}(h)\Delta W_n\right).$$

Proof.
a) Since the Itô stochastic integral has expectation zero, the estimates $E|d_n^X| = E|d_n^Y| = 0$ follow.
b) By definition

$$E(d_n^X)^2 = (\frac{\sigma}{2r+\nu})^2 E\left(\int_{t_n}^{t_{n+1}} (\alpha_{12}(t_{n+1}-s) - \alpha_{12}(h))dW_s\right)^2.$$

Then, by the Itô's isometry we have

$$= (\frac{\sigma}{2r+\nu})^2 \int_{t_n}^{t_{n+1}} (\alpha_{12}(t_{n+1}-s) - \alpha_{12}(h))^2 ds.$$

But by the mean value theorem

$$= (\frac{\sigma}{2r+\nu})^2 \int_{t_n}^{t_{n+1}} ((n+1)h - s - h)^2 (\alpha'_{12}(\xi(s)))^2 ds,$$

for some $t_{n+1} - s < \xi(s) < h$, for the differentiable function $\alpha_{12}(x) = e^{rx} - e^{(-r-\nu)x})$ we have $|\alpha'_{12}(\xi(s))| \leq \leq (2r+\nu)e^{rh}$. Then

$$\leq (\frac{\sigma}{2r+\nu})^2 \int_{t_n}^{t_{n+1}} (nh - s)^2 ((2r+\nu)e^{rh})^2 ds$$

$$= \sigma^2 e^{2rh} \int_{t_n}^{t_{n+1}} (n^2h^2 - 2nhs + s^2) ds = \sigma^2 e^{2rT} h^3/3,$$

for any $h < c_0/2r$ since $\int_{t_n}^{t_{n+1}} (n^2h^2 - 2nhs + s^2)ds = h^3/3$.
In the same manner, by definition

$$E(d_n^Y)^2 = (\frac{\sigma}{2r+\nu})^2 E\left(\int_{t_n}^{t_{n+1}} (\alpha_{22}(t_{n+1}-s) - \alpha_{22}(h))dW_s\right)^2.$$

then, by the Itô's isometry we have

$$= (\frac{\sigma}{2r+\nu})^2 \int_{t_n}^{t_{n+1}} (\alpha_{22}(t_{n+1}-s) - \alpha_{22}(h))^2 ds$$

But by the mean value theorem

$$= (\frac{\sigma}{2r+\nu})^2 \int_{t_n}^{t_{n+1}} ((n+1)h - s - h)^2 (\alpha'_{22}(\xi(s)))^2 ds,$$

for some $t_{n+1} - s < \xi < h$ and for the differentiable function $\alpha_{22}(x) = re^{rx} + (r+\nu)e^{(-r-\nu)x})$. Since the function $|\alpha'_{22}(x)|$ is an increasing function, $\alpha'_{22}(\xi(s)) \leq r^2 e^{rh} - (r+\nu)^2 e^{(-r-\nu)h} \leq r^2 e^{rh} \leq (2r+\nu)e^{rh})$. Then

$$\leq (\frac{\sigma}{2r+\nu})^2 \int_{t_n}^{t_{n+1}} (nh - s)^2 ((2r+\nu)e^{rh})^2 ds$$

$$= \sigma^2 e^{2rh} \int_{t_n}^{t_{n+1}} (n^2h^2 - 2nhs + s^2) ds = \sigma^2 e^{2rT} h^3/3.$$

Now, let us find an estimate for $|d_n^X d_n^Y|$. But by the fact that expectation of product of independent increments is zero, we have

$$|d_n^X d_n^Y| \leq (\frac{\sigma}{2r+\nu})^2 \int_{t_n}^{t_{n+1}} (\alpha_{12}(t_{n+1} - s) - \alpha_{12}(h))(\alpha_{22}(t_{n+1} - s) - \alpha_{22}(h)) ds.$$

But by the mean value theorem, we obtain

$$|d_n^X d_n^Y| \leq (\frac{\sigma}{2r+\nu})^2 \int_{t_n}^{t_{n+1}} ((n+1)h - s - h)^2 |\alpha'_{12}(\psi(s))||\alpha'_{22}(\xi(s))| ds$$

for some $t_{n+1} - s < \psi(s) < h$ and $t_{n+1} - s < \xi(s) < h$. Hence,

$$\leq (\frac{\sigma}{2r+\nu})^2 \int_{t_n}^{t_{n+1}} (nh - s)^2 ((2r+\nu)e^{rh})(2r+\nu)e^{rh} ds$$

$$= r\sigma^2 e^{2rh} \int_{t_n}^{t_{n+1}} (n^2h^2 - 2nhs + s^2) ds = \sigma^2 e^{2rT} h^3/3.$$

Corollary 2. Let the positive real numbers p and c_p be as in Corollary 1. If we take in Lemma 2 the step size h with $h < (c_p)^{1/p}/2r$, then
a) $E[(d_n^X)^2] = O(h^{3-p})$, $E[(d_n^Y)^2] = O(h^{3-p})$ and $E[|d_n^X d_n^Y|] = O(h^{3-p})$, where the upper bounds for the estimates do not depend on σ, h, and T. If we take, for example $p = 1.2$, then we get the case $c_{1.2} = 0.6043$. Hence, for any $h < 0.6572/2r$
b) $E[(d_n^X)^2] = O(h^{1.8})$, $E[(d_n^Y)^2] = O(h^{1.8})$ and $E[|d_n^X d_n^Y|] = O(h^{1.8})$.
We now indicate the global mean-square error of the stochastic exponential integrators (11) and (12).

Theorem 1. Consider the numerical solution of (13), the method (11) and (12). Then, the mean-square errors of the numerical scheme satisfy
a) $(E|X_n - X_{t_n}|^2)^{1/2} \leq C_3(T)h$,
b) $(E|Y_n - Y_{t_n}|^2)^{1/2} \leq C_4(T)h$,
for some constants $C_3(T)$ and $C_4(T)$.

Proof. The recursive relation for the solution of linear part is

$$\begin{pmatrix} X_{t_{n+1}} \\ Y_{t_{n+1}} \end{pmatrix} = e^{Ah} \begin{pmatrix} X_{t_n} \\ Y_{t_n} \end{pmatrix} + \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-s)} \begin{pmatrix} 0 \\ \sigma \dot{W}_s \end{pmatrix} ds.$$

Using equations (11) and (12), we have

$$E_{n+1} = e^{Ah} E_n + d_n,$$

where $E_n = \begin{pmatrix} e_n^X \\ e_n^Y \end{pmatrix} = \begin{pmatrix} X_{t_n} - X_n \\ Y_{t_n} - Y_n \end{pmatrix}$ and $d_n = \begin{pmatrix} d_n^X \\ d_n^Y \end{pmatrix}$. Using the mathematical induction, we obtain the formula

$$E_{n+1} = e^{A(n+1)h} E_0 + \sum_{j=0}^{n} e^{A(n-j)h} d_n = \sum_{j=0}^{n} e^{A(n-j)h} d_j,$$

since $E_0 = \vec{0}$. Hence,

$$E[(e_{n+1}^X)^2] = (\frac{1}{2r+\nu})^2 E \left[\sum_{j=0}^{n} \left(\alpha_{11}((n-j)h) d_j^X + \alpha_{12}((n-j)h) d_j^Y \right) \right]^2$$

$$= (\frac{1}{2r+\nu})^2 E \sum_{j=0}^{n} \sum_{i=0}^{n} \left(\alpha_{11}((n-j)h) d_j^X + \alpha_{12}((n-j)h) d_j^Y \right) \left(\alpha_{11}((n-i)h) d_i^X + \alpha_{12}((n-i)h) d_i^Y \right)$$

since expectation of product of independent increments is zero, we have

$$= (\frac{1}{2r+\nu})^2 \sum_{j=0}^{n} \left((\alpha_{11}((n-j)h))^2 E((d_j^X)^2) + (\alpha_{12}((n-j)h))^2 E((d_j^Y)^2) \right)$$

$$+2(\frac{1}{2r+\nu})^2 \sum_{j=0}^{n} \left(\alpha_{11}((n-j)h) \alpha_{12}((n-j)h) E(d_j^X d_j^Y) \right)$$

$$= (\frac{1}{2r+\nu})^2 \sum_{j=0}^{n} \left(\alpha_{11}((n-j)h) + \alpha_{12}((n-j)h) \right)^2 O(h^3)$$

$$= (\frac{1}{2r+\nu})^2 \sum_{j=0}^{n} \left((r+\nu) e^{rjh} + r e^{(-r-\nu)jh} + e^{rjh} - e^{(-r-\nu)jh} \right)^2 O(h^3)$$

$$\leq (\frac{1}{2r+\nu})^2 \sum_{j=0}^{n} \left((2r+\nu+1) e^{rjh} \right)^2 O(h^3)$$

$$\leq (\frac{2r+\nu+1}{2r+\nu})^2 T e^{2rT} O(h^2).$$

Similarly, we get

$$E[(e_{n+1}^Y)^2] = (\frac{1}{2r+\nu})^2 E \left[\sum_{j=0}^{n} \left(\alpha_{21}((n-j)h) d_j^X + \alpha_{22}((n-j)h) d_j^Y \right) \right]^2$$

$$= (\frac{1}{2r+\nu})^2 \sum_{j=0}^{n} \left(\alpha_{21}((n-j)h) + \alpha_{22}((n-j)h) \right)^2 O(h^3)$$

$$= (\frac{1}{2r+\nu})^2 \sum_{j=0}^{n} \left(r(r+\nu) e^{rjh} - r(r+\nu) e^{(-r-\nu)jh} + r e^{rjh} + (r+\nu) e^{(-r-\nu)jh} \right)^2 O(h^3)$$

$$= (\frac{1}{2r+\nu})^2 \sum_{j=0}^{n} \left(r(r+\nu+1) e^{rjh} + (1-r)(r+\nu) e^{(-r-\nu)jh} \right)^2 O(h^3)$$

since $0 < r < 1$, we have

$$\leq (\frac{1}{2r+\nu})^2 \sum_{j=0}^{n} \Big((r(r+\nu+1)+(1-r)(r+\nu))e^{rjh}\Big)^2 O(h^3) \leq Te^{2rT}O(h^2).$$

This completes the proof of the theorem.

Corollary 3. Consider the numerical solution of (13), the method (11) and (12). Let $1 < p < 1.5$ and let the positive real number c_p be as in Corollary 1. In Theorem 1 if we take the step size h with $h < (c_p)^{1/p}/(2rj)$ for any $j = 1, 2, 3, ..., n$ and using Corollary 2, therefore the mean-square errors of the numerical scheme satisfy the convergence estimates
a) $(E|X_n - X_{t_n}|^2)^{1/2} \leq C_3 h^{(3-2p)/2}$
b) $(E|Y_n - Y_{t_n}|^2)^{1/2} \leq C_4 h^{(3-2p)/2}$
for some constants C_3 and C_4 independent of T.

Proof. By following the proof of Theorem 1, we have

$$E[(e^X_{n+1})^2]$$

$$\leq (\frac{1}{2r+\nu})^2 \sum_{j=0}^{n}(2r+\nu+1)^2 e^{2rjh}O(h^{3-p}) \leq (\frac{2r+\nu+1}{2r+\nu})^2 \frac{1}{(2r)^p} O(h^{3-2p})(1+\sum_{j=1}^{n}\frac{1}{j^p}).$$

Since the infinite series $\sum_j \frac{1}{j^p}$ converges for $p > 1$, we have

$$(E|X_n - X_{t_n}|^2)^{1/2} \leq C_3 h^{(3-2p)/2}.$$

But this estimate is independent of T.
Estimate b) for the velocity component is obtained in a similar way.

Numerical Results

For the comparison of the numeric solution of the difference equation and the analytical solution of the differential equation, the error terms are computed by the following formulation:

$$E_h = \frac{1}{N_{sim}} \left(\sum_{j=1}^{N_{sim}} (X_n - X_{t_n})^2 \right)^{1/2}. \tag{18}$$

Maintaining the same notation that has been used in the second section, we represent the analytical solution of system of equations (6) by X_{t_n}, and numerical solutions of the problem based on the equations (11)-(12) by X_n. The error terms are recorded for various values of h, i.e. size of the step in time. The results are shown in the Table 1 for $h = 0.1$, $h = 0.01$, $h = 0.001$ and $h = 0.0001$, respectively. In all of these numerical experiments, the number of simulations N_{sim} is kept constant at $10,000$. Hence, each numerical problem has been solved based on $10,000$ different sample paths for the process of Standard Brownian motion, W_t. As one could easily see from Table 1 and the way that the error is computed in equation (18) the convergence between the numerical and the analytical solutions is measured in the sense of pointwise convergence with respect to the time variable. Each row in the table measures the difference between the numerical and the analytical solution for a specific time point between $t = 0$ and $t = 1$. Finally, for each sample path this difference is computed, squared, summed, square rooted and averaged based on the number of simulations used, which is $10,000$, to arrive at the final value of the error term. This final step is the typical way of computing the error for Monte Carlo Simulation applications which is often called in the literature as the root mean square error.

Table 1

Comparison of the errors for the approximate solution of problem

Point in Time/Step Size	$h = 0.1$	$h = 0.01$	$h = 0.001$	$h = 0.0001$
t = 0.1	6.3198e-04	7.4114e-05	1.0097e-06	2.9488e-07
t = 0.2	0.0022	6.8840e-04	1.1159e-04	5.0846e-05
t = 0.3	0.0038	4.7510e-04	5.1366e-04	6.6956e-05
t = 0.4	0.0123	0.0025	1.9910e-04	2.8405e-05
t = 0.5	0.0120	0.0013	2.7540e-04	2.2375e-05
t = 0.6	0.0161	0.0064	0.0017	8.1783e-04
t = 0.7	0.0244	0.0061	6.4270e-04	9.8445e-05
t = 0.8	0.0511	0.0093	0.0019	9.1302e-04
t = 0.9	0.0616	0.0157	0.0077	0.0031
t = 1.0	0.0829	0.0033	0.0012	5.4713e-04

Some of the rows in Table 1 are highlighted in order to emphasize the order one convergence which is theoretically proved in *Theorem 1.* It is clear that for each cell in the Table the number of steps is multiplied by 10, hence the size of the step is divided by 10. It is expected that the error term goes down by a factor of 10 as one goes from left to right on each row. If the first row is considered, highlighted light blue, roughly the error terms are divided by 10 at every step going from left to right. If one carefully looks that that first highlighted row, he would see that every step there is one more digit that is 0. First row corresponds to the error term at $t = 0.1$. Similar observations can also be made about the other rows, especially on the pink highlighted row that corresponds to $t = 0.6$ and the yellow highlighted row which corresponds to $t = 1$. Figure 1 shows the behaviour of $E[X_t^2]$ computed along $10,000$ sample paths for a step size $h = 0.001$ on the time interval $[0, 100]$ along the numerical solution given by the previous section.

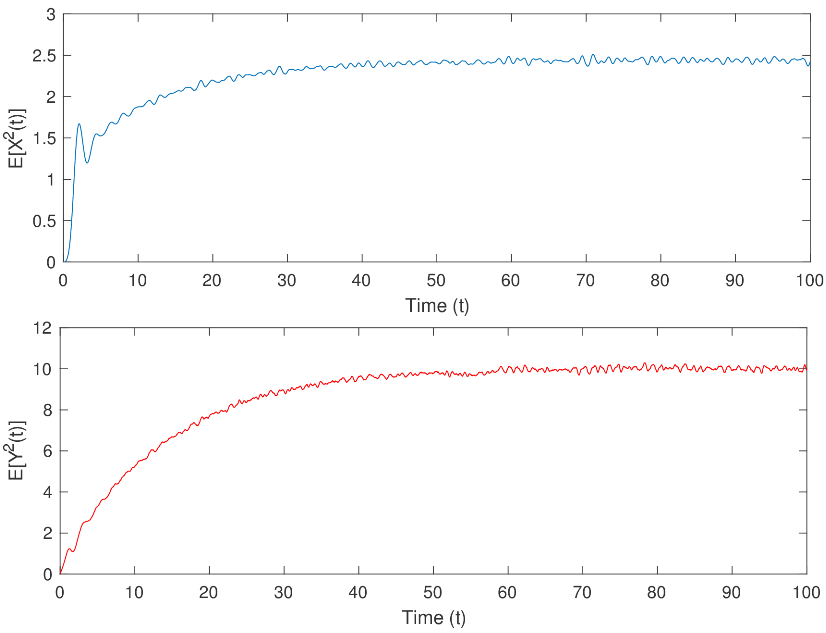

Figure 1. The convergence of expected value of the squared position and velocity functions

As $T \to \infty$, the numerical solution converges to the limit value 2.44, and the velocity converges to the value 9.92. [15] does the same numerical exercise with the same model parameters and initial conditions. [15] obtains a very similar result for the solution. Here, the numerical experiment has been extended to the velocity also. For further details on the physical interpretation of this result, the reader is referred to look at [16].

At least, this numerical experiment can be thought as a test of stability. In Table 1 error terms beyond $t = 1$ is not reported. One could be interested in the question that what happens to the numerical solution as the time grows. This is a partial answer to that question that the proposed numerical scheme is stable.

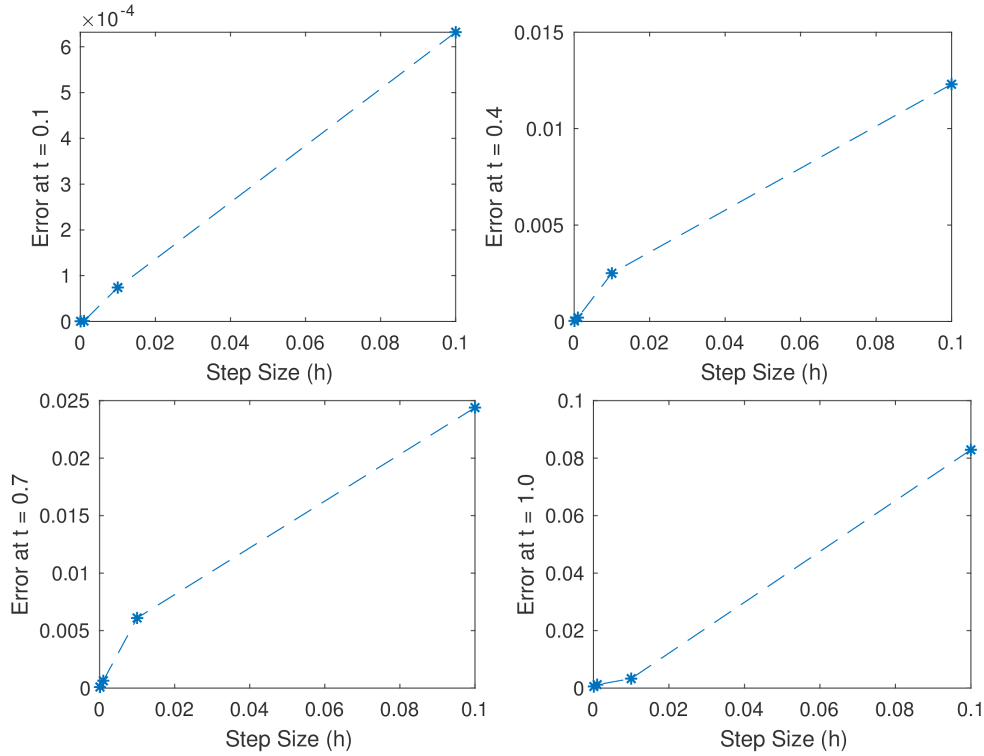

Figure 2. Error measured as the difference between the exact solution and the numerical solution at particular points in time between 0 and 1. Step sizes used vary from $h = 0.1$ down to $h = 0.0001$

Finally, let us have a look at the mean-square errors of the numerical scheme offered in the previous section. Fig. 2 illustrates the point wise mean-square errors at various times between $t = 0$ and $t = 1$ of the numerical scheme for the initial values $x_0 = 0$, $y_0 = 0$, and the parameters $\nu = 0.05$, $\sigma = 1$ and $M = 10,000$. The step size h ranges from 0.1 down to 0.0001. We observe a first order of convergence both in the position and in the velocity. This is the same mean-square order of convergence as the one offered in [15]. For the plots the log scale has been avoided intentionally. To emphasize the order $|h|$ convergence the original cale has been kept and the almost straight lines are observed as a result. Of course, these error terms are only for some specific values of t, for more detailed values for the error terms please also see Table 1.

Conclusion

In this study, a new explicit numerical scheme has been constructed for a specific Langevin-type equation. The main mathematical tool behind this construction is the variation-of-constants formulation. The convergence rate for one step has been established to be $3/2$ for the linear Langevin-type equation. As a result of this, the convergence rate at any step has established to be of order 1. In the main theorem of the paper, Theorem 1, the upper bounds for the convergence analysis depend on the upper limit of the time interval, T. In a later corollary, these upper bounds have been updated to versions that are also independent of the the upper limit of the time interval, T.

The proposed numerical scheme have been applied to the non-linear version of the Langevin-type equation. The theoretical results that have been proven for the linear case have been verified also by the non-linear case numerically. The stability of the numerical scheme has been shown numerically and graphically. Similar results have been obtained in the literature, but with semi-implicit numerical schemes. Just as strong results have been provided with explicit and easy to implement numerical difference equations. All of the numerical experiments have been in line with the existing literature, and occasional extensions, such as the stability of the velocity term, have been provided.

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Ланжевен типті теңдеуінің сандық схемасы туралы

Мақалада Ланжевен типті сандық теңдеулері үшін тұрақтыны варияциялау формуласына негізделген сандық тәсілі ұсынылған. Сызықты және сызықты емес жағдайлары жеке қарастырылған. Жинақты болуының дәлелдеуі сызықты жағдай үшін көрсетілген, ал сандық есептеуі сызықты емес жағдай үшін орындалған. Сандық схема үшін, бірінші ретті жинақтылығы теориялық және сандық түрде көрсетілген. Сандық схеманың орнықтылығы сандық түрде көрсетілген және графикалық түрде бейнеленген.

Кілт сөздер: айырымдық схемасы, стохастикалық осцилляторлары, Ланжевен теңдеуі, тұрақты вариациясы.

М. Акат, Р. Кошкер, А. Сирма

О численных схемах для уравнений типа Ланжевена

В статье предложен численный подход, основанный на формуле вариации констант для численных уравнений дискретизации типа Ланжевена. Линейные и нелинейные случаи рассмотрены отдельно. Доказательства сходимости были предоставлены для линейного случая, а численная реализация выполнена для нелинейного случая. Сходимость первого порядка для численной схемы показана теоретически и численно. Устойчивость численной схемы показана численно и изображена графически.

Ключевые слова: разностные схемы, стохастические осцилляторы, уравнение Ланжевена, вариация постоянных.

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A remark on elliptic differential equations on manifold

For elliptic boundary value problems of nonlocal type in Euclidean space, the well posedness has been studied by several authors and it has been well understood. On the other hand, such kind of problems on manifolds have not been studied yet. Present article considers differential equations on smooth closed manifolds. It establishes the well posedness of nonlocal boundary value problems of elliptic type, namely Neumann-Bitsadze-Samarskii type nonlocal boundary value problem on manifolds and also Dirichlet-Bitsadze-Samarskii type nonlocal boundary value problem on manifolds, in Hölder spaces. In addition, in various Hölder norms, it establishes new coercivity inequalities for solutions of such elliptic nonlocal type boundary value problems on smooth manifolds.

Keywords: differential equations on manifolds, well-posedness, self-adjoint positive definite operator.

Introduction

In the study of partial differential equations, the importance of the well-posedness (coercivity inequalities) is well known (see, for example [1–3]). Many researchers has been studied extensively the well-posedness of nonlocal boundary value problems of elliptic type partial differential equations in the Euclidean space, which is a flat manifold, (see, e.g. [4–18] and the references therein).

In the present article, we consider differential equations on smooth closed manifolds. We establish the well-posedness of nonlocal boundary value problems Hölder spaces. Furthermore, in various Hölder norms we establish new coercivity estimates for the solutions of such boundary value problems for elliptic equations.

Preliminaries

This section provides the basic definitions and fact about the Laplacian on Riemannian manifolds. The reader is referred to [19, 20] and the references therein for more information and unexplained subjects.

A Riemannian manifold is a pair $(\mathcal{M}, g)$, where $\mathcal{M}$ is a smooth manifold and to each $x \in \mathcal{M}$ $\langle \cdot, \cdot \rangle_{g(x)} : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$ is a positive definite symmetric non-degenerate bilinear form such that for all smooth vector fields $X, Y \in \Gamma_{C^\infty}(T\mathcal{M})$, $x \longmapsto \langle X(x), Y(x) \rangle_{g(x)}$ is smooth.

In the local coordinates $(x_1, \ldots, x_n)$, $\left\{ \left(\frac{\partial}{\partial x^1}\right)_x, \ldots, \left(\frac{\partial}{\partial x^n}\right)_x \right\}$ is the corresponding basis of tangent space $T_x\mathcal{M}$, $g_{ij} = \left\langle \left(\frac{\partial}{\partial x^i}\right)_x, \left(\frac{\partial}{\partial x^j}\right)_x \right\rangle_{g(x)}$, and g^{ij} are the entries of the inverse matrix of (g_{ij}).

$\nabla_g : \mathscr{C}^\infty(\mathcal{M}) \to \Gamma_{\mathscr{C}^\infty}(T\mathcal{M})$ is the *gradient* operator defined by

$$\langle \nabla_g \varphi, X \rangle_g = d\varphi(X)$$

for every $\varphi \in \mathscr{C}^{\infty}(\mathcal{M})$, $X \in \Gamma_{\mathscr{C}^{\infty}}(T\mathcal{M})$. In local coordinates $(x_1, \ldots, x_n)$, the gradient $\nabla_g \varphi$ is equal to

$$\sum_{i,j=1}^{n} g^{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_j}.$$

From the fact $d(\varphi + \psi) = d\varphi + d\psi$ for every $\varphi, \psi \in \mathscr{C}^1(\mathcal{M})$ it follows that $\nabla_g(\varphi + \psi) = \nabla_g \varphi + \nabla_g \psi$. The fact that $d(\varphi \cdot \psi) = \varphi \cdot d\psi + \psi \cdot d\varphi$ results $\nabla_g(\varphi \cdot \psi) = \varphi \cdot \nabla_g \psi + \psi \cdot \nabla_g \varphi$.

If $\omega \in \Omega^n(\mathcal{M})$ is an $n-$form and X is a vector field on $\mathcal{M}$, then $\iota_X \omega \in \Omega^{n-1}(\mathcal{M})$ is the $(n-1)-$form defined by

$$\iota_X \omega (X_1, \ldots, X_{n-1}) = \omega(X, X_1, \ldots, X_{n-1}).$$

Here, $X_1, \ldots, X_{n-1}$ are vector fields on the Riemaniann manifold $\mathcal{M}$. From the fact that $d(\iota_X \omega) \in \in \Omega^n(\mathcal{M})$ it follows that $d(\iota_X \omega) = \operatorname{div}_\omega(X)\omega$ for some number $\operatorname{div}_\omega(X)$.

Recall that $\operatorname{div}_g : \Gamma_{\mathscr{C}^{\infty}}(T\mathcal{M}) \to \mathscr{C}^{\infty}(\mathcal{M})$ is the *divergence* operator defined by

$$d(\iota_X \omega_g) = \operatorname{div}_g(X)\omega_g \text{ for every } X \in \Gamma_{\mathscr{C}^{\infty}}(T\mathcal{M}),$$

where $\omega_g \in \Omega^n(\mathcal{M})$ denotes the volume element obtained from the metric g. In local coordinates $(x_1, \ldots, x_n)$, for $X = \sum\limits_{j=1}^{n} b_j \frac{\partial}{\partial x_j} \in \Gamma_{\mathscr{C}^{\infty}}(T\mathcal{M})$ divergence becomes

$$\operatorname{div}_g(X) = \frac{1}{\sqrt{\det g}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(b_i \sqrt{\det g} \right). \tag{1}$$

Note that if $X, Y \in \Gamma_{C^{\infty}}(T\mathcal{M})$ and $\omega \in \Omega^n(\mathcal{M})$, then $\iota_{X+Y}\omega = \iota_X \omega + \iota_Y \omega$. By this fact, we have $\operatorname{div}_g(X+Y) = \operatorname{div}_g(X) + \operatorname{div}_g(Y)$ Moreover, from (1) it follows that for $\varphi \in \mathscr{C}^{\infty}(M)$

$$\operatorname{div}_g(\varphi X) = \varphi \operatorname{div}_g X + \langle \nabla_g \varphi, X \rangle_g.$$

The Laplace operator Δ_g on smooth functions $\mathscr{C}^{\infty}(\mathcal{M})$ is defined by

$$\Delta_g = -\operatorname{div}_g \circ \nabla_g$$

is the *Laplace-Beltrami* operator on $(\mathcal{M}, g)$.

Note that for any $\varphi, \psi \in \mathscr{C}^{\infty}(\mathcal{M})$

$$\begin{aligned} &\Delta_g(\varphi + \psi) = \Delta_g \varphi + \Delta_g \psi, \\ &\Delta_g(\varphi \cdot \psi) = \psi \Delta_g \varphi + \varphi \Delta_g \psi - 2\langle \nabla_g \varphi, \nabla_g \psi \rangle_g. \end{aligned}$$

In local coordinates $(x_1, \ldots, x_n)$, we have

$$\Delta_g = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{\det g} \frac{\partial}{\partial x_j} \right).$$

For example, let us consider the $n-$spere

$$\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}; x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

in geodesic polar coordinates, to be more precise $\xi : (0, \pi)^{n-1} \times (0, 2\pi) \to \mathbb{S}^n$,

$$\begin{aligned} &x_1 = \cos\theta_1, \\ &x_2 = \sin\theta_1 \cos\theta_2, \\ &x_3 = \sin\theta_1 \sin\theta_2 \cos\theta_3, \\ &\vdots \\ &x_n = \sin\theta_1 \sin\theta_2 \cdots \cos\theta_n, \\ &x_{n+1} = \sin\theta_1 \sin\theta_2 \cdots \sin\theta_n, \end{aligned} \tag{2}$$

where $0<\theta_1,\theta_2,\ldots,\theta_{n-1}<\pi$, $0<\theta_n<2\pi$. Then, we get

$$g_{\mathbb{S}^n}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots\cdots \\ 0 & \sin^2\theta_1 & 0 & 0 & 0 & \cdots\cdots \\ 0 & 0 & \sin^2\theta_1\sin^2\theta_2 & 0 & 0 & \cdots\cdots \\ 0 & 0 & 0 & \ddots & 0 & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots\cdots \\ 0 & 0 & 0 & 0 & 0 & \sin^2\theta_1\cdots\sin^2\theta_{n-1} \end{bmatrix},$$

$$\sqrt{\det g_{\mathbb{S}^n}}=\prod_{\ell=1}^{n-1}(\sin\theta_\ell)^{n-\ell}.$$

Moreover, the Laplace-Beltrami operator $\Delta_{\mathbb{S}^n}$ in these coordinates becomes

$$-\frac{1}{\prod\limits_{\ell=1}^{n-1}(\sin\theta_\ell)^{n-\ell}}\sum_{j=1}^{n}\frac{\partial}{\partial\theta_j}\left(a_j(\theta_1,\ldots,\theta_n)\frac{\partial}{\partial\theta_j}\right), \tag{3}$$

where $a_1=1$ and for $j=2,\ldots,n$, $a_j=\dfrac{\prod\limits_{\ell=1}^{n-1}(\sin\theta_\ell)^{n-\ell}}{\prod\limits_{i=1}^{j-1}\sin^2\theta_i}$.

We recall Stokes' Theorem and Divergence Theorem for manifolds.

Theorem 1. [Stokes′ Theorem] *Assume $\mathcal{M}$ is an oriented smooth compact n-manifold with boundary and $\alpha\in\Omega^{n-1}(\mathcal{M})$ have compact support. Denoting by $\iota:\partial\mathcal{M}\to\mathcal{M}$ the inclusion map, $\iota^*\alpha\in\Omega^{n-1}(\partial\mathcal{M})$. Then $\int_{\partial\mathcal{M}}\iota^*\alpha=\int_{\mathcal{M}}d\alpha$, or for short, $\int_{\partial\mathcal{M}}\alpha=\int_{\mathcal{M}}d\alpha$.*

Theorem 2. [Divergence Theorem] *Suppose $\mathcal{M}$ is a Riemannian manifold and X is a C^1-vector field on $\mathcal{M}$. Then,*

$$\int_{\mathcal{M}}\operatorname{div}_g(X)\,dV_g=\int_{\partial\mathcal{M}}\langle X,\nu\rangle_g\,d\sigma_g.$$

Here, div_g, dV_g, and ν denote respectively the divergence operator on $(\mathcal{M},g)$, the natural volume element on $(\mathcal{M},g)$, and the unit vector normal to $\partial\mathcal{M}$.

From these results it follows

Theorem 3. [Green′s Theorem] *For a compact Riemannian manifold $(\mathcal{M},g)$ with boundary $\partial\mathcal{M}$, if $\psi\in\mathscr{C}^1\left(\overline{\mathcal{M}}\right)$ and $\varphi\in\mathscr{C}^2\left(\overline{\mathcal{M}}\right)$, then the following equality is valid:*

$$\int_{\mathcal{M}}\psi\cdot\Delta_{\mathcal{M}}\phi\,dV_g=\int_{\mathcal{M}}\langle\nabla_g\psi,\nabla_g\phi\rangle\,dV_g-\int_{\partial\mathcal{M}}\psi\frac{\partial\phi}{\partial\nu}d\sigma_g.$$

Here, ∇_g denotes the gradient operator on the Riemannian manifold $(\mathcal{M},g)$.

Green's Theorem yields

Theorem 4. [19] If $(\mathcal{M},g)$ is a closed (i.e. compact without a boundary) Riemannian manifold, then

1 *(Formal self-adjointness): $\langle\psi,\Delta_{\mathcal{M}}\phi\rangle_{\mathscr{L}_2(\mathcal{M},dV_g)}=\langle\phi,\Delta_{\mathcal{M}}\psi\rangle_{\mathscr{L}_2(\mathcal{M},dV_g)}$.*

2 *(Positivity): $\langle\Delta_{\mathcal{M}}\phi,\phi\rangle_{\mathscr{L}_2(\mathcal{M},dV_g)}\geq 0$.*

Here, $\mathscr{L}_2(\mathcal{M},dV_g)$ is the Hilbert space

$$\{f:\mathcal{M}\to\mathbb{R};\langle\phi,\phi\rangle_{\mathscr{L}_2(\mathcal{M},dV_g)}:=\int_{\mathcal{M}}\phi^2(x)\,dV_g(x)<\infty\}.$$

Recall that eigenvalues of the Laplacian on $n-$sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ are $\lambda_\ell = \ell(\ell+n-1)$, $\ell = 0, 1, 2, \ldots$. The corresponding eigenfunctions are restrictions of harmonic polynomials to the sphere.

Elliptic differential equations on manifolds

Neumann-Bitsadze-Samarskii type nonlocal boundary value problem on manifold

Let $(a_i, b_i) \subset (0, \pi)$, $i = 1, \ldots, n-1$ and $(a_n, b_n) \subset (0, 2\pi)$. We consider the domain

$$\Omega = \xi((a_1, b_1) \times \cdots \times (a_{n-1}, b_{n-1}) \times (a_n, b_n)) \subset \mathbb{S}^n, \tag{4}$$

where $\xi : (0, \pi)^{n-1} \times (0, 2\pi) \to \mathbb{S}^n$ is the geodesic polar parametrization (2).

$$\begin{cases} -u_{tt}(t,x) + \Delta_{\mathbb{S}^n} u(t,x) + \delta u(t,x) = f(t,x), \quad x \in \Omega, \quad 0 < t < 1, \\ u_t(0,x) = 0, \quad u_t(1,x) = \sum\limits_{i=1}^{p} \beta_i u_t(\lambda_i, x), \quad x \in \Omega, \\ \sum\limits_{i=1}^{p} |\beta_i| \le 1, \quad 0 \le \lambda_1 < \cdots < \lambda_p < 1, \quad \dfrac{\partial u}{\partial \overrightarrow{n}}(t,x) \mid_{x \in \partial\Omega} = 0. \end{cases} \tag{5}$$

Here, $\Delta_{\mathbb{S}^n}$ is the Laplace-Beltrami operator on the Riemannian manifold $(\mathbb{S}^n, g_{\mathbb{S}^n})$ and $\delta > 0$.

We prove

Theorem 5. For the solutions of problem (5), the following coercivity estimate holds:

$$\|u_{tt}\|_{\mathscr{C}^\alpha(\mathscr{L}_2(\Omega, dV_g))} + \|u\|_{\mathscr{C}^\alpha\left(\mathscr{W}_2^2(\Omega, dV_g)\right)} \le \frac{K(\delta, \lambda_p)}{\alpha(1-\alpha)} \|f\|_{\mathscr{C}^\alpha(\mathscr{L}_2(\Omega, dV_g))}.$$

Here, K is independent of $f(t,x)$.

Let us consider Equation (5) as the following nonlocal boundary value problem of Bitsadze Samarskii type

$$\begin{cases} -U''(t) + \mathbf{L}U(t) = F(t), \quad 0 \le t \le 1, \\ U_t(0) = 0, \ U_t(1) = \sum\limits_{i=1}^{p} \beta_i U_t(\lambda_i), \\ \sum\limits_{i=1}^{p} |\beta_i| \le 1, \ 0 \le \lambda_1 < \cdots < \lambda_p < 1 \end{cases}$$

in $\mathscr{L}_2(\Omega, dV_g)$ with the self adjoint and positive definite operator $\mathbf{L} = \Delta_{\mathbb{S}^n} + \delta I$. Here, I denotes the identity operator.

The proof of Theorem 5 is based on the symmetry property of $\mathbf{L}$, Theorem 6 with $H = \mathscr{L}_2(\Omega, dV_g)$ and Theorem 7 on the coercivity inequality for the solution of elliptic differential problem in $\mathscr{L}_2(\Omega, dV_g)$.

Theorem 6. [17] Let A be a self-adjoint positive definite operator with dense domain $D(A)$ in a Hilbert space H. Let $\varphi, \psi \in E_\alpha\left(D\left(A^{1/2}\right), H\right)$. Then the following elliptic type differential problem

$$\begin{cases} -v_{tt}(t,x) + Av(t) = g(t), \quad 0 < t < 1, \\ v_t(0) = \varphi, \quad v_t(1) = \sum\limits_{i=1}^{p} \beta_i v_t(\lambda_i) + \psi, \\ \sum\limits_{i=1}^{p} |\beta_i| \le 1, \quad 0 \le \lambda_1 < \cdots < \lambda_p < 1 \end{cases} \tag{6}$$

is well-posed in Hölder space $\mathscr{C}^{\alpha}(H)$ and for the solutions of (6) the following coercivity inequality holds:

$$\|v''\|_{\mathscr{C}^{\alpha}(H)} + \|Av\|_{\mathscr{C}^{\alpha}(H)} \leq K(\delta)\left[\left\|A^{1/2}\varphi\right\|_{H} + \left\|A^{1/2}\psi\right\|_{H}\right] + \frac{K(\delta,\lambda_p)}{\alpha(1-\alpha)}\|g\|_{\mathscr{C}^{\alpha}(H)}.$$

Theorem 7. The solutions of the following elliptic differential problem

$$\begin{cases} \Delta_{\mathbb{S}^n} u(\xi(\overrightarrow{\theta})) = \omega(\xi(\overrightarrow{\theta})), \ \overrightarrow{\theta} = (\theta_1,\ldots,\theta_n) \in (a_1,b_1)\times\cdots\times(a_n,b_n), \\ \frac{\partial u(\xi(\overrightarrow{\theta})}{\partial \overrightarrow{n}} = 0, \ \overrightarrow{\theta} \text{ in boundary of } [a_1,b_1]\times\cdots\times[a_n,b_n] \end{cases}$$

satisfy the coercivity inequality

$$\sum_{i=1}^{n} \|u_{\theta_i\theta_i}\|_{\mathscr{L}_2(\Omega,dV_g)} \leq K_1||\omega||_{\mathscr{L}_2(\Omega,dV_g)}.$$

The proof of Theorem 7 is based on the following theorem.

Theorem 8. [8] For the solutions of the elliptic differential problem

$$\begin{cases} A^{\xi}u(\xi) = \omega(\xi), \ \ \xi \in (\alpha_1,\beta_1)\times\cdots\times(\alpha_n,\beta_n), \\ \frac{\partial u(\xi)}{\partial \overrightarrow{n}} = 0, \ \ \xi \text{ in boundary } [\alpha_1,\beta_1]\times\cdots\times[\alpha_n,\beta_n] \end{cases}$$

the following coercivity inequality

$$\sum_{i=1}^{n} \|u_{\xi_i\xi_i}\|_{\mathscr{L}_2((\alpha_1,\beta_1)\times\cdots\times(\alpha_n,\beta_n))} \leq K_2||\omega||_{\mathscr{L}_2((\alpha_1,\beta_1)\times\cdots\times(\alpha_n,\beta_n))}$$

is valid. Here, $A^{\xi} = \sum\limits_{r=1}^{n} \frac{\partial}{\partial\xi_r}\left(a_r(\xi)\frac{\partial}{\partial\xi_r}\right)$ *and* $a_r(\xi) \geq a > 0$, $r = 1,\ldots,n$.

Proof of Theorem 7. Clearly, the image $\xi(\overrightarrow{\theta})$ of boundary of the $n-$cube $[a_1,b_1]\times\cdots\times[a_n,b_n]$ is the boundary of Ω. This parametrization maps $(a_1,b_1)\times\cdots\times(a_n,b_n)$ to the interior of Ω. Let $u:\Omega\to\mathbb{R}$ be so that $\frac{\partial u}{\partial\nu}$ vanishes on the boundary of Ω. Then, $v = u\circ\xi : [a_1,b_1]\times\cdots\times[a_n,b_n] \to \mathbb{R}$ and $\frac{\partial v}{\partial \nu}$ vanishes on the boundary of the cube $[a_1,b_1]\times\cdots\times[a_n,b_n]$. Here, ν is the outward unit normal to the boundary.

For some constants $k,K>0$, on Ω we have $0 < k \leq \prod\limits_{\ell=1}^{n-1}(\sin\theta_\ell)^{n-\ell} \leq K$.

Equation (3) and Theorem 8 yield

$$\begin{aligned}
\int_{\Omega} |\Delta_{\mathbb{S}^n} u(x)|^2\, dV_g(x) &= \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \frac{\left\{\sum\limits_{j=1}^{n}\frac{\partial}{\partial\theta_j}\left(a_j(\overrightarrow{\theta})\frac{\partial u\circ\xi(\overrightarrow{\theta})}{\partial\theta_j}\right)\right\}^2}{\prod\limits_{\ell=1}^{n-1}(\sin\theta_\ell)^{n-\ell}} d\theta_n\cdots d\theta_1 \\
&\geq \frac{1}{K}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\left\{\sum_{j=1}^{n}\frac{\partial}{\partial\theta_j}\left(a_j(\overrightarrow{\theta})\frac{\partial u\circ\xi(\overrightarrow{\theta})}{\partial\theta_j}\right)\right\}^2 d\theta_n\cdots d\theta_1 \\
&= \frac{1}{K}\left\|A^{(\theta_1,\ldots,\theta_n)}u\circ\xi\right\|^2_{\mathscr{L}_2((a_1,b_1)\times\cdots\times(a_n,b_n))} \\
&= \frac{1}{K}\left\|A^{(\theta_1,\ldots,\theta_n)}v\right\|^2_{\mathscr{L}_2((a_1,b_1)\times\cdots\times(a_n,b_n))} \\
&\geq \frac{1}{K\cdot K_2^2}\left(\sum_{i=1}^{n}\|v_{\theta_i\theta_i}\|_{\mathscr{L}_2((a_1,b_1)\times\cdots\times(a_n,b_n))}\right)^2.
\end{aligned}$$

Hence, we obtain

$$\left(\int\limits_{\Omega} |\Delta_{\mathbb{S}^n} u(x)|^2 \, dV_g(x)\right)^{1/2} \geq \frac{1}{\sqrt{K} K_2} \sum_{i=1}^{n} \|v_{\theta_i\theta_i}\|_{\mathscr{L}_2((a_1,b_1)\times\cdots\times(a_n,b_n))} \, . \tag{7}$$

For $i = 1, \ldots, n$, we have

$$\begin{aligned}
\|v_{\theta_i\theta_i}\|_{\mathscr{L}_2((a_1,b_1)\times\cdots\times(a_n,b_n))} &= \left(\int\limits_{a_1}^{b_1} \cdots \int\limits_{a_n}^{b_n} |v_{\theta_i\theta_i}(\theta_1,\ldots,\theta_n)|^2 \, d\theta_n \cdots d\theta_1\right)^{1/2} \\
&\geq \left(\int\limits_{a_1}^{b_1} \cdots \int\limits_{a_n}^{b_n} |v_{\theta_i\theta_i}(\theta_1,\ldots,\theta_n)|^2 \, \frac{\prod\limits_{\ell=1}^{n-1} (\sin\theta_\ell)^{n-\ell}}{K} d\theta_n \cdots d\theta_1\right)^{1/2} \\
&= \frac{1}{\sqrt{K}} \left(\int\limits_{a_1}^{b_1} \cdots \int\limits_{a_n}^{b_n} |v_{\theta_i\theta_i}(\theta_1,\ldots,\theta_n)|^2 \prod_{\ell=1}^{n-1} (\sin\theta_\ell)^{n-\ell} \, d\theta_n \cdots d\theta_1\right)^{1/2} \\
&= \frac{1}{\sqrt{K}} \left(\int\limits_{a_1}^{b_1} \cdots \int\limits_{a_n}^{b_n} |(u\circ\xi)_{\theta_i\theta_i}(\theta_1,\ldots,\theta_n)|^2 \prod_{\ell=1}^{n-1} (\sin\theta_\ell)^{n-\ell} \, d\theta_n \cdots d\theta_1\right)^{1/2} \\
&= \frac{1}{\sqrt{K}} \|u_{\theta_i\theta_i}\|_{\mathscr{L}_2(\Omega, dV_g)} \, .
\end{aligned} \tag{8}$$

Combining equations (7) and (8), we get

$$\left(\int\limits_{\Omega} |\Delta_{\mathbb{S}^n} u(x)|^2 \, dV_g(x)\right)^{1/2} \geq \frac{1}{K \cdot K_2} \sum_{i=1}^{n} \|u_{\theta_i\theta_i}\|_{\mathscr{L}_2(\Omega, dV_g)} \, .$$

This is the end of the proof of Theorem 7.

Dirichlet-Bitsadze-Samarskii type nonlocal boundary value problem on manifold

Assume $(\mathcal{M}, g)$ is a closed orientable Riemannian manifold (such as $n-$sphere $\mathbb{S}^n$, $n-$torus $\mathbb{T}^n$). Let us consider the mixed boundary value problem of Dirichlet-Bitsadze-Samarskii type

$$\begin{cases}
-u_{tt}(t,x) + \Delta_{\mathcal{M}} u(t,x) + \delta u(t,x) = f(t,x), \;\; x \in \mathcal{M}, \;\; 0 < t < 1, \\
u(0,x) = \varphi(x), \;\; u(1,x) = \sum\limits_{j=1}^{p} \alpha_j u(\lambda_j, x) + \psi(x), \;\; x \in \mathcal{M}, \\
0 < \lambda_1 < \cdots < \lambda_p < 1, \;\; \sum\limits_{j=1}^{p} |\alpha_j| \leq 1,
\end{cases} \tag{9}$$

where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on the Riemannian manifold $(\mathcal{M}, g)$.

We prove

Theorem 9. If $\varphi, \psi \in D(\mathbf{L})$, then for the solution of (9) we have the following coercivity inequality

$$\|u_{tt}\|_{\mathscr{C}_{01}^{\alpha}(\mathscr{L}_2(\mathcal{M},dV_g))} + \|\mathbf{L}u\|_{\mathscr{C}_{01}^{\alpha}(\mathscr{L}_2(\mathcal{M},dV_g))}$$
$$\leq K\left[\|\mathbf{L}\varphi\|_{\mathscr{L}_2(\mathcal{M},dV_g)} + \|\mathbf{L}\psi\|_{\mathscr{L}_2(\mathcal{M},dV_g)}\right] + \frac{K(\delta,\lambda_1,\lambda_p)}{\alpha(1-\alpha)}\|f\|_{\mathscr{C}_{01}^{\alpha}(\mathscr{L}_2(\mathcal{M},dV_g))}.$$

Here, $K(\delta,\lambda_1,\lambda_p)$ does not depend on $\varphi(x), \psi(x)$, and $f(t,x)$.

Let us consider problem (9) as the following nonlocal boundary value problem of Bitsadze Samarskii type

$$\begin{cases} -U''(t) + \mathbf{L}U(t) = F(t), \quad t \in (0,1), \\ U(0) = \varphi, \quad U(1) = \sum\limits_{j=1}^{p} \alpha_j U(\lambda_j) + \psi, \\ 0 < \lambda_1 < \cdots < \lambda_p < 1, \quad \sum\limits_{j=1}^{p} |\alpha_j| \leq 1 \end{cases} \tag{10}$$

in $\mathscr{L}_2(\mathcal{M}, dV_g)$ with the self-adjoint and positive definite operator $\mathbf{L} = \Delta_{\mathcal{M}} + \delta I$. Here, I denotes the identity operator, $\|U\|_{\mathscr{L}_2(\mathcal{M},dV_g)} = \left(\int_{\mathcal{M}} U^2(x)dV_g(x)\right)^{1/2}$, and dV_g denotes natural volume element of $\mathcal{M}$ obtained from metric tensor g.

The proof of Theorem 9 relies on the following theorem.

Theorem 10. [16] Assume A is a self-adjoint positive definite operator with dense $D(A) \subset H$ in a Hilbert space H and $\varphi, \psi \in D(A)$. Then, the following boundary value problem

$$\begin{cases} -v_{tt}(t,x) + Av(t) = f(t), \quad 0 < t < 1, \\ v(0) = \varphi, \quad v(1) = \sum\limits_{j=1}^{p} \alpha_j v(\lambda_j) + \psi, \\ 0 < \lambda_1 < \cdots < \lambda_p < 1, \quad \sum\limits_{j=1}^{p} |\alpha_j| \leq 1 \end{cases}$$

is well-posed in Hölder space $\mathscr{C}_{01}^{\alpha}(H)$. Moreover, the solutions of the problem satisfy the following coercivity inequality

$$\|v''\|_{\mathscr{C}_{01}^{\alpha}(H)} + \|Av\|_{\mathscr{C}_{01}^{\alpha}(H)} \leq K\left[\|A\varphi\|_H + \|A\psi\|_H\right] + \frac{K(\delta,\lambda_1,\lambda_p)}{\alpha(1-\alpha)}\|f\|_{\mathscr{C}_{01}^{\alpha}(,H)}.$$

Here, $K(\delta,\lambda_1,\lambda_p)$ is independent of of $\varphi(x), \psi(x)$, and $f(t,x)$. $\mathscr{C}_{01}^{\alpha}(H)\,(0 < \alpha < 1)$ denotes the Banach space which is the completion of of smooth funtions $v : [0,1] \to H$ with the following norm

$$\|v\|_{\mathscr{C}_{01}^{\alpha}(H)} = \|v\|_{\mathscr{C}(H)} + \sup_{0 \leq t < t+\tau \leq 1} \frac{(1-t)^{\alpha}(t+\tau)^{\alpha}\|v(t+\tau) - v(t)\|_H}{\tau^{\alpha}}$$

and $\|v\|_{\mathscr{C}(H)} = \max\limits_{0 \leq t \leq 1} \|v(t)\|_H$.

Dirichlet-Bitsadze-Samarskii type nonlocal boundary value problem on a relatively compact domain

For the domain Ω in (4), let us consider the Dirichlet-Bitsadze-Samarskii type mixed boundary value problem

$$\begin{cases} -u_{tt}(t,x) + \Delta_{\mathbb{S}^n} u(t,x) = f(t,x), \quad x \in \Omega, \quad t \in (0,1), \\ u(0,x) = \varphi(x), u(1,x) = \sum_{j=1}^{p} \alpha_j u(\lambda_j, x) + \psi(x), \quad x \in \Omega, \\ 0 < \lambda_1 < \cdots < \lambda_p < 1, \quad \sum_{j=1}^{p} |\alpha_j| \leq 1, \\ u(t,x) = 0, \quad x \in \partial\Omega, \end{cases} \tag{11}$$

where $\Delta_{\mathbb{S}^n}$ is the Laplace-Beltrami operator on the Riemannian manifold $(\mathbb{S}^n, g_{\mathbb{S}^n})$.

We have

Theorem 11. The solutions of nonlocal boundary value problem (11) satisfy following coercivity inequality

$$\begin{aligned} \|u_{tt}\|_{\mathscr{C}_{01}^{\alpha}(\mathscr{L}_2(\Omega,dV_g))} + \|u\|_{\mathscr{C}_{01}^{\alpha}\left(\mathscr{W}_2^2(\Omega,dV_g)\right)} \leq {} & K \left[\|\varphi\|_{\mathscr{W}_2^2(\Omega,dV_g)} + \|\psi\|_{\mathscr{W}_2^2(\Omega,dV_g)} \right] \\ & + \frac{K(\delta,\lambda_1,\lambda_p)}{\alpha(1-\alpha)} \|f\|_{\mathscr{C}_{01}^{\alpha}(\mathscr{L}_2(\Omega,dV_g))}, \end{aligned}$$

where $K(\delta,\lambda_1,\lambda_p)$ does not depend on $\varphi(x), \psi(x)$, and $f(t,x)$.

Let us consider problem (11) as the nonlocal boundary value problem (10) in the Hilbert space $H = \mathscr{L}_2(\Omega, dV_g)$ with the self-adjoint positive definite operator $\mathbf{L} = \Delta_{\mathbb{S}^n}$.

The proof of Theorem 11 is based on the symmetry properties of the operator $\mathbf{L}$ defined by formula (11), Theorem 10 with $H = \mathscr{L}_2(\Omega, dV_g)$, and the following result which is about the coercivity estimate for the solution of the elliptic type differential equation in $\mathscr{L}_2(\Omega, dV_g)$.

Theorem 12. For the following differential equation of elliptic type

$$\begin{cases} \Delta_{\mathbb{S}^n} u(\xi(\overrightarrow{\theta})) = \omega(\xi(\overrightarrow{\theta})), \ \overrightarrow{\theta} = (\theta_1, \ldots, \theta_n) \in (a_1,b_1) \times \cdots \times (a_n,b_n), \\ u(\xi(\overrightarrow{\theta}) = 0, \ \overrightarrow{\theta} \ \textit{in boundary of} \ [a_1,b_1] \times \cdots \times [a_n,b_n] \end{cases}$$

we have the following coercivity estimate

$$\sum_{i=1}^{n} \|u_{\theta_i\theta_i}\|_{\mathscr{L}_2(\Omega,dV_g)} \leq K_1 ||\omega||_{\mathscr{L}_2(\Omega,dV_g)}.$$

The proof of Theorem 12 relies on the following theorem.

Theorem 13. [8] For the solutions of the elliptic differential problem

$$\begin{cases} A^{\xi} u(\xi) = \omega(\xi), \quad \xi \in (\alpha_1,\beta_1) \times \cdots \times (\alpha_n,\beta_n), \\ u(\xi) = 0, \quad \xi \ \textit{in boundary} \ \ [\alpha_1,\beta_1] \times \cdots \times [\alpha_n,\beta_n] \end{cases}$$

the coercivity inequality

$$\sum_{r=1}^{n} \|u_{\xi_r\xi_r}\|_{\mathscr{L}_2((\alpha_1,\beta_1)\times\cdots\times(\alpha_n,\beta_n))} \leq K_2 |\omega||_{\mathscr{L}_2((\alpha_1,\beta_1)\times\cdots\times(\alpha_n,\beta_n))}$$

is valid. Here, $A^{\xi}=\sum\limits_{r=1}^{n}\frac{\partial}{\partial\xi_r}\left(a_r(\xi)\frac{\partial}{\partial\xi_r}\right)$ *and* $a_r(\xi)\geq a>0,\ r=1,\dots,n.$

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А. Ашыралыев, Я. Созен, Ф. Незенжи

Көпбейнедегі эллипстік дифференциалдық теңдеу туралы ескерту

Евклидтік кеңістігінде бейлокальді типті эллипстік шеттік есептері үшін қойылған есептің корректілігі бірнеше авторлармен жақсы және толық зерттелген. Басқа жағынан, осы мәселелер көпбейнеде зерттелмеген. Мақалада тегіс тұйық көпбейнеде дифференциалдық теңдеу қарастырылған. Эллипстік типті бейлокальді шеттік есептің корректілігі қойылады, нақтырақ айтатын болсақ көпбейнеде, Гольдер кеңістігіндегі көпбейнеде Дирихле-Бицадзе-Самарский түріндегі бейлокальді шеттік есебі. Сонымен қатар, әртүрлі Гольдер нормасында тегіс көпбейнеде бейлокальді типті эллипстік шеттік есебін шығару үшін мәжбүрлі жаңа теңсіздіктер анықталған.

Кілт сөздер: көпбейнедегі дифференциалдық теңдеу, корректілігі, өзіне-өзі түйіндес оң анықталған оператор.

А. Ашыралыев, Я. Созен, Ф. Незенжи

Замечание об эллиптических дифференциальных уравнениях на многообразии

Для эллиптических краевых задач нелокального типа в евклидовом пространстве корректность поставленной задачи была хорошо изучена несколькими авторами. С другой стороны, такие проблемы на многообразиях широко не изучены. В настоящей статье рассмотрены дифференциальные уравнения на гладких замкнутых многообразиях. Установлена корректность нелокальных краевых задач эллиптического типа, а именно нелокальной краевой задачи типа Неймана-Бицадзе-Самарского на многообразиях, а также нелокальной краевой задачи типа Дирихле-Бицадзе-Самарского на многообразиях в пространствах Гольдера. Кроме того, в различных нормах Гольдера установлены новые неравенства коэрцитивности для решений краевых задач эллиптического нелокального типа на гладких многообразиях.

Ключевые слова: дифференциальные уравнения на многообразиях, корректность, самосопряженный положительно определенный оператор.

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Basic reproduction number and effective reproduction number for North Cyprus for fighting Covid-19

The aim of this paper is to show how North Cyprus fought with Covid-19 by using R_0 and R_t, as herd immunity. For that purpose, we used a SEIR model for basic reproduction number, R_0, and calculated R_t values by using R_0 values. North Cyprus is the first country in Europe to free from Covid-19 epidemic. One of the most important reasons for this is that the government decided to tackle Covid-19 pandemic by using R_0 and R_t daily. For R_0, we constructed a new SEIR model by using real data for North Cyprus. From March 11, 2020 to May 15, 2020, R_0 varies from 0.65 to 2.38.

Keywords: Covid-19, Northern Cyprus, epidemics, mathematical model.

Introduction

Coronavirus is the virus that causes one of the most infectious diseases, Covid-19 (namely SARS-CoV-2). This highly fatal disease began in December 2019, in Wuhan, China [1]. Disease has been named by the World Health Organization, after a new coronavirus discovered from an infected patient by Chinese Center for Disease Control and Prevention (CDC) [2]. In a susceptible population, the main route of the transmission for Covid-19 is through small droplets from an infected person to other people [3].

Symptoms of this disease are very similar with influenza, such as high fever, dry cough, tiredness. Intensity of symptoms can range from very mild to severe. Infected people may have many symptoms or no symptoms at all [4]. Many countries around the world have brought many restrictions to prevent the spread of the disease. These restrictions include closure of workplaces, shops, restaurants and airports [5].

Cyprus is the third largest island located in the Mediterranean region. In the North side of Cyprus, the population is approximately 374000, and consists mainly of Turkish Cypriots [6]. In Northern Cyprus, the SARS-CoV-2 outbreak started with patient zero on March 9, 2020 [7]. SARS-CoV-2 entered the Northern Cyprus through the routes of Germany and England [7]. Since then the government took many restrictions to prevent the spread of disease. On March 10, 2020 all of the schools, including universities, were closed till March 15, 2020. Then closure was extended till the end of semester. Afterwards, on March 15, 2020, all businesses except markets, pharmacies and gas stations were closed. With all these restrictions, partial curfew and closure of the airport were announced by councilof ministers.

For infectious diseases, mathematical models can be constructed in order to study the infectiousness of the disease. SEIR model is one type of the mathematical models that contains four main compartments which are S, E, I, and R. Here S denotes susceptible, E denotes exposed (infected but not yet infectious), I denotes infectious and R denotes recovered individuals in that population [8]. We constructed a new SEIR model in order to calculate R_0 and R_t by using real data for North Cyprus [7].

The basic reproduction number, denoted by R_0, can be defined as the number of cases which are expected to occur on average in a homogeneous population as a result of infection by a single individual. The effective reproduction number, R_e, sometimes also denoted by R_t, is the number of people in a population who can be infected by an individual at any specific time. It changes as the population becomes increasingly immunized, either by individual immunity following infection or by vaccination, and also as people die [9, 10].

There are few differences between R_0 and R_t. The main difference is that in R_0 there are no immune individuals taken into account while in R_t we count immune individuals as well. During an epidemic R_0 can not reflect the change of epidemic in time but R_t can provide more information since it tracks the evolution of transmission. Another important difference is that R_0 works with daily cases. However, R_t generally works with death ratios [11, 12].

Currently, a total of 30025 tests have been conducted resulting in 108 Covid-19 positive cases in Northern Cyprus, of whom no patients are left under treatment. There are no individuals under quarantine for 26 days due to the risk of carrying the Covid-19. As a result, 104 of patients have recovered and 4 deaths have occurred [7]. North Cyprus is the first European Country that has become Covid-19 free in 37 days [13]. In addition, no new cases were seen for 75 days.

In this paper, firstly we define the basic reproduction number, R_0, and the effective reproduction number, R_t, as herd immunity. Then, we define the method and formulas which we have used in order to calculate R_0 and R_t values. With these values, we illustrate in figures the evolution of disease in North Cyprus. Lastly, we conclude our findings.

The basic reproduction number

An epidemiological definition of the basic reproduction number, denoted by R_0, is the expected number of secondary cases by a single individual who is infected in an entirely susceptible population [14, 15]. Estimating R_0 values in an epidemic can be helpful in order to see the infectiousness of the disease. For that purpose we generally use mathematical modeling to find a formula for R_0 of an epidemic [15, 16].

Basic reproduction number is calculated by using the parameters of the mathematical model [17]. Biological, social behaviour, and environmental factors can affect the basic reproduction number [18]. However, immunization is not an effect for R_0, which may occur naturally or by vaccination [19].

When $R_0 > 1$, outbreak is expected to continue. We expect an outbreak to end if $R_0 < 1$ or in other words the number of infected individuals are expected to decrease [18, 20]. Since $R_0 < 1$ means that each infected individual causes less than one new infection, this guarantees that the disease will die out under that circumstances [21]. Hence, we desire the value of R_0 to be less than 1.

Basic reproduction numbers for the previous pandemics are given in Fig. 1. SARS-CoV-2 has an average value 2.65 during this pandemic if we compare with the other pandemics. If we check the Fig. 1, we can easily see that measles, HIV, or even influenza (Autumn 1918) are more infectious than SARS-CoV-2.

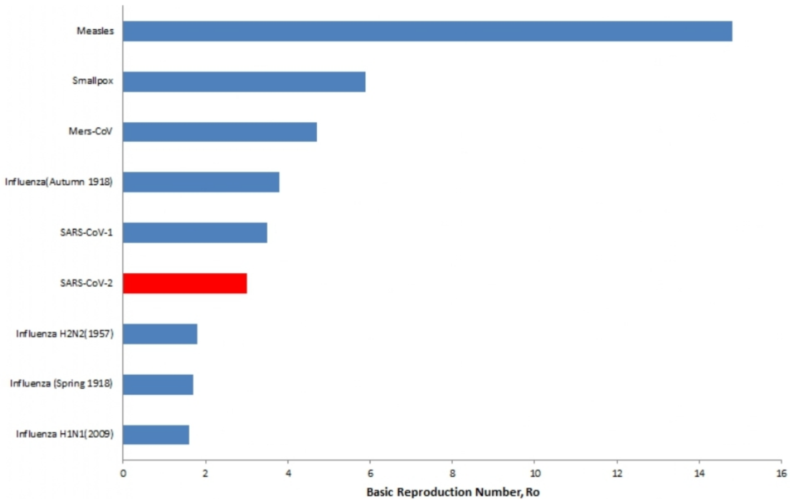


Figure 1. The basic reproduction numbers for pandemics comparing to SARS-Cov-2 (Covid-19)

The effective reproduction number

The Effective Reproduction Number, R_t, can be defined as the real average number of secondary cases infected by primary cases per time [10]. As in R_0, $R_t < 1$ means that epidemic will decline and the epidemic will spread if $R_t > 1$ [10]. In this paper, we will use the effective reproduction number as herd immunity.

When most of the population gain immunity to an infectious disease, this provides indirect protection, namely herd immunity, to those who don't have immunity to that disease. In other words, in a population, the greater the number of immune people means the lower likelihood that a susceptible individual will be infected [22]. There are two ways for gaining herd immunity; vaccines and infection. Since the vaccine of SARS-CoV-2 has not been found yet, we can analyze herd immunity idea only for infection [23].

There is a threshold that must be reached in order to say that the population has gained herd immunity. This threshold is called herd immunity threshold which is the percentage of the population that must be immune by getting infected [22]. Herd immunity threshold changes from disease to disease. The proportion of the population that needs to gain immunity to the disease to stop the spreading increase as the infectiousness of the disease increase [23].

In this paper, we calculate the effective reproduction number, R_t, as herd immunity. We attempt to analyze the herd immunity idea for Northern Cyprus.

Calculating the basic reproduction number and effective reproduction number

In order to calculate the basic reproduction number in Northern Cyprus we use a basic SEIR model where S is susceptible, E is exposed, I is infectious and R is the recovery compartment. This model was first introduced by William Ogilvy Kermack and Anderson Gray McKendrick in 1927.

In this paper, herd immunity, R_t, is calculated by the following formula

$$R_t = 1 - \frac{1}{R_0}, \tag{1}$$

where R_0 is the basic reproduction number.

We construct the following model for Covid-19

$$\begin{aligned}
\frac{dS}{dt} &= \pi - \lambda S, \\
\frac{dE}{dt} &= \lambda S - (\theta_1 + \theta_2)E, \\
\frac{dQ}{dt} &= \theta_1 E - (\delta_1 + \theta_3)Q, \\
\frac{dI_1}{dt} &= \theta_2 E + \theta_3 Q - (\delta_2 + \omega + \theta_4 + \alpha_1)I_1, \\
\frac{dI_2}{dt} &= \theta_4 I_1 - (\Phi + \alpha_2 + \delta_3)I_2, \\
\frac{dH}{dt} &= \omega I_1 + \Phi I_2 - (\delta_4 + \alpha_3)H, \\
\frac{dR}{dt} &= \delta_1 Q + \delta_2 I_1 + \delta_3 I_2 + \delta_4 H,
\end{aligned}$$

called a SEIR model with seven compartments which are explained in Table 1. By using this model and system, we calculated R_0 values for North Cyprus with real data. The formula for R_0 can be obtained by using the next generation matrix method. Then, with calculated R_0 values, we can find R_t values using formula (1).

The value of R_0 is calculated as

$$R_0 = \frac{(((b_1\beta\tau_1 + \beta\tau_2\theta_3)\,\theta_1 + \beta\tau_2 k_2\theta_2)\,k_3 + \omega\,\beta\tau_4\,(\theta_2 k_2 + \theta_3\theta_1))\,b_2 + \theta_4\,(\beta\tau_4\varphi + \beta\tau_3 k_3)\,(\theta_2 k_2 + \theta_3\theta_1)}{k_1 k_2 k_3 b_1 b_2}, \tag{2}$$

where the variables and parameters of the model are described in Table 1 and Table 2, respectively. Here

$$k_1 = \theta_1 + \theta_2, k_2 = \delta_1 + \theta_3, k_3 = \delta_4 + \alpha_3, b_1 = \delta_2 + \omega + \theta_4 + \alpha_1, b_2 = \varphi + \alpha_2 + \delta_3 \tag{3}$$

Table 1

Variables for the model

Variables	Descriptions
N	Total population of humans
S	Susceptible humans at the risk of having COVID-19 infection
E	Exposed humans
I_1	Infected humans with moderate infection
I_2	Infected humans with severe infection
Q	Human population under quarantine / isolation
H	Hospitalized humans
R	Recovered humans

Table 2

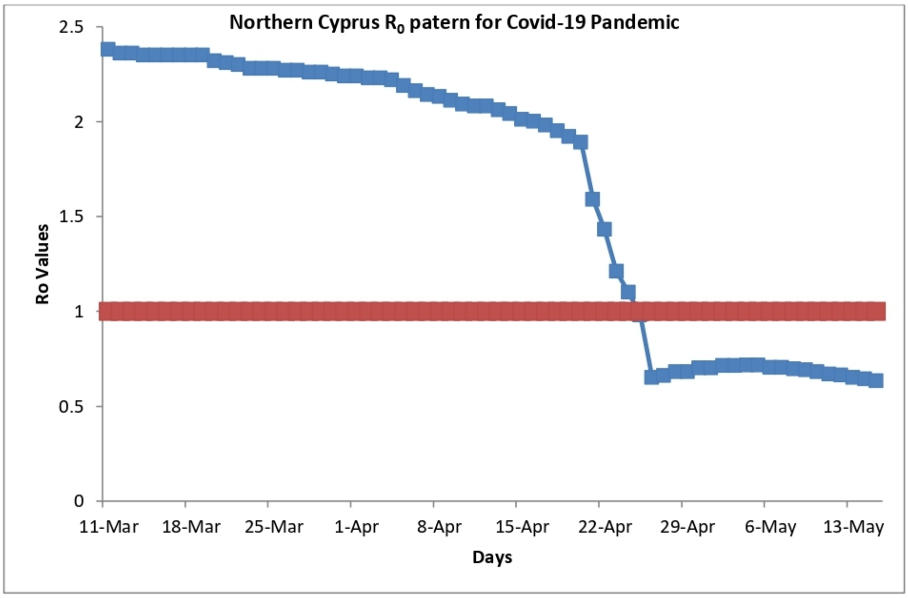
Parameters for the model and basic reproduction number

Parameters	Descriptions
π	Recruitment rate
β	Transmission rate
τ_i $(i = 1, 2, 3, 4)$	Parameters for increase / decrease on infectiousness in humans
θ_i $(i = 1, 2, 3, 4)$	Progression rates
ω	Hospitalization rate from I_1 class
ϕ	Hospitalization rate from I_2 class
α_i $(i = 1, 2, 3)$	Disease induced death rates
δ_i	Recovery rates

A formula (2) for R_0 is obtained by using the method which needs next generation matrix where finitely many distinct categories of individuals are introduced in a population. The method that uses next generation matrix for calculating R_0 was introduced by Diekmann et al. (1990) and van den Driessche and Watmough (2002) [24]. This method needs two matrices which can be obtained from the mathematical model. One matrix includes new infections of the disease taken from the system while the other matrix consists of the rest of the system [24, 25].

As we can see from Figure 2, while calculating R_0 and R_t values, we used daily cases between the dates March 11, 2020 and May 15, 2020. We can observe that after approximately April 27, 2020, R_0 value decreased below one. This means that disease is not infectious anymore in TRNC under taken restrictions, after that time.

On the other hand, if we look at the R_t values which were calculated by formula (1), we see that it is below one from the beginning. So, we can not make any comment by using R_t for North Cyprus.



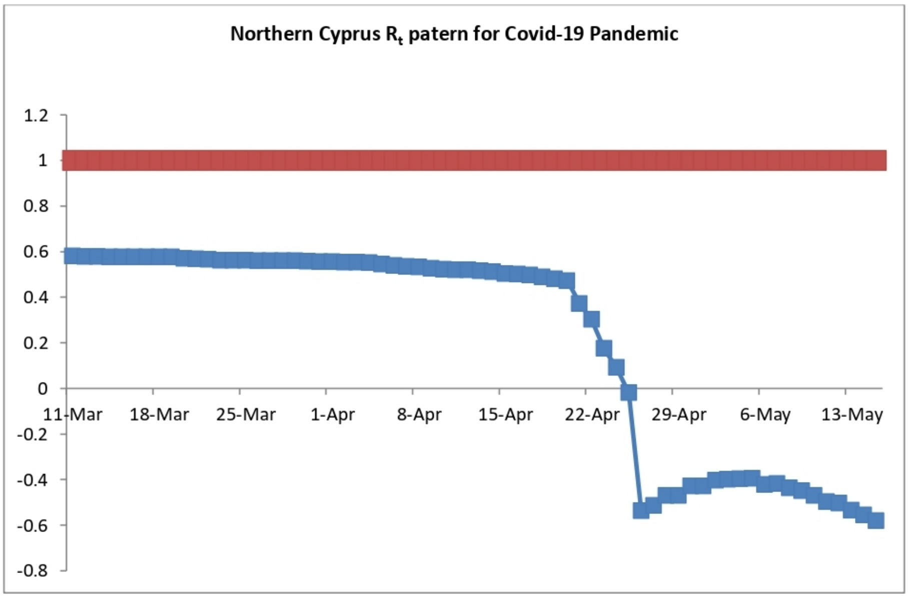


Figure 2. R_0 and R_t values from March 11, 2020 to May 15, 2020

Conclusions

Tackling of Covid-19 in North Cyprus has been compared with the other European countries. With using the model, we calculated the basic reproduction number that secondary cases of new infectious for Covid-19. Then we compared the infectiousness of the Covid-19 with the other pandemics, which can be seen in Figure 1.

Some of countries used effective reproduction number during the SARS-CoV-2 pandemic. In Figure 2, we gave two graphs that are showing the infectiousness in North Cyprus with using R_0 and R_t. Between April 17, 2020 - July 1, 2020 there were no new Covid-19 cases in North Cyprus. It can be seen in the Figure 2 that this was what we assumed for the progression of the disease in North Cyprus by using R_0 values that we have obtained from the formula (2).

Furthermore, we have monitored Covid-19 pandemic in North Cyprus with R_t as herd immunity. The second graph in Figure 2 illustrates that R_t values are less than one which shows us that there was no pandemic in North Cyprus. Although, both figures have similar behaviour, Figure 2 shows that R_0 is more effective than R_t. However, we can not generalize this result.

In Figure 3, we can see that the North Cyprus has the lowest death rate with highest recovery in Europe. Furthermore, North Cyprus is the leading country in Europe that it has almost 80000 tests around 1000000 population. As a result, we can say that North Cyprus has reached zero at the case of Covid-19 in 37 days.

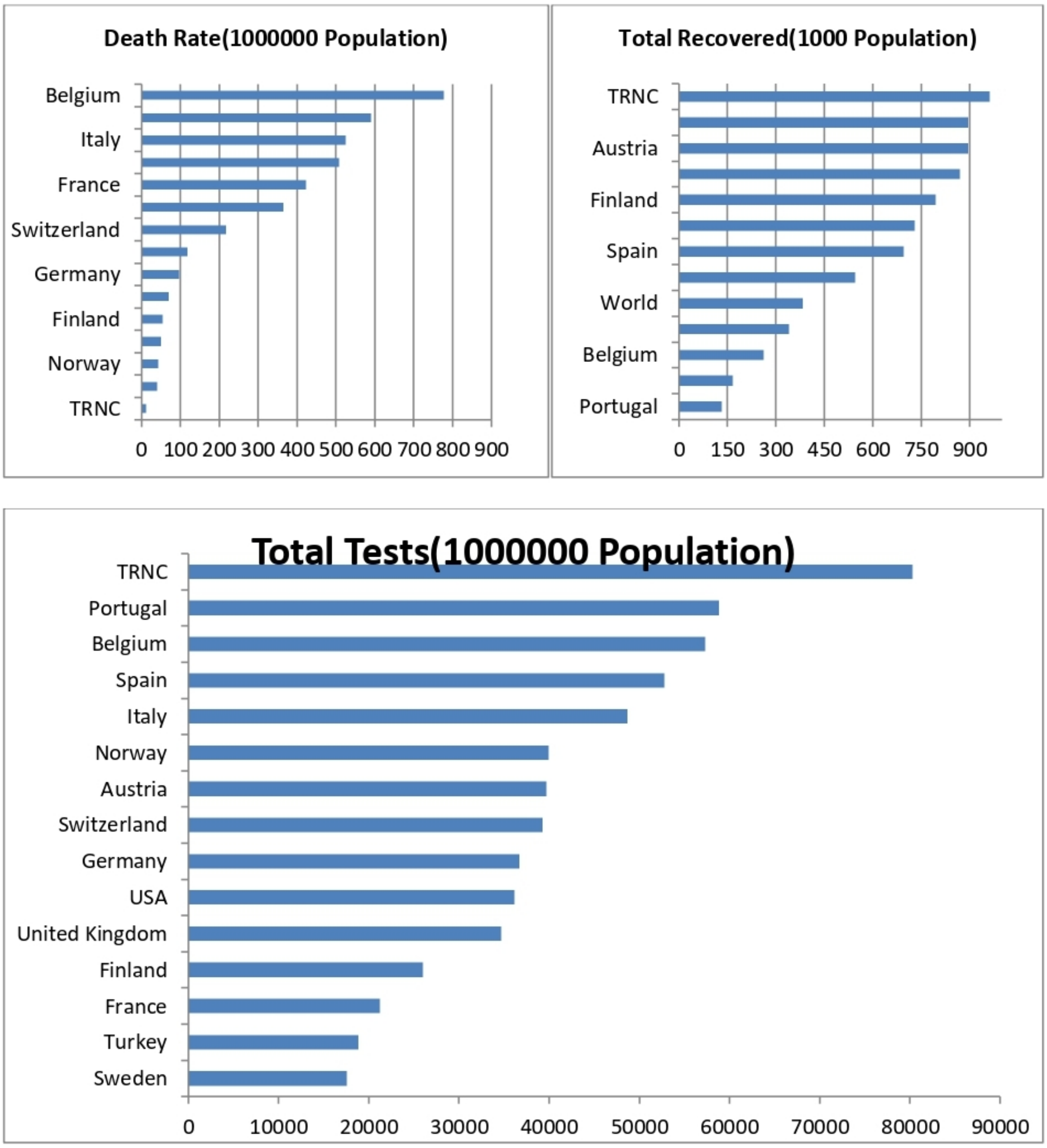


Figure 3. Comparison of North Cyprus with some other countries and world data between March 11, 2020 and May 15, 2020

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Э. Хинжал, Б. Каймакамзаде, Незихал Гокбулут

Ковид-19 пандемиясымен күресу үшін Солтүстік Кипрдің репродукциясының базалық нөмірі және репродукциясының тиімді нөмірі

Мақаланың мақсаты — ұжымдық иммунитет ретінде R0 және Rt қолдану арқылы Солтүстік Кипр Covid-19 пандемиясымен қалай күрескенін көрсету. Осы мақсатта R0 мәндерін қолдану арқылы Rt мәндері есептелді және R0 базалық нөмірін ойнату үшін SEIR моделі қолданылған. Солтүстік Кипр — Ковид-19 эпидемиясынан айыққан Еуропа елдері ішінен алғашқы ел болып табылады. Мұның маңызды себептерінің бірі - үкіметтің күнделікті R0 және Rt қолданып, Ковид-19 пандемиясымен күресу туралы шешім қабылдауы. Солтүстік Кипр үшін нақты мәліметтерді қолданып, R0 үшін SEIR жаңа моделін құрастырды. 2020 жылдың 11 наурызынан бастап 2020 жылдың 15 мамырына дейін R0 мәні 0,65- тен 2,38-ге дейін ауытқып тұрды.

Кілт сөздер: Ковид-19, Солтүстік Кипр, эпидемиялар, математикалық модель.

Э. Хинжал, Б. Каймакамзаде, Незихал Гокбулут

Базовый номер репродукции и эффективный номер репродукции Северного Кипра для борьбы с Ковид-19

Цель данной статьи — показать, как Северный Кипр боролся с Covid-19, используя R_0 и R_t, в качестве коллективного иммунитета. Для этого авторами использована модель SEIR для базового номера воспроизведения, R_0, и вычисление значения R_t, используя значения R_0. Северный Кипр является первой страной в Европе, которая избавилась от эпидемии Ковид-19. Одна из наиболее важных причин этого заключается в том, что правительство решило бороться с пандемией Covid-19, используя ежедневные R_0 и R_t. Для R_0 нами построена новая модель SEIR с использованием реальных данных для Северного Кипра. С 11 марта 2020 г. по 15 мая 2020 г. уровень R_0 в этой стране колеблется в пределах от 0,65 до 2,38.

Ключевые слова: Ковид-19, Северный Кипр, эпидемии, математическая модель.

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A note on well-posedness of source identification elliptic problem in a Banach space

We study the source identification problem for an elliptic differential equation in a Banach space. The exact estimates for the solution of source identification problem in Hölder norms are obtained. In applications, four elliptic source identification problems are investigated. Stability and coercive stability estimates for solution of source identification problems for elliptic equations are obtained.

Keywords: well-posedness, elliptic equations, positivity, coercive stability, source identification, exact estimates, boundary value problem.

Introduction

Several source identification problems for partial differential equations have been extensively investigated by many researchers (see [3, 4, 8–11, 14, 15, 17–19] and the bibliography herein). Well-posedness of nonclassical boundary value problems for various partial differential and difference equations was established in a number of publications (see [1]-[22] and references therein).

Large number of the source identification problems for an elliptic differential equations can be written as the source identification problem for the second order differential equation

$$\begin{cases} -u''(t) + Au(t) = f(t) + p, \quad 0 < t < 1, \\ \\ u(0) = u(1), \ u'(0) = u'(1), u(\lambda) = \xi, \lambda \in (0,1) \end{cases} \tag{1}$$

in an arbitrary Banach space E with a positive operator A. Here parameter $p \in E$ and abstract function $u : [0,1] \to E$ are unknown and element $\xi \in D(A)$ and abstract function $f : [0,1] \to E$ are given.

Let $E_1 \subset E$ and $F(E)$ be the Banach space of $E-$valued smooth functions on $[0,1]$. We say that the pair $\{u(t), p\}$ is the solution of the source identification problem (1) in $F(E) \times E_1$ if the following conditions are valid:

(i) $p \in E_1, u''(t) \in F(E), \ Au(t) \in F(E)$,

(ii) $\{u(t), p\}$ is satisfied the equation and all three conditions of (1).

In the present paper, theorem on well-posedness of the source identification problem (1) in Hölder spaces is established. In applications, stability and coercive stability estimates for solution of the four type of source identification problems for elliptic equations are obtained.

Stability and coercive stability estimates

Denote by $C_{01}^{\alpha,\alpha}(E)$ $(0<\alpha<1)$, the Banach space obtained by completion of the set of $E-$valued smooth functions $\varphi(t)$ defined on $[0,1]$ with values in E in the norm

$$\|\varphi\|_{C_{01}^{\alpha,\alpha}(E)}=\|\varphi\|_{C(E)}+\sup_{0\leq t<t+\tau\leq 1}\tau^{-\alpha}(1-t)^{\alpha}(t+\tau)^{\alpha}\left\|\varphi(t+\tau)-\varphi(t)\right\|_{E},$$

where $C(E)$ is the Banach space of all continuous functions $\varphi(t)$ defined on $[0,1]$ with values in E equipped with the norm

$$\|\varphi\|_{C(E)}=\max_{0\leq t\leq 1}\|\varphi(t)\|_{E}.$$

Assume that $v(t)$ is the solution of the nonlocal boundary value problem

$$\begin{cases}-v''(t)+Av(t) \;=\; f(t),\;\; 0<t<1,\\ \\ v(0)=v(1),\;\; v'(0)=v'(1).\end{cases}\tag{2}$$

Then, for the solution of problem (1) we have the following formulas

$$u(t)=v(t)+A^{-1}p,\tag{3}$$

$$p=A\xi-Av(\lambda).\tag{4}$$

Therefore, the following algorithm can be used to find the solution of problem (1):

(1) Find the solution $v(t)$ of nonlocal boundary value problem (2).

(2) Use (4) to obtain the source element p of source identification problem (1).

(3) Applying (3), obtain the solution $u(t)$ of source identification problem (1).

It is known that the operator $B=A^{\frac{1}{2}}$ is the strongly positive operator for any positive operator A. Therefore, the operator $-B$ will be a generator of an analytic semigroup $\exp -tB$ $(t\geq 0)$ with exponentially decreasing norm (see [7]), when $t\to\infty$, i.e. there exist some $M(B)\in[1,+\infty)$, $\alpha(B)\in(0,+\infty)$ such that the following estimates

$$\|\exp(-tB)\|_{E\to E}\leq M(B)\exp(-\alpha(B)t),\tag{5}$$

$$\|tB\;\exp(-tB)\|_{E\to E}\leq M(B)\exp(-\alpha(B)t)(t>0),\tag{6}$$

$$\|T\|_{E\to E}\leq M(B)\left(1-\exp(-\alpha(B))\right)^{-1}\tag{7}$$

are satisfied. Here $T=(I-\exp(-B))^{-1}$.

The solution of direct problem (2) is defined by (formula (1.7) [1])

$$v(t)=\tfrac{1}{2}B^{-1}T\exp(-\left(1-t\right)B)\int\limits_0^1\exp(-sB)f(s)ds\tag{8}$$

$$+\tfrac{1}{2}B^{-1}\int\limits_0^t\exp(-\left(t-s\right)B)f(s)ds\;+\tfrac{1}{2}B^{-1}\int\limits_t^1\exp(\left(t-s\right)B)f(s)ds$$

$$+\;\tfrac{1}{2}B^{-1}T\exp(-tB)\int\limits_0^1\exp(-\left(1-s\right)B)f(s)ds\;.$$

From (4) and (8), it follows that

$$p = A\xi - \tfrac{1}{2}BT\exp(-(1-\lambda)B)\int_0^1 \exp(-sB)f(s)ds \tag{9}$$

$$-\tfrac{1}{2}B\int_0^\lambda \exp(-(\lambda-s)B)f(s)ds \;-\tfrac{1}{2}B\int_\lambda^1 \exp((\lambda-s)B)f(s)ds$$

$$-\tfrac{1}{2}BT\exp(-B\lambda)\int_0^1 \exp(-(1-s)B)f(s)ds.$$

Finally, by using formulas (8), (3) and (9), we can obtain $u(t)$.

Now, we formulate result on well-posedness of the source identification problem (1) in the space $C_{01}^{\alpha,\alpha}(E)$.

Theorem 1. Assume that $\xi\in D(A)$ and $f(t)\in C_{01}^{\alpha,\alpha}(E), 0<\alpha<1$. For the solution $\{u(t),p\}$ of the source identification problem (1) the following stability inequality

$$\|u\|_{C(E)}+\left\|A^{-1}p\right\|_E\le M\;\left[\|\xi\|_E+\|f\|_{C(E)}\right] \tag{10}$$

and coercive inequality

$$\|u''\|_{C_{01}^{\alpha,\alpha}(E)}+\|Au\|_{C_{01}^{\alpha,\alpha}(E)}+\|p\|_E\le M[\|A\xi\|_E+\tfrac{1}{\alpha(1-\alpha)}\|f\|_{C_{01}^{\alpha,\alpha}(E)}], \tag{11}$$

hold, where M is independent of α,ξ and $f(t)$.

The proof of Theorem 1 is based on the formula (3) and estimates (5) and (7) on the Theorem on well-posedness of the nonlocal boundary value problem (2) [1].

Note that same results can be established for the solutions of the general source identification problems

$$\begin{cases}-u''(t)+Au(t)=f(t)+p, & 0<t<1,\\ u(0)=\sum\limits_{j=1}^N a_ju(t_j)+\varphi,\ u'(0)=u'(1)+\psi, u(\lambda)=\xi, \lambda\in(0,1)\,, & \end{cases}$$

where $0<t_1<...<t_N\le 1$, if the operator

$$I-e^{-2B}-\sum_{j=1}^N a_j\left(e^{-t_jB}-e^{-(2-t_j)B}-e^{-(1-t_j)B}+e^{-(1+t_j)B}\right)$$

has a bounded inverse in E and

$$\begin{cases}-u''(t)+Au(t)=f(t)+p, & 0<t<1,\\ u(0)=u(1)+\varphi,\ u'(0)=\sum\limits_{j=1}^N a_ju'(s_j)+\psi, u(\lambda)=\xi, \lambda\in(0,1)\,, & \end{cases}$$

where $0<s_1<...<s_N\le 1$, if the operator

$$\left(I-e^{-B}\right)^2-\sum_{j=1}^N a_j\left(e^{-s_jB}+e^{-(2-s_j)B}-e^{-(1-s_j)B}-e^{-(1+s_j)B}\right)$$

has a bounded inverse in E .

Applications

In this section, we consider the applications of Theorem 1. First, we study the source identification problem for the two dimensional elliptic differential equation with nonlocal boundary conditions

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) = f(t,x) + p(x), \\ 0 < t < 1, 0 < x < l, \\ u(0,x) = u(1,x), u_t(0,x) = u_t(1,x),\ u(\lambda,x) = \xi(x), 0 \le x \le l, \\ u(t,0) = u(t,l),\ \ u_x(t,0) = u_x(t,l),\ \ 0 \le t \le 1, \end{cases} \tag{12}$$

where $a(x)$, $\xi(x)$ and $f(t,x)$ are given sufficiently smooth functions and $a(x) > 0,\ 0 < \lambda < 1, \delta > 0$ is a sufficiently large number. Assume that all compatibility conditions are satisfied.

We introduce the Banach spaces $C^{\beta}[0,l]$ $(0 < \beta < 1)$ of all continuous functions $\varphi(x)$ satisfying a Hölder condition for which the following norms are finite

$$\| \varphi \|_{C^{\beta}[0,l]} = \| \varphi \|_{C[0,l]} + \sup_{0 \le x < x+\tau \le l} \frac{|\varphi(x+\tau) - \varphi(x)|}{\tau^{\beta}},$$

where $C[0,l]$ is the space of the all continuous functions $\varphi(x)$ defined on $[0,l]$ with the usual norm

$$\| \varphi \|_{C[0,l]} = \max_{0 \le x \le l} |\varphi(x)|.$$

Theorem 2. For the solution of the source identification problem (12) the following stability and coercive stability estimates hold:

$$\|u\|_{C(C^{\beta}[0,l])} \le M(\beta) \left[\|\xi\|_{C^{\beta}[0,l]} + \|f\|_{C(C^{\beta}[0,l])} \right],$$

$$\|u\|_{C_{01}^{2+\alpha,\alpha}(C^{\beta}[0,l])} + \|u\|_{C_{01}^{\alpha,\alpha}(C^{\beta+2}[0,l])} + \|p\|_{C^{\beta}[0,l]}$$

$$\le \frac{M(\beta)}{\alpha(1-\alpha)} \|f\|_{C_{01}^{\alpha,\alpha}(C^{\beta}[0,l])} + M(\beta) \|\xi\|_{C^{\beta+2}[0,l]},$$

where $M(\beta)$ is independent of α, $\xi(x)$ and $f(t,x), 0 < \alpha < 1,\ 0 < \beta < 1$.

The proof of Theorem 2 is based on the Theorem 1 and the positivity of the elliptic operator A in $C^{\beta}[0,l]$ [7].

Second, we investigate the source identification problem on the range $\{0 \le t \le 1, x \in R^n\}$

$$\begin{cases} -u_{tt}(t,x) + \sum\limits_{|l|=2m} a_l(x) \frac{\partial^{|l|}}{\partial x_1^{l_1} \ldots \partial x_n^{l_n}} u(t,x) + \delta u(t,x) = f(t,x) + p(x), \\ 0 < t < 1, x \in R^n, \\ u(0,x) = u(1,x), u_t(0,x) = u_t(1,x),\ u(\lambda,x) = \xi(x), x \in R^n \end{cases} \tag{13}$$

for the $2m-$order multidimensional elliptic equation, where $a_l(x)$ $(l = (l_1, ..., l_n), |l| = 0, ..., 2m)$ and $\xi(x)$ are known sufficiently smooth functions, $a_l(x) > 0$, and $0 < \lambda < 1$, $\delta > 0$ are given real numbers. Assume that all compatibility conditions are satisfied and the symbol

$$F^x(\zeta) = \sum_{|l|=2m} a_l(\zeta)\ (i\zeta_1)^{l_1} \ldots (i\zeta_n)^{l_n},\ \zeta = (\zeta_1, \ldots, \zeta_n) \in R^n$$

of the differential operator

$$F^x = \sum_{|l|=2m} a_l(\zeta) \frac{\partial^{|l|}}{\partial x_1^{l_1} ... \partial x_n^{l_n}} \tag{14}$$

acting on functions in the space R^n, satisfies the inequalities

$$0 \le M_1 |\zeta|^{2m} \le (-1)^m F^x(\zeta) \le M_2 |\zeta|^{2m} < \infty,$$

for $\zeta \neq 0$.

Theorem 3. For the solution of the source identification problem (13) the following stability and coercive stability estimates are satisfied:

$$\|u\|_{C(C^\mu(R^n))} \le M(\mu) \left[\|\xi\|_{C^\mu(R^n)} + \|f\|_{C(C^\mu(R^n))} \right],$$

$$\|u\|_{C_{01}^{2+\alpha,\alpha}(C^\mu(R^n))} + \sum_{|l|=2m} \left\| \frac{\partial^{|l|} u}{\partial x_1^{l_1} ... \partial x_n^{l_n}} \right\|_{C_{01}^{\alpha,\alpha}(C^\mu(R^n))} + \|p\|_{C^\mu(R^n)}$$

$$\le \frac{M(\mu)}{\alpha(1-\alpha)} \|f\|_{C_{01}^{\alpha,\alpha}(C^\mu(R^n))} + M(\mu) \sum_{|l|=2m} \left\| \frac{\partial^{|l|} \xi}{\partial x_1^{l_1} ... \partial x_n^{l_n}} \right\|_{C^\mu(R^n)}$$

where $M(\mu)$ is independent of α, $\xi(x)$ and $f(t,x), 0 < \alpha < 1,\ 0 < \mu < 1$.

The proof of Theorem 3 is based on the Theorem 1 and the positivity of the elliptic operator A^x in $C^\mu(R^n)$ [7] and the coercivity estimate for an operator A^x in $C^\mu(R^n)$ [8].

Third, let $\Omega = (0,1)^n$ be the open cube in R^n with suitable boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$, we study the source identification problem

$$\begin{cases} -u_{tt}(t,x) - \sum\limits_{k=1}^{n} a_k(x) u_{x_k x_k}(t,x) + \delta u(t,x) = f(t,x) + p(x), \\ x = (x_1, ..., x_n) \in \Omega, 0 < t < 1, \\ u(0,x) = u(1,x), u_t(0,x) = u_t(1,x),\ u(\lambda, x) = \xi(x), x \in \overline{\Omega}, \\ u(t,x) = 0,\ 0 \le t \le 1,\ x \in S \end{cases} \tag{15}$$

for the multidimensional elliptic equation. Here $a_r(x)$ $(x \in \Omega)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$ $(x \in \overline{\Omega})$ are given sufficiently smooth functions, and $0 < \lambda < T$, $\delta > 0$ are known numbers. Assume that all compatibility conditions are satisfied.

Denote by $C_{01}^{\beta}(\overline{\Omega})(\beta = (\beta_1, ..., \beta_n), \beta_i, 1 \le i \le n)$, the Banach spaces of continuous functions satisfying a Hölder condition with weight $x_k^{\beta_k}(1 - x_k - h_k)^{\beta_i}, 0 \le x_k < x_k + h_k \le 1, 1 \le k \le n$ and the indicator β which equipped with the corresponding norm

$$\|f\|_{C_{01}^{\beta}(\overline{\Omega})} = \|f\|_{C(\overline{\Omega})} + \sup_{\substack{0 \le x_k < x_k + h_k \le 1, \\ 1 \le k \le n}} |f(x+h) - f(x)| \prod_{k=1}^{n} \left(\frac{x_k}{h_k} \right)^{\beta_{ki}} (1 - x_k - h_k)^{\beta_k}.$$

It is well known that the differential expression

$$A^x u = -\sum_{k=1}^{n} a_k\ u_{x_k x_k} + \delta u \tag{16}$$

defines a positive operator A^x acting on $C_{01}^{\beta}(\overline{\Omega})$ with domain $D(A^x) \subset C_{01}^{2+\beta}(\overline{\Omega})$ and satisfying the boundary condition $u = 0$ on S.

Theorem 4. For the solution of the source identification problem (15) the following stability and coercive stability estimates hold

$$\|u\|_{C(C_{01}^{\mu}(\overline{\Omega}))} \leq M(\mu) \left[\|f\|_{C(C_{01}^{\mu}(\overline{\Omega}))} + \|\xi\|_{C_{01}^{\mu}(\overline{\Omega})} \right], \tag{17}$$

$$\|u\|_{C_{01}^{2+\alpha,\alpha}(C_{01}^{\mu}(\overline{\Omega}))} + \sum_{k=1}^{n} \|u_{x_k x_k}\|_{C_{0T}^{\alpha,\alpha}(C_{01}^{\mu}(\overline{\Omega}))} + \|p\|_{C_{01}^{\mu}(\overline{\Omega})}$$

$$\leq \frac{M(\mu)}{\alpha(1-\alpha)} \|f\|_{C_{0T}^{\alpha,\alpha}(C_{01}^{\mu}(\overline{\Omega}))} + M(\mu) \|\xi\|_{C_{01}^{\mu+2}(\overline{\Omega})}, \tag{18}$$

$$0 < \alpha < 1, \mu = (\mu_1, ..., \mu_n),\ 0 < \mu_i < 1,\ 1 \leq i \leq n,$$

where $M(\mu)$ is independent of α, $\xi(x)$ and $f(t,x)$.

Fourth, in $[0,1] \times \Omega$, we consider the source identification problem

$$\begin{cases} -u_{tt}(t,x) - \sum_{k=1}^{n} a_k(x) u_{x_k x_k}(t,x) + \delta u(t,x) = f(t,x) + p(x), \\ x \in \Omega, 0 < t < 1, \\ u(0,x) = u(1,x), u_t(0,x) = u_t(1,x),\ u(\lambda,x) = \xi(x), x \in \overline{\Omega}, \\ \frac{\partial}{\partial \vec{n}} u(t,x) = 0,\ 0 \leq t \leq 1,\ x \in S \end{cases} \tag{19}$$

for the multidimensional elliptic equation. Assume that all compatibility conditions are satisfied. The differential expression (16) defines a positive operator A^x acting on $C_{01}^{\beta}(\overline{\Omega})$ with domain $D(A^x) \subset C_{01}^{2+\beta}(\overline{\Omega})$ and satisfying the boundary condition $\frac{\partial u}{\partial \vec{n}} = 0$ on S. Therefore, by using Theorem 1, we can get the following result.

Theorem 5. For the solution of the source identification problem (19) the stability and coercive stability estimates (17) and (18) respectively are valid.

Conclusion

In the present paper, the well-posedness of the source identification problem for the abstract elliptic equation in Banach spaces is investigated. The exact estimates for the solution of this problem in Hölder norms are established. In future investigation, absolute stable difference schemes for approximately solution of the source identification problem for elliptic differential equations will be constructed and investigated.

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Банах кеңiстiгiнде дереккөздi сәйкестендiруде эллипстiк есептiң корректiлiгi туралы ескерту

Банах кеңiстiгiнде эллипстiк дифференциалдық теңдеу үшiн дереккөздi сәйкестендiру мәселесi қарастырылған. Хелдер нормасында дереккөздердi сәйкестендiру есебiн шешу үшiн дәл бағамы алынды. Қосымшаларда дереккөздi сәйкестендiрудiң төрт эллипстiк есебi зерттелген. Эллипстiк теңдеу үшiн дереккөздердi сәйкестендiру есебiн шешу үшiн мәжбүрлi орнықтылық және орнықтылық бағамы алынған.

Кiлт сөздер: корректiлiгi, эллипстiк теңдеу, позитивтi, мәжбүрлi орнықтылық, дереккөздi сәйкестендiру, дәл бағамы, шеттiк есеп.

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Замечание о корректности эллиптической задачи идентификации источника в банаховом пространстве

Исследована проблема идентификации источника для эллиптического дифференциального уравнения в банаховом пространстве. Получены точные оценки для решения задачи идентификации источника в нормах Хелдера. В приложениях исследованы четыре эллиптических задачи идентификации источника. Получены оценки устойчивости и коэрцитивной устойчивости для решения задач идентификации источника для эллиптических уравнений.

Ключевые слова: корректность, эллиптические уравнения, позитивность, коэрцитивная устойчивость, идентификация источника, точные оценки, краевая задача.

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On the stable difference scheme for the time delay telegraph equation

The stable difference scheme for the approximate solution of the initial boundary value problem for the telegraph equation with time delay in a Hilbert space is presented. The main theorem on stability of the difference scheme is established. In applications, stability estimates for the solution of difference schemes for the two type of the time delay telegraph equations are obtained. As a test problem, one-dimensional delay telegraph equation with nonlocal boundary conditions is considered. Numerical results are provided.

Keywords: difference schemes, delay telegraph equations, stability.

Introduction

Time delays appear in a diversity of science and engineering, such as biology, physics, chemistry, dynamical processes. The delay term can cause oscillatory instabilities and chaos. However, to find more realistic solutions to the problems encountered in life, the delay term should be taken into consideration in mathematical modeling. Many scientists have worked to solve such problems (see [1-10]).

Telegraph equation is mostly interested in physical systems. Many physicists, engineers and mathematicians have studied on telegraph equation without time delay (see [11-18]) paranthesis is missed. Operator theory is used in [19] for the investigation of stability of the initial value problem for the telegraph equation in a Hilbert space. Ashyralyev, Agirseven and Turk in [20] studied the stability of the initial value problem for the telegraph differential equation with time delay

$$\begin{cases} \frac{d^2u(t)}{dt^2}+\alpha\frac{du(t)}{dt}+Au(t)=aAu([t]),\ t>0, \\ u(0)=\varphi, \quad u'(0)=\psi \end{cases} \tag{1}$$

in a Hilbert space H with a self-adjoint positive definite operator A, $A\geq\delta I$, φ and ψ are elements of $D(A)$ and $[t]$ denotes the greatest-integer function, here $\delta>\frac{\alpha^2}{4}$ and $0<a<1$.

In the present paper, the first order of accuracy stable two-step difference scheme

$$\begin{cases} \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\frac{u_{k+1}-u_k}{\tau}+Au_{k+1}=aAu_{\left[\frac{k-mN}{N+1}\right]N+mN}, \\ N\tau=1,\ (m-1)N+1\leq k\leq mN-1,\ m=1,2,..., \\ u_0=\varphi, \quad \left((1+\alpha\tau)I+\tau^2A\right)\frac{u_1-u_0}{\tau}=\psi, \\ \left((1+\alpha\tau)I+\tau^2A\right)\frac{u_{mN+1}-u_{mN}}{\tau}=\frac{u_{mN}-u_{mN-1}}{\tau}, \quad m=1,2,... \end{cases} \tag{2}$$

for the solution of the problem (1) is constructed. The main theorem on stability estimates for the solution of difference problem (2) is established. In applications, stability estimates for the solution of the difference scheme for the two type of the time delay telegraph equations are established. As a test problem, an initial-boundary value problem for one-dimensional delay telegraph equations with nonlocal boundary conditions is considered. Numerical results are given.

The stability of difference scheme (2)

Throughout this paper, the operator B is defined by the formula

$$B = A - \frac{\alpha^2}{4} I.$$

It is easy to show that for $\delta > \frac{\alpha^2}{4}$, the operator B is a self-adjoint positive definite operator in a Hilbert space H with $B \geq (\delta - \frac{\alpha^2}{4})I$. Operator functions R and $\widetilde{R}$ are given by formulas

$$Ru = \left(\left(1 + \frac{\alpha\tau}{2}\right) I - i\tau B^{1/2}\right)^{-1} u, \quad \widetilde{R}u = \left(\left(1 + \frac{\alpha\tau}{2}\right) I + i\tau B^{1/2}\right)^{-1} u.$$

Lemma 1. The following estimates hold:

$$\|B^{-1/2}\|_{H \to H} \leq \frac{1}{\sqrt{\delta - \frac{\alpha^2}{4}}}, \tag{3}$$

$$\|R\|_{H \to H} \leq 1, \quad \|\tau B^{1/2} R\|_{H \to H} \leq 1, \quad \| \widetilde{R}\|_{H \to H} \leq 1, \|\tau B^{1/2} \ \widetilde{R}\|_{H \to H} \leq 1, \tag{4}$$

$$\left\|\tau B^{1/2}((1 + \alpha\tau)I + \tau^2 A)^{-1}\right\|_{H \to H} \leq 1. \tag{5}$$

The proof of Lemma 1 is based on the spectral representation of the self-adjoint positive definite operator B in Hilbert space H (see [21]).

Theorem 1. For the solution of difference problem (2), the following estimates hold:

$$\max_{1 \leq k \leq N} \|u_k\|_H \leq b \|\varphi\|_H + \left\|B^{-1/2}\psi\right\|_H, \tag{6}$$

$$\max_{1 \leq k \leq N} \left\|B^{-1/2} \frac{u_k - u_{k-1}}{\tau}\right\|_H \leq c \|\varphi\|_H + d \left\|B^{-1/2}\psi\right\|_H, \tag{7}$$

$$\max_{mN+1 \leq k \leq (m+1)N} \|u_k\|_H \leq b \max_{(m-1)N \leq k \leq mN} \|u_k\|_H + \max_{(m-1)N+1 \leq k \leq mN} \left\|B^{-1/2} \frac{u_k - u_{k-1}}{\tau}\right\|_H, \ m = 1, 2, ..., \tag{8}$$

$$\max_{mN+1 \leq k \leq (m+1)N} \left\|B^{-1/2} \frac{u_k - u_{k-1}}{\tau}\right\|_H \leq c \max_{(m-1)N \leq k \leq mN} \|u_k\|_H + d \max_{(m-1)N+1 \leq k \leq mN} \left\|B^{-1/2} \frac{u_k - u_{k-1}}{\tau}\right\|_H, \ m = 1, 2, ..., \tag{9}$$

where

$$b = |a| + |1 - a|d, \quad c = |1 - a| \frac{\delta}{\delta - \frac{\alpha^2}{4}}, \quad d = 1 + \frac{\frac{\alpha}{2}}{\sqrt{\delta - \frac{\alpha^2}{4}}}.$$

Proof. Difference problem (2) can be rewritten as the equivalent initial value problem for the second order difference equations with operator coefficients

$$\begin{cases} \left((1+\alpha\tau)\, I+\tau^2 A\right) u_{k+1}-(2+\alpha\tau)\, u_k+u_{k-1}=a\tau^2 A u_{\left[\frac{k-mN}{N+1}\right]N+mN}, \\ N\tau=1,\ (m-1)N+1\leq k\leq mN-1,\ m=1,2,... \\ u_0=\varphi, \quad u_1=\varphi+\tau((1+\alpha\tau)I+\tau^2 A)^{-1}\psi, \\ u_{mN+1}=u_{mN}+R\widetilde{R}(u_{mN}-u_{mN-1}),\ m=1,2,... \end{cases}$$

Let $1\leq k\leq N$. It is clear that

$$u_1=\varphi+\tau B^{1/2}R\widetilde{R}B^{-1/2}\psi$$

and

$$B^{-1/2}\frac{u_1-u_0}{\tau}=((1+\alpha\tau)I+\tau^2 A)^{-1}B^{-1/2}\psi=R\widetilde{R}B^{-1/2}\psi.$$

Then, using the triangle inequality and estimate (5), we get

$$\|u_1|_H\leq\|\varphi\|_H+\|B^{-1/2}\psi\|_H$$

and

$$\|B^{-1/2}\frac{u_1-u_0}{\tau}\|_H\leq\|\varphi\|_H+\|B^{-1/2}\psi\|_H.$$

Therefore, they follow the estimates (6) and (7) for $k=1$. Now, we prove estimates (6) and (7) for $2\leq k\leq N$. We have that (see [21])

$$u_k=R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left(R^{k-1}-\widetilde{R}^{k-1}\right)u_0+\left(\widetilde{R}-R\right)^{-1}\left(\widetilde{R}^{k}-R^{k}\right)u_1$$
$$+\sum_{j=1}^{k-1}R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left(\widetilde{R}^{k-j}-R^{k-j}\right)a\tau^2 A u_{\left[\frac{j-mN}{N+1}\right]N+mN}. \tag{10}$$

Using the formula (10) and the following identities

$$\left(I-\widetilde{R}\right)(I-R)=\tau^2 AR\widetilde{R},\ \left(\widetilde{R}-R\right)^{-1}=\left(-2i\tau B^{1/2}\right)^{-1}R^{-1}\widetilde{R}^{-1},$$

we get

$$u_k=\left\{a+(1-a)\,\frac{i}{2}\left(B^{-1/2}\left(-\frac{\alpha}{2}I-iB^{1/2}\right)R^{k-1}\right.\right.$$
$$\left.\left.-B^{-1/2}\left(-\frac{\alpha}{2}I+iB^{1/2}\right)\widetilde{R}^{k-1}\right)\right\}\varphi+\frac{i}{2}\left(\widetilde{R}^{k}-R^{k}\right)B^{-1/2}\psi \tag{11}$$

and

$$B^{-1/2}\frac{u_{k+1}-u_k}{\tau}=\left\{(1-a)\,\frac{i}{2}B^{-1/2}\left(B^{-1/2}\left(-\frac{\alpha}{2}I-iB^{1/2}\right)\left(-\frac{\alpha}{2}I+iB^{1/2}\right)R^{k}\right.\right.$$
$$\left.\left.-B^{-1/2}\left(-\frac{\alpha}{2}I+iB^{1/2}\right)\left(-\frac{\alpha}{2}I-iB^{1/2}\right)\widetilde{R}^{k}\right)\right\}\varphi$$
$$+\frac{i}{2}B^{-1/2}\left\{\left(-\frac{\alpha}{2}I-iB^{1/2}\right)\widetilde{R}^{k+1}-\left(-\frac{\alpha}{2}I+iB^{1/2}\right)R^{k+1}\right\}B^{-1/2}\psi. \tag{12}$$

Applying the formulas (11) and (12), using the triangle inequality and the estimates (3) and (4), we obtain

$$\|u_k\|_H \leq \left(|a| + |1-a|\left(1+\frac{\alpha}{2}\frac{1}{\sqrt{\delta^2-\frac{\alpha^2}{4}}}\right)\right)\|\varphi\|_H + \left\|B^{-1/2}\psi\right\|_H \tag{13}$$

and

$$\left\|B^{-1/2}\frac{u_{k+1}-u_k}{\tau}\right\|_H \leq |1-a|\frac{\delta}{\delta-\frac{\alpha^2}{4}}\|\varphi\|_H + \left(1+\frac{\alpha}{2}\frac{1}{\sqrt{\delta^2-\frac{\alpha^2}{4}}}\right)\left\|B^{-1/2}\psi\right\|_H. \tag{14}$$

From (13) and (14), they follow the estimates (6) and (7) for $2 \leq k \leq N$.

Now, let $mN+1 \leq k \leq (m+1)N$ for $m = 1, 2, 3....$ It is clear that

$$u_{mN+1} = u_{mN} + \tau B^{1/2} R\widetilde{R}\left(B^{-1/2}\frac{u_{mN}-u_{mN-1}}{\tau}\right) \tag{15}$$

and

$$B^{-1/2}\frac{u_{mN+1}-u_{mN}}{\tau} = R\widetilde{R}B^{-1/2}\frac{u_{mN}-u_{mN-1}}{\tau}. \tag{16}$$

Applying formulas (15), (16) and using triangle inequality and estimates (3) and (4), we get

$$\|u_{mN+1}\|_H \leq \|u_{mN}\|_H + \left\|B^{-1/2}\frac{u_{mN}-u_{mN-1}}{\tau}\right\|_H \tag{17}$$

and

$$\left\|B^{-1/2}\frac{u_{mN+1}-u_{mN}}{\tau}\right\|_H \leq \|u_{mN}\|_H + \left\|B^{-1/2}\frac{u_{mN}-u_{mN-1}}{\tau}\right\|_H. \tag{18}$$

So, from these estimates they follow the estimates (8) and (9) for $k = mN$, respectively. Now, we will prove estimates (6) and (7) for $mN+2 \leq k \leq (m+1)N,\ m = 1, 2, ...$ We have that (see [21])

$$\begin{aligned} u_k = {} & R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left(R^{k-mN-1}-\widetilde{R}^{k-mN-1}\right)u_{mN} + \left(\widetilde{R}-R\right)^{-1}\left(\widetilde{R}^{k-mN}-R^{k-mN}\right)u_{mN+1} \\ & + \sum_{j=mN+1}^{k-1} R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left(\widetilde{R}^{k-j}-R^{k-j}\right)a\tau^2 A u_{\left[\frac{j-mN}{N+1}\right]N+mN} \end{aligned} \tag{19}$$

for the solution of the difference problem (2). Using formula (19), we get

$$\begin{aligned} u_k = {} & \left[a + (1-a)\frac{i}{2}B^{-1/2}\left[\left(\frac{-\alpha}{2}I - iB^{1/2}\right)R^{k-mN-1} - \left(\frac{-\alpha}{2}I + iB^{1/2}\right)\widetilde{R}^{k-mN-1}\right]\right]u_{mN} \\ & + \frac{i}{2}B^{-1/2}\left(R\widetilde{R}\right)^{-1}\left[\widetilde{R}^{k-mN} - R^{k-mN}\right]\left(\frac{u_{mN+1}-u_{mN}}{\tau}\right). \end{aligned} \tag{20}$$

Applying the formula (20) and using triangle inequality, we get

$$\|u_k\|_H \leq b\,\|u_{mN}\|_H + \left\|B^{-1/2}\left(\frac{u_{mN+1}-u_{mN}}{\tau}\right)\right\|_H. \tag{21}$$

From (21) it follows the estimate (8). Using (20), we obtain

$$B^{-1/2}\frac{u_{k+1}-u_k}{\tau}=\left[(1-a)\,\frac{i}{2}\left[B^{-1}AR^{k-mN}-B^{-1}A\widetilde{R}^{k-mN}\right]\right]u_{mN}$$
$$+\frac{i}{2}\left[\left(-\frac{\alpha}{2}B^{-1/2}-i\right)R^{-1}\widetilde{R}^{k-mN}-\left(-\frac{\alpha}{2}B^{-1/2}+i\right)\widetilde{R}^{-1}R^{k-mN}\right]B^{-1/2}\left(\frac{u_{mN+1}-u_{mN}}{\tau}\right).\quad(22)$$

Now, applying (22) and using triangle inequality, we get

$$\left\|B^{-1/2}\frac{u_{k+1}-u_k}{\tau}\right\|_H\le|1-a|\,\frac{\delta}{\delta-\frac{\alpha^2}{4}}\,\|u_{mN}\|_H+\left(1+\frac{\frac{\alpha}{2}}{\sqrt{\delta-\frac{\alpha^2}{4}}}\right)\left\|B^{-1/2}\left(\frac{u_{mN+1}-u_{mN}}{\tau}\right)\right\|_H.\quad(23)$$

From (23) it follows the estimate (9). Therefore, the proof of Theorem 1 is completed. □

By applying operator $B^{1/2}$, in the same manner of proof of Theorem 1, we can obtain the following stability results.

Theorem 2. For the solution of difference problem (2), the following estimates hold:

$$\max_{1\le k\le N}\left\|B^{1/2}u_k\right\|_H\le b\left\|B^{1/2}\varphi\right\|_H+\|\psi\|_H,\quad(24)$$

$$\max_{1\le k\le N}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H\le c\left\|B^{1/2}\varphi\right\|_H+d\,\|\psi\|_H,\quad(25)$$

$$\max_{mN+1\le k\le(m+1)N}\left\|B^{1/2}u_k\right\|_H\le b\max_{(m-1)N\le k\le mN}\left\|B^{1/2}u_k\right\|_H$$
$$+\max_{(m-1)N+1\le k\le mN}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H,\ m=1,2,...,\quad(26)$$

$$\max_{mN+1\le k\le(m+1)N}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H\le c\max_{(m-1)N\le k\le mN}\left\|B^{1/2}u_k\right\|_H$$
$$+d\max_{(m-1)N+1\le k\le mN}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H,\ m=1,2,....\quad(27)$$

Applications

Now, we consider the applications of abstract Theorem 1 and Theorem 2.

As first application, we consider the initial value problem for the delay telegraph equations with nonlocal boundary conditions

$$\begin{cases} u_{tt}(t,x)+\alpha u_t(t,x)-(a(x)u_x(t,x))_x+\delta u(t,x)\\ =a\left(-(a(x)u_x([t],x)))_x+\delta u([t],x)\right),\ 0<t<\infty,\ 0<x<l,\\ u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x),\ 0\le x\le l,\\ u(t,0)=u(t,l),\ u_x(t,0)=u_x(t,l),\ 0\le t<\infty.\end{cases}\quad(28)$$

Problem (28) has a unique smooth solution $u(t,x)$ for smooth funtions $a(x)\ge a_0>0,\ (x\in(0,l))$, $a(l)=a(0)$, $\delta>0$, $\varphi(x)$, $\psi(x)$, $(x\in[0,l])$ and $0<a<1$. This allows us to reduce the problem (28) to the initial value problem (1) in a Hilbert space $H = L_2[0,l]$ with a self-adjoint positive definite operator A^x defined by the formula (28).

The discretization of problem (28) is carried out in two steps. In the first step, we define the grid space

$$[0,l]_h = \{x = x_n : x_n = nh,\ 0 \leq n \leq M,\ Mh = l\}.$$

We introduce the Hilbert spaces $L_{2h} = L_2([0,l]_h)$ and $W_{2h}^1 = W_2^1([0,l]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ defined on $[0,l]_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in[0,l]_h} \left|\varphi^h(x)\right|^2 h\right)^{1/2},$$

$$\left\|\varphi^h\right\|_{W_{2h}^1} = \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x\in[0,l]_h} \left|\varphi_{x,j}^h(x)\right|^2 h\right)^{1/2},$$

respectively. To the differential operator A^x defined by (28), we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x) = \{-(a(x)\varphi_{\overline{x}})_{x,n} + \delta\varphi_n\}_1^{M-1} \tag{29}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ satisfying the conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we reach the initial value problem

$$\begin{cases} \frac{d^2u^h(t,x)}{dt^2} + \alpha\frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = aA_h^x u^h([t],x), \\ 0 < t < \infty,\ x \in [0,l]_h, \\ u^h(0,x) = \varphi^h(x),\ u_t^h(0,x) = \psi^h(x),\ x \in [0,l]_h. \end{cases} \tag{30}$$

In the second step, we replace (30) with the difference scheme (2) and we get

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2} + \alpha\frac{u_{k+1}^h(x)-u_k^h(x)}{\tau} + A_h^x u_{k+1}^h(x) = aA_h^x u^h_{\left[\frac{k-mN}{N+1}\right]N+mN}(x), \\ t_k = k\tau,\quad x \in [0,l]_h,\ N\tau = 1,\ (m-1)N+1 \leq k \leq mN-1,\quad m = 1,2,..., \\ u_0^h(x) = \varphi^h(x),\ \left((1+\alpha\tau)I_h + \tau^2 A_h^x\right)\frac{u_1^h(x)-u_0^h(x)}{\tau} = \psi^h(x),\ x \in [0,l]_h, \\ \left((1+\alpha\tau)I_h + \tau^2 A_h^x\right)\frac{u_{mN+1}^h(x)-u_{mN}^h(x)}{\tau} = \frac{u_{mN}^h(x)-u_{mN-1}^h(x)}{\tau}, m = 1,2,...,\ x \in [0,l]_h. \end{cases} \tag{31}$$

Theorem 3. Suppose that $\delta > \frac{\alpha^2}{4}$. Then, for the solution $\left\{u_k^h(x)\right\}_0^N$ of problem (31) the following stability estimates hold:

$$\max_{1\leq k\leq N} \|u_k^h\|_{L_{2h}} \leq M_1\left\{\|\varphi^h\|_{L_{2h}} + \|\psi^h\|_{L_{2h}}\right\},$$

$$\max_{1\leq k\leq N} \|u_k^h\|_{W_{2h}^1} + \max_{1\leq k\leq N}\left\|\frac{u_k^h - u_{k-1}^h}{\tau}\right\|_{L_{2h}} \leq M_2\left\{\|\varphi^h\|_{W_{2h}^1} + \|\psi^h\|_{L_{2h}}\right\},$$

$$\max_{mN+1\le k\le (m+1)N}\|u_k^h\|_{L_{2h}}$$
$$\le M_3\left\{\max_{(m-1)N\le k\le mN}\|u_k^h\|_{L_{2h}}+\max_{(m-1)N+1\le k\le mN}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}\right\},\ m=1,2,...,$$

$$\max_{mN+1\le k\le (m+1)N}\|u_k^h\|_{W_{2h}^1}+\max_{mN+1\le k\le (m+1)N}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}$$
$$\le M_4\left\{\max_{(m-1)N\le k\le mN}\|u_k^h\|_{W_{2h}^1}+\max_{(m-1)N+1\le k\le mN}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}\right\},\ m=1,2,...,$$

where M_1, M_2, M_3 and M_4 do not depend on $\varphi^h(x)$ or $\psi^h(x)$.

Proof. Difference scheme (31) can be written in abstract form

$$\begin{cases} \frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}+\alpha\frac{u_{k+1}^h-u_k^h}{\tau}+A_h^x u_{k+1}^h=aA_h^x u_{\left[\frac{k-mN}{N+1}\right]N+mN}^h, \\ t_k=k\tau,\quad N\tau=1,\ (m-1)N+1\le k\le mN-1,\quad m=1,2,..., \\ u_0^h=\varphi^h,\ \left((1+\alpha\tau)I_h+\tau^2A_h^x\right)\frac{u_1^h-u_0^h}{\tau}=\psi^h, \\ \left((1+\alpha\tau)I_h+\tau^2A_h^x\right)\frac{u_{mN+1}^h-u_{mN}^h}{\tau}=\frac{u_{mN}^h-u_{mN-1}^h}{\tau},\ m=1,2,... \end{cases}$$

in a Hilbert space L_{2h} with self-adjoint positive definite operator $A_h=A_h^x$ by formula (29). Here, $u_k^h=u_k^h(x)$ is unknown abstract mesh function defined on $[0,l]_h$ with the values in $H=L_{2h}$. Therefore, estimates of Theorem 3 follow from estimates (6), (7), (8) and (9), respectively. □

For second application of abstract Theorem 1 and Theorem 2, let $\Omega\subset R^n$ be an open bounded domain with smooth boundary S, $\overline{\Omega}=\Omega\cup S$. In $[0,\infty)\times\Omega$, we consider the initial-boundary value problem for the delay telegraph equations

$$\begin{cases} u_{tt}(t,x)+\alpha u_t(t,x)-\sum\limits_{r=1}^{n}(a_r(x)u_{x_r}(t,x))_{x_r}=a\left(-\sum\limits_{r=1}^{n}(a_r(x)u_{x_r}([t],x))_{x_r}\right), \\ x=(x_1,...,x_n)\in\Omega,\ 0<t<\infty, \\ u(0,x)=\varphi(x),\ \frac{\partial u(0,x)}{\partial t}=\psi(x),\ x\in\overline{\Omega}, \\ u(t,x)=0,\ x\in S,\ 0\le t<\infty, \end{cases} \tag{32}$$

where $a_r(x),\ (x\in\Omega),\ \varphi(x),\ \psi(x),\ (x\in\overline{\Omega})$ are given smooth functions and $a_r(x)>0$ and $0<a<1$.

We introduce the Hilbert space $L_2(\overline{\Omega})$, the space of all integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})}=\left\{\int\cdots\int_{x\in\overline{\Omega}}|f(x)|^2\,dx_1...dx_n\right\}^{\frac{1}{2}}.$$

The discretization of problem (32) is carried out in two steps. In the first step, we define the grid space

$$\overline{\Omega}_h = \{x = x_r = (h_1 j_1, \cdots, h_n j_n),\ j = (j_1, \cdots, j_n),\ 0 \le j_r \le N_r,\ N_r h_r = 1,\ r = 1, \cdots, n\},$$
$$\Omega_h = \overline{\Omega}_h \cap \Omega,\ S_h = \overline{\Omega}_h \cap S.$$

We introduce the Hilbert spaces $L_{2h} = L_2(\overline{\Omega}_h)$ and $W^1_{2h} = W^1_2(\overline{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 r_1, ..., h_n r_n)\}$ defined on $\overline{\Omega}_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in\overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_n\right)^{1/2},$$

$$\left\|\varphi^h\right\|_{W^1_{2h}} = \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x\in\overline{\Omega}_h} \sum_{r=1}^{m} \left|\varphi^h_{x_r,j_r}(x)\right|^2 h_1 \cdots h_n\right)^{1/2},$$

respectively. To the differential operator A^x defined by (32), we assign the difference operator A^x_h by the formula

$$A^x_h u^h = -\sum_{r=1}^{n} \left(a_r(x) u^h_{x_r}\right)_{x_r,j_r},$$

where A^x_h is known as self-adjoint positive definite operator in L_{2h}, acting in the space of grid functions $u^h(x)$ satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. With the help of the difference operator A^x_h, we arrive at the following initial value problem

$$\begin{cases} \frac{d^2u^h(t,x)}{dt^2} + \alpha\frac{du^h(t,x)}{dt} + A^x_h u^h(t,x) = aA^x_h u^h([t], x), \\ 0 < t < \infty,\ x \in \Omega_h, \\ u^h(0,x) = \varphi^h(x), u^h_t(0,x) = \psi^h(x),\ x \in \overline{\Omega}_h \end{cases} \tag{33}$$

for an infinite system of ordinary differential equations.

In the second step, we replace (33) with the difference scheme (2) and we get

$$\begin{cases} \frac{u^h_{k+1}(x)-2u^h_k(x)+u^h_{k-1}(x)}{\tau^2} + \alpha\frac{u^h_{k+1}(x)-u^h_k(x)}{\tau} + A^x_h u^h_{k+1}(x) = aA^x_h u^h_{\left[\frac{k-mN}{N+1}\right]N+mN}(x), \\ t_k = k\tau,\quad x \in \Omega_h,\ N\tau = 1,\ (m-1)N + 1 \le k \le mN - 1,\quad m = 1, 2, ..., \\ u^h_0(x) = \varphi^h(x),\ \left((1+\alpha\tau) I_h + \tau^2 A^x_h\right) \frac{u^h_1(x)-u^h_0(x)}{\tau} = \psi^h(x),\ x \in \overline{\Omega}_h, \\ \left((1+\alpha\tau) I_h + \tau^2 A^x_h\right) \frac{u^h_{mN+1}(x)-u^h_{mN}(x)}{\tau} = \frac{u^h_{mN}(x)-u^h_{mN-1}(x)}{\tau},\ x \in \overline{\Omega}_h,\ m = 1, 2, ... \end{cases} \tag{34}$$

Theorem 4. Suppose that $\delta > \frac{\alpha^2}{4}$. Then, for the solution $\left\{u^h_k(x)\right\}_0^N$ of problem (34) the following stability estimates hold:

$$\max_{1\le k\le N} \|u^h_k\|_{L_{2h}} \le M_5 \left\{\|\varphi^h\|_{L_{2h}} + \|\psi^h\|_{L_{2h}}\right\},$$

$$\max_{1\le k\le N}\|u_k^h\|_{W_{2h}^1}+\max_{1\le k\le N}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}\le M_6\left\{\|\varphi^h\|_{W_{2h}^1}+\|\psi^h\|_{L_{2h}}\right\},$$

$$\max_{mN+1\le k\le (m+1)N}\|u_k^h\|_{L_{2h}}$$
$$\le M_7\left\{\max_{(m-1)N\le k\le mN}\|u_k^h\|_{L_{2h}}+\max_{(m-1)N+1\le k\le mN}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}\right\},\ m=1,2,...,$$

$$\max_{mN+1\le k\le (m+1)N}\|u_k^h\|_{W_{2h}^1}+\max_{mN+1\le k\le (m+1)N}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}$$
$$\le M_8\left\{\max_{(m-1)N\le k\le mN}\|u_k^h\|_{W_{2h}^1}+\max_{(m-1)N+1\le k\le mN}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}\right\},\ m=1,2,...,$$

where M_5, M_6, M_7 and M_8 do not depend on $\varphi^h(x)$ or $\psi^h(x)$.

Proof. Difference scheme (34) can be written in abstract form (2) in a Hilbert space $L_{2h}=L_2(\overline{\Omega}_h)$ with self-adjoint positive definite operator $A_h=A_h^x$ by formula (33). Here, $u_k^h=u_k^h(x)$ is unknown abstract mesh function defined on $\overline{\Omega}_h$ with the values in $H=L_{2h}$. Therefore, estimates of Theorem 4 follow from estimates (6), (7), (8) and (9) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in [22]. □

Theorem 5. For the solutions of the elliptic difference problem

$$A_h^x u^h(x)=\omega^h(x),\ x\in\Omega_h,\ u^h(x)=0,\ x\in S_h,$$

the following coercivity inequality holds:

$$\sum_{r=1}^{n}\left\|u^h{}_{x_r\overline{x_r}}\right\|_{L_{2h}}\le M_9||\omega^h||_{L_{2h}},$$

where M_9 does not depend on h and ω^h.

Numerical results

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of telegraph differential equations play an important role in applied mathematics. In this section the first order of accuracy difference scheme for the solution of the initial boundary value problem for one dimensional telegraph differential equation with nonlocal boundary conditions is presented.

We consider the initial-boundary value problem

$$\begin{cases} u_{tt}(t,x)+2u_t(t,x)-u_{xx}(t,x)+u(t,x)=0.001\left(-u_{xx}([t],x)+u([t],x)\right),\\ 0<t<\infty,\ 0<x<\pi,\\ u(t,x)=e^{-t}\sin(2x),\ -1\le t\le 0,\ 0\le x\le\pi,\\ u(t,0)=u(t,\pi),\ u_x(t,0)=u_x(t,\pi),\ 0\le t<\infty\end{cases}\tag{35}$$

for the delay telegraph differential equation with nonlocal conditions.

By using step by step method and Fourier series method, it can be shown that the exact solution of the problem (35) is

$$u(t,x)=T_n(t)\sin(2x),\ n-1\le t\le n,\ n=1,2,...,$$

where

$$T_1(t)=\frac{999}{1000}e^{-t}\cos(2t)-\frac{1}{2000}e^{-t}\sin(2t)+\frac{1}{1000},$$

$$T_{n+1}(t)=T_n(n)e^{-t}\cos(2t)+\frac{T_n(n)+T_n'(n)}{2}e^{-t}\sin(2t)$$
$$+\frac{T_n(n)}{2000}\left(2-2e^{-(t-n)}\cos(2(t-n))-e^{-(t-n)}\sin(2(t-n))\right),\ n=1,2....$$

Using first order of accuracy difference scheme for the approximate solutions of problem (35), we get the following system of equations

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+2\frac{u_n^{k+1}-u_n^k}{\tau}-\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+u_n^{k+1} \\ =0.001\left(-\frac{u_{n+1}^{\left[\frac{k-mN}{N+1}\right]N+mN}-2u_n^{\left[\frac{k-mN}{N+1}\right]N+mN}+u_{n-1}^{\left[\frac{k-mN}{N+1}\right]N+mN}}{h^2}+u_n^{\left[\frac{k-mN}{N+1}\right]N+mN}\right), \\ t_k=k\tau,\ N\tau=1,\ mN+1\le k\le(m+1)N-1,\ m=0,1,2,..., \\ x_n=nh,\ Mh=\pi,\ 1\le n\le M-1, \\ u_n^0=\sin(2nh),\ (1+2\tau)\frac{u_n^1-u_n^0}{\tau}+\tau\left(-\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}+u_n^1\right) \\ +\tau\left(\frac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2}-u_n^0\right)=-\sin(2nh),\ 0\le n\le M, \\ (1+2\tau)\frac{u_n^{mN+1}-u_n^{mN}}{\tau}+\tau\left(-\frac{u_{n+1}^{mN+1}-2u_n^{mN+1}+u_{n-1}^{mN+1}}{h^2}+u_n^{mN+1}\right) \\ +\tau\left(\frac{u_{n+1}^{mN}-2u_n^{mN}+u_{n-1}^{mN}}{h^2}-u_n^{mN}\right)=\frac{u_n^{mN}-u_n^{mN-1}}{\tau},\ 0\le n\le M,\quad m=1,2,..., \\ u_0^k=u_M^k,\quad u_1^k-u_0^k=u_M^k-u_{M-1}^k,\ mN\le k\le(m+1)N,\ m=0,1,2,.... \end{cases} \tag{36}$$

We can rewrite system (36) in the matrix form

$$\begin{cases} CU^{k+1}+DU^k+EU^{k-1}=\varphi\left(U^{\left[\frac{k-mN}{mN+1}\right]N+mN}\right),\ k=1,2,3,... \\ U^0=\begin{bmatrix}0\\ \sin(2h)\\ \cdot\\ \cdot\\ \cdot\\ \sin(2(M-1)h)\\ 0\end{bmatrix}_{(M+1)\times1},\ U^1=F^{-1}G\begin{bmatrix}0\\ \sin(2h)\\ \cdot\\ \cdot\\ \cdot\\ \sin(2(M-1)h)\\ 0\end{bmatrix}_{(M+1)\times1}, \\ U^{mN+1}=F^{-1}HU^{mN}-F^{-1}U^{mN-1},\ m=1,2,..., \end{cases} \tag{37}$$

where C, D, E, F, G and H are $(M+1)\times(M+1)$ matrices, $\varphi\left(U^{\left[\frac{k-mN}{mN+1}\right]N+mN}\right)$ and U^r, $r=k,k\pm1$ are $(M+1)\times 1$ column vectors defined by

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1 \\ a & b & a & 0 & . & 0 & 0 & 0 & 0 \\ 0 & a & b & a & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & a & b & a & 0 \\ 0 & 0 & 0 & 0 & . & 0 & a & b & a \\ 1 & -1 & 0 & 0 & . & 0 & 0 & -1 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$D=\begin{bmatrix} 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & c & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & c & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & c & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 & c & 0 \\ 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$E=\begin{bmatrix} 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & d & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & d & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & d & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 & d & 0 \\ 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$F=\begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1 \\ e & p & e & 0 & . & 0 & 0 & 0 & 0 \\ 0 & e & p & e & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & e & p & e & 0 \\ 0 & 0 & 0 & 0 & . & 0 & e & p & e \\ 1 & -1 & 0 & 0 & . & 0 & 0 & -1 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$G=\begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1 \\ e & s & e & 0 & . & 0 & 0 & 0 & 0 \\ 0 & e & s & e & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & e & s & e & 0 \\ 0 & 0 & 0 & 0 & . & 0 & e & s & e \\ 1 & -1 & 0 & 0 & . & 0 & 0 & -1 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1 \\ e & g & e & 0 & . & 0 & 0 & 0 & 0 \\ 0 & e & g & e & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & e & g & e & 0 \\ 0 & 0 & 0 & 0 & . & 0 & e & g & e \\ 1 & -1 & 0 & 0 & . & 0 & 0 & -1 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$\varphi\left(U^{\left[\frac{k-mN}{mN+1}\right]N+mN}\right) = \begin{bmatrix} 0 \\ \varphi_1^k \\ . \\ . \\ . \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1}, \quad U^r = \begin{bmatrix} U_0^r \\ U_1^r \\ . \\ . \\ . \\ U_{M-1}^r \\ U_M^r \end{bmatrix}_{(M+1)\times 1}, \text{ for } r = k, k \mp 1,$$

where

$$\varphi_n^k = 0.001\left(-\frac{u_{n+1}^{\left[\frac{k-mN}{N+1}\right]N+mN} - 2u_n^{\left[\frac{k-mN}{N+1}\right]N+mN} + u_{n-1}^{\left[\frac{k-mN}{N+1}\right]N+mN}}{h^2} + u_n^{\left[\frac{k-mN}{N+1}\right]N+mN}\right) \text{ for } k = 1, 2, ...,$$

$m = 0, 1, 2, ..., 1 \le n \le M - 1$.

Here, we denote $a = -1/h^2$, $b = 1/\tau^2 + 2/\tau + 2/h^2 + 1$, $c = -2/\tau^2 - 2/\tau$, $d = 1/\tau^2$, $e = -\tau^2/h^2$, $p = 1 + 2\tau + \tau^2 + 2\tau^2/h^2$, $s = 1 + \tau + \tau^2 + 2\tau^2/h^2$ and $g = 2 + 2\tau + \tau^2 + 2\tau^2/h^2$.

Hence, we have a second order of difference equation with matrix coefficients. We find the numerical solutions for different values of N and M and here, u_n^k represents the numerical solutions of the difference scheme at (t_k, x_n). For $N = M = 40$, $N = M = 80$ and $N = M = 160$ in $t \in [0, 1]$, $t \in [1, 2]$ and $t \in [2, 3]$, the errors computed by the following formula are given in Table 1.

$$E_M^N = \max_{\substack{mN+1 \le k \le (m+1)N,\ m = 0,1,... \\ 0 \le n \le M}} \left| u(t_k, x_n) - u_n^k \right|.$$

Table 1

Errors of Difference Scheme (36)

	N=M=40	N=M=80	N=M=160
$t \in [0, 1]$	0.045895	0.023073	0.011568
$t \in [1, 2]$	0.042967	0.021574	0.010810
$t \in [2, 3]$	0.019786	0.010107	0.0051085

As it is seen in Table 1, the errors in the first order of accuracy difference scheme decrease approximately by a factor of 1/2 when the values of M and N are doubled.

Conclusion

In this study, we consider the initial-boundary value problem for telegraph equations with time delay in a Hilbert space. Theorem on stability estimates for the solution of the first order of accuracy difference scheme is established. In practice, stability estimates for the solution of the difference schemes for the two type of the time delay telegraph equations are obtained. As a test problem, one-dimensional delay telegraph equation with nonlocal boundary conditions is considered. Numerical solutions of this problem are provided.

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Кідіртпелі телеграф теңдеуі үшін орнықты айырымдық схемасы туралы

Гильберт кеңістігінде кешігулі телеграф теңдеуі үшін бастапқы-шеттік есебінің жуықтау шешімінің орнықты айырымдық схемасы ұсынылған. Айырымдық схемасының орнықтылығы туралы негізгі теоремасы берілген. Қосымшасында уақыт кідіртпесі бар телеграф теңдеуінің екі түрі үшін айырымдық схемасының шешімінің орнықтылық бағамы алынды. Тестілік есебі ретінде, бейлокальді шарттарымен берілген кідіртпелі телеграф бірөлшемді теңдеуі қарастырылды. Сандық есептеулері мақалада көрсетілген.

Кілт сөздер: айырымдық схемасы, кешігулі телеграф теңдеуі, орнықтылық.

А. Ашыралыев, К. Турк, Д. Агирсевен

Об устойчивой разностной схеме для уравнения телеграфа с задержкой

Представлена устойчивая разностная схема для приближенного решения начально-краевой задачи для телеграфного уравнения с запаздыванием в гильбертовом пространстве. Установлена основная теорема об устойчивости разностной схемы. В приложениях получены оценки устойчивости решения разностных схем для двух типов телеграфных уравнений с временной задержкой. В качестве тестовой задачи рассмотрено одномерное уравнение задержки телеграфа с нелокальными условиями. Численные результаты приведены в статье.

Ключевые слова: разностные схемы, уравнения телеграфа с запаздыванием, устойчивость.

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A note on the hyperbolic-parabolic identification problem with involution and Dirichlet boundary condition

In the present paper, a source identification problem for hyperbolic-parabolic equation with involution and Dirichlet condition is studied. The stability estimates for the solution of the source identification hyperbolic-parabolic problem are established. The first order of accuracy stable difference scheme is constructed for the approximate solution of the problem under consideration. Numerical results are given for a simple test problem.

Keywords: source identification problem, hyperbolic-parabolic differential equation, difference scheme, stability.

Introduction

Partial differential equations with unknown source terms are used to model the behaviour of real-life systems in many different areas of science and technology. They have been studied extensively by many researchers (see, e.g., [1]-[13] and the references therein). Numerous source identification problems for hyperbolic-parabolic equations and the corresponding difference schemes for their approximate solutions were previously studied by the authors (see [14]-[18]. Partial differential equations with the involution have been recently investigated in [19]-[22] However, source identification problems for hyperbolic-parabolic equation with involution have not been investigated.

The present paper is devoted to the study of source identification problems for hyperbolic-parabolic differential and difference equations with involution. The stability of these source identification problems is established. Numerical results are presented.

Stability of differential equation

We consider the space-dependent source identification problem

$$\begin{cases} u_{tt}(t,x) - \big(a(x)u_x(t,x)\big)_x - \beta\big(a(-x)u_x(t,-x)\big)_x + \delta u(t,x) \\ \qquad = p(x) + f(t,x),\ -\ell < x < \ell,\ 0 < t < 1, \\ u_t(t,x) - \big(a(x)u_x(t,x)\big)_x - \beta\big(a(-x)u_x(t,-x)\big)_x + \delta u(t,x) \\ \qquad = p(x) + g(t,x),\ -\ell < x < \ell,\ -1 < t < 0, \\ u(0^+,x) = u(0^-,x),\ u_t(0^+,x) = u_t(0^-,x),\ -\ell \le x \le \ell, \\ u(t,-\ell) = u(t,\ell) = 0,\ -1 \le t \le 1, \\ u(-1,x) = \varphi(x),\ u(1,x) = \psi(x),\ -\ell \le x \le \ell \end{cases} \tag{1}$$

for one-dimensional hyperbolic-parabolic differential equation with involution. Throughout this paper, we will assume that $\overline{a} \ge a(x) = a(-x) \ge \underline{a} > 0$, $x \in (-\ell,\ell)$ and $\underline{a} - \overline{a}|\beta| \ge 0$. Under compatibility conditions problem (1) has a unique smooth solution $\big(u(t,x),p(x)\big)$ for the given smooth functions $a(x)$, $\varphi(x)$, $\psi(x)$, $x\in[-\ell,\ell]$, $f(t,x)$, $(t,x)\in(0,1)\times(-\ell,\ell)$, $g(t,x)$, $(t,x)\in(-1,0)\times(-\ell,\ell)$ and constant $\delta>0$.

Let the Sobolev space $W_2^2[-\ell,\ell]$ be defined as the set of all functions $v(x)$ defined on $[-\ell,\ell]$ such that $v(x)$ and the second order derivative function $v''(x)$ are both locally integrable in $L_2[-\ell,\ell]$, equipped with the norm

$$\|v(x)\|_{W_2^2[-\ell,\ell]}=\left(\int\limits_{-\ell}^{\ell}|v(x)|^2\,dx\right)^{1/2}+\left(\int\limits_{-\ell}^{\ell}\left|v''(x)\right|^2dx\right)^{1/2}.$$

Theorem 1. Suppose that $\varphi,\psi\in W_2^2[-\ell,\ell]$. Let function $f(t,x)$ be continuously differentiable in t on $[0,1]\times[-\ell,\ell]$ and function $g(t,x)$ be continuously differentiable in t on $[-1,0]\times[-\ell,\ell]$. Then the solution of the identification problem (1) satisfies the stability estimates

$$\|u\|_{C([-1,1],L_2[-\ell,\ell])}+\left\|(A^x)^{-1}p\right\|_{L_2[-\ell,\ell]}$$
$$\le M_1(\delta)\Big[\|\varphi\|_{L_2[-\ell,\ell]}+\|\psi\|_{L_2[-\ell,\ell]}+\|f\|_{C([0,1],L_2[-\ell,\ell])}+\|g\|_{C([-1,0],L_2[-\ell,\ell])}\Big],$$
$$\|u\|_{C^{(2)}([0,1],L_2[-\ell,\ell])}+\|u\|_{C^{(1)}([-1,0],L_2[-\ell,\ell])}+\|u\|_{C([-1,1],W_2^2[-\ell,\ell])}+\|p\|_{L_2[-\ell,\ell]}$$
$$\le M_2(\delta)\Big[\|\varphi\|_{W_2^2[-\ell,\ell]}+\|\psi\|_{W_2^2[-\ell,\ell]}+\|f\|_{C^{(1)}([0,1],L_2[-\ell,\ell])}+\|g\|_{C^{(1)}([-1,0],L_2[-\ell,\ell])}\Big],$$

where $M_1(\delta)$ and $M_2(\delta)$ do not depend on $\varphi(x)$, $\psi(x)$, $f(t,x)$ and $g(t,x)$.

Proof. Problem (1) can be written in the following abstract form

$$\begin{cases} u''(t)+Au(t)=p+f(t),\ 0<t<1,\\ u'(t)+Au(t)=p+g(t),\ -1<t<0,\\ u(0^+)=u(0^-),\ u'(0^+)=u'(0^-),\\ u(-1)=\varphi,\ u(1)=\psi \end{cases}$$

in a Hilbert space $L_2[-\ell,\ell]$ with self-adjoint positive definite operator $A=A^x$ defined by the formula

$$A^xu(x)=-\big(a(x)u_x(x)\big)_x-\beta\big(a(-x)u_x(-x)\big)_x+\delta u(x) \tag{2}$$

with the domain $D(A^x)=\Big\{u\in W_2^2[-\ell,\ell]\ \Big|\ u(-\ell)=u(\ell)=0\Big\}$. Here, $f(t)=f(t,x)$ and $g(t)=g(t,x)$ are given abstract functions, $u(t)=u(t,x)$ is unknown function and $p=p(x)$ is the unknown element of $L_2[-\ell,\ell]$. Therefore, the proof of Theorem 1 is based on the self-adjointness and positive definiteness of the space operator A^x (see [19]).

Stability of difference scheme

Now, we study the stable difference scheme for the approximate solution of identification problem (1). The discretization of source identification problem (1) is carried out in two steps.

In the first step, the spatial discretization is carried out. We define the grid space

$$[-\ell,\ell]_h=\left\{x=x_n\ \middle|\ x_n=nh,\ -M\le n\le M,\ Mh=\ell\right\}.$$

We introduce the Hilbert space $L_{2h}=L_2([-\ell,\ell]_h)$ of the grid functions $\varphi^h(x)=\{\varphi^n\}_{-M}^{M}$ defined on $[-\ell,\ell]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_{2h}}=\left(\sum_{x\in[-\ell,\ell]_h}\left|\varphi^h(x)\right|^2h\right)^{1/2}.$$

To the differential operator A^x defined by the formula (2), we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x)=\Big\{-\big(a(x)\varphi_{\bar{x}}^n\big)_x-\beta\big(a(-x)\varphi_{\bar{x}}^{-n}\big)_x+\delta\varphi^n\Big\}_{-M+1}^{M-1}, \tag{3}$$

acting in the space of grid functions $\varphi^h(x)=\{\varphi^n\}_{-M}^{M}$ and satisfying the conditions $\varphi_{-M}=\varphi_M=0$. Here

$$\varphi_{\bar{x}}^n=\frac{\varphi^n-\varphi^{n-1}}{h},\ -M+1\le n\le M,\quad \varphi_x^n=\frac{\varphi^{n+1}-\varphi^n}{h},\ -M\le n\le M-1.$$

It is well-known that A_h^x, defined by (3), is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, the first discretization step results in the following identification problem

$$\begin{cases} u_{tt}^h(t,x)+A_h^x u^h(t,x)=p^h(x)+f^h(t,x),\ x\in[-\ell,\ell]_h,\ 0<t<1,\\ u_t^h(t,x)+A_h^x u^h(t,x)=p^h(x)+g^h(t,x),\ x\in[-\ell,\ell]_h,\ -1<t<0,\\ u^h(0^+,x)=u^h(0^-,x),\ u_t^h(0^+,x)=u_t^h(0^-,x),\ x\in[-\ell,\ell]_h,\\ u^h(-1,x)=\varphi^h(x),\ u^h(1,x)=\psi^h(x),\ x\in[-\ell,\ell]_h. \end{cases} \tag{4}$$

In the second step, we replace the identification problem (4) with the following first order of accuracy difference scheme

$$\begin{cases} \dfrac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_{k+1}^h(x)=p^h(x)+f_k^h(x),\ 1\le k\le N-1,\ x\in[-\ell,\ell]_h,\\ \dfrac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x u_k^h(x)=p^h(x)+g_k^h(x),\ -N+1\le k\le 0,\ x\in[-\ell,\ell]_h,\\ f_k^h(x)=f^h(t_k,x),\ 1\le k\le N-1,\ g_k^h(x)=g(t_k,x),\ -N+1\le k\le 0,\quad x\in[-\ell,\ell]_h,\\ u_1^h(x)-u_0^h(x)=u_0^h(x)-u_{-1}^h(x),\ u_{-N}^h(x)=\varphi^h(x),\ u_N^h(x)=\psi^h(x),\ x\in[-\ell,\ell]_h, \end{cases} \tag{5}$$

where $\tau=1/N$ and $t_k=k\tau$, $-N\le k\le N$.

Theorem 2. Let τ and h be sufficiently small numbers. For the solution $\left\{\left\{u_k^h(x)\right\}_{-N}^{N},p^h(x)\right\}$ of problem (5) the following stability estimates

$$\max_{-N\le k\le N}\|u_k\|_{L_{2h}}+\left\|(A_h^x)^{-1}p^h\right\|_{L_{2h}}$$
$$\le \tilde{M}_1(\delta)\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\max_{-N+1\le k\le 0}\left\|g_k^h\right\|_{L_{2h}}+\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\|_{L_{2h}}+\max_{-N+1\le k\le 0}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}+\max_{-N\le k\le N}\left\|u_k^h\right\|_{W_{2h}^2}+\left\|p^h\right\|_{L_{2h}}$$
$$\le \tilde{M}_2(\delta)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|g_0^h\right\|_{L_{2h}}+\max_{-N+1\le k\le -1}\left\|\frac{g_k^h-g_{k-1}^h}{\tau}\right\|_{L_{2h}}\right.$$
$$\left.+\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\frac{f_k^h-f_{k-1}^h}{\tau}\right\|_{L_{2h}}\right]$$

hold, where $\tilde{M}_1(\delta)$ and $\tilde{M}_2(\delta)$ do not depend on τ, h, f_k^h, $1\le k\le N-1$, g_k^h, $-N+1\le k\le 0$, $\varphi^h(x)$ and $\psi^h(x)$.

Proof. Difference scheme (5) can be written in the following abstract form

$$\begin{cases} \dfrac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2} + A_h u_{k+1}^h = p^h + f_k^h, \ 1 \leq k \leq N-1, \\ \dfrac{u_k^h - u_{k-1}^h}{\tau} + A_h u_k^h = p^h + g_k^h, \ -N+1 \leq k \leq 0, \\ u_1^h - u_0^h = u_0^h - u_{-1}^h, \ u_{-N}^h = \varphi^h, u_N^h = \psi^h \end{cases}$$

in a Hilbert space L_{2h} with operator $A_h = A_h^x$ defined by formula (3). Here, $f_k^h = f_k^h(x)$ and $g_k^h = g_k^h(x)$ are given abstract functions, $u_k^h = u_k^h(x)$ is unknown mesh function and $p^h = p^h(x)$ is the unknown mesh element of L_{2h}. Therefore, the proof of Theorem 2 is based on the self-adjointness and positive definiteness of the space operator A_h in L_{2h} [23].

Numerical experiments

When the analytical methods do not work properly, the numerical methods for obtaining the approximate solutions of partial differential equations play an important role in applied mathematics. In this section, we will use the first order of accuracy difference scheme to approximate the solution of a simple test problem. We will apply a procedure of modified Gauss elimination method to solve the problem. Finally, the error analysis of first order of accuracy difference scheme will be given.

We consider the identification problem with the Dirichlet condition

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) - \dfrac{1}{2}\big(u_x(t,-x)\big)_x + u(t,x) = p(x) + f(t,x), \ x \in (-\pi,\pi), \ t \in (0,1), \\ u_t(t,x) - u_{xx}(t,x) - \dfrac{1}{2}\big(u_x(t,-x)\big)_x + u(t,x) = p(x) + g(t,x), \ x \in (-\pi,\pi), \ t \in (-1,0), \\ u(-1,x) = \varphi(x), \ u(1,x) = \psi(x), \ x \in [-\pi,\pi], \\ u(t,-\pi) = u(t,\pi) = 0, \ t \in [-1,1] \end{cases} \tag{6}$$

for one-dimensional hyperbolic-parabolic equation with involution, where

$$\begin{aligned} f(t,x) &= \left(\frac{1}{2}\cos t - 1\right)\sin x, \ x \in (-\pi,\pi), \ t \in (0,1), \\ g(t,x) &= \left(\frac{3}{2}\cos t - \sin t - 1\right)\sin x, \ x \in (-\pi,\pi), \ t \in (-1,0), \\ \varphi(x) &= \cos 1 \sin x, \ \psi(x) = \cos 1 \sin x, \ x \in [-\pi,\pi]. \end{aligned}$$

The exact solution of problem (6) is the pair of functions

$$\big(u(t,x), p(x)\big) = \big(\cos t \sin x, \sin x\big), \ -\pi \leq x \leq \pi, \ -1 \leq t \leq 1.$$

We define the set $[-1,1]_\tau \times [-\pi,\pi]_h$ of all grid points as following:

$$[-1,1]_\tau \times [-\pi,\pi]_h = \big\{(t_k, x_n) \bigm| t_k = k\tau, -N \leq k \leq N, \ N\tau = 1, \ x_n = nh, -M \leq n \leq M, \ Mh = \pi\big\}.$$

For the numerical solution of source identification problem (6), we construct the first order of accuracy difference scheme in t

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}-\dfrac{u_{-n+1}^{k+1}-2u_{-n}^{k+1}+u_{-n-1}^{k+1}}{2h^2}+u_n^{k+1} \\ \quad =p_n+f(t_k,x_n),\ 1\le k\le N-1,\ -M+1\le n\le M-1, \\ \dfrac{u_n^k-u_n^{k-1}}{\tau}-\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}-\dfrac{u_{-n+1}^k-2u_{-n}^k+u_{-n-1}^k}{2h^2}+u_n^k \\ \quad =p_n+g(t_k,x_n),\ -N+1\le k\le 0,\ -M+1\le n\le M-1, \\ u_n^1-u_n^0=u_n^0-u_n^{-1},\ u_n^{-N}=\varphi(x_n),\ u_n^N=\psi(x_n),\ -M\le n\le M, \\ u_{-M}^k=u_M^k=0,\ -N\le k\le N, \end{cases} \tag{7}$$

where u_n^k and p_n denote the numerical approximations of $u(t,x)$ at $(t,x)=(t_k,x_n)$ and $p(x)$ at $x=x_n$, respectively. The solution of difference scheme (7) can be found in the form

$$\begin{aligned} &u_n^k=v_n^k-v_n^N+\psi(x_n),\ -M\le n\le M,\ -N\le k\le N, \\ &p_n=\psi(x_n)-\frac{\psi(x_{n+1})-2\psi(x_n)+\psi(x_{n-1})}{h^2}-\frac{\psi(x_{-n+1})-2\psi(x_{-n})+\psi(x_{-n-1})}{2h^2} \\ &\qquad +\frac{v_{n+1}^N-2v_n^N+v_{n-1}^N}{h^2}+\frac{v_{-n+1}^N-2v_{-n}^N+v_{-n-1}^N}{2h^2}-v_n^N,\ -M+1\le n\le M-1, \end{aligned}$$

where $\left\{\left\{v_n^k\right\}_{k=-N}^{N}\right\}_{n=-M}^{M}$ is the solution of the following nonlocal boundary value problem

$$\begin{cases} \dfrac{v_n^{k+1}-2v_n^k+v_n^{k-1}}{\tau^2}-\dfrac{v_{n+1}^{k+1}-2v_n^{k+1}+v_{n-1}^{k+1}}{h^2}-\dfrac{v_{-n+1}^{k+1}-2v_{-n}^{k+1}+v_{-n-1}^{k+1}}{2h^2}+v_n^{k+1} \\ \quad =f(t_k,x_n),\ 1\le k\le N-1,-M+1\le n\le M-1, \\ \dfrac{v_n^k-v_n^{k-1}}{\tau}-\dfrac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2}-\dfrac{v_{-n+1}^k-2v_{-n}^k+v_{-n-1}^k}{2h^2}+v_n^k \\ \quad =g(t_k,x_n),\ -N+1\le k\le 0,\ -M+1\le n\le M-1, \\ v_n^1-v_n^0=v_n^0-v_n^{-1},\ v_n^N-v_n^{-N}=\psi(x_n)-\varphi(x_n),\ -M\le n\le M, \\ v_{-M}^k=v_M^k=0,\ -N\le k\le N. \end{cases} \tag{8}$$

To obtain the solution of difference scheme (8), we first rewrite it in the matrix form

$$\begin{cases} AV_{n+1}+BV_n+AV_{n-1}+CV_{-n+1}+DV_{-n}+CV_{-n-1}=F_n,\ -M+1\le n\le M-1, \\ V_{-M}=V_M=\tilde{0}, \end{cases} \tag{9}$$

where $\tilde{0}$ is $(2N+1)\times 1$ zero vector and

$$A=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & b & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & b \end{bmatrix}_{(2N+1)\times(2N+1)} \qquad F_n=\begin{bmatrix} \psi(x_n)-\varphi(x_n) \\ \tau g\left(t_{-N+1},x_n\right) \\ \vdots \\ \tau g\left(t_0,x_n\right) \\ 0 \\ \tau^2 f\left(t_1,x_n\right) \\ \vdots \\ \tau^2 f\left(t_{N-1},x_n\right) \end{bmatrix}_{(2N+1)\times 1}$$

$$C=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & q & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & q & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & r & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & r \end{bmatrix}_{(2N+1)\times(2N+1)} \qquad V_n=\begin{bmatrix} v_n^{-N} \\ v_n^{-N+1} \\ \vdots \\ v_n^0 \\ v_n^1 \\ v_n^2 \\ \vdots \\ v_n^N \end{bmatrix}_{(2N+1)\times 1}$$

$$B=\begin{bmatrix} -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & c & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & c & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & -2 & d & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & d \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$D=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & s & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & s & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \sigma & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \sigma \end{bmatrix}_{(2N+1)\times(2N+1)} \qquad V_{-n}=\begin{bmatrix} v_{-n}^{-N} \\ v_{-n}^{-N+1} \\ \vdots \\ v_{-n}^0 \\ v_{-n}^1 \\ v_{-n}^2 \\ \vdots \\ v_{-n}^N \end{bmatrix}_{(2N+1)\times 1}$$

with $a=-\dfrac{\tau}{h^2}$, $b=-\dfrac{\tau^2}{h^2}$, $c=1+\dfrac{2\tau}{h^2}+\tau$, $d=1+\dfrac{2\tau^2}{h^2}+\tau^2$, $q=-\dfrac{\tau}{2h^2}$, $r=-\dfrac{\tau^2}{2h^2}$, $s=\dfrac{\tau}{h^2}$, $\sigma=\dfrac{\tau^2}{h^2}$. Next, we rewrite the system (9) as following

$$\begin{cases} \tilde{A}Z_{n+1}+\tilde{B}Z_n+\tilde{A}Z_{n-1}=\phi_n, \ 1\le n\le M-1, \\ \tilde{C}Z_1+\tilde{B}Z_0=\phi_0, \\ Z_M=\tilde{0}, \end{cases} \tag{10}$$

where $\tilde{A}=\begin{bmatrix} A & C \\ C & A \end{bmatrix}$, $\tilde{B}=\begin{bmatrix} B & D \\ D & B \end{bmatrix}$ and $\tilde{C}=\tilde{A}+\begin{bmatrix} C & A \\ A & C \end{bmatrix}$ are $(4N+2)\times(4N+2)$ matrices, $Z_n=\begin{bmatrix} V_n \\ V_{-n} \end{bmatrix}$ and $\phi_n=\begin{bmatrix} F_n \\ F_{-n} \end{bmatrix}$ are $(4N+1)\times 1$ column vectors. Now, the matrix equation (10) can be solved by using the modified Gauss elimination method [24]. We seek a solution of the matrix equation (10) in the following form:

$$\begin{cases} Z_n=\alpha_{n+1}Z_{n+1}+\beta_{n+1}, \ n=M-1,\ldots,2,1, \\ Z_M=\tilde{0}, \end{cases}$$

where α_n are $(4N+2)\times(4N+2)$ square matrices and β_n are $(4N+2)\times 1$ column vectors, calculated by

$$\begin{cases} \alpha_{n+1} = -\big(\tilde{B}+\tilde{A}\alpha_n\big)^{-1}\tilde{A} \\ \beta_{n+1} = \big(\tilde{B}+\tilde{A}\alpha_n\big)^{-1}(\phi_n - \tilde{A}\beta_n) \end{cases}$$

for $n=1,2,\ldots,M-1$. Here, $\alpha_1 = -\tilde{B}^{-1}\tilde{C}$ and $\beta_1 = \tilde{B}^{-1}\phi_0$.

The numerical solutions of the first order of accuracy difference scheme (7) are computed for different values of M and N by using the algorithm described above. We measure the error between the exact solution and numerical solution by

$$\|E_u\|_\infty = \max_{\substack{-N+1\le k\le N-1 \\ -M+1\le n\le M-1}} \left|u(t_k,x_n) - u_n^k\right|, \qquad \|E_p\|_\infty = \max_{-M+1\le n\le M-1} |p(x_n)-p_n|,$$

where $u(t_k,x_n)$ is the exact value of $u(t,x)$ at (t_k,x_n) and $p(x_n)$ is the exact value of source $p(x)$ at $x=x_n$; u_n^k and p_n represent the corresponding numerical solutions. Table 1 shows the errors between the exact solution of the problem (6) and the numerical solutions computed by using the first order of accuracy scheme. We observe that the scheme has the first order convergence as it is expected to be.

Таблица 1: The errors between the exact solution of the problem (6) and the numerical solutions computed by using the first order of accuracy difference scheme (7) for different values of $h=\dfrac{\pi}{M}$ and $\tau=\dfrac{1}{N}$.

	$\|E_p\|_\infty$	Order	$\|E_u\|_\infty$	Order
$N=M=20$	4.9976×10^{-2}	-	3.6439×10^{-2}	-
$N=M=40$	2.5072×10^{-2}	0.9951	1.8518×10^{-2}	0.9765
$N=M=80$	1.2558×10^{-2}	0.9975	9.3355×10^{-3}	0.9881
$N=M=160$	6.2845×10^{-3}	0.9987	4.6871×10^{-3}	0.9940
$N=M=320$	3.1436×10^{-3}	0.9994	2.3484×10^{-3}	0.9970

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Шекаралық Дирихле шарттарымен және инволяциясымен сәйкестендірілген гиперболалы-параболалық есебі туралы ескерту

Мақалада инволюциясымен және Дирихле шартымен берілген гиперболалы-параболалық теңдеу үшін дереккөзді сәйкестендіру мәселесі зерттелген. Дереккөзі сәйкестендірілген гиперболалы-параболалық есебі шешімінің орнықтылық бағамы алынған. Қарастырылған есептің жуық шешімі үшін бірінші ретті дәлдікпен орнықты айырымдық схемасы құрастырылды. Қарапайым тестік есептері үшін сандық нәтижелері берілген.

Кілт сөздер: дереккөзді сәйкестендіру есебі, гиперболалы-параболалық дифференциалдық теңдеу, айырымдық схема, орнықтылық.

М. Ашыралыев, М.А. Ашыралыева, А. Ашыралыев

Замечание о гиперболо-параболической задаче идентификации с инволюцией и граничным условием Дирихле

В статье исследована проблема идентификации источника для гиперболо-параболического уравнения с инволюцией и условием Дирихле. Получены оценки устойчивости решения гиперболо-параболической задачи идентификации источника. Построена устойчивая разностная схема первого порядка точности для приближенного решения рассматриваемой задачи. Приведены численные результаты для простой тестовой задачи.

Ключевые слова: задача идентификации источника, гиперболо-параболическое дифференциальное уравнение, разностная схема, устойчивость.

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A note on the parabolic identification problem with involution and Dirichlet condition

A space source of identification problem for parabolic equation with involution and Dirichlet condition is studied. The well-posedness theorem on the differential equation of the source identification parabolic problem is established. The stable difference scheme for the approximate solution of this problem is presented. Furthermore, stability estimates for the difference scheme of the source identification parabolic problem are presented. Numerical results are given.

Keywords: well-posedness, elliptic equations, positivity, coercive stability, source identification, exact estimates, boundary value problem.

Introduction

The theory and applications of source identification problems for partial differential equations have been studied by many authors (see, e.g., [1–9] and the references given therein). Numerous source identification problems for hyperbolic-parabolic equations and their applications have been investigated too (see, e.g., [10–13] and the references given therein). In the last decade, partial differential equations with involutions were investigated in [14–18]. However, source identification problems for parabolic equations with involution have not been well-investigated.

The present paper is devoted to study a space source of identification problem for parabolic equation with involution and Dirichlet condition. The well-posedness theorem on the differential equation of the source identification parabolic problem is proved. The stable difference schemes for the approximate solution of this problem are constructed. Furthermore, stability estimates for the difference schemes of the source identification parabolic problem are established. Numerical results are provided.

Well-posedness of differential problem

We consider the space source identification problem

$$\begin{cases} u_t(t,x) - \left(a(x)u_x\left(t,x\right)\right)_x - \beta\left(a(-x)u_x\left(t,-x\right)\right)_x + \delta u(t,x) \\ = p(x) + f(t,x),\ -l < x < l,\ 0 < t < T, \\ u(t,-l) = u(t,l) = 0,\ 0 \le t \le T, \\ u(0,x) = \varphi(x), u(T,x) = \psi(x),\ -l \le x \le l \end{cases} \tag{1}$$

for the one dimensional parabolic differential equation with involution. Problem (1) has a unique solution $(u(t,x), p(x))$ for the smooth functions $f(t,x)$ $(t \in (0,T) \times (-l,l))$, $a \ge a\left(x\right) = a\left(-x\right) \ge \delta > 0$, $\delta - a\left|\beta\right| \ge 0$ $\left(x \in (-l,l)\right)$, and $\varphi(x),\ \psi(x),\ x \in [-l,l]$.

In the present paper $C_0^\alpha\left([0,T],H\right)$ $(0<\alpha<1)$ stands for Banach spaces of all abstract continuous functions $\varphi(t)$ defined on $[0,T]$ with values in H satisfying a Hölder condition with weight t^α for which the following norm is finite

$$\|\varphi\|_{C_0^\alpha([0,T],H)}=\|\varphi\|_{C([0,T],H)}+\sup_{0\le t<t+\tau\le T}\frac{(t+\tau)^\alpha\left\|\varphi(t+\tau)-\varphi(t)\right\|_H}{\tau^\alpha}.$$

Here, $C\left([0,T],H\right)$ stands for the Banach space of all abstract continuous functions $\varphi(t)$ defined on $[0,T]$ with values in H equipped with the norm

$$\|\varphi\|_{C([0,T],H)}=\max_{0\le t\le T}\|\varphi(t)\|_H\,.$$

Theorem 1. Suppose that $\varphi,\psi\in W_2^2\left[-l,l\right]$. Let $f(t,x)$ be continuously differentiable in t on $[0,T]\times[-l,l]$ function. Then the solutions of the identification problem (1) satisfy the stability estimates

$$\|u\|_{C([0,T],L_2[-l,l])}+\left\|(A^x)^{-1}p\right\|_{L_2[-l,l]}$$

$$\le M_1\left(\delta,\sigma,\beta,l\right)\left[\|\varphi\|_{L_2[-l,l]}+\|\psi\|_{L_2[-l,l]}+\|f\|_{C([0,T],L_2[-l,l])}\right],\tag{2}$$

$$\|u\|_{C^{(1)}([0,T],L_2[-l,l])}+\|u\|_{C\left([0,T],W_2^2[-l,l]\right)}+\|p\|_{L_2[-l,l]}$$

$$\le M_2\left(\delta,\sigma,\beta,l\right)\left[\|\varphi\|_{W_2^2[-l,l]}+\|\psi\|_{W_2^2[-l,l]}+\|f\|_{C^{(1)}([0,T],L_2[0,l])}\right].\tag{3}$$

Here $M_1\left(\delta,\sigma,\beta,l\right)$ and $M_2\left(\delta,\sigma,\beta,l\right)$ do not depend on $\varphi(x),\psi(x)$ and $f(t,x)$. The Sobolev space $W_2^2\left[-l,l\right]$ is defined as the set of all functions $u(x)$ defined on $[0,l]$ such that $u(x)$ and the second order derivative function $u''(x)$ are all locally integrable in $L_2[-l,l]$, equipped the norm

$$\|u\|_{W_2^2[-l,l]}=\left(\int_{-l}^{l}|u(x)|^2\,dx\right)^{\frac12}+\left(\int_{-l}^{l}\left|u''(x)\right|^2\,dx\right)^{\frac12}.$$

Proof. Problem (1) can be written in abstract form

$$\begin{cases}\frac{du(t)}{dt}+Au(t)=p+f(t),0<t<T,\\ u(0)=\varphi,u(T)=\psi\end{cases}\tag{4}$$

in a Hilbert space $H=L_2[-l,l]$ with self-adjoint positive definite operator $A=A^x$ defined by the formula

$$A^xu(x)=-\left(a(x)u_x(x)\right)_x-\beta\left(a(-x)u_x\left(-x\right)\right)_x+\delta u\left(x\right)\tag{5}$$

with the domain $D(A^x)=\left\{u\in W_2^2\left[-l,l\right]:u\left(-l\right)=u\left(l\right)=0\right\}$ [14]. The proof of Theorem 1 is based on the symmetry properties of this space operator A and on the following stability results.

Theorem 2 [5]. Assume that $\varphi,\psi\in D(A)$ and $f(t)$ be continuously differentiable in t on $[0,T]$ function. Then, for the solution $\{u(t),p\}$ of the source identification problem (4) the following stability inequalities hold:

$$\|u\|_{C([0,T],H)}+\left\|A^{-1}p\right\|_H\le M\;\left[\|\varphi\|_H+\|\psi\|_H+f_{C([0,T],H)}\right]\tag{6}$$

$$\|u\|_{C^{(1)}([0,T],H)}+\|Au\|_{C([0,T],H)}+\|p\|_H\le M\left[\|A\varphi\|_H+\|A\psi\|_H+\|f\|_{C^{(1)}([0,T],H)}\right],\tag{7}$$

where M is independent of φ,ψ and $f(t)$.

Moreover, we have the following coercive stability results.

Theorem 3. Suppose that $\varphi, \psi \in W_2^2[-l,l]$ and $f(t,x) \in C_0^\alpha([0,T], L_2[-l,l])$. Then the solutions of the identification problem ((1) satisfy coercive stability estimates

$$\|u_t\|_{C_0^\alpha([0,T],L_2[-l,l])} + \|u\|_{C_0^\alpha([0,T],W_2^2[-l,l])} + \|p\|_{L_2[-l,l]}$$

$$\leq M(\delta,\sigma,\alpha,\beta,l)\left[\|\varphi\|_{W_2^2[-l,l]} + \|\psi\|_{W_2^2[-l,l]} + \|f\|_{C_0^\alpha([0,T],L_2[-l,l])}\right],$$

where $M(\delta,\sigma,\alpha,\beta,l)$ is independent of $\varphi(x), \psi(x)$ and $f(t,x)$.

The proof of Theorem 3 is based on the following abstract Theorem on coercive stability of the identification problem (4) in $C_0^\alpha([0,T],H)$ spaces and on self-adjointness and positive definite of the unbounded operator A defined by formula (5) in $L_2[-l,l]$ space.

Theorem 4. Assume that $\varphi, \psi \in D(A)$ and $f(t)$ and $f \in C_0^\alpha([0,T],H)$ $(0 < \alpha < 1)$. Then, for the solution $\{u(t), p\}$ of the source identification problem (4) the following coercive stability inequalities hold:

$$\|u'\|_{C_0^\alpha([0,T],H)} + \|Au\|_{C_0^\alpha([0,T],H)} + \|p\|_H$$

$$\leq M\left[\|A\varphi\|_H + \|A\psi\|_H + \frac{1}{\alpha(1-\alpha)}\|f\|_{C([0,T],H)}\right],$$

where M is independent of φ, ψ and $f(t)$.

Stability of difference schemes

Now, we study the stable difference schemes for the approximate solution of identification problem (1). The discretization of source identification problem (1) is carried out in two stages. In the first stage, we define the grid space

$$[-l,l]_h = \{x = x_n : x_n = nh,\ -M \leq n \leq M,\ Mh = l\}.$$

We introduce the Hilbert spaces $L_{2h} = L_2([-l,l]_h)$ and $W_{2h}^2 = W_2^2([-l,l]_h)$ of the grid functions $\varphi^h(x) = \{\varphi^r\}_{-M}^M$ defined on $[-l,l]_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in[-l,l]_h}\left|\varphi^h(x)\right|^2 h\right)^{1/2}$$

and

$$\left\|\varphi^h\right\|_{W_{2h}^2} = \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x\in[-l,l]_h}\left|\left(\varphi^h\right)_{x\overline{x},j}\right|^2 h\right)^{1/2},$$

respectively. To the differential operator A generated by problem (5), we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x) = \{-(a(x)\varphi_{\overline{x}}(x))_{x,r} - \beta\left(a(-x)\varphi_{\overline{x}}(-x)\right)_{x,r} + \delta\varphi_r\}_{-M+1}^{M-1}, \tag{8}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_r\}_{-M}^M$ satisfying the conditions $\varphi_{-M} = \varphi_M = 0$.

It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we reach the identification problem

$$\begin{cases} u_t^h(t,x) + A_h^x u^h(t,x) = p^h(x) + f^h(t,x),\ x \in [-l,l]_h,\ 0 < t < T, \\ u^h(0,x) = \varphi^h(x), u^h(T,x) = \psi^h(x),\ x \in [-l,l]_h. \end{cases} \tag{9}$$

In the second stage, we replace identification problem (9) with a first order of accuracy difference scheme

$$\left\{\begin{array}{l} \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x u_k^h(x)=p^h(x)+f_k^h, f_k^h(x)=f(t_k,x), \\ t_k=k\tau, 1\leq k\leq N, N\tau=T, x\in[-l,l]_h, \\ u_0^h(x)=\xi^h(x), u_N^h(x)=\varphi^h(x),\ x\in[-l,l]_h. \end{array}\right. \tag{10}$$

Let $\alpha\in(0,1)$ is a given number and $C_\tau(H)$ and $C_\tau^\alpha(H)$ be Banach spaces of H-valued grid functions $w_\tau=\{w_k\}_{k=0}^N$ with the corresponding norms

$$\|w_\tau\|_{C_\tau(H)}=\max_{0\leq k\leq N}\|w_k\|_H,\ \|w_\tau\|_{C_\tau^\alpha(H)}=\sup_{1\leq k<k+n\leq N}(n\tau)^{-\alpha}(k\tau)^\alpha\|w_{k+n}-w_k\|_H+\|w_\tau\|_{C_\tau(H)}.$$

Theorem 5. For the solution $\left\{\left\{u_k^h(x)\right\}_0^N, p^h(x)\right\}$ of problem (10), the following stability estimates

$$\left\|\left\{u_k^h\right\}_0^N\right\|_{C_\tau(L_{2h})}+\left\|\left(A_h^x\right)^{-1}p^h\right\|_{L_{2h}}$$

$$\leq M_3(\delta,\sigma,\beta,l)\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}\right], \tag{11}$$

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_0^N\right\|_{C_\tau(L_{2h})}+\left\|\left\{u_k^h\right\}_0^N\right\|_{C_\tau(W_{2h}^2)}$$

$$\leq M_4(\delta,\sigma,\beta,l)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}+\max_{2\leq k\leq N}\left\|\left\{\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\}_2^N\right\|_{L_{2h}}\right] \tag{12}$$

hold, where $M_3(\delta,\sigma,\beta,l)$ and $M_4(\delta,\sigma,\beta,l)$ do not depend on $\tau, h, f_k^h,\ 1\leq k\leq N,\ \varphi^h(x)$ and $\psi^h(x)$.

Proof. Difference scheme (10) can be written in the following abstract forms

$$\left\{\begin{array}{l} \frac{u_k-u_{k-1}}{\tau}+Au_k=p+f_k, 1\leq k\leq N, \\ u_0=\varphi, u_N=\psi \end{array}\right. \tag{13}$$

in a Hilbert space $H=L_{2h}$ with operator $A=A_h^x$ by formula (8). Here, $f_k=f_k^h(x)$ is a given abstract mesh function, $u_k=u_k^h(x)$ is unknown mesh function and $p=p^h(x)$ is the unknown mesh element of L_{2h}. Therefore, the proof of Theorem 5 is based on the self- adjointness and positive definiteness of the space difference operator A in L_{2h} [14] and on the following stability results.

Theorem 6. [5]. For the solution $\left\{\{u_k\}_0^N, p\right\}$ of the source identification difference problem (13), the following stability inequalities hold:

$$\left\|\{u_k\}_0^N\right\|_{C_\tau(H)}+\left\|A^{-1}p\right\|_H$$

$$\leq M_5(\delta,\sigma,\beta,l)\left[\|\varphi\|_H+\|\psi\|_H+\left\|\{f_k\}_1^N\right\|_{C_\tau(H)}\right],$$

$$\left\|\left\{\frac{u_k-u_{k-1}}{\tau}\right\}_0^N\right\|_{C_\tau(H)}+\left\|\{Au_k\}_0^N\right\|_{C_\tau(H)}$$

$$\leq M_5(\delta,\sigma,\beta,l)\left[\|A\varphi\|_H+\|A\psi\|_H+\|f_1\|_H+\max_{2\leq k\leq N}\left\|\left\{\frac{1}{\tau}(f_k-f_{k-1})\right\}_2^N\right\|_H\right],$$

where $M_5(\delta,\sigma,\beta,l)$ is independent of φ,ψ and $f(t)$.

Moreover, we have the following coercive stability results.

Theorem 7. The solutions of the identification difference problem (10) satisfies coercive stability estimate

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_1^N\right\|_{C_\tau^\alpha(L_{2h})}+\left\|\left\{u_k^h\right\}_0^N\right\|_{C_\tau^\alpha(W_{2h}^2)}\leq$$

$$\leq M_5\left(\delta,\sigma,\beta,l\right)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau^\alpha(L_{2h})}\right],$$

where $M_5\left(\delta,\sigma,\beta,l\right)$ does not depends on $\tau,h,f_k^h,\ 1\leq k\leq N,\ \ \varphi^h\left(x\right)$ and $\psi^h\left(x\right).$

The proof of Theorem 7 is based on the self-adjointness and positive definiteness of the space difference operator A in L_{2h} [14] and on the following coercive stability results.

Theorem 8. For the solution $\left\{\left\{u_k\right\}_0^N,p\right\}$ of the source identification difference problem (13) the following coercive stability inequality holds:

$$\left\|\left\{\frac{u_k-u_{k-1}}{\tau}\right\}_1^N\right\|_{C_\tau^\alpha(H)}+\left\|\left\{u_k\right\}_0^N\right\|_{C_\tau^\alpha(H)}\leq M_6\left(\delta,\sigma,\beta,l\right)\left[\left\|A\varphi\right\|_H+\left\|A\psi\right\|_H+\left\|\left\{f_k\right\}_1^N\right\|_{C_\tau^\alpha(H)}\right],$$

where $M_6\left(\delta,\sigma,\beta,l\right)$ does not depends on $\tau,h,f_k,\ 1\leq k\leq N,\ \ \varphi$ and $\psi.$

Numerical experiment

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of partial differential equations play an important role in applied mathematics. We can say that there are many considerable works in the literature. In present section for the approximate solutions of a problem, we use the first order of accuracy difference scheme. We apply a procedure of modified Gauss elimination method to solve the problem. Finally, the error analysis of first order of accuracy difference scheme is given.

We consider the identification problem with the Dirichlet condition

$$\begin{cases} u_t\left(t,x\right)-u_{xx}\left(t,x\right)-\frac{1}{2}u_{x,x}\left(t,-x\right)+u(t,x) \\ =p\left(x\right)-\sin x+\cos t\sin x+\frac{3}{2}\sin t\sin x,\quad x\in\left(-\pi,\pi\right),t\in\left(0,\pi\right), \\ u\left(0,x\right)=0,u\left(\pi,x\right)=0,x\in\left[-\pi,\pi\right], \\ u\left(t,-\pi\right)=u\left(t,\pi\right)=0,t\in\left[0,\pi\right] \end{cases} \tag{14}$$

for parabolic equation with involution. The exact solution pair of this problem is

$$\left(u\left(t,x\right),p\left(x\right)\right)=\left(\sin t\sin x,\sin x\right),-\pi\leq x\leq\pi,0\leq t\leq\pi.$$

Here and in future, we denote the set $\left[0,\pi\right]_\tau\times\left[-\pi,\pi\right]_h$ of all grid points

$$\left[0,\pi\right]_\tau\times\left[-\pi,\pi\right]_h=\left\{\left(t_k,x_n\right):t_k=k\tau,0\leq k\leq N,\right.$$

$$\left.N\tau=\pi,x_n=nh,-M\leq n\leq M,Mh=\pi\right\}.$$

For the numerical solution of SIP (14), we present the first order of accuracy difference scheme in t

$$\begin{cases} \tau^{-1}\left(u_n^k-u_n^{k-1}\right)-h^{-2}\left(u_{n+1}^k-2u_n^k+u_{n-1}^k\right) \\ -\frac{1}{2}h^{-2}\left(u_{-n+1}^k-2u_{-n}^k+u_{-n-1}^k\right)+u_n^k=p_n-\sin x_n \\ +\cos t_k\sin x_n+\frac{3}{2}\sin t_k\sin x_n,1\leq k\leq N,-M+1\leq n\leq M-1, \\ u_n^0=0,u_n^N=0,-M\leq n\leq M, \\ u_{-M}^k=u_M^k=0,0\leq k\leq N. \end{cases} \tag{15}$$

In the first step, we obtain $\left\{\left\{\omega_n^k\right\}_0^N\right\}_{n=-M}^M$ as solution of nonlocal BVP

$$\begin{cases} \tau^{-1}\left(\omega_n^k-\omega_n^{k-1}\right)-h^{-2}\left(\omega_{n+1}^k-2\omega_n^k+\omega_{n-1}^k\right) \\ -\frac{1}{2}h^{-2}\left(\omega_{-n+1}^k-2\omega_{-n}^k+\omega_{-n-1}^k\right)+\omega_n^k \\ =-\sin x_n+\cos t_k\sin x_n+\frac{3}{2}\sin t_k\sin x_n, 1\le k\le N, -M+1\le n\le M-1, \\ \omega_n^0-\omega_n^N=0, -M\le n\le M, \\ \omega_{-M}^k=\omega_M^k=0, 0\le k\le N. \end{cases} \tag{16}$$

Here and in future, ω_n^k denotes the numerical approximation of $\omega(t,x)$ at (t_k,x_n). For obtaining the solution of difference scheme (16), we rewrite it in the matrix form

$$\begin{cases} A\omega_{n+1}+B\omega_n+A\omega_{n-1}+C\omega_{-n+1}+D\omega_{-n}+C\omega_{-n-1}=f_n, \\ A\omega_{-n+1}+B\omega_{-n}+A\omega_{-n-1}+C\omega_{n+1}+D\omega_n+C\omega_{n-1}=f_{-n}, \end{cases}, \tag{17}$$

$$1\le n\le M-1, \begin{pmatrix} \omega_M \\ \omega_{-M} \end{pmatrix}=\begin{pmatrix} \overrightarrow{0} \\ \overrightarrow{0} \end{pmatrix},$$

where $\overrightarrow{0}$, w_s for $s=n,\ n\pm1$, and f_n are $(N+1)\times1$ column matrices, and $(N+1)\times(N+1)$ square matrices A,B,C,D are defined as follows:

$$A=\begin{bmatrix} 0 & 0 & 0 & . & 0 & 0 \\ 0 & -h^{-2} & 0 & . & 0 & 0 \\ 0 & 0 & -h^{-2} & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & -h^{-2} & 0 \\ 0 & 0 & 0 & . & 0 & -h^{-2} \end{bmatrix},$$

$$B=\begin{bmatrix} 1 & 0 & 0 & . & 0 & -1 \\ -\tau^{-1} & \tau^{-1}+2h^{-2}+1 & 0 & . & 0 & 0 \\ 0 & -\tau^{-1} & \tau^{-1}+2h^{-2}+1 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & \tau^{-1}+2h^{-2}+1 & 0 \\ 0 & 0 & 0 & . & -\tau^{-1} & \tau^{-1}+2h^{-2}+1 \end{bmatrix},$$

$$C=\frac{1}{2}A,\ D=-A.$$

Grouping the above expression (17) as

$$\begin{cases} A\omega_{n+1}+C\omega_{-n-1}+B\omega_n+D\omega_{-n}+A\omega_{n-1}+C\omega_{-n+1}=f_n, \\ C\omega_{n+1}+A\omega_{-n-1}+D\omega_n+B\omega_{-n}+C\omega_{n-1}+A\omega_{-n+1}=f_{-n} \end{cases},$$

and defining $z_n=\begin{pmatrix} w_n \\ w_{-n} \end{pmatrix}$ and $\phi_n=\begin{pmatrix} f_n \\ f_{-n} \end{pmatrix}$, the system can be written as

$$\begin{pmatrix} A & C \\ C & A \end{pmatrix} z_{n+1}+\begin{pmatrix} B & D \\ D & B \end{pmatrix} z_n+\begin{pmatrix} A & C \\ C & A \end{pmatrix} z_{n-1}=\phi_n,\ 1\le n\le M-1,\ z_M=\begin{pmatrix} \overrightarrow{0} \\ \overrightarrow{0} \end{pmatrix}. \tag{18}$$

For solving the system (18), we use the Gauss elimination method. Thus, let's define

$$z_n=\alpha_{n+1}z_{n+1}+\beta_{n+1}, n=M-1,...,1, z_M=\begin{pmatrix} \overrightarrow{0} \\ \overrightarrow{0} \end{pmatrix}. \tag{19}$$

where α_n $(1 \le n \le M)$ are $(2N+2)\times(2N+2)$ square matrices and β_n $(1 \le n \le M)$ are $(2N+2)\times 1$ column vectors, calculated as,

$$\begin{cases} \alpha_{n+1} = -\left(P\alpha_n + Q\right)^{-1} P, \\ \beta_{n+1} = \left(P\alpha_n + Q\right)^{-1}\left(R\phi_n - P\beta_n\right), \\ n = 1, ..., M-1, \end{cases} \tag{20}$$

where $P = \begin{pmatrix} A & C \\ C & A \end{pmatrix}$ and $Q = \begin{pmatrix} B & D \\ D & B \end{pmatrix}$ and R is $(2N+2)\times(2N+2)$ identity matrix.

First, we evaluate α_n and β_n $(1 \le n \le M)$. Since,

$$\phi_0 = \begin{pmatrix} f_0 \\ f_0 \end{pmatrix} = \begin{pmatrix} A\omega_1 + C\omega_{-1} \\ C\omega_1 + A\omega_{-1} \end{pmatrix} + \begin{pmatrix} B\omega_0 + D\omega_0 \\ D\omega_0 + B\omega_0 \end{pmatrix} + \begin{pmatrix} A\omega_{-1} + C\omega_1 \\ C\omega_{-1} + A\omega_1 \end{pmatrix},$$

we get

$$z_0 = \begin{pmatrix} \omega_0 \\ \omega_0 \end{pmatrix} = \begin{pmatrix} B & D \\ D & B \end{pmatrix}^{-1} \left\{ -\begin{pmatrix} A+C & A+C \\ A+C & A+C \end{pmatrix} z_1 + \phi_0 \right\}$$

and

$$\alpha_1 = -\begin{pmatrix} B & D \\ D & B \end{pmatrix}^{-1} \begin{pmatrix} A+C & A+C \\ A+C & A+C \end{pmatrix},$$

$$\beta_1 = \begin{pmatrix} B & D \\ D & B \end{pmatrix}^{-1} \phi_0.$$

Using the iteration (20), we obtain all α_n and β_n $(1 \le n \le M)$ values. Second, using the formula (19), we obtain z_n and the equality $z_n = \begin{pmatrix} w_n \\ w_{-n} \end{pmatrix}$ gives the values of ω_n.

In the second step, using [5, Equation 8], we get

$$p_n = \frac{\omega_{n+1}^N - 2\omega_n^N + \omega_{n-1}^N}{h^2} + \frac{1}{2}\frac{\omega_{-n+1}^N - 2\omega_{-n}^N + \omega_{-n-1}^N}{h^2} - \omega_n^N$$

for $-M+1 \le n \le M-1$.

In the last step, using formula (see, [5])

$$u_n^k = \omega_n^k - \omega_n^N, n = -M, -M+1, ..., M, k = 0, ..., N, \tag{21}$$

we obtain $\left\{ \{u_n^k\}_{k=0}^N \right\}_{n=-M}^M$.

Here, we compute the error between the exact solution and numerical solution by

$$\begin{cases} \|E_u\|_\infty = \max\limits_{0 \le k \le N, -M \le n \le M} \left| u(t_k, x_n) - u_n^k \right|, \\ \|E_p\|_\infty = \max\limits_{-M < n < M} \left| p(x_n) - p_n \right|, \end{cases} \tag{22}$$

where $u(t,x)$, $p(x)$ represent the exact solution, u_n^k represent the numerical solutions at (t_k, x_n) and p_n represent the numerical solutions at x_n. The numerical results are given in the Table 1.

Table 1.

Errors	$\|E_p\|_\infty$	$\|E_u\|_\infty$
$N=20, M=20$	0.1117	0.0195
$N=40, M=40$	0.0557	0.0101
$N=80, M=80$	0.0278	0.0052
$N=160, M=160$	0.0139	0.0026

Conclusion

In this paper, we considered a space source of identification problem for parabolic equation with involution and Dirichlet condition. The theoretical considerations that prove well-posedness theorem on the differential equation of the source identification parabolic problem and stability estimates for the difference schemes of the source identification parabolic problem were given. To support the theoretical results by a numerical experiment, we constructed a stable difference scheme for the approximate solution of the problem. Obtained results given in Table 1 also support the theoretical results.

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Дирихле шартымен және инволюциясымен сәйкестендірілген параболалық теңдеу туралы ескерту

Дирихле шартымен және инволюциясымен сәйкестендірілген параболалық теңдеу үшін кеңістіктік есептері зерттелген. Параболалық дифференциалдық теңдеу үшін дереккөзді сәйкестендіру есебінің корректілігі теоремасы құрылған. Осы есептің жуық шешімі үшін орнықты айырымдық схемасы көрсетілген. Сонымен қатар, дереккөзді сәйкестендіру параболалық теңдеуінің орнықтылық айырымдық схемасының бағамы берілген. Сандық нәтижелер келтірілген.

Кілт сөздер: корректілігі, эллипстік теңдеу, оң таңбалы, коэрцитивті орнықтылық, дереккөзді сәйкестендіру, дәл бағамы, шеттік есеп.

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Замечание о параболической проблеме идентификации с инволюцией и условием Дирихле

Исследована пространственная задача идентификации источника для параболического уравнения с инволюцией и условием Дирихле. Установлена теорема корректности задачи идентификации источника для параболического дифференциального уравнения. Представлена устойчивая разностная схема для приближенного решения этой задачи. Кроме того, даны оценки устойчивости разностной схемы параболической задачи идентификации источника. Приведены численные результаты.

Ключевые слова: корректность, эллиптические уравнения, положительность, коэрцитивная устойчивость, идентификация источника, точные оценки, краевая задача.

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