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МАЗМҰНЫ — СОДЕРЖАНИЕ — CONTENTS

МАТЕМАТИКА
MATHEMATICS

МЕХАНИКА
MECHANICS

ИНФОРМАТИКА
INFORMATICS

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On boundary value problems for essentially loaded parabolic equations in bounded domains

In the paper we study issues of a strong solution for "essentially" loaded differential equations of the parabolic type in bounded domains. Features of the problems under consideration: for example, in the $L_2(Q)$ space the corresponding differential operators are not closure operators, since firstly, the load does not obey the corresponding differential part of the considered operator, that is, for its differential part the load is not a weak perturbation. Secondly, it is obvious that load operators in the spaces $L_2(0,1)$ and $L_2(Q)$ are not closure operators. This indicates that it is impossible to directly investigate the issues of the strong solution to boundary value problems for non-closed loaded differential equations. However, the study of equations [1-4] give theoretical character, but also a clear applied [5-7] character.

Keywords: "essentially" loaded parabolic equations, Volterra integral equation, boundary value problem, strong solution, load operator.

1 Statement of boundary value problems

Statement of the first boundary value problem. Consider the following boundary value problem in the domain $Q = \{x,t|0 < x < 1, 0 < t < 2\pi\}$

$$L_1 u \equiv \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \alpha \cdot x \cdot \frac{\partial^2 u(x,t)}{\partial x^2}|_{x \equiv \overline{x}} = f(x,t), \{x,t\} \in Q; \tag{1}$$

$$u(0,t) = u(1,t) = 0, u(x,0) = u(x,2\pi), \tag{2}$$

where $\overline{x} \in (0,1)$ is a given point; $\alpha \in C$ is a given number;

$$f \in L_2\left(0, 2\pi; \dot{W}_2^1(0,1)\right) \tag{3}$$

is a given function.

Statement of the second boundary value problem. Consider the following boundary value problem in the domain $Q = \{x,t|0 < x < 1, 0 < t < 2\pi\}$

$$L_2 u \equiv \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \alpha(x) \cdot \frac{\partial^k u(x,t)}{\partial x^k}|_{x \equiv \overline{x}} = f(x,t), \{x,t\} \in Q; \tag{4}$$

$$u(0,t) = u(1,t) = 0, u(x,0) = u(x,2\pi), \tag{5}$$

where

$$\begin{cases} \overline{x} \in (0,1) \text{ is a fixed point;} \alpha \in W_2^{2m}(0,1), \\ f \in L_2\left(0, 2\pi; W_2^{2m}(0,1) \cap \dot{W}_2^m(0,1)\right) \text{ are the given functions,} \\ k \geq 2, m = \begin{cases} \frac{k}{2}, \text{ if k is an even number,} \\ \frac{k-1}{2} \text{ if k is an odd number.} \end{cases} \end{cases} \tag{6}$$

Remark 1. The loaded differential operator L_1 defined by problem (1) - (3) is not closed in the $L_2(Q)$ space, so for considering problem (1)-(3) we introduce the following an auxiliary problem:

$$L_3 u \equiv \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial^2 f}{\partial x^2}, \{x,t\} \in Q; \tag{7}$$

$$u(0,t) = u(1,t) = 0, u(x,0) = u(x,2\pi); \tag{8}$$

$$\frac{\partial^2 u(0,1)}{\partial x^2} = 0, \frac{\partial^2 u(1,t)}{\partial x^2} - \alpha \frac{\partial^2 u(\overline{x},t)}{\partial x^2} = 0. \tag{9}$$

Note that in the operator L_3 boundary value problem (7)-(9) (except the L_1 operator) is closed in the $L_2(Q)$ space. It is also obvious that boundary value problems (1)-(3) and (7)-(9) are connected. In fact, a regular solution to problem (7)-(9) is also a solution to problem (1)-(2). And visa versa, if the regular solution to problem (1)-(2) contains a derivative of the required order, then it is a regular solution to problem (7)-(9) [8].

Remark 2. For considering problem (4)-(6), in the domain Q we introduce a non-contiguous auxiliary problem

$$L_4 u \equiv \frac{\partial^{2m}}{\partial x^{2m}}\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) + \alpha^{(2m)}(x)\frac{\partial^k u(\overline{x},t)}{\partial x^k} = \frac{\partial^{2m} f}{\partial x^{2m}}; \tag{10}$$

$$u(0,t) = u(1,t) = 0, u(x,0) = u(x,2\pi); \tag{11}$$

$$\frac{\partial^2 u(0,t)}{\partial x^2} - \alpha(0)\frac{\partial^k u(\overline{x},t)}{\partial x^k} = 0; \frac{\partial^2 u(1,t)}{\partial x^2} - \alpha(1)\frac{\partial^k u(\overline{x},t)}{\partial x^k} = 0; \tag{12}$$

$$\frac{\partial^{j+1} u(0,t)}{\partial x^j \partial t} - \frac{\partial^{j+2} u(0,t)}{\partial x^{j+2}} + \alpha^{(j)}(0)\frac{\partial^k u(\overline{x},t)}{\partial x^k} = 0; \tag{13}$$

$$\frac{\partial^{j+1} u(1,t)}{\partial x^j \partial t} - \frac{\partial^{j+2} u(1,t)}{\partial x^{j+2}} + \alpha^j(1)\frac{\partial^k u(\overline{x},t)}{\partial x^k} = 0 \tag{14}$$

$$j = 1, \ldots, m-1.$$

Note that boundary problem (4)-(5) and (10)-(14) are connected. In fact, a regular solution of problem (10)-(14) is a solution to problem (4)-(5). And visa versa if a regular solution to problem (4)-(5) contains a derivative of the required order, then it is a regular solution to problem (10)-(14).

There are some necessary definitions [9].

Supposing that $\tilde{C} = \{u \mid u \in C^{2,1}_{X,t}(\overline{Q}), u_t, u_{xx} \in C^{2,0}_{x,t}(Q), \}$ and conditions (8)-(9) are implemented [9].

Definition 1. If there exists a sequence of functions $\{u_n(x,t)\}_{(n=3)}^{\infty} \subset \tilde{C}$ such that the following conditions 1^0 and 2^0 are implemented:

1^0. In $L_2(Q) \lim_{n\to\infty} u_n(x,t) = u(x,t)$;

2^0. In $L_2(Q) \lim_{n\to\infty} L_3 u_n(x,t) = \frac{\partial^2 f}{\partial x^2}$

then the function $u(x,t)$ is called a strong solution to boundary value problem (7)-(9).

Definition 2. A strong solution to boundary value problem (7)-(9) are called a strong solution to boundary value problem (1)-(2).

2 Theorems on uniqueness and existence of a strong solution

First we consider the first boundary value problem, and show that the following statements are valid.

Theorem 1. Let
$$\delta_s = 1 - \frac{\alpha \cdot sh\{\lambda\overline{x}\}}{sh\{\lambda\}} \neq 0, \forall_s \in v, \tag{15}$$
in the case, $v = \{s|s = 0; \pm1; \pm2; \dots\}, \lambda^2 = is$, $i = \sqrt{(-1)}$. Than for any function
$$f \in L_2\left(0, 2\pi; W_2^2(0,1) \cap \dot{W}_2^1(0,1)\right)$$
boundary value problem (1) - (2) has a strong solution $u(x,t)$.

Corollary 1. Let $\alpha \in R^1$. The statement in this case is true for Theorem 1, iff the following condition is valid
$$1 - \alpha\overline{x} \neq 0 \tag{16}$$
The statement is a simple consequence of following fact: in (15) the imaginary part of the expression $\frac{sh\{\lambda\overline{x}\}}{sh\{\lambda\}}$ is not equal to zero at any point $s \in v\backslash\{0\}$, since the real and imaginary parts of this expression have a value that is always different from zero.

Corollary 2. Let (16) be not satisfied, i.e. $1 - \alpha\overline{x} = 0$. Then the operator of boundary value problem (1)-(2) is equal to zero, and according to this, the function is equal to
$$w_0(x) = x(1 - x^2) \tag{17}$$
Proof of the first theorem. In proving this theorem, we refer to the proving by A.A. Desin [8]. We are looking for a solution to problem (7) - (9) based on the following series:
$$u(x,t) = \sum_{s\in v} u_s(x)e^{is\cdot t}, f(x,t) = \sum_{s\in v} f_s(x)e^{is\cdot t} \tag{18}$$
Then from boundary value problems (1) - (2) taking into account the Fourier coefficients defined from (18), we obtain boundary problems for an ordinary differential equation
$$\begin{cases} (isu_s(x) - u_s''(x) + \alpha x u_s''(\overline{x}) = f_s(x), x \in (0,1), \\ u_s(0) = u_s(1) = 0, \end{cases} \quad \forall_s \in v. \tag{19}$$
A unique solution to (19) can be represented as follows:
$$\begin{cases} u_s(x) = \alpha\delta_s^{-1}\left[\int_0^1 G_s(\overline{x},\xi)f_s(\xi)d\xi - \frac{1}{\lambda^2}f_s(\overline{x})\right] \times \\ \times\left[\frac{sh(\lambda x)}{sh(\lambda)} - x\right] + \int_0^1 G_s(x,\xi)f_s(\xi)d\xi, \forall_s \in v\backslash\{0\}; \\ u_0(x) = 6^{-1}\delta_0^{-1}\alpha x(x^2-1)f_0(\overline{x}) + \int_0^1 G_0(x,\xi)f_0\{\xi\}d\xi \end{cases} \tag{20}$$
where
$$G_s(x,\xi) = \begin{cases} \frac{sh(\lambda\xi)sh\{\lambda(1-x)\}}{\lambda sh(\lambda)}, 0 \leq \xi \leq x \leq 1, \\ \frac{sh(\lambda\xi)sh\{\lambda(1-\xi)\}}{\lambda sh(\lambda)}, 0 \leq x \leq \xi \leq 1, \end{cases} \quad \forall_s \in v \tag{21}$$
and
$$\delta_s = 1 - \frac{\alpha sh\{\lambda\overline{x}}{sh\{\lambda\}} \neq 0, \ \forall_s \in v \tag{22}$$
Expressions for $G_0(x,\xi)$ and δ_0 can be obtained directly at $s = 0$ or passing to the limit from formulas (21) and (22) as $\lambda \to 0 (s \to 0)$
$$G_0(x,\xi) = \begin{cases} \xi(1-x), 0 \leq \xi \leq x \leq 1, \\ x(1-\xi), 0 \leq x \leq \xi \leq 1, \end{cases}$$
$$\delta_0 = 1 - \alpha\overline{x}.$$

Formula (20) can define a regular solution to boundary value problem (7)-(9) for the Fourier coefficients with sufficient smoothness of the function $f_s(x)$. Therefore, for the correctness of the function $f_s(x)$ according to the functions $u_s(x)$ found on the basis of the formula (20), any combinations in the form

$$u^N(x,t) = \sum_{s=-N}^{s=N} u_s(x) e^{is\cdot x}$$

defines a regular solution to boundary value problem (7)-(9).

Based on formula (20) we obtain the following a'priori estimates

$$\|u_s(x)\|_{L_2(0,1)} \le K \cdot \|f_s''(x)\|_{L_2(0,1)}, s \in v, \tag{23}$$

where k is a constant that independent of s, so estimates (23) are constant relative to the s [10-12].

Furthermore, proving estimate (23),we establish that the following estimate for the Green function $G(x,\xi)$ is fair

$$\int_0^1 \int_0^1 |G_s(x,\xi)|^2 dx d\xi \le \frac{C}{|\lambda|^3} \le K = const, \forall_s \in v\backslash\{0\}, (\lambda^2 = is).$$

Really, taking into account that $\lambda = \lambda_1 + i\lambda_1$, we get

$$\int_0^1 |G_s(x,\xi)|^2 d\xi \le \frac{1}{|\lambda|^2 |sh\lambda|^2} \left[|sh\lambda(1-x)|^2 \int_0^x |sh\lambda\xi|^2 d\xi + |sh\lambda x|^2 \int_x^1 |sh\lambda(1-\xi)|^2 d\xi \right] =$$

$$= \frac{1}{2|\lambda|^2|sh\lambda|^2} \left[|sh\lambda(1-x)|^2 \int_0^x |ch2\lambda_1\xi - cos2\lambda_1\xi) d\xi + |sh\lambda x|^2 \int_x^1 |ch2\lambda_1(1-\xi) - cos2\lambda_1(1-\xi)) d\xi \right] =$$

$$= \frac{1}{8\lambda_1|\lambda|^2|sh\lambda|^2} \cdot \{[ch2\lambda_1(1-x) - cos2\lambda_1(1-x)](sh2\lambda_1 x - sin2\lambda_1 x) +$$
$$(ch2\lambda_1 x - cos2\lambda_1 x) \times [sh2\lambda_1(1-x) - sin2\lambda_1(1-x)]\} =$$
$$= \frac{1}{8\lambda_1|\lambda|^2|sh\lambda|^2} \cdot [sh2\lambda_1 x + sin2\lambda_1 x - ch2\lambda_1(1-x) sin2\lambda_1 x -$$
$$-sh2\lambda_1 x cos2\lambda_1(1-x) - ch2\lambda_1 x sin2\lambda_1(1-x) - sh2\lambda_1(1-x) cos2\lambda_1 x].$$

As a result, we get the following estimate

$$\int_0^1 \int_0^1 |G_s(x,\xi)|^2 dx d\xi = \frac{1}{8\lambda_1|\lambda|^2|sh\lambda|^2} \left(sh2\lambda_1 x + sin2\lambda_1 x - \frac{ch2\lambda_1}{\lambda_1} + \frac{cos2\lambda_1}{\lambda_1} \right) \le \frac{C}{|\lambda|^3},$$
$$\lambda_1 = Re\lambda = Im\lambda, 2|\lambda_1|^2 = |\lambda|^2.$$

To receive estimate (23) for $s \ne 0$ (20) we obtain the following equalities

$$\lambda^2 \cdot \frac{d^2 u_s}{dx^2} = -\lambda^2 f_s(x) + \lambda^4 \int_0^1 G_s(x,\xi) f_s(\xi) d\xi + \alpha \cdot \delta_s^{-1} \lambda^4 \frac{sh\lambda x}{sh\lambda} \cdot \left[\int_0^1 G_s(\overline{x},\xi) f_s(\xi) d\xi - \frac{f_s(\overline{x})}{\lambda^2} \right]; \tag{24}$$

$$\frac{d^4 u_s(x)}{dx^4} = -\frac{d^2 f_s(x)}{dx^2} + \lambda^4 \int_0^1 G_s(x,\xi) f_s(\xi) d\xi - \lambda^2 f_s(x) +$$
$$+\alpha \cdot \delta_s^{-1} \left[\int_0^1 G_s(\overline{x},\xi) f_s(\xi) d\xi - \frac{f_s(\overline{x})}{\lambda^2} \right] \cdot \lambda^4 \frac{sh\lambda x}{sh\lambda}. \tag{25}$$

For some terms of solution (20) and their derivatives (24)–(25) we get the following inequality

$$\left\|\lambda^2 f_s(x)\right\|^2_{L_2(0,1)} = \left\||\lambda|^2 f_s(x)\right\|^2_{L_2(0,1)} \le \left\| |\lambda|^{\frac{5}{2}} f_s(x) \right\|^2_{L_2(0,1)}.$$

We take into account $\lambda^2 = is, s = \pm1, \pm2, \ldots$

$$\int_0^1 \left| \lambda^4 \int_0^1 G_s(x,\xi) f_s(\xi) d\xi \right|^2 dx \le \left\| |\lambda|^{\frac{5}{2}} f_s \right\|^2_{L_2(0,1)} \times \int_0^1 \left\| |\lambda|^{\frac{3}{2}} G_s \right\|^2_{L_2(0,1)} dx \le \| |\lambda|^{\frac{5}{2}} f_s \|^2_{L_2(0,1)} \cdot |\lambda|^3 \cdot \frac{C^2}{|\lambda|^3} =$$

$$= C^2 \left\| |\lambda|^{\frac{5}{2}} f_s \right\|^2_{L_2(0,1)}$$

$$\int_0^1 \left| \lambda^4 \frac{sh\lambda x}{sh\lambda} \int_0^1 G_s(\overline{x},\xi) f_s(\xi) d\xi \right|^2 dx = \frac{|\lambda|^8}{|sh\lambda|^2} \int_0^1 |sh\lambda x|^2 dx \left(\int_0^1 |G_s(\overline{x},\xi) f_s(\xi)| d\xi \right)^2 \leq$$

$$\leq K_1 |\lambda|^4 \|f_s(x)\|^2_{L_2(0,1)} = K_1 \left\| |\lambda|^2 f_s(x) \right\|^2_{L_2(0,1)},$$

there, we are used the following

$$\int_0^1 |sh\lambda|^2 dx = \frac{1}{2|sh\lambda^2|} \int_0^1 (ch2\lambda_1 x - cos2\lambda_1 x) dx = \frac{sh2\lambda_1 - sin2\lambda_1}{4\lambda_1 |sh\lambda|^2} \leq \frac{C}{|\lambda|}.$$

We get estimate

$$\left\| \lambda^4 \frac{sh\lambda x}{sh\lambda} \cdot \frac{f_s(\overline{x})}{\lambda^2} \right\|^2_{L_{2(0,1)}} = \frac{|\lambda|^4}{|sh\lambda|^2} \int_0^1 |sh\lambda x|^2 dx \cdot \left(\int_0^{\overline{x}} |f_s'(\xi)| d\xi \right)$$

$$\leq C \cdot \left\| |\lambda|^{\frac{3}{2}} f_s'(x) \right\|^2_{L_2(0,1)} \leq C \cdot \left\| |\lambda|^{\frac{3}{2}} f_s(x) \right\|^2_{W_2^2(0,1)}$$

or

$$\left\| |\lambda|^2 \cdot \frac{d^2 u_s}{dx^2} \cdot \frac{f_s(\overline{x})}{\lambda^2} \right\|^2_{L_{2(0,1)}} \leq K_1 \left[|| |\lambda|^{\frac{5}{2}} f_s(x) \|^2_{L_2(0,1)} + \| |\lambda|^{\frac{3}{2}} f_s(x) \|^2_{W_2^2(0,1)} \right].$$

Taking into account (24), and terms on the right side of equality (25) are covered by the right side of the equality, the next assessment will not be difficult to obtain

$$\left\| \frac{d^4 u_s(x)}{dx^4} \cdot \frac{f_s(\overline{x})}{\lambda^2} \right\|^2_{L_{2(0,1)}} \leq K_2 \left[|| |\lambda|^{\frac{5}{2}} f_s(x) \|^2_{L_2(0,1)} + \| |\lambda|^{\frac{3}{2}} f_s(x) \|^2_{W_2^2(0,1)} + \| f_s(x) \|^2_{W_2^2(0,1)} \right].$$

Now, we can determine that the estimate is valid. Really,

$$||u_s(x)\|^2_{L_2(0,1)} \leq K \left[\| \frac{f_s(x)}{|\lambda|} \|^2_{L_{2(0,1)}} + \| \frac{f_s(x)}{|\lambda|^{\frac{3}{2}}} \|^2_{L_{2(0,1)}} + \| \frac{f_s(x)}{|\lambda|^2} \|^2_{L_{2(0,1)}} \right] \leq$$

$$K_1 \left\| \frac{f_s(x)}{|\lambda|} \right\|^2_{L_{2(0,1)}} \leq ||f_s(x)\|^2_{L_2(0,1)} \leq K_3 ||f_s''(x)\|^2_{L_2(0,1)},$$

for $s = 0$. Based on formula (20), this calculation is taken in a simple form.

Taking into account estimate (23), on the basis of the results [8] (p. 118-119) we proved uniqueness of strong solution to boundary value problem (7)-(9). The theorem is proved.

In addition, from the above estimates, the following uniform estimate for $s \in v$ is derived by the formula

$$\left\| |\lambda|^2 \cdot \frac{d^2 u_s(x)}{dx^2} \right\|^2_{L_{2(0,1)}} + \left\| \frac{d^4 u_s(x)}{dx^4} \right\|^2_{L_{2(0,1)}} \leq$$

$$\leq K \left[\left\| |\lambda|^{\frac{5}{2}} f_s(x) \right\|^2_{L_{2(0,1)}} + \left\| |\lambda|^{\frac{3}{2}} f_s(x) \right\|^2_{W^2_{2(0,1)}} + \| f_s(x) \|^2_{W^2_{2(0,1)}} \right], \tag{26}$$

in addition, For derivative the next estimate is valid

$$\left\| \frac{\partial^3 u}{\partial x^2 \partial e} \right\|_{L_2(Q)} \leq K \left[\| f(x,t) \|_{W^{\frac{5}{2}}_{2(0.2\pi; L_2(0,1))}} + \| | f_x(x,t) \|^2_{W^{\frac{3}{4}}_{2(0.2\pi; L_2(0,1))}} + \| f(x,t) \|_{L_{2(0.2\pi; W_2^2(0,1) \cap W_2^1(0,1))}} \right]$$

Estimate (26) presents that the strong solution to the boundary value problem has the differential property that is given by estimate (23).

Thus, in condition (3), the requirement for the function $f(x,t)$ can be replaced by the following: $-f \in L_2(0,2\pi; W_2^2(0,1))$. In this case (23), the estimate has the following form

$$\|u_s(x)\|_{L_2(0,1)} \leq K \cdot \|f_s(x)\|_{W^2_{2(0,1)}}, s \in v.$$

Definition 3. Let $u_n(x,t)_{n=1}^{\infty} \subset \widetilde{C}$ be a sequence of functions and

$$1^0.L_2(Q) in \lim_{n\to\infty} u_n(x,t) = u(x,t);$$

$$2^0.L_2(Q) in \lim_{n\to\infty} L_4 u_n(x,t) = \frac{\partial^{2m} f}{\partial x^{2m}}.$$

Then the function $u(x,t)$ is-(14) are called a strong solution to boundary value problem (10)-(14).

If

$$u \in C(\widetilde{Q}), u \in C(0,2\pi; C^{2m+2}(0,1) \cap C^{m+1}[0,1]), \frac{\partial u(x,t)}{\partial t} \in C(0,2\pi; C^{2m}(0,1) \cap C^{m}[0,1]),$$

we assume that the conditions $u \in \widetilde{C}$ and (11)-(14) are satisfied.

Definition 4. The strong solution to boundary value problem (10)-(14) is called a strong solution to boundary value problem (4)-(5).

From these definitions it follows that the domains of the closed operators L_2 and L_4 are equal.

We get the following

$$D(L_2) \equiv D(L_4) \equiv$$

$$\left\{ u | u \in L_2\left(0,2\pi; W_2^{2m+2}(0,1)\right), \frac{\partial u}{\partial t} \in L_2\left(0,2\pi; W_2^{2m}(0,1)\right), \right\} \times \; boundary\ conditions (11)-(14) \quad (27)$$

For the second boundary value problem, the following statement is valid [13-15].

Theorem 2. Let

$$\delta_s \equiv 1 + \frac{\partial^k}{\partial x^k}\int_0^1 G_s(\overline{x},\xi)\alpha(\xi)d\xi \neq 0, \forall_s \in v, \quad (28)$$

where $v = \{s | s = 0; \pm1; \pm2; \ldots\}, \lambda^2 = is, i = \sqrt{-1}, G_s(\overline{x},\xi)$ is the function defined by the formula from conditions (21), then for any $f \in L_2\left(0,2\pi; W_2^{2m}(0,1) \cap \dot{W}_2^{m}(0,1)\right), \alpha \in W_2^{2m}(0,1)$ the function $u(x,t)$ is a strong solution to boundary value problem (4)-(5). The proof of the second theorem is similar to the proof of the first theorem.

The validity of the statements follows from (27) and (28).

3 Conjugated problem

Consider conjugate problem (1)-(2),

$$L_1^*\psi \equiv -\frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial t^2} + \delta''(x-\overline{x}) \bigotimes \int_0^1 \alpha \cdot \xi \cdot \psi(\xi,t)d\xi = g(x,t), \{x,t\} \in Q, \quad (29)$$

$$\psi(0,t) = \psi(1,t) = 0; \psi(x,0) = \psi(x,2\pi), \overline{x} \in (0,1), \quad (30)$$

here it is taken into account that the value $supp\{\psi(x,1)\} \subseteq \overline{Q}$. A weak solution $\psi \in L_2(Q)$ of this problem we define the following integral equality: for any $\omega \in \widetilde{C}$ (from the first definition) $(\omega, L_1^*\psi) = (L_1, \omega, \psi) = (\omega, g)$.

First, we show that the operator L_1^* is conjugate with the operator L_1. To do this, it is enough to make sure that the following relation is valid

$$\int_0^{2\pi}\int_0^1 x\frac{\partial^2 u(\overline{x},t)}{(\partial x^2)}\psi(x,t)dxdt = \int_0^{2\pi}\int_0^1 \delta''(x-\overline{x})\left(\int_0^1 \xi\psi(\xi,t)d\xi\right)u(x,t)dxdt.$$

Indeed,

$$\int_0^{2\pi}\int_0^1 x\frac{\partial^2 u(\overline{x},t)}{(\partial x^2)}\psi(x,t)dxdt = \int_0^{2\pi}\int_0^1 x\psi(x,t)\left[\int_0^1 \delta(\xi-\overline{x})\frac{\partial^2 u(\xi,t)}{\partial \xi^2}d\xi\right]dxdt =$$

$$= \int_0^{2\pi}\int_0^1 \delta(\xi-\overline{x})\left(\int_0^1 \xi\psi(\xi,t)d\xi\right)\frac{\partial^2 u(x,t)}{(\partial x^2)}dxdt = \int_0^{2\pi}\delta(x-\overline{x})\left(\int_0^1 \xi\psi(\xi,t)d\xi\right)\frac{\partial u(x,t)}{(\partial x)}|_0^1 dt-$$

$$-\int_0^{2\pi}\int_0^1 \delta'(x-\overline{x})\left(\int_0^1 \xi\psi(\xi,t)d\xi\right)\frac{\partial u(x,t)}{\partial x}dxdt = -\int_0^1 \delta(x-\overline{x})\left(\int_0^1 \xi\psi(\xi,t)d\xi\right)u(x,t)|_0^1 dt+$$

$$+\int_0^{2\pi}\int_0^1 \delta'(x-\overline{x})\left(\int_0^1 \xi\psi(\xi,t)d\xi\right)u(x,t)dxdt = \int_0^{2\pi}\int_0^1 \delta''(x-\overline{x})\left(\int_0^1 \xi\psi(\xi,t)d\xi\right)u(x,t)dxdt.$$

Using the method of separation of variables from (29) - (30), we get the corresponding system of problems for the Fourier coefficient $\psi_s(x), s \in v\backslash\{0\}$

$$\begin{cases} -is\psi_s(x) - \psi_s''(x) + \delta(x-\overline{x})\int_0^1 \alpha\cdot\xi\cdot\psi_s(x)d\xi = g_s(x), x \in (0,1), \\ \psi_s(0) = \psi_s(1) = 0, \forall s \in \backslash\{0\}. \end{cases}$$

The solutions to these problems have the form: {the Fourier coefficient of the function $g(x,t)$ according to $g_s(x)$}

$$\psi_s(x) = \int_0^1 \widetilde{G}_S(x,\xi)g_s(\xi)d\xi + \int_0^1 \xi\psi_s(\xi)d\xi\cdot[\lambda^2\cdot\widetilde{G}_s(x,\overline{x})]$$

where

$$\widetilde{G}_S(x,\xi) = \begin{cases} \frac{sin\{\lambda\xi\}sin\{\lambda(1-x)\}}{\lambda\sin(\lambda)}, 0 \le \xi \le x \le 1, \\ \frac{sin\{\lambda\xi\}sin\{\lambda(1-\xi)\}}{\lambda sin(\lambda)}, 0 \le x \le \xi \le 1, \end{cases} \quad \forall s \in v \tag{31}$$

iff

$$\widetilde{\delta}_s = 1 + \alpha\overline{x} - \alpha\frac{sin(\lambda\overline{x})}{sin\lambda} \ne 0, \forall s \in v(\lambda^2 = is). \tag{32}$$

The expressions for $G_0(x,\xi)$ and δ_0 can be obtained directly or passing to the limit as $s \to 0(\lambda \to 0)$ in formulas (31) and (32):

$$\widetilde{G}_0(x,\xi) = \begin{cases} \xi(1-x), 0 \le \xi \le x \le 1; \\ x(1-\xi), 0 \le x \le \xi \le 1; \end{cases}$$

$$\widetilde{\delta}_0 = 1.$$

Remark 3. Let $\alpha \in R^1$. If the function $\omega_0(x)$ (17) given by (1)-(2) is orthogonal to all functions $g(x,t)$ from conjugate problem (29)-(30) (by Corollary 2) then $\omega_0(x)$ (17) is a univocal weak solution. In this case, condition (32) is valid for all $s \in v$.

Remark 4. If $g(x,t) \equiv 0$ then (29)-(30) has a unique solution $\psi(x,t) = \delta(x-\overline{x})$.

Note that to study the integral equation to which the problem for a parabolic equation has been reduced, we can use the Laplace transform by applying the model solution method [16].

References

1 Akhmanova, D.M., Dzhenaliev, M.T., & Ramazanov, M.I. On a particular second kind Volterra integral equation with a spectral parameter. *Siberian Mathematical Journal, 52, 1,* 1–10. DOI: 10.1134/S0037446606010010.

2 Amangaliyeva, M.M., Akhmanova, D.M., Dzhenaliev, M.T., & Ramazanov, M.I. (2011). Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity. *Differential Equations, 47, 2,* 231–243. DOI: 10.1134/S0012266111020091.

3 Dzhenaliev, M.T., & Ramazanov, M.I. (2006). On the boundary value problem for the spectrally loaded heat conduction operator. *Siberian Mathematical Journal, 47, 3,* 433-451. DOI: 10.1007/s11202-006-0056-z

4 Dzhenaliev, M.T., & Ramazanov, M.I. (2007). On a boundary value problem for a spectrally loaded heat operator: I *Differential Equations, 43, 4,* 513-524. DOI: 10.1134/S0012266107040106.

5 Dzhenaliev, M.T., & Ramazanov, M.I. (2007). On a boundary value problem for a spectrally loaded heat operator: II *Differential Equations, 43, 6,* 806-812. DOI: 10.1134/S0012266107060079.

6 Jenaliyev, M., & Ramazanov, M. (2016). On a homogeneous parabolic problem in an infinite corner domain. *AIP Conference Proceedings, 1759,* 020085. DOI: 10.1063/1.4959699.

7 Jenaliyev, M., Amangaliyeva, M., Kosmakova, M., & Ramazanov, M. (2014). About Dirichlet boundary value problem for the heat equation in the infinite angular domain. *Boundary Value Problems, 213,* 1–21. doi: 10.1186/s13661-014-0213-4.

8 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). Uniqueness and non-uniqueness of solutions of the boundary value problems of the heat equation. *AIP Conference Proceedings, 1676,* 020028. DOI: 10.1063/1.4930454.

9 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2014). On the spectrum of Volterra integral equation with the "incompressible"kernel. *AIP Conference Proceedings, 1611,* 127–132. DOI: 10.1063/1.4893816.

10 Akhmanova, D.M., Ramazanov, M.I., & Yergaliyev. M.G. (2018). On an integral equation of the problem of heat conduction with domain boundary moving by law of $t = x^2$. *Bulletin of the Karaganda University-Mathematics, 1 (89),* 15-19. DOI: 10.31489/2018M1/15-19.

11 Kosmakova, M.T., Ramazanov, M.I., Tokesheva, A.S., & Khairkulova, A.A. (2016). On the non-uniqueness of solution to the homogeneous boundary value problem for the heat conduction equation in an angular domain. *Bulletin of the Karaganda University-Mathematics, 4 (84),* 80-87. DOI: 10.31489/2016M4/80-87.

12 Kosmakova, M.T. (2016). On an integral equation of the Dirichlet problem for the heat equation in the degenerating domain. *Bulletin of the Karaganda University-Mathematics, 1 (81),* 62–67.

13 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. On a Volterra equation of the second kind with 'incompressible' kernel. *Advances in Difference Equations, 71,* 1–14. DOI: 10.1186/s13662-015-0418-6.

14 Amangaliyeva, M.M., Dzhenaliev, M.T., Kosmakova, M.T., & Ramazanov M.I. (2015). On one homogeneous problem for the heat equation in an infinite angular domain. *Siberian Mathematical Journal, Vol. 56, No. 6*, 982–995. DOI: 10.1134/S0037446615060038.

15 Sarsengeldin, Merey, Kharin, Stanislav, Rayev, Zhangir, & Khairullin, Yermek. (2018). Mathematical model of heat transfer in opening electrical contacts with tunnel effect. *Filomat, Vol. 32, No. 3*, 1003–1008. DOI: 10.2298/FIL1803003S

16 Akhmanova, D.M., Kervenev, K.E., & Baltabayeva, A.M. (2019). On singular integral equations with variable limits of integration. *Bulletin of the Karaganda University-Mathematics, 1 (93),* 8–18. DOI: 10.31489/2019M1/8-18

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Шектелген аймақтардағы елеулі жүктелген параболалық теңдеулерге арналған шекаралық есептер туралы

Мақалада шектелген аймақтағы елеулі жүктелген параболалық дифференциалдық теңдеулерге арналған әлді шешім сұрақтары зерттелген. Қарастырылған есептердің ерекшеліктері: мысалы, $L_2(Q)$ кеңістігіне сәйкес дифференциалдық операторлар тұйықтаушы болмайды, себебі, біріншіден, жүктеме қарастырылып отырған оператордың сәйкес дифференциалдық бөлігіне бағынбайды, яғни оның дифференциалдық бөлігі үшін әлсіз ауытқу болып табылмайды. Екіншіден $L_2(0, 1)$ және $L_2(Q)$ кеңістіктерінде жүктеме операторларының өздері тұйықтаушы операторлар болып табылмайтыны белгілі. Осының барлығы тұйықталмайтын жүктелген дифференциалдық теңдеулерге арналған әлді шешімді шекаралық есептер сұрақтарын тікелей зерттеу мүмкін емес екенін көрсетеді. Алайда [1-4] теңдеулерін зерттеу теориялық қана емес, анық қолданбалы [5-7] сипат береді.

Кілт сөздер: елеулі жүктелген дифференциалдық теңдеулер, Вольтерр интегралдық теңдеуі, шекаралық есеп, әлді шешім, оператор.

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О граничных задачах для существенно нагруженных параболических уравнений в ограниченных областях

В статье изучены вопросы сильного решения для существенно нагруженных дифференциальных уравнений параболического типа в ограниченных областях. Особенности рассматриваемых задач: например, в $L_2(Q)$ пространстве соответствующие дифференциальные операторы не являются операторами замыкания, поскольку, во-первых, нагрузка не подчиняется соответствующей дифференциальной части рассматриваемого оператора, то есть для его дифференциальной части не является слабым возмущением. Во-вторых, очевидно, что в пространствах $L_2(0,1)$ и $L_2(Q)$ операторы нагрузки сами не являются операторами замыкания. Все это указывает на то, что невозможно непосредственно исследовать вопросы сильного решения граничных задач для незамкнутых нагруженных дифференциальных уравнений. Однако исследование уравнений [1–4] дает не только теоретический, но и выраженный прикладной [5–7] характер.

Ключевые слова: существенно нагруженные дифференциальные уравнения, интегральное уравнение Вольтерра, граничная задача, сильное решение, оператор нагрузки.

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The Cauchy problem for the Navier-Stokes equations[1]

Ch. Fefferman in his works two problems for Navier-Stokes equations are set out: one of them is the Cauchy problem and he considers «only those solutions that are infinitely smooth functions are physically meaningful». In this article, the author received positive answers for the above problem of Ch. Fefferman. He proved the uniqueness and existence of smooth solutions of the Cauchy problem for the Navier-Stokes equations. The ratio between the pressure P and the kinetic energy density E, previously established by the author. is taken as the basis. As a result of in-depth studies of the Cauchy problem for the Navier-Stokes equations, it is shown that E is a bounded, continuous function that satisfies the Laplace equation and has continuous first-order derivatives with respect to t and all kinds of second derivatives with respect to the spatial variables $\mathbf{x}$ and is a regular harmonic function in the space R_3. An explicit form of E is found with the help of which the Navier-Stokes equations are reduced to a system of linear parabolic equations and the solutions are written out by the Fourier transform that are infinitely differentiable with respect to t and $\mathbf{x}$. The systems of equations for the curl-vector are found. Proven uniqueness, the existence of infinite smoothness. An estimate is obtained linking the curl-vectors with the Reynolds number.

Keywords: The Cauchy problem for the Navier-Stokes equations, the uniqueness and existence of smooth solutions of the Navier-Stokes equations, the harmonicity of the kinetic energy density, the equations for the vortex vector, the Cauchy problem for the curl-vector equations, the uniqueness and existence of smooth solutions of the equations curl-vectors

0.1 Some introductory information

Unsolved problems in the theory of Navier-Stokes equations homogeneous liquids are given in [1–2], [3] and others.

In a number of works [4]–[6] of the author, the results of some explored. The substantiation of the simplest principle is given in [4] maximum for three-dimensional Navier-Stokes equations, which allows get a positive answer to an unresolved problem O.A. Ladyzhenskaya in [1, 2].

In [5], based on the properties of solutions of the Navier-Stokes equations, the relation between the pressure and squared modulus of the velocity vector. Based on what the uniqueness of the weak and the existence of strong solutions to a problem from a class of functions

$$C((0,T];W_2^1(G)\cup C^1((0,T];W_2^2(G))$$

for the Navier-Stokes equations in bounded domain of G in whole time $t\in[0,T],\forall T<\infty$.

The justification of the method was given in [6] splitting for solving the Navier-Stokes equations. Shown the compactness of the solution sequence, thereby the existence of strong solutions to the three-dimensional Navier-Stokes equations in whole time.

The original Navier-Stokes equations are not equations of type Cauchy-Kovalevskaya. Using ratio $(P=-|\mathbf{U}|^2)\vee(P=0)$ from [5] the system of equations (1a) can be reduced to the Cauchy-Kovalevskaya type. We will study the Navier-Stokes equations (1a) taking into account the relation $P=-|\mathbf{U}|^2$, preserving the condition of incompressibility of the fluid.

The Cauchy problem for the Navier-Stokes equations with respect to the velocity vector $\mathbf{U}=(U_1,U_2,U_3)$ in the domain $Q=(0,\infty)\times R_3$ it will be written in the form [5]:

$$\frac{\partial\mathbf{U}}{\partial t}-\mu\Delta\mathbf{U}+(\mathbf{U},\nabla)\mathbf{U}-2\nabla E=\mathbf{f}(t,\mathbf{x}),\ \nabla\cdot\mathbf{U}=0, \tag{1a}$$

[1]The work was done on the personal initiative of the author.

$$\mathbf{U}(0,\mathbf{x})=\mathbf{\Phi}(\mathbf{x}), \tag{1b}$$

where $\mathbf{x}\in R_3$; $E=\frac{1}{2}|\mathbf{U}|^2$; $t\in(0,\infty)$.

Known [1] orthogonal decomposition $\mathbf{L}_2(Q)=\mathbf{G}(Q)\oplus\mathring{\mathbf{J}}(Q)$, moreover, the elements $\mathring{\mathbf{J}}(Q)$ at $\forall t$belong to $\mathring{\mathbf{J}}(R_3)$, and the elements $\mathbf{G}(Q)$ belong to the subspace $\mathbf{G}(R_3)$; $\mathring{\mathbf{J}}(R_3)$ — the space of solenoidal vectors, and $\mathbf{G}(R_3)$ consists of $\nabla\eta$, where η is a unique function in R_3. $\mathbf{L}_\infty(Q)-$ subspace $\mathbf{C}(Q)$. $W_p^k(B_R)$ is the Sobolev space.

In the plane $t=0$, we introduce the ball B_R (*imaginary, of course, since in the case of the Cauchy problem a homogeneous incompressible fluid fills all spaces* R_3) of radius $R\gg 1$ with center at origin of coordinates.

Input $\mathbf{f}$ and $\mathbf{\Phi}$ problems (1) satisfy the requirements:

$$\mathbf{i)}\,\mathbf{f}(t,\mathbf{x})\in\mathbf{C}^\infty(Q)\cap\mathring{\mathbf{J}}(Q),\Big|\frac{\partial^\gamma\mathbf{f}(t,\mathbf{x})}{\partial t^\gamma}\Big|\le \mathrm{g}_{\gamma\kappa}(1+\mathrm{t})^{-\kappa}\wedge\Big|\frac{\partial^\alpha\mathbf{f}(\mathrm{t},\mathbf{x})}{\partial \mathrm{x}_1^{\alpha_1}\partial \mathrm{x}_2^{\alpha_2}\partial \mathrm{x}_3^{\alpha_3}}\Big|\le \mathrm{d}_{\alpha\kappa}(1+|\mathbf{x}|)^{-\kappa};$$

$$\mathbf{ii)}\,\mathbf{\Phi}(\mathbf{x})\in\mathbf{C}^\infty(R_3)\cap\mathring{\mathbf{J}}(R_3),\ \Big|\frac{\partial^\alpha\mathbf{\Phi}(\mathbf{x})}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\Big|\le q_{\alpha\kappa}(1+|\mathbf{x}|)^{-\kappa},\alpha=\alpha_1+\alpha_2+\alpha_3,$$

where $\alpha_i\in\{0,1,...,\alpha\},\gamma,\ \kappa-$ positive integers. $\mathrm{g}_{\gamma\kappa},d_{\alpha\kappa},q_{\alpha\kappa}$ -positive constants. □

0.2 On the harmonically of the kinetic energy density E

Theorem 1. If the input data of the problem (1) satisfies the requirements **i)**, **ii)**, then for the solutions of the problem (1) the estimate

$$\|\mathbf{U}\|_{C(0,\infty;L_\infty(R_3))}\le q_{04}\|\mathbf{\Phi}\|_{\mathbf{L}_\infty(R_3)}+d_{2,4}\|\mathbf{f}\|_{C(0,\infty;L_\infty(R_3))}\equiv A_1, \tag{2}$$

$$\big\|E\big\|_{C(0,\infty;L_\infty(R_3))}\le A_1,\quad E=\frac{1}{2}|\mathbf{U}|^2,\ d_{2,4}=\mathrm{g}_{02}\mathrm{q}_{04}. \tag{3}$$

Proof. We write a formula from vector algebra

$$(\mathbf{U},\nabla)\mathbf{U}-\nabla E=[rot\mathbf{U},\mathbf{U}]$$

using this formula of the equation (1a) we rewrite

$$\frac{\partial\mathbf{U}}{\partial t}-\mu\Delta\mathbf{U}-\frac{1}{2}\nabla|\mathbf{U}|^2=-[rot\mathbf{U},\mathbf{U}]+\mathbf{f}(t,\mathbf{x}). \tag{4}$$

We multiply the equation (4) by the vector function $\mathbf{U}\ne 0$, then, taking into account the property $[rot\mathbf{U},\mathbf{U}]\perp\mathbf{U}$ we get

$$\frac{\partial\mathbf{U}}{\partial t}-\mu\Delta\mathbf{U}-\nabla E=\mathbf{f}(t,\mathbf{x}). \tag{5}$$

Acting by the operator *div* on (5), we have

$$\Delta E=0. \tag{6}$$

Lemma 1. There is a relation

$$-\big(\Delta\mathbf{U},\mathbf{U}\big)\ge 0. \tag{7}$$

Proof. By painting ΔE and doing a little counting, we find

$$\Delta E=div\nabla\big(\frac{1}{2}|\mathbf{U}|^2\big)=\big(\Delta\mathbf{U},\mathbf{U}\big)+\sum_{\alpha=1}^{3}\big(\nabla U_\alpha\big)^2=0.$$

Whence the inequality (7) follows.

Multiply the equation (4) by a vector function $p|\mathbf{U}|^{2(p-1)}\mathbf{U}$ and taking into account the property $[rot\mathbf{U},\mathbf{U}]\perp\mathbf{U}$ integrate over domain of B_R

$$\frac{1}{2}\frac{d}{dt}\int\limits_{B_R}|\mathbf{U}|^{2p}\,\mathbf{dx}-p\mu\int\limits_{B_R}\big(\Delta\mathbf{U},\mathbf{U}\big)|\mathbf{U}|^{2(p-1)}\mathbf{dx}-$$

$$-\frac{1}{2}\int\limits_{B_R}\mathbf{U}\nabla|\mathbf{U}|^{2p}\mathbf{dx}\ =p\int\limits_{B_R}(|\mathbf{U}|^{2(p-1)}\mathbf{U}\mathbf{f}\mathbf{dx}. \tag{8}$$

Each term (8) is simplified accordingly. When evaluating the second term in the left-hand side, we take into account (7). Third $\int\limits_{B_R} \mathbf{U}\nabla|\mathbf{U}|^{2p}\mathbf{dx} = 0$, due to the orthogonality [1] of the spaces $\mathring{\mathbf{J}}(B_R)$ and $\mathbf{G}(B_R)$. The right-hand side, estimated by Holder inequality, we get:

$$\frac{1}{2}\frac{d}{dt}\int\limits_{B_R}\left|\mathbf{U}\right|^{2p}d\mathbf{x} \leq p\Big(\int\limits_{B_R}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\Big)^{\frac{2p-1}{2p}}\Big(\int\limits_{B_R}\left|\mathbf{f}\right|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}}. \tag{9}$$

Both parts (9), dividing by a positive value $p\Big(\int\limits_{B_R}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\Big)^{\frac{2p-1}{2p}}$, we have

$$\frac{d}{dt}\Big(\int\limits_{B_R}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}} \leq \Big(\int\limits_{B_R}\left|\mathbf{f}\right|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}}.$$

Choosing an arbitrary $t \in (0,\infty)$ and integrating the last time inequality ranging from 0 to t, find

$$\Big(\int\limits_{B_R}\left|\mathbf{U}(t,\mathbf{x})\right|^{q}d\mathbf{x}\Big)^{\frac{1}{q}} \leq \Big(\int\limits_{B_R}\left|\mathbf{\Phi}(\mathbf{x})\right|^{q}d\mathbf{x}\Big)^{\frac{1}{q}} + \int\limits_0^t\Big(\int\limits_{B_R}\left|\mathbf{f}(\tau,\mathbf{x})\right|^{q}d\mathbf{x}\Big)^{\frac{1}{q}}d\tau, \forall q = 2p, p \in N.$$

Or

$$\left\|\mathbf{U}(t)\right\|_{L_q(B_R)} \leq \left\|\mathbf{\Phi}(\mathbf{x})\right\|_{L_q(B_R)} + \int\limits_0^t\left\|\mathbf{f}(\tau)\right\|_{L_q(B_R)}d\tau,$$

since it is inequality valid for any q, we put $q = \infty$ and take into account the property **i**) of the well-known vector function

$$|\mathbf{f}| \leq \mathrm{g}_{0\kappa}(1+\mathrm{t})^{-\kappa},\ \kappa = 2$$

then

$$\left\|\mathbf{U}(t)\right\|_{L_\infty(B_R)} \leq \left\|\mathbf{\Phi}\right\|_{L_\infty(B_R)} + \sup_{t\geq 0}\left\|\mathbf{f}(t)\right\|_{L_\infty(B_R)},\ t \in (0,\infty),$$

as you can see, the right-hand side is independent of the time t and the inequality holds for all $t \in (0,\infty)$, thereby the left side continuous in t, i.e.

$$\left\|\mathbf{U}(t)\right\|_{C(0,\infty;L_\infty(B_R))} \leq \left\|\mathbf{\Phi}\right\|_{L_\infty(B_R)} + \left\|\mathbf{f}\right\|_{C(0,\infty;L_\infty(B_R))}.$$

From where, using the input properties **i**), **ii**) we have

$$\left\|\mathbf{U}(t)\right\|_{C(0,\infty;L_\infty(B_R))} \leq \frac{8\pi}{3}(1-\frac{R^2}{(1+R)^3})\Big(\left\|\mathbf{\Phi}\right\|_{L_\infty(B_R)} + \left\|\mathbf{f}\right\|_{C(0,\infty;L_\infty(B_R))}\Big).$$

Hence, for $R \to \infty$ we arrive at the proof inequalities (2), (3) of the theorem 1. □

Next, we differentiate the equations (5) with respect to time t, and initial conditions for it are found from the systems of equations themselves (5), т. е.

$$\mathbf{U}_t(0,\mathbf{x}) = \mu\Delta\mathbf{\Phi}(\mathbf{x}) + \frac{1}{2}\nabla|\mathbf{\Phi}|^2 + \mathbf{f}(0,\mathbf{x}) \equiv \mathbf{\Phi_1}(\mathbf{x}),$$

then we have the extended Cauchy problem for $\mathbf{U}_t$,

$$\frac{\partial\mathbf{U}_t}{\partial t} - \mu\Delta\mathbf{U}_t - \nabla E_t = \mathbf{f}_t(t,\mathbf{x}), \tag{10a}$$

$$\mathbf{U}_t(0,\mathbf{x}) = \mathbf{\Phi}_1(\mathbf{x}). \tag{10b}$$

Problem (10) is no different from problems (5), (1b), only in place of the vector functions $\mathbf{U}$ and the functions E stand for them corresponding derivatives $\mathbf{U}_t$ и E_t.

Multiply the equation (10a) by the vector function $p|\mathbf{U}_t|^{2(p-1)}\mathbf{U}_t$ and integrate over the ball B_R, then

$$\frac{1}{2}\frac{d}{dt}\int\limits_{B_R}|\mathbf{U}_t|^{2p}\,\mathbf{dx}-p\mu\int\limits_{B_R}\big(\Delta\mathbf{U}_t,\mathbf{U}_t\big)|\mathbf{U}_t|^{2(p-1)}\mathbf{dx}-$$

$$-\frac{1}{2}\int\limits_{B_R}\mathbf{U}_t\nabla|\mathbf{U}_t|^{2p}\mathbf{dx}\;=p\int\limits_{B_R}|\mathbf{U}_t|^{2(p-1)}\mathbf{U}_t\mathbf{f}_t\mathbf{dx}. \tag{11}$$

Proof. We denote $\mathbf{v}=\mathbf{U}_t$.
Lemma 2. There is an inequality (see [1])

$$-\big(\Delta\mathbf{v},\mathbf{v}\big)\geq\lambda_1(\mathbf{v},\mathbf{v}), \tag{12}$$

since the operator $-\Delta$ in the finite domain B_R positive definite, i.e. $-\big(\Delta\mathbf{v},\mathbf{v}\big)=\lambda^2(\mathbf{v},\mathbf{v})$, where $\lambda_1=\min\lambda^2$.

Each term (11) simplify accordingly. The second term in the left-hand side is estimated taking into account (12) using the following inequality chains:

$$-p\mu\int\limits_{B_R}\big(\Delta\mathbf{U}_t,\mathbf{U}_t\big)|\mathbf{U}_t|^{2(p-1)}\mathbf{dx}\geq-p\mu\sup_{t\geq0}\||\mathbf{U}_t|^{2(p-1)}\|_{L_\infty(B_R)}\int\limits_{B_R}\big(\Delta\mathbf{U}_t,\mathbf{U}_t\big)\mathbf{dx}\geq$$

$$\geq p\mu\lambda_1\sup_{t\geq0}\||\mathbf{U}_t|^{2(p-1)}\|_{L_\infty(B_R)}\int\limits_{B_R}\big(\mathbf{U}_t,\mathbf{U}_t\big)\mathbf{dx}\geq0.$$

Third term $\int\limits_{B_R}\mathbf{U}_t\nabla|\mathbf{U}_t|^{2p}\mathbf{dx}=0$ due to the orthogonality of spaces $\mathring{\mathbf{J}}(B_R)$ and $\mathbf{G}(B_R)$. The right side is estimated by inequality Holder:

$$\frac{1}{2}\frac{d}{dt}\int\limits_{B_R}\big|\mathbf{U}_t\big|^{2p}d\mathbf{x}\leq p\Big(\int\limits_{B_R}\big|\mathbf{U}_t\big|^{2p}d\mathbf{x}\Big)^{\frac{2p-1}{2p}}\Big(\int\limits_{B_R}\big|\mathbf{f}_t\big|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}}.$$

From where, arguing as well as in the previous case, we come to the statement:
Theorem 2. If the input data to the original problem (1) satisfies the requirements **i**), **ii**), then for the solutions of the problem (10) the following estimates are valid:

$$\|\,\mathbf{U}\,\|_{\mathbf{C}^1(0,\infty;\mathbf{L}_\infty(R_3))}\leq q_{2,4}\,\|\,\mathbf{\Phi_1}\,\|_{\mathbf{L}_\infty(R_3)}+d_{2,4}\,\|\,\mathbf{f}\,\|_{\mathbf{C}^1(0,\infty;\mathbf{L}_\infty(R_3))}\equiv A_2,$$

$$\big\|E\big\|_{\mathbf{C}^1(0,\infty;\mathbf{L}_\infty(R_3))}\leq A_2,\;\; d_{2,4}=\mathrm{g}_{2,2}\mathrm{q}_{04}.$$

Next, we introduce the differential operator

$$D^\alpha\cdot=\frac{\partial^\alpha\cdot}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}},\;\;\alpha=\alpha_1+\alpha_2+\alpha_3,\;\;\alpha=\overline{1,3};\;\;\alpha_i\in\{0,1,2,3\},$$

For example, when $\alpha=1,\;D\cdot=\frac{\partial\cdot}{\partial x_1}\vee\frac{\partial\cdot}{\partial x_2}\vee\frac{\partial\cdot}{\partial x_3}$. Acting by the operator D on the problem (10) we obtain the extended Cauchy problem with respect to vector functions $\mathbf{U}_{tx_i}$:

$$\frac{\partial D\mathbf{U}_t}{\partial t}-\mu\Delta D\mathbf{U}_t-\nabla DE_t=D\mathbf{f}_t(t,\mathbf{x}), \tag{13a}$$

$$D\mathbf{U}_t(0,\mathbf{x})=D\mathbf{\Phi}_1(\mathbf{x}), \tag{13b}$$

Theorem 3. If the input of the problem (1) satisfies the requirements **i**), **ii**), then for solutions to the problem (13) the following estimates are valid:

$$\|\mathbf{U}\|_{C^1(0,\infty;\mathbf{W}^1_\infty(B_R))}\leq\|\mathbf{\Phi}_1\|_{\mathbf{W}^1_\infty(B_R)}+\|\mathbf{f}\|_{C^1(0,\infty;\mathbf{W}^1_\infty(B_R))}\equiv A_3.$$

$$\big\|E\big\|_{C^1(0,\infty;\mathbf{W}^1_\infty(B_R))} \le A_3. \tag{14}$$

Proof. We denote $\mathbf{v} = D\mathbf{U}_t$. Multiply the equation (13a) by a vector function $p|D\mathbf{U}_t|^{2(p-1)}D\mathbf{U}_t$ and integrate domains B_R and simplify each term of the result, as in the proof of theorem 2, the second term from the left side taking into account (12), and the third term $\int\limits_{B_R} D\mathbf{U}_t\nabla|D\mathbf{U}_t|^{2p}\mathbf{dx} = 0$, due to the orthogonality of spaces $\mathring{\mathbf{J}}(B_R)$ and $\mathbf{G}(B_R)$. We estimate the right-hand side by Holder's inequality and in the end we get the estimate:

$$\frac{1}{2}\frac{d}{dt}\int\limits_{B_R}\big|D\mathbf{U}_t\big|^{2p}d\mathbf{x} \le p\Big(\int\limits_{B_R}\big|D\mathbf{U}_t\big|^{2p}d\mathbf{x}\Big)^{\frac{2p-1}{2p}}\Big(\int\limits_{B_R}\big|D\mathbf{f}_t\big|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}}. \tag{15}$$

The inequality (15) is no different from (9), only in the places of the function under the integrals, in this case there are their derivatives, that is, $D\mathbf{U}_t$. Therefore, arguing literally, as after the inequality (9) of the theorem 1, we find the estimates (14). The theorem 3 is proved.
Corollary 1.

$$\|\mathbf{U}(t,\mathbf{x})\|_{\mathbf{C}^1(0,\infty;C(R_3))} \le q_{2\kappa},\ q_{2\kappa} - const, \tag{16a}$$

$$\big\|E\big\|_{C^1(0,\infty;C(R_3))} \le q_{2\kappa}. \tag{16b}$$

Proof. From the estimate (14), using embedding theorems Sobolev [7; 64], we find

$$\|\mathbf{U}\|_{\mathbf{C}^1(0,\infty;\mathbf{C}(B_R))} \le d_1\|\mathbf{\Phi}\|_{\mathbf{C}(B_R)} + d_2\|\mathbf{f}\|_{\mathbf{C}^1(0,\infty;\mathbf{C}(B_R))}, \tag{17}$$

where $d_1 = \frac{M\|\mathbf{\Phi}\|_{\mathbf{W}^1_\infty(B_R)}}{\|\mathbf{\Phi}\|_{\mathbf{C}(B_R)}}$, $d_2 = \frac{M\|\mathbf{f}\|_{C^1(0,\infty;\mathbf{W}^1_\infty(B_R)}}{\|\mathbf{f}\|_{C^1(0,\infty;\mathbf{C}(B_R))}}$, M- constant of the embedding theorem. From the inequality (17), taking into account the requirement **i**), **ii**) for input data, we find

$$\|\mathbf{U}(t,\mathbf{x})\|_{\mathbf{C}^1(0,\infty;C(B_R))} \le q_{2\kappa}\frac{8\pi}{3}(1 - \frac{R^2}{(1+R)^3}) = d_{1,4}, \tag{18}$$

$$\big\|E\big\|_{C^1(0,\infty;\mathbf{C}(B_R))} \le d_{1,4},\ \kappa = 4.$$

From here, passing to the limit at $R \to \infty$ we come to inequalities (16). The corollary 1 is proved.

Further, acting by the operator D^α on the problem (10) sequentially for $\alpha = 2,3$ we get extended Cauchy problems with respect to a vector function $\mathbf{U}_{tx_ix_j}$, $\mathbf{U}_{tx_ix_jx_k}$:

$$\frac{\partial D^\alpha\mathbf{U}_t}{\partial t} - \mu\Delta D^\alpha\mathbf{U}_t - \nabla D^\alpha E_t = D^\alpha\mathbf{f}_t(t,\mathbf{x}), \tag{19a}$$

$$D^\alpha\mathbf{U}_t(0,\mathbf{x}) = D^\alpha\mathbf{\Phi}_1(\mathbf{x}), \quad \alpha = 2,3. \tag{19b}$$

Theorem 4. If the input to the problem (1) satisfies the requirements **i**), **ii**), then for the solutions of the extended problems (19), the estimates:

$$\|\mathbf{U}\|_{C^1(0,\infty;\mathbf{W}^\alpha_\infty(B_R))} \le \|\mathbf{\Phi}_1\|_{\mathbf{W}^\alpha_\infty(B_R)} + \|\mathbf{f}\|_{C^1(0,\infty;\mathbf{W}^\alpha_\infty(B_R))} \equiv A_\alpha, \big\|E\big\|_{C^1(0,\infty;\mathbf{W}^\alpha_\infty(B_R))} \le A_\alpha.\ \alpha = 2,3, \tag{20}$$

From estimates (20), using embedding theorems and the requirement **i**), **ii**) for the input, we obtain the inequalities are similar (17), (18), then moving from there to the limit as $R \to \infty$ we find the estimates (21), (22) in Corollary 2:
Corollary 2.

$$\Big(\|\mathbf{U}(t,\mathbf{x})\|_{\mathbf{C}^1(Q)} \le q_{3\kappa}\Big) \wedge \Big(\big\|E\big\|_{C^1(Q)} \le q_{3\kappa}\Big),\ \alpha = 2,\ q_{m\kappa} - const. \tag{21}$$

$$\Big(\|\mathbf{U}(t,\mathbf{x})\|_{\mathbf{C}^1(0,\infty;C^2(R_3))} \le q_{4\kappa}\Big) \wedge \Big(\big\|E\big\|_{C^1(0,\infty;C^2(R_3))} \le q_{4\kappa}\Big),\ \alpha = 3. \tag{22}$$

Note that the number of all possible derivatives of the third the order of the vector function $D^3\mathbf{U}_t$ in spatial $\mathbf{x}$ is equal to ten.

As a result, the following main

Theorem 5. From the theorems 1–4 and Corollaries 1, 2 followed by boundedness, continuity and continuity of the first time derivative t vector functions $\mathbf{U}$ and kinetic energy density E, as well as the continuity of various derivatives of the first and second orders in the spatial variables $\mathbf{x}$ and satisfies the Laplace equation (6) for all $t \in (0, \infty)$, thus the function E is regular harmonic function in the finite domain B_R. From the general theory harmonic functions (h.f.) [8], [9] it follows that h.f. It has derivatives of any order and according to the statements proved to The function E also belongs to this class. Then in the domain B_R for each $t \in (0, \infty)$ based on the corollary of the Poisson formula and Harnack inequalities positive harmonic function E is constant over the spatial variables $\mathbf{x}$ for every $t \in (0, \infty)$.

0.3 On the existence and uniqueness of smooth solutions of the Navier-Stokes equations

From the theorem 5 it follows that the harmonic function $E(t, \mathbf{x})$ is constant inside the ball B_R right up to spherical ball surface ∂B_R, i.e.

$$E(t, \mathbf{x})\big|_{\partial B_R} = \frac{1}{2}(1+t)^{-\kappa}|\mathbf{\Phi}(R)|^2, \tag{23}$$

where $\kappa-$positive integer. Then the harmonic function E can be determined from the Dirichlet problem for the Laplace equation in the exterior of the sphere ∂B_R of radius R with constant boundary condition (23):

$$\Delta E(t, \mathbf{x}) = 0, \qquad E(t, \mathbf{x})\big|_{\partial B_R} = \frac{1}{2}(1+t)^{-\kappa}|\mathbf{\Phi}(R)|^2, \ \ \forall t \in (0, \infty).$$

It is known [8; 231] that the solution to this problem can be written with using the Poisson formula:

$$E(t, \mathbf{x}) = \frac{|\mathbf{\Phi}(R)|^2}{8\pi(1+t)^{\kappa}} \int\limits_{\partial B_R} \frac{\rho^2 - R^2}{Rr^3} d(\partial B_R), \ \ R < \rho < \infty. \tag{24}$$

Hence, since $\rho > R$ we find the function E

$$E(t, \mathbf{x}) = \frac{|\mathbf{\Phi}(R)|^2}{2(1+t)^{\kappa}} \frac{R}{\rho}, \qquad \rho = |\mathbf{x}|,$$

which is a continuous harmonic function of the form:

$$E(t, \mathbf{x}) = \begin{cases} \frac{|\mathbf{\Phi}(R)|^2}{2(1+t)^{\kappa}}, & \mathbf{x} \in \bar{B}_R, \\ \\ \frac{|\mathbf{\Phi}(R)|^2}{2(1+t)^{\kappa}} \frac{R}{\rho}, & \mathbf{x} \in R_3 \backslash B_R, \ \forall t \in (0, \infty), \end{cases}$$

which has continuous derivatives of all orders outside the sphere B_R.

Where from

$$\nabla E(t, \mathbf{x}) = \begin{cases} 0, & \mathbf{x} \in \bar{B}_R, \\ \\ c(t)\nabla(\frac{1}{\rho}), & \mathbf{x} \in R_3 \backslash B_R, \ \forall t \in (0, \infty), \end{cases} \tag{25}$$

where $c(t) = \frac{|\mathbf{\Phi}(R)|^2 R}{2(1+t)^{\kappa}}$.

Now from a non-linear system of Navier-Stokes equations (5), taking into account (25), we arrive at a linear system equations of parabolic type, i. e., to a system of disengaged heat equations with known right-hand sides

$$\mathbf{f}^{\mathrm{g}}(t,\mathbf{x})=\begin{cases}\mathbf{f}, \quad \mathbf{x}\in\bar{B}_R,\\ \\ \mathbf{f}+c\nabla(\frac{1}{\rho}),\ \mathbf{x}{\in}R_3\backslash B_R,\ \forall t\in(0,\infty),\end{cases}$$

then the Cauchy problem for the obtained systems of equations taking into account initial conditions (1b) can be written as:

$$\frac{\partial\mathbf{U}}{\partial t}-\mu\Delta\mathbf{U}=\mathbf{f}^{\mathrm{g}}(t,\mathbf{x}), \tag{26a}$$

$$\mathbf{U}(0,\mathbf{x})=\mathbf{\Phi}(\mathbf{x}). \tag{26b}$$

Where do we get the uniqueness solution to the problem (26), using Poisson formula obtained and justified by the Fourier transform for the heat equation, for example, in [8]:

$$U_\alpha(t,\mathbf{x})=(2\sqrt{\pi})^3\int_0^t\int_{R_3}\frac{1}{(t-\tau)^{\frac{3}{2}}}\exp\big(-\frac{r^2}{4\mu(t-\tau)}\big)f^{\mathrm{g}}_\alpha(\tau,\mathbf{y})d\mathbf{y}d\tau+$$

$$+\frac{1}{(2\sqrt{(\pi t)})^3}\int_{R_3}\exp\big(-\frac{r^2}{4\mu t}\big)\Phi_\alpha(\mathbf{y})d\mathbf{y},\ r=|\mathbf{x}-\mathbf{y}|,\ \alpha=1,2,3. \tag{27}$$

For $t>0$ the function $U_\alpha(t,\mathbf{x})$ is infinite differentiable with respect to t and spatial variables $\mathbf{x}$ and that all derivatives can be obtained using differentiation, the Poisson formula (27) under the sign integral. □

0.4 On the estimation of the curl vector of a problem (1)

Multiply the Navier-Stokes equations (1a) by $2\mathbf{U}$ and integrate over the domain B_R

$$\frac{d}{dt}\int_{B_R}|\mathbf{U}|^2d\mathbf{x}-2\mu\int_{B_R}(\Delta\mathbf{U},\mathbf{U})d\mathbf{x}-\int_{B_R}\mathbf{U}\nabla Ed\mathbf{x}=2\int_{B_R}\mathbf{U}\mathbf{f}d\mathbf{x}. \tag{28}$$

From where we transform the second term on the left with integration by parts

$$-2\mu\int_{B_R}(\Delta\mathbf{U},\mathbf{U})d\mathbf{x}=2\mu\int_{B_R}\sum_{\alpha=1}^{3}(\nabla U_\alpha)^2d\mathbf{x}-\mu\int_{\partial B_R}\frac{\partial}{\partial\mathbf{n}}\sum_{\alpha=1}^{3}U_\alpha^2d\mathbf{x}=$$

$$2\mu\int_{B_R}\sum_{\alpha=1}^{3}(\nabla U_\alpha)^2d\mathbf{x}-2\mu\int_{\partial B_R}\frac{\partial E}{\partial\mathbf{n}}d\mathbf{x}=2\mu\int_{B_R}\sum_{\alpha=1}^{3}(\nabla U_\alpha)^2d\mathbf{x},$$

because $\int_{\partial B_R}\frac{\partial E}{\partial\mathbf{n}}d\mathbf{x}=0$ by harmonic property functions E, where ∂B_R is the spherical surface of the ball B_R (imaginary, of course). Third term $\int_{B_R}\mathbf{U}\nabla Ed\mathbf{x}=0$ by virtue of orthogonality of spaces $\mathring{\mathbf{J}}(B_R)$ and $\mathbf{G}(B_R)$. The right side (28) is estimated by Cauchy-Bunyakovsky inequality and as a result we get:

$$\frac{d}{dt}\int_{B_R}|\mathbf{U}|^2d\mathbf{x}+2\mu\int_{B_R}\sum_{\alpha=1}^{3}(\nabla U_\alpha)^2d\mathbf{x}\le2\Big(\int_{B_R}|\mathbf{U}|^2d\mathbf{x}\Big)^{\frac{1}{2}}\Big(\int_{B_R}|\mathbf{f}|^2d\mathbf{x}\Big)^{\frac{1}{2}}. \tag{29}$$

From here

$$\frac{d}{dt}\int\limits_{B_R}|\mathbf{U}|^2d\mathbf{x}\le 2\Big(\int\limits_{B_R}|\mathbf{U}|^2d\mathbf{x}\Big)^{\frac{1}{2}}\Big(\int\limits_{B_R}|\mathbf{f}|^2d\mathbf{x}\Big)^{\frac{1}{2}}.$$

Whence it follows that

$$\frac{d}{dt}\parallel\mathbf{U}(t)\parallel_{L_2(B_R)}\le\parallel\mathbf{f}(t)\parallel_{L_2(B_R)}.$$

We integrate the last inequality ranging from 0 to t and will find

$$\sup_{t>0}\parallel\mathbf{U}(t)\parallel_{L_2(B_R)}\le\parallel\mathbf{\Phi}\parallel_{L_2(B_R)}+\sup_{t>0}\parallel\mathbf{f}(t)\parallel_{L_2(B_R)}\equiv A_4. \tag{30}$$

Now, integrating (29) over $t\in(0,\infty)$ and taking into account (30), we find

$$\int\limits_0^t\sum_{\alpha=1}^3\|\nabla U_\alpha(\tau)\|^2_{L_2(B_R)}d\tau\le\frac{1}{\mu}\Big(\parallel\mathbf{\Phi}\parallel^2_{L_2(B_R)}+A_4\sup_{t>0}\|\mathbf{f}(t)\|_{L_2(B_R)}\Big). \tag{31}$$

Lemma 3. Occurs

$$\|rot\mathbf{U}(t)\|^2_{L_2(B_R)}=\sum_{\alpha=1}^3\|\nabla U_\alpha(t)\|^2_{L_2(B_R)},\forall t\in(0,\infty). \tag{32}$$

Proof. Follows from identity

$$\sum_{\alpha=1}^3\frac{\partial\mathbf{U}}{\partial x_\alpha}\nabla U_\alpha=\sum_{\alpha=1}^3(\nabla U_\alpha)^2-(rot\mathbf{U})^2.$$

It suffices to integrate this identity over the domain B_R with orthogonality of spaces $\mathring{\mathbf{J}}(B_R)$ and $\mathbf{G}(B_R)$. From (31), using (32),we obtain an estimate for the curl vector

$$\int\limits_0^t\|rot\mathbf{U}(\tau)\|^2_{L_2(B_R)}d\tau\le Re\Big(\parallel\mathbf{\Phi}\parallel^2_{L_2(B_R)}+A_4\sup_{t>0}\|\mathbf{f}(t)\|_{L_2(B_R)}\Big). \tag{33}$$

where $Re-$ is the Reynolds number. Hence it is not difficult to notice that with the $Re\to\infty$ curl vector is destroyed. □

References

1 Ladyzenskaja, O. (1969). *The Mathematical Theory of Viscous Incompressible Flows.* (2d ed.). Gordon and Breach, New York,

2 Ladyzhenskaya, O.A. (2003). The 6th problem of millennium: the Navier-Stokes equation, existence and smoothness. *UMN,* ***58***:*2(350),* 45–78.

3 Fefferman, Ch. (2000). Existence and smoothness of the Navier-Stokes equation. *claymath.org.* Retrieved from http://claymath. org/Millenium Prize Problems/Navier-Stokes Equations. Cambridge, MA:Clay Mathematics Institute.

4 Akysh (Akishev), A.Sh. (2016). The simplest maximum principle for Navier-Stokes equations. *Bulletin of the Karagandy University, No. 2(83),* 8–11 [in Russian].

5 Akysh (Akishev), A.Sh. (2018). *T*he natural solvability of the Navier-Stokes equations. *Bulletin of the Karagandy University, No. 4(92),* 8–14. Retrieved from http:www.ksu.kz.

6 Akysh (Akishev), A.Sh. (2019).Splitting methods and the existence of a strong solution of the Navier-Stokes equations. *Bulletin of the Karagandy University, No. 2(94),* 8–14. Retrieved from http:www.ksu.kz

7 Sobolev, S.L. (1950). *Some applications of functional analysis in mathematical physics. Publishing House of Leningrad State University.*

8 Mikhlin, S.G. (1977). *Linear partial differential equations Moscow. Higher school.*

9 Courant, R. (1962). *Partial Differential Equations.* New York. London.

Ә.Ш. Ақыш (Ақышев)

Навье-Стокс теңдеулеріне Коши есебі

Ch. Feffermann жұмыстарында Навье-Стокс теңдеулеріне (НСТ) екі есеп қойылған, оның біреуі Коши есебі және ол «физикалық тұрғыдан ойластырылған тек шексіз тегіс функциялар болып табылатын шешімдер» деп тұжырымдайды. Автордың осы мақаласында Ch. Feffermannның жоғарыдағы аталған есебіне оң жауап алынған. Навье-Стокс теңдеулері үшін Коши есебінің жалқы шексіз тегіс шешуінің барлығы дәлелденген. Нәтижесінде автордың ертеректе көрсеткен, қысым мен кинетикалық энергияның арасындағы байланысқа негізделген. Навье-Стокс теңдеулері үшін Коши есебін тереңірек зерттеу нәтижесінде E функциясының тұйық үзіліссіздігі Лаплас теңдеуін қанағаттандырады және t бойынша бірінші, ал кеңістік айнымалылары $\mathbf{x}$ бойынша екінші туындыларының барлығының үзіліссіздігі көрсетіліп және R_3 кеңістігінде регуляр гармониялық функция екендігі көрсетілген. $E-$нің айқын түрі табылып, оның көмегімен Навье-Стокс теңдеулері жылдамдық векторының құраушылары бойынша сызықты параболалық теңдеулерге келтіріліп, Фурье түрлендіруінің әдісімен есептің t және кеңістік айнымалылары $\mathbf{x}$ бойынша шексіз тегіс дәл шешуі табылған. Құйын векторының Рейнольдс санымен байланыстыратын бағалау алынған.

Кілт сөздер: Навье-Стокс теңдеулері үшін Коши есебі, кинетикалық энергия тығыздығының R_3 кеңістігінде регуляр гармониялығы, құйын векторын Рейнольдс санымен байланыстыратын бағалау.

А.Ш. Акыш (Акишев)

Задача Коши для уравнений Навье-Стокса

В работах Ch. Feffermana ставятся две задачи для уравнений Навье-Стокса: одной из них является задача Коши, и он считает «физически осмысленными только те решения, которые являются бесконечно гладкими функциями». В данной статье автор получил положительные ответы на упомянутую выше задачу Ch. Feffermana. Им доказаны единственность и существование гладких решений задачи Коши для уравнений Навье-Стокса. За основу взято соотношение между давлением P и плотностью кинетической энергии E, ранее установленное автором. В результате углубленных исследований задачи Коши для уравнений Навье-Стокса показано, что E — ограниченная, непрерывная функция, удовлетворяющая уравнению Лапласа, имеющая непрерывные производные первого порядка по t и всевозможные вторые производные по пространственным переменным $\mathbf{x}$ и являющаяся регулярной гармонической функцией в пространстве R_3. Найден явный вид E, с помощью которого уравнения Навье-Стокса сведены к системе линейных параболических уравнений и выписаны решения преобразованием Фурье, бесконечно дифференцируемые по t и $\mathbf{x}$. Получена оценка, связывающая векторвихря с числом Рейнольдса.

Ключевые слова: задача Коши для уравнений Навье-Стокса, гармоничность плотности кинетической энергии, единственность и существование гладких решений уравнений Навье-Стокса, оценка, связывающая векторвихря с числом Рейнольдса.

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The first with displacement problem for a third-order parabolic-hyperbolic equation and the effect of inequality of characteristics as data carriers of the Tricomi problem

As part of this scientific work, we study a displacement boundary value problem for a third-order parabolic-hyperbolic type equation with a third-order parabolic equation backward in time and a wave equation in the domain of hyperbolicity. As one of the boundary conditions we have a linear combination including variable coefficients of the sought function on the characteristic lines AC and BC. The present paper reports following results: inequality between characteristics of AC and BC lines limiting the hyperbolic part Ω_1 of the domain Ω as carriers of data for the Tricomi problem as $0 \leq x \leq 2\pi$, as a matter of fact, the solvability of the Tricomi problem with data on the characteristic line BC does not imply the solvability of the Tricomi problem with data on the AC; necessary and sufficient conditions for the existence and uniqueness of a regular solution to the problem under study are found. Under certain conditions for the given functions, the solution to the problem under study is written out explicitly. It is shown that under violation of the necessary conditions established in this paper the homogeneous problem has innumerable linearly independent solutions, while the set of solutions to the corresponding inhomogeneous problem can exist only with additional conditions.

Keywords: mixed type equation, third-order parabolic-hyperbolic equation, Tricomi problem, Tricomi method, first with displacement problem, Green's function, Fredholm's integral equation of the second kind.

Problem Statement. Results Summary

In a Euclidean plane with independent variables $x$and y consider the equation

$$0 = \begin{cases} u_{xx} - u_{yy} - f_1, & y < 0, \\ u_{xxx} + u_y - f_2, & y > 0, \end{cases} \tag{1}$$

where $f_1 = f_1(x, y)$, $f_2 = f_2(x, y)$ – are specified functions, $u = u(x, y)$ – is a sought function.

Equation (1) as $y < 0$ coincides with the inhomogeneous wave equation

$$u_{xx} - u_{yy} = f_1(x, y), \tag{2}$$

while as $y > 0$ it coincides with the backward in time nonhomogeneous equation

$$u_{xxx} + u_y = f_2(x, y), \tag{3}$$

of the third order with multiple characteristics [1; 9] of the parabolic type [2; 72].

Equation (1) is considered in the domain Ω, bounded by characteristic lines $AC: \ x + y = 0$ and $CB: \ x - y = 2\pi$ of equation (2) as $y < 0$ starting at the point $C = (\pi, -\pi)$ and passing through the points $A = (0, 0)$ and $B = (2\pi, 0)$ respectively, and by a rectangle with vertices at A, B, $A_0 = (0, h)$, $B_0 = (2\pi, h)$, $h > 0$, as $y > 0$. Denote $\Omega_1 = \Omega \cap \{y < 0\}$, $\Omega_2 = \Omega \cap \{y > 0\}$, $J = \{(x, 0): \ 0 < x < r\}$, $\Omega = \Omega_1 \cup \Omega_2 \cup J$.

Assume that a *regular* solution to equation (1) in the domain Ω is the function $u = u(x, y)$ of the class $C(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega_1) \cap C_x^3(\Omega_2)$, $u_x, \ u_y \in L_1(J)$ satisfying equation (1).

Problem 1. Find a regular solution to equation (1) in the domain Ω satisfying the conditions

$$u(0,y)=\varphi_1(y),\ u(2\pi,y)=\varphi_2(y),\ u_x(2\pi,y)=\varphi_3(y),\quad 0\le y<h, \tag{4}$$

$$\alpha(x)\,u[\theta_0(x)]+\beta(x)\,u[\theta_\pi(x)]=\psi(x),\quad 0\le x\le 2\pi, \tag{5}$$

where $\theta_0(x)=\left(\frac{x}{2};\ -\frac{x}{2}\right)$, $\theta_\pi(x)=\left(\frac{x}{2}+\pi;\ \frac{x}{2}-\pi\right)$ – are the points for intersection of the characteristic lines of equation (2), starting at $(x,0)$ with characteristics of AC and BC respectively; $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$; $\alpha(x)$, $\beta(x)$, $\psi(x)$ – are the specified functions and what is more $\alpha^2(x)+\beta^2(x)\ne 0\ \ \forall\ x\in[0,2\pi]$.

Formulated problem (1), (4), (5) belongs to the class of A.M. Nakhushev nonlocal boundary value problems with displacement [3].

For the first time, a problem with a boundary condition relating the values of the desired function on two characteristic lines from different families in the hyperbolic part of the domain for the Lavrentiev-Bitsadze equation was formulated and studied in [4].

The concept for a boundary value problem with displacement was introduced in [5], [6], and a number of nonlocal boundary value problems with various types of displacements were studied for hyperbolic, degenerate hyperbolic, and mixed type equations. In particular, the posing of the first Darboux problem for wave equation (2) with an initial condition

$$u(x,0)=\tau(x),\quad 0\le x\le 1 \tag{6}$$

and non-local condition (5) was generalized in [5]. It was shown that the conditions: $\alpha^2(1)+\beta^2(0)\ne 0$, $\alpha(x)\ne\beta(x)\ \ \forall\ x\in[0,1]$, $\alpha(x),\beta(x),\tau(x),\psi(x)\in C[0,1]\cap C^2\,]0,1[$ for the given functions $\alpha(x)$, $\beta(x)$, $\tau(x)$, $\psi(x)$ ensures the correctness of the investigated problem with displacement.

In [6], a method of posing of the nonlocal displacement boundary value problems for a degenerate hyperbolic equation of the form

$$(-y)^m u_{xx}-u_{yy}=0,\quad m=const>0 \tag{7}$$

with the Riemann-Liouville fractional operator. Criteria were found for the unique solvability of the problem with conditions (6) and

$$\alpha(x)\,D_{0x}^{1-\varepsilon}u[\theta_0(x)]+\beta(x)\,D_{xr}^{1-\varepsilon}u[\theta_r(x)]=\psi(x),\quad 0<x<r$$

for equation (7), where $\theta_0(x)$, $\theta_r(x)$ were defined as the intersection points of the characteristic lines of equation (7), as above, and what is more $2(m+2)\varepsilon=m$.

In [7], the first and second Darboux problems were studied for the class of degenerate hyperbolic equations. Sufficient conditions for the given functions providing solvability to the problems were established. It was also shown that the Darboux problem with the following data:

$$u_y(x,0)=\nu(x),\ u(x,y)|_{AC}=\psi(x),\quad 0<x<r$$

is well posed for the equation

$$y^2u_{xx}-u_{yy}+u_x=0, \tag{8}$$

considered in the domain D, bounded by the characteristic lines

$$AC:\ 2x-y^2=0,\ \ BC:\ 2x+y^2=2r,\quad 0\le x\le r$$

and the segment $I\equiv AB$ of the straight line $y=0$.

$$u_y(x,0)=\nu(x),\ \ u(x,y)|_{AC}=\psi(x),\quad 0<x<r.$$

At the same time, the homogeneous Darboux problem for equation (8) with

$$u_y(x,0)=0,\ \ u(x,y)\,|_{BC}\ =0,\quad 0<x<r$$

has nonzero solutions of the form $u(x,\,y)=g\left(x+\frac{1}{2}\,y^2\right)-g(r)$, where $g=g(x)$ is an arbitrary function of the class $g(x)\in C^1\left[\frac{r}{2},\,r\right]\cap C^2\left]\frac{r}{2},\,r\right[$, which indicates the inequality between characteristic lines AC and BC as data carriers of the second Darboux problem for the equation (8) .

The displacement boundary value problems have found their important application in mathematical modeling of biological processes, and transonic gas dynamics. Similar nonlocal boundary conditions arise in the study of heat and mass transfer in capillary-porous bodies, in the mathematical modeling of gas dynamics and nonlocal physical processes, in the study of cell propagation processes, in the theory of electromagnetic field propagation into inhomogeneous medium [2, 8, 9]. Comprehensive bibliographies of scientific literature devoted to the study of boundary value problems with displacements is presented in [3], [10–18], as well as in thesis [19–23].

The displacement boundary value problem with condition (5) for a second-order parabolic-hyperbolic type equation with a heat equation in the parabolic domain was studied in [24]; also a necessary and sufficient condition for the existence of a unique regular solution to the problem under study was found.

In this paper, we study a displacement boundary value problem for a third-order parabolic-hyperbolic type equation (1) with a third-order parabolic equation backward in time and a wave equation in the domain of hyperbolicity. As one of the boundary conditions we have a linear combination including variable coefficients of the sought function on the characteristic lines AC and BC. The present paper reports following results: inequality between characteristics of AC and BC lines limiting the hyperbolic part Ω_1 of the domain Ω as carriers of data for the Tricomi problem as $0\le x\le 2\pi$. As a matter of fact, the solvability of the Tricomi problem with data on the characteristic line BC does not imply the solvability of the Tricomi problem with data on the AC. Necessary and sufficient conditions for the existence and uniqueness of a regular solution to the problem under study are found. Under certain conditions for the given functions, the solution to the problem under study is written out explicitly. It is shown that under violation of the necessary conditions established in this paper the homogeneous problem has innumerable linearly independent solutions, while the set of solutions to the corresponding inhomogeneous problem can exist only with additional conditions. Among the works closely related to our research there are [25–31].

Problem 1 as $\alpha(x)\equiv 0$

The study of problem 1 we begin as $\alpha(x)\equiv 0$, $\beta(x)\ne 0\ \forall\ x\in[0,2\pi]$. The following theorem is true.

Theorem 1. Assume that for the given functions $\alpha(x)$, $\beta(x)$, $\psi(x)$, $\varphi_i(y)$, $i=\overline{1,3}$, $f_1(x,y)$ and $f_2(x,\,y)$

$$\alpha(x)\equiv 0,\ \beta(x)\ne 0\quad \forall\ x\in[0,2\pi]\,, \tag{9}$$

$$\beta(x)\,,\ \psi(x)\in C^1\,[0,2\pi]\,, \tag{10}$$

$$\varphi_1(y)\,,\ \varphi_2(y)\,,\ \varphi_3(y)\in C\,[0,h]\,, \tag{11}$$

$$f_1(x,y)\in C\left(\bar\Omega_1\right),\ f_2(x,y)\in C\left(\bar\Omega_2\right). \tag{12}$$

be satisfied.

Therefore there exists a unique solution to problem 1.

Indeed, let there exist a solution to problem (1), (4), (5) and let

$$u(x,0)=\tau(x)\,,\quad 0\le x\le 2\pi;\ \ u_y(x,0)=\nu(x)\,,\quad 0<x<2\pi. \tag{13}$$

Passing to the limit as $y \to +0$ in the equation (1), accepting notation (13), we obtain the first basic relation between $\tau(x)$ and $\nu(x)$, transferred from the parabolic domain Ω_2 to the line $y=0$:

$$\tau'''(x) + \nu(x) = f_2(x,0), \quad 0 < x < 2\pi. \tag{14}$$

Employing boundary conditions (4) as $y \to +0$ we can get

$$\tau(0) = \varphi_1(0), \ \tau(2\pi) = \varphi_2(0), \ \tau'(2\pi) = \varphi_3(0). \tag{15}$$

Next we can find basic relation between $\tau(x)$ and $\nu(x)$, transferred from the hyperbolic part Ω_1 of the domain Ω to the line of the type changing $y=0$. Let condition (9) of Theorem 1 be satisfied. In this case, the studied problem (1), (4), (5) becomes one of the analogues of the Tricomi problem for equation (1) with (4) and

$$u(x,y)|_{BC} = u[\theta_\pi(x)] = \frac{\psi(x)}{\beta(x)}, \quad 0 \le x \le 2\pi. \tag{16}$$

To find the relationship between $\tau(x)$ and $\nu(x)$ let us use representation of the solution to problem (13) for equation (2) [32; 59]:

$$u(x,y) = \frac{\tau(x+y) + \tau(x-y)}{2} + \frac{1}{2}\int\limits_{x-y}^{x+y} \nu(t)\,dt - \frac{1}{2}\int\limits_0^y \int\limits_{x-y+s}^{x+y-s} f_1(t,s)\,dtds. \tag{17}$$

By formulae (17) we can find:

$$u[\theta_\pi(x)] = u\left(\frac{x+2\pi}{2}, \frac{x-2\pi}{2}\right) = \frac{\tau(2\pi)+\tau(x)}{2} - \frac{1}{2}\int\limits_x^{2\pi} \nu(t)\ dt + \frac{1}{2}\int\limits_{\frac{x}{2}-\pi}^{0} \int\limits_{2\pi+s}^{x-s} f_1(t,\ s)\,dt\,ds. \tag{18}$$

Substituting value $u[\theta_\pi(x)]$ from (18) into (16) we can get

$$\tau(2\pi) + \tau(x) - \int\limits_x^{2\pi} \nu(t)\ dt + \int\limits_{\frac{x}{2}-\pi}^{0} \int\limits_{2\pi+s}^{x-s} f_1(t,\ s)\,dt\,ds = \frac{2\,\psi(x)}{\beta(x)},$$

hence

$$\nu(x) = -\tau'(x) + \left(\frac{2\psi(x)}{\beta(x)}\right)' - \int\limits_{\frac{x}{2}-\pi}^{0} f_1(x-s,s)\,ds. \tag{19}$$

Relation (19) is the basic relation between the sought functions $\tau(x)$ and $\nu(x)$ transferred from the hyperbolic part Ω_1 of the domain Ω to the line $y=0$ of the type changing of equation (1) as $\alpha(x) \equiv 0$, $\beta(x) \ne 0 \quad \forall\, x \in [0, 2\pi]$.

By relations (14) and (19) for the sought function $\tau = \tau(x)$ we arrive at the finding a solution to the equation

$$\tau'''(x) - \tau'(x) = f_2(x,0) + \int\limits_{\frac{x}{2}-\pi}^{0} f_1(x-s,s)\,ds - \left(\frac{2\,\psi(x)}{\beta(x)}\right)', \quad 0 < x < 2\pi, \tag{20}$$

satisfying conditions (15). A solution to problem (20), (15), under conditions (9)–(12) for given functions exists, is unique, and can be written out by the formula:

$$\tau(x) = \frac{1}{4\pi^2}\left[(2\pi - x)^2 + 2\int_0^{2\pi} G(x,\, t)\ (t - 2\pi)\, dt\right]\varphi_1(0) +$$

$$+\frac{1}{4\pi^2}\left[x(4\pi - x) - 2\int_0^{2\pi} G(x,t)(t - 2\pi)\, dt\right]\varphi_2(0) +$$

$$+\frac{1}{2\pi}\left[x(x - 2\pi) + 2\int_0^{2\pi} G(x,t)(t - \pi)\, dt\right]\varphi_3(0) + \int_0^{2\pi} G(x,t)\, f_2(t,0)\, dt +$$

$$+\int_0^{2\pi} G(x,t)\int_{\frac{t}{2}-\pi}^{0} f_1(t - s,\, s)\, ds\, dt - 2\int_0^{2\pi} G(x,t)\left(\frac{\psi(t)}{\beta(t)}\right)' dt,$$

where $G(x,t) = -\frac{1}{1-ch(2\pi)}\begin{cases}(1 - ch\, t)(1 - ch\,(2\pi - x)) - (1 - ch\,(x - t))(1 - ch\,(2\pi)), & 0 \le x < t,\\ (1 - ch\, t)(1 - ch\,(2\pi - x)), & t < x \le 2\pi.\end{cases}$

Problem 1 as $\beta(x) \equiv 0$

Assume that further specified functions $\alpha(x)$ and $\beta(x)$ are such that

$$\beta(x) \equiv 0,\ \alpha(x) \ne 0 \quad \forall\, x \in [0, 2\pi]. \tag{21}$$

The following theorem is true.

Theorem 2. Let condition (21) *be satisfied for the given functions* $\alpha(x)$ *and* $\beta(x)$, *and* $\alpha(x) \in C^1[0,\, 2\pi]$. *This implies that the homogeneous problem corresponding to the problem under study 1 has innumerable linearly independent solutions, while the inhomogeneous problem* (1), (4), (5) *is solvable if and only if the additional condition is satisfied.*

Indeed let condition (21) be satisfied for $\alpha(x)$ and $\beta(x)$. Then problem 1 becomes the Tricomi problem for equation (1) with conditions (4) and

$$u(x,y)\,|_{AC} = u[\theta_0(x)] = \frac{\psi(x)}{\alpha(x)}, \quad 0 \le x \le 2\pi. \tag{22}$$

By (17) with condition (22) we can find:

$$u[\theta_0(x)] = u\left(\frac{x}{2}, -\frac{x}{2}\right) = \frac{\tau(0) + \tau(x)}{2} + \frac{1}{2}\int_x^0 \nu(t)\, dt - \frac{1}{2}\int_0^{-x/2}\int_{x+s}^{-s} f_1(t,\, s)\, dt ds. \tag{23}$$

Substituting value $u[\theta_0(x)]$ from (23) into condition (22) using differentiation, we arrive at a fundamental relation

$$\nu(x) = \tau'(x) - \int_{-x/2}^{0} f_1(x + s,\, s)\, ds - 2\left(\frac{\psi(x)}{\alpha(x)}\right)'. \tag{24}$$

Taking out the function $\nu(x)$ from (14) and (24) we can find a solution to the equation

$$\tau'''(x) + \tau'(x) = F(x), \tag{25}$$

satisfying conditions (15), where $F(x) = f_2(x,0) + \int\limits_{-x/2}^{0} f_1(x+s,s)\,ds + 2\left(\frac{\psi(x)}{\alpha(x)}\right)'$.

Problem (25), (15) corresponds to the homogeneous problem

$$\tau'''(x) + \tau'(x) = 0, \tag{26}$$

$$\tau(0) = 0,\ \tau(2\pi) = 0,\ \tau'(2\pi) = 0. \tag{27}$$

The homogeneous problem (26), (27) corresponding to problem (25), (15) has the nonzero solution

$$\tau(x) = c(1 - \cos x), \quad c = const.$$

The solution to the inhomogeneous problem (25), (15) in this case exists only under the additional condition for the given functions

$$\varphi_2(0) - \varphi_1(0) = \int\limits_0^{2\pi} (1 - \cos t)\,F(t)\,dt. \tag{28}$$

If condition (28) is satisfied, then the set of solutions to problem (25), (15) is written out by the formula:

$$\tau(x) = \int\limits_0^x [1 - \cos x \cos t]\,F(t)\,dt + \int\limits_x^{2\pi} \sin x\,\sin t\,F(t)\,dt + \varphi_1(0)\cos x + \varphi_3(0) + c(1 - \cos x).$$

It follows from the above that the characteristic lines AC and BC limiting the hyperbolic part Ω_1 of the domain Ω are not equal as carriers of data for the Tricomi problem as $0 \le x \le 2\pi$. And generally speaking, the solvability of the Tricomi problem with data on the characteristic line BC does not imply the solvability of the Tricomi problem with data on AC.

Mean value theorem

Now find in general the basic relationship between $\tau(x)$ and $\nu(x)$ transferred from the hyperbolic part Ω_1 of the domain Ω on the line of type changing $y = 0$. For this purpose, prove the following lemma (theorem) on the mean value for an inhomogeneous one-dimensional wave equation (2).

Lemma **1**. *Any regular solution to equation* (2) *satisfying the condition $u(x,0) = \tau(x)$ possesses the following property*

$$u[\theta_0(x)] + u[\theta_\pi(x)] = u(x,0) + u(\pi,-\pi) + \frac{1}{2}\int\limits_{-\pi}^{0}\int\limits_{-s}^{2\pi+s} f_1(t,s)\,dtds -$$

$$-\frac{1}{2}\int\limits_{\frac{x}{2}-\pi}^{0}\int\limits_{x-s}^{2\pi+s} f_1(t,s)\,dtds - \frac{1}{2}\int\limits_{-x/2}^{0}\int\limits_{-s}^{x+s} f_1(t,s)\,dtds. \tag{29}$$

Indeed, taking into account formulas (18) and (23), we find

$$u[\theta_0(x)] + u[\theta_\pi(x)] = \tau(x) + \frac{\tau(0) + \tau(2\pi)}{2} - \frac{1}{2}\int\limits_0^{2\pi} \nu(t)\,dt -$$

$$-\frac{1}{2}\int\limits_{0}^{\frac{x}{2}-\pi}\int\limits_{2\pi+s}^{x-s} f_1(t,s)\,dtds-\frac{1}{2}\int\limits_{0}^{-x/2}\int\limits_{x+s}^{-s} f_1(t,s)\,dtds. \tag{30}$$

By (17) as $(x,y)=(\pi,-\pi)$ it is easy to show that

$$\frac{\tau(0)+\tau(2\pi)}{2}-\frac{1}{2}\int\limits_{0}^{2\pi}\nu(t)\,dt=u(\pi,-\pi)+\frac{1}{2}\int\limits_{0}^{-\pi}\int\limits_{2\pi+s}^{-s} f(t,s)\,dtds. \tag{31}$$

By (30) and (31) we arrive at (29).

Now we employ formula (29) to take forward steps. Find the value of $u(\pi;-\pi)$. Using boundary condition (5) as x = 0 and in view of the first condition of (15), we find

$$\alpha(0)\varphi_1(0)+\beta(0)u(\pi;-\pi)=\psi(0),$$

whence as $\beta(0)\neq 0$ find

$$u(\pi;-\pi)=\frac{\psi(0)-\alpha(0)\varphi_1(0)}{\beta(0)}. \tag{32}$$

Similarly as $x=2\pi$ and $\alpha(2\pi)\neq 0$ by (5) and (15) find

$$u(\pi;-\pi)=\frac{\psi(2\pi)-\beta(2\pi)\varphi_2(0)}{\alpha(2\pi)}. \tag{33}$$

Thus, if $\alpha^2(2\pi)+\beta^2(0)\neq 0$, the value of the sought function $u(x,y)$ at the point $C=(\pi;-\pi)$ is found by formulas (32) or (33). Assume, for example, that $\alpha(2\pi)\neq 0$. Therefore, equality (29) can be rewritten as follows

$$u[\theta_0(x)]+u[\theta_\pi(x)]=\tau(x)+F_1(x), \tag{34}$$

where $F_1(x)=\frac{\psi(2\pi)-\beta(2\pi)\varphi_2(0)}{\alpha(2\pi)}+\frac{1}{2}\left(\int\limits_{-\pi}^{0}\int\limits_{-s}^{2\pi+s}-\int\limits_{\frac{x}{2}-\pi}^{0}\int\limits_{x-s}^{2\pi+s}-\int\limits_{-x/2}^{0}\int\limits_{-s}^{x+s}\right)f_1(t,s)\,dtds.$

Problem 1 as $\alpha(x)\equiv\beta(x)$

Next consider the case as $\alpha(x)\equiv\beta(x)\ \forall x\in[0,r]$. The following theorem is true.

Theorem 3. Let the given functions $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$; $\alpha(x)$, $\beta(x)$, $\psi(x)$; $f_1(x,y)$, $f_2(x,y)$ be such that:

$$\alpha(x)\equiv\beta(x)\neq 0,\quad \forall\, x\in[0,2\pi], \tag{35}$$

$$\alpha^2(0)+\alpha^2(2\pi)\neq 0; \tag{36}$$

$$\varphi_1(y),\ \varphi_2(y),\ \varphi_3(y)\in C[0,h]\cap C^1\,]0,h[\,; \tag{37}$$

$$\alpha(x),\ \psi(x)\in C^1[0,2\pi]\cap C^3\,]0,2\pi[\,; \tag{38}$$

$$f_1(x,y)\in C^1\left(\bar{\Omega}_1\right),\ f_2(x,y)\in C\left(\bar{\Omega}_2\right). \tag{39}$$

Then there exists a unique solution to Problem 1 that is regular in the domain Ω.

Indeed, by (5) in view of (34) and (35) find

$$\tau(x)=\frac{\psi(x)}{\alpha(x)}-F_1(x).$$

Whence under conditions (38), (39) by (14) we have

$$\nu(x) = f_2(x,0) - \tau'''(x) = f_2(x,0) - \left(\frac{\psi(x)}{\alpha(x)} - F_1(x)\right)'''.$$

With values found for $\tau(x)$ and $\nu(x)$ the solution to the initial problem (1), (4), (5) in the domain Ω_1 is written out by formula (17). While the solution to the boundary value problem in the domain Ω_2 for equation (3) with boundary conditions (4) and initial condition $u(x,0) = \tau(x)$ is written out as below:

$$u(x,y) = \frac{1}{\pi}\left\{\int_0^y G(x,-y;\,0,-\eta)\,\varphi_3(\eta)\,d\eta - \int_0^y G_{\xi\xi}(x,-y;\,0,-\eta)\,\varphi_1(\eta)\,d\eta + \right.$$

$$\left. + \int_0^y G_{\xi\xi}(x,-y;\,r,-\eta)\,\varphi_2(\eta)\,d\eta + \int_0^{2\pi} G(x,-y;\,\xi,0)\,\tau(\xi)\,d\xi + \int_0^y\int_0^r G(x,-y;\,\xi,-\eta)\,f(\xi,\eta)\,d\xi d\eta\right\}, \tag{40}$$

where $G(x,y;\,\xi,\eta) = U(x,y;\,\xi,\eta) - W(x,y;\,\xi,\eta) -$ Green's function of the operator, $U(x,y;\,\xi,\eta)$ and $W(x,y;\,\xi,\eta)$ are fundamental solutions to equation (2) [1; 135].

Thus, in contrast to the problem with conditions (5) *and for a strictly hyperbolic equation* (2) *[6] the problem with displacement* (4) - (5) *for equation* (1) *is uniquely solvable even as* $\alpha(x) \equiv \beta(x) \neq 0$ $\forall\, x \in [0,2\pi]$ *with the functions* $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$; $\alpha(x)$, $\beta(x)$, $\psi(x)$; $f_1(x,y)$, $f_2(x,y)$ *possessing properties* (36)–(39).

Problem 1, general case

Further assume that $\alpha(x) \neq \beta(x)\ \forall x \in [0,2\pi]$. The following uniqueness theorem holds for a regular solution to the problem (1), (4), (5).

Theorem 4. Let the following conditions:

$$\alpha(x),\,\beta(x) \in C^1[0,2\pi] \tag{41}$$

$$\alpha^2(x) + \beta^2(x) \neq 0 \quad \forall\, x \in [0,2\pi], \tag{42}$$

$$\alpha^2(2\pi) + \beta^2(0) \neq 0, \tag{43}$$

$$\alpha(x) \neq \beta(x) \quad \forall\, x \in [0,2\pi] \tag{44}$$

$$\left[\frac{\alpha(x)+\beta(x)}{\alpha(x)-\beta(x)}\right]' > 0 \quad \forall\, x \in [0,2\pi]. \tag{45}$$

be satisfied for the given functions $\alpha(x)$ *and* $\beta(x)$.

Then the solution to problem 1 is unique within the required class.

Proof. Under condition (44) of (5) and (34) arrive at the following system of linear algebraic equations

$$\begin{cases} u[\theta_0(x)] + u[\theta_\pi(x)] = \tau(x) + F_1(x), \\ \alpha(x)\,u[\theta_0(x)] + \beta(x)\,u[\theta_\pi(x)] = \psi(x) \end{cases} \tag{46}$$

for the unknown $u[\theta_0(x)]$ and $u[\theta_\pi(x)]$. When solving (46) we find that

$$u[\theta_0(x)] = \frac{\beta(x)}{\beta(x)-\alpha(x)}\tau(x) + \frac{\beta(x)\,F_1(x) - \psi(x)}{\beta(x)-\alpha(x)}. \tag{47}$$

On the other hand, by formula (17)

$$u\left[\theta_0(x)\right]=\frac{\varphi_1(0)+\tau(x)}{2}-\frac{1}{2}\int\limits_0^x \nu(s)\,ds+\frac{1}{2}\int\limits_{-x/2}^{0}\int\limits_{-t}^{x+t} f_1(s,t)\,dsdt. \tag{48}$$

Substituting the value of $u\left[\theta_0(x)\right]$ (48) into (47), and differentiating the resulting equality, we arrive at

$$\nu(x)=\left[a(x)\,\tau(x)\right]'-F_2(x), \tag{49}$$

where $a(x)=\frac{\alpha(x)+\beta(x)}{\alpha(x)-\beta(x)}$, $F_2(x)=\int\limits_{-x/2}^{0} f_1(x+t,t)\,dt+2\left[\frac{\beta(x)F_1(x)-\psi(x)}{\alpha(x)-\beta(x)}\right]'$.

Relation (49) is the basic relation between $\tau(x)$ and $\nu(x)$ when conditions (41)-(44) are met.

For the homogeneous problem $\left(\varphi_i(y)=f_j(x,y)=\psi(x)\equiv 0,\ i=\overline{1,3},\ j=\overline{1,2}\right)$ corresponding to the initial problem (1), (4), (5) consider the integral

$$J=\int\limits_0^{2\pi}\tau(x)\,\nu(x)\,dx.$$

In view of relation (14) under conditions (15) we have that the integral in question

$$J=\int\limits_0^{2\pi}\tau(x)\,\nu(x)\,dx=-\int\limits_0^{2\pi}\tau(x)\,\tau'''(x)\,dx=-\frac{1}{2}\left[\tau'(0)\right]^2\leq 0. \tag{50}$$

And in view of relation (49) we have

$$J=\int\limits_0^{r}\tau(x)\,\nu(x)\,dx=\int\limits_0^{r}\tau(x)\left[a(x)\,\tau(x)\right]'dx=\frac{1}{2}\int\limits_0^{r}a'(x)\,\tau^2(x)\,dx. \tag{51}$$

Provided that conditions (41)-(45) of Theorem 2 are satisfied, by inequalities (50) and (51) we have that $\tau(x)\equiv 0$. Moreover, by relations (14) or (49) we have that $\nu(x)\equiv 0$. Then by formula (17) we can conclude $u(x,y)\equiv 0$ in Ω_1, while by (40) $u(x,y)\equiv 0$ in Ω_2. Thus, it is shown that the homogeneous problem corresponding to (1), (4), (5) under the conditions of Theorem 2 has only a trivial solution $u(x,y)\equiv 0$ in Ω that implies the uniqueness of a regular solution to the investigated *problem 1.*

Theorem 5. Let the conditions (11), (12), (41), (42), (43), (44), (45) *be satisfied for the given functions* $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$; $\alpha(x)$, $\beta(x)$, $\psi(x)$; $f_1(x,y)$, $f_2(x,y)$ *and let:*

$$\psi(x)\in C^1\left[0,2\pi\right]. \tag{52}$$

Then there is a regular solution to Problem 1.

Proof. To prove Theorem 5 return again to relations (14), (15) and (49). Take out from (14) and (49) the sought function$\nu(x)$, and find for $\tau(x)$ a solution to the ordinary second-order differential equation of the form

$$\tau'''(x)+a(x)\,\tau'(x)+a'(x)\,\tau(x)=f_2(x,0)+F_2(x),\quad 0<x<2\pi, \tag{53}$$

satisfying conditions (15).

By integrating equation (53) three times from x до to 2π we arrive at the integral equation

$$\tau(x) - \frac{1}{4\pi^2}\int\limits_0^{2\pi} K(x,t)\ a(t)\ \tau(t)\,dt = F_3(x), \tag{54}$$

where $K(x,t) = \begin{cases} (2\pi - x)^2 t - 4\pi^2 (t - x), & 0 \le x < t, \\ (2\pi - x)^2 t, & t < x \le 2\pi, \end{cases}$

$$F_3(x) = \left(1 - \frac{x}{2\pi}\right)^2 \varphi_1(0) + \frac{x(4\pi x - x)}{4\pi^2}\varphi_2(0) + \frac{x(x - 2\pi)}{2\pi}\varphi_3(0) +$$

$$+\frac{(2\pi - x)^2}{8\pi^2}\int\limits_0^{2\pi} t^2\ [f_2(t,0) + F_2(t)]\,dt - \frac{1}{2}\int\limits_x^{2\pi}(t - x)^2 [f_2(t,0) + F_2(t)]\,dt$$

corresponding to problem (53), (15).

Equation (54) is a Fredholm integral equation of the second kind with the kernel $K(x,t) \in \in C([0,2\pi] \times [0,\ 2\pi])$ and with the right-hand side $F_3(x) \in C^1[0,2\pi]$. The unique solvability of equation (54) under conditions (41) - (45) involving the functions $\alpha(x)$ and $\beta(x)$ follows from the uniqueness theorem proved above. Properties (11), (12) and (52) imply that the solution $\tau = \tau(x)$ to equation (54) belongs to the class $\tau(x) \in C[0,2\pi] \cap C^3\,]0,2\pi[$.

Problem 1 as $\left[\frac{\alpha(x)+\beta(x)}{\alpha(x)-\beta(x)}\right]' \equiv 0$

Finally, consider the case as $a'(x) = \left[\frac{\alpha(x)+\beta(x)}{\alpha(x)-\beta(x)}\right]' \equiv 0 \quad \forall\, x \in [0,2\pi]$, i.e.

$$a(x) = \frac{\alpha(x) + \beta(x)}{\alpha(x) - \beta(x)} = a = const \quad \forall\, x \in [0,2\pi]. \tag{55}$$

Under condition (55) from (53) we can arrive at the following problem for $\tau(x)$

$$\tau'''(x) + a\tau'(x) = f_2(x,0) + F_2(x), \quad 0 < x < 2\pi, \tag{56}$$

$$\tau(0) = \varphi_1(0),\ \tau(2\pi) = \varphi_2(0),\ \tau'(2\pi) = \varphi_3(0). \tag{57}$$

The solution to problem (56) - (57) is written out by the formula

$$\tau(x) = \frac{1}{4\pi^2}\left[(2\pi - x)^2 + 2a\int\limits_0^{2\pi}(2\pi - t)\ G(x,t)\,dt\right]\varphi_1(0) +$$

$$+\frac{1}{4\pi^2}\left[1 - (2\pi - x)^2 - 2a\int\limits_0^{2\pi}(2\pi - t)\ G(x,t)\,dt\right]\varphi_2(0) +$$

$$+\frac{1}{2\pi}\left[x^2 - 2\pi x + 2a\int\limits_0^{2\pi}(\pi - t)\ G(x,t)\,dt\right]\varphi_3(0) + \int\limits_0^{2\pi} G(x,t)\ [f_2(t,0) + F_2(t)]\,dt. \tag{58}$$

The function $G(x,t)$ in (58) is Green's function of the operator $L[\tau(x)] = \tau'''(x) + a\tau'(x)$ with condition (57), whose explicit form is determined depending on the sign of the number a by one of the formulas below:

$$G(x,t)=\frac{1}{a\left[1-ch\left(2\sqrt{-a}\pi\right)\right]}\begin{cases}\left[1-ch\left(\sqrt{-a}\,(2\pi-x)\right)\right]\left[1-ch\left(\sqrt{-a}\,t\right)\right]-\\ -\left[1-ch\left(2\sqrt{-a}\pi\right)\right]\left[1-ch\left(\sqrt{-a}\,(x-t)\right)\right], & 0\le x<t, \quad as\ a<0;\\ \left[1-ch\left(\sqrt{-a}\,(2\pi-x)\right)\right]\left[1-ch\left(\sqrt{-a}\,t\right)\right], & t<x\le 2\pi,\end{cases}$$

$$G(x,t)=\frac{1}{8\pi^2}\begin{cases}t^2\,(2\pi-x)^2-4\pi^2\,(t-x)^2, & 0\le x<t,\\ t^2\,(2\pi-x)^2, & t<x\le 2\pi\end{cases}\quad as\ a=0;$$

and

$$G(x,t)=\frac{1}{a\left[1-\cos\left(2\sqrt{a}\pi\right)\right]}\begin{cases}\left[1-\cos\left(\sqrt{a}\,t\right)\right]\left[1-\cos\left(\sqrt{a}\,(2\pi-x)\right)\right]-\\ -\left[1-\cos\left(2\sqrt{a}\pi\right)\right]\left[1-\cos\left(\sqrt{a}\,(x-t)\right)\right], & 0\le x<t,\\ \left[1-\cos\left(\sqrt{a}\,t\right)\right]\left[1-\cos\left(\sqrt{a}\,(2\pi-x)\right)\right], & t<x\le 2\pi,\end{cases}$$

as $a>0$ and $a\neq n^2$, $n\in N$.

In each cases considered above with the value found for the function $\tau(x)$ within the fundamental relations (14) or (49) we can also find a value for the function $\nu(x)$. At that, the solution of the initial problem (1), (4), (5) in the domain Ω_1 is written out by the d'Alembert formula (17), while in Ω_2 the solution to problem (3), (4) with $u(x,0)=\tau(x)$ is written out by formula (40).

Provided that $a(x)=\frac{\alpha(x)+\beta(x)}{\alpha(x)-\beta(x)}=a=n^2\quad\forall\,x\in[0,2\pi]$, $n\in N$ the homogeneous problem corresponding to problem (56), (57) has nonzero solutions $\tau(x)=c\,(1-\cos nx)$, $c=const$. The function $G(x,t)$ in this case does not exist, and a solution to problem (56)–(57) can exist with the additional condition

$$\int\limits_0^{2\pi}\left[F_2(t)+f_2(t,0)\right]\left[\cos(nt)-1\right]dt=n^2\left[\varphi_1(0)-\varphi_2(0)\right] \tag{59}$$

be satisfied.

As condition (59) is satisfied the solution to problem 1 in the domain $\Omega_1$is written out by the formula

$$u(x,y)=\frac{g(x+y)+g(x-y)}{2}+\frac{1}{2}\int\limits_{x-y}^{x+y}\nu(t)\,dt-\frac{1}{2}\int\limits_0^y\int\limits_{x-y+s}^{x+y-s}f_1(t,s)\,dtds,$$

while in the domain Ω_2 the solution is written out as below

$$u(x,y)=\frac{1}{\pi}\left\{\int\limits_0^y G(x,-y;\,0,-\eta)\,\varphi_3(\eta)\,d\eta-\int\limits_0^y G_{\xi\xi}(x,-y;\,0,-\eta)\,\varphi_1(\eta)\,d\eta+\right.$$

$$\left.+\int\limits_0^y G_{\xi\xi}(x,-y;\,r,-\eta)\,\varphi_2(\eta)\,d\eta+\int\limits_0^{2\pi}G(x,-y;\,\xi,0)\,g(\xi)\,d\xi+\int\limits_0^y\int\limits_0^r G(x,-y;\,\xi,-\eta)\,f(\xi,\eta)\,d\xi d\eta\right\},$$

where $g(x)$ is an arbitrary, fairly smooth function, and $G(x,y;\,\xi,\eta)=U(x,y;\,\xi,\eta)-W(x,y;\,\xi,\eta)$, as above, is the Green function of the operator $Lu=u_{xxx}-u_y$, $U(x,y;\,\xi,\eta)$ and $W(x,y;\,\xi,\eta)$ are fundamental solutions to equation (2) [1; 135].

References

1 Джураев Т.Д. Краевые задачи для уравнений смешанного и смешанно-составного типов / Т.Д. Джураев. — Ташкент: ФАН, 1979. — 239 с.

2 Нахушев А.М. Уравнения математической биологии / А.М. Нахушев. — М.: Высш. шк., 1995. — 301 с.

3 Нахушев А.М. Задачи со смещением для уравнений в частных производных / А.М. Нахушев. — М.: Наука, 2006. — 287 с.

4 Жегалов В.И. Краевая задача для уравнения смешанного типа с граничными условиями на обеих характеристиках и с разрывами на линии перехода / В.И. Жегалов // Ученые записки Казан. ун-та. — 1962. — Т. 122. — Кн. 3. — С. 3-16.

5 Нахушев А.М. О некоторых краевых задачах для гиперболических уравнений и уравнений смешанного типа / А.М. Нахушев // Дифференциальные уравнения. — 1969. — Т. 5. — № 1. — С. 44-59.

6 Нахушев А.М. Новая краевая задача для одного вырождающегося гиперболического уравнения / А.М. Нахушев // Докл. АН СССР. — 1969. — Т. 187. — № 4. — С. 736-739.

7 Нахушев А.М. О задаче Дарбу для вырождающихся гиперболических уравнений / А.М. Нахушев // Дифференциальные уравнения. — 1971. — Т. 7. — № 1. — С. 49-56.

8 Берс Л. Математические вопросы дозвуковой и околозвуковой газовой динамики / Л. Берс. — М.: Иностранная литература, 1961. — 208 с.

9 Франкль Ф.И. Избранные труды по газовой динамике / Ф.И. Франкль. — М.: Наука, 1973. — 712 с.

10 Салахитдинов М.С. Уравнения смешанно-составного типа / М.С. Салахитдинов. — Ташкент: Фан, 1974. — 156 с.

11 Репин О.А. Краевые задачи со смещением для уравнений гиперболического и смешанного типа / О.А. Репин. — Самара: Изд-во Самар. филиала Саратов. гос. экон. ун-та, 1992. — 161 с.

12 Кальменов Т.Ш. Краевые задачи для линейных уравнений в частных производных гиперболического типа / Т.Ш. Кальменов. — Шымкент: Гылая, 1993. — 328 с.

13 Жегалов В.И. Дифференциальные уравнения со старшими частными производными / В.И. Жегалов, А.Н. Миронов. — Казань: Казан. мат. общ-во, 2001. — 226 с.

14 Репин О.А. Краевые задачи для уравнений в частных производных с разрывными коэффициентами / О.А. Репин, А.А. Килбас, О.И. Маричев. — Самара: Изд-во Самар. гос. экон. ун-та, 2008. — 275 с.

15 Пулькина Л.С. Задачи с неклассическими условиями для гиперболических уравнений / Л.С. Пулькина. — Самара: Изд-во Самар. гос. экон. ун-та, 2012. — 194 с.

16 Сабитов К.Б. К теории уравнений смешанного типа / К.Б. Сабитов. — М.: Физматлит, 2014. — 304 с.

17 Сабитов К.Б. Прямые и обратные задачи для уравнений параболо-гиперболического типа / К.Б. Сабитов. — Уфа: Гилем, 2015. — 240 с.

18 Нахушева З.А. Нелокальные краевые задачи для основных и смешанного типов дифференциальных уравнений / З.А. Нахушева. — Нальчик: КБНЦ РАН, 2011. — 196 с.

19 Салахитдинов М.С. К теории уравнений третьего порядка смешанно-составного типа: дис. ... д-ра физ.-мат. наук / М.С. Салахитдинов. — Ташкент, 1967.

20 Нахушев А.М. К теории линейных краевых задач для гиперболических и смешанных уравнений второго порядка: дис. ... д-ра физ.-мат. наук / А.М. Нахушев. — Новосибирск, 1970. — 172 с.

21 Жегалов В.И. Исследование краевых задач со смещением для уравнений смешанного типа: дис. ... д-ра физ.-мат. наук / В.И. Жегалов. — Новосибирск, 1988.

22 Сабитов К.Б. Некоторые вопросы качественной и спектральной теории уравнений смешанного типа: дис. ... д-ра физ.-мат. наук / К.Б. Сабитов. — М., 1991. — 313 с.

23 Репин О.А. Краевые задачи для уравнений гиперболического и смешанного типов и дробное интегро-дифференцирование: дис. ... д-ра физ.-мат. наук / О.А. Репин. — Минск, 1998.

24 Балкизов Ж.А. Об одной краевой задаче типа задачи Трикоми для уравнения смешанного параболо-гиперболического типа второго порядка с тремя смещениями в гиперболической части области / Ж.А. Балкизов // Науч. ведомости Белгород. гос. ун-та. Сер. Математика. Физика. — 2019. — Т. 51. — № 1. — С. 5-14. — DOI 10.18413/2075-4639-2019-51-1-5-14.

25 Репин О.А. Нелокальная краевая задача для параболо-гиперболического уравнения с характеристической линией изменения типа / О.А. Репин // Дифференциальные уравнения. — 1992. — Т. 28. — № 1. — С. 173-176.

26 Килбас А.А. Задача со смещением для параболо-гиперболического уравнения / А.А. Килбас, О.А. Репин // Дифференциальные уравнения. — 1998. — Т. 34. — № 6. — С. 799-805.

27 Репин О.А. Нелокальная краевая задача для параболо-гиперболического уравнения с нехарактеристической линией изменения типа / О.А. Репин, С.В. Ефимова // Вестн. СамГТУ. Сер. физ.-мат. науки. — 2002. — Вып. 16. — С. 10-14.

28 Ефимова С.В. Задача с нелокальными условиями на характеристиках для уравнения влагопереноса / С.В. Ефимова, О.А. Репин // Дифференциальные уравнения. — 2004. — Т. 40. — № 10. — С. 1419-1422.

29 Мирсабуров М. О задаче с тремя вариантами смещений для уравнения смешанного типа / М. Мирсабуров, С.Т. Чориева // Изв. вузов. Сер. Математика. — 2016. — № 4. — С. 32-45.

30 Ефимова С.В. Нелокальная задача для гиперболического уравнения, вырождающегося внутри области / С.В. Ефимова // Вестн. Самар. гос. техн. ун-та. Сер. физ.-мат. науки. — 2005. — Вып. 34. — С. 194-196.

31 Балкизов Ж.А. Краевая задача со смещением для модельного уравнения параболо-гиперболического типа третьего порядка / Ж.А. Балкизов // Вестн. КРАУНЦ. Сер. Физ.-мат. науки. — 2018. — № 3(23). — С. 19-26. DOI: 10.18454/2079-6641-2018-23-3-19-26.

32 Тихонов А.Н. Уравнения математической физики / А.Н. Тихонов, А.А. Самарский. — М.: Наука, 1977. — 736 с.

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Үшінші ретті парабола-гиперболалық типті теңдеу үшін ығысумен берілген бірінші теңдеу және Трикоми есебінің тасымалдаушысы ретіндегі сипаттаушылардың тең еместігінің әсері

Мақалада үшінші ретті парабола-гиперболалық типті гиперболалық облыста уақытқа қарсы және толқындық теңдеулі үшінші ретті параболалық теңдеулі біртекті емес теңдеу үшін бір шекаралық шарт есебінде AC және BC характеристикаларында ізделінді функцияның мәндерінен тәуелді айнымалы коэффициентті сызықтық комбинациясымен берілген шеттік есеп зерттелген. Келесі нәтижелер алынды: $0 \leq x \leq 2\pi$ болғанда Трикоми есебінің деректерін тасымалдаушылар сияқты Ω облысында Ω_1 бөлігін шектейтін AC және BC сипаттаушыларының тең мүмкіндікті еместігі көрсетілді және BC сипаттаушыларындағы деректерімен Трикоми есебінің шешуінен, жалпы алғанда, AC сипаттаушысындағы деректерімен Трикоми есебі шешілмейді; зерттеліп отырған есептің регулярлы шешуінің бар болуы және жалғыздығының қажетті және жеткілікті шарттары табылған. Берілген функцияға белгілі бір шарттарда зерттелетін есептің шешуі айқын түрде жазылды. Берілген функцияға жұмыста табылған қажетті шарттар бұзылса, зерттелетін есепке сәйкес біртекті есептің шексіз көп шешуі болатыны көрсетілген, сәйкес біртекті емес есептің шешулер жиыны тек қана берілген функцияларға қосымша талаптар болғанда ғана бар болады.

Кілт сөздер: аралас типті теңдеу, үшінші ретті парабола-гиперболалық теңдеу, Трикоми теңдеуі, ығысумен берілген бірінші теңдеу, Грин функциясы, екінші текті Фредгольмнің интегралдық теңдеуі.

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Первая задача со смещением для уравнения параболо-гиперболического типа третьего порядка и эффект неравноправия характеристик как носители данных задачи Трикоми

В статье исследована краевая задача со смещением для неоднородного уравнения параболо-гиперболического типа третьего порядка с параболическим уравнением третьего порядка с обратным ходом времени и волновым уравнением в области гиперболичности, когда в качестве одного из граничных условий задана линейная комбинация с переменными коэффициентами от значений искомой функции на характеристиках AC и BC. Получены следующие результаты: показано неравноправие характеристик AC и BC, ограничивающих гиперболическую часть Ω_1 области Ω как носители данных задачи Трикоми при $0 \leq x \leq 2\pi$, и из разрешимости задачи Трикоми с данными на характеристике BC, вообще говоря, не следует разрешимость задачи Трикоми с данными на характеристике AC; найдены необходимые и достаточные условия существования и единственности регулярного решения исследуемой задачи. При определенных условиях на заданные функции решение исследуемой задачи выписано в явном виде. Показано, что при нарушении найденных в работе необходимых условий на заданные функции, однородная задача, соответствующая исследуемой задаче, имеет бесчисленное множество линейно независимых решений, а множество решений соответствующей неоднородной задачи может существовать только при дополнительном требовании на заданные функции.

Ключевые слова: уравнение смешанного типа, параболо-гиперболическое уравнение третьего порядка, задача Трикоми, метод Трикоми, первая задача со смещением, функция Грина, интегральное уравнение Фредгольма второго рода.

References

1 Dzhuraev, T.D. (1979). *Kraevye zadachi dlia uravnenii smeshannoho i smeshanno-sostavnoho tipov [Boundary value problems for equations of mixed and mixed composite types].* Tashkent: FAN [in Russian].

2 Nahushev, A.M. (1995). *Uravneniia matematicheskoi biolohii [The equations of mathematical biology].* Moscow: Vysshaia shkola [in Russian].

3 Nahushev, A.M. (2006). *Zadachi so smeshcheniem dlia uravnenii v chastnykh proizvodnykh [Problems with displacement for partial differential equations].* Moscow: Nauka [in Russian].

4 Zhegalov, V.I. (1962). Kraevaia zadacha dlia uravneniia smeshannoho tipa s hranichnymi usloviiami na obeikh kharakteristikakh i s razryvami na linii perekhoda [The boundary value problem for a mixed type equation with boundary conditions on both characteristic lines and with gaps on the transition line]. *Uchenye zapiski Kazanskoho universiteta, 122, 3,* 3-16 [in Russian].

5 Nahushev, A.M. (1969). O nekotorykh kraevykh zadachakh dlia hiperbolicheskikh uravnenii i uravnenii smeshannoho tipa [On some boundary value problems for hyperbolic equations and equations of mixed type]. *Differential equations, 5, 1,* 44-59 [in Russian].

6 Nahushev, A.M. (1969). Novaia kraevaia zadacha dlia odnoho vyrozhdaiushchehosia hiperbolicheskoho uravneniia [A new boundary-value problem for one degenerate hyperbolic equation]. *Doklady Akademii nauk SSSR — Doklady of the USSR Academy of Sciences, 187, 4,* 736-739 [in Russian].

7 Nahushev, A.M. (1971). O zadache Darbu dlia vyrozhdaiushchikhsia hiperbolicheskikh uravnenii [On the Darboux problem for degenerate hyperbolic equations]. *Differential equations, 7, 1,* 49-56 [in Russian].

8 Bers, L. (1961). *Matematicheskie voprosy dozvukovoi i okolozvukovoi hazovoi dinamiki [Mathematical problems of subsonic and transonic gas dynamics].* Moscow: Inostrannaia literatura [in Russian].

9 Frankl', F.I. (1973). *Izbrannye trudy po hazovoi dinamike [Selected works on gas dynamics].* Moscow: Nauka [in Russian].

10 Salahitdinov, M.S. (1974). *Uravneniia smeshanno-sostavnoho tipa [Equations of mixed-composite type].* Tashkent: FAN [in Russian].

11 Repin, O.A. (1992). *Kraevye zadachi so smeshcheniem dlia uravnenii hiperbolicheskoho i smeshannoho tipa [Boundary value problems with displacement for equations of hyperbolic and mixed type].* Samara: Izdatelstvo Samarskoho filiala Saratovskoho hosudarstvennoho universiteta [in Russian].

12 Kal'menov, T.Sh. (1993). *Kraevye zadachi dlia lineinykh uravnenii v chastnykh proizvodnykh hiperbolicheskoho tipa [Boundary value problems for linear partial differential equations of hyperbolic type].* Shymkent: Hylaia [in Russian].

13 Zhegalov, V.I. & Mironov, A.N. (2001). *Differentsialnye uravneniia so starshimi chastnymi proizvodnymi [Differential equations with highest partial derivatives].* Kazan: Kazanskoe matematicheskoe obshchestvo [in Russian].

14 Repin, O.A., Kilbas, A.A. & Marichev, O.I. (2008). *Kraevye zadachi dlia uravnenii v chastnykh proizvodnykh s razryvnymi koeffitsientami [Boundary value problems for partial differential equations with discontinuous coefficients].* Samara: Iizdatelstvo Samarskoho hosudarstvennoho ekonomicheskoho universiteta [in Russian].

15 Pul'kina, L.S. (2012). *Zadachi s neklassicheskimi usloviiami dlia hiperbolicheskikh uravnenii [Problems with nonclassical conditions for hyperbolic equations].* Samara: Izdatelstvo Samarskoho universiteta [in Russian].

16 Sabitov, K.B. (2014). *K teorii uravnenii smeshannoho tipa [To the theory of equations of mixed type].* Moscow: Fizmatlit [in Russian].

17 Sabitov, K.B. (2015). *Priamye i obratnye zadachi dlia uravnenii parabolo-hiperbolicheskoho tipa [Direct and inverse problems for equations of parabolic-hyperbolic type].* Ufa: Hilem [in Russian].

18 Nahusheva, Z.A. (2011). *Nelokalnye kraevye zadachi dlia osnovnykh i smeshannoho tipov differentsialnykh uravnenii [Nonlocal boundary value problems for the basic and mixed types differential equations].* Nalchik: KBSC RAS [in Russian].

19 Salahitdinov, M.S. (1967). K teorii uravnenii treteho poriadka smeshanno-sostavnoho tipa [On the theory of third-order equations of mixed-composite type]. *Doctor's thesis.* Tashkent [in Russian]

20 Nahushev, A.M. (1970). K teorii lineinykh kraevykh zadach dlia hiperbolicheskikh i smeshannykh uravnenii vtoroho poriadka [On the theory of linear boundary value problems for hyperbolic and mixed second-order equations]. *Doctor's thesis.* Novosibirsk [in Russian].

21 Zhegalov, V.I. (1988). Issledovanie kraevykh zadach so smeshcheniem dlia uravnenii smeshannoho tipa [The study of boundary value problems with bias for equations of mixed type]. *Doctor's thesis.* Novosibirsk [in Russian].

22 Sabitov, K.B. (1991). Nekotorye voprosy kachestvennoi i spektralnoi teorii uravnenii smeshannoho tipa [On some questions of the qualitative and spectral theory of equations of mixed type]. *Doctor's thesis.* Moscow [in Russian].

23 Repin, O.A. (1998). Kraevye zadachi dlia uravnenii hiperbolicheskoho i smeshannoho tipov i drobnoe intehro-differentsirovanie [Boundary value problems for mixed hyperbolic types equations and fractional integro-differentiation]. *Doctor's thesis.* Minsk [in Russian].

24 Balkizov, Zh.A. (2019). Ob odnoi kraevoi zadache tipa zadachi Trikomi dlia uravneniia smeshannoho parabolo-hiperbolicheskoho tipa vtoroho poriadka s tremia smeshcheniiami v hiperbolicheskoi chasti oblasti [On a boundary value problem of Tricomi mixed type problem for an equation of the second order with three displacements in the hyperbolic part of the domain]. *Nauchnye vedomosti Belhorodskoho hosudarstvennoho universiteta. Seriia Matematika. Fizika — Scientific reports of Belgorod State University. Mathematics Series. Physics, 51, 1,* 5-14. DOI 10.18413/2075-4639-2019-51-1-5-14 [in Russian].

25 Repin, O.A. (1992). Nelokalnaia kraevaia zadacha dlia parabolo-hiperbolicheskoho uravneniia s kharakteristicheskoi liniei izmeneniia tipa [Nonlocal boundary value problem for a parabolic-hyperbolic equation with a characteristic line of type changing]. *Differential equations, 28, 1,* 173-176 [in Russian]

26 Kilbas, A.A. & Repin, O.A. (1998). Zadacha so smeshcheniem dlia parabolo-hiperbolicheskoho uravneniia [Problem with displacement for a parabolic-hyperbolic equation]. *Differential equations, 34, 6,* 799-805 [in Russian].

27 Repin, O.A. & Efimova, S.V. (2002). Nelokalnaia kraevaia zadacha dlia parabolo-hiperbolicheskoho uravneniia s nekharakteristicheskoi liniei izmeneniia tipa [Nonlocal boundary value problem for a parabolic-hyperbolic equation with an uncharacteristic line of type changing]. *Vestnik Samarskoho hosudarstvennoho universiteta. Seriia fiziko-matematicheskikh nauk — Bulletin of Samara State Technical University. Series Phys.-Math. Science, 16,* 10-14 [in Russian].

28 Efimova, C.V. & Repin, O.A. (2004). Zadacha s nelokalnymi usloviiami na kharakteristikakh dlia uravneniia vlahoperenosa [A problem with nonlocal conditions with characteristics for a moisture transfer equation]. *Differential equations, 40:10,* 1498-1502 [in Russian].

29 Mirsaburov, M. & Chorieva, S.T. (2016). O zadache s tremia variantami smeshchenii dlia uravneniia smeshannoho tipa [On a problem with three variants of displacements for a mixed type equation]. *Izvestiia vuzov. Seriia Matematika — News of Universities. Maths, 2016, 4,* 32-45 [in Russian].

30 Efimova, S.V. (2005). Nelokalnaia zadacha dlia hiperbolicheskoho uravneniia, vyrozhdaiushchehosia vnutri oblasti [Nonlocal problem for a hyperbolic equation degenerating inside a domain]. *Bulletin of Samara State Technical University. Series Phys.-Math. science, 34,* 194-196 [in Russian].

31 Balkizov, Zh.A. (2018). Kraevaia zadacha so smeshcheniem dlia modelnoho uravneniia parabolo-hiperbolicheskoho tipa treteho poriadka [A boundary value problem with an displacement for a model equation of parabolic-hyperbolic type of the third order]. *Vestnik KRAUNTS. Phiz.-mat. sciences, 3(23),* 19-26 [in Russian].

32 Tihonov, A.N. & Samarskii, A.A. (1977). *Uravneniia matematicheskoi fiziki [Equations of mathematical physics].* Moscow: Nauka, 736 [in Russian].

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Generalization of one theorem of F. Riesz to some other spaces

It is known from the analysis course that in order a function to serve as an undefined integral of a summable function, it is necessary and sufficient that it be absolutely continuous. Therefore, it is natural to raise the question of the characteristic of a function which is an undefined integral of the function included in $L_p, p > 1$. The answer is well known theorem of F.Riesz concerning the conditions of representability of a given function in the form of an integral with variable upper limit on the functions of Lebesgue spaces. In the one-dimensional and multi-dimensional case, many mathematicians have generalized this theorem for Lebesgue and Orlicz spaces. In this work we will prove theorem of F.Riesz for other functional spaces. Generalization of the theorem of F.Riesz to the case when subintegral function from the weighted Lebesgue spaces is obtained. Also, we prove a necessary condition for the above representation of a function $f \in L_p\varphi(L_p)$.

Keywords: function, functional spaces, integral, theorem of F.Riesz, weighted Lebesgue space.

1 Introduction

In the theory of functions, the following theorem of F.Riesz is known (see, for example, [1], page 225): for the function $F(x)$ $(a \le x \le b)$ to be representable as

$$F(x) = C + \int_a^x f(t)dt,$$

where $f(t) \in L_p(p > 1)$, it is necessary and sufficient that for every subdivision $[a;b]$ by points $a = x_0 < x_1 < < ... < x_n = b$ the inequality was executed

$$\sum_{i=1}^{n-1} \frac{|F(x_{i+1}) - F(x_i)|^p}{(x_{i+1} - x_i)^{p-1}} \le K < \infty,$$

where K does not depend on the way $[a;b]$ is subdivided.

In the future, a number of authors have proposed various generalizations of this theorem [2–5]. We will prove this theorem for spaces, which are defined below.

Let $W(x)$ is a non-negative function. Through $L_{p,W}[a;b]$ we will designate the space of all measurable by Lebesgue on $[a;b]$ functions f, for which

$$\|f\|_{p,W} = \left(\int_a^b |f(x)|^p W(x)dx \right)^{\frac{1}{p}} < +\infty, 1 \le p < +\infty.$$

We assume that the function $W(x)$ satisfies A_p-condition [6] (or $W \in A_p$), if

$$\sup_{I \subset [a;b]} \left[\frac{1}{|I|} \int_I W(x)dx \right]^{\frac{1}{p}} \cdot \left[\frac{1}{|I|} \int_I (W(x))^{-\frac{1}{p-1}} dx \right]^{\frac{1}{p'}} < +\infty, \frac{1}{p} + \frac{1}{p'} = 1.$$

Let the function $\varphi(t)$ satisfies the following conditions [7]:

a) $\varphi(t)$ is an even, non-negative, non-decreasing on $[0, +\infty)$;

b) $\varphi(t^2) \le C\varphi(t)$, $t \in [0, \infty)$, $C \ge 1$;

c) $\frac{\varphi(t)}{t^\varepsilon} \downarrow$ on $(0, +\infty)$ for some $\varepsilon > 0$.

Measurable on $[a;b]$ function $f \in L_p\varphi(L_p)$, if

$$\int\limits_a^b |f(x)|^p \cdot \varphi\left(|f(x)|^p\right) dx < +\infty.$$

2 The results and their proofs

We proved the following theorem.

Theorem 1. For the function $F(x)(a \leq x \leq b)$ to be representable as:

$$F(x) = C + \int\limits_a^x f(t)dt, \tag{1}$$

where $f(t) \in L_{p,W}[a;b](p > 1)$, it is necessary and sufficient that for every subdivision $[a;b]$ by points $a = x_0 < x_1 < ... < x_n = b$ the inequality was executed

$$\sum_{k=0}^{n-1} \frac{|F(x_{k+1}) - F(x_k)|^p}{\left(\int\limits_{x_k}^{x_{k+1}} W^{\frac{-p'}{p}}(t)dt\right)^{p-1}} \leq K, \tag{2}$$

where K does not depend on the way $[a;b]$ is subdivided.

Proof. Let's prove the necessity of theorem. Suppose inequality (1) holds. Then by Holder's inequality:

$$|F(x_{k+1}) - F(x_k)| = \left|\int\limits_{x_k}^{x_{k+1}} f(t)dt\right| = \left|\int\limits_{x_k}^{x_{k+1}} f(t)W^{\frac{1}{p}}(t)W^{\frac{-1}{p}}(t)dt\right| \leq$$

$$\leq \left(\int\limits_{x_k}^{x_{k+1}} |f(t)|^p W(t)dt\right)^{\frac{1}{p}} \left(\int\limits_{x_k}^{x_{k+1}} W^{\frac{-p'}{p}}(t)dt\right)^{\frac{1}{p'}}, \forall k = 0, ..., n-1,$$

where $p' = \frac{p}{p-1}$.

We obtain:

$$\frac{|F(x_{k+1}) - F(x_k)|^p}{\left(\int\limits_{x_k}^{x_{k+1}} W^{\frac{-p'}{p}}(t)dt\right)^{p-1}} \leq \int\limits_{x_k}^{x_{k+1}} |f(t)|^p W(t)dt, k = 0, ..., n-1.$$

Therefore, folding of these inequalities, we will get:

$$\sum_{k=0}^{n-1} \frac{|F(x_{k+1}) - F(x_k)|^p}{\left(\int\limits_{x_k}^{x_{k+1}} W^{\frac{-p'}{p}}(t)dt\right)^{p-1}} \leq \int\limits_a^b |f(t)|^p W(t)dt = K.$$

The necessity of condition (2) is proved.

Now we prove sufficiency of the conditions (2). First of all, note that inequality (2) can only increase if we discard some components of its left part. Therefore, for any finite system of mutually not impose intervals $(a_k, b_k), (k = 1, 2, ..., n)$ contained in $[a, b]$ will be

$$\sum_{k=1}^{n} \frac{|F(b_k) - F(a_k)|^p}{\left(\int\limits_{a_k}^{b_k} W^{\frac{-p'}{p}}(t)dt\right)^{p-1}} \leq K.$$

But because of the Holder's inequality holds:

$$\sum_{k=1}^{n}|F(b_k)-F(a_k)|=\sum_{k=1}^{n}\frac{|F(b_k)-F(a_k)|}{\left(\int\limits_{a_k}^{b_k}W^{\frac{-p'}{p}}(t)dt\right)^{\frac{p-1}{p}}}\cdot\left(\int\limits_{a_k}^{b_k}W^{\frac{-p'}{p}}(t)dt\right)^{\frac{p-1}{p}}\leq$$

$$\leq\left(\sum_{k=1}^{n}\frac{|F(b_k)-F(a_k)|^p}{\left(\int\limits_{a_k}^{b_k}W^{\frac{-p'}{p}}(t)dt\right)^{p-1}}\right)^{\frac{1}{p}}\cdot\left(\sum_{k=1}^{n}\int\limits_{a_k}^{b_k}W^{\frac{-p'}{p}}(t)dt\right)^{\frac{1}{p'}}\leq\sqrt[p]{K}\cdot\left(\int\limits_{a}^{b}W^{\frac{-p'}{p}}(t)dt\right)^{\frac{1}{p'}}.$$

From the last inequality implies absolute continuity of the function $F(x)$. Then this function is representable in the form (1), where f is some summable function. It remains to prove that $f(x)\in L_{p,W}[a;b]$.

With this goal in expanding the segment $[a;b]$ into equal parts by the points $x_k^{(n)}=a+\frac{k}{n}(b-a), k=0,1,...,n$ let us introduce the function $f_n(t)$, believing:

$$f_n(t)=\frac{F(x_{k+1}^{(n)})-F(x_k^{(n)})}{x_{k+1}^{(n)}-x_k^{(n)}}\cdot\chi_k(t),$$

where $\chi_k(t)$ - a characteristic function of the interval $\left(x_k^{(n)},x_{k+1}^{(n)}\right)$.

At the division points we believe $f_n\left(x_k^{(n)}\right)=0, k=0,1,...,n.$

It is easy to see that almost everywhere will be:

$$\lim_{n\to\infty}|f_n(t)|^p\,W(t)=|f(t)|^p\,W(t).$$

Hence, by Fatou's theorem:

$$\int\limits_{a}^{b}|f(t)|^p\,W(t)dt\leq\sup_n\left\{\int\limits_{a}^{b}|f_n(t)|^p\,W(t)dt\right\}.$$

For $f_n(t)$, since $W\in A_p$ we get the following inequality:

$$\int\limits_{a}^{b}|f_n(t)|^p\,W(t)dt=\sum_{k=0}^{n-1}\int\limits_{x_k{}^{(n)}}^{x_{k+1}^{(n)}}\frac{\left|F(x_{k+1}^{(n)})-F(x_k^{(n)})\right|^p}{\left(x_{k+1}^{(n)}-x_k^{(n)}\right)^p}\cdot W(t)dt=$$

$$=\sum_{k=0}^{n-1}\frac{\left|F(x_{k+1}^{(n)})-F(x_k^{(n)})\right|^p}{\left(x_{k+1}^{(n)}-x_k^{(n)}\right)^p}\cdot\int\limits_{x_k{}^{(n)}}^{x_{k+1}^{(n)}}W(t)dt\leq$$

$$\leq C\cdot\sum_{k=0}^{n-1}\frac{\left|F(x_{k+1}^{(n)})-F(x_k^{(n)})\right|^p}{\left(x_{k+1}^{(n)}-x_k^{(n)}\right)^p}\cdot\frac{\left(x_{k+1}^{(n)}-x_k^{(n)}\right)^p}{\left(\int\limits_{x_k{}^{(n)}}^{x_{k+1}^{(n)}}W^{\frac{-p'}{p}}(t)dt\right)^{\frac{p}{p'}}}=$$

$$= C \cdot \sum_{k=0}^{n-1} \frac{\left| F(x_{k+1}^{(n)}) - F(x_k^{(n)}) \right|^p}{\left(\int\limits_{x_k{}^{(n)}}^{x_{k+1}^{(n)}} W^{\frac{-p'}{p}}(t)dt \right)^{p-1}} \leq C \cdot K.$$

And it became,

$$\int\limits_a^b |f(t)|^p \cdot W(t)dt < +\infty,$$

i.e. $f(t) \in L_{p,W}[a;b]$.

The theorem is proved.

Remark. In the case of $W(t) \equiv 1$, the theorem of F.Riesz follows from the proved theorem.

Now we will prove the necessary condition of representation (1), from the function of $L_p\varphi(L_p)$ space.

Theorem 2. If $F(x)$ can be represented as

$$F(x) = C + \int\limits_a^x f(t)dt,$$

where $f \in L_p\varphi(L_p)$, then for every subdivision of $[a;b]$ by points $a = x_0 < x_1 < ... < x_n = b$ the following inequality holds:

$$\sum_{k=0}^{n-1} \frac{|F(x_{k+1}) - F(x_k)|^p}{\left(\int\limits_{x_k}^{x_{k+1}} \varphi^{\frac{-1}{p-1}} \left(|f(t)|^p \right) dt \right)^{p-1}} \leq K.$$

Proof. Let $F(x)$ represented as

$$F(x) = C + \int\limits_a^x f(t)dt.$$

Then by Holder's inequality we get:

$$|F(x_{k+1}) - F(x_k)| = \left| \int\limits_{x_k}^{x_{k+1}} f(t)dt \right| = \left| \int\limits_{x_k}^{x_{k+1}} f(t)\varphi^{\frac{1}{p}} \left(|f(t)|^p \right) \varphi^{\frac{-1}{p}} \left(|f(t)|^p \right) dt \right| \leq$$

$$\leq \left(\int\limits_{x_k}^{x_{k+1}} |f(t)|^p \varphi \left(|f(t)|^p \right) dt \right)^{\frac{1}{p}} \left(\int\limits_{x_k}^{x_{k+1}} \varphi^{\frac{-1}{p-1}} \left(|f(t)|^p \right) dt \right)^{\frac{p-1}{p}}, \forall k = 0, ..., n-1.$$

Hence, for all $k = 0, ..., n-1$ we obtain

$$\frac{|F(x_{k+1}) - F(x_k)|^p}{\left(\int\limits_{x_k}^{x_{k+1}} \varphi^{\frac{-1}{p-1}} \left(|f(t)|^p \right) dt \right)^{p-1}} \leq \int\limits_{x_k}^{x_{k+1}} \varphi^{\frac{-1}{p-1}} \left(|f(t)|^p \right) dt.$$

Now adding up these inequalities, because $f \in L_p\varphi(L_p)$ we get

$$\sum_{k=0}^{n-1} \frac{|F(x_{k+1}) - F(x_k)|^p}{\left(\int\limits_{x_k}^{x_{k+1}} \varphi^{\frac{-1}{p-1}} \left(|f(t)|^p \right) dt \right)^{p-1}} \leq K.$$

Remark. In the case of $\varphi(t) \equiv 1$, the necessary part of the theorem of F.Riesz follows from the proved theorem.

References

1 Натансон И.П. Теория функций вещественной переменной / И.П. Натансон. — М.: Гостехиздат, 1950. — 480 с.

2 Медведев Ю.Т. Обобщение одной теоремы Ф. Рисса / Ю.Т. Медведев // Успехи математических наук. — 1953. — Т. VIII. — Вып. 6 (58). — С. 115-118.

3 Кудрявцев Л.Д. О p-вариации отображений и суммируемости степеней производной Радона-Никодима / Л.Д. Кудрявцев // Успехи математических наук. — 1955. — Т. 10, № 2. — С. 167-174.

4 Fuglede B. On a theorem of F.Riesz / B. Fuglede // Mathematica Scandinavica. — 1955. — Vol. 3. — P.283-302.

5 Kakochashvili G. On the theorem of F. Riesz in variable Lebesgue space / G. Kakochashvili, Sh. Zviadadze // Transactions of A.Razmadze Mathematical Institute. — 2016. — Vol. 170, Issue 1. — P. 56-61.

6 Muckenhoupt B. Weighted norm inequalities for the Hardy maximal functions / B. Muckenhoupt // Transactions of the American Mathematical Society. — 1972. — Vol. 162. — P. 207–226.

7 Bitimkhan S. Hardy-Littlewood theorem for series with general monotone coefficients / S. Bitimkhan // Bulletin of the Karaganda university-mathematics. — 2018. — 2(90). — P. 43-48.

С. Битимхан, Д.Т. Алибиева

Ф. Рисстің бір теоремасын кейбір басқа кеңістіктерге жалпылау

Функцияның қосындыланатын функциядан анықталмаған интеграл түрінде болуы үшін оның абсолютті үзіліссіз болуы қажетті және жеткілікті екені анализ курсынан белгілі. Осыған байланысты $L_p, p > 1$ кеңістігіне кіретін функцияның анықталмаған интегралы болатын функцияның сипаттамалық белгісі туралы сұрақ қойылуы заңды. Жауабы Ф.Рисстің берілген функцияның Лебег кеңістігі функциясынан алынған жоғарғы шегі айнымалы интеграл арқылы жазылу шартына қатысты теоремасында. Бірөлшемді және көпөлшемді жағдайларда көптеген математиктер бұл теореманы Лебег және Орлич кеңістіктерінде талдады. Осы жұмыста авторлар Ф. Рисс теоремасын басқа функционалдық кеңістіктер үшін дәлелдеген. Ф.Рисс теоремасының жалпыламасы интеграл астындағы функция салмақты Лебег кеңістігінен болған жағдайға алынды. Сонымен бірге, жоғарыдағы интегралдық жазылудың қажетті шарты $f \in L_p\varphi(L_p)$ функциясы үшін дәлелденді.

Кілт сөздер: функция, функциялық кеңістіктер, интеграл, Ф. Рисс теоремасы, салмақты Лебег кеңістігі.

С. Битимхан, Д.Т. Алибиева

Обобщение одной теоремы Ф. Рисса на некоторые другие пространства

Из курса анализа известно, что для того чтобы функция служила неопределенным интегралом суммируемой функции, необходимо и достаточно, чтобы она была абсолютно непрерывна. В связи с этим естественно поставить вопрос о характеристическом признаке функции, являющейся неопределенным интегралом функции, входящей в $L_p, p > 1$. Ответом служит известная теорема Ф.Рисса, касающаяся условий представимости заданной функции в виде интеграла с переменным верхним пределом от функции пространства Лебега. В одномерном и многомерном случаях многие математики обобщили эту теорему для пространств Лебега и Орлича. В настоящей работе авторами предпринята попытка доказать теорему Ф.Рисса для других функциональных пространств. Получено обобщение теоремы Ф.Рисса на случай, когда подынтегральная функция из весового пространства Лебега. Также доказано необходимое условие сказанного выше представления от функции $f \in L_p\varphi(L_p)$.

Ключевые слова: функция, функциональные пространства, интеграл, теорема Ф.Рисса, весовое пространство Лебега.

References

1 Natanson, I.P. (1950). *Funktsii veshchestvennoi peremennoi [The theory of functions of a real variable].* Moscow: Hostekhizdat [in Russian].

2 Medvedev, Yu.T. (1953). Obobshchenie odnoi teoremy F.Rissa [Generalization of one theorem of F. Riesz]. *Uspekhi matematicheskikh nauk — Successes of mathematical Sciences, VIII, 6(58)*, 115–118 [in Russian].

3 Kudriavtsev, L.D. (1955). O p-variatsii otobrazhenii i summiruemosti stepenei proizvodnoi Radona-Nikodima [About p-variation mappings and the summability of degree of the Radon-Nikodim derivative]. *Uspekhi matematicheskikh nauk —Successes of mathematical Sciences, 10, 2*, 167-174 [in Russian].

4 Fuglede, B. (1955). On a theorem of F.Riesz. *Mathematica Scandinavica, 3*, 283-302.

5 Kakochashvili, G.,& Zviadadze, Sh. (2016). On the theorem of F. Riesz in variable Lebesgue space. *Transactions of A. Razmadze Mathematical Institute, 170, 1*, 56-61.

6 Muckenhoupt, B. (1972). Weighted norm inequalities for the Hardy maximal functions. *Transactions of the American Mathematical Society, 162*, 207-226.

7 Bitimkhan, S. (2018). Hardy-Littlewood theorem for series with general monotone coefficients. *Bulletin of the Karaganda University.Mathematics series, 2(90)*, 43-48.

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A boundary jumps phenomenon in the integral boundary value problem for singularly perturbed differential equations

The article is devoted to the study of the asymptotic behavior of solving an integral boundary value problem for a third-order linear differential equation with a small parameter for two higher derivatives, provided that the roots of the "additional characteristic equation" have opposite signs. In the work are constructed the fundamental system of solutions, boundary functions for singularly perturbed homogeneous differential equation and are provided their asymptotic representations. An analytical formula of solution for a given singularly perturbed integral boundary value problem is obtained. Theorem about asymptotic estimates of solution is proved. For a singularly perturbed integral boundary value problem, the growth of the solution and its derivatives at the boundary points of this segment is obtained when the small parameter tends to zero. It is established that the solution of a singularly perturbed integral boundary value problem has initial jumps at both ends of this segment. In this case, we say that there is a phenomenon of boundary jumps, which is a feature of the considered singularly perturbed integral boundary value problem. Moreover, the orders of initial jumps were different. Namely, at the point $t = 0$, there is a phenomenon of the initial jump of the first order, and at the point $t = 1$, the order of the initial jump was equal to zero. The results obtained allow us to construct uniform asymptotic expansions of solutions of nonlinear singularly perturbed integral boundary value problems.

Keywords: singularly perturbed differential equation, asymptotic estimates, boundary functions, small parameter.

Introduction

Equations containing a small parameter in the highest derivatives are called singularly perturbed equations. Such equations are mathematical models of many applied problems. A significant contribution to the development of the theory of singularly perturbed equations was made by L. Schlesinger [1], G.D. Birkhoff [2], P. Noaillon [3], W. Wasow [4], A.N. Tikhonov [5, 6], M.I. Vishik, L.A. Lyusternik [7, 8], N.N. Bogolyubov, U.A. Mitropolsky [9], A.B. Vasilieva and V.F. Butuzov [10], Trenogin, V.A. [11], R.E. O'Malley [12], W. Eckhaus [13], K.W. Chang and F.A. Howes [14], J. Kevorkian and J.D. Cole [15], P.V. Kokotovic [16], S.A. Lomov [17], M.I. Imanaliev [18], K.A. Kassymov [19–21] and others.

Initial problems with singular initial conditions for a second-order nonlinear ordinary differential equation with a small parameter were first studied by M.I. Vishik and L.A. Lyusternik [8] and K.A. Kassymov [20]. They showed that the solution of the original problem with initial values leads to the solution of a degenerate equation with altered initial conditions when a small parameter approaches zero. Such problems became known as Cauchy problems with initial jumps. The most general cases of the Cauchy problem for singularly perturbed nonlinear systems of ordinary differential and integro-differential equations, as well as for differential equations in partial derivatives of a hyperbolic type, was studied by K.A. Kassymov. Then, singularly perturbed initial and boundary value problems with initial jumps have been studied in [22–30]. In this paper, we consider general integral boundary value problems for linear ordinary differential equations of the third order with a small parameter with two highest derivatives, when the roots of an additional characteristic equation have opposite signs. It is shown that there is a phenomenon of boundary jumps. Boundary value problems without integral boundary conditions for singularly perturbed differential and integro-differential equations have been considered in [31–33].

Statement of the problem and preliminaries

Consider the singularly perturbed differential equation

$$L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A_0(t)y'' + A_1(t)y' + A_2(t)y = F(t), \tag{1}$$

with integral boundary conditions

$$h_1 y(t,\varepsilon) \equiv y(0,\varepsilon) - \int_0^1 \sum_{i=0}^{1} a_i(x) y^{(i)}(x,\varepsilon)dx = \alpha_0,$$

$$h_2 y(t,\varepsilon) \equiv y'(0,\varepsilon) - \int_0^1 \sum_{i=0}^{1} b_i(x) y^{(i)}(x,\varepsilon)dx = \alpha_1, \tag{2}$$

$$h_3 y(t,\varepsilon) \equiv y(1,\varepsilon) - \int_0^1 \sum_{i=0}^{1} c_i(x) y^{(i)}(x,\varepsilon)dx = \beta,$$

where $\varepsilon > 0$ is a small parameter, $\alpha_0, \ \alpha_1, \ \beta$ are known constants independent of ε.

We will need the following assumptions:

C1) $A_i(t) \in C^2[0,1], \ i = \overline{0,2}, \ F(t) \in C[0,1]$.

C2) The roots $\mu_i(t), \ i = 1,2$ of "additional characteristic equation"$\mu^2(t) + A_0(t)\mu(t) + A_1(t) = 0$ satisfy the following inequalities $\mu_1(t) \le -\gamma_1 < 0, \ \ \mu_2(t) \ge \gamma_2 > 0$.

C3)

$$\overline{\Delta} \equiv \ y_{20}(1)a_1(1)\left(y_{30}(1) - \int_0^1 \sum_{i=0}^{1} c_i(x) y_{30}^{(i)}(x)dx\right) + y_{20}(1)\left(1 - c_1(1)\right)\cdot \left(1 - \int_0^1 \sum_{i=0}^{1} a_i(x) y_{30}^{(i)}(x)dx\right) \neq 0.$$

We consider homogeneous singularly perturbed equation associated with (1)

$$L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A_0(t)y'' + A_1(t)y' + A_2(t)y = 0. \tag{3}$$

The system of fundamental solutions of the homogeneous singular perturbed differential equation (3) is as follows

$$y_1^{(q)}(t,\varepsilon) = \frac{1}{\varepsilon^q} \exp\left(\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx\right)\left(\mu_1^q(t) y_{10}(t) + O(\varepsilon)\right), \ \ q = \overline{0,2},$$

$$y_2^{(q)}(t,\varepsilon) = \frac{1}{\varepsilon^q} \exp\left(-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx\right)\left(\mu_2^q(t) y_{20}(t) + O(\varepsilon)\right), \ \ q = \overline{0,2}, \tag{4}$$

$$y_3^{(q)}(t,\varepsilon) = y_{30}^{(q)}(t) + O(\varepsilon), \ \ q = \overline{0,2},$$

here $\mu_1(t), \ \mu_2(t)$ are roots of the additional characteristic equation $\mu^2(t) + A_0(t)\mu(t) + A_1(t) = 0$, the functions $y_{i0}(t), \, i = \overline{1,3}$ are defined by these problems

$$y_{i0}'(t) + \frac{3\mu_i(t)\mu_i'(t) + A_0(t)\mu_i'(t) + A_2(t)}{\mu_i^2(t)} y_{i0}(t) = 0, \ \ y_{i0}(0) = 1, \ \ i = 1,2,$$

$$A_1(t)y_{30}'(t) + A_2(t)y_{30}(t) = 0, \quad y_{30}(0) = 1.$$

The asymptotic formula of Wronskian consisting of a system of fundamental solutions is expressed as follows

$$W(t,\varepsilon) = \frac{1}{\varepsilon^3}\exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu_1(x)dx - \frac{1}{\varepsilon}\int\limits_t^1 \mu_2(x)dx\right)\cdot \tag{5}$$

$$\cdot\left(\mu_1(t)\mu_2(t)(\mu_2(t)-\mu_1(t))y_{10}(t)y_{20}(t)y_{30}(t) + O(\varepsilon)\right) \neq 0.$$

Let's enter the following functions

$$K_0(t,s,\varepsilon) = \frac{P_0(t,s,\varepsilon)}{W(s,\varepsilon)}, \quad K_1(t,s,\varepsilon) = \frac{P_1(t,s,\varepsilon)}{W(s,\varepsilon)}, \tag{6}$$

where $P_0(t,s,\varepsilon)$, $P_1(t,s,\varepsilon)$ are the third order determinant obtained from the Wronskian $W(s,\varepsilon)$ by replacing the third row with $y_1(t,\varepsilon), 0, y_3(t,\varepsilon)$ and $0, y_2(t,\varepsilon), 0$ respectively. Sum of $K_0(t,s,\varepsilon)$ and $K_1(t,s,\varepsilon)$ is the Cauchy function. Therefore, these functions have the following properties

1. With respect to the variable t satisfy equation (3), i.e.

$$L_\varepsilon K_0(t,s,\varepsilon) = 0, \quad L_\varepsilon K_1(t,s,\varepsilon) = 0, \quad t \in [0,1], \quad t \neq s.$$

2. When $t = s$ satisfy the conditions

$$K_0(s,s,\varepsilon) + K_1(s,s,\varepsilon) = 0, \quad K_0'(s,s,\varepsilon) + K_1'(s,s,\varepsilon) = 0, \quad K_0''(s,s,\varepsilon) + K_1''(s,s,\varepsilon) = 1.$$

By applying formulas (5), (6), for functions $K_0(t,s,\varepsilon), K_1(t,s,\varepsilon)$ are valid the following asymptotic representations as $\varepsilon \to 0$

$$K_0^{(i)}(t,s,\varepsilon) = \varepsilon^2\left(\frac{y_{30}^{(i)}(t)}{A_1(s)y_{30}(s)} - \frac{\mu_1^i(t)y_{10}(t)}{\varepsilon^i\mu_1(s)(\mu_2(s)-\mu_1(s))y_{10}(s)}\exp\left(\frac{1}{\varepsilon}\int\limits_s^t \mu_1(x)dx\right) + O(\varepsilon)\right),$$

$$t \geq s, \quad i = \overline{0,2}. \tag{7}$$

$$K_1^{(i)}(t,s,\varepsilon) = \varepsilon^2\left(\frac{\mu_2^i(t)y_{20}(t)}{\varepsilon^i\mu_2(s)(\mu_2(s)-\mu_1(s))y_{20}(s)}\exp\left(-\frac{1}{\varepsilon}\int\limits_t^s \mu_2(x)dx\right) + O(\varepsilon)\right), \; t \leq s, \; i = \overline{0,2}.$$

Let functions $\Phi_i(t,\varepsilon), i = \overline{1,3}$ are solutions of the following problem

$$L_\varepsilon \Phi_i(t,\varepsilon) = 0, \quad i = \overline{1,3}, \quad h_k\Phi_i(t,\varepsilon) = \delta_{ki}, \quad k = \overline{1,3}, \tag{8}$$

where δ_{ki} is Kronecker symbol.

Functions $\Phi_i(t,\varepsilon), i = \overline{1,3}$ are called *boundary functions* and can be represented in the form

$$\Phi_i(t,\varepsilon) = \frac{\Delta_i(t,\varepsilon)}{\Delta(\varepsilon)}, \quad i = \overline{1,3}, \tag{9}$$

where

$$\Delta(\varepsilon) = \begin{vmatrix} h_1y_1(t,\varepsilon) & h_1y_2(t,\varepsilon) & h_1y_3(t,\varepsilon) \\ h_2y_1(t,\varepsilon) & h_2y_2(t,\varepsilon) & h_2y_3(t,\varepsilon) \\ h_3y_1(t,\varepsilon) & h_3y_2(t,\varepsilon) & h_3y_3(t,\varepsilon) \end{vmatrix},$$

$\Delta_i(t,\varepsilon)$ is the determinant obtained from $\Delta(t,\varepsilon)$ by replacing the i-th row by the fundamental system of solutions $y_1(t,\varepsilon)$,$y_2(t,\varepsilon)$,$y_3(t,\varepsilon)$ of the equation $L_\varepsilon y = 0$. By taking account formulas (2), (4), we get asymptotic representation for determinant $\Delta(\varepsilon)$:

$$\Delta(\varepsilon) = \frac{1}{\varepsilon}(\mu_1(0)\overline{\Delta} + O(\varepsilon)), \tag{10}$$

where $\overline{\Delta}$ has the form as in condition (C3).

For boundary functions $\Phi_i^{(j)}(t,\varepsilon)$, $j=\overline{0,2}$, $i=\overline{1,3}$ from (9) in view (4), (10) we obtain asymptotic representation as $\varepsilon\to 0$:

$$\Phi_1^{(j)}(t,\varepsilon)=\frac{1}{\varepsilon^{j-1}}\exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu_1(x)dx\right)\frac{\mu_1^j(t)y_{10}(t)M_{11}}{\mu_1(0)\overline{\Delta}}-$$

$$-\frac{1}{\varepsilon^j}\exp\left(-\frac{1}{\varepsilon}\int\limits_t^1 \mu_2(x)dx\right)\frac{\mu_2^j(t)y_{20}(t)\left(y_{30}(1)-\int\limits_0^1\sum\limits_{i=0}^1 c_i(x)y_{30}^{(i)}(x)dx\right)}{\overline{\Delta}}+$$

$$+\frac{y_{20}(1)\left(1-c_1(1)\right)y_{30}^{(j)}(t)}{\overline{\Delta}}+O\left(\varepsilon+\frac{1}{\varepsilon^{j-2}}\exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu_1(x)dx\right)+\frac{1}{\varepsilon^{j-1}}\exp\left(-\frac{1}{\varepsilon}\int\limits_t^1 \mu_2(x)dx\right)\right),$$

$$\Phi_2^{(j)}(t,\varepsilon)=\frac{1}{\varepsilon^{j-1}}\exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu_1(x)dx\right)\frac{\mu_1^j(t)y_{10}(t)}{\mu_1(0)}+\frac{1}{\varepsilon^{j-1}}\exp\left(-\frac{1}{\varepsilon}\int\limits_t^1 \mu_2(x)dx\right)\frac{\mu_2^j(t)y_{20}(t)M_{22}}{\mu_1(0)\overline{\Delta}}-$$

$$-\varepsilon\frac{M_{23}y_{30}^{(j)}(t)}{\mu_1(0)\overline{\Delta}}+O\left(\varepsilon^2+\frac{1}{\varepsilon^{j-2}}\exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu_1(x)dx\right)+\frac{1}{\varepsilon^{j-2}}\exp\left(-\frac{1}{\varepsilon}\int\limits_t^1 \mu_2(x)dx\right)\right),\quad j=\overline{0,2},\ (11)$$

$$\Phi_3^{(j)}(t,\varepsilon)=\frac{1}{\varepsilon^{j-1}}\exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu_1(x)dx\right)\frac{\mu_1^j(t)y_{10}(t)M_{31}}{\mu_1(0)\overline{\Delta}}+$$

$$+\frac{1}{\varepsilon^j}\exp\left(-\frac{1}{\varepsilon}\int\limits_t^1 \mu_2(x)dx\right)\frac{\mu_2^j(t)y_{20}(t)\left(1-\int\limits_0^1\sum\limits_{i=0}^1 a_i(x)y_{30}(x)^{(i)}dx\right)}{\overline{\Delta}}+$$

$$+\frac{a_1(1)y_{20}(1)y_{30}^{(j)}(t)}{\overline{\Delta}}+O\left(\varepsilon+\frac{1}{\varepsilon^{j-2}}\exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu_1(x)dx\right)+\frac{1}{\varepsilon^{j-1}}\exp\left(-\frac{1}{\varepsilon}\int\limits_t^1 \mu_2(x)dx\right)\right),$$

where

$$M_{11}=\begin{vmatrix} -b_1(1)y_{20}(1) & y'_{30}(0)-\int\limits_0^1\sum\limits_{i=0}^1 b_i(x)y_{30}^{(i)}(x)dx \\ y_{20}(1)(1-c_1(1)) & y_{30}(1)-\int\limits_0^1\sum\limits_{i=0}^1 c_i(x)y_{30}^{(i)}(x)dx \end{vmatrix},$$

$$M_{22}=\begin{vmatrix} 1+a_1(0) & 1-\int\limits_0^1\sum\limits_{i=0}^1 a_i(x)y_{30}^{(i)}(x)dx \\ c_1(0) & y_{30}(1)-\int\limits_0^1\sum\limits_{i=0}^1 c_i(x)y_{30}^{(i)}(x)dx \end{vmatrix},$$

$$M_{23}=\begin{vmatrix} 1+a_1(0) & -a_1(1)y_{20}(1) \\ c_1(0) & y_{20}(1)(1-c_1(1)) \end{vmatrix},$$

$$M_{31}=\begin{vmatrix} -a_1(1)y_{20}(1) & 1-\int\limits_0^1\sum\limits_{i=0}^1 a_i(x)y_{30}^{(i)}(x)dx \\ -b_1(1)y_{20}(1) & y'_{30}(0)-\int\limits_0^1\sum\limits_{i=0}^1 b_i(x)y_{30}^{(i)}(x)dx \end{vmatrix}.$$

From (11) we obtain the following asymptotic estimations

$$|\Phi_1^{(j)}(t,\varepsilon)| \le C + \frac{C}{\varepsilon^{j-1}} e^{-\gamma_1 \frac{t}{\varepsilon}} + \frac{C}{\varepsilon^j} e^{-\gamma_2 \frac{1-t}{\varepsilon}}, \ \ j=\overline{0,2},$$

$$|\Phi_2^{(j)}(t,\varepsilon)| \le C\varepsilon + \frac{C}{\varepsilon^{j-1}} e^{-\gamma_1 \frac{t}{\varepsilon}} + \frac{C}{\varepsilon^{j-1}} e^{-\gamma_2 \frac{1-t}{\varepsilon}}, \ \ j=\overline{0,2}, \tag{12}$$

$$|\Phi_3^{(j)}(t,\varepsilon)| \le C + \frac{C}{\varepsilon^{j-1}} e^{-\gamma_1 \frac{t}{\varepsilon}} + \frac{C}{\varepsilon^j} e^{-\gamma_2 \frac{1-t}{\varepsilon}}, \ \ j=\overline{0,2}.$$

Main result. We seek the solution of the problem (1), (2) in the form

$$y(t,\varepsilon) = \sum_{i=1}^{3} C_i \Phi_i(t,\varepsilon) + \frac{1}{\varepsilon^2} \int_0^t K_0(t,s,\varepsilon) F(s) ds - \frac{1}{\varepsilon^2} \int_t^1 K_1(t,s,\varepsilon) F(s) ds, \tag{13}$$

where $\Phi_i(t,\varepsilon), i=\overline{1,3}$ are boundary functions, $K_0(t,s,\varepsilon)$, $K_1(t,s,\varepsilon)$ are auxiliary functions expressed by formula (6), $C_i, i=\overline{1,3}$ are unknown constants.

Now, we determine the unknown constants $C_i, \ i=\overline{1,3}$ in (13). For determining these constants we substitute (13) into (2). Then, taking into account (8), we find that

$$C_1 = \alpha_0 - h_1 P(t,\varepsilon), \ \ C_2 = \alpha_1 - h_2 P(t,\varepsilon), \ \ C_3 = \beta - h_3 P(t,\varepsilon) \tag{14}$$

where

$$P(t,\varepsilon) = \frac{1}{\varepsilon^2} \int_0^t K_0(t,s,\varepsilon) F(s) ds - \frac{1}{\varepsilon^2} \int_t^1 K_1(t,s,\varepsilon) F(s) ds. \tag{15}$$

The effect on the operator h_1 to function $P(t,\varepsilon)$ is characterized by the following expression

$$h_1 P(t,\varepsilon) \equiv P(0,\varepsilon) - \int_0^1 \sum_{i=0}^{1} a_i(x) P^{(i)}(x,\varepsilon) dx = -\frac{1}{\varepsilon^2} \int_0^1 K_1(0,s,\varepsilon) F(s) ds -$$

$$- \int_0^1 a_0(x) \left(\frac{1}{\varepsilon^2} \int_0^x K_0(x,s,\varepsilon) F(s) ds - \frac{1}{\varepsilon^2} \int_x^1 K_1(x,s,\varepsilon) F(s) ds \right) dx -$$

$$- \int_0^1 a_1(x) \left(\frac{1}{\varepsilon^2} \int_0^x K_0'(x,s,\varepsilon) F(s) ds - \frac{1}{\varepsilon^2} \int_x^1 K_1'(x,s,\varepsilon) F(s) ds \right) dx =$$

$$= -\frac{1}{\varepsilon^2} \int_0^1 K_1(0,s,\varepsilon) F(s) ds - \int_0^1 F(s) \left(\int_s^1 \frac{1}{\varepsilon^2} \sum_{i=0}^{1} a_i(x) K_0^{(i)}(x,s,\varepsilon) \, dx - \right.$$

$$\left. - \int_0^s \frac{1}{\varepsilon^2} \sum_{i=0}^{1} a_i(x) K_1^{(i)}(x,s,\varepsilon) dx \right) ds = -\frac{1}{\varepsilon^2} \int_0^1 \left(K_1(0,s,\varepsilon) + \right.$$

$$\left. + \int_s^1 \sum_{i=0}^{1} a_i(x) K_0^{(i)}(x,s,\varepsilon) dx - \int_0^s \sum_{i=0}^{1} a_i(x) K_1^{(i)}(x,s,\varepsilon) dx \right) F(s) ds.$$

Then from (14) the constant C_1 defined by the formula

$$C_1=\alpha_0+\frac{1}{\varepsilon^2}\int\limits_0^1\left(K_1(0,s,\varepsilon)+\int\limits_s^1\sum_{i=0}^1 a_i(x)K_0^{(i)}(x,s,\varepsilon)dx-\int_0^s\sum_{i=0}^1 a_i(x)K_1^{(i)}(x,s,\varepsilon)dx\right)F(s)ds. \quad (16)$$

Using formula (7) to (16), we get for the constant C_1 the following asymptotic estimation as $\varepsilon\to 0$:

$$C_1=\alpha_0+\int\limits_0^1\left(a_1(s)+\int\limits_s^1\sum_{i=0}^1 a_i(x)\frac{y_{30}^{(i)}(x)dx}{y_{30}(s)}\right)\frac{F(s)}{A_1(s)}ds+O(\varepsilon). \quad (17)$$

In this way, the effect on the operators $h_2,\ h_3$ to the function $P(t,\varepsilon)$, we define the constants $C_2,\ C_3$:

$$C_2=\alpha_1+\frac{F(0)}{\mu_2^2(0)(\mu_2(0)-\mu_1(0))}+\int\limits_0^1\left(b_1(s)+\int\limits_s^1\sum_{i=0}^1 b_i(x)\frac{y_{30}^{(i)}(x)}{y_{30}(s)}dx\right)\frac{F(s)}{A_1(s)}ds+O(\varepsilon), \quad (18)$$

$$C_3=\beta-\int\limits_0^1\left(\frac{y_{30}(1)}{y_{30}(s)}+c_1(s)-\int\limits_s^1\sum_{i=0}^1 c_i(x)\frac{y_{30}^{(i)}(x)}{y_{30}(s)}dx\right)\frac{F(s)}{A_1(s)}ds+O(\varepsilon). \quad (19)$$

Substituting (7) into (15), we have the asymptotic representation of the function $P^{(j)}(t,\varepsilon),\ j=\overline{0,2}$ as $\varepsilon\to 0$:

$$P^{(j)}(t,\varepsilon)=\frac{\mu_1^{j-2}(t)-\mu_2^{j-2}(t)}{\varepsilon^{j-1}(\mu_2(t)-\mu_1(t))}F(t)+\int\limits_0^t\frac{y_{30}^{(j)}(t)F(s)}{A_1(s)y_{30}(s)}ds-$$

$$-\frac{\mu_1^j(t)y_{10}(t)F(0)}{\varepsilon^{j-1}\mu_1^2(0)(\mu_2(0)-\mu_1(0))}e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_1(x)dx}+\frac{\mu_2^j(t)y_{20}(t)F(1)}{\varepsilon^{j-1}\mu_2^2(1)y_{20}(1)(\mu_2(1)-\mu_1(1))}e^{-\frac{1}{\varepsilon}\int\limits_t^1\mu_2(x)dx}+O(\varepsilon),\quad j=\overline{0,2}. \quad (20)$$

Thus, the following theorem holds.

Theorem 1. Let the conditions (C1)-(C3) are valid. Then integral boundary value problem (1), (2) on the interval $[0,1]$ has an unique solution and expressed by the formula

$$y(t,\varepsilon)=\sum_{i=1}^3 C_i\Phi_i(t,\varepsilon)+P(t,\varepsilon), \quad (21)$$

where $\Phi_i(t,\varepsilon),i=\overline{1,3}$ are boundary functions, $P(t,\varepsilon)$ is defined by the formula (15), $C_i,i=\overline{1,3}$ have the form (14) and are expressed by the asymptotic formulas (17), (18), (19).

Theorem 2. If conditions (C1)-(C3) are valid, then solution for integral boundary value problem (1), (2) hold the following asymptotic estimates as $\varepsilon\to 0$:

$$|y^{(j)}(t,\varepsilon)|\le C\left(|\alpha_0|+\varepsilon|\alpha_1|+|\beta|+\max_{0\le t\le 1}|F(t)|\right)+\frac{C}{\varepsilon^{j-1}}|\mu_1^{j-2}(t)-\mu_2^{j-2}(t)|\max_{0\le t\le 1}|F(t)|+$$

$$+\frac{C}{\varepsilon^{j-1}}\left(|\alpha_0|+|\alpha_1|+|\beta|+\max_{0\le t\le 1}|F(t)|\right)e^{-\gamma_1\frac{t}{\varepsilon}}+$$

$$+\frac{C}{\varepsilon^j}\left(|\alpha_0|+\varepsilon|\alpha_1|+|\beta|+\max_{0\le t\le 1}|F(t)|\right)e^{-\gamma_2\frac{1-t}{\varepsilon}},j=\overline{0,2}, \quad (22)$$

where $C>0$ is a constant independent of ε.

Proof. By applying formulas (17)–(19), (12), (20) in (21), we get asymptotic representations of the solution of the problem (1), (2) as $\varepsilon \to 0$:

$$y^{(j)}(t,\varepsilon) = \left(\alpha_0 + \int_0^1 \left(a_1(s) + \int_s^1 \sum_{i=0}^{1} a_i(x)\frac{y_{30}^{(i)}(x)}{y_{30}(s)}dx\right)\frac{F(s)}{A_1(s)}ds + O(\varepsilon)\right)\cdot$$

$$\cdot\left(\frac{\mu_1^j(t)y_{10}(t)M_{11}}{\varepsilon^{j-1}\cdot\mu_1(0)\overline{\Delta}}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx} + \frac{\mu_2^j(t)y_{20}(t)\left(y_{30}(1) + \int_0^1 \sum_{i=0}^{1} c_i(x)y_{30}^{(i)}(x)dx\right)}{\varepsilon^j\cdot\overline{\Delta}}e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx} + \right.$$

$$\left.+\frac{y_{20}(1)y_{30}^{(j)}(t)(1-c_1(1))}{\overline{\Delta}} + O\left(\varepsilon + \frac{1}{\varepsilon^{j-2}}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx} + \frac{1}{\varepsilon^{j-1}}e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx}\right)\right) +$$

$$+\left(\alpha_1 + \frac{F(0)}{\mu_2^2(0)(\mu_2(0)-\mu_1(0))} + \int_0^1 \left(b_1(s) + \int_s^1 \sum_{i=0}^{1} b_i(x)\frac{y_{30}^{(i)}(x)}{y_{30}(s)}dx\right)\frac{F(s)}{A_1(s)}ds + O(\varepsilon)\right)\cdot$$

$$\cdot\left(-\frac{\mu_1^j(t)y_{10}(t)M_{21}}{\varepsilon^{j-1}\cdot\mu_1(0)\overline{\Delta}}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx} + \frac{\mu_2^j(t)y_{20}(t)M_{22}}{\varepsilon^{j-1}\cdot\mu_1(0)\overline{\Delta}}e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx} - \varepsilon\frac{M_{23}y_{30}^{(j)}(t)}{\mu_1(0)\overline{\Delta}} + \right.$$

$$\left.+O\left(\varepsilon^2 + \frac{1}{\varepsilon^{j-2}}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx} + \frac{1}{\varepsilon^{j-1}}e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx}\right)\right) + \tag{23}$$

$$+\left(\beta - \int_0^1 \left(\frac{y_{30}(1)}{y_{30}(s)} - c_1(s) - \int_s^1 \sum_{i=0}^{1} c_i(x)\frac{y_{30}^{(i)}(x)}{y_{30}(s)}dx\right)\frac{F(s)}{A_1(s)}ds + O(\varepsilon)\right)\cdot$$

$$\cdot\left(\frac{\mu_1^j(t)y_{10}(t)M_{31}}{\varepsilon^{j-1}\cdot\mu_1(0)\overline{\Delta}}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx} + \frac{\mu_2^j(t)y_{20}(t)\left(1 - \int_0^1 \sum_{i=0}^{1} a_i(x)y_{30}^{(i)}(x)dx\right)}{\varepsilon^j\cdot\overline{\Delta}}e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx} + \right.$$

$$\left.+\frac{a_1(1)y_{20}(1)y_{30}^{(j)}(t)}{\overline{\Delta}} + O\left(\varepsilon + \frac{1}{\varepsilon^{j-2}}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx} + \frac{1}{\varepsilon^{j-1}}e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx}\right)\right) + \int_0^t \frac{y_{30}^{(j)}(t)F(s)}{A_1(s)y_{30}(s)}ds +$$

$$+\frac{\mu_1^{j-2}(t) - \mu_2^{j-2}(t)}{\varepsilon^{j-1}(\mu_2(t)-\mu_1(t))}F(t) - \frac{\mu_1^j(t)y_{10}(t)F(0)}{\varepsilon^{j-1}\cdot\mu_1^2(0)(\mu_2(0)-\mu_1(0))}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx} +$$

$$\frac{\mu_2^j(t)y_{20}(t)F(1)}{\varepsilon^{j-1}\cdot\mu_1^2(1)y_{20}(1)(\mu_2(1)-\mu_1(1))}e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx} + O(\varepsilon).$$

From asymptotic representations (23), we obtain asymptotic estimations (22). Theorem 2 is proved.

The theorem 2 implies that the solution of the problem (1), (2) at point $t = 0$ has *the phenomenon of the first order initial jump* and at point $t = 1$ has *the phenomenon of the zero order initial jump*, i.e.

$$y(0,\varepsilon) = O(1),\ y'(0,\varepsilon) = O(1),\ y''(0,\varepsilon) = O\left(\frac{1}{\varepsilon}\right)$$

and

$$y(1,\varepsilon) = O(1),\ y'(1,\varepsilon) = O\left(\frac{1}{\varepsilon}\right),\ y''(1,\varepsilon) = O\left(\frac{1}{\varepsilon^2}\right).$$

In this case, we say that the solution of the boundary value problem (1), (2) has *the phenomenon of the boundary jumps.*

Conclusion

In this paper, we consider a three-point boundary value problem for a third-order linear differential equation with a small parameter at two highest derivatives when the roots of the "additional characteristic equation" have negative signs. Theorem about asymptotic estimates of solution is proved. It is established that the solution of this integral boundary value problem has the phenomenon of boundary jumps. This means that the points of the initial jump are not only the left, but also the right point of the segment. The results allow us to construct uniform asymptotic expansions of solutions of boundary value problems with boundary jumps with any degree of accuracy with respect to a small parameter.

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References

1 Schlesinger L. Uber asymptotische darstellungen der losungen linearer differential systeme als funktionen eines parameters / L. Schlesinger // Mathematische Annalen. — 1907. — Vol. 63, No. 3. — P. 277-300.

2 Birkhoff G.D. On the asymptotic character of the solutions of certain linear differential equations containing a parameter / G.D. Birkhoff // Transactions of the American Mathematical Society. — 1908. — Vol. 9, No. 2. — P. 219-231.

3 Noaillon Developpements asymptotiques dans les equations differentielles lineaires a parameter variable / Noaillon // Mem. Soc. Sci. Liege. — 1912. — Vol. 3, No. 11. — P. 197.

4 Wasow W. Singular perturbations of boundary value problems for nonlinear differential equations of the second order / W. Wasow // Comm. On Pure and Appl. Math. — 1956. — No. 9. — P. 93-113.

5 Тихонов А.Н. О зависимости решений дифференциальных уравнений от малого параметра / А.Н. Тихонов // Матем. сб. — 1948. — Т. 22, Вып. 2. — С. 193-204.

6 Тихонов А.Н. О системах дифференциальных уравнений, содержащих параметры / А.Н. Тихонов // Матем. сб. — 1950. — Т. 27, Вып. 69. — С. 147-156.

7 Vishik M.I. Regular degeneration and boundary layer for linear differential equations with small parameter multiplying the highest derivatives / M.I. Vishik, L.A. Lyusternik // Usp. Mat. Nauk. — 1957. — No. 12. — P. 3-122; Amer. Math. Soc. Transl. — 1962. — Vol. 20, No. 2. — P. 239-364.

8 Vishik M.I. On the initial jump for non-linear differential equations containing a small parameter / M.I. Vishik, L.A. Lyusternik // Doklady Akademii Nauk SSSR. — 1960. — Vol. 132, No. 6. — P. 1242-1245.

9 Bogoliubov N.N. The method of integral manifolds in nonlinear mechanics, In Contributions to Differential Equations / N.N. Bogoliubov, Yu.A. Mitropolsky // Interscience-John Wiley. — 1963. — No. 2. — P. 123-196.

10 Vasil'eva A.B. The boundary function method for singular perturbation problems / A.B. Vasil'eva, A.B. Butuzov, L. Kalachev // SIAM Studiesin Applied Mathematics. — Philadeplhia, 1995.

11 Trenogin V.A. Development and applications of the asymptotic method of Liusternik and Vishik / V.A. Trenogin // Usp. Mat. Nauk. — 1970. — No. 25. — P. 123-156; Transl. in Russ. Math. Surv. — No. 25. — P. 119-156.

12 O'Malley R. Singular perturbations methods for ordinary differential equations / R. O'Malley. — Berlin, Heidelberg: Springer-Verlag, 1991.

13 Eckhaus W. Fundamental concepts of matching / W. Eckhaus // SIAM Rev. — 1994. — No. 36. — P. 431-439.

14 Chang K. Nonlinear Singular Perturbation Phenomena: Theory and Application / K. Chang, K. Howes. — New York: Springer-Verlag, 1984.

15 Kevorkian J. Singular perturbation methods in applied mathematics / J. Kevorkian, J.D. Cole. — Berlin, Heidelberg: Springer-Verlag, 1981.

16 Kokotovic P.V. Applications of singular perturbation techniques to control problems / P.V. Kokotovic // SIAM Rev. — 1984. — No. 26. — P. 501-550.

17 Lomov S.A. Introduction to the general theory of singular perturbations / S.A. Lomov. — American Mathematical Society, Providence, 1992.

18 Иманалиев М.И. Асимптотические методы в теории сингулярно возмущенных интегро-дифференциальных систем / М.И. Иманалиев. — Фрунзе: Илим, 1972.

19 Касымов К.А. Об асимптотике решения задачи Коши с большими начальными условиями для нелинейного обыкновенного дифференциального уравнения, содержащего малый параметр / К.А. Касымов // Успехи математических наук. — 1962. — Т. 17, Вып. 5. — P. 187–188.

20 Касымов К.А. О задаче с начальным скачком для нелинейных систем дифференциальных уравнений, содержащих малый параметр / К.А. Касымов // Докл. АН СССР. — 1968. — Т. 179, Вып. 2. — P. 275-278.

21 Касымов К.А. Асимптотика решения задачи с начальными скачками для системы дифференциальных уравнений гиперболического типа с малым параметром при производной / К.А. Касымов // Докл. АН СССР. — 1971. — Т. 196, Вып. 2. — С. 274-277.

22 Абильдаев Е.А. Асимптотические оценки решений сингулярно возмущенных краевых задач с начальными скачками для линейных дифференциальных уравнений / Е.А. Абильдаев, К.А. Касымов // Дифференциальные уравнения. — 1992. — Т. 28, Вып. 10. — P. 1659-1668.

23 Kassymov K.A. Estimates of solutions of the Cauchy problem with an arbitrary-order initial jump for linear singularly perturbed integro-differential equations / K.A. Kassymov, M.K. Dauylbaev // Differential Equations. — 1999. — Vol. 35, No. 6. — P. 822-830.

24 Dauylbaev M.K. The asymptotic behavior of solutions to singularly perturbed nonlinear integro-differential equations / M.K. Dauylbaev // Siberian Mathematical Journal. — 2000. — Vol. 41, No. 1. — P. 49-60.

25 Kassymov K.A. Asymptotic behavior of solutions of linear singularly perturbed general separated boundary-value problems with initial jump / K.A.Kassymov, D.N. Nurgabyl // Ukrainian Mathematical Journal. — 2003. — Vol. 55, No. 11. — P. 1777-1792.

26 Kassymov K.A. Asymptotic estimates of solution of a singularly perturbed boundary value problem with an initial jump for linear differential equations / K.A.Kassymov, D.N. Nurgabyl // Differential Equations. — 2004. — Vol. 40, No. 5. — P. 641-651.

27 Kassymov K.A. Asymptotic estimates for the solutions of boundary-value problems with initial jump for linear differential equations with small parameter in the coefficients of derivatives / K.A.Kassymov, D.N. Nurgabyl, A.B. Uaissov // Ukrainian Mathematical Journal. — 2013. — Vol. 65, No. 5. — P. 694-708.

28 Nurgabul D. Asymptotic estimates for the Solution of a Restoration Problem with Initial Jump / D. Nurgabul // Journal of Applied Mathematics.– Vol.2014. — Article ID 956402.

29 Dauylbayev M.K. The initial jumps of solutions and integral terms in singular BVP of linear higher order integro-differential equations / M.K. Dauylbayev, N. Atakhan // Miskolc Math. Notes. — 2015. — Vol. 16, No. 2. — P. 747-761.

30 Akhmet M. A singularly perturbed differential equation with piecewise constant argument of generalized type / M. Akhmet, M. Dauylbayev, A. Mirzakulova // Turkish Journal of Mathematics. — 2018. — Vol. 42, No. 4. — P. 1680 - 1685.

31 Касымов К.А. Асимптотические оценки решения краевой задачи для сингулярно возмущенных линейных дифференциальных уравнений третьего порядка / К.А. Касымов, Ж.У. Шарипова // Вестн. КазГУ им. С.М. Кирова. Сер. мат. — 1993. — Вып. 1. — P. 146-150.

32 Касымов К.А. Представление решения краевой задачи для линейного дифференциального уравнения с малым параметром при старших производных / К.А. Касымов, Д.А. Жакыпбекова, Д.Н. Нургабыл // Вестн. КазНУ им. аль-Фараби. Сер. мат., мех., инф. — 2001. — Вып. 3. — P. 73–78.

33 Dauylbaev M.K. Boundary-value problems with initial jumps for singularly perturbed integro-differential equations / M.K. Dauylbaev, A.E. Mirzakulova // Journal of Mathematical Sciences. — 2017. — Vol. 222, No. 3. — P. 214-225.

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Сингулярлы ауытқыған дифференциалдық теңдеулерге арналған интегралдық шеттік есептегі шекаралық секірістер құбылысы

Мақала қосымша сипаттаушы теңдеудің түбірлері қарама-қарсы болған жағдайдағы екі жоғарғы туындыларының алдында кіші параметрі бар үшінші ретті сызықты дифференциалдық теңдеу үшін шекаралы секірісті жалпы интегралды шеттік есебін зерттеуге арналған. Жұмыста қосымша сипаттаушы теңдеудің түбірлері қарама-қарсы болған жағдайдағы сингулярлы ауытқыған біртекті дифференциалдық теңдеудің іргелі шешімдер жүйесі құрылған. Іргелі шешімдер жүйесі арқылы сингулярлы ауытқыған біртекті дифференциалдық теңдеудің $K_i(t, s, \varepsilon), i = 0, 1$ көмекші функциялары және шекаралық функциялары берілген. Және олардың асимптотикалық сипаттары мен бағалаулары келтірілген. Берілген сингулярлы ауытқыған жалпы интегралды шеттік есеп шешімінің аналитикалық формуласы алынған. Шешімнің асимптотикалық бағалауы туралы теорема дәлелденген. Сингулярлы ауытқыған жалпы интегралды шеттік есеп шешімі кесіндінің екі жақ шетінде де бастапқы секіріске ие болатыны анықталған. Зерттеу нәтижесінде есеп шешімінің сол жақ және оң жақ нүктелерінде әртүрлі ретті бастапқы секіріс құбылыстарын және алынған нәтижелердің қорытындысында берілген шеттік есептің шешімінің $t = 0$ нүктесінде бірінші ретті, ал $t = 1$ нүктесінде нөлінші ретті бастапқы секірістері бар екендігі анықталды. Алынған нәтижелер сызықты емес сингулярлы ауытқыған интегралды шеттік есептер шешімдерінің біркелкі асимптотикалық жіктелуін құруға мүмкіндік береді.

Кілт сөздер: сингулярлы ауытқыған дифференциалдық теңдеу, асимптотикалық бағалау, шекаралық функциялар, кіші параметр.

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Явление граничных скачков в интегральной краевой задаче для сингулярно возмущенных дифференциальных уравнений

Статья посвящена исследованию асимптотического поведения решения интегральной краевой задачи для линейного дифференциального уравнения третьего порядка с малым параметром при двух старших производных при условии, когда корни «дополнительного характеристического уравнения» имеют противоположные знаки. В работе построена фундаментальная система решений сингулярно возмущенного однородного дифференциального уравнения с учетом противоположности знаков корней «дополнительного характеристического уравнения». Затем с помощью фундаментальной системы решений строятся вспомогательные функции и граничные функции сингулярно возмущенного однородного дифференциального уравнения. Получены асимптотические представления и оценки вспомогательных и граничных функций. Получена аналитическая формула решения рассматриваемой сингулярно возмущенной интегральной краевой задачи. Доказана теорема об асимптотических оценках решения. Для сингулярно возмущенной интегральной краевой задачи получен рост решения и его производных в граничных точках данного отрезка при стремлении малого параметра к нулю. Установлено, что решение сингулярно возмущенной интегральной краевой задачи имеет начальные скачки на обоих концах данного отрезка. В этом случае мы говорим, что имеет место явление граничных скачков, что является особенностью рассматриваемой сингулярно возмущенной интегральной краевой задачи. Причем порядки начальных скачков оказались разными. А именно: в точке $t = 0$ имеет место явление начального скачка первого порядка, а в точке $t = 1$ порядок начального скачка оказался равным нулю. Полученные результаты позволяют построить равномерные асимптотические разложения решений нелинейных сингулярно возмущенных интегральных краевых задач.

Ключевые слова: сингулярно возмущенное дифференциальное уравнение, асимптотические оценки, граничные функции, малый параметр.

References

1 Schlesinger, L. (1907). Uber asymptotische darstellungen der losungen linearer differential systeme als funktionen eines parameters. *Mathematische Annalen, 63(3),* 277-300.

2 Birkhof, G.D. (1908). On the asymptotic character of the solutions of certain linear differential equations containing a parameter. *Transactions of the American Mathematical Society, 9(2),* 219-231.

3 Noaillon. (1912). Developpements asymptotiques dans les equations differentielles lineaires a parameter variable. *Mem. Soc. Sci. Liege., 3(11),* 197.

4 Wasow, W. (1956). Singular perturbations of boundary value problems for nonlinear differential equations of the second order. *Comm. On Pure and Appl. Math., 9,* 93-113.

5 Tikhonov, A.N. (1948). O zavisimosti reshenii differentsialnykh uravnenii ot maloho parametra. *Matematicheskii sbornik, 22(2),* 193-204 [in Russian].

6 Tikhonov, A.N. (1950). O sistemakh differentsialnykh uravnenii, soderzhashchikh parametry. *Matematicheskii sbornik, 27(69),* 147-156 [in Russian].

7 Vishik, M.I., & Liusternik, L.A. (1957, 1962). Regular degeneration and boundary layer for linear differential equations with small parameter multiplying the highest derivatives. *Usp. Mat. Nauk, 12,* 3-122 [in Russian]; *Amer. Math. Soc. Transl., 20, 2,* 239-364.

8 Vishik, M. I., & Liusternik, L.A. On the initial jump for non-linear differential equations containing a small parameter. *Doklady Akademii Nauk SSSR, 132(6),* 1242-1245.

9 Bogoliubov, N.N., & Mitropolsky, Yu.A. (1963). The method of integral manifolds in nonlinear mechanics, In Contributions to Differential Equations. *Interscience-John Wiley, 2,* 123-196.

10 Vasil'eva, A. B., Butuzov, A. B., & Kalachev, L. (1995). The boundary function method for singular perturbation problems. *SIAM Studiesin Applied Mathematics, Philadeplhia.*

11 Trenogin, V.A. (1970). Development and applications of the asymptotic method of Liusternik and Vishik. *Usp. Mat. Nauk, 25,* 123-156; *Transl. in Russ. Math. Surv. 25,* 119-156.

12 O'Malley, R. (1991). *Singular perturbations methods for ordinary differential equations.* Berlin, Heidelberg: Springer-Verlag.

13 Eckhaus, W. (1994). Fundamental concepts of matching. *SIAM Rev., 36,* 431-439.

14 Chang, K., & Howes, K. (1984). *Nonlinear Singular Perturbation Phenomena: Theory and Application.* New York: Springer Verlag.

15 Kevorkian, J., & Cole, J.D. (1981). *Singular perturbation methods in applied mathematics.* Berlin, Heidelberg: Springer-Verlag.

16 Kokotovic P.V. (1984). Applications of singular perturbation techniques to control problems. *SIAM Rev, 26,* 501-550.

17 Lomov, S. A. (1992). *Introduction to the general theory of singular perturbations.* American Mathematical Society, Providence.

18 Imanaliev, M.I. (1972). *Asimptoticheskie metody v teorii sinhuliarno vozmushchennykh intehro-differentsialnykh sistem.* Frunze: Ilim [in Russian].

19 Kassymov, K.A. (1962). Ob asimptotike resheniia zadachi Koshi s bolshimi nachalnymi usloviiami dlia nelineinoho obyknovennoho differentsialnoho uravneniia, soderzhashcheho malyi parametr. *Uspekhi matematicheskikh nauk, 17(5),* 187, 188 [in Russian].

20 Kassymov, K.A. (1968). O zadache s nachalnym skachkom dlia nelineinykh sistem differentsialnykh uravnenii, soderzhashchikh malyi parametr. *DAN SSSR, 179(2),* 275-278 [in Russian].

21 Kassymov, K.A. (1971). Asimptotika resheniia zadachi s nachalnymi skachkami dlia sistemy differentsialnykh uravnenii hiperbolicheskoho tipa s malym parametrom pri proizvodnoi. *DAN SSSR, 196(2),* 274-277 [in Russian].

22 Abildaev, E.A., & Kassymov, K.A. (1992). Asimptoticheskie otsenki reshenii sinhuliarno vozmuschennykh kraevikh zadach s nachalnimi skachkami dlia lineinikh differentsialnikh uravnenii. *Differentsialnie uravneniia, 28(10),* 1659-1668 [in Russian].

23 Kassymov, K.A., & Dauylbaev, M.K. (1999). Estimates of solutions of the Cauchy problem with an arbitrary-order initial jump for linear singularly perturbed integro-differential equations. *Differential Equations, 35(6),* 822-830.

24 Dauylbaev, M.K. (2000). The asymptotic behavior of solutions to singularly perturbed nonlinear integro-differential equations. *Siberian Mathematical Journal, 41(1),* 49-60.

25 Kassymov, K.A., & Nurgabyl, D.N. (2003). Asymptotic behavior of solutions of linear singularly perturbed general separated boundary-value problems with initial jump. *Ukrainian Mathematical Journal, 55(11),* 1777-1792.

26 Kassymov, K.A., & Nurgabyl, D.N. (2004). Asymptotic estimates of solution of a singularly perturbed boundary value problem with an initial jump for linear differential equations. *Differential Equations, 40(5),* 641-651.

27 Kassymov, K.A., Nurgabyl D.N., & Uaissov A.B. (2013). Asymptotic estimates for the solutions of boundary-value problems with initial jump for linear differential equations with small parameter in the coefficients of derivatives. *Ukrainian Mathematical Journal, 65(5),* 694-708.

28 Nurgabul D. (2014). Asymptotic estimates for the Solution of a Restoration Problem with Initial Jump *Journal of Applied Mathematics, Vol.2014, Article ID 956402.*

29 Dauylbayev, M.K., & Atakhan, N. (2015). The initial jumps of solutions and integral terms in singular BVP of linear higher order integro-differential equations. *Miskolc Math. Notes, 16(2),* 747-761.

30 Akhmet, M., Dauylbayev, M., & Mirzakulova, A. (2018). A singularly perturbed differential equation with piecewise constant argument of generalized type. *Turkish Journal of Mathematics, 42(4),* 1680-1685.

31 Kassymov, K.A., & Sharipova, Zh.U. (1993). Asimptoticheskie otsenki resheniia kraevoi zadachi dlia sinhuliarno vozmushchennykh lineinykh differentsialnykh uravnenii treteho poriadka. *Vestnik KazHU imeni S.M. Kirova. Seriia matematicheskaia, 1,* 146-150 [in Russian].

32 Kassymov, K.A., Zhakipbekova, D.A., & Nurgabyl, D.N. (2001). Predstavlenie resheniia kraevoi zadachi dlia lineinoho differentsialnoho uravneniha s malym parametrom pri starshikh proizvodnykh. *Vestnik KazNU imeni al'-Farabi. Seriia matematika, mekhanika, informatika, 3,* 73-78 [in Russian].

33 Dauylbaev, M.K., & Mirzakulova, A.E. (2017). Boundary-value problems with initial jumps for singularly perturbed integro-differential equations. *Journal of Mathematical Sciences, 222(3),* 214-225.

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UDC 51

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On the parallel surfaces of the non-developable surfaces

In the differential geometry of curves and surfaces, the curvatures of curves and surfaces are often calculated and results are given. In particular, the results given by using the apparatus of the curve-surface pair are important in terms of what kind of surface the surface indicates. In this study, some relationships between curvatures of the parallel surface pair (X, X^r) via structure functions of non-developable ruled surface $X(u, v) = a(u) + vb(u)$ are established such that $a(u)$ is striction curve of non-developable surface and $b(u)$ is a unit spherical curve in E^3. Especially, it is examined whether the non-developable surface X^r is minimal surface, linear Weingarten surface and Weingarten surface. X and its parallel X^r are expressed on the Helicoid surface sample. It is indicated on the figure with the help of SWP. Moreover, curvatures of curve-surface pairs (X, a) and (X^r, β) are investigated and some conclusions are obtained.

Keywords: parallel surfaces, non-developable ruled surface, striction line, Gaussian curvature, mean curvature, curvatures of curve-surface pair.

Introduction

The parallel surfaces have an important place in the theory of surfaces. A parallel surface can be defined as the locus of points at a non-zero constant distance throughout normal of surface from a regular surface [1].

A surface composed by a singly infinite system of straight lines is called a ruled surface. A developable ruled surface is a special ruled surface with the property that it has the same tangent plane at all points on one and the same straight line. We know that a ruled surface is a developable ruled surface if and only if its Gaussian curvature K is zero [2; 89]. If $K \neq 0$, the ruled surface is non-developable [3; 32]. In 3-dimensional Euclidean space, a regular curve is defined by its curvature κ and torsion τ and also a curve-surface pair is defined by its curvatures κ_g, κ_n and τ_g, where κ_g, κ_n and τ_g are geodesic curvature, asymptotic curvature and geodesic torsion, respectively. The relations between the curvatures of a curve-surface pair and the curvatures of the curve can be seen in many papers [4–8].

We denote a regular parameter surface with the parameters u and v in E^3 by $X(u, v)$ and a non-developable ruled surface by

$$X(u, v) = a(u) + vb(u), \tag{1}$$

where $b^2(u) = 1$ and the parameter u is the arc length parameter of $b(u)$ as a unit spherical curve in E^3. Here, if $a'(u).b'(u) = 0$, $a(u)$ is striction line of ruled surface [9-10].

In this paper, firstly, we obtain the parallel surface $X^r(u, v)$ of the non-developable ruled surface $X(u, v)$. Then, we calculate Gaussian and mean curvature of the parallel surface $X^r(u, v)$. Later, we determine relations between Gaussian and mean curvatures of parallel surface pair by means of structure functions of non-developable surface. Furthermore, we show that $X^r(u, v)$ is a Weingarten surface. Finally, we give some theorems and results by calculating geodesic curvature, asymptotic curvature and geodesic torsion of curve-surface pairs (X, a) and(X^r, β).

Preliminaries

Let $a : I \to X$ be a unit speed curve lying on X such that X is a regular surface in Euclidean 3-space. We know that the Frenet frame $\{T, N, B\}$ correspond at each point of the curve $a(u)$ because $a(u)$ is a space curve, where u is arc length parameter. Other than this frame, we can talk about frame called Darboux frame of $a(u)$ in E^3. The Darboux frame is denoted by $\{T, Y, n\}$ under the conditions that T is the unit tangent vector of $a(u)$, n is the unit normal of X and $Y = n \times T$.

Definition 2.1. The Darboux derivative formulas can be defined using the following matrix:

$$\begin{pmatrix} T' \\ Y' \\ n' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ Y \\ n \end{pmatrix},$$

where κ_g is defined as geodesic curvature, κ_n is defined as normal curvature and τ_g is defined as geodesic torsion. Furthermore, it is known that [11; 248]

$$\kappa_g = \left\langle a''(u), Y \right\rangle, \tag{2}$$

$$\kappa_n = \left\langle a''(u), n \right\rangle, \tag{3}$$

$$\tau_g = -\left\langle n', Y \right\rangle, \tag{4}$$

and also

a) $a(u)$ is a geodesic curve $\Leftrightarrow \kappa_g = 0$.

b) $a(u)$ is a asymptotic curve $\Leftrightarrow \kappa_n = 0$.

c) $a(u)$ is a line of curvature $\Leftrightarrow \tau_g = 0$.

Definition 2.2. Let X be a surface in E^3 with unit normal n. Parallel surface X^r of X is given by $X^r = \{P + rn_P : P \in X, r \in R \;\; and \;\; r = constant\}$, where for $r \in R$, $f(P) = P + rn_P$ defines a new surface X^r. For all P on X, $n^{f(p)} = n^p$ [1] *Theorem 2.3.* Let (X, X^r) be a parallel surface pair. Suppose that the Gauss curvatures of X and X^r be denoted by K and K^r and the mean curvature of X and X^r be denoted by H and H^r, respectively. Then, we can write [12; 212]

$$K^r = \frac{K}{1 - 2rH + r^2K}, \tag{5}$$

$$H^r = \frac{H - rK}{1 - 2rH + r^2K}. \tag{6}$$

Definition 2.4. A ruled surface in E^3 may therefore be represented in the form

$$X(a, b) : I \times E \to E^3,$$

$$(u, v) \to X(a, b)(u, v) = a(u) + vb(u)$$

such that $a : I \to E^3$, $b : I \to E^3$ are differentiable transformations. Here, $a(u)$ is called base curve and $b(u)$ is called the director curve [13; 190]

As stated previously, Gauss curvature is zero, the ruled surface is developable ruled surface. Otherwise, the surface is non-developable [14].

Definition 2.5. Suppose that the non-developable ruled surface $X(u, v)$ is given by the equation (1) in E^3. Let $a(u)$ be the striction line of the $X(u, v)$ and $b(u)$ be a unit spherical curve, where u is the arc length parameter of $b(u)$. Then, if we write as $x(u) = b(u)$, $x'(u) = \alpha(u)$, and $y(u) = \alpha(u) \times x(u)$, the spherical Frenet formulas of the curve $b(u)$ can be given by

$$x'(u) = \alpha(u),$$

$$\alpha'(u) = -x(u) + k_g(u)y(u),$$

$$y'(u) = -k_g(u)\alpha(u),$$

where $k_g(u)$ is called the spherical curvature function and $\{x(u), \alpha(u), y(u)\}$ is called the spherical Frenet frame of $b(u)$ [9-10].

Definition 2.6. Suppose that $X(u,v)$ is given by the equation (1) in E^3 and $a(u)$ is the striction line of $X(u,v)$ under the condition $a'(u) = \lambda(u)x(u) + \mu(u)y(u)$. Then, the surface $X(u,v)$ can be given by the triple $\{k_g(u), \lambda(u), \mu(u)\}$ in E^3. Here, $k_g(u)$, $\lambda(u)$ and $\mu(u)$ are defined as structure functions of the surface $X(u,v)$ in E^3 [9-10].

Definition 2.7. Suppose that $X(u,v)$ is given by the equation (1) in E^3 and $a\,(u)$ is the striction line of $X(u,v)$ under the condition $a'(u) = \lambda(u)x(u) + \mu(u)y(u)$. Here, $\{\alpha(u), x(u) = b(u), y(u)\}$ is the spherical Frenet frame of $b(u)$. If $\lambda(u) \neq 0$, $X(u,v)$ is described as pitched ruled surface [9–10].

Let $X(u,v)$ be given by equation (1). In this case, the coefficients of the first fundamental form of $X(u,v)$

$$E = \lambda^2(u) + \mu^2(u) + \nu^2,$$

$$F = \lambda(u),$$

$$G = 1.$$

The unit normal of the surface $X(u,v)$ is

$$n = \frac{-\mu(u)\alpha(u) + vy(u)}{\sqrt{\mu^2(u) + v^2}}.$$

The second fundamental quantities of $X(u,v)$ are

$$e = \frac{-\left(\lambda(u) - k_g(u)\mu(u)\right)\mu(u) + \left(\mu'(u) + k_g(u)v\right)v}{\sqrt{\mu^2\,(u) + v^2}},$$

$$f = \frac{-\mu(u)}{\sqrt{\mu^2(u) + v^2}},$$

$$g = 0.$$

As a result of these calculations,

$$K(u,v) = \frac{-\mu^2(u)}{\left(\mu^2(u) + v^2\right)^2} \tag{7}$$

and

$$H(u,v) = \frac{k_g(u)v^2 + \mu'(u)v + k_g(u)\mu^2(u) + \lambda(u)\mu(u)}{2\sqrt{\left(\mu^2(u) + v^2\right)^3}}, \tag{8}$$

where K and H are the Gauss and mean curvature of $X(u,v)$, respectively [9-10].

Proposition 2.8. Suppose that the surface $X(u,v)$ is given by (1) such that $\lambda(u)$, $\mu(u)$ and $k_g(u)$ are the structure functions of $X(u,v)$. If λ, μ and k_g are constants, the surface $X(u,v)$ is a Weingarten surface [10]

The curvatures of the parallel surface pairs

Suppose that $X\,(u,v)$ is given by (1) in E^3. By definition of parallel surface, we obtain

$$X^r\,(u,v) = X\,(u,v) + r\,n\,(u,v)\,,$$

$$X^r\left(u,v\right)=a\left(u\right)+\nu b\left(u\right)+r\left[\frac{-\mu\left(u\right)\alpha\left(u\right)+vy\left(u\right)}{\sqrt{\mu^2\left(u\right)+v^2}}\right].$$

Theorem 3.1. Suppose that $X\left(u,v\right)$ is given by (1) in E^3 and $X^r\left(u,v\right)$ is parallel surface of the surface $X(u,v)$. Then, the relationships between Gauss and mean curvature of $X(u,v)$ and $X^r\left(u,v\right)$ are given, respectively, by

$$K^r=\frac{-\mu^2}{\left(\mu^2+v^2\right)^2-r\sqrt{\mu^2+v^2}\left(k_gv^2+\mu'v+k_g\mu^2+\lambda\mu\right)-r^2\mu^2} \tag{9}$$

and

$$H^r=\frac{\sqrt{\mu^2+v^2}\left(k_gv^2+\mu'v+k_g\mu^2+\lambda\mu\right)+2r\mu^2}{2(\mu^2+v^2)^2-2r\sqrt{\mu^2+v^2}\begin{pmatrix}k_gv^2+\mu'v\\+k_g\mu^2+\lambda\mu\end{pmatrix}-2r^2\mu^2}. \tag{10}$$

Proof. Combining the equations (5)-(8), we can easily obtain the equations (9)-(10).

Corollary 3.2. If $k_g=\lambda=0$ and μ is a constant, $k_gv^2+\mu'v+k_g\mu^2+\lambda\mu=0$. This mean that, $H=0$. In this case, X is minimal but X^r is not minimal surface. Because, $H^r\neq 0$ under the conditions that $k_g=\lambda=0$ and μ is a constant.

Corollary 3.3. X is not a linear Weingarten surface. Because, $a\neq 0$ and $b\neq 0$ are not constants satisfying $aH+bK=1$. Similarly, X^r is also not linear Weingarten surface.

Example 3.4. Let us consider that $X(u,v)=a(u)+vb(u)=(v\cos u,v\sin u,u)$ is a helicoid surface. Here, we choose $a(u)=(0,0,u),b(u)=(\cos u,\sin u,0)=x(u)$. Hence, we obtain

$$b'(u)=x'(u)=(-\sin u,\cos u,0)=\alpha(u),$$

$$y(u)=\alpha(u)\times x(u)=(0,0,-1),$$

$$\alpha'(u)=(-\cos u,-\sin u,0)=-x(u)+k_gy(u), \tag{11}$$

$$y'(u)=0=k_g\alpha(u). \tag{12}$$

From the equations (11)-(12), we find $k_g=0$. Moreover,

$$a'(u)=(0,0,1)=\lambda x+\mu y=\lambda(\cos u,\sin u,0)+\mu(0,0,-1) \tag{13}$$

and from the equation (13), we find $\lambda=0$ and $\mu=-1$. Hence, we obtain unit surface normal as follows:

$$n=\frac{1}{\sqrt{1+v^2}}(-\sin u,\cos u,-v)$$

In this case, we can write

$$X^r(u,v)=(v\cos u,v\sin u,u)+\frac{r}{\sqrt{1+v^2}}(-\sin u,\cos u,-v),$$

where $X^r(u,v)$ is parallel surface of $X(u,v)$. For $r=1$, the images of $X(u,v)$ and $X^r(u,v)$ are shown in Figure 1.

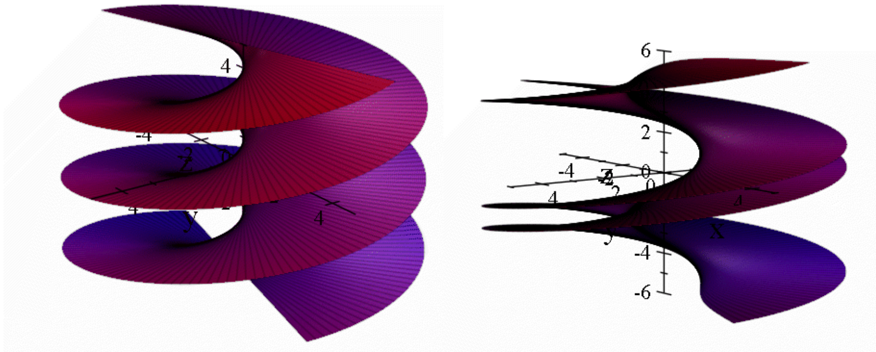

Figure 1. $X(u;v)$ and $X^r(u;v)$.

Theorem 3.5. Suppose that any non-developable ruled surface is given by (1) in E^3 such that $a'(u) = \lambda(u)x(u) + \mu(u)y(u)$, Then, $X^r(u,v)$ is a Weingarten surface if and only if λ, μ and k_g are constants.

Proof. It is known that if $X(u,v)$ is a Weingarten surface,

$$K_u H_v = K_v H_u, \tag{14}$$

where K and H are Gauss and mean curvature of $X(u,v)$, respectively. In this case, from the equations (9)–(10), we find the following partial derivatives with respect to u and v:

$$K_u^r = \frac{-\mu\left(\begin{array}{c} \left(2v^4 - 2\mu^4\right)\mu'\sqrt{\mu^2+v^2} \\ +r\left(\begin{array}{c} (v^3\mu + v\mu^3)\mu'' + \left(-2v^3 - v\mu^2\right)(\mu')^2 \\ +\left(\begin{array}{c} -2v^4k_g - v^2k_g\mu^2 \\ +k_g\mu^4 - v^2\mu\lambda \end{array}\right)\mu' \\ +\left(\mu^2+v^2\right)\left(\left(\mu^2+v^2\right)k_g' + \lambda'\mu\right)\mu \end{array}\right) \end{array}\right)}{\sqrt{\mu^2+v^2}\left(r\sqrt{\mu^2+v^2}w - \mu^4 + \left(r^2 - 2v^2\right)\mu^2 - v^4\right)^2}, \tag{15}$$

$$K_v^r = \frac{-\mu^2\left(\begin{array}{c} 3k_g\mu^2v + 3k_gv^3 - 4\mu^2\sqrt{\mu^2+v^2}v \\ -4\sqrt{\mu^2+v^2}v^3 + \mu'\mu^2 + 2\mu'v^2 + \lambda\mu v \end{array}\right)}{\sqrt{\mu^2+v^2}\left(\sqrt{\mu^2+v^2}w - \mu^4 + \left(r^2 - 2v^2\right)\mu^2 - v^4\right)^2}, \tag{16}$$

$$H_u^r = \frac{1}{2}\left(\begin{array}{c} \left(\begin{array}{c} \sqrt{\mu^2+v^2}w \\ +2r\mu^2 \end{array}\right)\left(\begin{array}{c} -r\sqrt{\mu^2+v^2}t - \frac{rw\mu\mu'}{\sqrt{\mu^2+v^2}} \\ +4\left(\mu^2+v^2\right)\mu\mu' - 2r^2\mu\mu' \end{array}\right) \\ +\left(\begin{array}{c} \left(\frac{w\mu\mu'}{\sqrt{\mu^2+v^2}} + \sqrt{\mu^2+v^2}t + 4r\mu\mu'\right) \\ \left(\left(\mu^2+v^2\right)^2 - r\sqrt{\mu^2+v^2}w - r^2\mu^2\right) \end{array}\right) \end{array}\right), \tag{17}$$

$$H_v^r = \frac{1}{2}\left(\begin{array}{c} \left(\begin{array}{c} \frac{wv}{\sqrt{\mu^2+v^2}} \\ +\sqrt{\mu^2+v^2}\left(2k_gv + \mu'\right) \end{array}\right)\left(\begin{array}{c} \left(\mu^2+v^2\right)^2 \\ -r\sqrt{\mu^2+v^2}w - r^2\mu^2 \end{array}\right) \\ +\left(\sqrt{\mu^2+v^2}w + 2r\mu^2\right)\left(\begin{array}{c} 4\left(\mu^2+v^2\right)v - \frac{rwv}{\sqrt{\mu^2+v^2}} \\ -r\sqrt{\mu^2+v^2}\left(2k_gv + \mu'\right) \end{array}\right) \end{array}\right), \tag{18}$$

where $w = \lambda\mu + v^2k_g + \mu^2k_g + v\mu', t = \mu\lambda' + \lambda\mu' + 2\mu k_g\mu' + v^2{k_g}' + \mu^2{k_g}' + v\mu''$. From the equation (14), $K_u^rH_v^r - K_v^rH_u^r = 0$. Here, if we use the equations (15)-(18) and make the necessary calculations, we obtain that all the structure functions of $X(u,v)$ are constants.

Hence, from the Proposition 2.8. , we can write the following result:

Corollary 3.6. $X^r(u,v)$ is a Weingarten surface if and only if $X(u,v)$ is a Weingarten surface.

The curvatures of the parallel curve-surface pairs

The striction line $a(u)$ on $X(u,v)$ generates a Darboux frame by the vector fields $\{T,n,Y\}$, the unit tangent, the principal normal and their cross product, respectively. Hence, for $n=\frac{-\mu\alpha+\nu y}{\sqrt{\mu^2+v^2}}$

$$T=\lambda x+\mu y,$$

$$Y=n\times T=\mu^2 x-\lambda v\alpha-\lambda\mu y.$$

Using the equations (2)-(4) geodesic curvature of curve-surface pair (a,X)

$$\kappa_g=\left\langle a'',Y\right\rangle=\left\langle \lambda' x+(\lambda-\mu k_g)\,\alpha+\mu' y,\mu^2 x-\lambda\mu y\right\rangle,$$

$$\kappa_g=\mu\left(\lambda'\mu-\lambda\mu'\right), \tag{19}$$

asymptotic curvature of curve-surface pair (a,X)

$$\kappa_n=\left\langle a'',n\right\rangle=\left\langle \lambda' x+(\lambda-\mu k_g)\alpha+\mu' y,-\alpha\right\rangle$$

$$\kappa_n=\mu k_g-\lambda, \tag{20}$$

and geodesic torsion of curve-surface pair (a,X)

$$\tau_g=-\left\langle n',Y\right\rangle=-\left\langle x-k_g y,\mu^2 x-\lambda\mu y\right\rangle,$$

$$\tau_g=-\mu(\mu+\lambda k_g). \tag{21}$$

From the equations (19)–(21) and Definition 2.1, the following theorems can be written:

Theorem 4.1. Suppose that $X(u,v)$ is given by (1) in E^3 such that $a'(u)=\lambda x+\mu y$. $a(u)$ is geodesic curve if and only if $\frac{\lambda}{\mu}$ is a constant, where λ and μ are the structure functions of $X(u,v)$.

Proof. If the equation (19) equals to zero, $a(u)$ is a geodesic curve. In this case, we get

$$\kappa_g=\mu\left(\lambda'\mu-\lambda\mu'\right)=0.$$

Here, since $\mu\neq 0$, $\lambda'\mu-\lambda\mu'=0$. If we solve this differential equation, we obtain that $\frac{\lambda}{\mu}$ is a constant. This finishes the proof.

Theorem 4.2. Suppose that $X(u,v)$ is given by (1) in E^3 such that $a'(u)=\lambda x+\mu y$. $a(u)$ is asymptotic curve if and only if $k_g=\frac{\lambda}{\mu}$, where λ, μ and k_g are the structure functions of $X(u,v)$.

Proof. From the Definition 2.1., if the equation (20) equals to zero, $a(u)$ is asymptotic curve. In this case, we obtain

$$\kappa_n=\mu k_g-\lambda=0.$$

From here, we can easily get $k_g=\frac{\lambda}{\mu}$.

Theorem 4.3. Suppose that $X(u,v)$ is given by (1) in E^3 such that $a'(u) = \lambda x + \mu y$. $a(u)$ is line of curvature if and only if $k_g = -\frac{\mu}{\lambda}$, where λ, μ and k_g are the structure functions of $X(u,v)$.

Proof. From the Definition 2.1., if the equation (21) equals to zero, $a(u)$ is line of curvature. In this case, we can write

$$\tau_g = -\mu\left(\mu + \lambda k_g\right) = 0.$$

Since $\mu \neq 0$ in this last equation, we get $k_g = -\frac{\mu}{\lambda}$.

Now, the above calculations will be found for the parallel surface. By considering definition of parallel surface, image on parallel surface of the striction line $a(u)$ can be given by

$$\beta(u) = a(u) + r\,n.$$

In this case, we write Darboux frame elements T^r, Y^r, n^r of the parallel curve-surface pair

$$T^r = \frac{\left(1 + r(\lambda - \mu k_g)\right)T + r(\mu + \lambda k_g)Y}{\sqrt{\left(1 + r(\lambda - \mu k_g)\right)^2 + \left(r(\mu + k_g\lambda)\right)^2}},$$

$$Y^r = \frac{\left(-r(\mu + \lambda k_g)\right)T + \left(1 + r(\lambda - \mu k_g)\right)Y}{\sqrt{\left(1 + r(\lambda - \mu k_g)\right)^2 + \left(r(\mu + k_g\lambda)\right)^2}},$$

$$n^r = n,$$

and also we obtain geodesic curvature, asymptotic curvature and geodesic torsion, respectively as following:

$$\kappa_g^r = \frac{\begin{array}{l} -\,r^2(\mu + k_g\lambda)\left[(\lambda' - \mu' k_g - \mu {k'}_g) - (\lambda'\mu - \mu'\lambda)(\mu + k_g\lambda)\right] \\ -\,\left[1 + r(\lambda - \mu k_g)\right]^2(\lambda'\mu - \lambda\mu') \\ -\,r(1 + r(\lambda - \mu k_g))(\mu' + {k'}_g\lambda + k_g\lambda') \end{array}}{\sqrt{\left(1 + r(\lambda - \mu k_g)\right)^2 + r^2(\mu + k_g\lambda)^2}},$$

$$\kappa_n^r = -(\lambda - \mu k_g)\left(1 + r(\lambda - \mu k_g)\right) - r(\mu + k_g\lambda)^2,$$

$$\tau_g^r = \frac{-(\mu + k_g\lambda)}{\sqrt{\left(1 + r\left(\lambda - \mu k_g\right)\right)^2 + r^2(\mu + k_g\lambda)^2}}.$$

Then, we have the following theorems:

Theorem 4.4. Suppose that $X(u,v)$ is given by (1) in E^3 such that $a'(u) = \lambda x + \mu y$. $a(u)$ is line of curvature if and only if the image on parallel surface of $a(u)$ is line of curvature $\left(\tau_g = 0 \Leftrightarrow \tau_g^r = 0\right)$.

Theorem 4.5. Suppose that $X(u,v)$ is given by (1) in E^3 such that $a'(u) = \lambda x + \mu y$. If $a(u)$ is an asymptotic curve, then

$$\kappa_g^r = \frac{(\lambda'\mu - \lambda\mu')\left(1 + r^2(\mu + \lambda k_g)^2\right) - r(\mu' + \lambda {k'}_g + k_g\lambda')}{\sqrt{1 + r^2(\mu + \lambda k_g)^2}},$$

$$\kappa_n^r = -r(\mu + \lambda k_g)^2,$$

$$\tau_g^r = \frac{-(\mu + \lambda k_g)}{\sqrt{1 + r^2(\mu + \lambda k_g)^2}}.$$

Theorem 4.6. Suppose that $X(u,v)$ is given by (1) in E^3 such that $a'(u) = \lambda x + \mu y$. If $a(u)$ is line of curvature, then,

$$\kappa_g^r = \pm(\lambda'\mu - \lambda\mu')\left(1 + r(\lambda - \mu k_g)\right),$$

$$\kappa_n^r = (-\lambda + \mu k_g)(1 + r\lambda - r k_g\mu),$$

$$\tau_g^r = \tau_g.$$

Theorem 4.6. immediately gives the following results:

Corollary 4.7. Let a non-developable ruled surface be given by (1) such that λ, μ and k_g are the structure functions and the curve $a(u)$ be both the striction line and the line of curvature. Then, the image on parallel surface of $a(u)$ is asymptotic curve if and only if $k_g = \frac{\lambda}{\mu}(\kappa_n = 0 \Leftrightarrow \kappa_n^r = 0)$.

Corollary 4.8. Let a non-developable ruled surface be given by (1) such that λ, μ and k_g are the structure functions and the curve $a(u)$ be both the striction line and the line of curvature. Then, the image on parallel surface of $a(u)$ is a geodesic curve if and only if $a(u)$ is a geodesic curve $(\frac{\lambda}{\mu} = constant)$.

Theorem 4.9. Suppose that $X(u,v)$ is given by (1) in E^3 such that $a'(u) = \lambda x + \mu y$. If $\kappa_n = \frac{1}{r}$, then,

$$\kappa_g^r = \mp r(\lambda'\mu - \lambda\mu')(\mu + \lambda k_g),$$
$$\kappa_n^r = -r(\mu + \lambda k_g)^2,$$
$$\tau_g^r = -\frac{1}{r}.$$

Corollary 4.10. Let a non-developable ruled surface be given by (1) and λ, μ and k_g be the structure functions of this surface. For $\kappa_n = \frac{1}{r}$, the image on parallel surface of $a(u)$ is geodesic curve if and only if $a(u)$ is geodesic curve $(\frac{\lambda}{\mu} = constant)$.

References

1 Craig T. Note on parallel surfaces / T. Craig // Journal fur die Reine und Angewandte Mathematik. — 1883. — No. 94. — P. 162–170.

2 Kühnel W. Differential geometry, Curves-Surfaces-Manifolds // American Mathematical Society. — 2006.

3 Hicks N.J. Nots on Differential Geometry // Van Nostrand Reinhold. — 1971.

4 Kaya F.E. Harmonic curvature of the curve-surface pair under Mobius transformation / F.E. Kaya // International Journal of Physical Sciences. — 2013. — 8. — No. 21. — P. 1133–1142.

5 Kızıltuğ S. On the curves lying on parallel surfaces in the Euclidean 3-space / S. Kızıltuğ Ö. Tarakci, Y. Yayli // Journal of Advanced Research in Dynamical and Control Systems. — 2013. — 5. — No. 3. — P. 26–35.

6 Körpınar T. New inextensible flows of principal normal spherical image / T. Körpınar // Asian-European Journal of Mathematics. — 2018. — 11. — No. 1. — P. 1850001.

7 Özkaldı S. Constant angle surfaces and curves in E^3 / S.Özkaldı, Y. Yayli // International Electronic Journal of Geometry. — 2011. — 4. — No. 1. — P. 70–78.

8 Ünlütürk Y. Characteristic properties of the parallel ruled surfaces with Darboux frame in Euclidean 3-space / Y.Ünlütürk, M.Çimdiker, C. Ekici // Communication in Mathematical Modeling and Applications. — 2016. — No. 1. — P. 26–43.

9 Liu H. Invariants of non-developable ruled surfaces in Euclidean 3-space / H. Liu, Y. Yu, S.D. Jung // Contrib. Algebra Geom. — 2014. — No. 55. — P. 189–199.

10 Yu Y. Structure and characterization of ruled surfaces in Euclidean 3-space / Y. Yu, H. Liu, S.D. Jung // Applied Mathematics and Computation — 2014. — No. 233. — P. 252259.

11 O'Neill B. Elementary Differential Geometry // Academic Press. — 1996.

12 Carmo M. Differential Geometry of Curves and Surfaces // Prentice-Hall. — 1976.

13 Struik D.J. Lectures on Classical Differential Geometry // Dover Publication. — 1988.

14 Yoon D.W. On Non-Developable Ruled Surfaces in Euclidean 3-Spaces / D.W. Yoon // Indian J. pure appl. Math. — 2007. — 38. — No. 4. — P. 281–290.

А. Сакмак, Ю. Яйли

Жаймаланбайтын беттерге параллель беттер жайлы

Қисықтар мен беттердің дифференциалдық геометриясында қисықтар мен беттердің қисаюы көп қарастырылған және нәтижелер келтірілген. Атап айтқанда, қисық бет жұбының аппаратын қолдану арқылы алынған нәтижелер беттің қандай түрін нұсқайтындығында маңызды. Зерттеуде $X(u,v) = a(u) + vb(u)$ кеңейтілген сызықтық бетінің құрылымдық функциялары арқылы (X, X^r) жұптар беттерінің параллельдік қисықтықтарының арасындағы қандай да бір өзара байланыс $a(u)$ жаймаланбайтын беттің үйкелу қисығы, ал $b(u)$ E^3 В-тегі бірлік сфералық қисық болатындай етіп қойылған. Дербес жағдайда, X^r жаймаланбайтын беті минималды бет, Вейнгартен сызықтық беті және Вейнгартен беті бола ала ма, осы жағдай зерттелген. X және оның X^r параллелі геликоид бетінің образында келтірілген. Суретте бұл (SWP) беттік толқындарды қолдайтын плазма көмегімен келтірілген. Сонымен қатар, (X, a) және (X^r, β) қисық-бет жұбының қисықтықтары зерттелген және нәтижелер алынған.

Кілт сөздер: параллель беттер, жаймаланбайтын сызықтық бет, үйкелу сызығы, Гаусс қисығы, орташа қисықтық, қисық-бет жұбының қисықтығы.

А. Сакмак, Ю. Яйли

О параллельных поверхностях неразвертывающихся поверхностей

В дифференциальной геометрии кривых и поверхностей искривления кривых и поверхностей часто рассчитываются и даются результаты. В частности, результаты, полученные с использованием аппарата пары кривая — поверхность, важны с точки зрения того, на какого рода поверхность указывает поверхность. В этом исследовании некоторые взаимосвязи между кривизной параллельной поверхности пары (X, X^r) через структурные функции неразвертывающейся линейчатой поверхности $X(u,v) = a(u) + vb(u)$ устанавливаются таким образом, что $a(u)$ является кривой трения неразвертывающейся поверхности, а $b(u)$ — единичной сферической кривой в E^3. В частности, исследуется, является ли неразвертывающаяся поверхность X^r минимальной поверхностью, линейной поверхностью Вейнгартена и поверхностью Вейнгартена. X и ее параллель X^r выражены на образце поверхности геликоида. На рисунке это показано с помощью плазмы, поддерживаемой поверхностными волнами (SWP). Кроме того, исследованы кривизны пар кривая — поверхность (X, a) и (X^r, β) и получены некоторые результаты.

Ключевые слова: параллельные поверхности, неразвертывающаяся линейчатая поверхность, линия трения, кривизна Гаусса, средняя кривизна, кривизна пары кривая – поверхность.

References

1 Craig T. (1883). Note on parallel surfaces. *Journal fur die Reine und Angewandte Mathematik*, 94, 162–170.

2 Kühnel W. (2006). *Differential geometry, Curves-Surfaces-Manifolds* / American Mathematical Society.

3 Hicks N. J. (1971). *Nots on Differential Geometry* / Van Nostrand Reinhold.

4 Kaya F. E. (2013). Harmonic curvature of the curve-surface pair under Mobius transformation. *International Journal of Physical Sciences*, 8, 21, 1133–1142.

5 Kızıltuğ S., Tarakci Ö. & Yayli Y. (2013). On the curves lying on parallel surfaces in the Euclidean 3-space. *Journal of Advanced Research in Dynamical and Control Systems*, 5, 3, 26–35.

6 Körpınar T. (2018). New inextensible flows of principal normal spherical image. *Asian-European Journal of Mathematics* 11, 1, 1850001.

7 Özkaldı S. & Yayli Y. (2011). Constant angle surfaces and curves in E^3. *International Electronic Journal of Geometry*, 4, 1, 70–78.

8 Ünlütürk Y, Çimdiker M. & Ekici C. (2016). Characteristic properties of the parallel ruled surfaces with Darboux frame in Euclidean 3-space. *Communication in Mathematical Modeling and Applications*, 1, 26–43.

9 Liu H, Yu Y. & Jung S. D. (2014). Invariants of non-developable ruled surfaces in Euclidean 3-space. *Contrib. Algebra Geom.*, 55, 189–199.

10 Yu Y, Liu H. & Jung S. D. (2014). Structure and characterization of ruled surfaces in Euclidean 3-space. *Applied Mathematics and Computation*, 233, 252259.

11 O'Neill B. (1996). *Elementary Differential Geometry*, Academic Press.

12 Carmo M. (1976). *Differential Geometry of Curves and Surfaces* / Prentice-Hall.

13 Struik D.J. (1988). *Lectures on Classical Differential Geometry*, Dover Publication.

14 Yoon D.W. (2007). On Non-Developable Ruled Surfaces in Euclidean 3-Spaces, *Indian J. pure appl. Math.*, 38, 4, 281–290.

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On Solonnikov-Fasano Problem for the Burgers Equation

The paper is devoted to the questions of solvability in the Sobolev classes for boundary value problem for the Burgers equation with boundary conditions of the Solonnikov-Fasano type in degenerating domain with degenerate point at the origin. By applying the Galerkin methods and a priori estimates we prove the Existence and Uniqueness Theorems for the solution of the considered boundary value problem, as well as its regularity with increasing smoothness of given functions.

Keywords: Burgers equation, boundary value problem, Sobolev classes, degenerating domain, Galerkin methods, a priori estimates.

Introduction

Studying of the Burgers equation has a long history, part of which is given in the works [1] and [2], and also in the books [3] and [4].

In the works [1] and [2] were studied the solvability of the boundary value problems for the Burgers equation in a non-rectangular domain. If in [1] it is required that it (a non-degenerate domain) be transformed by regularly replacing (independent) variables into a rectangular domain; then in work [2] this requirement is excluded (the domain of independent variables degenerates at the initial moment of time). In Sobolev spaces, by using the Faedo-Galerkin methods and a priori estimates the existence and uniqueness of a regular solution of the boundary value problems under consideration are established.

The paper [5] studies in the angular domain the boundary value problem for the heat equation with the time derivative under boundary conditions. It is also noted there that the case of an nonhomogeneous boundary value problem "... is useful for study of some problems with free boundaries". For example, for single-phase problem "... Stefan under the following assumptions: the liquid phase with a positive temperature $u(x,t)$ occupies the segment $0 < x < s(t)$, at $x = 0$ a positive heat flow is given, and free boundary $x = s(t)$ starts at the solid boundary $x = 0$, i.e. the conditions$s(0) = 0$ is satisfied". Note that in the paper [5] the theorem on the unique solvability of the considered boundary value problem in weight Holder spaces is established.

The range of application of boundary value problems for parabolic type equations in a domain with a boundary that varies in time is quite wide. Problems of this kind arise: in the study of thermal processes in electrical contacts [6], in the processes of ecology and medicine [7], in the solution of some problems of hydromechanics [8], thermomechanics during heat stroke [9], and so on.

Voluminous literature is devoted to the study of the solvability of linear and nonlinear parabolic equations in cylindrical domains. However, regard to nonlinear boundary value problems in degenerate non-cylindrical domains, they have been studied relatively little.

For angular domains in the Lebesgue classes, we studied boundary value problems of heat conduction and established theorems on their solvability by reducing to the Volterra singular integral equations of the second kind [10], [11].

In [12] we studied various cases of nonhomogeneous boundary. In these cases, it is shown that takes place both unique solvability and non-unique solvability for the corresponding boundary value problems.

In this paper we study in the Sobolev classes issues of solvability of the boundary value problem for the Burgers equation in angular (degenerate) domain with time derivatives in boundary conditions (in some sense an analogue of the Solonnikov-Fasano problem [5] for the Burgers equation). In Sec. 1, we give the statement of the boundary value problem, with respect to which in Sec. 2 we construct a sequence of boundary value problems in non-degenerate domains. Here, using a transformation of independent variables, we come to a family of problems in the corresponding rectangular domains. A number of theorems on their unique solvability are formulated. Section 3 establishes a priori estimates for solving the above boundary value problems. In the same section, we formulate the main result of the work in the form of a theorem for the initial nonlinear boundary value problems in a degenerate triangular domain. The proofs of these theorems are given in sections 4 and 5. The work finishes with a brief conclusion.

1 Statement of the boundary value problem

Let $Q_{xt_1} = \{x, t_1 \mid 0 < x < t_1,\ 0 < t_1 < T_1 < \infty\}$ be a triangle domain, one of the vertices of which is at the origin, and Ω_{t_1} be a cross section of the domain Q_{xt_1} for fixed temporary variable $t_1 \in (0, T_1)$. In domain Q_{xt_1} we consider the following boundary value problem for the Burgers equation:

$$\partial_{t_1} u + u\partial_x u - \nu\partial_x^2 u = f,\ \ \nu > 0, \tag{1}$$

$$[\partial_{t_1} u - \partial_x u(x, t_1)]\,|_{x=0} = 0,\ [\partial_{t_1} u + 2\partial_x u(x, t_1)]\,|_{x=t_1} = 0, \tag{2}$$

where

$$f \in L_2(Q_{xt_1}) \cap C(\overline{Q}_{xt_1}). \tag{3}$$

In this paper we study the question of the existence and uniqueness of a solution to a boundary value problem (1)–(2) in the Sobolev space (throughout the paper the notation of spaces corresponds to those adopted in the book [13]):

$$u \in H^{2,1}(Q_{xt_1})/X_{xt_1} \equiv \left\{L_2(0, T_1; H^2(0, t_1)) \cap H^1(0, T_1; L_2(0, t_1))\right\}/X_{xt_1}, \tag{4}$$

where (for space V) V/X_{xt_1} is a is the quotient space over the subspace X_{xt_1} consisting of all possible constants $k = \text{const}$, defined on the set Q_{xt_1}.

2 On a family of auxiliary boundary value problems in quadrangular domains (in the form of trapezoids)

To the problem (1)–(4) we will put a family of boundary value problems, each of which is considered in the domain representing the corresponding trapezoid.

So, let $n \in \mathbb{N}^* \equiv \{n \in \mathbb{N} : n \geq n_1, 1/n_1 < T_1\}$, $Q^n_{xt_1} = \{x, t_1 :\ 0 < x < t_1,\ 1/n < t_1 < T_1 < \infty\}$ be a trapezoid, and Ω_{t_1} be a cross section of a trapezoid for a given $t_1 \in (1/n, T_1)$. Note that at the point $t_1 = 1/n$ domain $Q^n_{xt_1}$ no longer degenerates to a point, in addition, between the original domain Q_{xt_1} and the domains $Q^n_{xt_1}$ take place strict inclusions $Q^{n_1}_{xt_1} \subset Q^{n_1+1}_{xt_1} \subset ... \subset Q_{xt_1}$ and, obviously that $\lim\limits_{n\to\infty} Q^n_{xt_1} = Q_{xt_1}$.

In the non-degenerating domain $Q^n_{xt_1}$ (for each finite $n \in \mathbb{N}^*$) we consider a boundary value problem:

$$\partial_{t_1} u_n + u_n\partial_x u_n - \nu\partial_x^2 u_n = f_n, \tag{5}$$

$$[\partial_{t_1} u - \partial_x u(x, t_1)]\,|_{x=0} = 0,\ [\partial_{t_1} u_n + 2\partial_x u_n(x, t_1)]\,|_{x=t_1} = 0,\ u_n(x, t_1)|_{t_1=1/n} = 0, \tag{6}$$

$$f_n \in L_2(Q^n_{xt_1}) \cap C(\overline{Q}_{xt_1}). \tag{7}$$

We want to transform the boundary value problem (5)–(7) so that it would be placed in a rectangular domain. To do this we will transform the independent variables: let's move from variables $\{x, t_1\}$ to variables $\{y, t\}$. We have that

$$x = \frac{y}{n-t},\ t_1 = \frac{1}{n-t};\ y = \frac{x}{t_1},\ t = n - \frac{1}{t_1}; \tag{8}$$

$Q^n_{yt} = \{y, t : 0 < y < 1,\ 0 < t < T\}$ is a rectangular domain, and Ω is a cross section of the rectangle Q^n_{yt} for any fixed $t \in [0, T]$,

$$t_1 = 1/n \Leftrightarrow t = 0,\ t_1 = T_1 \Leftrightarrow t = T = n - \frac{1}{T_1}.$$

Since

$$\tilde{u}_n(y, t) \triangleq u_n\left(\frac{y}{n-t}, \frac{1}{n-t}\right),\ \ \tilde{f}_n(y, t) = f_n\left(\frac{y}{n-t}, \frac{1}{n-t}\right), \tag{9}$$

then for derivative with respect to t_1 from function $u_n(x, t_1)$ (9) we have that

$$\partial_{t_1} u_n(x, t_1) = (n-t)^2 \partial_t \tilde{u}_n(y, t) - (n-t) y\, \partial_y \tilde{u}_n(y, t).$$

Now we find derivatives from function $u_n(x, t_1)$ (9) with respect to variable x :

$$\partial_x u_n = (n-t)\, \partial_y \tilde{u}_n, \quad \partial^2_x u_n = (n-t)^2 \partial^2_y \tilde{u}_n.$$

We write down the boundary value problem (5)–(7) in domain Q^n_{yt}:

$$\partial_t \tilde{u}_n + (n-t)^{-1} \tilde{u}_n \partial_y \tilde{u}_n - \nu \partial^2_y \tilde{u}_n - y(n-t)^{-1} \partial_y \tilde{u}_n = (n-t)^{-2} \tilde{f}_n, \tag{10}$$

$$\left[\partial_t \tilde{u}_n - (n-t)^{-1} \partial_y \tilde{u}_n(y, t)\right]\big|_{y=0} = 0,\ \left[\partial_t \tilde{u}_n + (n-t)^{-1} \partial_y \tilde{u}_n\right]\big|_{y=1} = 0,\ \ 0 < t < T, \tag{11}$$

$$\tilde{u}_n(y, 0) = 0,\ \ y \in \Omega = \{y : 0 < y < 1\}. \tag{12}$$

Remark 1. The relations (8)–(9) are one-to-one, i.e. reversible. In the future, we will use this property.

Instead of (10)–(12) in domain Q^n_{yt} we will consider more general boundary value problem:

$$\partial_t \tilde{u}_n + \alpha_n(t) \tilde{u}_n \partial_y \tilde{u}_n - \nu \partial^2_y \tilde{u}_n - \gamma_n(y, t) \partial_y \tilde{u}_n = \beta_n(t) \tilde{f}_n,\ (\nu > 0), \tag{13}$$

$$[\partial_t \tilde{u}_n - \varepsilon_n(t) \partial_y \tilde{u}_n(y, t)]\,|_{y=0} = 0,\ [\partial_t \tilde{u}_n + \delta_n(t) \partial_y \tilde{u}_n]\,|_{y=1} = 0,\ \ 0 < t < T, \tag{14}$$

$$\tilde{u}_n(y, 0) = 0,\ \ y \in \Omega = \{y : 0 < y < 1\}. \tag{15}$$

where given continuous functions $\alpha_n(t)$, $\beta_n(t)$, $\gamma_n(y, t)$, $\delta_n(t)$ for each fixed number $n \in \mathbb{N}^*$ satisfy the conditions

$$\begin{array}{ll} \alpha_{1n} \le \alpha_n(t) \le \alpha_{2n},\ \beta_{1n} \le \beta_n(t) \le \beta_{2n},\ \delta_{1n} \le |\delta_n(t)| \le \delta_{2n}, & \forall\, t \in [0, T], \\ \varepsilon_{1n} \le |\varepsilon_n(t)| \le \varepsilon_{2n},\ |\gamma_n(y, t)| \le \gamma_{1n},\ |\partial_y \gamma_n(y, t)| \le \gamma_{1n}, & \forall\, \{y, t\} \in Q^n_{yt}, \end{array} \tag{16}$$

with given positive constants $\alpha_{1n},\ \alpha_{2n},\ \beta_{1n},\ \beta_{2n},\ \delta_{1n},\ \delta_{2n},\ \varepsilon_{1n},\ \varepsilon_{2n},\ \gamma_{1n}$.

Remark 2. For the coefficients of the boundary value problem (10)–(12) conditions (16) take place and accordingly take the form:

$$0 < \frac{1}{n} = \alpha_{1n} \le \alpha_n(t) = \frac{1}{n-t} \le \alpha_{2n} = T_1,\ 0 < \frac{1}{n^2} = \beta_{1n} \le \beta_n(t) = \frac{1}{(n-t)^2} \le \beta_{2n} = T_1^2,$$

$$0<\frac{2}{n}=\delta_{1n}\le\delta_n(t)=\frac{2}{n-t}\le\delta_{2n}=T_1,\ 0<\frac{1}{n}=\varepsilon_{1n}\le\varepsilon_n(t)=\frac{1}{n-t}\le\varepsilon_{2n}=T_1,$$

$$|\gamma_n(y,t)|=\frac{y}{n-t}\le\gamma_{1n}=T_1,\ \ |\partial_y\gamma_n(y,t)|=\frac{1}{n-t}\le\gamma_{1n}=T_1.$$

The following theorem holds.

Theorem 1. Let $n\in\mathbb{N}^*$ be a fixed number. Thus, if

$$\tilde f_n\in L_2(Q^n_{yt})\cap C(\overline{Q}^n_{yt})$$

and

$$\alpha_n(t),\ \beta_n(t),\ \delta_n(t),\ \gamma_n(y,t)$$

satisfy the conditions (16), then boundary value problem (13)–(15) has a unique solution

$$\tilde u_n\in H^{2,1}(Q^n_{yt})\equiv L_2(0,T;H^2(0,1))\cap H^1(0,T;L_2(0,1)),\ \tilde u_n(1,t),\ \tilde u_n(0,t)\in H^1(0,T),$$

which satisfies the estimate:

$$\|\tilde u_n\|_{H^{2,1}(Q^n_{yt})}+\|\tilde u_n(1,t)\|_{H^1(0,T)}+\|\tilde u_n(0,t)\|_{H^1(0,T)}\le K\left(\|\tilde f_n(y,t)\|_{L_2(Q^n_{yt})},\nu,B_n\right),\qquad(17)$$

where $K\,(0,\nu,B_n)=0$, $B_n=\{\alpha_{2n},\beta_{2n},\gamma_{1n},\delta_{2n},\varepsilon_{2n}\}$.

From Theorem (1) as a corollary, we obtain the following statement.

Theorem 2. Let $n\in\mathbb{N}^*$ be a fixed number. Thus, if $\tilde f_n\in L_2(Q^n_{yt})\cap C(\overline{Q}^n_{yt})$, then boundary value problem (10)–(12) has a unique solution

$$\tilde u_n\in H^{2,1}(Q^n_{yt})\equiv L_2(0,T;H^2(0,1))\cap H^1(0,T;L_2(0,1)),\ \tilde u_n(1,t),\ \tilde u_n(0,t)\in H^1(0,T),$$

which satisfies the estimate:

$$\|\tilde u_n\|_{H^{2,1}(Q^n_{yt})}+\|\tilde u_n(1,t)\|_{H^1(0,T)}+\|\tilde u_n(0,t)\|_{H^1(0,T)}\le K\left(\|\tilde f_n(y,t)\|_{L_2(Q^n_{yt})},\nu,B\right),\qquad(18)$$

where $K\,(0,\nu,B)=0$, $B=\{T_1,T_1^2,T_1,T_1,T_1\}$.

On the basis of Lemmas 1, 3 and 4 established in Section 3 below, the proof of the theorem 1 can be carried out by the Galerkin method (for example, like in [13]).

We give the correspondence of function spaces in terms of independent variables $\{y,t\}\in Q^n_{yt}$ and $\{x,t_1\}\in Q^n_{xt_1}$:

$$\tilde f_n\in L_2(Q^n_{yt})\cap C(\overline{Q}^n_{yt})\Leftrightarrow f_n\in L_2(Q^n_{xt_1})\cap C(\overline{Q}^n_{xt_1}),\qquad(19)$$

$$\tilde u_n(y,t)\in H^{2,1}(Q^n_{yt})\Leftrightarrow u_n(x,t_1)\in H^{2,1}(Q^n_{xt_1})\equiv L_2(1/n,T_1;H^2(0,t_1))\cap H^1(1/n,T_1;L_2(0,t_1)).\quad(20)$$

Further, taking into account the correspondence of spaces (19)–(20), in accordance with Theorem 2, as well as transformation formulas (8)–(9), we can formulate the following statement:

Theorem 3. Let $n\in\mathbb{N}^*$ be a fixed number. Thus, if $f_n\in L_2(Q^n_{xt_1})\cap C(\overline{Q}^n_{xt_1})$ (19), then boundary value problem (5)–(7) has a unique solution $u_n\in H^{2,1}(Q^n_{xt_1})$ (20),

$$u_n(t_1,t_1),\ u_n(0,t_1)\in H^1(1/n,T_1),$$

which satisfies the estimate:

$$\|u_n\|_{H^{2,1}(Q^n_{xt_1})}+\|u_n(t_1,t_1)\|_{H^1(1/n,T_1)}+\|u_n(0,t_1)\|_{H^1(1/n,T_1)}\le K_0\left(\|f_n(x,t_1)\|_{L_2(Q^n_{xt_1})},\nu,B\right),\quad(21)$$

where $K_0\,(0,\nu,B)=0$, $B=\{T_1,T_1^2,T_1,T_1,T_1\}$.

Theorem 3 is proved in Section 4.

3 A priori estimates for solving the problem (13)–(15). Statement of the main result

Here we establish a number of lemmas on the basis of which by using the Galerkin method we can prove the theorems (1)–(2) formulated in the previous section.

Lemma 1. There exists a positive constant K_1 independent of $\tilde{u}_n(y,t)$, such that for all $t \in (0,T]$ the following inequality holds:

$$\|\tilde{u}_n(y,t)\|^2_{L_2(0,1)} + \|\tilde{u}_n(1,t)\|^2_{L_2(0,1)} + \|\tilde{u}_n(0,t)\|^2_{L_2(0,1)} +$$

$$+\int\limits_0^t \|\partial_y \tilde{u}_n(y,\tau)\|^2_{L_2(0,1)} d\tau \le K_1\left(\|\tilde{f}_n(y,t)\|_{L_2(Q^n_{yt})}, \nu, B_n\right),$$

where $K_1\left(0,\nu,B_n\right)=0$, $B_n=\{\alpha_{2n},\beta_{2n},\gamma_{1n},\delta_{2n},\varepsilon_{2n}\}$.

Proof. Multiplying equation (13) scalarly in $L_2(0,t_1)$ by $\tilde{u}_n(x,t_1)$ and taking into account (14), we have that

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}_n(y,t)\|^2_{L_2(0,1)} + \nu\|\partial_y\tilde{u}_n(y,t)\|^2_{L_2(0,1)} =$$

$$= -\alpha_n(t)\left(\tilde{u}_n(y,t)\partial_y\tilde{u}_n(y,t),\tilde{u}_n(y,t)\right) + \left(\gamma_n(y,t)\partial_y\tilde{u}_n(y,t),\tilde{u}_n(y,t)\right) +$$

$$+\beta_n(t)\left(\tilde{f}_n(y,t),\tilde{u}_n(y,t)\right) + \nu\tilde{u}_n(y,t)\partial_y\tilde{u}_n(y,t)\Big|_{y=0}^{y=1} =$$

$$= -\alpha_n(t)\left(\tilde{u}_n(y,t)\partial_y\tilde{u}_n(y,t),\tilde{u}_n(y,t)\right) + \left(\gamma_n(y,t)\partial_y\tilde{u}_n(y,t),\tilde{u}_n(y,t)\right) +$$

$$+\beta_n(t)\left(\tilde{f}_n(y,t),\tilde{u}_n(y,t)\right) - \frac{\nu}{\delta_n(t)}\cdot\frac{d}{dt}\left|\tilde{u}_n(1,t)\right|^2 - \frac{\nu}{\varepsilon_n(t)}\cdot\frac{d}{dt}\left|\tilde{u}_n(0,t)\right|^2,$$

or

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}_n(y,t)\|^2_{L_2(0,1)} + \frac{\nu}{\delta_{2n}}\frac{d}{dt}\left|\tilde{u}_n(1,t)\right|^2 + \frac{\nu}{\varepsilon_{2n}}\frac{d}{dt}\left|\tilde{u}_n(0,t)\right|^2 + \nu\|\partial_y\tilde{u}_n(y,t)\|^2_{L_2(0,1)} \le$$

$$\le \alpha_{2n}\left|\left(\tilde{u}_n(y,t)\partial_y\tilde{u}_n(y,t),\tilde{u}_n(y,t)\right)\right| + \gamma_{1n}\left|\left(\partial_y\tilde{u}_n(y,t),\tilde{u}_n(y,t)\right)\right| + \beta_{2n}\left|\left(\tilde{f}_n(y,t),\tilde{u}_n(y,t)\right)\right|. \quad (22)$$

Since

$$\left(\tilde{u}_n(y,t)\partial_y\tilde{u}_n(y,t),\tilde{u}_n(y,t)\right) = \frac{1}{3}\left[\tilde{u}_n(y,t)\right]^3\Big|_{y=0}^{y=1} \le \frac{1}{3}\left[\left|\tilde{u}_n(1,t)\right|^3 + \left|\tilde{u}_n(0,t)\right|^3\right], \quad (23)$$

then using inequalities (23) and

$$\gamma_{1n}\left|\left(\partial_y\tilde{u}_n(y,t),\tilde{u}_n(y,t)\right)\right| \le \frac{\nu}{2}\|\partial_y\tilde{u}_n(y,t)\|^2_{L_2(0,1)} + \frac{\gamma^2_{1n}}{2\nu}\|\tilde{u}_n(y,t)\|^2_{L_2(0,1)},$$

$$\beta_{2n}\left|\left(\tilde{f}_n(y,t),\tilde{u}_n(y,t)\right)\right| \le \frac{1}{2}\|\tilde{f}_n(y,t)\|^2_{L_2(0,1)} + \frac{\beta^2_{2n}}{2}\|\tilde{u}_n(y,t)\|^2_{L_2(0,1)},$$

and integrating the relation (22) from 0 to t, taking into account (15) we get that

$$\tilde{v}_n(t) + \int\limits_0^t \|\partial_y\tilde{u}_n(y,\tau)\|^2_{L_2(0,1)} d\tau \le A_3 + A_2\int\limits_0^t\left[\tilde{v}_n(\tau) + \left(\tilde{v}_n(\tau)\right)^{3/2}\right]d\tau. \quad (24)$$

where

$$\tilde{v}_n(t) = \|\tilde{u}_n(y,t)\|^2_{L_2(0,1)} + \left|\tilde{u}_n(1,t)\right|^2 + \left|\tilde{u}_n(0,t)\right|^2,$$

$$A_1 = \min\left\{1;\frac{2\nu}{\delta_{2n}};\frac{2\nu}{\varepsilon_{2n}};\nu\right\}, \quad A_1A_2 = \max\left\{\frac{2\alpha_{2n}}{3};\frac{\gamma^2_{1n}}{\nu}+\beta^2_{2n}\right\},$$

$$A_1A_3 = \|\tilde{f}_n(y,\tau)\|^2_{L_2(Q_{yt})}.$$

From (24) we will have the following inequalities:

$$\tilde{v}_n(t) \le A_3 + A_2 \int\limits_0^t \left[\tilde{v}_n(\tau) + (\tilde{v}_n(\tau))^{3/2}\right] d\tau, \;\; t \in (0,T], \tag{25}$$

$$\int\limits_0^t \|\partial_y \tilde{u}_n(y,\tau)\|^2_{L_2(0,1)} d\tau \le A_3 + A_2 \int\limits_0^t \left[\tilde{v}_n(\tau) + (\tilde{v}_n(\tau))^{3/2}\right] d\tau, \;\; t \in (0,T].$$

For inequality (25) we will apply the following Lemma 2 from the work of Bihari I. [14] which we cite in its original formulation.

Lemma 2. Let $Y(t)$, $F(t)$ be a positive continuous functions $a \le t \le b$ and $k \ge 0$, $M \ge 0$ (constants), further $\omega(v)$ be a non-negative non-decreasing continuous function for $v \ge 0$. Then from the inequality

$$Y(t) \le V(t) \equiv k + M \int\limits_a^t F(\tau)\omega(Y(\tau))d\tau \;\; (a \le \tau \le b)$$

it follows the inequality

$$Y(t) \le V(t) \le G^{-1}\left(G(k) + M\int\limits_a^t F(\tau)d\tau\right) \;\; (a \le t \le b'), \tag{26}$$

where $b' \le b$,

$$G(v) = \int\limits_{v_0}^{v} \frac{dw}{\omega(w)} \;\; (v_0 > 0, \; v \ge 0)$$

and $v = G^{-1}(\psi)$ is an inverse function for $G(v)\colon v \to \psi$ $\left(G^{-1}(\psi)\colon \psi \to v\right.$ exists due to monotonicity $\left.G(v)\right)$.

It is obvious that variable t may belong to a sub-interval (a, b') from (a, b), so that the argument $\psi = G(k) + M\int\limits_a^t F(\tau)d\tau$ would belong to the function domain $G^{-1}(\psi)$. Therefore, it may turn out that condition (26) will be satisfied only for $a \le t \le b'$ with some definable $b' \le b$.

In our case we have

$$Y(t) = \tilde{v}_n(t), \; k = A_3, \; M = A_2, \; F(t) \equiv 1, \; \omega(v) = v + v^{3/2}, \; a = 0, \; b = T. \tag{27}$$

First of all note that by (27) $\omega(v) : (0,\infty) \to (0,\infty)$ is a strictly increasing function. We calculate the integral

$$\psi = G(v) = \int\limits_{v_0}^{v} \frac{dw}{\omega(w)} = \int\limits_{v_0}^{v} \frac{dw}{w + w^{3/2}} = \ln\left[\frac{w}{(1+2\sqrt{w})^2}\right]\Bigg|_{w=v_0}^{w=v}. \tag{28}$$

Taking into account (28), for the value ψ we have

$$\psi = \ln\left[\frac{v}{(1+2\sqrt{v})^2}\right] = \ln\left[\frac{A_3}{(1+2\sqrt{A_3})^2}\right] + A_2 t = G(A_3) + A_2 t. \tag{29}$$

Further, to find the inverse function $G^{-1}: \psi \to v$ it is necessary to solve the following algebraic equation with respect to v:

$$\frac{v}{(1+2\sqrt{v})^2} = \zeta, \text{ where } \zeta = \exp\{\psi\} \geq 0. \tag{30}$$

We reduce the equality (30) to the following quadratic equation ($v = z^2$)

$$(1-4\zeta)z^2 - 4\zeta z - \zeta = 0. \tag{31}$$

For the roots z_1 and z_2 of equation (31) the inverse functions $G^{-1}: \psi \to v$ will correspond:

$$v_1 = \begin{cases} 1/16, & \text{if } 1-4\zeta = 0, \\ \frac{\zeta}{(1-2\sqrt{\zeta})^2}, & \text{if } 1-4\zeta \neq 0, \end{cases} \qquad v_2 = \frac{\zeta}{(1+2\sqrt{\zeta})^2} \text{ for } \zeta \geq 0, \tag{32}$$

where $\zeta = \exp\{\psi\} \geq 0$.

The first inverse function from (32) is non-negative on $[0, \infty)$, but is not suitable for our purposes, since it has a discontinuity of the second kind at $1 - 2\sqrt{\zeta} = 0$. The second is devoid of this feature, it is continuous and bounded everywhere on $[0, \infty)$. Hence,

$$v = G^{-1}(\psi) = \frac{\exp\{\psi\}}{(1+2\exp\{\psi/2\})^2}. \tag{33}$$

Now from (33) and (29) we have that

$$v = G^{-1}(\psi) = G^{-1}(G(A_3) + A_2 t) = G^{-1}\left(\ln\left[\frac{A_3}{(1+2\sqrt{A_3})^2}\right] + A_2 t\right),$$

i.e.

$$v = \frac{A^2 \exp\{A_2 t\}}{4\,(1 + A\exp\{A_2 t/2\})}, \quad \text{where } 0 \leq A \equiv \frac{2\sqrt{A_3}}{1+2\sqrt{A_3}} < \infty.$$

Now, applying the inequality (26) from Lemma 2 for (25), we have the estimate

$$\tilde{v}_n(t) \leq \frac{A^2 \exp\{A_2 t\}}{4\,(1 + A\exp\{A_2 t/2\})} \leq$$

$$\leq \max_{0 \leq t \leq T} \frac{A^2 \exp\{A_2 t\}}{4\,(1 + A\exp\{A_2 t/2\})} \equiv C_1\left(\|f_\nu(y,t)\|_{L_2(Q_{yt})}, \nu, B_n\right), \quad t \in (0, T]. \tag{34}$$

It remains to get the estimate for the summand $\int\limits_0^t \|\partial_y \tilde{u}_n(y,\tau)\|^2_{L_2(0,1)} d\tau$. On the basis of estimate (34) we get

$$\int\limits_0^t \|\partial_y \tilde{u}_n(y,\tau)\|^2_{L_2(0,1)} d\tau \leq C_2\left(\|f_\nu(y,t)\|_{L_2(Q_{yt})}, \nu, B_n\right), \quad t \in (0, T]. \tag{35}$$

Note that constants C_1 and C_2 in estimates (34)–(35) satisfy conditions

$$C_1\,(0, \nu, B_n) = 0, \quad C_2\,(0, \nu, B_n) = 0.$$

Therefore, estimates (34)–(35) complete the proof of the Lemma 1.

Lemma 3. For the positive constant K_2 independent of $\tilde{u}_n(y,t)$, for all $t \in (0, T]$ the following inequality holds:

$$\|\partial_y \tilde{u}_n(y,t)\|^2_{L_2(0,1)} + \int\limits_0^t |\partial_y \tilde{u}_n(1,\tau)|^2 d\tau + \int\limits_0^t |\partial_y \tilde{u}_n(0,\tau)|^2 d\tau +$$

$$+\int_0^t \|\partial_y^2\tilde{u}_n(y,\tau)\|^2_{L_2(0,1)}d\tau \le K_2\left(\|\tilde{f}_n(y,t)\|_{L_2(Q^n_{yt})},\nu,B_n\right), \tag{36}$$

where $K_2\,(0,\nu,B_n)=0$, $B_n=\{\alpha_{2n},\beta_{2n},\gamma_{1n},\delta_{2n},\varepsilon_{2n}\}$.

Proof. Multiplying equation (13) scalarly in $L_2(0,t_1)$ by $-\partial_x^2\tilde{u}_n(x,t_1)$ and taking into account (14), we have that

$$\frac{1}{2}\frac{d}{dt}\|\partial_y\tilde{u}_n(y,t)\|^2_{L_2(0,1)}+\nu\|\partial_y^2\tilde{u}_n(y,t)\|^2_{L_2(0,1)}=$$

$$=\alpha_n(t)\left(\tilde{u}_n(y,t)\partial_y\tilde{u}_n(y,t),\partial_y^2\tilde{u}_n(y,t)\right)-\left(\gamma_n(y,t)\partial_y\tilde{u}_n(y,t),\partial_y^2\tilde{u}_n(y,t)\right)-$$

$$-\beta_n(t)\left(\tilde{f}_n(y,t),\partial_y^2\tilde{u}_n(y,t)\right)+\partial_t\tilde{u}_n(y,t)\partial_y\tilde{u}_n(y,t)\Big|_{y=0}^{y=1}=$$

$$=\alpha_n(t)\left(\tilde{u}_n(y,t)\partial_y\tilde{u}_n(y,t),\partial_y^2\tilde{u}_n(y,t)\right)-\left(\gamma_n(y,t)\partial_y\tilde{u}_n(y,t),\partial_y^2\tilde{u}_n(y,t)\right)-$$

$$-\beta_n(t)\left(\tilde{f}_n(y,t),\partial_y^2\tilde{u}_n(y,t)\right)-\frac{1}{\delta_{2n}}\left|\partial_t\tilde{u}_n(1,t)\right|^2-\frac{1}{\varepsilon_{2n}}\left|\partial_t\tilde{u}_n(0,t)\right|^2,$$

or

$$\frac{1}{2}\frac{d}{dt}\|\partial_y\tilde{u}_n(y,t)\|^2_{L_2(0,1)}+\frac{1}{\delta_{2n}}\left|\partial_t\tilde{u}_n(1,t)\right|^2+\frac{1}{\varepsilon_{2n}}\left|\partial_t\tilde{u}_n(0,t)\right|^2+$$

$$+\nu\|\partial_y^2\tilde{u}_n(y,t)\|^2_{L_2(0,1)}\le\alpha_{2n}\left|\left(\tilde{u}_n(y,t)\partial_y\tilde{u}_n(y,t),\partial_y^2\tilde{u}_n(y,t)\right)\right|+$$

$$+\gamma_{1n}\left|\left(\partial_y\tilde{u}_n(y,t),\partial_y^2\tilde{u}_n(y,t)\right)\right|+\beta_{2n}\left|\left(\tilde{f}_n(y,t),\partial_y^2\tilde{u}_n(y,t)\right)\right|. \tag{37}$$

First we consider estimate of nonlinear summand from (37). First, we have

$$\left|\left(\tilde{u}_n(y,t)\partial_y\tilde{u}_n(y,t),\partial_y^2\tilde{u}_n(y,t)\right)\right|\le\|\tilde{u}_n(y,t)\|_{L_4(0,1)}\|\partial_y\tilde{u}_n(y,t)\|_{H^1(0,1)}\|\partial_y\tilde{u}_n(y,t)\|_{L_4(0,1)}. \tag{38}$$

Next, given the interpolation inequality from ([15], Theorems 5.8–5.9, p.140–141)

$$\alpha_{2n}\|\partial_y\tilde{u}_n(y,t)\|_{L_4(0,1)}\le C\|\partial_y\tilde{u}_n(y,t)\|^{1/2}_{H^1(0,1)}\|\partial_y\tilde{u}_n(y,t)\|^{1/2}_{L_2(0,1)},\ \forall\,\partial_y\tilde{u}_n(y,t)\in H^1(0,1),$$

from (38) we get

$$\alpha_{2n}\left|\left(\tilde{u}_n(y,t)\partial_y\tilde{u}_n(y,t),\partial_y^2\tilde{u}_n(y,t)\right)\right|\le$$

$$\le C\|\tilde{u}_n(y,t)\|_{L_4(0,1)}\|\partial_y\tilde{u}_n(y,t)\|^{3/2}_{H^1(0,1)}\|\partial_y\tilde{u}_n(y,t)\|^{1/2}_{L_2(0,1)}\le$$

$$\le\frac{\nu}{8}\|\partial_y^2\tilde{u}_n(y,t)\|^2_{L_2(0,1)}+\left[\frac{\nu}{8}+C_2\|\tilde{u}_n(y,t)\|^4_{L_4(0,1)}\right]\|\partial_y\tilde{u}_n(y,t)\|^2_{L_2(0,1)}. \tag{39}$$

Here we used Young's inequality $(p^{-1}+q^{-1}=1)$:

$$|AB|=\left|\left(a^{1/p}A\right)\left(a^{1/q}\frac{B}{a}\right)\right|\le\frac{a}{p}\left|A\right|^p+\frac{a}{qa^q}\left|B\right|^q,$$

where

$$A=\|\partial_y\tilde{u}_n(y,t)\|^{3/2}_{H^1(0,1)},\quad B=C\,\|\tilde{u}_n(y,t)\|_{L_4(0,1)}\,\|\partial_y\tilde{u}_n(y,t)\|^{1/2}_{L_2(0,1)},\quad a=\frac{\nu}{6},\ \ p=\frac{4}{3},\ \ q=4.$$

Next, for the last two summands from (37) we have:

$$\gamma_{1n}\left|\left(\partial_y\tilde{u}_n(y,t),\partial_y^2\tilde{u}_n(y,t)\right)\right|\le\frac{\nu}{8}\|\partial_y^2\tilde{u}_n(y,t)\|^2_{L_2(0,1)}+C_3\|\partial_y\tilde{u}_n(y,t)\|^2_{L_2(0,1)},$$

$$\beta_{2n}\left|\left(\tilde{f}_n(y,t),\partial_y^2\tilde{u}_n(y,t)\right)\right| \le \frac{\nu}{4}\|\partial_y^2\tilde{u}_n(y,t)\|^2_{L_2(0,1)} + C_4\|\tilde{f}_n(y,t)\|^2_{L_2(0,1)}. \tag{40}$$

From (37), (39)–(40) we get

$$\frac{d}{dt}\|\partial_y\tilde{u}_n(y,t)\|^2_{L_2(0,1)} + \frac{2}{\delta_{2n}}\left|\partial_t\tilde{u}_n(1,t)\right|^2 + \frac{2}{\varepsilon_{2n}}\left|\partial_t\tilde{u}_n(0,t)\right|^2 + \nu\|\partial_y^2\tilde{u}_n(y,t)\|^2_{L_2(0,1)} \le$$

$$\le 2C_4\|\tilde{f}_n(y,t)\|^2_{L_2(0,1)} + \left[\frac{\nu}{4} + 2C_2\|\tilde{u}_n(y,t)\|^4_{L_4(0,1)} + 2C_3\right]\|\partial_y\tilde{u}_n(y,t)\|^2_{L_2(0,1)}, \tag{41}$$

or, integrating (41) on t from 0 to t, we obtain

$$\|\partial_y\tilde{u}_n(y,t)\|^2_{L_2(0,1)} + \frac{2}{\delta_{2n}}\int\limits_0^t\left|\partial_\tau\tilde{u}_n(1,\tau)\right|^2d\tau + \frac{2}{\varepsilon_{2n}}\int\limits_0^t\left|\partial_\tau\tilde{u}_n(0,\tau)\right|^2d\tau +$$

$$+\nu\int\limits_0^t\|\partial_y^2\tilde{u}_n(y,\tau)\|^2_{L_2(0,1)}d\tau \le 2C_4\|\tilde{f}_n(y,t)\|^2_{L_2(Q^n_{yt})} + \int\limits_0^t A_5(\tau)\|\partial_y\tilde{u}_n(y,\tau)\|^2_{L_2(0,1)}d\tau, \tag{42}$$

where

$$A_4 = 2C_4,\ \ A_5(t) = \frac{\nu}{4} + 2C_2\|\tilde{u}_n(y,t)\|^4_{L_4(0,1)} + 2C_3.$$

From inequality (42) in the same way as in the proof of the Lemma 1 we get the required estimate (36). Lemma 3 is completely proved.

Lemma 4. For the positive constants $\{K_3, K_4, K_5\}$ independent of $\{\tilde{u}_n(y,t),\ \tilde{u}_n(1,t),\ \tilde{u}_n(0,t)\}$, for all $t\in(0,T]$ the following inequalities hold:

$$\|\partial_t\tilde{u}_n(y,t)\|^2_{L_2(Q^n_{yt})} \le K_3\left(\|\tilde{f}_n(y,t)\|_{L_2(Q^n_{yt})},\nu,B_n\right),$$

$$\|\partial_t\tilde{u}_n(1,t)\|^2_{L_2(0,1)} \le K_4\left(\|\tilde{f}_n(y,t)\|_{L_2(Q^n_{yt})},\nu,B_n\right),$$

$$\|\partial_t\tilde{u}_n(0,t)\|^2_{L_2(0,1)} \le K_5\left(\|\tilde{f}_n(y,t)\|_{L_2(Q^n_{yt})},\nu,B_n\right),$$

where $K_3\left(0,\nu,B_n\right)=0$, $K_4\left(0,\nu,B_n\right)=0$, $K_5\left(0,\nu,B_n\right)=0$, $B_n=\{\alpha_{2n},\beta_{2n},\gamma_{1n},\delta_{2n},\varepsilon_{2n}\}$.

Proof. The statement of the Lemma 4 follows from Lemma 1 and Lemma 3, as well as from equation (13) and boundary conditions (14).

Therefore, applying the Galerkin method [13], and using the Lemmas (1), (3) and (4) we directly obtain the validity of the statement of Theorem 1 and a priori estimate (17), and with them the validity of the Theorem 2 and a priori estimate (18).

Now we can formulate the main result of our work.

Theorem 4 (Main result). Let $f(x,t_1)\in L_2(Q_{xt_1})\cap C(\overline{Q}_{xt_1})$. Then problem (1)–(2) has a unique solution (4)

$$u(x,t_1)\in H^{2,1}(Q_{xt_1})/X_{xt_1}.$$

Moreover, traces of the solution satisfy the conditions $u(t_1,t_1)$, $u(0,t_1)\in H^1(0,T_1)$.

Proof of the Theorem 4 will be given below (section 3).

4 Proof of the Theorem 3

Here we establish a series of lemmas, on the basis of which we will prove the Theorem 3 formulated in Section 2. The following three lemmas are consequences of the lemmas 1, 3–4, respectively.

Lemma 5. There exists a positive constant K_1 independent of $u_n(x,t_1)$, such that for all $t_1 \in (1/n, T_1]$ the following inequality holds:

$$\|u_n(x,t_1)\|^2_{L_2(0,t_1)} + \|u_n(t_1,t_1)\|^2_{L_2(0,t_1)} + \|u_n(0,t_1)\|^2_{L_2(0,t_1)} +$$

$$+\int\limits_{1/n}^{t_1} \|\partial_x u_n(x,\tau_1)\|^2_{L_2(0,t_1)} d\tau_1 \le K_1 \left(\|f_n(x,t_1)\|_{L_2(Q^n_{xt_1})}, \nu, B\right),$$

where $K_1(0,\nu,B) = 0$, $B = \{T_1, T_1^2, T_1, T_1, T_1\}$.

Lemma 6. For a positive constant K_2 independent of $u_n(x,t_1)$, for all $t_1 \in (1/n, T_1]$ the following inequality holds:

$$\|\partial_x u_n(x,t_1)\|^2_{L_2(0,t_1)} + |\partial_x u_n(t_1,t_1)|^2 + |\partial_x u_n(0,t_1)|^2 +$$

$$+\int\limits_{1/n}^{t_1} \|\partial_x^2 u_n(x,\tau_1)\|^2_{L_2(1/n,t_1)} d\tau_1 \le K_2 \left(\|f_n(x,t_1)\|_{L_2(Q^n_{xt_1})}, \nu, B\right),$$

where $K_2(0,\nu,B) = 0$, $B = \{T_1, T_1^2, T_1, T_1, T_1\}$.

Lemma 7. For positive constants $\{K_3, K_4, K_5\}$ independent of $\{u_n(x,t_1),\, u_n(t_1,t_1),\, u_n(0,t_1)\}$, for all $t_1 \in (1/n, T_1]$ the following inequalities hold:

$$\|\partial_{t_1} u_n(x,t_1)\|^2_{L_2(Q^n_{xt_1})} \le K_3 \left(\|f_n(x,t_1)\|_{L_2(Q^n_{xt_1})}, \nu, B\right),$$

$$\|\partial_{t_1} u_n(t_1,t_1)\|^2_{L_2(0,t_1)} \le K_4 \left(\|f_n(x,t_1)\|_{L_2(Q^n_{xt_1})}, \nu, B\right),$$

$$\|\partial_t u_n(0,t_1)\|^2_{L_2(0,t_1)} \le K_5 \left(\|f_n(x,t_1)\|_{L_2(Q^n_{xt_1})}, \nu, B\right),$$

where $K_3(0,\nu,B) = 0$, $K_4(0,\nu,B) = 0$, $K_5(0,\nu,B) = 0$, $B = \{T_1, T_1^2, T_1, T_1, T_1\}$.

Based on the statements of the Lemmas 5–7, using the Galerkin method [13], we establish the validity of the Theorem 3.

5 Proof of the Theorem 4

The proof of Theorem 4 is based on Theorem 3. In boundary value problems (5)–(7) each element of sequences

$\{u_n(x,t_1),\ f_n(x,t_1),\ \{x,t_1\} \in Q^n_{xt_1};\ u_n(t_1,t_1),\ u_n(0,t_1),\ t_1 \in$ $(1/n, T_1);\ n \in \mathbb{N}^*\}$ continue with zero, respectively, over the entire triangle domain Q_{xt_1} and the interval $(0, T_1)$. As a result, we obtain a sequence of functions denoted by

$$\left\{\widetilde{u_n(x,t_1)},\ \widetilde{f_n(x,t_1)},\ \widetilde{u_n(t_1,t_1)},\ \widetilde{u_n(0,t_1)},\ n \in \mathbb{N}^*\right\}. \tag{43}$$

Obviously, each four functions from the sequence (43) satisfies the boundary value problem (1)–(2) according to the statement of the Theorem 3. In addition, we note that estimate (21) will be strengthened if its on right side $\|\widetilde{f_n(x,t_1)}\|_{L_2(Q_{xt_1})}$ is replaced to expression $\|f(x,t_1)\|_{L_2(Q_{xt_1})}$, since

$$K_0\left(\|\widetilde{f_n(x,t_1)}\|_{L_2(Q_{xt_1})}, \nu, B\right) \le K_0\left(\|f(x,t_1)\|_{L_2(Q_{xt_1})}, \nu, B\right),$$

where $K_0(0,\nu,B) = 0$, $B = \{T_1, T_1^2, T_1, T_1, T_1\}$.

Therefore, we obtain a bounded sequence of functions (43), from which we can be extract a weakly convergent sequence, i.e. (for this subsequence, we keep the notation n for the index). We have that

$$\widetilde{u_n(x,t_1)} \to w(x,t_1) \ \text{ weakly in } \ H^{2,1}(Q_{xt_1}), \tag{44}$$

$$\widetilde{u_n(t_1,t_1)} \to w(t_1,t_1) \ \text{ weakly in } \ H^{1}(0,T_1), \tag{45}$$

$$\widetilde{u_n(0,t_1)} \to w(0,t_1) \ \text{ weakly in } \ H^{1}(0,T_1). \tag{46}$$

Since from (44) it follows that

$$\widetilde{u_n(x,t_1)} \to w(x,t_1) \ \text{ strongly in } \ L_2(Q_{xt_1}), \tag{47}$$

then by (44)–(47) we can pass to the limit as $n \to \infty$ in integral identities

$$0 = \int\limits_{Q_{xt_1}} \left[\partial_{t_1}\widetilde{u_n(x,t_1)} + \widetilde{u_n(x,t_1)}\partial_x\widetilde{u_n(x,t_1)} - \nu\partial_x^2\widetilde{u_n(x,t_1)} - \widetilde{f_n(x,t_1)}\right]\varphi(x,t_1)\,dx\,dt_1 \to$$

$$\to \int\limits_{Q_{xt_1}} \left[\partial_{t_1}w(x,t_1) + w(x,t_1)\partial_x w(x,t_1) - \nu\partial_x^2 w(x,t_1) - f(x,t_1)\right]\varphi(x,t_1)\,dx\,dt_1, \ \ \forall\,\varphi \in L_2(Q_{xt_1}), \tag{48}$$

$$0 = \int\limits_0^{T_1} \left[\partial_{t_1}\widetilde{u_n(x,t_1)} - \partial_x\widetilde{u_n(x,t_1)}\right]\Big|_{x=0}\varphi_0(t_1)\,dt_1 \to$$

$$\to \int\limits_0^{T_1} \left[\partial_{t_1}w(x,t_1) - \partial_x w(x,t_1)\right]\big|_{x=0}\varphi_0(t_1)\,dt_1 \ \ \forall\,\varphi_0 \in L_2(0,T_1), \tag{49}$$

$$0 = \int\limits_0^{T_1} \left[\partial_{t_1}\widetilde{u_n(x,t_1)} - \partial_x\widetilde{u_n(x,t_1)}\right]\Big|_{x=t_1}\varphi_1(t_1)\,dt_1 \to$$

$$\to \int\limits_0^{T_1} \left[\partial_{t_1}w(x,t_1) - \partial_x w(x,t_1)\right]\big|_{x=t_1}\varphi_1(t_1)\,dt_1 \ \ \forall\,\varphi_1 \in L_2(0,T_1). \tag{50}$$

So, we have established that the boundary value problem (1)–(2) has a weak solution $w(x,t_1)$ in the sense of integral identities (48)–(50).

Now we show the uniqueness. Let boundary value problem (1)–(2) has two different solutions $u^{(1)}(x,t_1)$ и $u^{(2)}(x,t_1)$. Then their difference $u(x,t_1) = u^{(1)}(x,t_1) - u^{(2)}(x,t_1)$ will satisfy the homogeneous boundary value problem:

$$\partial_{t_1}u + u\partial_x u - \nu\partial_x^2 u = 0, \ \ \nu > 0, \tag{51}$$

$$\left[\partial_{t_1}u - \partial_x u(x,t_1)\right]|_{x=0} = 0, \ \left[\partial_{t_1}u + 2\partial_x u(x,t_1)\right]|_{x=t_1} = 0. \tag{52}$$

We establish that the boundary value problem (51)–(52) нwill not have a non-trivial solution that differs from the constant. It is clear that

$$u(x,t_1) \in L_\infty(0,T_1;H^1(0,t_1)), \ \ u(t_1,t_1) \ \text{ and } \ u(0,t_1) \in L_\infty(0,T_1). \tag{53}$$

Multiplying equation (51) by function $u(x,t_1)$ scalarly in $L_2(0,t_1)$ and taking into account (52), we obtain

$$\frac{1}{2}\frac{d}{d\,t_1}\|u(x,t_1)\|^2_{L_2(0,t_1)}+\nu\frac{d}{d\,t_1}\,|u(t_1,t_1)|^2+\nu\frac{d}{d\,t_1}\,|u(0,t_1)|^2+\nu\|\partial_x u(x,t_1)\|^2_{L_2(0,t_1)}=$$

$$=-\left([u(x,t_1)]^2,\partial_x u^{(1)}(x,t_1)\right)-\left(u^{(2)}(x,t_1),u(x,t_1)\partial_x u(x,t_1)\right). \tag{54}$$

Integrating by parts

$$\int\limits_0^{t_1}[u(x,t_1)]^2\partial_x u^{(1)}(x,t_1)\,dx=\,|u(x,t_1)|^2u^{(1)}(x,t_1)\Big|_{x=0}^{x=t_1}-2\int\limits_0^{t_1}u^{(1)}(x,t_1)u(x,t_1)\partial_x u(x,t_1)\,dx,$$

from (54) we have that

$$\frac{1}{2}\frac{d}{d\,t_1}\|u(x,t_1)\|^2_{L_2(0,t_1)}+\nu\frac{d}{d\,t_1}\,|u(t_1,t_1)|^2+\nu\frac{d}{d\,t_1}\,|u(0,t_1)|^2+\nu\|\partial_x u(x,t_1)\|^2_{L_2(0,t_1)}=$$

$$=u^{(1)}(t_1,t_1)|u(t_1,t_1)|^2-u^{(1)}(0,t_1)|u(0,t_1)|^2+\int\limits_0^{t_1}\left[2u^{(1)}(x,t_1)-u^{(2)}(x,t_1)\right]u(x,t_1)\partial_x u(x,t_1)\,dx. \tag{55}$$

Now we estimate the right side of the relation (55). Using the (53), we obtain

$$u^{(1)}(t_1,t_1)|u(t_1,t_1)|^2-u^{(1)}(0,t_1)|u(0,t_1)|^2+\int\limits_0^{t_1}\left[2u^{(1)}(x,t_1)-u^{(2)}(x,t_1)\right]u(x,t_1)\partial_x u(x,t_1)\,dx\le$$

$$\le\|u^{(1)}(t_1,t_1)\|_{L\infty(0,T_1)}|u(t_1,t_1)|^2+\|u^{(1)}(0,t_1)\|_{L\infty(0,T_1)}|u(0,t_1)|^2+$$

$$+\frac{1}{4\nu}\left[2\|u^{(1)}(x,t_1)\|_{L\infty(Q_{xt_1})}+\|u^{(2)}(x,t_1)\|_{L\infty(Q_{xt_1})}\right]^2\|u(x,t_1)\|^2_{L_2(0,t_1)}+\nu\|\partial_x u(x,t_1)\|^2_{L_2(0,t_1)}.$$

From here and from (55) we get

$$v(t)\le C\int\limits_0^{t_1}v(\tau_1)d\tau_1,\quad \text{т.е.}\quad v(t)\equiv 0,\ \ \forall\,t_1\in(0,T_1],$$

where

$$v(t)=\min\{1,2\nu\}\left[\|u(x,t_1)\|^2_{L_2(0,t_1)}+|u(t_1,t_1)|^2+|u(0,t_1)|^2\right],$$

$$C=\min\{1,2\nu\}\max\left\{\frac{1}{2\nu}\left[2\|u^{(1)}(x,t_1)\|_{L\infty(Q_{xt_1})}+\|u^{(2)}(x,t_1)\|_{L\infty(Q_{xt_1})}\right]^2,\right.$$

$$\left.2\|u^{(1)}(t_1,t_1)\|_{L\infty(0,T_1)},2\|u^{(1)}(0,t_1)\|_{L\infty(0,T_1)}\right\}.$$

Therefore, the uniqueness of the solution of the boundary value problem (1)–(2) is established, and together with it the Theorem 4 is completely proved.

Conclusion

In the paper, we established in Sobolev classes the solvability theorems for boundary value problem for the Burgers equation in a degenerating domain with degenerate point at the origin.

The results of the work can be generalized to case when we have a domain of independent variables $Q_{xt_1} = \{x, t_1 : 0 < x < \varphi(t_1),\ 0 < t_1 < T_1 < \infty\}$, presented in a curved triangle whose moving side can change according to the rule $x = \varphi(t_1),\ t_1 \in [0, T_1]$, and the condition $\varphi(0) = 0$ is satisfied. In addition, from the function $\varphi(t_1)$ some natural conditions are required. For example, function $\varphi(t_1)$ must satisfy the following two conditions: 1^0. in a sufficiently small time interval $(0, t_1^*)$ function $\varphi(t_1)$would have a representation $\varphi(t_1) = \mu\, t_1$, where μ is a given positive constant (in our work it was equal to unity); 2^0. on the interval $[t_1^*, T_1]$ function $\varphi(t_1)$ would be continuous differentiable and would have the property of monotonicity, providing a one-to-one transformation from independent variables $\{x, t_1\}$ to variables $\{y, t\}$.

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References

1 Benia Y. Existence of solutions to Burgers equations in domains that can be transformed into rectangles / Y. Benia, B.K. Sadallah // Electronic Journal of Differential Equations. — 2016. — 157. — P. 1–13.

2 Benia Y. Existence of solutions to Burgers equations in a non-parabolic domain / Y. Benia, B.K. Sadallah // Electronic Journal of Differential Equations. — 2018. — 20. — P. 1–13.

3 Burgers J.M. The nonlinear diffusion equation. Asymptotic solutions and statistical problems / J.M. Burgers // Advances in Difference Equations. — Boston, USA: D. Reidel Publishing Company, Dordrecht-Holland. — 1974. — 174 p.

4 Вишик М.И. Математические задачи статистической гидромеханики / М.И. Вишик, А.В. Фурсиков. — М.: Наука, 1980. — 440 с.

5 Солонников В.А. Об одномерной параболической задаче, возникающей при изучении некоторых задач со свободными границами / В.А. Солонников, А. Фазано // Записки научного семинара ПОМИ. — 2000. — 269. — С. 322–338.

6 Ким Е.И. Математические модели тепловых процессов в электрических контактах / Е.И. Ким, В.Т. Омельченко, С.Н. Харин. — Алма-Ата: Наука, 1977. — 236 с.

7 Митропольский Ю.А. Задачи со свободными границами для нелинейного эволюционного уравнения в проблемах металлургии, медицины, экологии / Ю.А. Митропольский, А.А. Березовский, Т.А. Плотницкий // Укр. мат. журн. — 1992. — 44. — № 1. — С. 67–75.

8 Веригин Н.Н. Об одном классе гидромеханических задач для областей с подвижными границами / Н.Н. Веригин // Динамика жидкости со свободными границами. — СО АН СССР: — 1980. — 46. — С. 23–32.

9 Карташов Э.М. Проблема теплового удара в области с движущейся границей на основе новых интегральных соотношений / Э.М. Карташов // Изв. РАН. Энергетика. — 1997. — 4. — С. 122–137.

10 Amangaliyeva M.M. About Dirichlet boundary value problem for the heat equation in the infinite angular domain / M.M. Amangaliyeva, M.T. Jenaliyev, M.T. Kosmakova, M.I. Ramazanov // Boundary Value Problems. — 2014. — 213. — 21 p.

11 Amangaliyeva M.M. On one homogeneous problem for the heat equation in an infinite angular domain / M.M. Amangaliyeva, M.T. Jenaliyev, M.T. Kosmakova, M.I. Ramazanov // Sib. Math. Jour. — 2015. — Vol. 56.— 6. — P. 982–995.

12 Amangaliyeva, M.M. On the solvability of nonhomogeneous boundary value problem for the Burgers equation in the angular domain and related integral equations. / M.M. Amangaliyeva, M.T. Jenaliyev, M.T. Kosmakova, M.I. Ramazanov // Springer Proceedings in Mathematics and Statistics. — 2017. — 216. — P. 123–141.

13 Lions J.L. Problemes aux limites non homogenes et applications. / J.L. Lions, E. Magenes // Dunod, Paris. — 1968. — vol. 1. — 374 p.

14 Bihari I. A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations / I. Bihari // Acta Math. Acad. Scient. Hung. VII. — 1956. — 1. — P. 81–94.

15 Adams R. A. Sobolev spaces (2-nd Edition) / R. A. Adams, J. J. F. Fournier. — Elsevier, Amsterdam. — 2003. — vol. 140. — 320 p.

М.Т. Дженалиев, М.И. Рамазанов, А.А. Асетов

Бюргерс теңдеуі үшін Солонников-Фазано есебі туралы

Мақала нүктеге жойылатын облыста, Солонников-Фазано типті шекаралық шарттармен берілген Бюргерс теңдеуі үшін шекаралық есептің соболевтік кластарында шешілуі сұрақтарына арналған. Облыстың жойылу нүктесі координаталар басында орналасқан. Галеркин және априорлық бағалаулар әдістерін қолдану арқылы қарастырылып отырған шекаралық есептің шешімінің бар болуы және жалғыздығы туралы теоремалар, сонымен қатар берілген функциялардың тегістігі артқанда есептің регулярлығы дәлелденген.

Кілт сөздер: Бюргерс теңдеуі, шекаралық есеп, соболевтік кластар, жойылатын облыс, Галеркин әдісі, априорлық бағалаулар.

М.Т. Дженалиев, М.И. Рамазанов, А.А. Асетов

О задаче Солонникова-Фазано для уравнения Бюргерса

Работа посвящена вопросам разрешимости в соболевских классах граничной задачи для уравнения Бюргерса с граничными условиями типа Солонникова-Фазано в вырождающейся области. Точка вырождения области находится в начале координат. Использованием методов Галеркина и априорных оценок доказаны теоремы о существовании и единственности решения рассматриваемой граничной задачи, а также его регулярность при повышении гладкости заданных функций.

Ключевые слова: уравнение Бюргерса, граничная задача, соболевские классы, вырождающаяся область, метод Галеркина, априорные оценки.

References

1 Benia, Y. & Sadallah, B.-K. (2016). Existence of solutions to Burgers equations in domains that can be transformed into rectangles. *Electronic Journal of Differential Equations, 157,* 1–13.

2 Benia, Y. & Sadallah, B.-K. (2018). Existence of solutions to Burgers equations in a non-parabolic domain. *Electronic Journal of Differential Equations, 20,* 1–13.

3 Burgers, J.M. (1974). *The nonlinear diffusion equation. Asymptotic solutions and statistical problems. Advances in Difference Equations.* Boston, USA: D. Reidel Publishing Company, Dordrecht-Holland, 174.

4 Vishik, M.I. & Fursikov, A.V. (1980). *Matematicheskie zadachi statisticheskoi hidrodinamiki [Mathematical problems of statistical hydromechanics].* Moscow: Nauka [in Russian].

5 Solonnikov, V.A. & Fasano, A. (2000). Ob odnomernoi parabolicheskoi zadache, voznikaiushchei pri izuchenii nekotorykh zadach so svobodnymi hranitsami [One-dimensional parabolic problem arising in the study of some free boundary problems]. *Zapiski nauchnho seminara POMI, 269 (2000),* 322–338 [in Russian].

6 Kim, E.I., Omelchenko, V.T. & Kharin, S.N. (1977). *Matematicheskie modeli teplovykh protsessov v elektricheskikh kontaktakh [Mathematical models of thermal processes in electrical contacts].* Alma-Ata: Nauka [in Russian].

7 Mitropolsky, Yu.A., Berezovsky, A.A. & Plotnitskiy, T.A. (1992). Zadachi so svobodnymi hranitsami dlia nelineinoho evoliutsionnoho uravneniia v problemakh metallurhii, meditsiny, ekolohii [Problems with free boundaries for the nonlinear evolution equation in metallurgy, medicine, and ecology]. *Ukrainskii matematicheskii zhurnal, 44, No. 1,* 67–75 [in Russian].

8 Verigin, N.N. (1980). Ob odnom klasse hidromekhanicheskikh zadach dlia oblastei s podvizhnymi hranitsami [On one class of hydromechanical problems for domains with moving boundaries]. *Dinamika zhidkosti so svobodnymi hranitsami, 46,* 23–32 [in Russian].

9 Kartashov, E.M. (1997). Problema teplovoho udara v oblasti s dvizhushcheisia hranitsei na osnove novykh intehralnykh sootnoshenii [The problem of heat stroke in a domain with a moving boundary based on new integral relations]. *Izvestiia RAS. Energetics, 4,* 122–137 [in Russian].

10 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T. & Ramazanov, M.I. (2014). About Dirichlet boundary value problem for the heat equation in the infinite angular domain. *Boundary Value Problems, 213,* 21 p.

11 Amangaliyeva, M.M., Dzhenaliev, M.T., Kosmakova, M.T. & Ramazanov, M.I. (2015). On one homogeneous problem for the heat equation in an infinite angular domain *Sib. Math. Jour. Vol. 56, 6,* 982–995.

12 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T. & Ramazanov, M.I. (2017). On the solvability of nonhomogeneous boundary value problem for the Burgers equation in the angular domain and related integral equations. *Springer Proceedings in Mathematics and Statistics, 216,* 123–141.

13 Lions J.-L. & Magenes E. (1968). Problemes aux limites non homogenes et applications, Vol. 1. *Dunod, Paris,* 374.

14 Bihari, I. (1956). A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations. *Acta Math. Acad. Scient. Hung. VII, 1,* 81–94.

15 Adams, R.A. & Fournier, J.J.F. (2003). Sobolev spaces (2-nd Ed.), Vol. 140. *Elsevier, Amsterdam,* 320.

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Solvability of a semi-periodic boundary value problem for a third order differential equation with mixed derivative

This article is devoted to the study of the solvability of a semi-periodic boundary value problem for an evolution equation of the pseudoparabolic type. Nonlocal problems for high order partial differential equations have been investigated by many authors [1–4]. A certain interest in the study of these problems is caused in connection with their applied values. These problems include highly porous media with a complex topology, and first of all, soil and ground. Such equations can also describe long waves in dispersed systems. To solve this problem, new functions are introduced in the work and the method of a parameterizations applied [5]. Then the boundary value problem for a third order differential equation is reduced to a periodic boundary value problem for a family of systems of ordinary differential equations [6–18]. New constructive algorithms for finding an approximate solution are proposed and in terms of the initial data, coefficient-like signs of the unique solvability of the problem under study are obtained.

Keywords: partial differential equation, third-order pseudoparabolic equation, algorithm, approximate solution.

Introduction

On $\Omega = [0,\omega] \times [0,T]$ we consider the semi-periodic boundary value problem

$$\frac{\partial^3 u}{\partial x^2 \partial t} = A(x,t)\frac{\partial^2 u}{\partial x^2} + B(x,t)\frac{\partial u}{\partial t} + C(x,t)u + f(x,t), (x,t) \in \Omega, \tag{1}$$

$$u(x,0) = u(x,T), \quad x \in [0,\omega], \tag{2}$$

$$u(0,t) + \frac{\partial u(0,t)}{\partial x}x = \varphi(x,t), \quad (x,t) \in \Omega, \tag{3}$$

where $(n \times n)$ - are the matrices $A(x,t), B(x,t), C(x,t)$, n-vector functions $f(x,t), \varphi(x,t)$ continuous on Ω, here $\|u(x,t)\| = \max\limits_{i=\overline{1,n}} |u_i(x,t)|$, $\quad \|A(x,t)\| = \max\limits_{i=\overline{1,n}} \sum\limits_{j=1}^{n} |a_{ij}(x,t)|$.

Let $C(\Omega, R^n)$ - be the function space $u : \Omega \to R^n$ continuous on Ω, with the norm

$$\|u\|_0 = \max_{(x,t)\in\overline{\Omega}} \|u(x,t)\|.$$

A function $u(x,t) \in C(\Omega, R^n)$, having partial derivatives

$$\frac{\partial u^2(x,t)}{\partial x^2} \in C(\Omega, R^n), \quad \frac{\partial u(x,t)}{\partial t} \in C(\Omega, R^n), \quad \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \in C(\Omega, R^n)$$

is called a solution to problem (1)–(3) if it satisfies system (1) for all $(x,t) \in \Omega$, and conditions (2), (3).

To find a solution, we introduce the functions $v(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$, $w(x,t) = \frac{\partial u(x,t)}{\partial t}$ and reduce problem (1)–(3) to a family of periodic boundary value problems for a system of ordinary differential equations of the form

$$\frac{\partial v}{\partial t} = A(x,t)v + B(x,t)w + C(x,t)u + f(x,t), \quad (x,t) \in \Omega, \tag{4}$$

$$v(x,0) = v(x,T), \quad x \in [0,\omega], \tag{5}$$

functional relationships

$$w(x,t) = \varphi_t'(x,t) + \int\limits_0^x \int\limits_0^\xi \frac{\partial v(\xi_1,t)}{\partial t} d\xi_1 d\xi, \tag{6}$$

$$u(x,t) = \varphi(x,t) + \int\limits_0^x \int\limits_0^\xi v(\xi_1,t) d\xi_1 d\xi. \tag{7}$$

To solve problem (4)–(7) we apply the method of a parametrization.

By the step $h > 0 : Nh = T$ we make fragmentation $[0,T) = \bigcup\limits_{r=1}^{N} [(r-1)h, rh),\ N = 1,2,\ldots$. Moreover, the area Ω is divided into N parts. By $v_r(x,t)$, $u_r(x,t)$ we denote, respectively, the restriction of the function $v(x,t)$, $u(x,t)$ in $\Omega_r = [0,\omega] \times [(r-1)h, rh)$, $\quad r = \overline{1,N}$.

By $\lambda_r(x)$ we denote the value of the function $v_r(x,t)$ at $t = (r-1)h$, i.e. $\lambda_r(x) = v_r(x,(r-1)h)$ and make the replacement $\widetilde{v}_r(x,t) = v_r(x,t) - \lambda_r(x), r = \overline{1,N}$. We obtain an equivalent boundary value problem with unknown functions $\lambda_r(x)$:

$$\frac{\partial \widetilde{v}_r}{\partial t} = A(x,t)\widetilde{v}_r + A(x,t)\lambda_r(x) + B(x,t)w_r + (x,t)u_r + f(x,t), \tag{8}$$

$$\widetilde{v}_r(x,(r-1)h) = 0, \quad x \in [0,\omega], \quad r = \overline{1,N}, \tag{9}$$

$$\lambda_1(x) - \lambda_N(x) - \lim_{t\to T-0} \widetilde{v}_N(x,t) = 0, \quad x \in [0,\omega], \tag{10}$$

$$\lambda_s(x) + \lim_{t\to sh-0} \widetilde{v}_s(x,t) - \lambda_{s+1}(x) = 0, \quad x \in [0,\omega], \quad s = \overline{1,N-1}. \tag{11}$$

$$w_r(x,t) = \varphi_t'(x,t) + \int\limits_0^x \int\limits_0^\xi \frac{\partial \widetilde{v}_r(\xi_1,t)}{\partial t} d\xi_1 d\xi, \tag{12}$$

$$u_r(x,t) = \varphi(x,t) + \int\limits_0^x \int\limits_0^\xi \lambda_r(\xi_1) d\xi_1 d\xi + \int\limits_0^x \int\limits_0^\xi \widetilde{v}_r(\xi_1,t) d\xi_1 d\xi, \quad (x,t) \in \Omega_r, \quad r = \overline{1,N}, \tag{13}$$

where (11) is the condition for combining functions in the internal lines of the partition. Problem (8), (9) for fixed $\lambda_r(x), w_r(x,t), u_r(x,t)$ is a one-parameter family of Cauchy problems for systems of ordinary differential equations, where $x \in [0,\omega]$, and is equivalent to the integral equation

$$\widetilde{v}_r(x,t) = \int\limits_{(r-1)h}^{t} A(x,\tau)\widetilde{v}_r(x,\tau)d\tau + \int\limits_{(r-1)h}^{t} A(x,\tau)d\tau \cdot \lambda_r(x) + \int\limits_{(r-1)h}^{t} F(x,\tau,w_r,u_r)d\tau, \tag{14}$$

where

$$\int\limits_{(r-1)h}^{t} F(x,\tau,w_r,u_r)d\tau = \int\limits_{(r-1)h}^{t} B(x,\tau)w_r(x,\tau)d\tau + \int\limits_{(r-1)h}^{t} C(x,\tau)u_r(x,\tau)d\tau + \int\limits_{(r-1)h}^{t} f(x,\tau)d\tau.$$

Instead of $\widetilde{v}_r(x,\tau)$ we substitute the corresponding right-handed part of (14) and by repeating this process ν ($\nu = 1,2,\ldots$) times we obtain

$$\widetilde{v}_r(x,t) = D_{\nu r}(x,t)\lambda_r(x) + F_{\nu r}(x,t,w_r,u_r) + G_{\nu r}(x,t,\widetilde{v}_r), \quad r = \overline{1,N}, \tag{15}$$

where

$$D_{\nu r}(x,t)=\sum_{j=0}^{\nu-1}\int\limits_{(r-1)h}^{t}A(x,\tau_1)d\tau_1\ldots\int\limits_{(r-1)h}^{\tau_j}A(x,\tau_{j+1})d\tau_{j+1}\ldots d\tau_1,$$

$$F_{\nu r}(x,t,w_r,u_r)=\int\limits_{(r-1)h}^{t}\big[B(x,\tau_1)w_r(x,\tau_1)+C(x,\tau_1)u_r(x,\tau_1)+f(x,\tau_1)\big]d\tau_1+$$

$$+\sum_{j=1}^{\nu-1}\int\limits_{(r-1)h}^{t}A(x,\tau_1)\ldots\int\limits_{(r-1)h}^{\tau_{j-1}}A(x,\tau_j)\int\limits_{(r-1)h}^{\tau_j}\big[w_r(x,\tau_{j+1})+C(x,\tau_{j+1})u_r(x,\tau_{j+1})+f(x,\tau_{j+1})\big]d\tau_{j+1}d\tau_j\ldots d\tau_1,$$

$$G_{\nu r}(x,t,\widetilde{v}_r)=\int\limits_{(r-1)h}^{t}A(x,\tau_1)\ldots\int\limits_{(r-1)h}^{\tau_{\nu-2}}A(x,\tau_{\nu-1})\int\limits_{(r-1)h}^{\tau_{\nu-1}}A(x,\tau_\nu)\widetilde{v}_r(x,\tau_\nu)d\tau_\nu d\tau_{\nu-1}\ldots d\tau_1,$$

$\tau_0=t, r=\overline{1,N}$. Passing to the limit as $t\to rh-0$ in (15) we have

$$\lim_{t\to rh-0}\widetilde{v}_r(x,t)=D_{\nu r}(x,rh)\lambda_r(x)+F_{\nu r}(x,rh,w_r,u_r)+G_{\nu r}(x,rh,\widetilde{v}_r),$$

$x\in[0,\omega], r=\overline{1,N}$. Substituting in (10), (11) instead of $\lim\limits_{t\to rh-0}\widetilde{v}_r(x,t), r=\overline{1,N}$, the corresponding right-handed parts for unknown functions $\lambda_r(x), r=\overline{1,N}$, we obtain the system of functional equations:

$$Q_\nu(x,h)\lambda(x)=-F_\nu(x,h,w,u)-G_\nu(x,h,\widetilde{v}), \tag{16}$$

where $Q_\nu(x,h)=$

$$=\begin{vmatrix} I & 0 & \ldots & 0 & -[I+D_{\nu N}(x,Nh)]\\ I+D_{\nu 1}(x,h) & -I & \ldots & 0 & 0\\ 0 & I+D_{\nu 2}(x,2h) & \ldots & 0 & 0\\ 0 & 0 & \ldots & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & I+D_{\nu,N-1}(x,(N-1)h) & -I \end{vmatrix},$$

$$F_\nu(x,h,w,u)=(-F_{\nu N}(x,Nh,w_N,u_N),F_{\nu 1}(x,h,w_1,u_1),\ldots,F_{\nu,N-1}(x,(N-1)h,w_{N-1},u_{N-1})),$$

$$G_\nu(x,h,\widetilde{v})=(-G_{\nu N}(x,Nh,\widetilde{v}_N),G_{\nu 1}(x,h,\widetilde{v}_1),\ldots,G_{\nu,N-1}(x,(N-1)h,\widetilde{v}_{N-1})),$$

$I-$ is the unit matrix of dimension n.

To find a system of five functions $\{\lambda_r(x),\widetilde{v}_r(x,t),w_r(x,t),\ u_r(x,t)\}$, $r=\overline{1,N}$, we have a closed system consisting of equations (16), (15), (12) and (13).

Assuming the invertibility of the matrix $Q_\nu(x,h)$ for all $x\in[0,\omega]$, from equation (16), where

$$\widetilde{v}_r(x,t)=0, u_r(x,t)=\varphi(x,t), w_r(x,t)=\varphi'_t(x,t),$$

we find $\lambda^{(0)}(x)=(\lambda_1^{(0)}(x),\lambda_2^{(0)}(x),\ldots,\lambda_N^{(0)}(x))'$:

$$\lambda^{(0)}(x)=-[Q_\nu(x,h)]^{-1}\big\{F_\nu(x,h,\varphi'_t,\varphi)+G_\nu(x,h,0)\big\}.$$

Using equation (15), for $\lambda_r(x)=\lambda_r^{(0)}(x)$ we find the functions $\{\widetilde{v}_r^{(0)}(x,t)\}$, $r=\overline{1,N}$, i.e.

$$\widetilde{v}_r^{(0)}(x,t)=D_{\nu r}(x,t)\lambda_r^{(0)}(x)+F_{\nu r}(x,t,\varphi'_t,\varphi)+G_{\nu r}(x,t,0).$$

The functions $w_r^{(0)}(x,t), u_r^{(0)}(x,t), r=\overline{1,N}$, are determined from the relations

$$w_r^{(0)}(x,t)=\varphi_t'(x,t)+\int\limits_0^x\int\limits_0^\xi \frac{\partial \widetilde{v}_r^{(0)}(\xi_1,t)}{\partial t}d\xi_1 d\xi,$$

$$u_r^{(0)}(x,t)=\varphi(x,t)+\int\limits_0^x\int\limits_0^\xi \lambda_r^{(0)}(\xi_1)d\xi_1 d\xi+\int\limits_0^x\int\limits_0^\xi \widetilde{v}_r^{(0)}(\xi_1,t)d\xi_1 d\xi, \quad (x,t)\in\Omega_r.$$

For the initial approximation of the problem (8)-(13) we take the system $(\lambda_r^{(0)}(x),\ \widetilde{v}_r^{(0)}(x,t),\ u_r^{(0)}(x,t)),\ r=\overline{1,N}$ and successive approximations are constructed according to the following algorithm:

Step 1. A) Assuming that $w_r(x,t)=w_r^{(0)}(x,t), \quad u_r(x,t)=u_r^{(0)}(x,t), \quad r=\overline{1,N}$, are first approximations in $\lambda_r(x), \widetilde{v}_r(x,t), r=\overline{1,N}$, we find by solving problem (8)-(11). By taking

$$\lambda_r^{(1,0)}(x)=\lambda_r^{(0)}(x), \quad \widetilde{v}_r^{(1,0)}(x,t)=\widetilde{v}_r^{(0)}(x,t),$$

the system couple $\{\lambda_r^{(1)}(x), \widetilde{v}_r^{(1)}(x,t)\}, r=\overline{1,N}$, we find as the limit of the sequence $\lambda_r^{(1,m)}(x), \widetilde{v}_r^{(1,m)}(x,t)$, are defined the next way:

Step 1.1. Assuming the invertibility of the matrix $Q_\nu(x,h)$, $x\in[0,\omega]$, equation (16), where

$$\widetilde{v}_r(x,t)=\widetilde{v}_r^{(1,0)}(x,t),$$

we find $\lambda^{(1,1)}(x)=(\lambda_1^{(1,1)}(x),\lambda_2^{(1,1)}(x),\ldots,\lambda_N^{(1,1)}(x))'$:

$$\lambda^{(1,1)}(x)=-[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,w^{(0)},u^{(0)})+G_\nu(x,h,\widetilde{v}^{(1,0)})\Big\}.$$

Substituting the found $\lambda_r^{(1,1)}(x)$, $r=\overline{1,N}$, in (15) we find

$$\widetilde{v}_r^{(1,1)}(x,t)=D_{\nu r}(x,t)\lambda_r^{(1,1)}(x)+F_{\nu r}(x,t,w^{(0)},u^{(0)})+G_{\nu r}(x,t,\widetilde{v}^{(1,0)}).$$

Step 1.2 From equation (16), where $\widetilde{v}_r(x,t)=\widetilde{v}_r^{(1,1)}(x,t)$, we define

$$\lambda^{(1,2)}(x)=-[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,w^{(0)},u^{(0)})+G_\nu(x,h,\widetilde{v}^{(1,1)})\Big\}.$$

By using expression (15), again, we find the functions $\{\widetilde{v}_r^{(1,2)}(x,t)\}$, $r=\overline{1,N}$,

$$\widetilde{v}_r^{(1,2)}(x,t)=D_{\nu r}(x,t)\lambda_r^{(1,2)}(x)+F_{\nu r}(x,t,w^{(0)},u^{(0)})+G_{\nu r}(x,t,\widetilde{v}^{(1,1)}).$$

At the $(1,m)$ step, we obtain the system of couple $\{\lambda_r^{(1,m)}(x), \widetilde{v}_r^{(1,m)}(x,t)\}, r=\overline{1,N}$.

Let's suppose that the solution of problem (8)–(11) is a sequence of systems of couples $\{\lambda_r^{(1,m)}(x),\ \widetilde{v}_r^{(1,m)}(x,t)\}$ is defined and for $m\to\infty$ converges to continuous, respectively, on $x\in[0,\omega]$, $(x,t)\in\Omega_r$ functions $\lambda_r^{(1)}(x), \quad \widetilde{v}_r^{(1)}(x,t) \quad, r=\overline{1,N}$.

B) The functions $w_r^{(1)}(x,t), u_r^{(1)}(x,t), r=\overline{1,N}$, are determined from the relations

$$w_r^{(1)}(x,t)=\varphi_t'(x,t)+\int\limits_0^x\int\limits_0^\xi \frac{\partial \widetilde{v}_r^{(1)}(\xi_1,t)}{\partial t}d\xi_1 d\xi,$$

$$u_r^{(1)}(x,t)=\varphi(x,t)+\int\limits_0^x\int\limits_0^\xi \lambda_r^{(1)}(\xi_1)d\xi_1 d\xi+\int\limits_0^x\int\limits_0^\xi \widetilde{v}_r^{(1)}(\xi_1,t)d\xi_1 d\xi, \quad (x,t)\in\Omega_r.$$

Step 2. A) Assuming that

$$w_r(x,t) = w_r^{(1)}(x,t), \quad u_r(x,t) = u_r^{(1)}(x,t), \quad r = \overline{1,N},$$

are the second approximations in $\lambda_r(x), \widetilde{v}_r(x,t), r = \overline{1,N}$, we find solving problem (8)–(11). Taking

$$\lambda_r^{(2,0)}(x) = \lambda_r^{(1)}(x), \quad \widetilde{v}_r^{(2,0)}(x,t) = \widetilde{v}_r^{(1)}(x,t),$$

the system of couples $\{\lambda_r^{(2)}(x),\ \widetilde{v}_r^{(2)}(x,t)\}$, $r = \overline{1,N}$, we find as the limit of the sequence $\lambda_r^{(2,m)}(x)$, $\widetilde{v}_r^{(2,m)}(x,t)$, that defines in the following way:

Step 2.1 Assuming the matrix $Q_\nu(x,h)$, $x \in [0,\omega]$, is invertible, from equation (16), where

$$\widetilde{v}_r(x,t) = \widetilde{v}_r^{(2,0)}(x,t),$$

we find $\lambda^{(2,1)}(x) = (\lambda_1^{(2,1)}(x), \lambda_2^{(2,1)}(x), ..., \lambda_N^{(2,1)}(x))'$:

$$\lambda^{(2,1)}(x) = -[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,w^{(1)},u^{(1)}) + G_\nu(x,h,\widetilde{v}^{(2,0)})\Big\}.$$

By substituting the found $\lambda_r^{(2,1)}(x)$, $r = \overline{1,N}$, in (15) we find

$$\widetilde{v}_r^{(2,1)}(x,t) = D_{\nu r}(x,t)\lambda_r^{(2,1)}(x) + F_{\nu r}(x,t,w^{(1)},u^{(1)}) + G_{\nu r}(x,t,\widetilde{v}^{(2,0)}).$$

Step 2.2 From equation (16), where

$$\widetilde{v}_r(x,t) = \widetilde{v}_r^{(2,1)}(x,t),$$

we define

$$\lambda^{(2,2)}(x) = -[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,w^{(1)},u^{(1)}) + G_\nu(x,h,\widetilde{v}^{(2,1)})\Big\}.$$

Using expression (15), again, we find the functions $\{\widetilde{v}_r^{(2,2)}(x,t)\}$, $r = \overline{1,N}$:

$$\widetilde{v}_r^{(2,2)}(x,t) = D_{\nu r}(x,t)\lambda_r^{(2,2)}(x) + F_{\nu r}(x,t,w^{(1)},u^{(1)}) + G_{\nu r}(x,t,\widetilde{v}^{(2,1)}).$$

At the $(2,m)$ step, we obtain the system of couples

$$\{\lambda_r^{(2,m)}(x), \widetilde{v}_r^{(2,m)}(x,t)\}, r = \overline{1,N}.$$

Let's suppose that the solution to problem (8)–(11) is a sequence of systems of couples $\{\lambda_r^{(2,m)}(x)$, $\widetilde{v}_r^{(2,m)}(x,t)\}$ are defined and at $m \to \infty$ converges to $\{\lambda_r^{(2)}(x),\ \widetilde{v}_r^{(2)}(x,t)\}$, $\quad r = \overline{1,N}$.

B) The functions $w_r^{(2)}(x,t)$, $\quad u_r^{(2)}(x,t)$, $\quad r = \overline{1,N}$, are determined from the ratios

$$w_r^{(2)}(x,t) = \varphi_t'(x,t) + \int\limits_0^x\int\limits_0^\xi \frac{\partial \widetilde{v}_r^{(2)}(\xi_1,t)}{\partial t}d\xi_1 d\xi,$$

$$u_r^{(2)}(x,t) = \varphi(x,t) + \int\limits_0^x\int\limits_0^\xi \lambda_r^{(2)}(\xi_1)d\xi_1 d\xi + \int\limits_0^x\int\limits_0^\xi \widetilde{v}_r^{(2)}(\xi_1,t)d\xi_1 d\xi, \quad (x,t) \in \Omega_r.$$

By continuing the process, at the step k we obtain the system $\{\lambda_r^{(k)}(x),\ \widetilde{v}_r^{(k)}(x,t),\ w_r^{(k)}(x,t),\ u_r^{(k)}(x,t)\}$, $r = \overline{1,N}$.

The conditions of the following statement provide feasibility and convergence of the proposed algorithm, as well as unique solvability problems (8)–(13).

Theorem 1. Let's suppose that for some $h > 0 : Nh = T, N = 1, 2, ...,$ and ν, $\nu \in \mathbb{N}$, $(nN \times nN)$ the matrix $Q_\nu(x, h)$ is invertible for all $x \in [0, \omega]$ and the inequalities are carried out
1) $\|[Q_\nu(x, h)]^{-1}\| \leq \gamma_\nu(x, h)$;
2) $q_\nu(x, h)\frac{(\alpha(x)h)^\nu}{\nu!} \leq \mu < 1$, where $q_\nu(x, h) = 1 + \gamma_\nu(x, h)\sum\limits_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!}$.
Then there is a unique solution of problem (8)–(13) and the estimates are valid

$$a)\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^*(x,t)}{\partial t} - \frac{\partial\widetilde{v}_r^{(k)}(x,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^*(x) - \lambda_r^{(k)}(x)\| + \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+p)}(x,t) - \widetilde{v}_r^{(k)}(x,t)\|\Big\} \leq$$

$$\leq d_0(x)\sum_{j=k-1}^{\infty}\frac{1}{j!}\Big(\int_0^x d_0(\xi)d\xi\Big)^j\int_0^x\int_0^\xi\int_0^{\xi_1}\max\Big\{d_2(\xi_2), d_1(\xi_2)\Big\}d\xi_2 d\xi_1 d\xi\max\Big\{\|\varphi_t'\|_0, \|\varphi\|_0, \|f\|_0\Big\},$$

$$b)\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^*(x,t) - w_r^{(k)}(x,t)\|, \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^*(x,t) - u_r^{(k)}(x,t)\|\Big\} \leq$$

$$\leq\int_0^x\int_0^\xi\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^*(\xi_1,t)}{\partial t} - \frac{\partial\widetilde{v}_r^{(k)}(\xi_1,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^*(\xi_1) - \lambda_r^{(k)}(\xi_1)\| + \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(\widetilde{\xi},t) - \widetilde{v}_r^{(k)}(\widetilde{\xi},t)\|\Big\}d\xi, \quad k = 1, 2, \dots$$

where

$$\alpha(x) = \max_{t\in[0,T]}\|A(x,t)\|, \quad \beta(x) = \max_{t\in[0,T]}\|B(x,t)\|, \quad \sigma(x) = \max_{t\in[0,T]}\|C(x,t)\|,$$

$$\rho_1(x) = \beta(x) + \sigma(x) + 1, \quad \rho_2(x) = \alpha(\xi)\Big(1 + q_\nu(\xi,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(\xi)h)^j}{j!}\Big) + 1,$$

$$\rho_3(x) = \frac{(q_\nu(x,h) + \gamma_\nu(x,h))h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}}{1 - q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}},$$

$$d_0(x) = \max\Big\{\int_0^x\rho_2(\xi)[\beta(\xi) + \sigma(\xi)]d\xi, \rho_3(x)[\beta(x) + \sigma(x)]\Big\},$$

$$d_1(x) = \frac{1 + \gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}{1 - q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}q_\nu(x,h) + \gamma_\nu(x,h)\Big]\Big[h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)\int_0^x\rho_1(\xi)\rho_2(\xi)d\xi +$$

$$+h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)\int_0^x\int_0^\xi\rho(\xi_1)\theta_\nu(\xi_1,h)d\xi_1 d\xi + \frac{(\alpha(x)h)^\nu}{\nu!}\rho_1(x)q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!},$$

$$d_2(x) = \int_0^x\rho_2(\xi)\Big[\beta(\xi)\int_0^\xi\rho_2(\xi_1)\rho_1(\xi_1)d\xi_1 + \sigma(\xi_1)\int_0^\xi\int_0^{\xi_1}\rho(\xi_2)\theta_\nu(\xi_2,h)d\xi_2 d\xi_1\Big]d\xi.$$

Proof. We have the following inequality

$$\|F_\nu(x,h,w,u)\| \leq h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\big[\beta(x)\|w_r(x,t)\|+\sigma(x)\|u_r(x,t)\|+\|f(x,t)\|\big],$$

$$\|G_\nu(x,h,\widetilde{v})\| \leq \frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r(x,t)\|,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|D_{\nu r}(x,t)\| \leq \sum_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!}.$$

The following estimates follow from the zero step of the algorithm:

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(0)}(x)\| \leq \rho_1(x)\gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\max\Big\{\|\varphi'_t\|_0,\|\varphi\|_0,\|f\|_0\Big\},$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(0)}(x,t)\| \leq$$

$$\leq \sum_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!}\max_{r=\overline{1,N}}\|\lambda_r{}^{(0)}(x)\| + [\beta(x)+\sigma(x)+1]h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\max\Big\{\|\varphi'_t\|_0,\|\varphi\|_0,\|f\|_0\Big\}\leq$$

$$\leq \rho_1(x)q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\max\Big\{\|\varphi'_t\|_0,\|\varphi\|_0,\|f\|_0\Big\},$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(0)}(x,t)-\varphi'_t(x,t)\| \leq \int_0^x\rho_2(\xi)\rho_1(\xi)d\xi\max\Big\{\|\varphi'_t\|_0,\|\varphi\|_0,\|f\|_0\Big\},$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(0)}(x,t)-\varphi(x,t)\| \leq \int_0^x\int_0^\xi\rho(\xi_1)\theta_\nu(\xi_1,h)d\xi_1 d\xi\max\Big\{\|\varphi'_t\|_0,\|\varphi\|_0,\|f\|_0\Big\}.$$

The following estimates are valid:

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\| \leq$$

$$\leq \gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(0)}(x,t)-\varphi'_t(x,t)\|+$$

$$+\gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(0)}(x,t)-\varphi(x,t)\|+$$

$$+\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(0)}(x,t)\|,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\| \leq$$

$$\leq q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(0)}(x,t)-\varphi'_t(x,t)\|+$$

$$+q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(0)}(x,t)-\varphi(x,t)\|+$$

$$+q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(0)}(x,t)\|.$$

Select the inequality

$$\Delta^{(1,1)}(x)=\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\|\le$$

$$\le\theta_\nu(x,h)\beta(x)\int_0^x\rho_1(\xi)\rho_2(\xi)d\xi\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\}+$$

$$+\theta_\nu(x,h)\sigma(x)\int_0^x\int_0^\xi\rho_1(\xi_1)\theta_\nu(\xi_1,h)d\xi_1d\xi\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\}+$$

$$+\theta_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\rho_1(x)q_\nu(x,h)\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\}.$$

Thus,

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,m+1)}(x)-\lambda_r{}^{(1,m)}(x)\|\le$$

$$\le\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m)}(x,t)-\widetilde{v}_r^{(1,m-1)}(x,t)\|,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m+1)}(x,t)-\widetilde{v}_r^{(1,m)}(x,t)\|\le$$

$$\le q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m)}(x,t)-\widetilde{v}_r^{(1,m-1)}(x,t)\|.$$

Due to the inequality $q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}<1$ follows the uniform convergence $v_r^{(1,\,m+1)}(x,\,t)$, at $(x,t)\in\Omega_r$, to $v_r^{(1)}(x,t)$ and the convergence of a sequence of systems of functions $\lambda_r^{(1,m+1)}(x)$ to continuous $x\in[0,\omega]$ functions $\lambda_r^{(1)}(x)$ for all $r=\overline{1,N}$:

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m+1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|\le$$

$$\le\sum_{j=0}^m\left[q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\right]^j\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|.$$

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,m+1)}(x)-\lambda_r{}^{(1,0)}(x)\|\le$$

$$\le\sum_{j=0}^m\left[q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\right]^j\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|+$$

$$+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\|.$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m+1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,m+1)}(x)-\lambda_r{}^{(1,0)}(x)\|\le$$

$$\leq \sum_{j=0}^{m}\left[q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\right]^j\left[1+\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\right]\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|+$$

$$+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\|.$$

Moving to the limit at $m\to\infty$, we obtain estimates:

$$\Delta^{(1)}(x)=\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1)}(x,t)-\widetilde{v}_r^{(0)}(x,t)\|+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1)}(x)-\lambda_r{}^{(0)}(x)\|\leq$$

$$\leq d_1(x)\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\}.$$

$$\widetilde{\Delta}^{(1)}(x)=\max_{r=\overline{1,N}}\sup\left\|\frac{\partial\tilde{v}_r^{(1)}(\xi,t)}{\partial t}-\frac{\partial\tilde{v}_r^{(0)}(\xi,t)}{\partial t}\right\|\leq d_2(x)\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\}.$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(1)}(x,t)-w_r^{(0)}(x,t)\|\leq\int_0^x\int_0^\xi\widetilde{\Delta}^{(1)}(\xi_1)d\xi_1d\xi,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(1)}(x,t)-u_r^{(0)}(x,t)\|\leq\int_0^x\int_0^\xi\Delta^{(1)}(\xi_1)d\xi_1d\xi.$$

For difference systems $\lambda_r{}^{(k+1)}(x)-\lambda_r{}^{(k)}(x)$, $\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(k)}(x,t)$, $w_r^{(k+1)}(x,t)-w_r^{(k)}(x,t)$, $u_r^{(k+1)}(x,t)-\,-u_r^{(k)}(x,t)$, $r=\overline{1,N}$, $k=1,2,\ldots$ valid estimates:

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1,1)}(x)-\lambda_r{}^{(k+1,0)}(x)\|\leq$$

$$\leq\gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(k)}(x,t)-w_r^{(k-1)}(x,t)\|+$$

$$+\gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,1)}(x,t)-\widetilde{v}_r^{(k+1,0)}(x,t)\|\leq$$

$$\leq q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(k)}(x,t)-w_r^{(k-1)}(x,t)\|+$$

$$+q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1,m+1)}(x)-\lambda_r{}^{(k+1,m)}(x)\|\leq$$

$$\leq\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m)}(x,t)-\widetilde{v}_r^{(k+1,m-1)}(x,t)\|,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m+1)}(x,t)-\widetilde{v}_r^{(k+1,m)}(x,t)\|\leq$$

$$\leq q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m)}(x,t)-\widetilde{v}_r^{(k+1,m-1)}(x,t)\|.$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m+1)}(x,t)-\widetilde{v}_r^{(k+1,0)}(x,t)\|\leq$$

$$\leq\sum_{j=0}^{m}\Big[q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\Big]^j\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,1)}(x,t)-\widetilde{v}_r^{(k+1,0)}(x,t)\|.$$

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1,m+1)}(x)-\lambda_r{}^{(k+1,0)}(x)\|\leq$$

$$\leq\sum_{j=0}^{m-1}\Big[q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\Big]^j\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,1)}(x,t)-\widetilde{v}_r^{(k+1,0)}(x,t)\|+$$

$$+\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1,1)}(x)-\lambda_r{}^{(k+1,0)}(x)\|.$$

Moving to the limit at $m\to\infty$, we obtain estimates:

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(k)}(x,t)\|\leq$$

$$\leq\frac{q_\nu(x,h)h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(k)}(x,t)-w_r^{(k-1)}(x,t)\|+$$

$$+\frac{q_\nu(x,h)h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|,\tag{17}$$

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1)}(x)-\lambda_r{}^{(k)}(x)\|\leq$$

$$\leq\frac{\gamma_\nu(x,h)h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(k)}(x,t)-w_r^{(k-1)}(x,t)\|+$$

$$+\frac{\gamma_\nu(x,h)h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|.\tag{18}$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(k+1)}(x,t)-w_r^{(k)}(x,t)\|\leq\int\limits_0^x\int\limits_0^\xi\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\left\|\frac{\partial v_r^{(k-1)}(\xi_1,t)}{\partial t}-\frac{\partial v_r^{(k)}(\xi_1,t)}{\partial t}\right\|d\xi_1d\xi,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k+1)}(x,t)-u_r^{(k)}(x,t)\|\leq\int\limits_0^x\int\limits_0^\xi\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|v_r^{(k+1)}(\xi_1,t)-z_r^{(k)}(\xi_1,t)\|d\xi_1d\xi,$$

Summing, respectively, the left and right parts of inequalities (17), (18) we have

$$\Delta^{(k+1)}(x)=\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(k)}(x,t)\|+\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1)}(x)-\lambda_r{}^{(k)}(x)\|\leq$$

$$\leq \rho_3(x)\beta(x) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|w_r^{(k)}(x,t) - w_r^{(k-1)}(x,t)\| +$$

$$+\rho_3(x)\sigma(x) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(k)}(x,t) - u_r^{(k-1)}(x,t)\|. \tag{19}$$

$$\widetilde{\Delta}^{(k+1)}(x) = \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \left\| \frac{\partial \tilde{v}_r^{(1)}(\xi,t)}{\partial t} - \frac{\partial \tilde{v}_r^{(0)}(\xi,t)}{\partial t} \right\| \leq$$

$$\leq \int_0^x \rho_2(\xi)\beta(\xi) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|w_r^{(k)}(\xi,t) - w_r^{(k-1)}(\xi,t)\| d\xi +$$

$$+ \int_0^x \rho_2(\xi)\sigma(\xi) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(k)}(\xi,t) - u_r^{(k-1)}(\xi,t)\| d\xi. \tag{20}$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|w_r^{(k+1)}(x,t) - w_r^{(k)}(x,t)\| \leq \int_0^x \int_0^\xi \widetilde{\Delta}^{(k+1)}(\xi_1) d\xi_1 d\xi,$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(k+1)}(x,t) - u_r^{(k)}(x,t)\| \leq \int_0^x \int_0^\xi \Delta^{(k+1)}(\xi_1) d\xi_1 d\xi.$$

For the function $\left\{\widetilde{\Delta}^{(k+1)}(x), \Delta^{(k+1)}(x)\right\}$ based on (19), (20) we establish the inequality

$$\max\{\widetilde{\Delta}^{(k+1)}(x), \Delta^{(k+1)}(x)\} \leq$$

$$\leq \max\left\{ \int_0^x \rho_2(\xi)[\beta(\xi)+\sigma(\xi)]d\xi, \rho_3(x)[\beta(x)+\sigma(x)] \right\} \int_0^x \int_0^\xi \int_0^{\xi_1} \max\{\widetilde{\Delta}^{(k)}(\xi_2), \Delta^{(k)}(\xi_2)\} d\xi_2 d\xi_1 d\xi \leq$$

$$\leq d_0(x) \int_0^x \int_0^\xi \int_0^{\xi_1} \max\{\widetilde{\Delta}^{(k)}(\xi_2), \Delta^{(k)}(\xi_2)\} d\xi_2 d\xi_1 d\xi, \tag{21}$$

$$\max\left\{\widetilde{\Delta}^{(k+1)}(x), \Delta^{(k)}(x)\right\} \leq \frac{d_0(x)}{(k-1)!} \left(\int_0^x d_0(\xi) d\xi \right)^{k-1} \int_0^x \int_0^\xi \int_0^{\xi_1} \max\{\widetilde{\Delta}^{(1)}(\xi_2), \Delta^{(1)}(\xi_2)\} d\xi_2 d\xi_1 d\xi.$$

Establish inequalities

$$\max\left\{ \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \left\| \frac{\partial \widetilde{v}_r^{(k+p)}(x,t)}{\partial t} - \frac{\partial \widetilde{v}_r^{(k)}(x,t)}{\partial t} \right\|, \right.$$

$$\left. \max_{r=\overline{1,N}} \|\lambda_r^{(k+p)}(x) - \lambda_r^{(k)}(x)\| + \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(k+p)}(x,t) - \widetilde{v}_r^{(k)}(x,t)\| \right\} \leq$$

$$\leq \max\left\{\widetilde{\Delta}^{(k+p)}(x), \Delta^{(k)}(x)\right\} + \max\left\{\widetilde{\Delta}^{(k+p-1)}(x), \Delta^{(k+p-1)}(x)\right\} + ... + \max\left\{\widetilde{\Delta}^{(1)}(x), \Delta^{(1)}(x)\right\} \leq$$

$$\leq d_0(x)\sum_{j=k-1}^{k+p-2}\frac{1}{j!}\Big(\int_0^x d_0(\xi)d\xi\Big)^j\int_0^x\int_0^\xi\int_0^{\xi_1}\max\{\widetilde{\Delta}^{(1)}(\xi_2),\Delta^{(1)}(\xi_2)\}d\xi_2 d\xi_1 d\xi\leq$$

$$\leq d_0(x)\sum_{j=k-1}^{k+p-2}\frac{1}{j!}\Big(\int_0^x d_0(\xi)d\xi\Big)^j\int_0^x\int_0^\xi\int_0^{\xi_1}\max\Big\{d_2(\xi_2),d_1(\xi_2)\Big\}d\xi_2 d\xi_1 d\xi\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\},$$

$$\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(k+p)}(x,t)-w_r^{(k)}(x,t)\|,\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k+p)}(x,t)-u_r^{(k)}(x,t)\|\Big\}\leq$$

$$\leq\int_0^x\int_0^\xi\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^{(k+p)}(\xi_1,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{(k)}(\xi_1,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^{(k+p)}(\xi_1)-\lambda_r^{(k)}(\xi_1)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+p)}(\xi_1,t)-\widetilde{v}_r^{(k)}(\xi_1,t)\|\Big\}d\xi_1 d\xi,$$

proceeding to the limit for $p\to\infty$, for all $(x,t)\in\Omega_r$, $r=\overline{1,N}$, obtain the estimates of Theorem 1.

We prove uniqueness. Let there be $(\lambda_r^{**}(x)+\widetilde{v}_r^{**}(x,t), w_r^{**}(x,t), u_r^{**}(x,t))$, $r=\overline{1,N}$, another solution to the boundary value problem (8)–(13).

Similarly to relation (21) for the differences ${\lambda_r}^*(x)-{\lambda_r}^{**}(x)$, $\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{**}(x,t)$, $\dfrac{\partial\widetilde{v}_r^*(x,t)}{\partial t}-\dfrac{\partial\widetilde{v}_r^{**}(x,t)}{\partial t}$, $r=\overline{1,N}$, for all $(x,t)\in\overline{\Omega}$ we get:

$$\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^*(x,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{**}(x,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^*(x)-\lambda_r^{**}(x)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{**}(x,t)\|,\Big\}\leq$$

$$\leq d_0(x)\int_0^x\int_0^\xi\int_0^{\xi_1}\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^*(\xi_2,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{**}(\xi_2,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^*(\xi_2)-\lambda_r^{**}(\xi_2)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(\xi_2,t)-\widetilde{v}_r^{**}(\xi_2,t)\|\Big\}d\xi_2 d\xi_1 d\xi\leq$$

$$\leq d_0(x)\frac{x^3}{6}\int_0^x\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial v_r^*(\xi,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{**}(\xi,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^*(\xi)-\lambda_r^{**}(\xi)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(\xi,t)-\widetilde{v}_r^{**}(\xi,t)\|\Big\}d\xi.$$

Using the Bellman-Gronwall inequality [19] we have

$$\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^*(x,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{**}(x,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^*(x)-\lambda_r^{**}(x)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{**}(x,t)\|,\Big\}=0.$$

Whence it follows that $\widetilde{v}_r^*(x,t)=\widetilde{v}_r^{**}(x,t),\quad \lambda_r{}^*(x)=\lambda_r{}^{**}(x),\quad \dfrac{\partial\widetilde{v}_r^*(x,t)}{\partial t}=\dfrac{\partial\widetilde{v}_r^{**}(x,t)}{\partial t},\quad r=\overline{1,N}$. From the inequalities

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^*(x,t)-w_r^{**}(x,t)\|\le\int\limits_0^x\int\limits_0^\xi\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^*(\xi_1,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{**}(\xi_1,t)}{\partial t}\Big\|d\xi_1d\xi,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^*(x,t)-u_r^{**}(x,t)\|\le$$

$$\le\int\limits_0^x\int\limits_0^\xi(\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(\xi_1,t)-\widetilde{v}_r^{**}(\xi_1,t)\|+\max_{r=\overline{1,N}}\|\lambda_r^*(\xi_1)-\lambda_r^{**}(\xi_1)\|)d\xi_1d\xi$$

we have $w_r^*(x,t)=w_r^{**}(x,t),\quad u_r^*(x,t)=u_r^{**}(x,t),\quad r=\overline{1,N},\quad$ for all $(x,t)\in\Omega_r$. Theorem 1 is proved.

Because of the equivalence of problems (1)–(3) and (8)–(13) Theorem 1 implies

Theorem 2. Let's suppose that the conditions of Theorem 1 are satisfied. Then problem (1)–(3) has a unique solution $u^*(x,t)$ and the estimate are valid

$$\max\left\{\left\|\frac{\partial u^*(x,t)}{\partial t}-\frac{\partial u^{(k)}(x,t)}{\partial t}\right\|_0,\|u^*(x,t)-u^{(k)}(x,t)\|_0\right\}\le$$

$$\le\int\limits_0^x\int\limits_0^\xi d_0(\xi_1)\sum_{j=k-1}^{\infty}\frac{1}{j!}\Big(\int\limits_0^{\xi_1}d_0(\xi_2)d\xi_2\Big)^j\int\limits_0^{\xi_1}\max\Big\{d_2(\xi_2),d_1(\xi_2)\Big\}d\xi_2d\xi_1d\xi\,\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\}.$$

References

1 Нахушев А.М. Уравнения математической биологии/ А.М. Нахушев. — М.: Высш. шк., 1995. — 301 с. Zbl 0991.35500.

2 Kozhanov A.I. Composite Type Equations and Inverse Problems (Inverse and Iii-Posed Problems) / A.I. Kozhanov // VSP: Utrecht, 1999. — 171 p. DOI: 10.1515/9783110943276.

3 Kozhanov A.I. Pseudoparabolic and pseudohyperbolic equations in noncylindrical time domains / A.I. Kozhanov, G.A. Lukina // Mathematical notes of NEFU. — 2019. — 26. — No. 3. — P. 15-30. DOI: 10.25587/SVFU.2019.17.12.002.

4 Kozhanov A.I. Boundary-value problems for some higher-order nonclassical differential equations / A.I. Kozhanov, N.R. Pinigina // Mathematical notes. — 2017. — 101. — P. 467–474. DOI: 10.1134/S00014 34617030087.

5 Джумабаев Д.С. Признаки однозначной разрешимости линейной краевой задачи для обыкновенных дифференциальных уравнений / Д.С. Джумабаев // Журн. вычисл. мат. и мат. физики. — 1989. — № 1(29). — С. 34–46. DOI: 10.1016/0041-5553(89)90038-4.

6 Dzhumabaev D.S. On one approach to solve the linear boundary value problems for Fredholm integro-differential equations / D.S. Dzhumabaev // Journal of Computational and Applied Mathematics. — 2016. — 294. — P. 342–357. DOI: 10.1016/j.cam.2015.08.023.

7 Dzhumabaev D.S. Well-posedness of nonlocal boundary value problem for a system of loaded hyperbolic equations and an algorithm for finding its solution / D.S. Dzhumabaev // Journal of Mathematical analysis and Applications. — 2018. — 461. — No. 1. — P. 817–836. DOI: 10.1016/j.jmaa.2017.12.005.

8 Assanova A.T. On a Nonlocal Problem with Integral Conditions for the System of Hyperbolic Equations / A. T. Assanova//Differential equations. — 2018. — 54. — No.2. — P.201–214. DOI: 10.1134/S0012266118020076.

9 Орумбаева Н.Т. Об одном алгоритме нахождения решения периодической краевой задачи для системы квазилинейных гиперболических уравнений / Н.Т. Орумбаева // Сибир. электрон. мат. изв. — 2013. — 10. — С. 464–474. DOI: 10.17377/semi.2013.10.036.

10 Orumbayeva N. On a method of finding a solution of semi-periodic boundary value problem for hyperbolic equations / N. Orumbayeva, B. Shayakhmetova // AIP Conference Proceedings. — 2016. — Vol. 1759. — P. 020121. DOI: 10.1063/1.4959735.

11 Orumbaeva N.T. On solvability of non-linear semi-periodic boundary-value problem for system of hyperbolic equations / N.T. Orumbaeva//Russian Mathematics (Iz. VUZ). — 2016. — 60. — P. 23–37. DOI: 10.3103/ S1066369X16090036.

12 Orumbaeva N.T. Algorithms for finding a solution to the initial boundary value problems for differential equations in partial derivatives / N.T. Orumbaeva // Bulletin of the Karaganda University. Series Mathematics. — 2016. — 2. — No. 82. — P.107–112. ISSN 2518-7929 (Print), ISSN 2663–5011 (Online).

13 Orumbaeva N.T. On a Solution of a Nonlinear Semi-periodic Boundary Value Problem for a Differential Equation with Arbitrary Functions / N.T. Orumbayeva, G. Sabitbekova // Springer Proceedings in Mathematics & Statistics — 2017. — 216. — P. 158–163.

14 Orumbaeva N.T. A boundary value problem for nonlinear differential equation with arbitrary functions / N.T. Orumbayeva, G. Sabitbekova // Bulletin of the Karaganda University. Series Mathematics. — 2017. — 1. — No. 85. — P. 71–76. DOI: 10.31489/2017M1/71-76.

15 Kosmakova M.T. Problems of heat conduction with different boundary conditions in noncylindrical domains [Electronic resource] / M.T. Kosmakova, N.T. Orumbayeva, N.K. Medeubaev and Zh.M. Tuleutaeva // AIP Conference Proceedings. — 2018. — Vol.1997. — P.020071. — Access mode: https://aip. scitation.org/doi/abs/10.1063/1.5049065.

16 Assanova A.T. Well-posedness of a periodic boundary value problem for the system of hyperbolic equations with delayed argument/ A.T. Assanova, N.B. Iskakova and N.T. Orumbayeva.//Bulletin of the Karaganda University-Mathematics. — 2018. — 1. No. 89. — P.8–14. DOI: 10.31489 /2018M1/8-14.

17 Orumbayeva N.T. On the solvability of the duo-periodic problem for the hyperbolic equation system with a mixed derivative, N.T. Orumbayeva and A.B. Keldibekova // Bulletin of the Karaganda University. Series Mathematics. — 2019. — 1. — No. 93. — P. 59–71. DOI: 10.31489 /2019M1/59-71.

18 Assanova A.T. On the well-posedness of periodic problems for the system of hyperbolic equations with finite time delay, A.T. Assanova, N.B. Iskakova and N.T. Orumbayeva // Mathematical Methods in the Applied Sciences. — 2020. — 43. — No. 2. — P. 881–902. DOI: 10.1002/mma.5970.

19 Треногин В.А. Функциональный анализ: учеб. пос. для студ. вузов / В.А. Треногин. — М.: Наука, 1980. — 495 с.

А.Б. Кельдибекова

Аралас туындысы бар үшінші ретті дифференциалдық теңдеу үшін жартылай периодты шеттік есептің шешілуі

Мақала псевдопараболалық типтегі эволюциялық теңдеу үшін жартылай периодты шеттік есептің шешілуін зерттеуге арналған. Жоғары ретті дербес туындылы дифференциалдық теңдеулер үшін локальды емес есептерді көптеген авторлар зерттеген [1–4]. Мұндай есептерді зерттеуге деген қызығушылық олардың қолданбалы мағынасына байланысты туындады. Осындай есептерге күрделі топологиясы бар қатты кеуекті орта, бірінші кезекте топырақ және жер жатады. Сонымен бірге, мұндай теңдеулер дисперсиялық жүйелердегі ұзын толқындарды да сипаттауы мүмкін. Мақалада осындай есептерді шешу үшін жаңа функциялар енгізілген және параметризациялау әдісі қолданылған [5]. Онда үшінші реттік дифференциалдық теңдеу үшін шеттік есеп, қарапайым дифференциалдық теңдеулер жүйелер тобы үшін периодтық шеттік есебіне келтіріледі [6–18]. Жуық шешімді табудың жаңа конструктивтік алгоритмі ұсынылған және бастапқы есеп терминдері негізінде зерттеліп отырған есептің бірмәнді шешілуінің коэффициенттік белгілері алынған.

Кілт сөздер: дербес туынды теңдеу, үшінші ретті псевдопараболалық теңдеу, алгоритм, жуық шешім.

А.Б. Кельдибекова

Разрешимость полупериодической краевой задачи для дифференциального уравнения третьего порядка со смешанной производной

Статья посвящена исследованию разрешимости полупериодической краевой задачи для эволюционного уравнения типа псевдопараболических. Нелокальные задачи для дифференциальных уравнений с частными производными высокого порядка были исследованы многими авторами [1–4]. Определенный интерес к изучению данных задач вызван в связи с их прикладными значениями. К таким задачам относятся сильно пористые среды со сложной топологией, и, в первую очередь, почва и почвогрунт. Также такие уравнения могут описывать длинные волны в дисперсных системах. Для решения данной задачи в работе введены новые функции и применен метод параметризации [5]. Тогда краевая задача для дифференциального уравнения третьего порядка сводится к периодической краевой задаче для семейства систем обыкновенных дифференциальных уравнений [6–18]. Предложены новые конструктивные алгоритмы нахождения приближенного решения, и в терминах исходных данных получены коэффициентные признаки однозначной разрешимости исследуемой задачи.

Ключевые слова: уравнение в частных производных, псевдопараболическое уравнение третьего порядка, алгоритм, приближенное решение.

References

1 Nahushev, A.M. (1995). *Uravneniia matematicheskoi biolohii [Equations of mathematical biology]*. Moscow: Vysshaia shkola. Zbl 0991.35500 [in Russian].

2 Kozhanov, A.I. (1999). Composite Type Equations and Inverse Problems (Inverse and Iii-Posed Problems). *VSP: Utrecht.* DOI: 10.1515/9783110943276

3 Kozhanov, A.I. & Lukina, G.A. (2019). Pseudoparabolic and pseudohyperbolic equations in noncylindrical time domains. *Mathematical notes of NEFU, 3*, 26, 15–30. DOI: 10.25587/SVFU. 2019.17.12.002.

4 Kozhanov, A.I. & Pinigina, N.R. (2017). Boundary-value problems for some higher-order nonclassical differential equations. *Mathematical notes,* 101, 467–474. DOI: 10.1134/S0001434617030087.

5 Dzhumabaev, D.S. (1989). Priznaki odnoznachnoi razreshimosti lineinoi kraevoi zadachi dlia obyknovennykh differentsialnykh uravnenii [Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation]. *Zhurnal vychislitelnoi matematiki i matematicheskoi fiziki — Computational Mathematics and Mathematical Physics, 29, 1,* 34–46. DOI: 10.1016/0041-5553(89)90038-4 [in Russian].

6 Dzhumabaev, D.S. (2016). On one approach to solve the linear boundary value problems for Fredholm integro-differential equations. *Journal of Computational and Applied Mathematics, 294*, 342–357. DOI: 10.1016/j.cam.2015.08.023.

7 Dzhumabaev, D.S. (2018). Well-posedness of nonlocal boundary value problem for a system of loaded hyperbolic equations and an algorithm for finding its solution. *Journal of Mathematical analysis and Applications, 461*, 1, 817–836. DOI: 10.1016/j.jmaa.2017.12.005.

8 Assanova, A.T. (2018). On a Nonlocal Problem with Integral Conditions for the System of Hyperbolic Equations. *Differential equations, 54*, 2, 201–214. DOI: 10.1134/S0012266118020076.

9 Orumbaeva, N.T. (2013). Ob odnom alhoritme nakhozhdeniia resheniia periodicheskoi kraevoi zadachi dlia sistemy kvazilineinykh hiperbolicheskikh uravnenii [On an algorithm of finding periodical boundary value problem for system of the quasi-linear of hyperbolic equations]. *Siberian Electronic Mathematical Reports, 10*, 464–474 [in Rusian]. DOI: 10.17377/semi.2013.10.036.

10 Orumbaeva, N. & Shayakhmetova, B. (2016). On a method of finding a solution of semi-periodic boundary value problem for hyperbolic equations. *AIP Conference Proceedings, 1759* 020121. DOI: 10.1063/1.4959735.

11 Orumbaeva, N.T. (2016). On solvability of non-linear semi-periodic boundary-value problem for system of hyperbolic equations. *Russian Mathematics (Iz. VUZ), 9*, 26–41. DOI: 10.3103/S106 6369X16090036.

12 Orumbaeva, N.T. (2016). Algorithms for finding a solution to the initial boundary value problems for differential equations in partial derivatives. *Bulletin of the Karaganda University. Series Mathematics, 2*, 82, 107–112. ISSN 2518-7929 (Print), ISSN 2663–5011 (Online).

13 Orumbaeva, N.T. & Sabitbekova, G. (2017). On a Solution of a Nonlinear Semi-periodic Boundary Value Problem for a Differential Equation with Arbitrary Functions. *Springer Proceedings in Mathematics & Statistics, 216*, 158–163. Retrieved from https://doi.org/10.1007/978-3-319-67053-9_14

14 Orumbaeva, N.T. & Sabitbekova, G. (2017). A boundary value problem for nonlinear differential equation with arbitrary functions, *Bulletin of the Karaganda University. Series Mathematics, 1, 85,* 71–76. DOI: 10.31489/2017M1/71-76.

15 Kosmakova, M.T., Orumbayeva, N.T., Medeubaev N.K. & Tuleutaeva Zh.M. (2018). Problems of heat conduction with different boundary conditions in noncylindrical domains. textitAIP Conference Proceedings, V, 1997, 020071. Retrieved from https://aip.scitation.org/doi/abs/10.1063/1.5049065.

16 Assanova, A.T., Iskakova, N.B. & Orumbayeva, N.T. (2018). Well-posedness of a periodic boundary value problem for the system of hyperbolic equations with delayed argument. *Bulletin of the Karaganda University. Serise Mathematics, 1*, 89, 8—14. DOI: 10.31489/2018M1/8-14.

17 Orumbayeva, N.T. & Keldibekova, A.B. (2019). On the solvability of the duo-periodic problem for the hyperbolic equation system with a mixed derivative. *Bulletin of the Karaganda University. Series Mathematics, 1*, 93, 59—71. DOI:10.31489/2019M1/59-71.

18 Assanova, A.T., Iskakova, N.B. & Orumbayeva, N.T. (2020). On the well-posedness of periodic problems for the system of hyperbolic equations with finite time delay. *Mathematical Methods in the Applied Sciences, 43*, 2, 881–902 DOI: 10.1002/mma.5970.

19 Trenogin, V.A. (1980). *Funktsionalnyi analiz [Functional analysis].* Moscow: Nauka [in Russian].

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On the solution to a two-dimensional boundary value problem of heat conduction in a degenerating domain

The article considers a homogeneous boundary-value problem for the heat equation in the non-cylindrical domain, namely, in an inverted pyramid with a vertex at the origin of coordinates, two faces of which lie in coordinate planes.A solution to the problem is sought in the form of a sum of generalized thermal potentials. There is a need to study the system of two Volterra integral equations of the second kind with singularities of the kernel. It is assumed that densities (heat intensity) depend only on a time variable, i.e. the density in each time section is considered constant. As a result, the system of integral equations is reduced to the homogeneous Volterra integral equation of the second kind. It is shown that this equation is uniquely solvable in the class of continuous functions.

Keywords: equation of heat conduction, Volterra integral equation, degenerating domain, thermal potential.

Introduction

It is shown [1–3] that solving a homogeneous problem for the heat equation in the angular domain $G = \{(x;\ t): \quad t > 0, \quad 0 < x < t\}$ is reduced to solving the Volterra integral equation of the second kind with a kernel

$$K(t,\,\tau) = \frac{1}{2a\sqrt{\pi}} \left\{ \frac{t+\tau}{(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right) + \frac{1}{(t-\tau)^{\frac{1}{2}}} \exp\left(-\frac{t-\tau}{4a^2}\right) \right\}. \tag{1}$$

In these Refs, as well as in Refs [4–5] it is shown that the kernels of the integral equations are "incompressible", that is, the norm of the integral operator acting in the class of continuous functions is equal to unity.

In all works, the boundary of the domain moves at a constant velocity. Attempts to study the solvability of boundary value problems for the heat equation in non-cylindrical domains with a variable velocity of changing the boundary were made in works [6].

We also note that boundary value problems for a spectrally loaded parabolic equation reduce to this kind of singular integral equations, when the load line moves according to the law $x = t$ or $x^2 = t$ [7–11] and problems for essentially loaded equation of heat conduction [12].

In Ref [13] we have also investigated the Volterra integral equation with a singular kernel that differ from kernel (1). A norm of an integral operator acting in classes of continuous functions is equal to 3 [14].

In [15], the two-dimensional Dirichlet problem for the heat equation with respect to the spatial variable in an infinite dihedral angle was also considered. Using the Fourier transformation, the problem was reduced to a one-dimensional boundary value problem with the parameter. In [16] the boundary value problem for the heat equation was considered in an inverted cone. Assuming that the isotropy property is fulfilled in the angular coordinate (axial symmetry), we have studied the problem for the heat equation in polar coordinates, to which the two-dimensional problem in the spatial variable is reduced.

Now we are studying a homogeneous boundary value problem for the heat equation in the non-cylindrical domain, namely, in an inverted pyramid with a vertex at the origin of coordinates. As in papers [1–16], the boundary value problem of heat equation is considered in a degenerating domain, and the problem is also reduced to the Volterra integral equation. But a kernel of the obtained integral equation is differs from those considered by us earlier.

1 Formulation of the problem

In the domain (Fig. 1) $Q = \{(x, y; t), (x, y) \in D; t > 0\}$, we consider a problem: find a solution to the equation

$$\frac{\partial u}{\partial t} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0, \tag{2}$$

satisfying the condition on a lateral surface of the pyramid:

$$u|_{\Gamma} = 0. \tag{3}$$

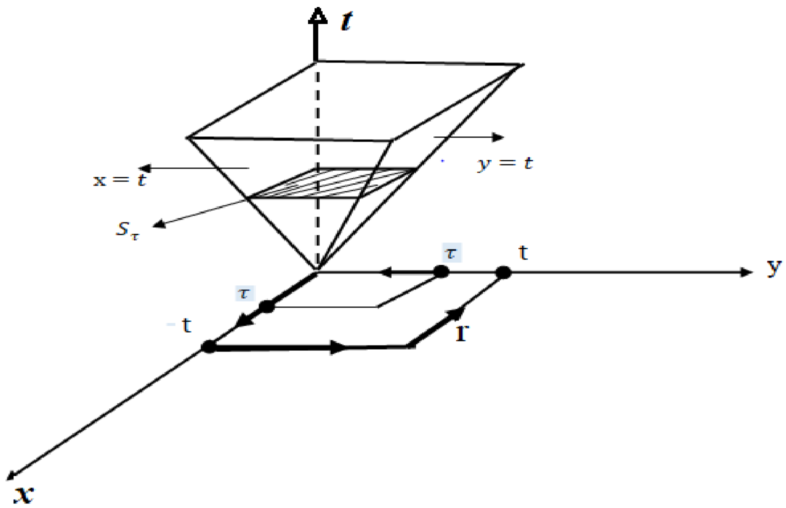

Figure 1. Domain Q

where $D = \{(x, y), 0 < x < t, 0 < y < t\}, \partial D = \Gamma$, (Fig. 2)

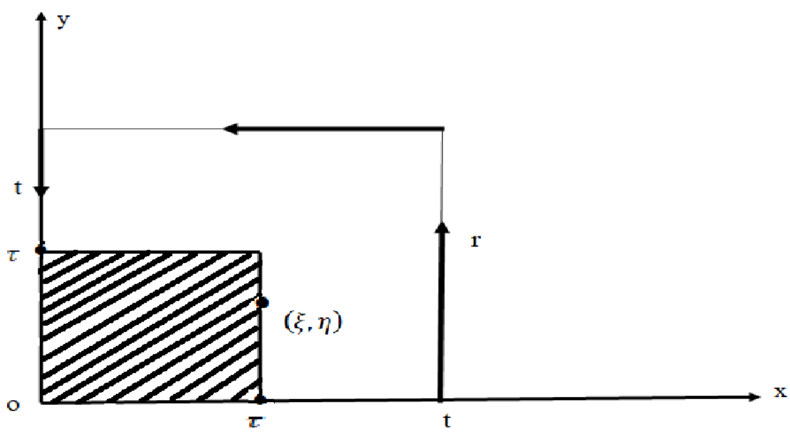

Figure 2. Domain D

2 Reducing the boundary value problem to a system of Volterra integral equations

We seek a solution to problem (2)–(3) using thermal potentials.

As is known, the thermal potential of the double layer has the form [17]:

$$W(x,y;t) = \frac{1}{2\pi}\int_0^t d\tau \int_\Gamma \frac{\psi(\sigma,\tau)}{t-\tau}\cdot\frac{\partial}{\partial \overline{n}}\exp\left(-\frac{r^2}{4a^2(t-\tau)}\right)d\sigma, \tag{4}$$

where an arc length σ of the contour Γ is counted from some fixed point, and $\psi(\sigma,\tau)$ is a density (intensity) is a function of a variable point $\sigma=(\xi,\,\eta)$ of the contour Γ and of the parameter τ.

$r=\sqrt{(x-\xi)^2+(y-\eta)^2}$ indicates the distance from the point (x,y) to a variable point σ of the contour Γ, $\overline{n}$ is a direction of the external normal at the variable integration point. It's obvious that $W(x,y;t)$ satisfies the heat equation (2).

We will seek a solution to problem (2)–(3) in the form of a sum of generalized thermal potentials

$$u(x,y,t) = \frac{1}{4a^2\pi}\int_0^t d\tau\int_0^\tau \frac{x}{(t-\tau)^2}\exp\left(-\frac{x^2+(y-\eta)^2}{4a^2(t-\tau)}\right)\mu_1(\eta,\tau)\,d\eta +$$

$$+\frac{1}{4a^2\pi}\int_0^t d\tau\int_0^\tau \frac{x-\tau}{(t-\tau)^2}\exp\left(-\frac{(x-\tau)^2+(y-\eta)^2}{4a^2(t-\tau)}\right)\mu_2(\eta,\tau)\,d\eta +$$

$$+\frac{1}{4a^2\pi}\int_0^t d\tau\int_0^\tau \frac{y}{(t-\tau)^2}\exp\left(-\frac{(x-\xi)^2+y^2}{4a^2(t-\tau)}\right)\varphi_1(\xi,\tau)\,d\xi +$$

$$+\frac{1}{4a^2\pi}\int_0^t d\tau\int_0^\tau \frac{y-\tau}{(t-\tau)^2}\exp\left(-\frac{(x-\xi)^2+(y-\tau)^2}{4a^2(t-\tau)}\right)\varphi_2(\xi,\tau)\,d\xi, \tag{5}$$

where $\mu_i(x,y;t)\,,\quad \varphi_i(x,y;t)\,,\quad i=1,\ 2$, are functions to be defined.

Note that expression (5) follows from formula (4) by directly calculating the normal derivative.

We use the well-known property of the generalized thermal potential of a double layer [18].

The function $W(x,y;t)$ is discontinuous at the contour Γ, and the following formulas hold:

$$W_i(x_0,y_0;t) = \lim_{(x,y)\to(x_0,y_0)} W_i(x_i,y_i;t) = W(x_0,y_0;t) + \frac{1}{2}\psi(x_0,y_0;t)\,,$$

$$W_l(x_0,y_0;t) = \lim_{(x,y)\to(x_0,y_0)} W_i(x_l,y_l;t) = W(x_0,y_0;t) - \frac{1}{2}\psi(x_0,y_0;t)\,,$$

if $\psi(x,y;t)$ is a continuous function, where (x_0,y_0) is a point of the boundary Γ, a point (x_i,y_i) lies inside the domain, and a point (x_l,y_l) lies outside the domain.

From the representation (5) and from the properties of the generalized thermal potential of the double layer, we obtain

$$\lim_{x\to 0+} u(x,y,t) = \frac{1}{2}\mu_1(y,t) + \frac{1}{4a^2\pi}\int_0^t d\tau\int_0^\tau \frac{-\tau}{(t-\tau)^2}\exp\left(-\frac{\tau^2+(y-\eta)^2}{4a^2(t-\tau)}\right)\mu_2(\eta,\tau)\,d\eta +$$

$$+\frac{1}{4a^2\pi}\int_0^t d\tau\int_0^\tau \frac{y}{(t-\tau)^2}\exp\left(-\frac{\xi^2+y^2}{4a^2(t-\tau)}\right)\varphi_1(\xi,\tau)\,d\xi +$$

$$+\frac{1}{4a^2\pi}\int_0^t d\tau\int_0^\tau \frac{y-\tau}{(t-\tau)^2}\exp\left(-\frac{\xi^2+(y-\tau)^2}{4a^2(t-\tau)}\right)\varphi_2(\xi,\tau)\,d\xi = 0. \tag{6}$$

$$\lim_{x\to\tau-0} u(x,y,t) = -\frac{1}{2}\mu_2(y,t) + \frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{\tau}{(t-\tau)^2}\exp\left(-\frac{\tau^2+(y-\eta)^2}{4a^2(t-\tau)}\right)\mu_1(\eta,\tau)\,d\eta+$$
$$+\frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{y}{(t-\tau)^2}\exp\left(-\frac{(\tau-\xi)^2+y^2}{4a^2(t-\tau)}\right)\varphi_1(\xi,\tau)\,d\xi+$$
$$+\frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{y-\tau}{(t-\tau)^2}\exp\left(-\frac{(\tau-\xi)^2+(y-\tau)^2}{4a^2(t-\tau)}\right)\varphi_2(\xi,\tau)\,d\xi = 0. \quad (7)$$

$$\lim_{y\to 0+} u(x,y,t) = \frac{1}{2}\varphi_1(x,t) + \frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{x}{(t-\tau)^2}\exp\left(-\frac{x^2+\eta^2}{4a^2(t-\tau)}\right)\mu_1(\eta,\tau)\,d\eta+$$
$$+\frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{x-\tau}{(t-\tau)^2}\exp\left(-\frac{(x-\tau)^2+\eta^2}{4a^2(t-\tau)}\right)\mu_2(\eta,\tau)\,d\eta+$$
$$+\frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{-\tau}{(t-\tau)^2}\exp\left(-\frac{(x-\xi)^2+\tau^2}{4a^2(t-\tau)}\right)\varphi_2(\xi,\tau)\,d\xi = 0. \quad (8)$$

$$\lim_{y\to\tau-0} u(x,y,t) = -\frac{1}{2}\varphi_2(x,t) + \frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{x}{(t-\tau)^2}\exp\left(-\frac{x^2+(\tau-\eta)^2}{4a^2(t-\tau)}\right)\mu_1(\eta,\tau)\,d\eta+$$
$$+\frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{x-\tau}{(t-\tau)^2}\exp\left(-\frac{(x-\tau)^2+(\tau-\eta)^2}{4a^2(t-\tau)}\right)\mu_2(\eta,\tau)\,d\eta+$$
$$+\frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{\tau}{(t-\tau)^2}\exp\left(-\frac{(x-\xi)^2+\tau^2}{4a^2(t-\tau)}\right)\varphi_1(\xi,\tau)\,d\xi = 0. \quad (9)$$

We get a system of four equations with four unknown functions.

If into equations (8) and (9) the variable x is replaced by the variable y and the integration variable ξ is replaced by η, then we get that these equations coincide with equations (6) and (7), and $\mu_i(y,t) = \varphi_i(y,t)$, $(i = 1,2)$.

Thus, it is possible to solve a system of two equations with two unknown functions $\mu_1(y,t)$ and $\mu_2(y,t)$. For this, it is enough into equations (6) and (7) to replace the integration variable ξ with the variable η and to replace $\varphi_i(\eta,\tau)$, respectively, with $\mu_i(\eta,\tau)$, $(i = 1,2)$.

As a result, we get:

$$\frac{1}{2}\mu_1(y,t) = \frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{\tau}{(t-\tau)^2}\exp\left(-\frac{\tau^2+(y-\eta)^2}{4a^2(t-\tau)}\right)\mu_2(\eta,\tau)\,d\eta-$$
$$-\frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{y}{(t-\tau)^2}\exp\left(-\frac{\eta^2+y^2}{4a^2(t-\tau)}\right)\mu_1(\eta,\tau)\,d\eta-$$
$$-\frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{y-\tau}{(t-\tau)^2}\exp\left(-\frac{\eta^2+(y-\tau)^2}{4a^2(t-\tau)}\right)\mu_2(\eta,\tau)\,d\eta, \quad (10)$$

$$\frac{1}{2}\mu_2(y,t) = \frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{\tau}{(t-\tau)^2}\exp\left(-\frac{\tau^2+(y-\eta)^2}{4a^2(t-\tau)}\right)\mu_1(\eta,\tau)\,d\eta+$$
$$+\frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{y}{(t-\tau)^2}\exp\left(-\frac{(\tau-\eta)^2+y^2}{4a^2(t-\tau)}\right)\mu_1(\eta,\tau)\,d\eta+$$
$$+\frac{1}{4a^2\pi}\int_0^t d\tau \int_0^\tau \frac{y-\tau}{(t-\tau)^2}\exp\left(-\frac{(\tau-\eta)^2+(y-\tau)^2}{4a^2(t-\tau)}\right)\mu_2(\eta,\tau)\,d\eta. \quad (11)$$

3 Case of a constant density (intensity) of heat propagation

We assume the following.

Let the densities (heat intensity) $\mu_1(\eta,\tau)$ and $\mu_2(\eta,\tau)$ not depend on the first variable, i.e. the density in each section S_τ (Fig. 2) are constant (and depends only on the variable τ), then we write equations (10) and (11) in the form:

$$\frac{1}{2}\mu_1(y,t)=\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{\tau^2}{4a^2(t-\tau)}\right)\mu_2(\tau)\left\{\frac{1}{2a\sqrt{\pi}}\int_0^\tau\frac{\exp\left(-\frac{(y-\eta)^2}{4a^2(t-\tau)}\right)}{\sqrt{t-\tau}}d\eta\right\}d\tau-$$

$$-\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{y}{(t-\tau)^{3/2}}\exp\left(-\frac{y^2}{4a^2(t-\tau)}\right)\mu_1(\tau)\left\{\frac{1}{2a\sqrt{\pi}}\int_0^\tau\frac{\exp\left(-\frac{\eta^2}{4a^2(t-\tau)}\right)}{\sqrt{t-\tau}}d\eta\right\}d\tau-$$

$$-\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{y-\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{(y-\tau)^2}{4a^2(t-\tau)}\right)\mu_2(\tau)\left\{\frac{1}{2a\sqrt{\pi}}\int_0^\tau\frac{\exp\left(-\frac{\eta^2}{4a^2(t-\tau)}\right)}{\sqrt{t-\tau}}d\eta\right\}d\tau. \tag{12}$$

$$\frac{1}{2}\mu_2(y,t)=\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{\tau^2+(y-\eta)^2}{4a^2(t-\tau)}\right)\mu_1(\tau)\left\{\frac{1}{2a\sqrt{\pi}}\int_0^\tau\frac{\exp\left(-\frac{(y-\eta)^2}{4a^2(t-\tau)}\right)}{\sqrt{t-\tau}}d\eta\right\}d\tau+$$

$$+\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{y}{(t-\tau)^{3/2}}\exp\left(-\frac{y^2}{4a^2(t-\tau)}\right)\mu_1(\tau)\left\{\frac{1}{2a\sqrt{\pi}}\int_0^\tau\frac{\exp\left(-\frac{(\tau-\eta)^2}{4a^2(t-\tau)}\right)}{\sqrt{t-\tau}}d\eta\right\}d\tau+$$

$$+\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{y-\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{(y-\tau)^2}{4a^2(t-\tau)}\right)\mu_2(\tau)\left\{\frac{1}{2a\sqrt{\pi}}\int_0^\tau\frac{\exp\left(-\frac{(\tau-\eta)^2}{4a^2(t-\tau)}\right)}{\sqrt{t-\tau}}d\eta\right\}d\tau. \tag{13}$$

We calculate the internal integrals in (12) and (13).

$$\frac{1}{2a\sqrt{\pi}}\int_0^\tau\frac{1}{\sqrt{t-\tau}}\exp\left(-\frac{(y-\eta)^2}{4a^2(t-\tau)}\right)d\eta=\left\|z=\frac{y-\eta}{2a\sqrt{t-\tau}};dz=-\frac{d\eta}{2a\sqrt{t-\tau}}\right\|=$$

$$=\frac{1}{\sqrt{\pi}}\int_{\frac{y-\tau}{2a\sqrt{t-\tau}}}^{\frac{y}{2a\sqrt{t-\tau}}}e^{-z^2}dz=\frac{1}{2}\left[erf\left(\frac{y}{2a\sqrt{t-\tau}}\right)-erf\left(\frac{y-\tau}{2a\sqrt{t-\tau}}\right)\right];$$

$$\frac{1}{2a\sqrt{\pi}}\int_0^\tau\frac{1}{\sqrt{t-\tau}}\exp\left(-\frac{\eta^2}{4a^2(t-\tau)}\right)d\eta=\left\|z=\frac{\eta}{2a\sqrt{t-\tau}};dz=\frac{d\eta}{2a\sqrt{t-\tau}}\right\|=$$

$$=\frac{1}{\sqrt{\pi}}\int_0^{\frac{\tau}{2a\sqrt{t-\tau}}}e^{-z^2}dz=\frac{1}{2}erf\left(\frac{\tau}{2a\sqrt{t-\tau}}\right);$$

$$\frac{1}{2a\sqrt{\pi}}\int_0^\tau\frac{1}{\sqrt{t-\tau}}\exp\left(-\frac{(\tau-\eta)^2}{4a^2(t-\tau)}\right)d\eta=\left\|z=\frac{\tau-\eta}{2a\sqrt{t-\tau}};dz=-\frac{d\eta}{2a\sqrt{t-\tau}}\right\|=$$

$$=\frac{1}{\sqrt{\pi}}\int_0^{\frac{\tau}{2a\sqrt{t-\tau}}}e^{-z^2}dz=\frac{1}{2}erf\left(\frac{\tau}{2a\sqrt{t-\tau}}\right).$$

Substituting these values into (12)–(13), we have:

$$\frac{1}{2}\mu_1(y,t)=\frac{1}{4a\sqrt{\pi}}\int_0^t\frac{\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{\tau^2}{4a^2(t-\tau)}\right)\left[erf\left(\frac{y}{2a\sqrt{t-\tau}}\right)+erf\left(\frac{y-\tau}{2a\sqrt{t-\tau}}\right)\right]\mu_2(\tau)\,d\tau$$

$$-\frac{1}{4a\sqrt{\pi}}\int_0^t \frac{y}{(t-\tau)^{3/2}}\exp\left(-\frac{y^2}{4a^2(t-\tau)}\right)erf\left(\frac{\tau}{2a\sqrt{t-\tau}}\right)\mu_1(\tau)\,d\tau-$$

$$-\frac{1}{4a\sqrt{\pi}}\int_0^t \frac{y-\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{(y-\tau)^2}{4a^2(t-\tau)}\right)erf\left(\frac{\tau}{2a\sqrt{t-\tau}}\right)\mu_2(\tau)\,d\tau; \tag{14}$$

$$\frac{1}{2}\mu_2(y,t)=\frac{1}{4a\sqrt{\pi}}\int_0^t \frac{\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{\tau^2}{4a^2(t-\tau)}\right)\left[erf\left(\frac{y}{2a\sqrt{t-\tau}}\right)+erf\left(\frac{y-\tau}{2a\sqrt{t-\tau}}\right)\right]\mu_1(\tau)\,d\tau+$$

$$+\frac{1}{4a\sqrt{\pi}}\int_0^t \frac{y}{(t-\tau)^{3/2}}\exp\left(-\frac{y^2}{4a^2(t-\tau)}\right)erf\left(\frac{\tau}{2a\sqrt{t-\tau}}\right)\mu_1(\tau)\,d\tau-$$

$$+\frac{1}{4a\sqrt{\pi}}\int_0^t \frac{y-\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{(y-\tau)^2}{4a^2(t-\tau)}\right)erf\left(\frac{\tau}{2a\sqrt{t-\tau}}\right)\mu_2(\tau)\,d\tau. \tag{15}$$

Since we assumed that the heat intensity (density) depends only on the variable t, then in equalities (14), (15) the variable y must be considered equal t

$$\mu_1(t)=\frac{1}{2a\sqrt{\pi}}\int_0^t \frac{\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{\tau^2}{4a^2(t-\tau)}\right)\left[erf\left(\frac{t}{2a\sqrt{t-\tau}}\right)+erf\left(\frac{\sqrt{t-\tau}}{2a}\right)\right]\mu_2(\tau)\,d\tau-$$

$$-\frac{1}{2a\sqrt{\pi}}\int_0^t \frac{t}{(t-\tau)^{3/2}}\exp\left(-\frac{t^2}{4a^2(t-\tau)}\right)erf\left(\frac{\tau}{2a\sqrt{t-\tau}}\right)\mu_1(\tau)\,d\tau-$$

$$-\frac{1}{2a\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{3/2}}\exp\left(-\frac{t-\tau}{4a^2}\right)erf\left(\frac{\tau}{2a\sqrt{t-\tau}}\right)\mu_2(\tau)\,d\tau; \tag{16}$$

$$\mu_2(t)=\frac{1}{2a\sqrt{\pi}}\int_0^t \frac{\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{\tau^2}{4a^2(t-\tau)}\right)\left[erf\left(\frac{t}{2a\sqrt{t-\tau}}\right)+erf\left(\frac{\sqrt{t-\tau}}{2a}\right)\right]\mu_1(\tau)\,d\tau+$$

$$+\frac{1}{2a\sqrt{\pi}}\int_0^t \frac{t}{(t-\tau)^{3/2}}\exp\left(-\frac{t^2}{4a^2(t-\tau)}\right)erf\left(\frac{\tau}{2a\sqrt{t-\tau}}\right)\mu_1(\tau)\,d\tau+$$

$$+\frac{1}{2a\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{3/2}}\exp\left(-\frac{t-\tau}{4a^2}\right)erf\left(\frac{\tau}{2a\sqrt{t-\tau}}\right)\mu_2(\tau)\,d\tau. \tag{17}$$

Adding the equations (16) and (17) we obtain the following homogeneous integral equation

$$\mu(t)-\int_0^t K(t,\tau)\,\mu(\tau)\,d\tau=0, \tag{18}$$

where $\mu(t)=\mu_1(t)+\mu_2(t)$,

$$K(t,\tau)=\frac{1}{2a\sqrt{\pi}}\frac{\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{\tau^2}{4a^2(t-\tau)}\right)\left\{erf\left(\frac{t}{2a\sqrt{t-\tau}}\right)+erf\left(\frac{\sqrt{t-\tau}}{2a}\right)\right\}.$$

Since

$$-\frac{\tau^2}{4a^2(t-\tau)}=-\frac{t-\tau}{4a^2}+\frac{t}{4a^2}-\frac{t-\tau}{4a^2(t-\tau)},$$

we rewrite the equation (18) in the form:

$$\psi(t)-\int_0^t K_1(t,\tau)\,\psi(\tau)\,d\tau=0, \tag{19}$$

where

$$\psi(t) = \exp\left\{\frac{t}{4a^2}\right\}\mu(\tau),$$

$$K_1(t,\tau) = \frac{1}{2a\sqrt{\pi}}\exp\left\{\frac{t}{4a^2}\right\}\ \frac{\tau}{(t-\tau)^{3/2}}\ \exp\left\{-\frac{t\tau}{4a^2(t-\tau)}\right\}\left[erf\left(\frac{t}{2a\sqrt{t-\tau}}\right)+erf\left(\frac{\sqrt{t-\tau}}{2a}\right)\right].$$

Let us estimate the integral

$$\int_0^t K_1(t,\tau)\,d\tau \geq 0.$$

$$0 \leq \int_0^t K_1(t,\tau)\,d\tau \leq \frac{1}{a\sqrt{\pi}}\exp\left(\frac{t}{4a^2}\right)J(t). \tag{20}$$

We introduce the replacement

$$z = \frac{t}{2a\sqrt{t-\tau}}, t-\tau = \frac{t^2}{4a^2z^2}, \tau = t - \frac{t^2}{4a^2z^2}, d\tau = \frac{t^2}{2a^2z^3}dz.$$

$$\frac{\tau}{t-\tau} = \left(t - \frac{t^2}{4a^2z^2}\right)\cdot\frac{4a^2z^2}{t^2} = \frac{4a^2}{t}\cdot z^2 - 1;$$

$$\tau = 0 \Rightarrow z = \frac{\sqrt{t}}{2a}; \qquad \tau \to t \Rightarrow z \to +\infty.$$

Then

$$J(t) = \int_0^t \frac{\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{4a^2(t-\tau)}\right\}d\tau = \int_{\frac{\sqrt{t}}{2a}}^{+\infty}\left(t-\frac{t^2}{4a^2z^2}\right)\cdot\frac{8a^3z^3t^2}{t^3\cdot 2a^2z^3}\times$$

$$\times\exp\left\{-\frac{t}{4a^2}\left(\frac{4a^2}{t}z^2-1\right)\right\}dz = 4at\cdot\exp\left\{\frac{t}{4a^2}\right\}\cdot\int_{\frac{\sqrt{t}}{2a}}^{+\infty}\left(1-\frac{t}{4a^2}\cdot\frac{1}{z^2}\right)e^{-z^2}dz =$$

$$= 4at\cdot\exp\left\{\frac{t}{4a^2}\right\}\cdot\int_{\frac{\sqrt{t}}{2a}}^{+\infty}z^{-2}\left(z^2-\frac{t}{4a^2}\right)e^{-z^2}dz = \left\|z^2 = x, z = \sqrt{x}, dz = \frac{dx}{\sqrt{x}}\right\| =$$

$$= 4at\cdot\exp\left\{\frac{t}{4a^2}\right\}\cdot\int_{\frac{t}{4a^2}}^{+\infty}x^{-\frac{3}{2}}\left(x-\frac{t}{4a^2}\right)\cdot e^{-x}dx.$$

We have used the formula 2.3.6(6) from [19], when

$$\left\|\begin{array}{l}\alpha = -\frac{1}{2}, \beta = 2, p = 1,\\ \alpha+\beta-1 = \frac{1}{2}, \alpha+\beta = \frac{3}{2}\end{array}\right\|$$

Then

$$J(t) = 4at\cdot\exp\left\{\frac{t}{4a^2}\right\}\cdot\frac{\sqrt{t}}{2a}\cdot\exp\left\{-\frac{t}{4a^2}\right\}\cdot\psi\left(2;\frac{3}{2};\frac{t}{4a^2}\right).$$

We use the formula 7.11.4.(8) from [20] and formula II.8 from [21]. Then

$$J(t) = 2t\sqrt{t}2^{\frac{3}{2}}\frac{2a}{\sqrt{t}}\exp\left(\frac{t}{8a^2}\right)\cdot D_{-3}\left(\frac{\sqrt{2}\sqrt{t}}{2a}\right) =$$

$$= 8\sqrt{2}\,at\,\exp\left(\frac{t}{8a^2}\right)\frac{1}{2}\exp\left(-\frac{t}{8a^2}\right)\frac{d}{dz}\left(\exp\left(\frac{z^2}{4}\right)\left\{\frac{z}{2}\cdot erfc\left(\frac{z}{\sqrt{2}}\right)-\frac{2}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\right\}\ \right)_{z=\frac{\sqrt{2}\sqrt{t}}{2a}} =$$

$$= 4\sqrt{2}\,at\,\exp\left(\frac{z^2}{4}\right)\left\{\frac{z^2}{4}erfc\left(\frac{z}{\sqrt{2}}\right)-\frac{z}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{4}\right)+\right.$$

$$+\frac{1}{2}erfc\left(\frac{z}{\sqrt{2}}\right)-\frac{z}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)+\frac{2z}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\Bigg\}_{z=\frac{\sqrt{2t}}{2a}}=$$

$$=4\sqrt{2}\,at\,\exp\left(\frac{t}{8a^2}\right)\left\{\left(\frac{t}{8a^2}+\frac{1}{2}\right)erfc\left(\frac{\sqrt{t}}{2a}\right)+\frac{\sqrt{t}}{a\sqrt{\pi}}\exp\left(-\frac{t}{4a^2}\right)\right\}.$$

Thus,

$$J(t)=4\sqrt{2}\,at\,\exp\left(\frac{t}{8a^2}\right)\left\{\left(\frac{t}{8a^2}+\frac{1}{2}\right)\cdot erfc\left(\frac{\sqrt{t}}{2a}\right)+\frac{\sqrt{t}}{a\sqrt{\pi}}\exp\left(-\frac{t}{4a^2}\right)\right\} \tag{21}$$

We substitute the expression (21) into inequality (20):

$$0\le\int_0^t K_1(t,\tau)\,d\tau\le\frac{4\sqrt{2}}{a\sqrt{\pi}}\exp\left(\frac{t}{4a^2}\right)\cdot at\cdot\exp\left(\frac{t}{8a^2}\right)\cdot\left\{\left(\frac{t}{8a^2}+\frac{1}{2}\right)erfc\left(\frac{\sqrt{t}}{2a}\right)+\right.$$

$$\left.+\frac{\sqrt{t}}{a\sqrt{\pi}}\cdot\exp\left(-\frac{t}{4a^2}\right)\right\}$$

or

$$0\le\int_0^t K_1(t,\tau)\,d\tau\le\frac{4\sqrt{2}}{\sqrt{\pi}}\cdot t\cdot\left(\left(\frac{t}{8a^2}+\frac{1}{2}\right)\exp\left\{\frac{3t}{8a^2}\right\}\cdot erfc\left(\frac{\sqrt{t}}{2a}\right)+\frac{\sqrt{t}}{a\sqrt{\pi}}\exp\left\{\frac{t}{8a^2}\right\}\right). \tag{22}$$

Taking the limit at $t\to 0$ from (22), we obtain

$$\lim_{t\to 0}\int_0^t K(t,\tau)=0.$$

4 Main results

Thus, the following lemma is proved.

Lemma 1. Integral equation (19) has a unique solution $\psi(t)\equiv 0$ in the class of continuous functions at $t\in[0,\ T]$, $\quad 0<T<+\infty$.

Since

$$\psi(t)=\exp\left\{\frac{t}{4a^2}\right\}\mu(\tau)$$

and $\mu(t)=\mu_1(t)+\mu_2(t)$, then the system of equations (16) – (17) is also uniquely solvable.

Further. The functions $\mu_1(t)$ and $\mu_2(t)$ are the density of thermal potentials under the assumption that the heat intensity (density) depends only on the variable t. Тогда из $\psi(t)\equiv 0$ следует, что $\mu_1(t)=\mu_2(t)\equiv 0$.

Lemma 2. Boundary value problem (2)–(3) in the domain Q is uniquely solvable at a constant density (intensity) of heat propagation.

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References

1 Jenaliyev, M., Amangaliyeva, M., Kosmakova, M., & Ramazanov, M. (2014). About Dirichlet boundary value problem for the heat equation in the infinite angular domain. *Boundary Value Problems, 213,* 1–21. doi: 10.1186/s13661-014-0213-4

2 Amangaliyeva, M.M., Dzhenaliev, M.T., Kosmakova, M.T., & Ramazanov M.I. (2015). On one homogeneous problem for the heat equation in an infinite angular domain. *Siberian Mathematical Journal, Vol. 56, No. 6*, 982–995. DOI: 10.1134/S0037446615060038

3 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). Uniqueness and non-uniqueness of solutions of the boundary value problems of the heat equation. *AIP Conference Proceedings, 1676,* 020028. DOI: 10.1063/1.4930454

4 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2014). On the spectrum of Volterra integral equation with the "incompressible" kernel. *AIP Conference Proceedings, 1611,* 127-132. DOI: 10.1063/1.4893816

5 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. On a Volterra equation of the second kind with 'incompressible' kernel. *Advances in Difference Equations, 71,* 1–14. DOI: 10.1186/ s13662-015-0418-6.

6 Kosmakova, M.T. (2016). On an integral equation of the Dirichlet problem for the heat equation in the degenerating domain. *Bulletin of the Karaganda University-Mathematics, 1 (81),* 62-67.

7 Amangaliyeva, M.M., Akhmanova, D.M., Dzhenaliev, M.T., & Ramazanov, M.I. (2011). Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity. *Differential Equations, 47, 2,* 231–243. DOI: 10.1134/S0012266111020091

8 Dzhenaliev, M.T., & Ramazanov, M.I. (2006). On the boundary value problem for the spectrally loaded heat conduction operator. *Siberian Mathematical Journal, 47, 3,* 433-451. DOI: 10.1007 /s11202-006-0056-z

9 Dzhenaliev, M.T., & Ramazanov, M.I. (2007). On a boundary value problem for a spectrally loaded heat operator: I *Differential Equations, 43, 4,* 513-524. DOI: 10.1134/S0012266107040106

10 Dzhenaliev, M.T., & Ramazanov, M.I. (2007). On a boundary value problem for a spectrally loaded heat operator: II *Differential Equations, 43, 6,* 806-812. DOI: 10.1134/S0012266107060079

11 Akhmanova, D.M., Ramazanov, M.I., & Yergaliyev. M.G. (2018). On an integral equation of the problem of heat conduction with domain boundary moving by law of $t = x^2$. *Bulletin of the Karaganda University-Mathematics, 1 (89),* 15-19. DOI: 10.31489/2018M1/15-19.

12 Ramazanov, M.I., Kosmakova, M.T., Romanovsky, V.G., Zhanbusinova, B.H., & Tuleutaeva, Z.M. (2018). Boundary value problems for essentially-loaded parabolic equation. *Bulletin of the Karaganda University-Mathematics, 4 (92),* 79-86. DOI: 10.31489/2018M4/79-86

13 Kosmakova, M.T., Romanovski, V.G., Orumbayeva, N.T., Tuleutaeva, Zh.M., & Kasymova, L.Zh. (2019). On the integral equation of an adjoint boundary value problem of heat conduction. *Bulletin of the Karaganda University-Mathematics, 3 (95),* 33–43. doi: 10.31489/2019M2/33-43.

14 Kosmakova, M.T., Akhmanova, D.M., Tuleutaeva, Zh.M., & Kasymova, L.Zh. Solving a nonhomogeneous integral equation with the variable lower limit. *Bulletin of the Karaganda University-Mathematics, 4 (96),* 52–57. doi: 10.31489/2019M4/52-57.

15 Kosmakova, M.T., Orumbayeva, N.T., Medeubaev, N.K., & Tuleutaeva, Zh.M. (2018). Problems of Heat Conduction with Different Boundary Conditions in Noncylindrical Domains. *AIP Conference Proceedings, 1997,* UNSP 020071-1. DOI: 10.1063/1.5049065

16 Kosmakova, M.T., Tanin, A.O. & Tuleutaeva, Zh.M. (2020). Constructing the fundamental solution to a problem of heat conduction. *Bulletin of the Karaganda University-Mathematics, 1 (97),* 68-78. DOI 10.31489/2020M1/68-78

17 Smirnov, V.I. (1964). *A Course of Higher Mathematics. Integral equations and partial differential equations, Vol. 4, Part 2.* Engl. transl. of 3-st ed. Pergamon Press: Oxford, [U.S.A. ed. distributed by Addison-Wesley Pub. Co., Reading, Mass.].

18 Tikhonov, A.N., & Samarskii, A.A. (2011). *Equations of the mathematical physics.* reprint edn., translated from the Russian by A.R.M Robson and P.Basu; Dover Publications: New York, USA.

19 Prudnikov, A.P., Brychkov, Yu.A., & Marichev, O.I. (1998). *Integrals and Series: Elementary Functions.* (N.M. Queen, Trans) CRC: New York, USA.

20 Prudnikov, A.P., Brychkov, Yu.A., & Marichev, O.I. (1989). *Integrals and Series: More Special Functions, Vol. 3.* Gordon and Breach: New York-London.

21 Prudnikov, A.P., Brychkov, Yu.A., & Marichev, O.I. (1998). *Integrals and Series: Special Functions, Vol. 2.* Taylor&Francis Ltd: London.

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Жойылатын облыстағы жылуөткізгіштіктің екі өлшемді шеттік есебінің шешуіне

Жұмыста цилиндрлік емес облыстағы жылуөткізгіштік теңдеуі үшін біртекті шеттік есеп қарастырылған, оның ішінде, төбесі координаталар басы болатын, екі жағы қоординаталық жазықтықтарда жататын төңкерілген пирамида. Есеп шешуі жалпыланған жылу потенциалдарының қосындысы түрінде іздестірілген. Ядросының сингулярлығы бар екінші текті екі интегралды Вольтерр теңдеулер жүйесін зерттеу қажеттілігі туындайды. Тығыздық (жылу қарқындылығы) тек уақытша айнымалыға тәуелді деп болжанады, яғни әрбір уақытша қимадағы тығыздық тұрақты болып саналады. Нәтижесінде интегралдық теңдеулер жүйесі екінші текті Вольтердің біртекті интегралдық теңдеуіне келтірілген. Үздіксіз функциялар класында бұл теңдеудің тек бір ғана жолмен шешілетіні көрсетілген.

Кілт сөздер: жылу өткізгіштік теңдеуі, Вольтердің интегралдық теңдеуі, жойылатын облыс, жылу потенциалы.

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К решению двумерной граничной задачи теплопроводности в вырождающейся области

В статье рассмотрена однородная краевая задача для уравнения теплопроводности в нецилиндрической области, а именно, в перевернутой пирамиде с вершиной в начале координат, две грани которой лежат в координатных плоскостях. Решение задачи ищется в виде суммы обобщенных тепловых потенциалов. Возникает необходимость исследования системы двух интегральных уравнений Вольтерра второго рода с сингулярностями ядра. Плотности (интенсивность тепла) предполагаются зависящими только от временной переменной, т.е. плотность в каждом временном сечении считается постоянной. В итоге система интегральных уравнений сведена к однородному интегральному уравнению Вольтерра второго рода. Показано, что это уравнение разрешимо единственным образом в классе непрерывных функций.

Ключевые слова: уравнение теплопроводности, интегральное уравнение Вольтерра, вырождающаяся область, тепловой потенциал.

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On the spectral properties of a class of high-order differential operators with operator coefficients

In this paper, we study one class of high-order differential operators. The main feature of these operators is their nonsemi-boundedness. The dependence of operator coefficients on variables creates additional difficulties in the study. In the course of the study, the conditions for the existence of a solution and separability were first found. Also studied are the questions of the smoothness of solutions and on the spectrum of boundary value problems for unbounded differential equations with variable operator coefficients

Keywords: a differential operator, nonsemi-boundedness, an operator coefficient, a spectrum.

1 Introduction. Formulation of the problem. Statement of the main results

Boundary value problems for differential equations with operator coefficients are studied in the papers of B.M. Levitan [1], M. Otelbaev [2], B.A. Suvorchenkov [3], V.I. Gorbachuk, M.L. Gorbachuk [4], I.M. Gehtman [5], V.A. Mikhlets [6], P.A. Mishnevsky [7], K.N. Ospanov [8] and others. Note that in all these papers are studied differential operators with operator coefficients of even, first, and third order, i.e. the so-called semi-bounded differential operators and operators with a coercive estimate

$$\sum_{|\alpha|\leq 2m} |a_\alpha(x)s^\alpha| \leq c|p(x,s)|,$$

where $p(x,s) = \sum_{|\alpha|\leq 2m} a_\alpha(x)s^\alpha, x \in R^n$, and $s \in \mathbb{R}^n$ and ρ is a polynomial differential operator $Lu = \sum_{|\alpha|\leq 2m} D_x^\alpha u, \alpha = (\alpha_1, \alpha_2, ..., \alpha_n), |\alpha| = \sum_{i=0}^n \alpha_i$,

$$D_x^\alpha = ((\frac{1}{i}\cdot\frac{\partial u}{\partial x_1})^{\alpha_1}, ..., (\frac{1}{i}\cdot\frac{\partial u}{\partial x_n})^{\alpha_n}), i^2 = -1.$$

However, in applications often appear differential equations with operator coefficients that do not satisfy the above conditions. For example, in particular, unbounded differential equations with operator coefficients that arise in the theory of differential equations of hyperbolic and mixed types. This case was first studied systematically in the paper of M.B. Muratbekov [9]. In this paper of the author, the case was studied when the operator potential is independent of variables. It is known that a completely different situation arises in the study of differential equations with variable operator coefficients, i.e. when operator coefficients depend on variables. In this case, the main difficulty lies in the fact that the spectrum of the operator coefficient depends on variables, and therefore, the expansion of an arbitrary function in a series of eigenfunctions becomes impossible. Therefore, the well-known methods used in the works of the above authors turn out to be little adapted when studying the questions of separability, smoothness of solutions, and the spectrum of boundary value problems for unbounded differential equations with variable operator coefficients. this paper is devoted to these pressing issues.

We believe that our results are of theoretical interest and can find application in the spectral theory of differential operators, in quantum mechanics and gas dynamics.

Let H is the Hilbert abstract separable space. Denote by $H_1 = L_2(R, H)$ the Hilbert space obtained by the completion of the set of compactly supported infinitely smooth vector functions $C_o^\infty(R, H)$ defined on $R = (-\infty, +\infty)$ with value in H by norm

$$\|u(y)\|_{H_1} = \Big(\int\limits_{-\infty}^{+\infty,} \|u(y)\|_H^2 dy\Big)^{\frac{1}{2}}$$

which corresponds to the scalar product

$$< u(y), v(y) >_{H_1} = \int\limits_{-\infty}^{+\infty,} < u(y), v(y) >_H dy.$$

In a given space the following differential equation is considered

$$Lu \equiv -u''(y) + k(y)A(y)u + ia(y)A^\alpha(y) + c(y)u = f \in H_1, \tag{1}$$

Here A is a positive definite self-adjoint variable-dependent operator with completely continuous inverse operator, $y \in H$, $\alpha \in [\frac{1}{2}, 1], k(y)$ is a piecewise continuous and bounded function in R, $k(0) = 0$ and $yk(y) > 0$ at $y \neq 0$.

By L we denote the closed operator corresponding to equation (1) in H_1. By the solution of equation (1) we mean the function $u \in H_1$ if there exists a sequence $\{u_n\}_{n=1}^{\infty,} \in C_o^\infty(R, H)$ such that

$$\|u_n - u\|_{H_1} \to 0, \|Lu_n - f\|_{H_1} \to 0$$

as

$$n \to \infty.$$

Hence it is easy to verify that finding a unique solution to equation (1) means proving the invertibility of the operator L for all $f \in H_1$.

Further statements of the results are given in the language of operators and we will use the results of [9].

Theorem 1. Let the following conditions are fulfilled:

a) $|a(y)| \geq \delta_0 > 0$ is a continuous functions in R;

b) $\sup\limits_{|y-t|\leq 1} \frac{a(y)}{a(t)} \leq c_0 < \infty$; $\sup\limits_{|y-t|\leq 1} \frac{c(y)}{c(t)} \leq c_1 < \infty$;

c) $\sup\limits_{|y-t|\leq 1} \|(A^\alpha(y) - A^\alpha(t))A^{-\alpha}\|_H \leq o(1), \alpha \in [\frac{1}{2}, 1]$;

d) $c(y) \leq c_0 a^2(y)$, for all $y \in R$, c_0 is any constant. Then for the operator $L + \lambda E$ for sufficiently large $\lambda > 0$ there exists a bounded inverse operator $(L + \lambda E)^{-1}$.

Theorem 2. Let conditions a)-c) are fulfilled. Then the estimate

$$\|u'(y)\|_{H_1} + \|ia(y)A^\alpha(y)u\|_{H_1} + \|c(y)u\|_{H_1} \leq c\|Lu\|_{H_1},$$

holds for all $u \in D(L)$, where $c > 0$ is a constant independent of $u(y)$.

The following new results were obtained in the work:

– conditions are found on the coefficients of high-order differential operators with operator coefficients that provide the following properties: a) discreteness of the spectrum; b) spectrum continuity;

– a criterion is obtained for the discreteness of the spectrum of a high-order differential operator with operator coefficients;

– the relation $L^{-1} \in \sigma_\rho$ is proved, where $1 < \rho < \infty$;

– the class of high-order unbounded differential operators with operator coefficients whose resolvents are Hilbert-Schmidt operators is indicated.

In qualitative spectral analysis, a special place is given to the study of the existence of the spectrum. In the case of its existence, problems of discreteness and continuity of the spectrum are considered. Among the papers that are similar in theme and influenced these studies, we note the papers of B.M. Levitan, I.S. Sargsyan, A.G. Kostyuchenko, M. Otelbaev, T.Sh. Kalmenov, E.I. Moiseev, C.M. Ponomarev, M.B. Muratbekov, A.S. Berdyshev, K.N. Ospanov, K.Kh. Boymatov, W.N. Everitt, M. Girtz et al.

It is known that spectral analysis of differential operators studies the nature of the spectrum depending on the behavior of the coefficients, boundary conditions, and region geometry. As an example, the last case includes the following facts: in a bounded domain, the spectrum of an elliptic operator with smooth coefficients is always discrete, and in an unbounded domain, the spectrum of the same operator with a bounded coefficient is continuous.

The most significant issue of spectral theory in the study of the spectrum depending on the behavior of the coefficients is a sign of discreteness of the spectrum. The first significant result in this direction is the Molchanov criterion on compactness of the resolvent of the singular Sturm-Liouville equation. This result was then disseminated to an operator of Schrodinger type by M.Sh. Birman and B.S. Pavlov, V.G. Maz'ya, M. Otelbaev and R. Oinarov, M.G. Gasimov obtained a criterion for the compactness of their embedding in Lebesgue space studying the topologies of energy spaces of elliptic operators. Based on this approach, the result of A.M. Molchanov was extended to new classes of semi-bounded differential operators whose energy spaces are embedded in some Sobolev weighted spaces.

Now questions arise about the discreteness and continuity of the spectrum of unbounded differential operators. Here, a significant difficulty is the question of the smoothness of elements from the domain of definition of the operator in order to extract the necessary information regarding the structure of the spectrum. These questions have not been investigated for the operator below.

By L we denote the closure in the norm $H_1 = L_2(R, H)$ of the differential operator

$$Lu \equiv (-1)^m u^{(2m)}(y) + k(y)Au + ia(y)A^{\alpha}u + c(y)u, \tag{2}$$

defined on the set $C_o^{\infty}(R, H)$, where m is a positive integer, $k(y)$ is a piecewise continuous and bounded function in $\mathbb{R}$, A is a some non-negative self-adjoint operator in the Hilbert space H with a completely continuous resolvent.

We shall assume that the coefficients $a(y), c(y)$ satisfy:

i) $|a(y)| \geq \delta_0 > 0, c(y) \geq \delta > 0$ are continuous functions in R.

Theorem 3. Let the condition i) is fulfilled and $c(y)$ is a bounded function and let $\lambda = 0$ be an eigenvalue of the operator A with finite multiplicity. Then the continuous spectrum of L is not empty.

Theorem 4. Let the condition i) is fulfilled. Then the discrete spectrum of L is not empty if the equality

$$\lim_{|y|\to\infty} \int\limits_{y}^{y+\omega} c(t)dt = \infty. \tag{3}$$

holds.

Theorem 5. Let the condition i) is fulfilled and let A be a positive definite operator with completely continuous inverse. Then the spectrum of L is discrete iff

$$\lim_{|y|\to\infty} \int\limits_{y}^{y+\omega} c(t)dt = \infty,$$

or

$$\lim_{|y|\to\infty}\int_{y}^{y+\omega}|a(t)|dt=\infty$$

for all $\omega>0$. To prove the above theorems, we will use the following auxiliary statements and estimates.

2 On a one-dimensional high-order differential operator

Consider the differential operator

$$lu=(-1)^m u^{(2m)}(y)+c(y)u,$$

initially defined on $C_o^\infty(R)$, with further closure of this operator in $L_2(R)$.

The following lemmas hold.

Lemma 1. Let the condition i) is fulfilled and let $c(y)$ be a limited function. Then the spectrum of the operator l is purely continuous.

Proof. Denote by l_t the operator defined by

$$l_t=(-1)^m u^{(2m)}+tu$$

on $C_o^\infty(R)$, where $b=\sup\limits_{t\in R}c(t)$. The operator l_t admits closure in $L_2(R)$. Introduce new metrics in the domains $D(l),D(l_t)$ of the operators l,l_t, believing

$$|u|_l=<lu,u>,u\in D(l);|u|_{l_t}=<l_tu,u>,u\in D(l_b);$$

and close the domains $D(l),D(l_t)$ in these metrics.

The resulting new Hilbert space we denoted by H_l,H_{l_b}.

It is easy to see that $H_{l_t}\subset H_l$ и $|u|_{l_t}\geq|u|_l$. Therefore, we assume that $l_t\geq l$. Where l and l_t are positive operators.

From general compactness theorems it follows: if the spectrum of the operator l is discrete, then the spectrum of the operator l_t is also discrete; if the spectrum of the operator l_t is continuous, then the spectrum of the operator l is also continuous.

It is known from the spectral theory of differential operators that the spectrum of l_t is continuous, then the spectrum of l is also continuous. The lemma is proved.

Consider the operator

$$l_tu=(-1)^m u^{(2m)}+(tk(y)+it^\alpha a(y)+c(y))u,$$

where $u\in D(l_t)$.

Lemma 2. Let the condition i) is fulfilled. Then the estimate

$$c\|l_tu\|_2^2\geq\int_{-\infty}^{+\infty,}[|u^{(m)}|^2+c((y)+|t|^\alpha|a(y)|)|u|^2]dy,$$

holds for all $u\in D(l_t)$, where $c>0$ is a constant independent of t,u.

Proof. Here and below, without loss of generality, we assume $|a(y)|\geq1,c(y)\geq1$. Consider the scalar product

$$<l_tu,u>=\int_{-\infty}^{+\infty,}[|u^{(m)}|^2+(tk(y)+it^\alpha a(y)+c(y))|u|^2]dy, \tag{4}$$

where $u\in C_o^\infty(R)$.

Further, since $a(y)$ does not change sign, we have:

$$| < l_t, u > | \geq |t^{\alpha}| \int\limits_{-\infty}^{+\infty,} |a(y)||u|^2 dy. \tag{5}$$

sing the Cauchy inequality with $\varepsilon > 0$ from inequality (5) we obtain

$$\frac{1}{2\varepsilon}\|l_t u\|_2^2 + \frac{\varepsilon}{2}\|u\|_2^2 \geq \frac{1}{2}|t|^{\alpha} \int\limits_{-\infty}^{+\infty,} |a(y)|^2|u|^2 dy + \frac{1}{2}\delta_0|t|^{\alpha}\|u\|_2^2.$$

From this inequality, by virtue of the condition i) and $d < t < \infty$ we obtain the following estimate

$$\frac{1}{2\varepsilon}\|l_t u\|_2^2 \geq \frac{1}{2}|t|^{\alpha} \int\limits_{-\infty}^{+\infty,} |a(y)|^2|u|^2 dy. \tag{6}$$

Further, it follows from equality (4) that

$$| < l_t u, u > | = \left| \int\limits_{-\infty}^{+\infty,} [|u^{(m)}|^2 + (tk(y) + it^{\alpha}a(y) + c(y))|u|^2]dy \right| \geq$$

$$\geq \left| \int\limits_{-\infty}^{+\infty,} [|u^{(m)}|^2 + (tk(y) + c(y))|u|^2]dy \right| \geq$$

$$\geq | \int\limits_{-\infty}^{+\infty,} [|u^{(m)}|^2 + c(y)|u|^2]dy| - |t| \left| \int\limits_{-\infty}^{+\infty,} [|k(y)||u|^2 dy \right|.$$

Here, according to the Cauchy condition with $\varepsilon > 0$, it is easy to verify that

$$\frac{1}{2\varepsilon}\|l_t u\|_2^2 \geq \frac{1}{2} \int\limits_{-\infty}^{+\infty,} (|u^{(m)}|^2 + c(y)|u|^2)dy - |t| \int\limits_{-\infty}^{+\infty,} |k(y)||u|^2 dy. \tag{7}$$

Using the Cauchy inequality and condition i), from inequality (5) we obtain

$$\|l_t u\|_2^2 \geq |t|^{2\alpha}\delta_0^2\|u\|_2^2 \tag{8}$$

Combining (7) and (8), we find

$$c\|l_t u\|_2^2 \geq \frac{1}{2} \int\limits_{-\infty}^{+\infty,} (|u^{(m)}|^2 + c(y)|u|^2)dy. \tag{9}$$

From inequalities (6) and (9), we have

$$c(\varepsilon, \delta_0)\|l_t u\|_2^2 \geq \int\limits_{-\infty}^{+\infty,} [|u^{(m)}|^2 + (c(y) + |t|^{\alpha}|a(y)|)|u|^2]dy,$$

where $c(\varepsilon, \delta_0) > 0$.

Lemma 3. Let the condition i) is fulfilled. Then the estimate

$$\|l_t^{-1}\|_2^2 \le \frac{c}{t^{2\alpha}},$$

holds, where $0 < d < t, c > 0$ are independent of $t, \alpha \in [\frac{1}{2}, 1)$.

The proof of Lemma 3 follows from Lemma 1.2.2. of paper [9].

Lemma 4. Let the condition i) is fulfilled. Then the operator l_t^{-1} is completely continuous iff the equality (3) holds.

Proof. We will use the method proposed in the paper of M.B. Muratbekov [10] for mixed type operators.

Necessity. Suppose that the condition of Lemma 4 is not satisfied. Then there exists a sequence of intervals $Q_d(y_j) \subset R$ such that

$$sup \int\limits_{Q_d(y_i)} c(t)dt < c,$$

where $d > 0$, i.e. when the interval $Q_d(y_j)$, goes to infinity with keeping length.

Let $\omega(y) \in C_0^\infty(Q(0))$. consider set of functions such that

$$u_j(y) = \omega(y - y_i).$$

For these functions, it is easy to establish the following inequality

$$\|(-1)^m u_j^{(2m)}(y) + (tk(y) + it^\alpha a(y) + c(y))u_j\|_2 \le c < \infty, \tag{10}$$

holds as $o < t < N$, where N is a finite number, c is independent of j.

It is not difficult to verify it follows from Lemma 3 that

$$\|l_t^{-1}\| \to 0$$

as $t \to \infty$.

The last property was taken into account in the proof of inequality (10).

It follows from inequality (10) that

$$F_j(y) \in L_2(R), suppF_j(y) \subset Q_d(y_i),$$

where $F_j = (-1)^m u_j^{(2m)} + (tk(y) + it^\alpha a(y) + c(y))u$.

Now it is easy to show that the sequence $F_j(y)$ converges weakly to zero. Indeed

$$| < F_j(y), v > | = \left| \int\limits_{-\infty}^{+\infty,} F_j(y)v(y)dy \right| = \left| \int\limits_{Q_d(y_j)} F_j(y)v(y)dy \right| \le$$

$$\le \left(\int\limits_{Q_d(y_j)} F_j^2(y))dy \right)^{\frac{1}{2}} \cdot \left(\int\limits_{Q_d(y_j)} v^2(y))dy \right)^{\frac{1}{2}} \le c \left(\int\limits_{Q_d(y_j)} v^2(y))dy \right)^{\frac{1}{2}} \tag{11}$$

for any $v \in L_2(R)$. Obviously $\int\limits_{Q_d(y_j)} v^2(y)dy \to 0$ as $j \to \infty$, since $v \in L_2(R)$. Hence and (11) it follows that the sequence $\{F_j(y)\} \to 0$ converges weakly at $j \to \infty$.

It is easy to verify that

$$\|u_j\|_2 = c > 0. \tag{12}$$

Since, if the operator l_t^{-1} is compact, then the sequence $\{u_j\}$ should converge to zero by norm $L_2(R)$. And this is impossible due to (12). The necessity is proved.

Sufficiency. Repeating the calculations and arguments used in the first part, we have

$$R(l_t^{-1}) \subset L_2^m(R, c(y)),$$

where $L_2^m(R, c(y))$ is a replenishment $C_0^\infty(R)$ by norm

$$\|u\|_{L_2^m(R,C(y))} = \left(\int_{-\infty}^{\infty,} (|u^{(m)}|^2 + c(y)|u|^2) dy \right)^{\frac{1}{2}}.$$

By the results of [9, 10], any bounded set in $L_2^m(R, c(y))$ is a compact in $L_2(R)$ if and only if the condition of [9, 10] are satisfied, i.e.

$$C^*(y) \to \infty, \tag{13}$$

as $|y| \to \infty$, where $C^*(y) = inf\{d^{-1} : d^{1-2m}\} \geq \int\limits_{y-\frac{d}{2}}^{y+\frac{d}{2}} c(t)dt.$

It follows that it is sufficient to prove the equivalence of conditions (13) and (3).

Suppose that (13) is not satisfied. Then there exists a sequence of points $y_n, n = 0, 1, 2, ...$ and constants c such that $c^*(y_n) \leq c$.

By virtue of equality

$$d_n^{1-2m} = \int_{y-\frac{d_n}{2}}^{y+\frac{d_n}{2}} c(t)dt,$$

it follows from the definition of $c^*(y)$, we obtain that there exist intervals Δ_n, which go to infinity, keeping the length and

$$\int_{\Delta_n} c(t)dt < c_1 < \infty.$$

The last inequality shows that condition (3) is not satisfied.

Conversely, let condition (3) not be satisfied. Then there exist some disjoint intervals $\triangle_n$ which go to infinity with keeping length.

From the definition $c^*(y)$ we obtain $c^*(y_n) \leq c$, where y_n is a centre of $\triangle_n$. This means that (13) is not satisfied, therefore (3) and (13) are equivalent. The sufficiency of Lemma 4 is proved.

Lemma 5. Let the condition i) is fulfilled and $d < t < \infty, d > 0$. Then the operator l_t^{-1} is completely continuous iff for any $\omega > 0$:

$$\lim_{|y|\to\infty} \int_y^{y+\omega} c(t)dt = \infty, \tag{14}$$

or

$$\lim_{|y|\to\infty} \int_y^{y+\omega} a(t)dt = \infty, \tag{15}$$

Proof. Note that the coefficient of $i(i^2 = -1)$ under any t does not vanish, since t varies on the interval (d, ∞).

This means that when studying the spectral properties of the operator l_t^{-1} we must take into account the behavior of both coefficients $a(y)$ and $c(y)$.

We consider first the case (14). In this case, repeating the arguments and calculations used in the proof of Lemma 4, we obtain the proof of Lemma 5.

Now we will consider the case (15).

Necessity. To prove this, suppose the contrary; let the conditions of the lemma not be satisfied. Then there exists a sequence of intervals $Q_d(y_i) \subset R$ such that

$$sup \int\limits_{Q_d(y_i)} |a(y)|dt < c,$$

where $d > 0$. The interval $Q_d(y_i)$ goes to the infinity keeping length.

Let $\omega(x) \in C_0^\infty(Q(0))$. Consider a set of functions such that $u_j(y) = \omega(y - y_j)$. Then it is easy to estimate

$$\|(-1)^m u_j^{(2m)} + (tk(y) + it^\alpha a(y) + c(y))u_j\|_2^2 \leq c < \infty,$$

where c is independent of j.

Let

$$F_j = (-1)^m u_j^{(2m)} + (tk(y) + it^\alpha(a(y)) + c(y))u, suppF_j(y) \subset Q_d(y_i).$$

Hence $F_j(y)$ is converge weakly to zero.

The proof of necessity completes same as in Lemma 4.

Sufficiency. It follows from the results of Lemma 2 that

$$R(l_t^{-1}) \subseteq W_{2,a,c}^m(R), \tag{16}$$

where $W_{2,a,c}^m(R)$ is the function space with norm

$$\|u : W_{2,a,c}^m(R)\| = \left(\int\limits_{-\infty}^{\infty} [|u^{(m)}|^2 + (|a(y)| + c(y))|u|^2]dy \right)^{\frac{1}{2}}.$$

It is easy to verify that

$$W_{2,a,c}^m(R) \subset W_{2,a}^m(R) \tag{17}$$

Indeed, let $u(x) \in W_{2,a,c}^m(R)$. Then the estimate

$$\int\limits_{-\infty}^{+\infty} [|u^{(m)}|^2 + |a(y)||u|^2]dy \leq \int\limits_{-\infty}^{+\infty} [|u^{(m)}|^2 + (|a(y)| + c(y))|u|^2]dy$$

holds. The last estimate proves inclusion (17). From here and (16) we have

$$R(l_t^{-1}) \subseteq W_{2,a}^m(R). \tag{18}$$

Further, from (18), using arguments similar to those of Lemma 4, we obtain a proof of sufficiency.

Proof of continuity and discreteness theorems.

Proof of Theorem 3. Denote by $\{e_n\}$ complete orthonormal system of eigenvectors of the operator A. Then the equality

$$u(y) = \sum_{n=1}^{\infty} u_n(y)e_n,$$

$$\|u(y)\|_{H_1}^2 = \sum_{n=1}^{\infty} \|u_n(y)\|_2^2$$

holds for all $u(y) \in H$.

Immediately following equality is easily verified

$$Au = \sum_{n=1}^{\infty} \lambda_n u_n(y) e_n;$$

$$A^\alpha u = \sum_{n=1}^{\infty} \lambda_n^\alpha u_n(y) e_n.$$

This shows that the separation of the variables of the spectral problem

$$Lu = \lambda u$$

reduces to the following spectral problems

$$-u_n^{(2m)}(y) + (k(y)\lambda_n + ia(y)\lambda_n^\alpha + c(y))u_n = \lambda u_n(y), \qquad (19)$$

$$n = 1, 2, 3, ...$$

If λ is a spectrum point of L, then λ is the spectrum point of one of the operators (19). And vice versa, if λ is a spectrum point of one of the operators (19), then λ is the spectrum point of L.

Now, if we use the assumption of Theorem 1, that $\lambda = 0$ is an eigenvalue of finite multiplicity, then Theorem 3 follows easily from Lemma 1.

Proof of Theorem 4. Similarly reasoning and using Lemma 2 we obtain the proof of Theorem 4.

Proof of Theorem 5. In Theorem 5, we assumed that the operator A is the positive definite self-adjoint with a completely continuous inverse, and this is due to the fact that the smallest eigenvalue of this operator is nonzero. Now the theorem being proved follows from Lemma 4.

3 On the properties of a resolvent of a single unbounded high-order differential operator with an operator coefficient

Let H is the separable Hilbert space. Denote by $C_0^\infty(R, H)$ is the set of infinitely smooth compactly supported functions defined on $R(-\infty, +\infty)$ with value in H.

Consider the differential operator

$$Lu \equiv (-1)^m u^{(2m)}(y) + k(y)Au + ia(y)A^\alpha u + c(y)u,$$

where $m = 1, 2, ..., u(y) \in C_0^\infty(R, H), A$ is the positive definite self-adjoint operator in the Hilbert space H with a completely continuous resolvent, $\alpha \in [\frac{1}{2}, 1), k(y)$ is the piecewise continuous and bounded function in $R, k(0) = 0$ and $yk(y) > 0$ at $y \neq 0$.

Let the conditions

1) $|a(y)| \geq \delta_0 > 0, c(y) \geq \delta > 0$ is continuous functions in R,

2) $\sup\limits_{|x-t|\leq 1} \frac{a(t)}{a(x)} \leq c < \infty, \sup\limits_{|x-t|\leq 1} \frac{c(t)}{c(x)} \leq c_0 < \infty;$

3) $0 < \delta_1 \leq \frac{a^2(y)}{c(y)}$ at $y \in R$ hold.

The following theorems hold.

Theorem 6. Let conditions 1) -3) are fulfilled. Then the resolvent of the operator L belong to σ_p if $p > 1$ and

$$\sum_{j=1}^{\infty} \int_{-\infty}^{\infty} Q^{\frac{m(-p+1)}{2}}(j, y)dy < \infty,$$

where σ_p is the set of all completely continuous operators such that $\|A\|_{\sigma_p}^p = \sum\limits_{n=1}^{\infty} S_n^p(A) < \infty, S_n(A)$ is an eigenvalues of $\sqrt{A^*A}$, $Q(t, y) = |a(y)it^\alpha + c(y)|^2$.

Theorem 7. Let conditions 1) -3) are fulfilled and

$$\sum_{n=1}^{\infty} n^l \lambda_n^{-1} < \infty,$$

for all $0 \le l < 1 + s, s > 0$, where λ_n is an eigenvalues of A. Then the resolvent of the operator L is a Hilbert-Schmidt operator if $a^{-m}(y) \in L_1(R)$.

To prove the above theorems, we use the following auxiliary statements and estimates.

Consider the operator defined by the equality

$$l_t = (-1)^m u^{(2m)} + (tk(y) + it^\alpha a(y) + c(y))u$$

in $L_2(R), d < t < \infty$.

It is known from the results of the first part that, under conditions 1)-3), there exists a resolvent l_t and the estimate

$$\left\| u^{(m)} \right\|_2^2 + \| it^\alpha a(y)u\|_2^2 + \|c(y)u\|_2^2 \le c\left(\|l_t u\|_2^2 + \|u\|_2^2 \right), \tag{20}$$

holds for all $u \in D(l_t)$, where $c > 0$ is independent of u and t.

Let A be a completely continuous operator. It is known that the eigenvalues of the operator $(A^*A)^{\frac{1}{2}}$ are called $s-$numbers of A. Nonzero $s-$numbers will be numbered in decreasing order, taking into account their multiplicity, so

$$s_j(A) = \lambda_j((A^*A)^{\frac{1}{2}}), j = 1, 2, ...$$

We introduce the following function $N(\lambda) = \sum\limits_{s_j > \lambda} 1$ is the number s_j greater than $\lambda > 0$. Let

$$M = \{u \in L_2(R) : \|l_t u\|_2^2 + \|u\|_2^2 \le 1\}.$$

Denote by d_k is the Kolmogorov k-width of the set M in $L_2(R)$.

By definition

$$d_k = \inf_{\{\varphi_k\}} \sup_{u \in M} \inf_{v \in \varphi_k} \|u - v\|_2,$$

where "infimum"takes over all subspaces φ_k dimensionality $\le k$.

Lemma 6. Suppose that the conditions of Theorem 4 are satisfied. Then the estimate

$$d_k \le c^2 \tilde{d}_k,$$

holds, where $\tilde{d}_k$ is the Kolmogorov k-width of the set

$$\tilde{M} = \{u \in L_2(R) : \|u^{(2m)}\|_2^2 + \|c(y)u\|_2^2 + \|t^\alpha a(y)u\|_2^2 \le 1\},$$

where $c > 0$ is any constant.

Proof. It follows from the hypothesis of the lemma that the estimate (20) holds for all $u \in D(l_t)$. Hence

$$\|u^{(m)}\|_2^2 + \|t^\alpha a(y)u\|_2^2 + \|c(y)u\|_2^2 \le c^2(\|l_t u\|_2^2 + \|u\|_2^2) \le c^2,$$

for all $u \in M$, where $c > 0$ is a constant independent of t and u.

Therefore $M \subseteq \tilde{M}_{c^2}$.

Now, using the property of the widths, we have

$$d_k \le c^2 \tilde{d}_k.$$

Lemma 6 is proved.

Lemma 7. Let the conditions of Theorem 4 is satisfied. Then the estimate

$$N(\lambda) \leq \tilde{N}(c^{-2}\lambda),$$

holds, where $N(\lambda) = \sum\limits_{d_k > \lambda} 1$ is the number of width d_k greater than $\lambda > 0$, $\tilde{N}(\lambda) = \sum\limits_{d_k > \lambda} 1$ is the number of width $\tilde{d_k}$ greater than $\lambda > 0$.

Proof. By virtue of Lemma 6 we have

$$N(\lambda) = \sum_{d_k > \lambda} 1 \leq \sum_{c^2 \tilde{d_k} > \lambda} 1 = \sum_{\tilde{d_k} > c^{-2}} \lambda 1 = \tilde{N}(c^{-2}\lambda).$$

Lemma 7 is proved.

Denote by $\tilde{N}(\lambda) = \sum\limits_{\lambda_j > \lambda} 1$ number of singular numbers $s_j (j = 1, 2, ...)$ of l_t^{-1} greater than $\lambda > 0$.

Lemma 8. Let the conditions of Lemma 7 be satisfied, then the estimate

$$\tilde{N}(\lambda) \leq c\lambda^{-\frac{1}{m}} mes(y \in R : Q^{\frac{m}{2}}(t, y) \leq c\lambda^{-1}),$$

holds, where c is a constant independent of $Q(t, y)$.

Proof. It is known

$$s_{j+1}(l_t^{-1}) = d_j, j = 1, 2, ...,$$

where d_j, j is the width of M.

Denote by $L^m_{2,Q(t,y)}$ space obtained by replenishment $C_0(R)$ relative to the norm

$$\left| u, L^m_{2,Q(t,y)} \right| = \left(\int\limits_{-\infty}^{+\infty} |u^{(m)}|^2 + Q(t, y)|u|^2 dy \right)^{\frac{1}{2}}.$$

It is clear that $\tilde{N} \subset L^m_{2,Q(t,y)}$.

Now the proof of the lemma follows from Lemmas 6 and 7 and the results of [10].

Proof of the main theorems.

Denote by s_{ji} singular numbers of $l_j, j = 1, 2, ...$. It is easy to verify the inequality

$$\sum_j \sum_i s_{ji} \leq c \sum_j \sum_i d_{ji}$$

holds. Let

$$F(\lambda) = N(\lambda^{-1}),$$

where $N(\cdot)$ is the function of the distribution widths $d_{ji}, i = 0, 1, 2, ...$ greater than $\lambda > 0$. Note that $F(\lambda) = 0$ if $\lambda \leq \delta_{j0} = \frac{1}{d_{j0}}$.

Since $F(\lambda_j) = j$ at $\lambda_j = d_{ji}^{-1}$, then

$$\sum_j \sum_{i=1}^{\infty} d_{ij}^p = \sum_j \lim_{N \to \infty} \sum_{i=0}^{N} d_{ji}^p = \sum_j \lim_{N \to \infty} \sum_{i=0}^{N} \lambda_{ji}^{-p} = \sum_j \lim_{N \to \infty} \int\limits_0^{\lambda_n} \lambda_j^{-p} dF(\lambda_j) = \sum_j \int\limits_0^{\infty} \lambda_j^{-p} dF(\lambda_j).$$

We transform the internal integral like follows.

Let $\alpha_i = d_{ji+1}^{-1}$, where $\{d_{ji+1}^{-1}\}_{i=0}^{\infty}$. Then integrating by parts, we have

$$\int\limits_0^{\alpha_i} \lambda_j^{-p} dF(\lambda_j) = \int\limits_{\delta_{j0}}^{\alpha_i} \lambda_j^{-p} dF(\lambda_j) = \lambda_j^{-p} F(\lambda_j) - \int\limits_0^{\alpha_i} \lambda_j^{-p-1} F(\lambda_j) d\lambda_j =$$

$$= \lambda_j^{-p} F(\alpha_i) - \delta_{jo}^{-p} F(\delta_i 0) - \int\limits_0^{\alpha_{ji}} \lambda_j^{-p-1} F(\lambda_j) d\lambda_j,$$

where $\delta_{j0} = \frac{1}{d_{j0}}$. So $F(\delta_{j0}) = 0$.

Due to the last equality, this equality takes the following form

$$\int\limits_0^{\alpha_{ji}} \lambda_j^{-p} dF(\lambda_i) = \alpha_{ji}^{-p} F(\alpha_{ji}) - \int\limits_0^{\alpha_i} \lambda_j^{-p-1} F(\lambda_j) d\lambda_j. \tag{21}$$

Using the condition of Theorem 6 we obtain

$$\int\limits_{-\infty}^{+\infty} Q^{\frac{m(-p+1)}{2}}(j,y)dy \geq \int\limits_{mes(y\in R: Q^{-\frac{m}{2}}(j,y)\geq c\lambda)} Q^{\frac{m(-p+1)}{2}}(j,y)dy \geq$$

$$\geq \int\limits_{mes(y\in R: Q^{-\frac{m}{2}}(j,y)\geq c\lambda)} Q^{\frac{-m(-p+1)}{2}}(j,y)dy =$$

$$= \int\limits_{mes(y\in R: Q^{-\frac{m}{2}}(j,y)\geq c\lambda)} Q^{-\frac{m}{2}\cdot(p-1)}dy \geq \lambda^{p-1} mes(y\in R: Q^{-\frac{m}{2}}(j,y)\geq c\lambda).$$

Hence

$$mes(y\in R: Q^{\frac{m}{2}}(j,y) \leq c\lambda^{-1}) \leq \frac{A}{\lambda^{p-1}} = A\lambda^{-(p-1)}$$

where $A = \int\limits_{-\infty}^{+\infty} Q^{\frac{m(-p+1)}{2}}(j,y)dy$.

From the last inequality and Lemma 8, we have

$$N(\lambda) \leq c\frac{A}{\lambda^{(p-1)+\frac{1}{m}}} = cA\lambda^{-p+1-\frac{1}{m}}.$$

Hence we obtain

$$d_{jk}^p \leq c\frac{A^{\frac{p}{(p-(1-\frac{1}{m}))}}}{(k+1)^{\frac{p}{[p-(1-\frac{1}{m})]}}}.$$

This inequality shows that outside the integral term of equality (21) is equal to zero as $k\to\infty$.

Now it remains to calculate the integral $\int\limits_0^{\infty} \lambda_j^{-p-1} F(\lambda_j) d\lambda_j$. Directly calculating, taking into account Lemma 8, we have

$$\int\limits_0^{\infty} \lambda_j^{-p-1} F(\lambda_j) d\lambda_j \leq c \int\limits_0^{\infty} \lambda_j^{-p-1} \lambda_j mes(y\in R: Q^{\frac{m}{2}}(t,y) \leq c\lambda) d\lambda_j =$$

$$= c_1 \int\limits_0^{\infty} \lambda_j^{-p} mes(y\in R: Q^{\frac{m}{2}}(t,y)\leq c\lambda_j)d\lambda_j = c_1 \int\limits_0^{\infty} mes(y\in R: Q^{\frac{m}{2}}(t,y)\leq c\lambda_j) d\lambda_j^{-p+1} =$$

$$= c_1\lambda_j^{-p+1} mes(y\in R: Q^{\frac{m}{2}}(t,y)\leq c\lambda_j) - c_1\int\limits_0^{\infty} \lambda_j^{-p+1} dmes(y\in R: Q^{\frac{m}{2}}(t,y)\leq c\lambda_j).$$

All outside integral terms disappear.

It remains to verify that

$$\int_0^{\infty} \lambda_j^{-p+1} dmes(y \in R : Q^{\frac{m}{2}}(t,y) \le c\lambda_j) = \int_0^{\infty} Q^{\frac{m}{2}\cdot(-p+1)}(t,y)dy. \tag{22}$$

Indeed, this follows from the fact that for any sequence of points

$$\delta_{j0}^2 \le \xi_0 \le \xi_1 \le \xi_2 \le ... \le \xi_k \le ...,$$

correspond to Darboux sums

$$\overline{S} = \sum_{k=1}^{\infty} M_k mes\Omega_k;$$

$$\underline{S} = \sum_{k=1}^{\infty} m_k mes\Omega_k,$$

where $\Omega_k = \{x \in R : \xi_{k-1} \le Q^{\frac{m}{2}}(t,y) \le \xi_k\}$,

$$M_k = \sup_{x\in\Omega_k} Q^{\frac{m}{2}\cdot(-p+1)}(t,y), m_k = \inf_{x\in\Omega_k} Q^{\frac{m}{2}\cdot(-p+1)}(t,y).$$

The inequality

$$\sum_{k=1}^{\infty} \xi_k^{-p+1} mes\Omega_k \le \underline{S} \le \overline{S} \le \sum_{k=1}^{\infty} \xi_{k-1}^{-p+1} mes\Omega_k \tag{23}$$

holds. If the right integral exists in (22), then, by virtue of (23), there exists the left integral and they are equal.

Theorem 6 is proved.

Proof of Theorem 7. We have

$$Q^{-\frac{m}{2}}(n,y) = \frac{1}{|i\lambda_n^{\alpha} a(y) + c(y)|^m} \le \frac{1}{\lambda_n^{m\alpha}|a(y)|^m},$$

$$n = 1, 2, 3, ...$$

This and Lemma 6 it follows that

$$\sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} Q^{-\frac{m}{2}}(n,y)dy \le \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{m\alpha}} \int_{-\infty}^{+\infty} \frac{1}{|a(y)|^m} dy = \int_{-\infty}^{+\infty} \frac{1}{|a(y)|^m} dy \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{m\alpha}}.$$

Now, using condition 2), from the last inequality we obtain the proof of Theorem 5.

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References

1 Левитан Б.М. Исследование функции Грина уравнения Штурма-Лиувилля с операторным коэффициентом / Б.М. Левитан // Мат. сб. — 1968. — Т. 76(118). — № 2. — С. 239–270.

2 Отелбаев М. Об условиях самосопряженности оператора Штурма-Лиувилля с операторным потенциалом / М. Отелбаев // Украин. мат. журн. — 1983. — Т. 28. — № 6. — С. 763–771.

3 Левитан Б.М. Достаточные условия дискретности спектра уравнения Штурма-Лиувилля с операторным потенциалом / Б.М. Левитан, Г.А. Суворченкова // Функциональный анализ и его приложения. — 1968. — Т. 2. — № 2. — С. 56–62.

4 Горбачук В.И. О некоторых классах граничных задач для уравнения Штурма-Лиувилля с операторным потенциалом / В.И. Горбачук, М.Л. Горбачук // Украин. мат. журн. — 1972. — Т. 23. — № 3. — С. 291–305.

5 Гехтман И.М. О спектре операторного уравнения Штурма-Лиувилля / И.М. Гехтман // Функциональный анализ и его приложения. — 1972. — Т. 6. — № 2. — С. 81–82.

6 Михалец В.А. Распределение собственных значений операторного уравнения Штурма-Лиувилля / В.А. Михалец // Изв. АН СССР. — 1977. — Т. 41. — № 3. — С. 607–619.

7 Мишневский П.А Функция Грина и асимптотическое поведение собственных значений операторной задачи Штурма-Лиувилля / П.А Мишневский // ДАН СССР. — 1987. — Т. 203. — № 4. — С. 726–765.

8 Муратбеков М.Б. Коэрцитивная разрешимость дифференциального уравнения нечетного порядка и ее приложения / М.Б. Муратбеков, М.М. Муратбеков, К.Н. Оспанов // Док. АН РФ. — 2010. — Т. 435. — № 3. — С. 1–3.

9 Муратбеков М.Б. Коэрцитивные оценки для одного дифференциального оператора высокого порядка / М.Б. Муратбеков // Дифференциальные уравнения. — 1981. — Т. 17. — № 5. — С. 893–901.

10 Muratbekov M.B. Two sides of the distribution function of s-values of class of mixed type differential operators / M.B. Muratbekov // Complex Variables and Elliptic Equations. — 2007. — Vol. 57. — № 12. — P. 1121–1144.

М.Б. Мұратбеков, С.Ж. Игисинов, Б.М. Мүсілімов, Р.Р. Мақұлбекова

Операторлы коэффициентті жоғары ретті дифференциалдық операторлардың бір класының спектральді қасиеттері туралы

Мақалада жоғары ретті дифференциалдық операторлардың бір класы зерттелген. Мұндай операторлардың негізгі ерекшелігі олардың жартылай шенелмеген болуында. Операторлы коэффициенттердің айнымалыларға тәуелді болуы зерттеуде қосымша қиындықтар туғызады. Зерттеу барысында ең алдымен шешімнің бар болу шарттары және бөліктенуі анықталған. Сонымен қатар айнымалы операторлы коэффициентті жартылай шенелмеген дифференциалдық теңдеулер үшін шешімнің тегістігі және шекаралық есептердің спектріне қатысты мәселелер зерттелген.

Кілт сөздер: дифференциалдық оператор, жартылай шенелмегендік, операторлы коэффициент, спектр.

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О спектральных свойствах одного класса дифференциальных операторов высокого порядка с операторными коэффициентами

В статье исследован один класс дифференциальных операторов высокого порядка. Главной особенностью данных операторов является их неполуограниченность. Зависимость операторных коэффициентов от переменных создает дополнительные трудности в исследовании. В ходе исследования сначала найдены условия существования решения и разделимости. Также изучены вопросы гладкости решений и о спектре краевых задач для неполуограниченных дифференциальных уравнений с переменными операторными коэффициентами.

Ключевые слова: дифференциальный оператор, неполуограниченность, операторный коэффициент, спектр.

References

1 Levitan, B.M. (1968). Issledovanie funktsii Hrina uravneniia Shturma-Liuvillia s operatornym koeffitsientom [Investigation of the Greens function of the Sturm-Liouville equation with an operator coefficient]. *Matematicheskii sbornik — Mathematical collection, 76(118), 2,* 239-270 [in Russian].

2 Otelbaev, M. (1983). Ob usloviiakh samosopriazhennosti operatora Shturma-Liuvillia s operatornym potentsialom [On self-adjointness conditions for the Sturm-Liouville operator with operator potential]. *Ukrainskii matematicheskii zhurnal — Ukrainian Mathematical Journal, 28, 6,* 763-771 [in Russian].

3 Levitan, B.M. & Suvorchenkova, G.A. (1968). Dostatochnye usloviia diskretnosti spektra uravneniia Shturma-Liuvillia s operatornym potentsialom [Sufficient conditions for the discreteness of the spectrum of the Sturm-Liouville equation with operator potential]. *Funktsionalnyi analiz i eho prilozheniia — Functional analysis and its applications, 2, 2,* 56-62 [in Russian].

4 Gorbachuk, V.I. & Gorbachuk, M.L. (1972). O nekotorykh klassakh hranichnykh zadach dlia uravneniia Shturma-Liuvillia s operatornym potentsialom [On some classes of boundary value problems for the Sturm-Liouville equation with operator potential]. *Ukrainskii matematicheskii zhurnal — Ukrainian Mathematical Journal, 23, 3,* 291-305 [in Russian].

5 Gekhtman, I.M. (1972). O spektre operatornoho uravneniia Shturma-Liuvillia [On the spectrum of the operator Sturm-Liouville equation]. *Funktsionalnyi analiz i eho prilozheniia — Functional analysis and its applications, 6, 2,* 81-82 [in Russian].

6 Mihalec, V.A. (1977). Raspredelenie sobstvennykh znachenii operatornoho uravneniia Shturma-Liuvillia [Distribution of eigenvalues of the operator Sturm-Liouville equation]. *Izvestiia AN SSSR — Proceedings of the USSR Academy of Sciences, 41, 3,* 607-619 [in Russian].

7 Mishnevskij, P.A. (1987). Raspredelenie sobstvennykh znachenii operatornoho uravneniia Shturma-Liuvillia [Green function and asymptotic behavior of the eigenvalues of the operator Sturm-Liouville problem]. *DAN SSSR — DAN USSR, 203, 4,* 726-765 [in Russian].

8 Muratbekov, M.B. & Muratbekov, M.M. & Ospanov, K.N. (2010). Koertsitivnaia razreshimost differentsialnoho uravneniia nechetnoho poriadka i ee prilozheniia [Coercive solvability of an odd-order differential equation and its applications]. *Doklady Akademii nauk RF — Reports of the Academy of Sciences of the Russian Federation, 435, 3,* 1–3 [in Russian].

9 Muratbekov, M.B. (1981). Koertsitivnye otsenki dlia odnoho differentsialnoho operatora vysokoho poriadka [Coercive estimates for one high-order differential operator]. *Differentsialnye uravneniia — Differential equations, 17, 5,* 893-901 [in Russian].

10 Muratbekov, M.B. (2007). Two sides of the distribution function of s-values of class of mixed type differential operators. *Complex Variables and Elliptic Equations, 57, 12,* 1121-1144.

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Multiperiodic solution of linear hyperbolic in the narrow sense system with constant coefficients

There is researched existential problem of a unique multiperiodic in all independent variables solution of a linear hyperbolic in the narrow sense system of differential equations with constant coefficients and its integral representation in vector-matrix form. To solve this problem, based on Cauchy's method of characteristics, a constructing methodology for solutions of initial problem system under consideration with various differentiation operators in vector fields directions of independent variables space has been developed based on projectors. Using this method, Cauchy problems for linear system with integral representation are solved. The introduced projectors by definition characteristic had significant value. By solving the main problem necessary and sufficient conditions for existence of multiperiodic solutions linear homogeneous systems other than trivial are established. The conditions are obtained for absence of nonzero multiperiodic solutions of these systems. In absence of nonzero multiperiodic solutions linear homogeneous systems, the main theorem on existence and uniqueness of multiperiodic solution linear nonhomogeneous system with derivation of its integral representation depending on projection operators is proved. The developed method has prospect of extending the results to quasilinear system under consideration, as well as to multidimensional vector $t = (t_1, ..., t_m)$ and multiperiodic matrices at partial derivatives of unknown vector-function.

Keywords: hyperbolic system in the narrow sense, multiperiodic solution, method of characteristics, projection operators, differentiation operators by vector fields, integral representation.

Introduction

Solving many problems of modern science and technology, one often has to deal with oscillatory processes, which are described by partial differential equations. In this regard, the study of oscillatory processes described by single and multifrequency periodic solutions of differential equations systems has important theoretical and applied value. It is known that the basis of theory oscillatory solutions of differential equations originates from the classical works of A.M. Lyapunov, A. Poincare, N.M. Krylov, N.N. Bogolyubov, Yu.A. Mitropolsky, A.M. Samoilenko, A.N. Kolmogorov, V.I. Arnold, Yu. Moser et al.

Methods for integrating systems of quasilinear differential equations with the same main part regarding this note are described in [1–6]. Note that the integration of quasilinear differential equations systems with different principal parts refers to little-studied problems in the sections of the theory partial differential equations. Therefore, the development of methods for solving problems of multiperiod solutions of such systems is at the initial stage of its development. It is known that the basis of theory multiperiodic solutions of partial differential equations systems with one differentiation operator was laid in [4–10]. Some ideas of the methods of these works, based on research [11–14], were extended in [15–17] to the study of problems on multiperiodic solutions of quasilinear equations systems with various differentiation operators along their characteristics.

In [15], the question of almost multiperiodic solutions of systems with small nonlinearity is studied, when the matrix of coefficients of the linear part has a triangular form, and the differentiation operators are row-wise different.

In the study [16], a quasilinear system with two differentiation operators is considered and the conditions for the existence of unique multiperiodic solution this system are established in the noncritical case.

In a note [17], multiperiodical in wide extent solutions of the periodic boundary value problem for linear systems that decompose into linear subsystems with various differentiation operators along the directions of vector fields of spatial variables are studied.

The issues of solutions of quasilinear equations systems that are almost periodic in time variable, the linear parts of which decompose into independent subsystems with its differentiation operators, were researched in the monograph [6] in terms of the matricant.

Splitting the linear part into independent subsystems is a very special case in which the problems under consideration are solved. Consequently, these tasks remain open to the general case.

In [18–20], the problems on multiperiodic solutions are investigated by introducing a projection operator. The aim of this paper is to substantiate the method of the projection operator for studying an initial and a multiperiodic problems the linear hyperbolic in the narrow sense systems with constant coefficients.

It is known that the basis of the general theory of partial differential equations systems are methods for studying linear systems. Moreover, in the oscillations theory of continuous medium of noninteracting particles, problems associated with the study of its vibrations mainly lead to the study of multiperiodic solutions of linear equations systems. In most cases, linear equations, in comparison with nonlinear ones, are considered to be studied quite widely and deeply. But these linear problems are so diverse that among them there are either poorly studied or generally unstudied until today. The latter also includes the problem that was posed above for linear systems.

Let the oscillatory process in the continuous medium be described by system of equations

$$\frac{\partial y}{\partial \tau} + A\frac{\partial y}{\partial t} = By + \varphi(\tau, t), \tag{1}$$

where $y = (y_1, ..., y_n)$ is unknown vector-function; $\tau \in (-\infty, +\infty) = R$ and $t \in R$; A and B are constant n-matrices; $\varphi(\tau, t)$ is n-vector function.

The initial-boundary value problems for system in the form (1) have been studied in various literature, in particular, in monographic works [1–3], with constant and variable coefficients in terms of solutions in the wide extent, and a detailed study has been carried out for systems in the scalar form.

In this paper, we consider the problem of existence and integral representation in the vector-matrix form of a unique (θ, ω)-periodic solution of system (1) with the following assumptions:

1^0. The matrix A has various real eigenvalues $\lambda_j = \lambda_j(A)$, $j = \overline{1, n}$:

$$\lambda_j \neq \lambda_k, \quad j, k = \overline{1, n}, \quad \lambda_j \in R. \tag{2}$$

System (1) under condition (2) is called hyperbolic in the narrow sense [2].

2^0. Matrix B satisfies the relation

$$det[Y(\theta) - E] \neq 0. \tag{3}$$

Here $Y(\tau) = exp[B\tau]$, E is identity matrix.

Under condition (3), homogeneous system corresponding to system (1) has no (θ, ω)-periodic solutions, except for the trivial one.

3^0. The vector-function $\varphi(\tau, t)$ has properties of (θ, ω)-periodicity and smoothness with respect to (τ, t) order $(0, 1)$:

$$\varphi(\tau + \theta, t + q\omega) = \varphi(\tau, t) \in C^{(0,1)}_{\tau,t}(R \times R), \quad q \in Z, \tag{4}$$

where θ and ω are rationally independent periods; Z is the set of integers; $C^{(0,1)}_{\tau,t}(R \times R)$ is the class of functions possessing the indicated smoothness properties in order $(0, 1)$.

Main Results

By linear replacement

$$y = Cx \tag{5}$$

with nondegenerate constant n-matrix C the system (1) is reduced to form

$$\frac{\partial x}{\partial \tau} + J\frac{\partial x}{\partial t} = Kx + f(\tau, t), \tag{6}$$

where $x = (x_1, ..., x_n)$; $C^{-1}AC = J \equiv diag[\lambda_1, ..., \lambda_n]$; $C^{-1}BC = K$ is constant n-matrix and $C^{-1}\varphi(\tau, t) = = f(\tau, t)$ is vector-function.

Matrix $X(\tau)$ in the form of

$$X(\tau) = exp[K\tau] = exp[C^{-1}BC\tau] = C^{-1}exp[B\tau]C = C^{-1}Y(\tau)C$$

is matricant of system

$$\frac{\partial x}{\partial \tau} + J\frac{\partial x}{\partial t} = Kx, \tag{7}$$

and $X(0) = E$.

Denote vector-function $f(\tau, t) = C^{-1}\varphi(\tau, t)$, the same like $\varphi(\tau, t)$, has properties (θ, ω)-periodicity and smoothness of order $(0, 1)$:

$$f(\tau + \theta, t + q\omega) = f(\tau, t) \in C^{(0,1)}_{\tau,t}(R \times R).$$

This can easily be verified on the basis of condition (4).

To study the main question, it is necessary to solve the problem for system (7) with the initial condition

$$x|_{\tau=\tau_0} = u(t) \in C^{(1)}_t(R). \tag{7^0}$$

Obviously, by virtue of the statement of problem (7)-(7^0), we are dealing with differentiation operators $D = (D_1, ..., D_n)$, that coordinate-wise act on vector-function $x(\tau, t) = (x_1(\tau, t), ..., x_n(\tau, t))$ with the property of smoothness $(1, 1))$ with respect to (τ, t) in the form

$$Dx(\tau, t) = (D_1x_1(\tau, t), ..., D_nx_n(\tau, t)). \tag{8}$$

As can be seen from system (7), on the left side these differentiation operators D_jx_j, $j = \overline{1, n}$ are defined as

$$D_jx_j \equiv \frac{\partial x_j}{\partial \tau} + \lambda_j\frac{\partial x_j}{\partial t} = 0, \quad j = \overline{1, n}, \tag{9}$$

which in the directions of corresponding vector fields

$$\frac{dt}{d\tau} = \lambda_j, \quad j = \overline{1, n} \tag{10}$$

act like a normal differentiation operator with respect to τ.

Each of the equations (10), which determines the direction of differentiation, can be called the characteristic equation of operator $D_j = \dfrac{\partial}{\partial \tau} + \lambda_j\dfrac{\partial}{\partial t}$. Then the general solution

$$t = \sigma + \lambda_j(\tau - s) \equiv h_j(\tau, s, \sigma), \quad j = \overline{1, n} \tag{11}$$

defines the characteristic $h_j(\tau, s, \sigma)$ of operator D_j coming from initial point $(s, \sigma) \in R \times R$.

Thus, for characteristics, we have a common notation $h(\tau, s, \sigma)$, which as values can take one of the known characteristics $h_j(\tau, s, \sigma)$, $j = \overline{1, n}$. Therefore, we have

$$h(\tau, s, \sigma) \in \{h_1(\tau, s, \sigma), ..., h_n(\tau, s, \sigma)\}, \tag{12}$$

where $h_j(\tau, s, \sigma)|_{\tau=s} = \sigma$.

Obviously, the operators D_j move on to the total derivative operator $\dfrac{d}{d\tau}$ with respect to τ, along the characteristics $t = h_j(\tau, s, \sigma)$ defined by relation (11) and we have

$$Dx|_{t=h(\tau,s,\sigma)} = (D_1x_1|_{t=h_1(\tau,s,\sigma)}, ..., D_nx_n|_{t=h_n(\tau,s,\sigma)}) =$$

$$= \big(\frac{d}{d\tau}x_1(\tau, h_1(\tau, s, \sigma)), ..., \frac{d}{d\tau}x_n(\tau, h_n(\tau, s, \sigma))\big) = \frac{d}{d\tau}x(\tau, h(\tau, s, \sigma)). \tag{13}$$

Thus, a relationship is established that expressed by relation (13) between the differentiation operator D acting by formula (8) and the total derivative $\dfrac{d}{d\tau}$ of the vector-function $x(\tau, t)$ along characteristics (11)–(12).

Next, to ensure output of function defined on characteristics $h(\tau, s, \sigma) \in \{h_1(\tau, s, \sigma), ..., h_n(\tau, s, \sigma)\}$ in the space of variables (τ, t), from the characteristics (11) we determine the first integrals

$$h_j(s, \tau, t) = \sigma, \quad j = \overline{1, n} \tag{14}$$

of characteristic systems (10) and we have

$$h(s, \tau, t) \in \{h_1(s, \tau, t), ..., h_n(s, \tau, t)\}.$$

It's obvious that

$$D_jh_j(s, \tau, t) = 0, \quad j = \overline{1, n}. \tag{15}$$

Moreover, if $H_j(t)$ an arbitrary differentiable function, then

$$D_jH_j(h_j(s, \tau, t)) = 0, (s, \tau, t) \in R \times R \times R, \quad j = \overline{1, n}. \tag{16}$$

Thus, if the function $x(\tau, t) = (x_1(\tau, t), ..., x_n(\tau, t))$ is defined along the characteristics (11)–(12):

$$(x_1(\tau, h_1(s, \tau, \sigma)), ..., x_n(\tau, h_n(s, \tau, \sigma))), \tag{17}$$

then in respect that (14) from function (17) we obtain the function $x(\tau, t)$ defined as $(\tau, t) \in R \times R$, by virtue of the relations

$$h_j(\tau, s, h_j(s, \tau, t)) = h_j(\tau, \tau, t) = t, \quad j = \overline{1, n}. \tag{18}$$

The rightness of relations (18) is easily verified on the basis of (11) and (14).

We begin the integration of system (7) from the system

$$\frac{\partial x}{\partial \tau} + J\frac{\partial x}{\partial t} = 0 \tag{19}$$

with unknown vector-function $x = (x_1, ..., x_n)$.

By virtue of (8)–(9) and (13) system (19) can be represented as

$$Dx = 0 \tag{20}$$

or in scalar form

$$D_jx_j = 0, \quad j = \overline{1, n}. \tag{21}$$

Then, by virtue of identities (15)–16), we have the basic solution

$$x(s,\tau,t) = (h_1(s,\tau,t),...,h_n(s,\tau,t))$$

with coordinates

$$x_j = h_j(s,\tau,t), \quad j=\overline{1,n},$$

satisfying the initial conditions

$$h_j(s,\tau,t)|_{\tau=s} = t$$

and general solution

$$x(s,\tau,t) = H(h(s,\tau,t)) \equiv (H_1(h_1(s,\tau,t)),...,H_n(h_n(s,\tau,t))) \tag{22}$$

with arbitrary differentiable n-vector-function $H(t) = (H_1(t),...,H_n(t))$.

Then the solution x of system (19) with the initial condition

$$x|_{\tau=\tau^0} = u(t) \tag{19^0}$$

and the vector-function $u(t) = (u_1(t),...,u_n(t)) \in C_t^{(1)}(R)$ is determined by relation (22) by choosing an arbitrary vector-function $H(t) = u(t)$.

Consequently,

$$x(\tau^0,\tau,t) = u(h(\tau^0,\tau,t)) \equiv (u_1(h_1(\tau^0,\tau,t)),...,u_n(h_n(\tau^0,\tau,t))) \tag{23}$$

represents a solution of the initial problem (19)–(19^0), otherwise, relation (23) can be called the zero of the operator D with initial condition (19^0).

Thus, we have the following statement.

Theorem 1. A solution of the initial problem (19)–(19^0) is determined by the relation (23).

Hence we have an obvious consequence.

Corollary. In the case of narrow hyperbolicity of system (1), the vector-function $y = Cx(\tau^0,\tau,t)$ is a solution of equation

$$\frac{\partial y}{\partial \tau} + A\frac{\partial y}{\partial t} = 0,$$

satisfying the initial condition $y|_{\tau=\tau^0} = Cu(t)$,
where C is the transformation matrix (5); $x(\tau^0,\tau,t)$ is the solution to problem (19)–(19^0).

Now we consider the system of homogeneous equation (7) with the initial condition (7^0).

Let the matrix be represented in the form $X(\tau) = \left[x_{jk}(\tau)\right]_1^n$, $k=\overline{1,n}$. Then, using the n-vector-functions $X_j(\tau) = (x_{j1}(\tau),...,x_{jn}(\tau))$, $j=\overline{1,n}$, composed from rows of matrix $X(\tau)$, we have its vector representation

$$X(\tau) = \begin{pmatrix} X_1(\tau) \\ ... \\ X_n(\tau) \end{pmatrix}. \tag{24}$$

Next, based on the initial vector-function $u(t) = (u_1(t),...,u_n(t)) \in C_t^{(1)}(R)$, we compose a matrix $U(h)$ defined along the first vector-integrals (12) with a representation of the form

$$U(h(\tau^0,\tau,t)) = \begin{pmatrix} u^T(h_1(\tau^0,\tau,t)) \\ ... \\ u^T(h_n(\tau^0,\tau,t)) \end{pmatrix} =$$

$$= \begin{pmatrix} (u_1(h_1(\tau^0,\tau,t)),...,u_n(h_1(\tau^0,\tau,t))) \\ ... \\ (u_1(h_n(\tau^0,\tau,t)),...,u_n(h_n(\tau^0,\tau,t))) \end{pmatrix} = \begin{pmatrix} U_1(h_1(\tau^0,\tau,t)) \\ ... \\ U_n(h_n(\tau^0,\tau,t)) \end{pmatrix}, \tag{25}$$

where u^T is transposed vector u.

Now we construct the vector-function $x(\tau^0, \tau, t)$ using the scalar product of vector components of matrices (24) and (25), in the form

$$x(\tau^0, \tau, t) = \begin{pmatrix} \langle X_1(\tau - \tau^0), U_1(h_1(\tau^0, \tau, t))\rangle \\ \dots \\ \langle X_n(\tau - \tau^0), U_n(h_n(\tau^0, \tau, t))\rangle \end{pmatrix}. \tag{26}$$

By a direct check, we verify that the vector-function (26) is a solution to the initial problem (7)–(7^0).

Now, to represent the solution (26) of system (7) as the product of matrix and initial vector-function (7^0), we introduce the operator $P = diag[P_1, ..., P_n]$, where P_j acts on the vector-function $u(t)$ defined along the first integrals in the form

$$P_j u_k(h(\tau^0, \tau, t)) = u_k(h_j(\tau^0, \tau, t)), \quad j, k = \overline{1, n}.$$

Therefore, solution (26) can be represented as

$$x(\tau^0, \tau, t) = PX(\tau - \tau^0)u(h(\tau^0, \tau, t)), \tag{27}$$

where the matrices on the right are determined by the relations

$$PX(\tau - \tau^0) = \Big[x_{ij}(\tau - \tau^0)P_i\Big]_{i,j=1}^{i,j=n}$$

and

$$PX(\tau - \tau^0)u(h(\tau^0, \tau, t)) =$$
$$= \Big[x_{ij}(\tau - \tau^0)P_i u_j(h(\tau^0, \tau, t))\Big]_{i,j=1}^{i,j=n} = \Big[x_{ij}(\tau - \tau^0)u_j(h_i(\tau^0, \tau, t))\Big]_{i,j=1}^{i,j=n}.$$

Indeed, the equivalence of relations (26) and (27) is visible from the following chains of transformations

$$PX(\tau - \tau^0)u(h(\tau^0, \tau, t)) = diag[P_1, ..., P_n]\Big(\big[x_{ij}(\tau - \tau^0)\big]_1^n u(h(\tau^0, \tau, t))\Big)_{i,j=1}^{i,j=n} =$$

$$= diag[P_1, ..., P_n]\left(\sum_{k=1}^{n} x_{ik}(\tau - \tau^0)u_k(h(\tau^0, \tau, t))\right)_{i=1}^{i=n} =$$

$$= \left(P_i \sum_{k=1}^{n} x_{ik}(\tau - \tau^0)u_k(h(\tau^0, \tau, t))\right)_{i=1}^{i=n} = \left(\sum_{k=1}^{n} x_{ik}(\tau - \tau^0)P_i u_k(h(\tau^0, \tau, t))\right)_{i=1}^{i=n} =$$

$$= \left(\sum_{k=1}^{n} x_{ik}(\tau - \tau^0)u_k(h_i(\tau^0, \tau, t))\right)_{i=1}^{i=n} =$$

$$= \Big(\langle X_i(\tau - \tau^0), U(h_i(\tau^0, \tau, t))\rangle\Big)_{i=1}^{i=n} = \begin{pmatrix} \langle X_1(\tau - \tau^0), U_1(h_1(\tau^0, \tau, t))\rangle \\ \dots \\ \langle X_n(\tau - \tau^0), U_n(h_n(\tau^0, \tau, t))\rangle \end{pmatrix}.$$

Thus, based on the projector P, the solution of problem (7)–(7^0) is determined by relation (27).

Lemma 1. The initial problem (7)–(7^0) has a unique solution $x(\tau^0, \tau, t)$, which with the help of matricant $X(\tau)$ and projector P is represented in the form of relation (27).

The existence of solution in the form (27) is justified above. Uniqueness follows from the existence of the matricant $X(\tau)$ of system (7).

Theorem 2. Let system (1) satisfies condition (2). Then the system

$$\frac{\partial y}{\partial \tau} + A\frac{\partial y}{\partial t} = By, \tag{28}$$

has a solution $y = Cx(\tau^0, \tau, t)$, where is the transformation matrix (5), $x(\tau^0, \tau, t)$ is the solution (27) of problem (7)–(7^0).

The proof of Theorem 2 follows from the reducibility of system (28) to system (7) based on transformation (5).

Now we consider the initial problem for the nonhomogeneous system (6).

It is easy to verify that the vector-function

$$x(\tau^0, \tau, t) = PX(\tau)u(h(\tau^0, \tau, t)) + \int_{\tau^0}^{\tau} PX(\tau)X^{-1}(s)f(s, h(s, \tau, t))ds \tag{29}$$

is a solution of system (6) with the initial condition $x|_{\tau=\tau_0} = u(t) \in C_t^{(1)}(R)$.

Based on the transformation (5) from the representation (29), we have a solution of system (1) in the shape of

$$y(\tau^0, \tau, t) = PCX(\tau)C^{-1}v(h(\tau^0, \tau, t)) + \int_{\tau^0}^{\tau} PCX(\tau)X^{-1}(s)C^{-1}\varphi(s, h(s, \tau, t))ds, \tag{30}$$

where $v(t) = Cu(t)$.

Obviously, $CX(\tau)C^{-1} = Y(\tau)$ is a matricant of the homogeneous system (28) corresponding to system (1).

Then we obtain a representation of solution (30) using the matricant $Y(\tau)$ in the form

$$y(\tau^0, \tau, t) = PY(\tau)v(h(\tau^0, \tau, t)) + \int_{\tau^0}^{\tau} PY(\tau)Y^{-1}(s)\varphi(s, h(s, \tau, t))ds, \tag{31}$$

satisfying the initial condition

$$y|_{\tau=\tau_0} = v(t) \in C_t^{(1)}(R). \tag{1^0}$$

When deriving formula (31), all its constituent parameters are uniquely determined.

Therefore, the solution (31) of system (1) with the initial condition $y|_{\tau=\tau_0} = v(t) = Cu(t)$ is unique, where $u(t) \in C_t^{(1)}(R)$.

Thus, the following Theorem 3 is proved.

Theorem 3. Let conditions (2) and (4) be satisfied. Then the initial problem (1)–(1^0) is uniquely solvable in the form of relation (31).

Note that the representation of solutions (31) implies the representations of solutions (23) and (27) of systems (19) and (7), respectively.

The essence of Theorem 3 is that the general solution of initial problem is defined as product of matricant and initial vector-function with certain directions of differentiation with respect to vector fields.

Multiperiodic solutions

Lemma 2. Suppose all eigenvalues $\lambda_j(A)$, $j=\overline{1,n}$ of matrix A are real and non-zero, moreover ω, $\lambda_j\theta$, $j=\overline{1,n}$, rationally incommensurable:

$$\lambda_j(A)\neq 0, \quad \frac{\omega}{\lambda_j\theta}\notin Q, \quad j=\overline{1,n}, \tag{32}$$

where Q is the set of rational numbers. Then, the (θ,ω)-periodic solutions of system (19), and, therefore, systems (20) and (21), are only constants.

Indeed, in order for the solution $x(\tau^0,\tau,t)$ be (θ,ω)-periodic, according to the structure of solutions (23) and the linearity characteristics $h_j(\tau^0,\tau,t)=t-\lambda_j(\tau-\tau^0)$, $j=\overline{1,n}$ with respect to τ and t the initial vector-function must be periodic with respect to t both with the period $p\omega$ and the period $q_j\lambda_j\theta$ with integers p and q_j. Then, by virtue of the incommensurability condition (32), $u(t)$ should only be constant.

Consider the set of solutions to system (7) with constant initial data $u=c$. Then we have solutions of the form

$$x(\tau)=X(\tau)c, \qquad c \text{ is constant.} \tag{33}$$

Theorem 4. Let condition (3) be satisfied. Then system (7) has only a zero (θ,ω)-periodic solution of the form (33).

Indeed, since (θ,ω)-periodic solutions of the form (33) satisfy the condition

$$x(\tau+\theta)=X(\tau+\theta)c=X(\tau)X(\theta)c=X(\tau)c=x(\tau),$$

then the vector c guaranteeing θ-periodicity of solution (33) is determined by the relation

$$\big[X(\theta)-E\big]c=0. \tag{34}$$

It follows from (34) that the (θ,ω)-periodic solution of the form (33) of system (7), under condition (3), is only a trivial solution.

In fact, since $Y(\tau)=CX(\tau)C^{-1}$, from condition (3) we have

$$\det\big[Y(\theta)-E\big]=\det\big[CX(\theta)C^{-1}-E\big]=\det C\det\big[X(\theta)-E\big]\det C^{-1}\neq 0.$$

Hence, $\det\big[X(\theta)-E\big]\neq 0$. Consequently, system (34) has only a zero solution, and from representation (33) we have $x=0$.

Theorem 4 is proved.

Theorem 4 can be formulated differently in the form of the following theorem.

Theorem $4'$. For the system (7) hasn't a nonzero θ-periodic solution it is necessary and sufficient that the following conditions are fulfilled

$$\det\big[X(\theta)-E\big]\neq 0. \tag{35}$$

Theorem 5. If the condition (3) is not satisfied; then for the solution (33) of system (7) with a constant vector c to be θ-periodic with respect to τ it is necessary and sufficient that the vector c be an eigenvector of the monodromy matrix $X(\theta)$ corresponding to its eigenvalue $\rho=1$.

Indeed, along with the nonzero solution (33), we consider the solution

$$x(\tau+\theta)=X(\tau+\theta)c \tag{33$'$}$$

of system (7). From the theory of periodic solutions it is known that in order for the two solutions (33) and (33$'$) to coincide everywhere, it is necessary and sufficient that feasibility of condition

$$x(\theta)=x(0).$$

Since $x(\theta) = X(\theta)c$ and $x(0) = c$, from this we have condition (34), which takes place in the presence of an eigenvalue $\rho = 1$ of the monodromy matrix $X(\theta)$.

Theorem 5 is completely proved.

Theorem 6. For system (7) to has (θ, ω)-periodic with respect to τ solutions it is necessary and sufficient that the system of functional-difference equations

$$PX(\theta)u(h(0, \theta, t)) = u(t) \tag{36}$$

to be solvable in the space of continuously differentiable ω-periodic n-vector-functions

$$u(t + \omega) = u(t) \in C_t^{(1)}(R) \tag{37}$$

with the norm $\|u\| = \sup\limits_{t \in R} |u(t)|$, where $|u(t)|$ is the Euclidean metric of vector u.

Indeed, at the same time with solution (27), we consider the solution

$$x(\tau + \theta, t + q\omega) = PX(\tau + \theta)u(h(0, \tau + \theta, t + q\omega)) =$$

$$= PX(\tau)X(\theta)u(h(0, \tau + \theta, t) + q\omega), \qquad q \in Z.$$

From the definition of the (θ, ω)-periodic solutions (27) of system (7) with respect to (τ, t), we have

$$PX(\tau)X(\theta)u(h(0, \tau + \theta, t) + q\omega) = PX(\tau)u(h(0, \tau, t)), \quad q \in Z. \tag{38}$$

Supposing $\tau = 0$, taking into account equality $Pu(h(0, 0, t)) = Pu(t) = u(t)$, from relation (38) we have $PX(\theta)u(h(0, \theta, t) + q\omega) = u(t)$. By virtue of property (37), we have $PX(\theta)u(h(0, \theta, t)) = u(t)$, that is, condition (36) is established. Thus, the necessity of Theorem 6 is proved. Conversely, along with the solution

$$x(\tau, t) = PX(\tau)u(h(0, \tau, t)), \tag{39}$$

consider a solution of system (7) in the form

$$\tilde{x}(\tau, t) = PX(\tau)X(\theta)u(h(0, \theta, (h(0, \tau, t))), \tag{40}$$

where the initial conditions of these solutions are identical as $\tau = 0$ by virtue of condition (36). Therefore, these solutions (39) and (40) coincide $\tilde{x}(\tau, t) \equiv x(\tau, t)$, and

$$\tilde{x}(\tau, t) = PX(\tau)X(\theta)u(h(0, \theta, (h(0, \tau, t)))) = PX(\tau + \theta)u(h(0, \theta, (h(\theta, \tau + \theta, t)))) =$$

$$= PX(\tau + \theta)u(h(0, \tau + \theta, t)) = x(\tau + \theta, t).$$

It follows from (36) that the solution (39) is θ-periodic.

Theorem 6 is completely proved.

Now we have the opportunity to generalize the Theorem 4' to the general case.

Theorem 7. Under condition (35), system (7) has no (θ, ω)-periodic solutions except zero.

We prove Theorem 7 by contradiction method, assuming existence of a nonzero (θ, ω)-periodic solution $x^*(\tau, t)$ with initial function $x^*(0, t) = u^*(t)$, and $u^*(t + \omega) = u^*(t) \in C_t^{(1)}(R)$, $|u^*| = \Delta^* > 0$.

Obviously, for $t = t'$ and $t = t''$ we have

$$\left|u^*(t') - u^*(t'')\right| = \left|\frac{\partial u^*(\tilde{t})}{\partial t}\right| \left|t' - t''\right| \leq l^* \left|t' - t''\right|,$$

where $\tilde{t} = t' + \alpha\,(t'' - t')$, $0 < \alpha < 1$, $l^* = \left|\dfrac{\partial u^*(\tilde{t})}{\partial t}\right|$.

Then, given that this solution is $p_j\theta$-periodic with respect to τ and $q_j\omega$-periodic with respect to t, on the basis of inequality (41) we have the estimate

$$|P_j u^*(h(0,p_j\theta,t)) - u^*(t)| = |u^*(t-\lambda_j p_j\theta) - u^*(t)| =$$

$$= |u^*(t+q_j\omega-\lambda_j p_j\theta) - u^*(t)| \leq l^* \left|q_j\omega - \lambda_j p_j\theta\right|, \quad q_j, p_j \in Z. \tag{41'}$$

Further, by virtue of rational incommensurable $\lambda_j p_j\theta$ and ω, can choose p_j and q_j so that the estimate

$$|q_j\omega - \lambda_j p_j\theta| < \delta$$

is satisfied for any constant $\delta > 0$.

Now, (θ,ω)-periodicity condition (36) for the solution $x^*(\tau,t)$ is written in the form

$$u^*(t) - PX(\theta)u^*(h(0,\theta,t)) = \left[u_i^*(t)\right]_{i=1}^n - \left[\sum_{j=1}^n x_{ij}(\theta)u_j^*(t-\lambda_i\theta)\right]_{i=1}^n =$$

$$= \left[u_i^*(t)\right]_{i=1}^n - \left[\sum_{j=1}^n x_{ij}(\theta)u_j^*(t)\right]_{i=1}^n + \left[\sum_{j=1}^n x_{ij}(\theta)u_j^*(t)\right]_{i=1}^n - \left[\sum_{j=1}^n x_{ij}(\theta)u_j^*(t-\lambda_i\theta)\right]_{i=1}^n =$$

$$= \left[\sum_{j=1}^n \{\delta_{ij} - x_{ij}(\theta)\}u_j^*(t)\right]_{i=1}^n - \left[\sum_{j=1}^n x_{ij}(\theta)\{u_j^*(t-\lambda_i\theta) - u_j^*(t)\}\right]_{i=1}^n = 0.$$

Therefore, from the last part of this identity we have

$$\left[E - X(\theta)\right]u^*(t) = \left[\sum_{j=1}^n x_{ij}(\theta)\{u_j^*(t-\lambda_i\theta) - u_j^*(t)\}\right]_{i=1}^n.$$

By condition (35), we obtain

$$u^*(t) = \left[E - X(\theta)\right]^{-1}\left[\sum_{j=1}^n x_{ij}(\theta)\{u_j^*(t-\lambda_i\theta) - u_j^*(t)\}\right]_{i=1}^n.$$

Based on inequality (41′), we have

$$|u^*(t)| = \left|\left[E - X(\theta)\right]^{-1}\right| \left[\sum_{j=1}^n |x_{ij}(\theta)| \left|u_j^*(t-\lambda_i\theta) - u_j^*(t)\right|\right]_{i=1}^n \leq$$

$$\leq \left|\left[E - X(\theta)\right]^{-1}\right| |\mathrm{mod}X(\theta)|\,\delta < \varepsilon.$$

Therefore, $u^* = 0$ which contradicts $|u^*| = \Delta^* > 0$.

The obtained contradiction proves the rightness of Theorem 7.

Theorem 8. If system (6) under condition (35) has a (θ,ω)-periodic solution, then it is unique.

Indeed, if the system under condition (35) has two different (θ,ω)-periodic solutions $x(\tau,t)$ and $\tilde{x}(\tau,t)$, then their difference $x(\tau,t) - \tilde{x}(\tau,t) = z(\tau,t)$ is a (θ,ω)-periodic solution of the corresponding homogeneous system (7). Under condition (35), it is trivial: $z(\tau,t) = 0$. Consequently, $x(\tau,t) \equiv \tilde{y}(\tau,t)$. The resulting contradiction proves Theorem 8.

Theorem 9. Under the conditions 1^0-3^0, system (6) admits a unique (θ,ω)-periodic solution $x^*(\tau,t)$, which is integrally represented by the formula

$$x^*(\tau,t) = [X^{-1}(\tau+\theta) - X^{-1}(\tau)]^{-1} P \int\limits_{\tau}^{\tau+\theta} X^{-1}(s) f_\theta(s,h(s,\tau,t))ds, \tag{42}$$

where vector-function $f_\theta(s,h(s,\tau,t))$ is determined by the relation

$$f_\theta(s,h(s,\tau,t)) = \begin{cases} f(s,h(s,\tau,t)), & \tau \le s \le 0, \\ f(s,h(s,\tau+\theta,t)), & 0 < s \le \tau+\theta. \end{cases} \tag{42'}$$

Indeed, suppose that (θ,ω)-periodic solution $x^*(\tau,t)$ has an initial function $x^*(0,t) = c^*$, where c^* is constant vector. Then, according to Theorem 3, it can be represented as

$$x^*(\tau,t) = X(\tau)c^* + PX(\tau) \int\limits_{\tau^0}^{\tau} X^{-1}(s) f(s,h(s,\tau,t))ds. \tag{43}$$

Along with this solution, we consider a solution

$$x^*(\tau+\theta,t) = X(\tau+\theta)c^* + PX(\tau+\theta) \int\limits_{\tau^0}^{\tau+\theta} X^{-1}(s) f(s,h(s,\tau+\theta,t))ds. \tag{44}$$

We write the system of representations (44) and (43) in the form

$$X^{-1}(\tau+\theta)x^*(\tau+\theta,t) = c^* + P \int\limits_{\tau^0}^{\tau+\theta} X^{-1}(s) f(s,h(s,\tau+\theta,t))ds, \tag{44'}$$

$$X^{-1}(\tau)x^*(\tau,t) = c^* + P \int\limits_{\tau^0}^{\tau} X^{-1}(s) f(s,h(s,\tau,t))ds.$$

Further, by replacing s with $s+\theta$ under the integral (44′), we obtain the system

$$X^{-1}(\tau+\theta)x^*(\tau+\theta,t) = c^* + P \int\limits_{\tau^0-\theta}^{\tau} X^{-1}(s) f(s,h(s,\tau+\theta,t))ds,$$

$$X^{-1}(\tau)x^*(\tau,t) = c^* + P \int\limits_{\tau^0}^{\tau} X^{-1}(s) f(s,h(s,\tau,t))ds.$$

Hence, taking into account that $x^*(\tau,t) = x^*(\tau+\theta,t)$, excluding the constant vector c^*, we have

$$\left[X^{-1}(\tau+\theta) - X^{-1}(\tau)\right] x^*(\tau,t) = P \int\limits_{\tau^0-\theta}^{\tau^0} X^{-1}(s) f(s,h(s,\tau,t))ds.$$

Consequently,

$$x^*(\tau,t) = \left[X^{-1}(\tau+\theta) - X^{-1}(\tau)\right]^{-1} P \int\limits_{\tau^0-\theta}^{\tau^0} X^{-1}(s) f(s,h(s,\tau,t))ds. \tag{45}$$

Since representation (45) is valid for all τ^0, it remains valid when replaced τ^0 with $\tau+\theta$, and the vector-function $f(s,h(s,\tau,t))$ is replaced by a vector function $f_\theta(s,h(s,\tau,t))$, which is determined by the relation (42′).

Therefore, we have the final representation of the (θ,ω)-periodic solution (42).

From the conditions 1^0-3^0 follows the fulfillment of all the requirements of Theorems 6-8. Therefore, system (6) has no other (θ,ω)-periodic solutions except (42).

Thus, when replacement (5), we obtain a solution of the main problem about the existence of a unique (θ,ω)-periodic solution $y^*(\tau,t)$ of system (1) in the form

$$y^*(\tau,t) = \left[Y^{-1}(\tau+\theta) - Y^{-1}(\tau)\right]^{-1} P \int\limits_{\tau}^{\tau+\theta} Y^{-1}(s)\varphi_\theta(s,h(s,\tau,t))ds. \quad (46)$$

where $Y(\tau) = CX(\tau)$, vector-function $\varphi_\theta(s,h(s,\tau,t))$ is determined by the relation

$$\varphi_\theta(s,h(s,\tau,t)) = \begin{cases} \varphi(s,h(s,\tau,t)), & \tau \le s \le 0, \\ \varphi(s,h(s,\tau+\theta,t)), & 0 < s \le \tau+\theta. \end{cases} \quad (46')$$

Note that, based on Theorem 9, we have a theorem on solving the main problem.

Theorem 10. Under the conditions 1^0-3^0, system (1) admits a unique (θ,ω)-periodic solution $y^*(\tau,t)$, which is integrally represented by the formula (46)–(46′).

In conclusion, we also note that the idea of work can be realized in the quasilinear case using the principle of compressed mappings.

References

1 Курант Р. Уравнения с частными производными / Р. Курант. — М.: Мир, 1964. — 830 с.

2 Рождественский Б.Л. Системы квазилинейных уравнений и их приложение к газовой динамике / Б.Л. Рождественский, Н.Н. Яненко. — М.: Наука, 1978. — 592 с.

3 Фарлоу С. Уравнения с частными производными для научных работников и инженеров / С. Фарлоу. — М.: Мир, 1985. — 384 с.

4 Харасахал В.Х. Почти периодические решения обыкновенных дифференциальных уравнений / В.Х. Харасахал. — Алма-Ата: Наука, 1970. — 200 с.

5 Умбетжанов Д.У. Почти многопериодические решения дифференциальных уравнений в частных производных / Д.У. Умбетжанов. — Алма-Ата: Наука, 1979. — 210 с.

6 Умбетжанов Д.У. Почти периодические решения эволюционных уравнений / Д.У. Умбетжанов. — Алма-Ата: Наука, 1990. — 184 с.

7 Vejvoda O. Partial differential equations: time-periodic solutions / O. Vejvoda, L. Herrmann, V. Lovicar, M. Sova, I. Straskaba, M. Stedry. — Springer Netherlands, 1982. — 358 p.

8 Сартабанов Ж.А. Периодты функциялар және кейбір қарапайым дифференциалдық теңдеулердің периодты шешімдері / Ж.А. Сартабанов. — Алматы: РБК, 2001. — 108 б.

9 Мухамбетова А.А. Устойчивость решений систем дифференциальных уравнений с многомерным временем / А.А. Мухабетова, Ж.А. Сартабанов. — Актобе: Принт А, 2007. — 168 с.

10 Кульжумиева А.А. Периодические решения систем дифференциальных уравнений с многомерным временем / А.А. Кульжумиева, Ж.А. Сартабанов. — Уральск: РИЦ ЗКГУ, 2001. — 168 с.

11 Sartabanov Z.A. The multi-period solution of a linear system of equations with the operator of differentiation along the main diagonal of the space of independent variables and delayed arguments / Z.A. Sartabanov // AIP Conference Proceedings. — 2017. — Vol. 1880, 040020. DOI:10.1063/1.5000636.

12 Sartabanov Z.A. Multiperiodic solutions of autonomous systems with operator of differentiation on the Lyapunov's vector field / Z.A. Sartabanov, B.Z. Omarova // AIP Conference Proceedings. — 2018. — Vol. 1997, 020041. DOI:10.1063/1.5049035.

13 Kulzhumiyeva A.A. On multiperiodic integrals of a linear system with the differentiation operator in the direction of the main diagonal in the space of independent variables / A.A. Kulzhumiyeva, Z.A. Sartabanov // Eurasian Mathematical Journal. — 2017. — Vol. 8, 1. — P. 67–75.

14 Kulzhumiyeva A.A. Integration of a linear equation with differential operator, corresponding to the main diagonal in the space of independent variables, and coefficients, constant on the diagonal / A.A. Kulzhumiyeva, Z.A. Sartabanov // Russian Mathematics. — 2019, Vol. 63, 6, C. 34–47. DOI: 10.3103/S1066369X19060045

15 Жакашбаев Б.Ж. О построении почти многопериодического решения одной системы уравнений в частных производных с треугольными матричными коэффициентами / Б.Ж. Жакашбаев, А.Б. Бержанов // Дифференциальные уравнения, теории функций и их приложения: сб. науч. тр. — Алма-Ата: Казах. гос. ун-т им. С.М. Кирова, 1986. — С. 38–40.

16 Кенжебаев К.К. Многопериодические решения квазилинейных гиперболических систем дифференциальных уравнений в частных производных / К.К. Кенжебаев, Ж.А. Сартабанов, А.У. Бекбауова // Матем. журн. — 2010. — Т. 10, № 1 (9). — С. 46–52.

17 Абдикаликова Г.А. О многопериодическом решении одной системы уравнений в частных производных / Г.А. Абдикаликова, А.Х. Жумагазиев // International Scientific Journal. — 2016. — № 4. — С. 6–9.

18 Сартабанов Ж.А. Многопериодические решения линейных систем с постоянными коэффициентами и с двумя операторами дифференцирования / Ж.А. Сартабанов, Г.А. Абдикаликова, А.Х. Жумагазиев // Проблемы дифференциальных уравнений, анализа и алгебры (1 ноября 2018 г.): материалы VIII Междунар. науч. конф. — Актобе: Актюб. регион. гос. ун-т им. К. Жубанова, 2018. — С. 89–95.

19 Сартабанов Ж.А. Об одном методе исследования многопериодического решения системы с различными операторами дифференцирования / Ж.А. Сартабанов, А.Х. Жумагазиев, Г.А. Абдикаликова // Традиционная Междунар. апр. мат. конф. в честь Дня работников науки РК и Workshop «Problems of modelling processes in electrical contacts», посвящ. 80-летнему юбилею акад. НАН РК С.Н. Харина: тез. докл. (3-5 апреля 2019 г.). — Алматы: Ин-т мат. и мат. моделирования КМ МОН РК, 2019. — С. 82–84.

20 Zhumagaziyev A.Kh. Multiperiodic solution of one hyperbolic system and its integral representation / A.Kh. Zhumagaziyev, Zh.A. Sartabanov, G.A. Abdikalikova // The Abstract Book of International Conference "Actual Problems of Analysis, Differential Equations and Algebra" (EMJ-2019), dedicated to the 10th anniversary of the Eurasian Mathematical Journal (October 16-19, 2019). — Nur-Sultan: L.N. Gumilyov Eurasian National University, 2019. — P. 99–101.

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Тұрақты коэффициентті сызықты тар мағынадағы гиперболалық жүйенің көппериодты шешімі

Тар мағыналы гиперболалық сызықты тұрақты коэффициентті дифференциалдық теңдеулер жүйесінің барлық тәуелсіз айнымалылары бойынша көппериодты шешімінің бары мен жалғыздығы және оның матрица-интегралдық бейнеде өрнектеу мәселелері зерттелген. Осындай мақсатпен қарастырылып отырған, тәуелсіз айнымалылар кеңістігіндегі векторлық өрістердің бағыттары бойынша әртүрлі дифференциалдау операторлы жүйелер үшін дифференциалдау және интегралдау жүретін характеристикаларын анықтайтын проекторларға сүйенген, Кошидің сипаттауыштар әдісі негізінде бастапқы есептерді шешу тәсілі түзілген. Осы әдістеме бойынша біртекті және біртекті емес сызықты жүйелер үшін Коши есептері шешілді және интегралдық бейнесі келтірілді. Осы тұста енгізілген характеристикаларды анықтаушы проекторлардың маңызы шешуші болды. Негізгі есепті шешу барысында дифференциалдық операторлардың көппериодты нөлдерімен қатар біртекті сызықты жүйелердің нөлден өзге көппериодты шешімдерінің бар болуының қажетті және жеткілікті шарттары тағайындалды. Олардың негізінде, осы жүйелердің нөлдік емес көппериодты шешімдері болмайтын шарттар алынды. Нәтижесінде, біртекті сызықты жүйелердің нөлдік емес көппериодты шешімдері болмайтын жағдайда біртекті емес сызықты жүйелердің көппериодты шешімінің бар және жалғыз болуы туралы негізгі теорема дәлелдендеуімен қатар, оның проекциялау операторларымен байланысты интегралдық өрнегі қорытылып шығарылды. Қолданылған әдістемемен алынған нәтижелерді қарастырылған жүйенің жалпыланған квазисызықты жағдайында да, сондай-ақ, көпөлшемді $t = (t_1, ...t_m)$ векторы үшін және белгісіз вектор-функцияның дербес туындыларының жанындағы коэффициенттері көппериодты матрицалар болған кезде де осы әдісті қолданып алуға болады.

Кілт сөздер: тар мағынадағы гиперболалық жүйе, көппериодты шешім, характеристикалар әдісі, проекциялау операторлары, векторлық өрістер бойынша дифференциалдау операторлары, интегралдық бейне.

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Многопериодическое решение линейной гиперболической в узком смысле системы с постоянными коэффициентами

Исследована задача о существовании и интегральном представлении в векторно-матричной форме единственного многопериодического по всем независимым переменным решения линейной гиперболической в узком смысле системы дифференциальных уравнений с постоянными коэффициентами. С целью решения поставленной задачи, на основе метода характеристик Коши, разработана методика построения решений начальной задачи для рассматриваемой системы с различными операторами дифференцирования по направлениям векторных полей пространства независимых переменных, основанная на проекторах, определяющих характеристики, по которым ведутся дифференцирование и интегрирование. По этой методике решены задачи Коши для линейной однородной и неоднородной систем с интегральным представлением. При этом существенное значение имели введенные проекторы по определению характеристик. По решению основной задачи, наряду с многопериодическими нулями операторов дифференцирования, установлены необходимые и достаточные условия существования многопериодических решений линейных однородных систем, отличных от тривиальных. Таким образом, получены условия отсутствия ненулевых многопериодических решений этих систем. В заключении, при отсутствии ненулевых многопериодических решений линейных однородных систем, доказана основная теорема о существовании и единственности многопериодического решения линейной неоднородной системы с выводом его интегрального представления, зависящего от операторов проектирования. Разработанная методика имеет перспективу распространения полученных результатов на квазилинейный случай рассматриваемой системы, а также на случаи многомерного вектора $t = (t_1, ..., t_m)$ и многопериодических матриц при частных производных искомой вектор-функции.

Ключевые слова: гиперболическая система в узком смысле, многопериодическое решение, метод характеристик, операторы проектирования, операторы дифференцирования по векторным полям, интегральное представление.

References

1 Kurant, R. (1964). *Uravneniia s chastnymi proizvodnymi [Partial differential equations].* Moscow: Mir [in Russian].

2 Rozhdestvenskij, B.L., & Janenko, N.N. (1978). *Sistemy kvazilineinykh uravnenii i ikh prilozhenie k hazovoi dinamike [Systems of quasilinear equations and their applications to gas dynamics].* Moscow: Nauka [in Russian].

3 Farlou, S. (1985). *Uravneniia s chastnymi proizvodnymi dlia nauchnykh rabotnikov i inzhenerov [Partial Differential Equations for Scientists and Engineers].* Moscow: Mir [in Russian].

4 Kharasakhal, V.Kh. (1970). *Pochti periodicheskie resheniia obyknovennykh differentsialnykh uravnenii [Almost periodic solutions of ordinary differential equations].* Alma-Ata: Nauka [in Russian].

5 Umbetzhanov, D.U. (1979). *Pochti mnohoperiodicheskie resheniia differentsialnykh uravnenii v chastnykh proizvodnykh [Almost multiperiodic solutions of partial differential equations].* Alma-Ata: Nauka [in Russian].

6 Umbetzhanov, D.U. (1990). *Pochti periodicheskie resheniia evoliutsionnykh uravnenii [Almost periodic solutions of evolution equations].* Alma-Ata: Nauka [in Russian].

7 Vejvoda, O., Herrmann, L., Lovicar, V., Sova, M., Straskaba, I., & Stedry M. (1982). *Partial differential equations: time-periodic solutions*. Springer Netherlands.

8 Sartabanov, Zh.A. (2001). *Periodty funktsiialar zhane keibir karapaiym differentsialdyk tendeuler-din periodty sheshimderi [Periodic functions and periodic solutions of some ordinary differential equations].* Alma-Ata: RBK [in Kazakh].

9 Mukhambetova, A.A., & Sartabanov, Zh.A. (2007). *Ustoichivost reshenii sistem differentsial-nykh uravnenii s mnohomernym vremenem [Stability of solutions of the systems of differential equations with multidimensional time].* Aktobe: Print A [in Russian].

10 Kulzhumieva, A.A., & Sartabanov, Zh.A. (2013). *Periodicheskie resheniia sistem differentsialnykh uravnenii s mnohomernym vremenem [Periodic solutions of the systems of differential equations with multidimensional time].* Uralsk: RITs ZKHU [in Russian].

11 Sartabanov, Z.A. (2017). The multi-period solution of a linear system of equations with the operator of differentiation along the main diagonal of the space of independent variables and delayed arguments. *AIP Conference Proceedings*, Vol. 1880, 040020. Doi:10.1063/1.5000636

12 Sartabanov, Z.A., & Omarova, B.Z. (2018). Multiperiodic solutions of autonomous systems with operator of differentiation on the Lyapunov's vector field. *AIP Conference Proceedings*, Vol. 1997, 020041. Doi:10.1063/1.5049035

13 Kulzhumiyeva, A.A., & Sartabanov, Z.A. (2017). On multiperiodic integrals of a linear system with the differentiation operator in the direction of the main diagonal in the space of independent variables. *Eurasian Mathematical Journal, Vol. 8, 1,* 67-75.

14 Kulzhumiyeva, A.A., & Sartabanov, Z.A. (2019). Integration of a linear equation with differential operator, corresponding to the main diagonal in the space of independent variables, and coefficients, constant on the diagonal. *Russian Mathematics, Vol. 63, 6,* 34-47. DOI: 10.3103/S1066369-X19060045

15 Zhakashbaev, B.Zh., & Berzhanov, A.B. (1986). O postroenii pochti mnohoperiodicheskoho resheniia odnoi sistemy uravnenii v chastnykh proizvodnykh s treuholnymi matrichnymi koeffitsientami [On the construction of an almost multiperiodic solution of a system of partial differential equations with triangular matrix coefficients]. *Differentsialnye uravneniia, teorii funktsii i ikh prilozheniia: sbornik nauchnykh trudov — Differential equations, function theory and their applications: the abstract book.* (p. 38-40). Alma-Ata: Kazakhskii hosudarstvennyi universitet imeni S.M. Kirova [in Russian].

16 Kenzhebaev, K.K., Sartabanov, Zh.A., & Bekbauova, A.U. (2010). Mnohoperiodicheskie resheniia kvazilineinykh hiperbolicheskikh sistem differentsialnykh uravnenii v chastnykh proizvodnykh [Multiperiodic solutions of quasilinear hyperbolic systems of partial differential equations]. *Matematicheskii zhurnal — Mathematical Journal, Vol. 10, 1,* 46-52 [in Russian].

17 Abdikalikova, G.A., & Zhumagaziyev, A.Kh. (2016). O mnohoperiodicheskom reshenii odnoi sistemy uravnenii v chastnykh proizvodnykh [On a multiperiodic solution of a system of partial differential equations]. *International Scientific Journal, 4,* 6-9 [in Russian].

18 Sartabanov, Zh.A., Abdikalikova, G.A., & Zhumagaziyev, A.Kh. (2018). Mnohoperiodicheskie resheniia lineinykh sistem s postoiannymi koeffitsientami i s dvumia operatorami differentsirovaniia [Multiperiodic solutions of linear systems with constant coefficients and with two differentiation operators]. Problems of differential equations, analysis and algebra: *VIII Mezhdunarodnaia nauchnaia konferentsiia (1 noiabria 2018 hoda) — Materials of VIII international scientific conference.* (pp. 89-95). Aktobe: Aktiubinskii rehionalnyi hosudarstvennyi universitet imeni K. Zhubanova [in Russian].

19 Sartabanov, Zh.A., Zhumagaziyev, A.Kh., & Abdikalikova, G.A. (2019). Ob odnom metode issledovaniia mnohoperiodicheskoho resheniia sistemy s razlichnymi operatorami differentsirovaniia [On one method of research of multiperiodic solution of system with various differentiation operators]. *Traditsionnaia mezhdunarodnaia aprelskaia matematicheskaia konferentsiia v chest Dnia rabotnikov nauki RK i Workshop «Problems of modelling processes in electrical contacts», posviashchennaia 80-letnemu yubileiu akademika NAN RK Stanislava Nikolaevicha Harina (3-5 aprelia 2019 hoda) – Traditional International April Scientific Conference, in honor of the Day of Science Workers of RK and Workshop «Problems of modelling processes in electrical conacts» dedicated to the 80th anniversary of the academician of the NAS RK S.N. Kharin.* (p. 82-84). Almaty: Institut matematiki i matematicheskoho modelirovaniia KN MON RK [in Russian].

20 Zhumagaziyev, A.Kh., Sartabanov, Zh.A., & Abdikalikova, G.A. (2019). Multiperiodic solution of one hyperbolic system and its integral representation *The Abstract Book of International Conference «Actual Problems of Analysis, Differential Equations and Algebra» (EMJ-2019), dedicated to the 10th anniversary of the Eurasian Mathematical Journal* (October 16-19, 2019). (pp. 99-101). Nur-Sultan: L.N. Gumilyov Eurasian National University.

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Bessel functions of two variables as solutions for systems of the second order differential equations

In this paper, the systems with solutions in the form of degenerate hypergeometric Humbert functions of two variables reduced to Bessel functions of two variables are established and studied. The connections between the Humbert and Bessel functions of two variables are revealed, their differential properties are investigated. The addition and multiplication theorems are proved. In future, these proven properties allow us to establish recurrent relations between degenerate hypergeometric functions of two variables, similarly to extend these properties to the case of many variables. The connection between type systems of Bessel and Whittaker is shown. Using the Frobenius-Latysheva method, the singularities of constructing normal-regular solutions of the newly established Bessel-type system are studied.

Keywords: Humbert function, system, Bessel function, properties, addition theorem, reducible, normal-regular.

Introduction

Applications of Bessel functions of one variable are very diverse. They are widely used in solving problems of acoustics, Radiophysics, hydrodynamics, nuclear and nuclear physics. In the theory of elasticity the solution in Bessel functions covers all spatial problems solved in spherical and cylindrical coordinates, various problems of vibrations of plates. There are also numerous publications, which study a large number of different problems relating to all important sections of mathematical physics. However, this work out has not received the development of the theory of Bessel functions of many variables. Although there are works where the properties of Bessel functions of many variables are studied, their relation to various special functions and orthogonal polynomials of many variables is analogous to the Bessel function of one variable. It is a special case of the degenerate hypergeometric Kummer function [1; 1]. It is known that the particular solution of the Kummer equation is a degenerate Gauss function $G(\alpha, \gamma; x)$:

$$\lim_{\tau\to 0} F(\alpha, \frac{1}{\varepsilon}; \gamma; \varepsilon x) = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m} \cdot \frac{x^m}{m!} = G(\alpha, \gamma; x) \tag{1.1}$$

obtained by the limit transition. Similarly, you can get the function

$$\lim_{\tau\to 0} F(\frac{1}{\varepsilon}, \frac{1}{\varepsilon}; \gamma; \varepsilon^2 x) = \sum_{m=0}^{\infty} \frac{1}{(\gamma)_m} \cdot \frac{x^m}{m!} = J(\gamma; x) \tag{1.2}$$

$J(\gamma; x)$ is called the function reduced to the Bessel function, since equality is just

$$J_k(x) = \frac{(\frac{x}{2})^k}{\Gamma(k+1)} \cdot J(k+1, -\frac{x^2}{2}) \tag{1.3}$$

and

$$x^2 \frac{d^2 J_k}{dx^2} + x \frac{dJ_k}{dx} + (x^2 - k^2) J_k = 0 \tag{1.4}$$

where (1.4) is the basic Bessel equation.

All known hypergeometric functions of two variables are solutions of some special systems consisting of two partial differential equations of the second order. In Horn list there are 34 such systems whose solutions are 34 hypergeometric functions of two variables. Out of them 20 are degenerate hypergeometric functions of two variables [2, 3]. The connection of Bessel functions of two variables with these degenerate functions of two variables is not studied properly. In several works [4; 138] it was proven that a function of M.P. Humbert $\Psi_2(\alpha,\gamma,\gamma';x,y)$ is the most closest to the Bessel functions [5; 129].

Definition 1.1. The degenerate hypergeometric Humbert function $\Psi_2(\alpha,\gamma,\gamma';x,y)$ two variables x and y is determined by a series of

$$\Psi_2(\alpha,\gamma,\gamma';x_1,x_2)=\sum_{m,n=0}^{\infty}\frac{(\lambda)_{m+n}}{(\gamma)_m(\gamma')_n}\cdot\frac{x_1^m}{m!}\cdot\frac{x_2^n}{n!} \tag{1.5}$$

The series (1.5) converges absolutely and uniformly if at $|x_1|<\varepsilon$, $|x_2|<\varepsilon$.

Theorem 1.1. Series (1.5) is a particular solution of the Horn system

$$\begin{cases} x_1Z_{x_1x_1}+(\gamma-x_1)Z_{x_1}-x_2Z_{x_2}-\lambda Z=0,\\ x_2Z_{x_2x_2}+(\gamma'-x_2)Z_{x_2}-x_1Z_{x_1}-\lambda Z=0,\end{cases} \tag{1.6}$$

which under the conditions of compatibility and integrability has four linearly independent partial solutions [6].

In the monograph of Appell and Kampe de Feriet [5; 124] there is a list of 23 (I-XXIII) degenerate hypergeometric functions of two variables derived from four Appell functions F_1-F_4 given by limit transition. Some of them coincide, despite the fact that they are obtained from various hypergeometric functions of Appell F_1-F_4. Five functions of them: (XIII), (XVI), (XVII), (XVIII) and (XXIII) are presented as a product of the function, which is reduced to the Bessel functions or functions of Bessel and Kummer.

Example 1. A number of (XVI)

$$\lim_{\tau\to 0}F_2(\frac{1}{\varepsilon},\frac{1}{\varepsilon},\frac{1}{\varepsilon};\gamma_1;\gamma_2;\varepsilon^2x_1,\varepsilon^2x_2)=$$

$$=\sum_{m,n=0}^{\infty}\frac{1}{(\gamma_1)_m(\gamma_2)_n}\cdot\frac{x_1^m}{m!}\cdot\frac{x_2^n}{n!}=J(\gamma_1;x_1)\cdot J(\gamma_2;x_2) \tag{1.7}$$

set by limit transition, where the functions $J(\gamma_j;x_j),(j=1,2)$ of the reduced to Bessel functions are a particular solution of the system

$$\begin{cases} x_1Z_{x_1x_1}+\gamma_1Z_{x_1}-Z=0,\\ x_2Z_{x_2x_2}+\gamma_2Z_{x_2}-Z=0,\end{cases} \tag{1.8}$$

obtained by limiting the transition from the Horn system (F_2).

The aim of this work is to establish and study systems with solutions in the form Bessel functions of two variables, using the Horn systems (1.6), to establish the connection of Bessel functions to the Humbert function and with other functions from the Horn list, to investigate their differential properties, addition and multiplication theorems, based on the properties of the Humbert function.

2. Properties of degenerate hypergeometric series reducible to Bessel functions of two variables.

In the previous paragraph we defined Kummer function (1.1), as the degenerated Gaussian function. A function reducible to the Bessel function of one variable [7; 21] was defined using the limit transition (1.2). Similarly, the limit transition is just

$$F_1(;\gamma;x)=1+\frac{1}{\gamma}x+\frac{1}{2!\gamma(\gamma+1)}x^2+\cdots=$$

$$= \lim_{\alpha\to\infty}[1 + \frac{\alpha}{1!\gamma}\frac{x}{\alpha} + \frac{\alpha(\alpha+1)}{2!\gamma(\gamma+1)}\frac{x^2}{\alpha^2} + \cdots] = J(\gamma; x) \tag{2.1}$$

We are interested in generalizing (2.1) in the case of a degenerate hypergeometric function of two variables.

Theorem 2.1. The degenerated hypergeometric Humbert function $\Psi_2(\lambda; \gamma_1, \gamma_2; x_1, x_2)$ of two variables by means of a limit transition is reduced to the form

$$\lim_{\lambda\to\infty} = \Psi_2(\lambda; \gamma_1, \gamma_2; x_1, x_2) = J(\gamma_1; x_1)J(\gamma_2; x_2) \tag{2.2}$$

where functions $J(\gamma_j; x_j), (j = 1, 2)$ are reduced to Bessel functions.

Proof. Indeed, there is the limit of a function of Humbert when $\lambda \to \infty$

$$\lim_{\lambda\to\infty} \Psi_2(\lambda; \gamma_1, \gamma_2; \frac{x_1}{\lambda}, \frac{x_2}{\lambda}) = \lim_{\lambda\to\infty}[1 + \frac{\lambda}{1!\gamma_1}\frac{x_1}{\lambda} + \frac{\lambda}{1!\gamma_2}\frac{x_2}{\lambda} + \frac{\lambda(\lambda+1)}{1!\gamma_1\gamma_2}\frac{x_1}{\lambda}\frac{x_2}{\lambda} +$$

$$+\frac{\lambda(\lambda+1)}{2!\gamma_1(\gamma_1+1)}\frac{x_1^2}{\lambda^2} + \frac{\lambda(\lambda+1)}{2!\gamma_2(\gamma_2+1)}\frac{x_2^2}{\lambda^2} + \cdots] = 1 + \frac{1}{1!\gamma_1}x_1 + \frac{1}{1!\gamma_2}x_2 + \frac{1}{1!\gamma_1\gamma_2}x_1x_2 +$$

$$+\frac{1}{2!\gamma_1(\gamma_1+1)}x_1^2 + \frac{1}{2!\gamma_2(\gamma_2+1)}x_2^2 + \cdots = (1 + \frac{1}{1!\gamma_1}x_1 + \frac{1}{2!\gamma_1(\gamma_1+1)}x_1^2 + \cdots)\cdot$$

$$\cdot(1 + \frac{1}{1!\gamma_2}x_2 + \frac{1}{2!\gamma_2(\gamma_2+1)}x_2^2 + \cdots) = J(\gamma_1; x_1)J(\gamma_2; x_2) = J(\gamma_1, \gamma_2; x_1, x_2). \tag{2.3}$$

Taking into account (2.1) and (2.2) we obtain a series of two variables

$$J(\gamma_1; x_1)J(\gamma_2; x_2) = \sum_{m_1,m_2=0}^{\infty} \frac{1}{(\gamma_1)_{m_1}(\gamma_2)_{m_2}} \cdot \frac{x_1^{m_1}}{m_1!} \cdot \frac{x_2^{m_2}}{m_2!}. \tag{2.4}$$

Theorem 2.2. The Bessel function of two variables of the first kind is represented as

$$J_{\gamma_1,\gamma_2}(x_1, x_2) = \sum_{m_1,m_2=0}^{\infty} \frac{(-1)^{m_1+m_2}}{m_1! \cdot m_2!\Gamma(\gamma_1 + m_1 + 1)\Gamma(\gamma_2 + m_2 + 1)} \cdot (\frac{x_1}{2})^{2m_1+\gamma_1} \cdot (\frac{x_2}{2})^{2m_2+\gamma_2} \tag{2.5}$$

Indeed, using (1.3) and the obtained functions by the limiting transition (2.3) and the series (1.7) we have

$$J_{\gamma_1,\gamma_2}(x_1, x_2) = \frac{(\frac{x_1}{2})^{\gamma_1}}{\Gamma(\gamma_1+1)} \cdot \frac{(\frac{x_2}{2})^{\gamma_2}}{\Gamma(\gamma_2+1)} \cdot J_{\gamma_1}(\gamma_1 + 1; -\frac{x_1^2}{2^2})J_{\gamma_2}(\gamma_2 + 1; -\frac{x_2^2}{2^2}) = \tag{2.6}$$

$$= \frac{(\frac{x_1}{2})^{\gamma_1}}{\Gamma(\gamma_1+1)} \cdot \frac{(\frac{x_2}{2})^{\gamma_2}}{\Gamma(\gamma_2+1)} \cdot J(\gamma_1 + 1, \gamma_2 + 1; -\frac{x_1^2}{2^2}, -\frac{x_2^2}{2^2}) =$$

$$= \frac{(\frac{x_1}{2})^{\gamma_1}}{\Gamma(\gamma_1+1)} \cdot \frac{(\frac{x_2}{2})^{\gamma_2}}{\Gamma(\gamma_2+1)} \cdot [1 - \frac{1}{1!(\gamma_1+1)} \cdot \frac{x_1^2}{2^2} - \frac{1}{1!(\gamma_2+1)} \cdot \frac{x_2^2}{2^2} + \frac{1}{2!(\gamma_1+1)(\gamma_1+2)} \cdot (\frac{x_1^2}{2^2})^2 +$$

$$+\frac{1}{1!(\gamma_1+1)(\gamma_2+1)} \cdot (\frac{x_1^2}{2^2}) \cdot (\frac{x_2^2}{2^2}) + \frac{1}{2!(\gamma_2+1)(\gamma_2+2)} \cdot (\frac{x_2^2}{2^2})^2 + \cdots] =$$

$$= \frac{(\frac{x_1}{2})^{\gamma_1}}{\Gamma(\gamma_1+1)} \cdot \frac{(\frac{x_2}{2})^{\gamma_2}}{\Gamma(\gamma_2+1)} \sum_{m_1,m_2=0}^{\infty} \frac{(-1)^{m_1+m_2}(\frac{x_1}{2})^{2m_1} \cdot (\frac{x_2}{2})^{2m_2}}{m_1! \cdot m_2!\Gamma(m_1+1)\Gamma(m_2+1)} =$$

$$= \sum_{m_1,m_2=0}^{\infty} \frac{(-1)^{m_1+m_2}}{m_1! \cdot m_2!\Gamma(\gamma_1 + m_1 + 1)\Gamma(\gamma_2 + m_2 + 1)} \cdot (\frac{x_1}{2})^{2m_1+\gamma_1} \cdot (\frac{x_2}{2})^{2m_2+\gamma_2}.$$

(2.6) shows the rightly of presentation (2.5).

Solutions of the system (1.8) are not difficult to build [8; 203].

Theorem 2.3. The system of differential equations (1.8) has four linearly independent partial solutions in the form of series, which are reduced to Bessel functions,

$$Z_1(x_1,x_2)=J(\gamma_1;x_1)J(\gamma_2;x_2)=\sum_{m,n=0}^{\infty}\frac{1}{(\gamma_1)_m(\gamma_2)_n}\cdot\frac{x_1^m}{m!}\cdot\frac{x_2^n}{n!}, \quad (2.7_1)$$

$$Z_2(x_1,x_2)=x_2^{1-\gamma_2}J(\gamma_1;x_1)J(2-\gamma_2;x_2), \quad (2.7_2)$$

$$Z_3(x_1,x_2)=x_1^{1-\gamma_1}J(2-\gamma_1;x_1)J(\gamma_2;x_2), \quad (2.7_3)$$

$$Z_4(x_1,x_2)=x_1^{1-\gamma_1}\cdot x^{1-\gamma_2}J(2-\gamma_1;x_1)J(2-\gamma_2;x_2), \quad (2.7_4)$$

Proof. Near the singularity $(0,0)$ we look for a solution in the form of a generalized power series of two variables

$$Z=x^\rho y^\sigma\sum_{m,n=0}^{\infty}A_{m,n}\cdot x^m\cdot y^n,\ (A_{0,0}\neq 0) \quad (2.8)$$

where $\rho,\sigma,A_{m,n}(m,n=0,1,2,...)$ are unknown constants.

In (2.8) unknown constants ρ and σ are determined from a system of defining equations regarding features $(0,0)$. It has four pairs of roots: $(0,0)$,$(0,1-\gamma_2)$,$(1-\gamma_1,0)$,$(1-\gamma_1,1-\gamma_2)$.

Unknown coefficients $A_{m,n}(m,n=0,1,2,...)$ are determined from the system of recurrent sequences,

$$\sum_{m,n=0}^{\infty}A^{(j)}_{\mu-m,\nu-n}\cdot f^{(j)}_{m,n}(\rho+\mu-m,\sigma+\nu-n)=0, \quad (2.9)$$

$(\mu,\nu=0,1,2...;j=1,2,...)$. Then, given (2.9) the values of the unknown constants, we obtain partial solutions of the form $(2.7_t)(t=\overline{1,4})$, where the first particular solution coincides with (2.4).

Various Appell F_1-F_4 functions are used to obtain series of the form (1.7).

Theorem 2.4. Row (XIII)

$$\lim_{\tau\to 0}F_2(\frac{1}{\varepsilon},\frac{1}{\varepsilon},\frac{1}{\varepsilon};\gamma_1;\gamma_2;\varepsilon^2x_1,\varepsilon^2x_2)=\sum_{m,n=0}^{\infty}\frac{1}{(\gamma)_{m+n}}\cdot\frac{x_1^m}{m!}\cdot\frac{x_2^n}{n!}=J(\gamma;x_1+x_2), \quad (2.10)$$

where the degenerate hypergeometric function $J(\gamma;x_1+x_2)$ reduced to the Bessel function of two variables is a particular solution of the system

$$\begin{cases}x_1Z_{x_1x_1}+x_2Z_{x_1x_2}+\gamma_1Z_{x_1}-Z=0,\\ x_2Z_{x_2x_2}+x_1Z_{x_1x_2}+\gamma_2Z_{x_2}-Z=0,\end{cases} \quad (2.11)$$

where $Z=Z(x_1,x_2)$ is the total unknown obtained by the limiting transition from the system

$$\begin{cases}x_1(1-\varepsilon^2x_1)Z_{x_1x_1}-x_1(1-\varepsilon^2x_1)Z_{x_1x_2}+\left[\gamma-(\frac{1}{\varepsilon}+\frac{1}{\varepsilon}+1)\varepsilon^2x_1\right]Z_{x_1}-\varepsilon x_2Z_{x_2}-Z=0,\\ x_2(1-\varepsilon^2x_2)Z_{x_2x_2}-x_2(1-\varepsilon^2x_2)Z_{x_1x_2}+\left[\gamma-(\frac{1}{\varepsilon}+\frac{1}{\varepsilon}+1)\varepsilon^2x_2\right]Z_{x_2}-\varepsilon x_2Z_{x_1}-Z=0.\end{cases} \quad (2.12)$$

By the Frobenius-Latysheva method, we establish that the joint system (2.11) obtained by the limiting transition from the system (2.12) under the conditions of compatibility and

$$1-\frac{x_2}{x_1}\cdot\frac{x_1}{x_2}=0 \quad (2.13)$$

has no more than three linearly independent solutions because (2.13) shows that the so-called integration condition fulfilled [9; 85].

Main results

2.1. Differential properties of the Humbert and Bessel function

The reasoning of the previous points shows that the Bessel function is mainly related to the Humbert function $\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)$, which is a particular solution of the Horn system (1.6). Based on the General theory of such systems, as stated in theorem 1.1, the following statement is true.

Theorem 2.5. The Horn system has four linearly independent solutions:

$$Z_1=\sum_{m_1,m_2=0}^{\infty}\frac{(\lambda)_{m_1+m_2}}{(\gamma_1)_{m_1}(\gamma_2)_{m_2}}\cdot\frac{x_1^{m_1}}{m_1!}\cdot\frac{x_2^{m_2}}{m_2!}=\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2), \tag{2.14}$$

$$Z_2=x_1^{1-\gamma_1}\cdot\Psi_2(\lambda+1-\gamma_1;2-\gamma_1,\gamma_2;x_1,x_2) \tag{2.15}$$

$$Z_3=x_2^{1-\gamma_2}\cdot\Psi_2(\lambda+1-\gamma_2;\gamma_1,2-\gamma_2;x_1,x_2) \tag{2.16}$$

$$Z_4=x_1^{1-\gamma_1}\cdot x_2^{1-\gamma_2}\cdot\Psi_2(\lambda+2-\gamma_1-\gamma_2;2-\gamma_1,2-\gamma_2;x_1,x_2). \tag{2.17}$$

As can be seen here, the first particular solution (2.14) defines the Humbert function Ψ_2. We find the derivatives of this function.

Theorem 2.6. Derivatives of Humbert variables x_1 and x_2 presented in the form:

1) by variables x_1 and x_2;

$$\begin{cases}\dfrac{\partial}{\partial x_1}\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)=\dfrac{\lambda}{\gamma_1}\Psi_2(\lambda+1;\gamma_1+1,\gamma_2;x_1,x_2),\\[2ex] \dfrac{\partial}{\partial x_2}\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)=\dfrac{\lambda}{\gamma_2}\Psi_2(\lambda+1;\gamma_1,\gamma_2+1;x_1,x_2),\end{cases} \tag{2.18}$$

2) higher derivatives;

$$\begin{cases}\dfrac{\partial^2}{\partial x_1\partial x_2}\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)=\dfrac{\lambda(\lambda+1)}{\gamma_1\gamma_2}\Psi_2(\lambda+2;\gamma_1+1,\gamma_2+1;x_1,x_2),\\[2ex] \dfrac{\partial^2}{\partial x_1^2}\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)=\dfrac{\lambda(\lambda+1)}{\gamma_1(\gamma_1+1)}\Psi_2(\lambda+2;\gamma_1+2,\gamma_2;x_1,x_2),\\[2ex] \dfrac{\partial^2}{\partial x_2^2}\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)=\dfrac{\lambda(\lambda+1)}{\gamma_2(\gamma_2+1)}\Psi_2(\lambda+2;\gamma_1,\gamma_2+2;x_1,x_2),\\[2ex] \dfrac{\partial^m}{\partial x_1^m}\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)=\dfrac{\lambda(\lambda+1)...(\lambda+m-1)}{\gamma_1(\gamma_1+1)...(\gamma_1+m-1)}\Psi_2(\lambda+m;\gamma_1+m,\gamma_2;x_1,x_2),\\[2ex] \dfrac{\partial^n}{\partial x_2^n}\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)=\dfrac{\lambda(\lambda+1)...(\lambda+n-1)}{\gamma_2(\gamma_2+1)...(\gamma_2+n-1)}\Psi_2(\lambda+n;\gamma_1,\gamma_2+n;x_1,x_2),\\[2ex] \dfrac{\partial^{m+n}}{\partial x_1^m\partial x_2^n}\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)=\dfrac{\lambda(\lambda+1)...(\lambda+m+n-1)}{\gamma_1(\gamma_1+1)...(\gamma_1+m-1)\cdot\gamma_2(\gamma_2+1)...(\gamma_2+n-1)}\cdot\\[2ex] \cdot\Psi_2(\lambda+m+n;\gamma_1+m,\gamma_2+n;x_1,x_2).\end{cases} \tag{2.19}$$

Similarly, it is possible to find derivatives of the other particular solutions of (2.15)-(2.17), using the kind of derivatives (2.18), (2.19).

2.2. Differential properties of a function that reduces to the Bessel function of two variables

The degenerate system (2.6) based on theorem 2.1 has four linearly independent partial solutions $(2.7_1) - (2.7_4)$. The first particular solution defines a series that reduces to the Bessel function of two variables

$$J(\gamma_1; x_1)J(\gamma_2; x_2) = \sum_{m,n=0}^{\infty} \frac{1}{(\gamma_1)_m(\gamma_2)_n} \cdot \frac{x_1^m}{m!} \cdot \frac{x_2^n}{n!}, \tag{2.20}$$

The derivative (2.20) can be found as products of two functions $J(\gamma_1; x_1)$ and $J(\gamma_2; x_2)$.

Theorem 2.7. Derivatives of functions reduced to the Bessel function of two variables are represented as

$$1. \frac{\partial}{\partial x_1}[J(\gamma_1; x_1)J(\gamma_2; x_2)] = \frac{1}{\gamma_1}[J(\gamma_1 + 1; x_1)J(\gamma_2; x_2)],$$

$$2. \frac{\partial}{\partial x_2}[J(\gamma_1; x_1)J(\gamma_2; x_2)] = \frac{1}{\gamma_2}[J(\gamma_1; x_1)J(\gamma_2 + 1; x_2)].$$

Second derivatives:

$$3. \frac{\partial^2}{\partial x_1 \partial x_2}[J(\gamma_1; x_1)J(\gamma_2; x_2)] = \frac{1}{\gamma_1\gamma_2}[J(\gamma_1 + 1; x_1)J(\gamma_2 + 1; x_2)],$$

$$4. \frac{\partial^2}{\partial x_1^2}[J(\gamma_1; x_1)J(\gamma_2; x_2)] = \frac{1}{(\gamma_1)_2}[J(\gamma_1 + 2; x_1)J(\gamma_2; x_2)],$$

$$5. \frac{\partial^2}{\partial x_2^2}[J(\gamma_1; x_1)J(\gamma_2; x_2)] = \frac{1}{(\gamma_2)_2}[J(\gamma_1; x_1)J(\gamma_2 + 2; x_2)].$$

Higher derivatives:

$$6. \frac{\partial^{m_1}}{\partial x_1^{m_1}}[J(\gamma_1; x_1)J(\gamma_2; x_2)] = \frac{1}{(\gamma_1)_{m_1}}[J(\gamma_1 + m_1; x_1)J(\gamma_2; x_2)],$$

$$7. \frac{\partial^{m_2}}{\partial x_2^{m_2}}[J(\gamma_1; x_1)J(\gamma_2; x_2)] = \frac{1}{(\gamma_2)_{m_2}}[J(\gamma_1; x_1)J(\gamma_2 + m_2; x_2)],$$

$$8. \frac{\partial^{m_1+m_2}}{\partial x_1^{m_1}\partial x_2^{m_2}}[J(\gamma_1; x_1)J(\gamma_2; x_2)] = \frac{1}{(\gamma_1)_{m_1}(\gamma_2)_{m_2}}[J(\gamma_1 + m_1; x_1)J(\gamma_2 + m_2; x_2)]. \tag{2.21}$$

Similarly are determined by the derivatives of the particular solutions of (2.7_2)–(2.7_4), in particular using (2.21). The main differential properties of the Bessel function of two variables were studied in the works [10; 23]. The differential properties of degenerate hypergeometric functions of one variable are given in the monographs of Lucy J. Slater [1; 15] and [11–13].

Let's consider the addition property of a degenerate hypergeometric function (2.10):

$$J(\gamma; x_1 + x_2) = \sum_{m,n=0}^{\infty} \frac{1}{(\gamma)_{m+n}} \cdot \frac{x_1^m}{m!} \cdot \frac{x_2^n}{n!}, \tag{2.10}$$

obtained as a particular solution of a degenerate hypergeometric system (2.11).

Theorem 2.8. For the degenerate hypergeometric function (2.11) there is an equality:

$$J(\gamma; x_1 + x_2) = \sum_{n=0}^{\infty} J^{(n)}(x_1) \cdot \frac{x_2^n}{n!}. \tag{2.22}$$

Proof. The formula is used to prove the theorem

$$f(x+y)=\sum_{n=0}^{\infty} f^{(n)}(x)\cdot\frac{y^n}{n!}, \tag{2.23}$$

applied by Lucy J.Slater [1; 22] in establishing the theorem of addition for the derivatives of the Kummer functions of $F_1[a;b;x]$. So, on the basis of (2.23) we obtain rightly (2.22):

$$J(\gamma;x_1+x_2)=\sum_{n=0}^{\infty} J^{(n)}(x_1)\cdot\frac{x_2^n}{n!}=J^{(0)}(\gamma;x_1)\frac{x_2^0}{0!}+J'(\gamma+1;x_1)\frac{x_2}{1!}+$$

$$+J''(\gamma+2;x_1)\frac{x_2^2}{2!}+...+J^{(n)}(\gamma+n;x_1)+...=\sum_{n=0}^{\infty}\frac{1}{(\gamma)_n}\cdot\frac{x_2^n}{n!}\cdot J^{(n)}(\gamma+n;x_1).$$

Theorem 2.9. For a degenerate hypergeometric function $J(\gamma;x_1,x_2)$, there is equality:

$$J(\gamma;x_1\cdot x_2)=\sum_{n=0}^{\infty}\frac{x_1^n(x_2-1)^n}{(\gamma)_n\cdot n!}\cdot J(\gamma+n;x_1).$$

The formula is used to prove the theorem

$$f(x_1\cdot x_2)=\sum_{n=0}^{\infty}\frac{x_1^n(x_2-1)^n}{n!}\cdot\frac{d^n}{dx^n}\left\{f(x_1)\right\}.$$

obtained from (2.23) by substitution x_2 for $(x_2-1)x_1$ and by Taylor's theorem.

Theorem 2.9 is related to the multiplication theorem for Kummer functions [1; 23].

3. Construction of normal-regular solutions of Bessel-type system

Problem statement. From the Horn system (1.6) by converting the form,

$$Z=exp(\frac{x_1}{2}+\frac{x_2}{2})x_1^{-\frac{\gamma_1}{2}}\cdot x_2^{-\frac{\gamma_2}{2}}\cdot U(x_1,x_2) \tag{3.1}$$

a system of Bessel-type is installed

$$\begin{cases} x_1^2\cdot U_{x_1x_1}-x_1x_2\cdot U_{x_2}+\left\{-\frac{1}{4}x_1^2-\frac{1}{2}x_1x_2+kx_1+\alpha(1-\alpha)\right\}U=0,\\ x_2^2\cdot U_{x_2x_2}-x_1x_2\cdot U_{x_1}+\left\{-\frac{1}{4}x_2^2-\frac{1}{2}x_1x_2+kx_2+\beta(1-\beta)\right\}U=0,\end{cases} \tag{3.2}$$

where $k=\alpha+\beta-\lambda$ and α,β,λ are some parameters, and $U=U(x_1,x_2)$ is general unknown.

Using the Frobenius-Latysheva method [14] it is required to prove that the solutions of the system (3.2) are functions that reduce to Bessel functions of two variables.

Theorem 3.1. The Bessel-type system (3.2) under the conditions of compatibility and integrability [6] has four linearly independent partial solutions

$$\begin{cases} U_1(x_1,x_2)=exp(-\frac{x_1}{2}-\frac{x_2}{2})\cdot x_1^{\alpha}\cdot x_2^{\beta}\cdot\Psi_2(\lambda,2\alpha,2\beta;x_1,x_2),\\ U_2(x_1,x_2)=exp(-\frac{x_1}{2}-\frac{x_2}{2})\cdot x_1^{\alpha}\cdot x_2^{1-\beta}\cdot\Psi_2(\lambda-2\beta+1,2\alpha,2\beta-2;x_1,x_2),\\ U_3(x_1,x_2)=exp(-\frac{x_1}{2}-\frac{x_2}{2})\cdot x_1^{1-\alpha}\cdot x_2^{\beta}\cdot\Psi_2(\lambda-2\alpha+1,2\alpha-2,2\beta;x_1,x_2),\\ U_4(x_1,x_2)=exp(-\frac{x_1}{2}-\frac{x_2}{2})\cdot x_1^{1-\alpha}\cdot x_2^{1-\beta}\cdot\Psi_2(\lambda-2\alpha-2\beta+1,2\alpha-2,2\beta-2;x_1,x_2),\end{cases} \tag{3.3}$$

which are expressed in terms of the degenerate Humbert hypergeometric function reduced when $\gamma_1=2\alpha,\gamma_2=2\beta$ to the Bessel function of two variables by the limit transition

$$\lim_{\lambda\to 0}\Psi_2(\lambda,2\alpha,2\beta;\lambda x_1,\lambda x_2)=\sum_{m,n=0}^{\infty}\frac{1}{(2\alpha)_m(2\beta)_n}\cdot\frac{x_1^m}{m!}\cdot\frac{x_2^n}{n!}. \tag{3.4}$$

The Frobenius-Latysheva method is used to prove the theorem [14; 160]. The studied system belongs to the Whittaker-type system [5; 132]. The application of the Frobenius-Latysheva method to the construction of a Whittaker-type system solution is described in the works [15; 27]. It was found out that its solutions are normal-regular species,

$$Z(x_1,x_2) = expQ(x_1,x_2)\cdot x_1^{\rho}\cdot x_2^{\sigma}\sum_{m,n=0}^{\infty} A_{m,n}\cdot x_1^m\cdot x_2^n\ ; (A_{0,0}\neq 0), \tag{3.5}$$

where $\rho,\sigma,A_{m,n}(m,n=0,1,2,...)$ are unknown constants; $Q=Q(x_1,x_2)$ polynomial of two variables:

$$Q(x_1,x_2) = \frac{\alpha_{p0}}{p}x_1^p + \frac{\alpha_{0p}}{p}x_2^p + ... + \alpha_{11}x_1x_2 + \alpha_{10}x_1 + \alpha_{01}x_2, \tag{3.6}$$

with unknown coefficients $\alpha_{p0},\alpha_{0p},...,\alpha_{11},\alpha_{10},\alpha_{01}$.

In the theory of ordinary differential equations with variable coefficients greater role will be played by the notion of rank $p=1+k$ introduced by H. Poincare, and the concept of antirank $m=-1-\chi$, introduced by L. Tome. Professor of Kiev University K. Ya. Latysheva used these concepts: to determine the polynomial Q, as well as in the classification of regular and irregular points of a given equation [16; 50].

The studied system (3.2) has a rank $p=1>0$ and antirank $m\leq 0$ [16, p.53]. Therefore, the singularity (∞,∞) is irregular , and the singularity $(0,0)$ is regular and there is a normal-regular solution of the form (3.5). The highest degree of the polynomial (3.6) is equal to the rank of the system, that is $p=1$. Then, the polynomial (3.6) turns into a polynomial of the first degree $Q(x_1,x_2)=\alpha_{10}x_1+\alpha_{01}x_2$ and its unknown coefficients α_{10} and α_{01} are determined from the auxiliary system obtained from (3.2) by the transformation in form (3.1):

$$U = exp(\alpha_{10}x_1+\alpha_{01}x_2)\cdot\Phi(x_1,x_2) \tag{3.7}$$

where $\Phi(x_1,x_2)$ is a new unknown function, by equating to zero coefficients at higher degrees of independent variables x_1 and x_2 :

$$f_{10}^{(1)}(\alpha_{10},\alpha_{01}) = \alpha_{10}^2 - \frac{1}{4} = 0,\ f_{01}^{(2)}(\alpha_{10},\alpha_{01}) = \alpha_{01}^2 - \frac{1}{4} = 0. \tag{3.8}$$

The resulting system of characteristic equations (3.8) has four pairs of roots:

$$(\alpha_{10}^{(1)},\alpha_{01}^{(1)}) = (\frac{1}{2},\frac{1}{2}), (\alpha_{10}^{(1)},\alpha_{01}^{(2)}) = (\frac{1}{2},-\frac{1}{2}),$$
$$(\alpha_{10}^{(2)},\alpha_{01}^{(1)}) = (-\frac{1}{2},\frac{1}{2}), (\alpha_{10}^{(2)},\alpha_{01}^{(2)}) = (-\frac{1}{2},-\frac{1}{2}). \tag{3.9}$$

In (3.9) only a pair $(\alpha_{10}^{(2)},\alpha_{01}^{(2)}) = (-\frac{1}{2},-\frac{1}{2})$ defines a joint system

$$\begin{cases} x_1^2\cdot\Phi_{x_1x_1} - x_1^2\cdot\Phi_{x_1} - x_1x_2\cdot\Phi_{x_2} + [kx_1+\alpha(1-\alpha)]\,\Phi = 0, \\ x_2^2\cdot\Phi_{x_1x_1} - x_2^2\cdot\Phi_{x_2} - x_1x_2\cdot\Phi_{x_1} + [kx_2+\beta(1-\beta)]\,\Phi = 0, \end{cases} \tag{3.10}$$

where $k=\alpha+\beta-\lambda$ and α,β,λ are some parameters, and $\Phi=\Phi(x_1,x_2)$ is unknown.

It has four linearly independent partial solutions, which are expressed in terms of the degenerate Humbert hypergeometric function $\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)$, $(\gamma_1=2\alpha,\gamma_2=2\beta)$:

$$\begin{cases} \Phi_1(x_1,x_2) = x_1^{\alpha}\cdot x_2^{\beta}\cdot\Psi_2(\lambda,2\alpha,2\beta;x_1,x_2), \\ \Phi_2(x_1,x_2) = x_1^{\alpha}\cdot x_2^{1-\beta}\cdot\Psi_2(\lambda-2\beta+1,2\alpha,2\beta-2;x_1,x_2), \\ \Phi_3(x_1,x_2) = x_1^{1-\alpha}\cdot x_2^{\beta}\cdot\Psi_2(\lambda-2\alpha+1,2\alpha-2,2\beta;x_1,x_2), \\ \Phi_4(x_1,x_2) = x_1^{1-\alpha}\cdot x_2^{1-\beta}\cdot\Psi_2(\lambda-2\alpha-2\beta+1,2\alpha-2,2\beta-2;x_1,x_2). \end{cases} \tag{3.11}$$

It should be noted that the system of defining equations relating the peculiarity $(0,0)$:

$$f_{00}^{(1)}(\rho,\sigma)=\rho(\rho-1)+\alpha(\alpha-1)=0,\ f_{00}^{(2)}(\rho,\sigma)=\sigma(\sigma-1)+\beta(\beta-1)=0, \tag{3.12}$$

has four pairs of roots

$$(\rho_1,\sigma_1)=(\alpha,\beta),\ (\rho_1,\sigma_2)=(\alpha,1-\beta),$$
$$(\rho_2,\sigma_1)=(1-\alpha,\beta),(\rho_2,\sigma_2)=(1-\alpha,1-\beta).$$

They identified the indicators of the series (3.11). We will draw some conclusions here.

Theorem 3.2. In order to have at least one solution of the form (3.5) for the auxiliary system obtained from the system (3.2) by transformation (3.7), it is necessary and sufficient to perform equality (3.8).

(3.8) has four pairs of roots. This is the first necessary condition for the existence of a normal-regular solution of the form (3.5) associated with the definition of unknown constants $\alpha_{p0},\alpha_{0p},...,\alpha_{11},\alpha_{10},\alpha_{01}$ polynomial $Q(x_1,x_2)$.

The second necessary condition is related to the definition of unknown constants $\rho,\sigma,A_{m,n}(m,n=0,1,2,...)$ in (3.5).

Theorem 3.3. To have a solution in the form of a generalized power series of two variables for the system (3.10), it is necessary that the pair $(\rho,\sigma,)$ to be the root of the system of defining equations regarding features (3.12) obtained by substituting b (3.10) instead of the unknown $\Phi(x_1,x_2)=x_1^{\rho}\cdot x_2^{\sigma}$.

The fulfillment of two necessary conditions ensures the existence of four normal-regular solutions of the form (3.3). The theorem is proved.

Summary.

1. Equality (3.4) on the basis of (2.2) and (2.3) when $\gamma_1=2\alpha$ and $\gamma_2=2\beta$ is represented as

$$J(\gamma_1,\gamma_2;x_1,x_2)=J(2\alpha,2\beta;x_1,x_2)=$$
$$=\sum_{m,n=0}^{\infty}\frac{1}{(2\alpha)_m(2\beta)_n}\cdot\frac{x_1^m}{m!}\cdot\frac{x_2^n}{n!}=J(2\alpha;x_1)J(2\beta;x_2), \tag{3.13}$$

Then, on the basis of (2.5) the Bessel function of two variables of the first kind, we obtain in the form

$$J_{2\alpha,2\beta}(x_1,x_2)=\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}}{m!\cdot n!\Gamma(2\alpha+m+1)\Gamma(2\beta+n+1)}\cdot(\frac{x_1}{2})^{2m+2\alpha}\cdot(\frac{x_2}{2})^{2n+2\beta}.$$

The course of proof as in (2.6).

2. The derivative (3.13) can be found as in 2.2. taking the meanings $\gamma_1=2\alpha$, $\gamma_2=2\beta$ into account. We give a General formula:

$$\frac{\partial^{m+n}}{\partial x_1^m\partial x_2^n}[J(2\alpha;x_1)J(2\beta;x_2)]=\frac{1}{(2\alpha)_m(2\beta)_n}[J(2\alpha+m;x_1)J(2\beta+n;x_2)].$$

Out of this we can derive various special cases of lower derivatives.

3. The solutions of the attached system (3.10) are expressed in terms of the Humbert hypergeometric function, which is reduced to the Bessel function of two variables by a limit transition (3.4). We have seen that in this case equality is true (3.13). Therefore,

$$\lim_{\lambda\to\infty}\Phi_j(x_1,x_2)=\lim_{\lambda\to\infty}[x_1^{\alpha}\cdot x_2^{\beta}\cdot\Psi(\lambda,2\alpha,2\beta;x_1,x_2)]=$$
$$=x_1^{\alpha}\cdot x^{\beta}\lim_{\lambda\to\infty}\Psi(\lambda,2\alpha,2\beta;x_1,x_2)=J(2\alpha,2\beta;x_1,x_2),$$

that is, in this case, in the limiting transition, all $\Phi_j(x_1,x_2),(j=\overline{1,4})$ are expressed through the function $J(2\alpha,2\beta;x_1,x_2)$. Similarly, all the particular solutions of system (3.2) are also expressed via (3.13).

Thus, we have established and studied a number of systems with solutions in the form of degenerate hypergeometric Humbert functions of two variables, which reduces to Bessel functions of two variables. The connection between these functions of two variables as solutions of systems of partial differential equations of the second order is revealed. Their properties, as well as differential properties, addition and multiplication theorems, have been investigated so far. Further, these properties make it possible to establish recurrent relations between these functions, as well as between degenerate hypergeometric functions of two and many variables as a whole. A system of Bessel-type has been installed and the features of the application of the method of Frobenius-Latysheva for the construction of normal-regular solutions installed by our system is shown. It is also shown that the Bessel-type system is related to the Whittaker-type system, the features of the solution of which are studied by M.P. Humbert [5; 132].

References

1 Слейтер Л.Дж. Вырожденные гипергеометрические функции / Л.Дж. Слейтер. — М.: Вычислительный центр, 1966. — 249 с.

2 Бейтмен Г. Высшие трансцендентные функции. — Ч. I. Гипергеометрическая функция. Функции Лежандра / Г. Бейтмен, А. Эрдейи. — М.: Наука, 1965. — 294 с.

3 Бейтмен Г. Высшие трансцендентные функции. — Ч. II. Функции Бесселя, функции параболического цилиндра, ортогональные многочлены / Г. Бейтмен, А. Эрдейи. — М.: Наука, 1974. — 295 с.

4 Tasmambetov Zh.N. Construction of normal-regular decisions of Bessel typed special system / Zh.N. Tasmambetov, M.Zh. Talipova // Abstracts of VI Congress of the Turkic world mathematical Society (October 2-5, 2017). — Astana, Kazakhstan. — P. 138-139.

5 Appell P. Fonctions hypergeometriques et hypesperiques / P. Appell, M.J. Kampe de Feriet. — Paris: Gauthier Villars, 1926. — 434p.

6 Wilczynski E.J. Projective differential geometry of Curves and Ruled surfaces / E.J. Wilczynski. — Leipzig: Leubner, 1966. — 120 p.

7 Кампе де Ферье Ж. Функции математической физики / Ж.Кампе де Ферье, Р. Кемпбелл, Г. Петьо, Т. Фогель. — М.: ГИФМЛ, 1963. — 162 с.

8 Тасмамбетов Ж.Н. О свойствах произведения функций Бесселя / Ж.Н.Тасмамбетов // Дифференциальные уравнения и математическая физика: тр. Междунар. науч. конф. — Алматы, 2014. — С. 203–207.

9 Тасмамбетов Ж.Н. Построение вырожденной гипергеометрической функции, сводящейся к функциям Бесселя двух переменных / Ж.Н. Тасмамбетов, Р.У. Жахина, А.Ж. Тасмамбетова // Наука, новые технологии и инновации Кыргызстана. — Бишкек, 2017. — № 7. — С. 85-89.

10 Aktas R. On a two-variable analogue of the Bessel functions / R.Aktas, A.Altin, B.Gekim // Journal of Inequalities and special functions. ISSN: 2217-4303. — Vol. 3. — Issue 4(2012). — P. 13-23.

11 Slater L.J. Generalized hypergeometric functions / L.J. Slater. — Cambridge: at the university Press, 1966. — 274 p.

12 Кратцер А. Трансцендентные функции / А. Кратцер, В. Франц. — М.: ИЛ, 1963. — 465 с.

13 Кузнецов Д. С. Специальные функции / Д.С. Кузнецов. — М.: Высш. шк., 1965. — 423 с.

14 Тасмамбетов Ж.Н. Построение нормальных и нормально-регулярных решений специальных систем дифференциальных уравнений в частных производных второго порядка / Ж.Н. Тасмамбетов. — Актобе: ИП Жанадилов С.Т., 2015. — 464 с.

15 Тасмамбетов Ж.Н. Об иррегулярных особых кривых систем типа Уиттекера / Ж.Н. Тасмамбетов // Вестн. СамГТУ. Сер. Физ.-мат. науки. — 2013. — № 4(33). — С. 25-33.

16 Латышева К.Я. Нормально-регулярные решения и их приложения / К.Я. Латышева, Н.И. Терещенко, Г. С.Орёл. — Киев: Вища шк., 1974. — 136 с.

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Екі айнымалының Бессель функциялары екінші ретті дербес туындылы дифференциалдық теңдеулер жүйесінің шешімі ретінде

Мақалада екі айнымалының Бессель функцияларына келтірілетін туындалған гипергеометриялық екі айнымалыларының Гумберт функциялары түріндегі шешімдері мен жүйелері орнатылған және зерттелген. Екі айнымалының Гумберт мен Бессель функциялары араларындағы байланыстар орнатылған олардың дифференциалдық қасиеттері зерттелген. Қосындылау және көбейту теоремалары дәлелденген. Әрі қарай олар бұл қасиеттер алдағы уақытта екі айнымалының туындалған гипергеометриялық функциялары араларында, Бессельдің екі айнымалының функциялары араларында өзара рекурренттік қатынастар орнатуға, әрі аталған қасиеттерді көп айнымалылар жағдайына таратуға мүмкіндік береді. Уиттекер және Бессель жүйелері араларындағы байланыстар көрсетілген. Фробениус-Латышева әдісінің көмегімен жаңадан құрылған Бессель текті жүйенің қалыпты-регулярлы шешімдерінің құрылуының ерекшеліктері зерттелген.

Кілт сөздер: Гумберт функциясы, жүйе, Бессель функциясы, қасиеттер, қосу теоремасы, Бессель функциясына келтірілген, қалыпты-регулярлы.

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Функции Бесселя двух переменных как решения систем дифференциальных уравнений в частных производных второго порядка

В статье установлены и изучены системы с решениями в виде вырожденных гипергеометрических функций Гумберта двух переменных, сводящиеся к функциям Бесселя двух переменных. Раскрыты связи между функциями Гумберта и Бесселя двух переменных, исследованы их дифференциальные свойства. Доказаны теоремы сложения и умножения. В дальнейшем они позволят установить рекуррентные соотношения между вырожденными гипергеометрическими функциями двух переменных, а также будут способствовать распространению этих свойств на случай многих переменных. Показана связь между системами типа Уиттекера и Бесселя. Методом Фробениуса-Латышевой изучены особенности построения нормально-регулярных решений вновь установленной системы типа Бесселя.

Ключевые слова: функция Гумберта, система, функция Бесселя, свойства, теорема сложения, сводящаяся к функциям Бесселя, нормально-регулярное.

References

1 Sleiter, L.Dz. (1966). *Vyrozhdennye hiperheometricheskie funktsii [Degenerated hypergeometric functions]*. Moscow: Vychislitelnyi tsentr [in Russian].

2 Beitmen, G. & Erdeii, A. (1965). *Vysshie transtsendentnye funktsii. Chast I. Hiperheometricheskaia funktsiia. Funktsiia Lezhandra [Higher Transcendental Functions.Part I. Hypergeometric functions. The functions of the Lezhanders]*. Moscow: Nauka [in Russian].

3 Beitmen, G. & Erdeii, A. (1974). *Vysshie transtsendentnye funktsii. Chast II. Funktsii Besselia, funktsii parabolicheskoho tsilindra, ortohonalnye mnohochleny [Higher Transcendental Functions. Part II.Bessel functions, functions of parabolic cylinder,the orthogonal polynoms].* Moscow: Nauka [in Russian].

4 Tasmambetov, Zh.N. & Talipova, M.Zh.(2017). Construction of normal-regular decisions of Bessel typed special system . Abstracts of VI Congress of the Turkic world mathematical Society. Astana, Kazakhstan.

5 Appell, P. & Kampe de Feriet, M.J.(1926). *Fonctions hypergeometriques et hypesperiques.* Paris: Gauthier Villars.

6 Wilczynski, E.J. (1966). *Projective differential geometry of Curves and Ruled surfaces.* Leipzig: Leubner.

7 Kampe de Feriet, M.J., Kambell, R., Peto,G. & Fohell, T. (1957). Funktsii matematicheskoi fiziki [Functions of mathematical physics]. Moscow:GIFML [in Russian].

8 Tasmambetov, Zh.N.(2014). O svoistvakh proizvedeniia funktsii Besselia [On the properties of the product of Bessel functions]. *Trudy Mezhdunarodnoi nauchnoi konferentsii — Proceedings of the International scientific conference.* Almaty [in Russian].

9 Tasmambetov, Zh. N., Zhakhina, R.U. & Tasmambetova, A.Zh. (2017). Postroenie vyrozhdennoi hiperheometricheskoi funktsii, svodiashcheisia k funktsiiam Besselia dvukh peremennykh [The Construction of the degenerated hypergeometric function reduces to the Bessel functions of two variables]. Bishkek, 7, 85-89 [in Russian].

10 Aktas, R., Altin, A. & Gekim, B. (2012). On a two variable analogue of the Bessel functions. *Journal of Inequalities and special functions. 3, 4,* 13-23.

11 Slater, L.J. (1966). *Generalized hypergeometric functions.* Cambridge: at the university Press.

12 Krattser, A. & Frants, V.(1963). *Transtsendentnye funktsii [Transcendental functions].* Moscow: IL [in Russian].

13 Kusnetsov, D.S. (1965). *Spetsialnye funktsii [Special functions].* Moscow: Vysshaia shkola [in Russian].

14 Tasmambetov, Zh.N.(2015). *Postroenie normalnykh i normalno-rehuliarnykh reshenii spetsialnykh sistem differentsialnykh uravnenii v chastnykh proizvodnykh vtoroho poriadka [Construction of normal and normally-regular solutions of special systems of partial equations of second order].* Aktobe: IP Zhanadilov S.T. [in Russian].

15 Tasmambetov, Zh.N.(2013). Ob irrehuliarnykh osobykh krivykh sistem tipa Uittekera [On irregular singular curves of systems of the Whittaker type]. *Vestnik SamHTU. Seriia fiziko-matematicheskie nauki, 4(33),* 25-33 [in Russian].

16 Latysheva, K.Ya., Tereschenko, N.I. & Orel, G.S. (1974). *Normalno-rehuliarnye resheniia i ikh prilozheniia [Normal regular solutions and their applications].* Kiev: Vishcha shkola [in Russian].

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Solution of inhomogeneous systems for differential equations in private derivatives of the third order

The possibilities of constructing inhomogeneous system solutions for partial differential equations of the third order have been studied. The construction of general and particular solutions corresponding to homogeneous system comprehensively investigated by using Frobenius-Latysheva method. Type of solutions near the special curves are established. The number of linearly independent partial solutions is determined. A theorem on the representation a general solution of inhomogeneous system is proved, and the application of uncertain coefficients method for such systems is revealed. On a concrete example, it is shown that the particular solutions of the inhomogeneous system constructed in this way are solutions of one inhomogeneous third-order equation obtained by adding the two equations of the considered example. One of particular solutions corresponding to homogeneous system relates to degenerate generalized hypergeometric series of Clausen type with two variables. Properties of generalized hypergeometric series are still poorly understood.

Keywords: inhomogeneous system, regular solution, singularity, method of undetermined coefficients, system, equations, theorem.

Introduction

Systems consisting of two partial differential equations of the second and third orders with a common unknown have long attracted the attention of mathematicians. The American mathematician E.J. Wilczynski used the system of second order to substantiate projective differential geometry [1]. Further research of such systems is associated with the study of generalized hypergeometric functions of two variables, in particular the four Appell hypergeometric functions $F_1 - F_4$ [2; 155-169], [3; 210-231].

J. Horn studied the convergence of all 34 hypergeometric series in two variables and established systems of partial differential equations of second order which they satisfy [4; 218-233].

In a number of works Zh.N. Tasmambetov [5, 6] proved that almost all Horn's systems are special cases of a regular joint system of second-order partial differential equations that consist of two equations

$$x^2g^{(0)}Z_{xx} + xyg^{(1)}Z_{xy} + y^2g^{(2)}Z_{yy} + xg^{(3)}Z_x + yg^{(4)}Z_y + g^{(5)}Z = 0\,,$$
$$y^2q^{(0)}Z_{yy} + xyq^{(1)}Z_{xy} + x^2q^{(2)}Z_{xx} + xq^{(3)}Z_x + yq^{(4)}Z_y + q^{(5)}Z = 0, \tag{1.1}$$

with coefficients in the form of polynomials

$$g^{(i)}(x,y) = a^{(i)}_{00} + a^{(i)}_{10}x^k,$$
$$q^{(i)}(x,y) = b^{(i)}_{00} + b^{(i)}_{01}y^k.(i = \overline{0,5}; k - integer).$$

The classification of their singular curves, the construction of solutions near singular curves and the existence of logarithmic solutions, etc. were considered.

For different values of k from (1.1) we get a number of interesting systems.

1. When $k = 0$ from (1.1) we obtain a system of Euler type [6; 242-249].

2. When $k = 1, a^{(5)}_{10} = 0, b^{(5)}_{01} = 0$, then we obtain a hypergeometric type system, since the solutions of such systems are the hypergeometric functions of two variables [5; 316-319].

3. Transformation

$$x^k = u, y^k = \upsilon. \tag{1.2}$$

leads the system to the previous view.

4. When $k \geq 2$ from system (1.1) as a special case, the systems are obtained whose solutions are the orthogonal polynomials of two variables. They are expressed through the functions of Appell. Specific examples of the application are shown in the works [2; 155-169], [7; 655-661].

5. The general condition for the compatibility of such systems is established [1]. In addition, the integrability condition must be satisfied

$$1 - \frac{g^{(1)}}{g^{(0)}} \cdot \frac{q^{(1)}}{q^{(0)}} \neq 0. \tag{1.3}$$

6. Under these important conditions, system (1.1) has four linearly independent particular solutions [1]. The overall solution is represented as the sum of these four solutions

$$Z(x, y) = \sum_{i=1}^{4} C_i Z_i(x, y). \tag{1.4}$$

If condition (1.4) is not satisfied, then the system has at most three linearly independent particular solutions. Until now, inhomogeneous systems of the form

$$\begin{aligned} &P^{(0)} Z_{xx} + P^{(1)} Z_{xy} + P^{(2)} Z_x + P^{(3)} Z_y + P^{(4)} Z = P^{(5)}(x, y), \\ &Q^{(0)} Z_{yy} + Q^{(1)} Z_{xy} + Q^{(2)} Z_x + Q^{(3)} Z_y + Q^{(4)} Z = Q^{(5)}(x, y), \end{aligned} \tag{1.5}$$

still insufficiently investigated, where $P^{(i)} = P^{(i)}(x, y), Q^{(i)} = Q^{(i)}(x, y)$ analytic functions of two variables. Although the works of Zh.N. Tasmambetov and M.Zh. Talipova [8, 9] are studied the possibilities of constructing solutions for inhomogeneous systems of the form (1.5) and some special cases of it. As in the ordinary case [10; 146], the rightness of assertion is proved [8].

Theorem 1.1. The general solution of the inhomogeneous system (1.5) is represented as the sum of the total solution (1.4):

$$\overline{Z} = \sum_{j=1}^{4} C_j Z_j(x, y), (j = 1, 2, 3, 4)$$

corresponding homogeneous system and particular solution $Z_0(x, y)$ of inhomogeneous system (1.5):

$$Z(x, y) = \overline{Z}(x, y) + Z_0(x, y) = \sum_{j=1}^{4} C_j Z_j(x, y) + Z_0(x, y).$$

Disseminate previous results obtained from second-order system case to the case with system consisting of two third-order equations. Determine the number of solutions corresponding to homogeneous system, near singular curves. Carry out classification of singular curves and establish the type of inhomogeneous system solutions. Develop specific examples.

Main results

2. Construction of homogeneous system solutions consisting two equations of third order

Problem statement. A nonhomogeneous regular system consisting two third-order partial differential equations near the singularity is considered.

$$x^3 g^{(0)} Z_{xxx} + x^2 y g^{(1)} Z_{xxy} + x^2 g^{(2)} Z_{xx} + xy g^{(3)} Z_{xy} + x g^{(4)} Z_x + y g^{(5)} Z_y + g^{(6)} Z = g^7(x, y),$$

$$y^3q^{(0)}Z_{yyy} + xy^2q^{(1)}Z_{xyy} + y^2q^{(2)}Z_{yy} + xyq^{(3)}Z_{xy} + xq^{(4)}Z_x + yq^{(5)}Z_y + q^{(6)}Z = q^7(x,y), \qquad (2.1)$$

where $Z = Z(x,y)$ total unknown, coefficient

$$g^{(i)} = g^{(i)}(x,y) = a_{00}^{(i)} + a_{10}^{(i)}x^k,$$

$$q^{(i)} = q^{(i)}(x,y) = b_{00}^{(i)} + b_{01}^{(i)}y^k, (i = \overline{0,6}) \qquad (2.2)$$

and the right parts $g^{(7)}(x,y), q^{(7)}(x,y)$ analytic functions or polynomials of two variables. Required to construct a general solution of inhomogeneous system (2.1) with coefficients in type (2.2) and show that it is represented as the sum of total solution $Z_0(x,y)$ corresponding to homogeneous system and particular solution of inhomogeneous system (2.1).

2.1. Construction of regular solutions corresponding to homogeneous system

Construction features of regular solutions corresponding to homogeneous system

$$x^3g^{(0)}Z_{xxx} + x^2yg^{(1)}Z_{xxy} + x^2g^{(2)}Z_{xx} + xyg^{(3)}Z_{xy} + xg^{(4)}Z_x + yg^{(5)}Z_y + g^{(6)}Z = 0,$$

$$y^3q^{(0)}Z_{yyy} + xy^2q^{(1)}Z_{xyy} + y^2q^{(2)}Z_{yy} + xyq^{(3)}Z_{xy} + xq^{(4)}Z_x + yq^{(5)}Z_y + q^{(6)}Z = 0, \qquad (2.3)$$

where $Z = Z(x,y)$ total unknown, not studied enough.

This system requires the establishment of a general method for constructing solutions near regular singularities $(0,0)$ and (∞,∞), determining the number of linearly independent solutions, as well as the classification of regular and irregular singularities, compatibility conditions and integrality. Systems (1.1) and (2.3) differ only in orders. Therefore, to construct a third-order system solution (2.3), it is advisable to use the Frobenius-Latysheva method [6], which has proved itself well in studying the second-order system (1.1).

The use of this method involves the fulfillment a number of conditions:

1. Suppose that system (2.3) is joint and the integrality condition is also represented in type (1.3). However, these concepts need further clarification.

2. Special curves at $k = 1$ determine by equating the coefficients at higher derivatives to zero Z_{xxx} and Z_{yyy}: $(0,0)$, $(0, -b_{00}^{(0)}/b_{01}^{(0)})$, $(-a_{00}^{(0)}/a_{01}^{(0)}, 0)$, $(-a_{00}^{(0)}/a_{01}^{(0)})$, $(-b_{00}^{(0)}/b_{01}^{(0)})$, $(0,\infty)$, $(\infty, 0)$, $(\infty, -b_{00}^{(0)}/b_{01}^{(0)})$, $(-a_{00}^{(0)}/a_{01}^{(0)}, \infty)$. As before, single out two pairs of features $(0,0)$ and (∞,∞) at building a solution.

3. In case under consideration, the coefficient (2.2) is reduced to the form of the previous case, using the transformation (1.2).

4. The solution near the feature $(0,0)$ is represented as

$$Z(x,y) = x^\rho y^\sigma \sum_{m,n=0}^{\infty} A_{m,n}x^m y^n, A_{0,0} \neq 0 \qquad (2.4)$$

a near the feature (∞,∞) in form

$$Z(x,y) = x^\rho y^\sigma \sum_{m,n=0}^{\infty} B_{m,n}x^{-m} y^{-n}, B_{0,0} \neq 0 \qquad (2.5)$$

where $\rho, \sigma, A_{m,n}, B_{m,n} (m,n = 0,1,2,3,...)$ unknown constants.

The application of the Frobenius-Latysheva method assumes [6] compilation characteristic functions system and determination systems of defining equations for the singularity $(0,0)$:

$$f_{00}^{(1)}(\rho,\sigma) = a_{00}^{(0)}\rho(\rho-1)(\rho-2) + a_{00}^{(1)}\rho(\rho-1)\sigma + a_{00}^{(2)}\rho(\rho-1) + a_{00}^{(3)}\rho\sigma + a_{00}^{(4)}\rho + a_{00}^{(5)}\sigma + a_{00}^{(6)} = 0,$$

$$f_{00}^{(2)}(\rho,\sigma) = b_{00}^{(0)}\sigma(\sigma-1)(\sigma-2) + b_{00}^{(1)}\sigma(\sigma-1)\rho + b_{00}^{(2)}\sigma(\sigma-1) + b_{00}^{(3)}\rho\sigma + b_{00}^{(4)}\rho + b_{00}^{(5)}\sigma + b_{00}^{(6)} = 0, \quad (2.6)$$

and concerning feature (∞, ∞) :

$$f_{10}^{(1)}(\rho,\sigma)=a_{10}^{(0)}\rho(\rho-1)(\rho-2)+a_{10}^{(1)}\rho(\rho-1)\sigma+a_{10}^{(2)}\rho(\rho-1)+a_{10}^{(3)}\rho\sigma+a_{10}^{(4)}\rho+a_{10}^{(5)}\sigma+a_{10}^{(6)}=0,$$

$$f_{01}^{(2)}(\rho,\sigma)=b_{01}^{(0)}\sigma(\sigma-1)(\sigma-2)+b_{01}^{(1)}\sigma(\sigma-1)\rho+b_{01}^{(2)}\sigma(\sigma-1)+b_{01}^{(3)}\rho\sigma+b_{01}^{(4)}\rho+b_{01}^{(5)}\sigma+b_{01}^{(6)}=0. \quad (2.7)$$

From (2.6) have been determined indicators of the series (2.4), and from (2.7) row indicators (2.5) as pairs (ρ_t, σ_t). It is important to determine the index t, since the number of such pairs allows determining the number of linearly independent particular solutions of system (2.3) near the singularities $(0, 0)$ and (∞, ∞).

Theorem 2.1. If the system (2.3) with coefficient type (2.2), where coefficient $a_{00}^{(0)} \neq 0$, $b_{00}^{(0)} \neq \neq 0, h = 1$ meet compatibility conditions. Then the system (2.3) near the singularity (0,0) has nine linearly independent regular particular solutions in type (2.4), where row indicators (ρ_t, σ_t), $(t = \overline{1,9})$ are determined from the system of defining equations (2.6), and unknown coefficients $A_{m,n}^{(t)}(m, n = 0, 1, ...;\ t = \overline{1,9})$ series (2.4) determined from recurrent sequence systems

$$\sum_{m,n=0}^{\infty} A_{\mu-m,\nu-n}^{(t)} f_{m,n}^{(j)}(\rho+\mu-m, \sigma+\nu-n)=0, \quad (2.8)$$

$(\mu, \nu = 0, 1, 2, ...,; j = 1, 2; t = \overline{1,9})$ obtained by substituting a series (2.4) to the original system (2.3) with coefficients of the form (2.2).

Proof. In general, to establish compatibility conditions is very difficult. If the system is hypergeometric type, then compatibility conditions are determined by the Kampe de Feriet method [2; 155-159]. The method for hypergeometric type of equations is shown in [11; 21]. Determine how many roots have the system of defining equations (2.6) and (2.7). To this end, write down system (2.6) in expanded form, using (2.5). From the second equation $f_{00}^{(2)}(\rho, \sigma) = 0$ we have discovered

$$\rho=\frac{b_{00}^{(0)}\sigma(\sigma-1)(\sigma-2)+b_{00}^{(2)}\sigma(\sigma-1)+b_{00}^{(5)}\sigma+b_{00}^{(6)}}{b_{00}^{(1)}\sigma(\sigma-1)+b_{00}^{(3)}\sigma+b_{00}^{(4)}}$$

and substituting in the first equation $f_{00}^{(1)}(\rho, \sigma) = 0$ systems (2.6), after exclusion σ get the ninth degree equation for ρ. In the case when only simple roots exist, it is possible to determine the nine roots of the resulting equation. In the same way, we define nine simple roots σ_t, $(t = \overline{1,9}$ and make of them nine pairs of roots (ρ_t, σ_t), $(t = \overline{1,9})$. These indicators correspond to nine linearly-independent particular solutions of the system (2.1) and (2.2), after determining unknown coefficient $A_{m,n}^{(t)}$, $(m, n = 0, 1, 2, ...; t = \overline{1,9})$ from the system of recurrent sequences (2.8). Similarly, we can verify that system (2.1) and (2.2) also has nine linearly independent particular solutions near the singularity (∞, ∞).

Theorem 2.2. If systems (2.1) and (2.2), where coefficient $a_{10}^{(0)} \neq 0, b_{01}^{(0)} \neq 0, h = 1$ conditions of compatibility and integrability are satisfied (1.3). When systems (2.1) near the singularities (∞, ∞) have nine linearly independent regular partial solutions in type (2.5), where a number of indicators $\rho_t, \sigma_t (t = \overline{1,9})$ determined from the system of defining equations (2.7), and unknown coefficients $B_{m,n}^{(t)}(m, n = 0, 1, 2, ...; t = \overline{1,9})$ series (2.4) determined from recurrent sequence systems

$$\sum_{m,n=0}^{\mu,\nu} B_{\mu-m,\nu-n}^{(t)} f_{m,n}^{(j)}(\rho-\mu+m, \sigma-\nu+n)=0,$$

$(\mu, \nu = 0, 1, 2, ...,; j = 1, 2; \mu - m \geq 0, \nu - n \geq 0, t = \overline{1,9})$ obtained by substituting series (2.4) into the original system (2.1), (2.2).

In Theorems 2.1 and 2.2, the conditions $a_{00}^{(0)} \neq 0, b_{00}^{(0)} \neq 0$ and $a_{10}^{(0)} \neq 0, b_{01}^{(0)} \neq 0$ essential since the ninth degree equations are relatively ρ and σ turns out only when they are non-zero. This shows that near regular singular curves $(0,0)$ and (∞,∞) there are nine regular linearly independent particular solutions $Z_t(x,y)$, $(t=\overline{1,9})$.

Theorem 2.3. Common solution of joint system (2.1), (2.2) in case of the integrability condition (1.3) is satisfied, is represented as the sum

$$\overline{Z}(x,y)=\sum_{i=1}^{9} C_i Z_i(x,y),(t=\overline{1,9}) \tag{2.9}$$

where $C_i(t=\overline{1,9})$ arbitrary constant.

Remark 2.1. Theorems 2.1 and 2.2 have been formulated for the case $h=1$. This is due to the classification of singular curves. They are true and generally, where $k\geq 2$. Then particular solutions are expressed through $Z=Z(x^k,y^k),(k\geq 2)$.

Remark 2.2. Theorem 2.3 is also valid when the coefficients of system (2.1) are analytic functions or polynomials of two variables.

2.2. Construction of inhomogeneous system solutions.

A theorem on the construction a general solution is formulated analogously to Theorem 1.1.

Theorem 2.4. The general solution of inhomogeneous system (2.1) is represented as the sum of total solution $Z(x,y)$ corresponding to homogeneous system (2.3) and a particular solution $Z_0(x,y)$ of inhomogeneous system (2.1):

$$Z(x,y)=\overline{Z}(x,y)+Z_0(x,y)=\sum_{t=1}^{9} C_t Z_t(x,y)+Z_0(x,y). \tag{2.10}$$

The form of the general solution (2.9) is established by Theorem 2.3. To construct a particular solution near the singularity $(0,0)$, we apply the method of undetermined coefficients generalized for the case of two variables series. To this end, a series of the form (2.4) representing a particular solution $Z(x,y)$ substitute into the inhomogeneous system (2.1) and obtain the system of Frobenius characteristic functions

$$x^\rho y^\sigma\{C_{0,0}f_{0,0}^{(j)}(\rho,\sigma)+[C_{1,0}f_{0,0}^{(j)}(\rho+1,\sigma)+C_{0,0}f_{1,0}^{(j)}(\rho,\sigma)]x+[C_{0,1}f_{0,0}^{(j)}(\rho,\sigma+1)+C_{0,0}f_{0,1}^{(j)}(\rho,\sigma)]y+$$

$$+[C_{1,1}f_{0,0}^{(j)}(\rho+1,\sigma+1)+C_{1,0}f_{0,1}^{(j)}(\rho+1,\sigma)+C_{0,1}f_{1,0}^{(j)}(\rho,\sigma+1)+C_{0,0}f_{1,1}^{(j)}(\rho,\sigma)]xy+...\}\equiv f_j(x,y)$$

where $f_1(x,y)=g^7(x,y), f_2(x,y)=q^7(x,y)$ and $f_{0,0}^{(j)}(\rho,\sigma),(j=1,2)$ determines the system of defining equations for the singularity $(0,0)$ of the form (2.6).

Further reasoning depends on form of right side representation $f_j(x,y),(j=1,2)$. Let them be represented as generalized power series of two variables in increasing degrees of independent variables x and y :

$$f_1(x,y)=g^{(7)}(x,y)=x^\alpha y^\beta\sum_{m,n=0}^{\infty} a_{m,n}x^m y^n,(a_{0,0}\neq 0)$$

$$f_2(x,y)=q^{(7)}(x,y)=x^\gamma y^\delta\sum_{m,n=0}^{\infty} b_{m,n}x^m y^n,(b_{0,0}\neq 0) \tag{2.11}$$

Then a series of form (2.4) representing a particular solution

$$Z_0(x,y)=x^\rho y^\sigma\sum_{m,n=0}^{\infty} C_{m,n}x^m y^n,(C_{0,0}\neq 0) \tag{2.12}$$

will be a formal particular solution only when uncertain coefficients $C_{m,n}(m,n=0,1,2,...)$ satisfy the following recurrent system

$$C_{0,0}f_{00}^{(j)}(\rho,\sigma)=\alpha_{0,0}^{(j)}$$

$$C_{1,0}f_{00}^{(j)}(\rho+1,\sigma)+C_{0,0}f_{10}^{(j)}(\rho,\sigma)=\alpha_{1,0}^{(j)}$$

$$C_{0,1}f_{00}^{(j)}(\rho,\sigma+1)+C_{0,0}f_{01}^{(j)}(\rho,\sigma)=\alpha_{0,1}^{(j)}$$

$$C_{1,1}f_{00}^{(j)}(\rho+1,\sigma+1)+C_{1,0}f_{01}^{(j)}(\rho+1,\sigma)+C_{0,1}f_{10}^{(j)}(\rho,\sigma+1)+C_{0,0}f_{11}^{(j)}(\rho,\sigma)=\alpha_{1,1}^{(j)}$$

$$C_{2,0}f_{00}^{(j)}(\rho+2,\sigma)+C_{1,0}f_{10}^{(j)}(\rho+1,\sigma)+C_{0,0}f_{20}^{(j)}(\rho,\sigma)=\alpha_{2,0}^{(j)}$$

$$C_{0,2}f_{00}^{(j)}(\rho,\sigma+2)+C_{0,1}f_{01}^{(j)}(\rho,\sigma+1)+C_{0,0}f_{02}^{(j)}(\rho,\sigma)=\alpha_{0,2}^{(j)}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2.13)$$

The recurrent system determines the coefficients $C_{m,n}(m,n=0,1,2,...)$ of series (2.12). When $j=1$ and $j=2$ it breaks down into two systems. When $j=1$ in the right part (2.13) $\alpha_{m,n}^{(1)}=a_{m,n}$ at $j=2$ coefficients $\alpha_{m,n}^{(2)}$ determines over $b_{m,n}(m,n=0,1,...)$, where $a_{m,n}$ and $b_{m,n}$ coefficients of generalized power series $g^{(7)}(x,y)$ and $q^{(7)}(x,y)$. Coefficient $C_{m,n}(m,n=0,1,2,...)$ determined at $j=1$ and $j=2$ of the two sequences of recurrent systems must be identical. From the recurrent system (2.13) they are determined only under the condition $(\alpha+k_1,\beta+k_1)$ and $(\gamma+k_2,\delta+k_2)$ where $k_j(j=1,2)$ any natural numbers, are not indicators of homogeneous system (2.3) solution. Series convergence $f_j(x,y)(j=1,2)$ involves the convergence of right-hand side series (2.12). When fulfilling the above conditions, particular solution $Z_0(x,y)$ the inhomogeneous system (2.1) with coefficients of the form (2.2), when the singularity $(0,0)$ can be constructed regularly.

Remark 2.3. If $(\alpha+k_1,\beta+k_1)$ and $(\gamma+k_2,\delta+k_2)$, where $k_j(j=1,2)$ any positive integers are indices of homogeneous system solution, then we obtain a more complicated «resonance» case. This case requires additional investigation.

Remark 2.4. If in coefficients (2.2) constant $k=1$, then in the recurrent sequence starting from $f_{1,1}^{(j)}(\rho,\sigma)$ all expressions $f_{2,0}^{(j)}(\rho,\sigma)$,$f_{0,2}^{(j)}(\rho,\sigma)$,$f_{3,0}^{(j)}(\rho,\sigma)$, ... will be zero.

Remark 2.5. The transformation (1.2) of the considered case will lead to a simpler form $k=1$.

Thus, based on the above reasoning, we can conclude that the statement is true.

Lemma 2.1. Let inhomogeneous system consisting of two third-order equations (2.1) with coefficients in type (2.2), where the right-hand sides $g^{(7)}(x,y)$ and $q^{(7)}(x,y)$ analytic functions of two variables regular near the singularity $(x=0,y=0)$. Then system's particular solution (2.1) has the form of the right-hand side (2.11), if $(\alpha+k_1,\beta+k_1)$ and $(\gamma+k_2,\delta+k_2)$ does not coincide with any pair of solution indicators corresponding to the homogeneous system (2.3) for any natural $k_j(j=1,2)$.

2.3. Construction and study properties of specific system solutions

J. Kampe de Feriet [2; 155-162] provides a method for constructing systems of third and fourth orders consisting of two equations, using the system

$$\sum_{j+k=0}^{j+k=\omega+1}(\rho_{j,k}-\alpha_{j,k}x)x^j y^k p_{j,k}=0,$$

$$\sum_{j+k=0}^{j+k=\omega+1}(\sigma_{j,k}-\beta_{j,k}y)x^j y^k p_{j,k}=0. \qquad (2.14)$$

This technique ensures the compatibility of two equations systems hypergeometric type (2.3). The solutions of such systems are generalized by hypergeometric functions of two variables. Consider a particular special case [2; 159] of such system and we will study the properties of its solutions.

Theorem 2.5. The system of partial differential equations consisting two equations of third order

$$x^2Z_{xxx}+xyZ_{xxy}+(\gamma+\delta+1)xZ_{xx}+\delta yZ_{xy}+\gamma\delta Z_x-Z=0,$$

$$y^2Z_{yyy}+xyZ_{xyy}+(\gamma+\delta^{'}+1)yZ_{yy}+\delta^{'}xZ_{xy}+\gamma\delta^{'}Z_y-Z=0, \tag{2.15}$$

has nine linearly independent particular solutions

$$Z_1(x,y)=F(\cdot;\delta,\delta^{'},\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{1}{(\delta)_m(\delta')_n(\gamma)_{m+n}}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!},$$

$$Z_2(x,y)=y^{1-\delta'}\{1+\frac{x}{\delta(\gamma+1-\delta')}+\frac{y}{(2-\delta)(\gamma+1-\delta')}+\frac{xy}{\delta(2-\delta')(\gamma+1-\delta')(\gamma+2-\delta')}+...\},$$

$$Z_3(x,y)=x^{1-\delta}\{1+\frac{x}{(2-\delta)(\gamma+1-\delta)}+\frac{y}{\delta(\gamma+1-\delta)}+\frac{xy}{\delta'(2-\delta)(\gamma+1-\delta)(\gamma+2-\delta)}+...\},$$

$$Z_4(x,y)=y^{1-\gamma}\{1+\frac{x}{\delta}+\frac{y}{(2-\gamma)(\delta'+1-\gamma)}+\frac{xy}{2\delta(2-\gamma)(\delta'+1-\gamma)}+\frac{x^2}{2^2\delta(\delta+1)}+...\},$$

$$Z_5(x,y)=x^{1-\delta}y^{1-\delta'}\{1+\frac{x}{(2-\delta)(\gamma+2-\delta-\delta')}+\frac{y}{(2-\delta')(\gamma+2-\delta-\delta')}+$$

$$+\frac{xy}{(2-\delta)(2-\delta')(\gamma+2-\delta-\delta')(\gamma+3-\delta-\delta')}+...\},$$

$$Z_6(x,y)=x^{1-\delta}y^{1-\gamma}\{1+\frac{x}{(2-\delta)^2}+\frac{y}{(2-\delta)(\delta'+1-\gamma)}+\frac{xy}{(2-\delta)^2(3-\delta)(2-\gamma)(1+\gamma-\delta')}+...\},$$

$$Z_7(x,y)=x^{1-\gamma}\{1+\frac{x}{(2-\gamma)(\delta+1-\gamma)}+\frac{y}{\delta'}+\frac{xy}{2\delta'(2-\gamma)(1+\delta-\gamma)}+...\},$$

$$Z_8(x,y)=x^{1-\gamma}y^{1-\delta'}\{1+\frac{x}{(2-\gamma)(2-\delta')(\delta+1-\gamma)}+\frac{y}{(2-\delta')^2}+$$

$$+\frac{xy}{(2-\delta')^2(2-\gamma)(\delta+1-\gamma)(3-\delta')}+...\},$$

$$Z_9(x,y)=x^{1-\gamma}y^{1-\gamma}\{1+\frac{x}{(2-\gamma)^2(1+\delta-\gamma)}+\frac{y}{(2-\gamma)^2(1+\delta-\gamma)}+$$

$$+\frac{xy}{(2-\gamma)^3(3-\gamma)(1+\delta-\gamma)(1+\delta'-\gamma)}+...\}. \tag{2.16}$$

Proof. The system is consistent in construction system (2.14). We will construct solutions using the Frobenius-Latvian method, based on the results of clause 2.1. The system of defining equations for the singularity $(0,0)$:

$$f_{00}^{(1)}(\rho,\sigma)=\rho(\rho-1)(\rho-2)+\rho(\rho-1)\sigma+(\gamma+\delta+1)\rho(\rho-1)+\delta\rho\sigma+\gamma\delta=0,$$

$$f_{00}^{(1)}(\rho,\sigma)=\sigma(\sigma-1)(\sigma-2)+\rho\sigma(\sigma-1)+(\gamma+\delta^{'}+1)\sigma(\sigma-1)+\delta^{'}\rho\sigma+\gamma\delta^{'}=0 \tag{2.17}$$

has nine pairs of roots:

$$1.(\rho_1=0,\sigma_1=0);2.(\rho_1=0,\sigma_2=1-\delta^{'});3.(\rho_2=1-\delta,\sigma_1=0);$$

$$4.(\rho_1=0,\sigma_3=1-\gamma);5.(\rho_2=1-\delta,\sigma_2=1-\delta^{'});6.(\rho_2=1-\delta,\sigma_3=1-\gamma);$$

$$7.(\rho_3=1-\gamma,\sigma_1=0);8.(\rho_3=1-\gamma,\sigma_2=1-\delta^{'});9.(\rho_3=1-\gamma,\sigma_3=1-\gamma).$$

Using the system of recurrent sequences (2.8), we determine the unknown coefficients of the series (2.4) sequentially substituting the values of the roots pairs $(\rho_t, \sigma_t)(t = \overline{1,9})$ of system defining equations (2.17).

In this way, we obtain nine linearly independent particular solutions (2.16). The theorem is proved.

Now we proceed to construct a particular solution of an inhomogeneous system

$$x^2 Z_{xxx} + xyZ_{xxy} + (\gamma + \delta + 1)xZ_{xx} + \delta y Z_{xy} + \gamma\delta Z_x - Z = g^{(7)}(x, y),$$

$$y^2 Z_{yyy} + xyZ_{xyy} + (\gamma + \delta' + 1)yZ_{yy} + \delta' xZ_{xy} + \gamma\delta' Z_y - Z = q^{(7)}(x, y), \tag{2.18}$$

where the right side has the type

$$g^{(7)}(x, y) = \gamma^2\delta\delta' + \frac{1}{\gamma\delta}x + \frac{1}{\gamma\delta'}y,$$

$$q^{(7)}(x, y) = \gamma^2\delta\delta' + \frac{1}{\gamma\delta}x + \frac{1}{\gamma\delta'}y. \tag{2.19}$$

Theorem 2.6. The general solution of the inhomogeneous system (2.18) with the right part (2.19) is represented as the sum (2.10) of the total solution $\overline{Z}(x, y)$ the corresponding homogeneous system (2.15) and a particular solution $Z_0(x, y)$ heterogeneous system (2.18).

Indeed, by virtue of Theorem 2.3, the general solution of the corresponding homogeneous system (2.15) is represented as the sum of nine linearly independent particular solutions $Z_j(x, y), (j = \overline{1,9})$ (2.16):

$$\overline{Z}(x, y) = \sum_{m,n=0}^{\infty} C_j Z_j(x, y), (j = \overline{1,9}).$$

It remains to build a particular solution $Z_0(x, y)$ of heterogeneous system (2.18) with the right part (2.19) using the method of uncertain coefficients described in clause 2.2 based on the right part of task $g^{(7)}(x, y)$ and $q^{(7)}(x, y)$ in view of

$$Z_0(x, y) = C_{0,0} + C_{1,0}x + C_{0,1}y. \tag{2.20}$$

Substituting (2.20) into (2.15) we determine the unknown coefficients: $C_{0,0} = -(\gamma^2\delta\delta' + 1)$, $C_{1,0} = -\frac{1}{\gamma\delta}$, $C_{0,1} = -\frac{1}{\gamma\delta'}$ obtain a particular solution of the inhomogeneous system (2.18) with the right side $g^{(7)}(x, y)$ and $q^{(7)}(x, y)$ in view of

$$Z_0(x, y) = -(\gamma^2\delta\delta' + 1) - \frac{x}{\gamma\delta} - \frac{y}{\gamma\delta'}. \tag{2.21}$$

Therefore, the general solution of inhomogeneous system (2.18) with the right-hand side (2.19) is represented as

$$Z(x, y) = \overline{Z}(x, y) + Z_0(x, y) = \sum_{j=1}^{9} C_j Z_j(x, y) - (\gamma^2\delta\delta' + 1) - \frac{x}{\gamma\delta} - \frac{y}{\gamma\delta'}, \tag{2.22}$$

where $Z_j(x, y), (j = \overline{1,9})$ particular solutions corresponding to homogeneous system (2.16).

It is easy to verify that the sum of two equations (2.18) also satisfy the particular solution of the inhomogeneous system (2.21).

Theorem 2.7. A particular solution of the inhomogeneous system (2.21) is also a solution of a third-order partial differential equation

$$x^2 Z_{xxx} + xyZ_{xxy} + y^2 Z_{yyy} + xyZ_{xyy} + (\gamma + \delta + 1)xZ_{xx} + (\gamma + \delta' + 1)yZ_{yy} +$$

$$+(\delta y+\delta' x)Z_{xy}+\gamma\delta Z_x+\gamma\delta' Z_y-2Z=2(\gamma^2\delta\delta'+\frac{x}{\gamma\delta}+\frac{y}{\gamma\delta'}) \tag{2.23}$$

obtained by adding the two equations of the inhomogeneous system (2.18).

Theorem 2.8. The general solution of the inhomogeneous system (2.22) is a solution of a third-order partial differential equation (2.23) obtained by adding two equations of the inhomogeneous system (2.18).

Conclusions: Thus, in this paper we have been studied the possibility of constructing solutions uncharted inhomogeneous system of differential equations in partial derivatives, consisting of two third-order equations.

1. To solve the corresponding homogeneous system (2.3) was applied the Frobenius-Latyshev method. Theorems 2.1 and 2.2 on the number of linearly independent solutions of the homogeneous system (2.3) have been proved. The main stages of building solutions by the Frobenius-Latyshev method are given. It was found that when the roots of the defining systems (2.6) and (2.7) with respect to characteristics $(0,0)$ and (∞,∞) simple, the homogeneous system has nine linearly independent particular solutions of species (2.4) or (2.5) near the singularities of $(0,0)$ and (∞,∞).

2. A theorem on the construction a general solution of the inhomogeneous system (2.1) with coefficients in type (2.2) is formulated. To this end, for such systems, for the first time the method of uncertain coefficients is used. Yu.I. Sikorskiy extended method of Frobenius-Latysheva to linear ordinary inhomogeneous differential equations [12, 13]. It has been shown that for solving various problems of thermoelasticity, the method of undetermined coefficients has an advantage over the method of arbitrary constant [14]. For example, when solving the inhomogeneous Bessel equation, particular solution is a linear combination of Lommel functions [15]. It should be noted that in this monograph thoroughly studied the possibility of constructing the solutions known classical ordinary differential equations with the right-hand side, the decisions of which are special functions and orthogonal polynomials in one variable. In the case of the studied systems, the research has not reached such a level.

3. A specific example is considered, where a homogeneous system is constructed by the method of J. Kampe de Feriet [2; 155-169]. Nine linearly independent solutions (2.16) obtained by the Frobenius-Latysheva. To build a general solution of system (2.18) with the right-hand side (2.19), the undetermined coefficients method is applied.

4. It is also shown that application of uncertain coefficients method allows to obtain solutions of one inhomogeneous partial differential equation of the third order (2.23) associated with the studied specific system (2.18).

5. The first particular solution of homogeneous system (2.15) relates to degenerate generalized hypergeometric series of two variables Clausen type

$$F(\cdot;\delta,\delta',\gamma;x,y)=1+\frac{1}{\delta\gamma}x+\frac{1}{\delta'\gamma}y+\frac{xy}{\delta\delta'\gamma(\gamma+1)}+\frac{1}{\delta(\delta+1)\gamma(\gamma+1)}\cdot\frac{x^2}{2!}+$$
$$+\frac{1}{\delta'(\delta'+1)\gamma(\gamma+1)}\cdot\frac{y^2}{2!}+... \tag{2.24}$$

The properties of this series remain understood. Consider the differential properties of the series (2.24).

Theorem 2.9. First m and n the of the series (2.24) are represented as:

$$1.\frac{\partial}{\partial x}F(\cdot;\delta,\delta',\gamma;x,y)=\frac{1}{\delta\gamma}F(\cdot;\delta+1,\delta',\gamma+1;x,y),$$

$$2.\frac{\partial}{\partial y}F(\cdot;\delta,\delta',\gamma;x,y)=\frac{1}{\delta'\gamma}F(\cdot;\delta,\delta'+1,\gamma+1;x,y),$$

$$3.\frac{\partial^2F(\cdot;\delta,\delta',\gamma;x,y)}{\partial x^2}=\frac{F(\cdot;\delta+2,\delta',\gamma+2;x,y)}{\delta(\delta+1)\gamma(\gamma+1))},$$

$$4. \frac{\partial^2 F(\cdot;\delta,\delta',\gamma;x,y)}{\partial y^2} = \frac{F(\cdot;\delta,\delta'+2,\gamma+2;x,y)}{\delta'(\delta'+1)\gamma(\gamma+1))},$$

$$5. \frac{\partial^2 F(\cdot;\delta,\delta',\gamma;x,y)}{\partial x\partial y} = \frac{F(\cdot;\delta+1,\delta'+1,\gamma+2;x,y)}{\delta\delta'\gamma(\gamma+1)},$$

$$6. \frac{\partial^{m_1} F(\cdot;\delta,\delta',\gamma;x,y)}{\partial x^{m_1}} = \frac{F(\cdot;\delta+m_1,\delta',\gamma+m_1;x,y)}{\delta(\delta+1)...(\delta+m_1)\gamma(\gamma+1)...(\gamma+2)},$$

$$7. \frac{\partial^{m_2} F(\cdot;\delta,\delta'+1,\gamma+m_2;x,y)}{\partial y^{m_2}} = \frac{F(\cdot;\delta,\delta'+m_2,\gamma+m_2;x,y)}{\delta'(\delta'+1)...(\delta'+m_2)\gamma(\gamma+1)...(\gamma+m_2)},$$

$$8. \frac{\partial^{m_1+m_2} F(\cdot;\delta,\delta',\gamma;x,y)}{\partial x^{m_1}\partial y^{m_2}} =$$

$$\frac{F(\cdot;\delta+m_1,\delta'+m_2,\gamma+m_1+m_2;x,y)}{\delta(\delta+1)...(\delta+m_1)\delta'(\delta'+1)...(\delta'+m_2)\gamma(\gamma+1)...(\gamma+m_1+m_2)}.$$

Confine ourselves with building a single solution corresponding to the indicator $(\rho_1 = 0, \sigma_1 = 0)$. Similarly, the differential properties of the remaining series in (2.16) can be derived. The output of these differential properties further facilitates the proof of theorem addition and multiplication, while others recurrence relations associated with degenerate generalized hypergeometric series.

References

1 Wilczynski E.J. Projective differential geometry of Curves and Ruled surfaces / E.J. Wilczynski. — Leipzig: Leubner, 1966. — 120 p.

2 Appell P. Fonctions hypergeometriques et hypesperiques / P. Appell, M.J. Kampe de Feriet. — Paris: Gauthier Villars, 1926. — 434 p.

3 Slater L.J. Hypergeometric functions / L.J. Slater, D. Lit. — Ph.D. Generalized: Cambridge At the university Press, 1966. — 292 p.

4 Beytmen G. Higher Transcendental Functions / G. Beytmen, A. Erdeyli.– M.: Science, 1965. — 294 p.

5 Тасмамбетов Ж.Н. Об одной системе дифференциальных уравнений в частных производных второго порядка / Ж.Н. Тасмамбетов // Украин. мат. журн. — 1992. — Т. 44, № 1. — С. 427–430.

6 Тасмамбетов Ж.Н. Построение нормальных и нормально-регулярных решений специальных систем дифференциальных уравнений в частных производных второго порядка / Ж.Н. Тасмамбетов. — Актобе: ИП Жандилдаева С.Т., 2015. — 464 с.

7 Тасмамбетов Ж.Н. Многочлены Лежандра двух переменных как решения приведенной системы в частных производных / Ж.Н. Тасмамбетов // Проблемы оптимизации сложных систем: материалы X Междунар.-азиат. шк.-сем. (25 июля – 5 августа 2014 г.). — Ч. 2. — С. 655-661. — Иссык-Куль, 2014.

8 Талипова М.Ж. Построение нормальных решений неоднородных систем дифференциальных уравнений в частных производных второго порядка: автореф. ... канд. дис. / М.Ж. Талипова. — Алматы, 2007. — 20 с.

9 Тасмамбетов Ж.Н. Нахождение нормально-регулярных решений неоднородной системы дифференциальных уравнений в частных производных второго порядка / Ж.Н. Тасмамбетов, М.Ж. Талипова // Изв. МОН НАН РК. Сер. физ.-мат. — 2003. — № 3. — С. 47–55.

10 Ince E.L. Ordinary Differential Equations / E.L. Ince. — London: New York & Bombay; Longmans, GreenCo, 1926. — 14 p.

11 Kampe de Feriet M.J. Fonctions de la Physique Mathematique Fascicule X / M.J. Kampe de Feriet, R.Kambell, Pet'o G., Fohell' T. — Sous la direction de. Centre national de la Recherchй scientifique, 1957. — 102 p.

12 Сикорский Ю.И. О неоднородных линейных дифференциальных уравнениях в регулярном случае / Ю.И. Сикорский, Н.И. Терещенко // Математическая физика: Респ. межведом. сб. — Киев, 1972. — С. 133–137.

13 Сикорский Ю.И. Об условиях существования замкнутых решений линейного неоднородного дифференциального уравнения с полиномиальными коэффициентами / Ю.И. Сикорский, Н.И. Терещенко // Украин. мат. журн. — 1973. — Т. 25, № 4. — С. 558–563.

14 Sidlyar M.M. Thermostressed state of a round Cylinder in case of interaction of the temperature field and deformations / M.M. Sidlyar, Yu.I. Sikorsky, N.I. Tereshenko. — Kiev: Reports of the Ukrainian Academy of Sciences.

15 Babister A.W. Trancendental function satisfying nonhomogeneous linear differential equations / A.W. Babister. — New York–London, 1967. — 414 p.

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Дербес туындылы үшінші ретті дифференциалдық біртектіемес жүйелер шешімі

Үшінші ретті дербес туындылы біртектіемес дифференциалдық теңдеулер жүйесінің шешімдерін құру мүмкіндіктері зерттелген. Фробениус-Латышева әдісімен сәйкес біртекті жүйенің жалпы және дербес шешімдерін құру жан-жақты қарастырылған. Сызықты-тәуелсіз дербес шешімдер саны анықталған. Мұндай жүйелерге анықталмаған коэффициенттер әдісін қолдану ерекешеліктері айқындалған және біртекті емес жүйенің жалпы шешімі туралы теорема дәлелденген. Нақты мысалда, осындай жолмен құрылған біртекті емес жүйенің дербес шешімі, қарастырылған мысалдағы теңдеулер жүйесінің екі теңдеуін қосындылаудан алынған үшінші ретті, біртекті емес бір теңдеудің де шешімі болатындығы көрсетілген. Сәйкес біртекті жүйенің дербес шешімдерінің бірі екі айнымалының Клаузен текті туындалған жалпыланған гипергеометриялық қатар түріне жатады. Мұндай жалпыланған гипергеометриялық қатарлардың қасиеттері әзірше аз зерттелген.

Кілт сөздер: біртекті емес жүйе, регулярлы шешім, ерекше нүктелер, белгісіз коэффициенттер әдісі, жүйе, теңдеулер, теорема.

Ж.Н. Тасмамбетов, Ж.К. Убаева

Решение неоднородных систем дифференциальных уравнений в частных производных третьего порядка

Изучены возможности построения решений неоднородной системы дифференциальных уравнений в частных производных третьего порядка. Методом Фробениуса-Латышевой всесторонне исследовано построение общего и частных решений соответствующей однородной системы. Установлены виды решения вблизи особых кривых. Определено количество линейно-независимых частных решений. Доказана теорема о представлении общего решения неоднородной системы и раскрыты особенности применения метода неопределенных коэффициентов для таких систем. На конкретном примере показано, что построенные таким образом частные решения неоднородной системы являются решениями и одного неоднородного уравнения третьего порядка, полученного путем сложения двух уравнений рассмотренного примера. Одно из частных решений соответствующей однородной системы относится к виду вырожденного обобщенного гипергеометрического ряда типа Клаузена двух переменных. Свойства таких обобщенных гипергеометрических рядов остаются мало изученными.

Ключевые слова: неоднородная система, регулярное решение, особенность, метод неопределенных коэффициентов, система, уравнения, теорема.

References

1 Wilczynski, E.J. (1966). *Projective differential geometry of Curves and Ruled surfaces.* Leipzig: Leubner.

2 Appell, P., & Kampe de Feriet, M.J. (1926). *Fonctions hypergeometriques et hypesperiques.* Paris: Gauthier Villars.

3 Slater, L.J., & Lit, D. (1966). Generalized Hypergeometric functions. Ph.D. Generalized: Cambridge at the university Press.

4 Beytmen, G., & Erdeyli, A. (1965). *Higher Transcendental Functions.* Moscow: Nauka.

5 Tasmambetov, Zh.N. (1992). Ob odnoi sisteme differentsialnykh uravnenii v chastnykh proizvodnykh vtoroho poriadka [On a system of partial differential equations of second order]. *Ukrainskii matematicheskii zhurnal — Ukrainian Mathematical Journal, 44, 3,* 427-430 [in Russian].

6 Tasmambetov, Zh.N. (2015). *Postroenie normalnykh i normalno-rehuliarnykh reshenii spetsialnykh sistem differentsialnykh uravnenii v chastnykh proizvodnykh vtoroho poriadka [Construction of normal and normally-regular solutions of special systems of partial equations of second order].* Aktobe: IP Zhanadildaeva S.T. [in Russian].

7 Tasmambetov, Zh.N. (2014). Mnohochleny Lenzhandra dvukh peremennykh kak resheniia privedennoi sistemy v chastnykh proizvodnykh [The Legendre polynomials of two variables as solutions of the reduced system in partial derivatives]. Problems of optimization of complex systems: *Acta Internationalis decima-Asian schola-seminario (25 Iulii–5 August 2014 hoda) — Proceedings of the tenth International-Asian school-seminar* (Part II, p. 655–661) [in Russian].

8 Talipova, M.Zh. (2007). Postroenie normalnykh reshenii neodnorodnykh sistem differentsialnykh uravnenii v chastnykh proizvodnykh vtoroho poriadka [Constructing normal solutions for inhomogeneous systems of second-order partial differential equations]. *Extended abstract of Doctor's thesis.* Almaty [in Russian].

9 Tasmambetov, Zh.N., & Talipova, M.Zh. (2003). Nakhozhdenie normalno-rehuliarnykh reshenii neodnorodnoi sistemy differentsialnykh uravnenii v chastnykh proizvodnykh vtoroho poriadka [Finding normally-regular solutions of an inhomogeneous system of second-order partial differential equations]. *Izvestiia MON NAN RK. Seriia fiziko-matematicheskaia — News of MON NAS RK. Physical-mathematical series, 3,* 47-55 [in Russian].

10 Ince, E.L. (1926). *Ordinary Differential Equations.* London: New York & Bombay; Longmans, GreenCo.

11 Kampe de Feriet, M.J., & Kambell, R., & Pet'o, G., & Fohell', T. (1957). Fonctions de la Physique Mathematique. Fascicule IX. Centre national de la Recherchй scientifique.

12 Sikorsky, Yu.I., & Tereshchenko, N.I. (1972). O neodnorodnykh lineinykh differentsialnykh uravneniiakh v rehuliarnom sluchae [On inhomogeneous linear differential equations in the regular case]. *Mathematicheskaia fizika: Respublikanskii mezhvedomstvennyi sbornik — Republican interdepartmental collection,* 133-137 [in Russian].

13 Sikorsky, Yu.I., & Tereshchenko, N.I. (1973). Ob usloviiakh sushchestvovaniia zamknutykh reshenii lineinoho neodnorodnoho differentsialnoho uravneniia s polinomialnymi koeffitsientami [On the conditions for the existence of closed decision by the linear inhomogeneous differential equation with polynomial coefficients]. *Ukrainskii matematicheskii zhurnal — Ukrainian Mathematical Journal, 25, 4,* 558-563 [in Russian].

14 Sidlyar, M.M., & Sikorsky, Yu.I., & Tereshenko, N.I. (1971). Thermostressed state of a round Cylinder in case of interaction of the temperature field and deformations. Kiev: Reports of the Ukrainian Academy of Sciences.

15 Babister, A.W. (1967). Trancendental function satisfying nonhomogeneous linear differential equations. New York-London.

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Model-theoretical questions of the Jonsson spectrum

In this paper, new concepts are defined in the framework of the study of Jonsson spectrum. We consider a spectrum with respect to the concept of cosemanticness, which is a generalization of elementary equivalence in the class of inductive, generally speaking, incomplete theories. Also, with the help of Jonsson spectrum, the actual directions of the study of Jonsson theories and their model classes are determined, namely, the study of classical questions of model theory, such as the completeness, model completeness, model companion of within the framework of the above conditions, which define a fairly wide subclass of inductive theories, and which Jonsson theories. Therefore, in studying the model-theoretical properties of Jonsson spectrum, we need to clarify the definition of those concepts that naturally arise when we move from the concept of elementary equivalence to the concept of cosemanticness, moreover, both theories and models. Some model-theoretical properties of the Jonsson spectrum are considered. When considering the Jonsson spectrum, all the tasks that are posed in this article make sense and their solution can be useful for solving related problems, because this problem is actively studied in the field of Jonsson theories.

Keywords: Jonsson theory, Jonsson spectrum, cosemanticness, completeness, similarity.

This article is devoted to the study of model-theoretical questions of the Jonsson spectrum. The concept of the Jonsson spectrum arose in the study of the Jonsson invariants of abelian groups and modules [1, 2]. This problem arises in the study of an arbitrary Jonsson theories of an arbitrary signature. It is well known that Jonsson theories, generally speaking, are not complete theories. In this regard, it becomes interesting to generalize the concept of elementary equivalence of two models. The role of this generalization is played by the concept of cosemanticness of two models. The cosemantic nature of models is a consequence of the cosemantic nature of Jonsson theories to the model classes of which these models belong. In connection with the concept of cosemantics, the question arises of the completeness of the theories under consideration, as well as their perfectness. Moreover, as is known by virtue of the criterion of perfectness [3; 158] Jonsson theories, if the Jonsson theory is perfect, the class of its existentially closed models is an elementary class. Perfect Jonsson theories are well-arranged in the sense that their semantic invariant, namely their center, is a model companion. But nevertheless, if we move from the Jonsson theory to an arbitrary Jonsson spectrum of one of the models of this theory, then this spectrum can contain both perfect and imperfect theories. Therefore, in studying the model-theoretical properties of Jonsson spectrum, we need to clarify the definition of those concepts that naturally arise when we move from the concept of elementary equivalence to the concept of cosemantics, moreover, both theories and models. It is clear that it would not be easy to proceed directly to the study of Jonsson spectrum in a general form due to the above difficulties. In this regard, we propose in a certain way to limit the studied class of problems to the framework that seems to us natural and interesting from the point of view of this problem.

Thus, the main purpose of this article is to identify certain concepts that are new in the study of Jonsson spectrum, and with their help we want to determine the actual directions of the study of Jonsson theories and their model classes.

All the above concepts are directly related to the study of Jonsson theories and their classes of models. This issue is being actively studied and in this article we cannot immediately cover all areas of these studies. Nevertheless, here is a list of links that are related to the concepts considered in this article [4–15].

We give the necessary definitions.

Let σ be some signature, L be the set of all formulas of signature σ, i. e. a language of this signature. Let $\mathcal{A}$ be an arbitrary model of signature σ, i. e. $\mathcal{A} \in Mod\,\sigma$. Let us call *the Jonsson spectrum* of model $\mathcal{A}$ a set:

$$JSp(\mathcal{A}) = \{T \mid T \text{ is Jonsson theory in language } \sigma \text{ and } \mathcal{A} \in Mod\,T\}.$$

Denote by $JSp_{\Gamma}(\mathcal{A}) = \{T \mid T \text{ is } \Gamma\text{-complete Jonsson theory in language } \sigma \text{ and } \mathcal{A} \in Mod\,T\}$, where Γ is type of a prenex prefix after reduction a set of all sentences of signature σ to a prenex normal form of.

The relation of cosemanticness on a set of theories is an equivalence relation. Then $JSp(\mathcal{A})/_{\bowtie}$ is the factor set of Jonsson spectrum of the model $\mathcal{A}$ with respect to $\bowtie$. Similarly, we can consider the factor set $JSp_{\Gamma}(\mathcal{A})/_{\bowtie}$.

Let $[T] \in JSp(\mathcal{A})/_{\bowtie}$. Since every theory $\Delta \in [T]$ has $\mathcal{C}_{\Delta} = \mathcal{C}_T$, then the semantic model of class $[T]$ will be called the semantic model of a theory T: $\mathcal{C}_{[T]} = \mathcal{C}_T$. The center of Jonsson class $[T]$ will be called an elementary theory $[T]^*$ of its semantic model $\mathcal{C}_{[T]}$, i. e. $[T]^* = \mathrm{Th}(\mathcal{C}_{[T]})$ and $[T]^* = \mathrm{Th}(\mathcal{C}_{\Delta})$ for every $\Delta \in [T]$.

Denote by $E_{[T]} = \bigcup\limits_{\Delta \in [T]} E_{\Delta}$ the class of all existentially closed models of class $[T] \in JSp(\mathcal{A})/_{\bowtie}$. Note that $\bigcap\limits_{\Delta \in [T]} E_{\Delta} \neq \varnothing$, since at least for every $\Delta \in [T]$ we have $\mathcal{C}_{[T]} \in E_{\Delta}$.

Let $\mathcal{A}$ and $\mathcal{B}$ are models of the same signature.

Definition 1. We will say that a model $\mathcal{A}$ is Jonsson elementarily equivalent to a model $\mathcal{B}$ ($\mathcal{A} \underset{J}{\equiv} \mathcal{B}$) if $JSp(\mathcal{A}) = JSp(\mathcal{B})$.

Considering the factorization, we can give the following definition.

Definition 2. We say that a model $\mathcal{A}$ is JSp-cosemantic to a model $\mathcal{B}$ ($\mathcal{A} \underset{JSp}{\bowtie} \mathcal{B}$) if $JSp(\mathcal{A})/_{\bowtie} = JSp(\mathcal{B})/_{\bowtie}$. Accordingly, we say that a model $\mathcal{A}$ is JSp-cosemantic to a model $\mathcal{B}$ regarding Γ and write down it $\mathcal{A} \overset{\Gamma}{\underset{JSp}{\bowtie}} \mathcal{B}$ if $JSp_{\Gamma}(\mathcal{A})/_{\bowtie} = JSp_{\Gamma}(\mathcal{B})/_{\bowtie}$.

Definition 3. The class $[T] \in JSp(\mathcal{A})/_{\bowtie}$ is called elementarily closed if $\forall [T]' \in JSp(\mathcal{A})/_{\bowtie}$: $[T]' \neq [T] \Rightarrow \Rightarrow E_{[T]} \cap E_{[T]'} = \emptyset$.

Definition 4. The class $[T] \in JSp(\mathcal{A})/_{\bowtie}$ is called locally convex if $Th_{\forall\exists}(\bigcap\limits_{\Delta \in [T]} E_{\Delta})$ is a Jonsson theory and convex if $Th_{\forall\exists}(A) \in JSp(\mathcal{A})$.

Definition 5. The class $[T] \in JSp(\mathcal{A})/_{\bowtie}$ is called companion-convex if the theory $\nabla = Th_{\forall\exists}(\bigcap\limits_{\Delta \in [T]} E_{\Delta})$ is a Jonsson theory and has a model companion.

We can define the completeness of the class $[T]$ as follows (Definition 4), and all four types of completeness are independent of each other and can combine. An interesting problem is the transfer of results from the Jonsson theory to the Jonsson spectrum, when the completeness of the Jonsson theory is replaced by the following types of completeness and their combinations.

Definition 6. The class $[T]$ is called a Γ-complete class if the following conditions are true:

1) $\forall \mathcal{A}, \mathcal{B} \in E_{[T]}$, $\mathcal{A} \equiv_{\Gamma} \mathcal{B}$;

2) $\forall \mathcal{A}, \mathcal{B} \in E_{[T]}$, $\mathcal{A} \underset{JSp}{\bowtie} \mathcal{B}$ and $\forall \Delta \in [T]$, $\Delta - \Gamma$-complete.

3) $\forall \varphi \in \Gamma$, $\forall \Delta \in [T]$, $\Delta \vdash \varphi$ or $\Delta \vdash \neg\varphi \Leftrightarrow \forall \mathcal{A}, \mathcal{B} \in Mod\Delta$, $\forall \Delta \in [T]$, $\mathcal{A} \equiv_{\Gamma} \mathcal{B}$;

4) $\forall \varphi \in \Gamma$, $\forall \mathcal{A}, \mathcal{B} \in E_{[T]}$, $\mathcal{A} \models \varphi \Leftrightarrow \mathcal{B} \models \varphi$.

It is well known that the concepts of completeness and model completeness do not coincide, but as shown by [3] in the case of a perfect Jonsson theory, these concepts coincide for the Jonsson theory under consideration. Therefore, in going over to the problem of the Jonsson spectrum, we must take into account that in the case of an imperfect class these concepts do not coincide.

Definition 7. The class $[T]$ is model complete $\leftrightharpoons \forall \Delta \in [T]$, Δ is model complete.

Theorem 1. The class $[T]$ is model complete, if $\forall \mathcal{A}, \mathcal{B} \in E_{[T]}$, $\forall$ monomorphism $f : \mathcal{A} \to \mathcal{B}$ is elementary if and only if $\forall \varphi \in L, \exists \psi \in \forall \cap \exists : [T] \vdash (\varphi \sim \psi)$ $[T] \vdash (\varphi \sim \psi) \leftrightharpoons \forall \Delta \in [T], \Delta \vdash (\varphi \sim \psi)$.

Lemma 1. If $\Delta \in [T]$ and Δ is imperfect, then $\exists \mathcal{B} \in E_\Delta$, $\mathcal{B} \in E_\Delta$ for some $\Delta' \in [T]$.

Definition 8. $[T]_1$, $[T]_2$ existentially mutually model complete ($[T]_1 \leftrightarrow [T]_2$) if $\forall \mathcal{B} \in E_{[T]_1}$, $\exists \mathcal{B}' \in E_{[T]_2} : \mathcal{B} \overset{\cong}{\to} \mathcal{B}'$ and the converse is true.

Lemma 2. $[T]_1 \leftrightarrow [T]_2 \Leftrightarrow Th_\forall(C_{[T]_1}) = Th_\forall(C_{[T]_2})$.

Let us consider some properties of the Jonsson spectrum at fixed completeness (a special case of Definition 6 (2)).

Let σ be an arbitrary signature, $\mathcal{A} \in Mod\sigma$, $JSp(\mathcal{A})/_{\bowtie}$, $\mathcal{A} \in ModT$;

$Mod[T] = \{\mathcal{A} \in Mod\sigma | \mathcal{A} \models T_i, \forall T_i \in [T]\}$.

$Mod(JSp(\mathcal{A})) = \{\mathcal{B} \in Mod\sigma | \mathcal{B} \models T_j, \forall T_j \in JSp(\mathcal{A})\}$.

$Mod(JSp(\mathcal{A})/_{\bowtie}) = \{\mathcal{B} \in Mod\sigma | \mathcal{B} \models [T], \forall [T] \in JSp(\mathcal{A})/_{\bowtie}\}$.

The E_T is elementary class $\Leftrightarrow$ T has a model companion.

Definition 9. $[T]$ has a model companion if any E_{T_i} are an elementary class, $T_i \in [T]$.

$\mathcal{A} \underset{JSp}{\bowtie} \mathcal{B} \Leftrightarrow JSp(\mathcal{A})/_{\bowtie} = JSp(\mathcal{B})/_{\bowtie}$.

$JSp(\mathcal{A})/_{\bowtie} = \{T | T - \text{Jonsson theory}, \mathcal{A} \models T\}$.

The $[T]$ is complete $\rightleftharpoons \forall \mathcal{A}, \mathcal{B} \in E_{[T]}$, $\mathcal{A} \underset{JSp}{\bowtie} \mathcal{B}$.

$E_{[T]} = \bigcup\limits_{i \in I} E_{T_i}$.

The $[T]$ is perfect $\rightleftharpoons C_{[T]}$ is saturated.

The $[T]$ is $\forall\exists$-complete $\rightleftharpoons \forall \mathcal{A}, \mathcal{B} \in E_{[T]}$, $\mathcal{A} \underset{JSp}{\overset{\forall\exists}{\bowtie}} \mathcal{B}$.

$A \underset{JSp}{\overset{\forall\exists}{\bowtie}} \mathcal{B} \rightleftharpoons JSp_{\forall\exists}(\mathcal{A})/_{\bowtie} \rightleftharpoons JSp_{\forall\exists}(\mathcal{B})/_{\bowtie}$.

$JSp_{\forall\exists}(\mathcal{A})/_{\bowtie} = \{T | T - \forall\exists - \text{complete Jonsson theory}, \mathcal{A} \models T\}$

Recall the definitions [3] of syntactic and semantic similarity of Jonsson theories.

Definition 10. Let T_1 and T_2 are arbitrary Jonsson theories. We say, that T_1 and T_2 are Jonsson's syntactically similar, if exists a bijection $f : E(T_1) \longrightarrow E(T_2)$ such that

1) restriction f to $E_n(T_1)$ is isomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;

2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;

3) $f(v_1 = v_2) = (v_1 = v_2)$.

Definition 11. The pure triple $\langle C, AutC, SubC \rangle$ is called the Jonsson semantic triple, where C is semantic model of T, $AutC$ is the automorphism group C, $SubC$ is a class of all subsets of the carrier C, which are carriers of the corresponding is existentially-closed submodels of C.

Definition 12. Two Jonsson theories T_1 and T_2 are called Jonsson's semantically similar if their Jonsson semantic triples are isomorphic as pure triples.

Given these definitions, we define syntactic and semantic similarities of the Jonsson spectrum.

Definition 13. Let $\mathcal{A} \in \text{Mod}\sigma_1$, $\mathcal{B} \in \text{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is J-syntactically similar to class $[T]_2$ and denote $[T]_1 \overset{S}{\rtimes} [T]_2$ if for any theory $\Delta \in [T]_1$ there is theory $\Delta' \in [T]_2$ such that $\Delta \overset{S}{\rtimes} \Delta'$.

Definition 14. The pure triple $\langle C, Aut(\mathcal{C}), \overline{E}_{[T]} \rangle$ is called the J-semantic triple for class $[T] \in JSp(A)/_{\bowtie}$, where C is the semantic model of $[T]$, $AutC$ is the group of all automorphisms of C, $\overline{E}_{[T]}$ is the class of isomorphically images of all existentially closed models of $[T]$.

Definition 15. Let $\mathcal{A} \in \text{Mod}\sigma_1$, $\mathcal{B} \in \text{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is J-semantically similar to class $[T]_2$ and denote $[T]_1 \underset{S}{\rtimes} [T]_2$ if their semantically triples are isomorphic as pure triples.

In the case when it is possible to determine a sufficiently good geometry on the subsets of the semantic model of a class, we can use the technique of strongly minimal Jonsson sets. If we consider the enrichment, which will be hereditary, then we can consider the technique of central types, which form an essential base in this geometry.

To determine pregeometry and, accordingly, geometry for the Jonsson spectrum, we will work with subsets of the semantic model of each cosemanticness class of a given Jonsson spectrum. The following definitions are given for fixed subsets X, Y, A, B of some semantic model $\mathcal{C}$ of the fixed class $[T] \in JSp(\mathcal{A})/_{\bowtie}$.

The following definitions (16–28) are taken from [11]. These definitions are consistent with the above conventions regarding the subsets of the semantic model of each class of cosemanticness of a given Jonsson spectrum.

Definition 16. If (X, cl) is a Jonsson pregeometry, we say that A is Jonsson independent if $a \notin cl(A \setminus \{a\})$ for all $a \in A$ and that B is a J-basis for Y if $B \subseteq Y$ is J-independent and $Y \subseteq acl(B)$.

Definition 17. We say that a J-pregeometry (X, cl) is J-geometry if $cl(\emptyset) = \emptyset$ and $cl(\{x\}) = \{x\}$ for any $x \in X$.

If (X, cl) is a J-pregeometry, then we can naturally define a J-geometry. Let $X_0 = X \setminus cl(\emptyset)$. Consider the relation $\sim$ on X_0 given by $a \sim b$ iff $cl(\{a\}) = cl(\{b\})$. By exchange, $\sim$ is an equivalence relation. Let $\widehat{X}$ be $X_0/_\sim$. Define $\widehat{cl}$ on $\widehat{X}$ by $\widehat{cl}(A/_\sim) = \{b/\sim : b \in cl(A)\}$.

Definition 18. Let (X, cl) be J-pregeometry. We say that (X, cl) is trivial if $cl(A) = Y_{a\in A} cl\{a\}$ for any $A \subseteq X$. We say that (X, cl) is modular if for any finite-dimensional closed $Jdim(A \cup B) = Jdim(A) + Jdim(B) - -Jdim(A \cap B)$.

We say that (X, cl) is locally modular if (X, cl_a) is modular for some $a \in X$.

Definition 19. We say that (X, cl) is modular if for any finite-dimensional closed $A, B \subseteq X$

$$dim(A \cup B) = dimA + dimB - dim(A \cap B)$$

Definition 20. If $X = \mathcal{C}$ and (X, cl) is a modular, then the Jonsson theory T is called modular.

We work actually with the following types of sets.

Definition 21. Let $X \subseteq \mathcal{C}$. We will say that a set X is $\nabla - cl$-Jonsson subset of C, if X satisfies the following conditions:

1) X is ∇-definable set (this means that there is a formula from ∇, the solution of which in the $\mathcal{C}$ is the set X, where $\nabla \subseteq L$, that is ∇ is a view of formula, for example $\exists, \forall, \forall\exists$ and so on.);

2) $cl(X) = M$, $M \in E_T$, where cl is some closure operator defining a pregeometry over $\mathcal{C}$ (for example $cl = acl$ or $cl = dcl$).

Definition 22. An enrichment $\overline{T}$ of the Jonsson theory T is said to be permissible if any ∇-type (it mean that ∇ subset of language L_σ and any formula from this type belongs to ∇) in this enrichment is definable in the framework of $\overline{T}_{\sigma'}$-stability, where $\sigma' = \{P\} \bigcup \{c\}$.

Definition 23. The Jonsson theory is said to be hereditary, if in any of its permissible enrichment, any expansion of it in this enrichment will be Jonsson theory.

Let $S^{(1)}_\nabla(X)$ be the set of all complete 1-types over the set X, formulas which belong to ∇. Let $X \subseteq M$, $M \in E_T$.

Definition 24. Type $p \in S^{(1)}_\nabla(X)$ is called essential if for any set Y, $Y \subseteq N$, $N \in E_T$, such that $X \subseteq Y$ in T exists only unique type $q \in S^{(1)}_\nabla(Y)$ and the type q is a J-nonforking extension of type p.

Let $p, q \in S^{(1)}_\nabla(X)$, $\mathfrak{A} \in E_T$ and $X \subseteq A$. The relation $p \leq_A q$ is means that for any model $\mathfrak{B} \in E_T$, such that $\mathfrak{B} \supseteq \mathfrak{A}$, from the realizability of q in $B \backslash A$ implies the realizability of p in $B \backslash A$. The relation $p \equiv q$ means that for any model $\mathfrak{A} \in E_T$, $X \subseteq A$, has $p \leq_A q$ and $q \leq_A p$. We denote the set $\{q | q \in S^{(1)}_\nabla(X), p \equiv q\}$ by $[p]$, and the set $\{[p] | p \in S^{(1)}_\nabla(X)\}$ denote by $S^{(1)}_\nabla[X]$. We write $[p] \leq_A [q]$, if $p \leq_A q$. The types p, q are called independent if for any $\mathfrak{A} \in E_T$, $X \subseteq A$, don't have a place neither $p \leq_A q$, nor $q \leq_A p$. If p and q are independent, then we say that $[p]$ and $[q]$ are independent.

The following definition gives the concept of a basis among the above types.

Definition 25. The set $B = \{[p_i] \in S^{(1)}_\nabla[X] | i \in I\}$ is called base for $S^{(1)}_\nabla[X]$ if:

(1) $[p_i]$ and $[q_j]$ independent for $i \neq j$;

(2) for any $[q] \in S^{(1)}_{\nabla}[X]$ and $\mathfrak{A} \in E_T$, $X \subseteq A$, exists $i \in I$ such that $[p_i] \leq_A [q]$.

Definition 26. The base of the theory T is the base for $S^{(1)}_{\nabla}[\varnothing]$ (if it exists). The base B of T is called essential if for any $[p] \in B$ exists an essential type $q \in [p]$.

Definition 27. We will call the essential base of the types of Jonsson theory T geometric if the following conditions are satisfied:

1) $\forall p \in S^{(1)}_{\nabla}(X)$, where $X \subseteq \mathcal{C}$, $\mathcal{C}$ as above and $(\mathcal{C}, cl)$ – J-geometry;

2) the concept of independence in the sense of geometry generated by a strongly minimal central type will coincide with the concept of independence $(\mathcal{C}, cl)$ – J-geometry (coincidence of the concept of a base in terms of strong minimality, pregeometry and central types that form an essential base, wherein the orbits of the central types are their solutions in the semantic model).

Definition 28. Let $\mathfrak{M}$ be an existentially closed model of T and $\varphi(\overline{x})$ be a non-algebraic ∇-formula.

1. The set $\varphi(\mathfrak{M})$ is called J-minimal in $\mathfrak{M}$ if for all ∇-formulas $\psi(\overline{x})$ the intersection $\varphi(\mathfrak{M}) \wedge \psi(\mathfrak{M})$ is either finite or cofinite in $\varphi(\mathfrak{M})$.

2. The formula $\varphi(\overline{x})$ is J-strongly minimal if $\varphi(\overline{x})$ defines a J-minimal set in all existentially closed extensions of $\mathfrak{M}$. In this case, we also call the definable set $\varphi(\mathfrak{M})$ is J-strongly minimal.

3. A non-algebraic type in $S^{(1)}_{\nabla}(T)$ containing a J-strongly minimal formula is called J-strongly minimal.

4. A Jonsson theory T is J-strongly minimal if any its existentially closed model is J-strongly minimal.

One of the interesting questions in the classical model theory is the characterization of algebraically prime models. The complexity of this issue is that the concept of an algebraically prime model does not have a syntactic characterization, as is the case with a prime model. Our proposal is the use of Jonsson sets and the application of the rheostat principle. In the case of considering the Jonsson spectrum, all these problems make sense and their solution may be useful for solving related problems.

Recall the basic definitions associated with different types of prime and atomic models in the study of Jonsson theories.

The following definitions (29–34) are taken from [12].

Definition 29. The set A is called $(\nabla_1, \nabla_2) - cl$ atomic in the theory T, if

1) $\forall a \in A, \exists \varphi \in \nabla_1$ such that for any formula $\psi \in \nabla_2$ follows that φ is a complete formula for ψ and $C \models \varphi(a)$;

2) $cl(A) = M, M \in E_T$,

and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$ atomic model of theory T.

Definition 30. The set A is called weakly $(\nabla_1, \nabla_2) - cl$ is atomic in T, if

1) $\forall a \in A, \exists \varphi \in \nabla_1$ such that in $C \models \varphi(a)$ for any formula $\psi \in \nabla_2$ follow that $T \models (\varphi \to \psi)$ whenever $\psi(x)$ of ∇_2 and $C \models \psi(a)$;

2) $cl(A) = M, M \in E_T$,

and obtained model M is said to be weakly $(\nabla_1, \nabla_2) - cl$ atomic model of theory T.

It is easy to understand that definitions 29 and 30 are naturally generalized the notion of atomicity and weak atomicity to be ∇_1-atom and weak ∇_1-atom for any tuple of finite length from set A.

Let $i \in \{1, 2\}$, $M_i = cl(A_i)$, where $A_i = (\nabla_1, \nabla_2)$ is a $cl-$ atomic set. $a_0, ..., a_{n-1} \in A_1$, $b_0, ..., b_{n-1} \in A_2$.

Definition 31. (i) $(M_1, a_0, ..., a_{n-1}) \Rightarrow_{\nabla} (M_2, b_0, ..., b_{n-1})$ means that for every formula $\varphi(x_1, ..., x_{n-1})$ of ∇, if $M_1 \models \varphi(\overline{a})$, then $M_2 \models \varphi(\overline{b})$.

(ii) $(M_1, \overline{a}) \equiv_{\nabla} (M_2, \overline{b})$ means that $(M_1, \overline{a}) \Rightarrow_{\nabla} (M_2, \overline{b})$ and $(M_1, \overline{b}) \Rightarrow_{\nabla} (M_1, \overline{a})$.

Definition 32. A set A will be called $(\nabla_1, \nabla_2) - cl$-algebraically prime in the theory T, if

1) If A is $(\nabla_1, \nabla_2) - cl$-atomic set in T;

2) $cl(A) = M, M \in AP_T$,

and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$ algebraically prime model of theory T.

From the definition of an algebraically prime set in the theory T follows that the Jonsson theory T which has an algebraically prime set is automatically existentially prime. It is easy to understand that an example of such a theory is the theory of linear spaces.

Recall that the model A of theory T is called core if it is isomorphically embedded in any model of a given theory and this isomorphism exactly one.

Definition 33. The set A will be called $(\nabla_1, \nabla_2) - cl$-core in the theory T, if

1) A is (∇_1, ∇_2) a cl - atomic set in the theory T;

2) $cl(A) = M$, where M is a core model of theory T

and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$ core model of theory T.

Definition 34. (a) A - $(\nabla_1, \nabla_2) - cl$-atomic set in theory T is said to be A - $(\nabla_1, \nabla_2) - cl$-$\Sigma$-nice-set in theory T, $\forall A'$: A' - $(\nabla_1, \nabla_2) - cl$-atomic set in theory T, if

1) $cl(A) = M \in E_T \cap AP_T$,

and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$-$\Sigma$-nice model of theory T.

2) for all $a_0, ..., a_{n-1} \in A$, $b_0, ..., b_{n-1} \in A'$, if $(M, a_0, ..., a_{n-1}) \Rightarrow_\exists (M', b_0, ..., b_{n-1})$, then $\forall a_n \in A$, $\exists b_n \in A'$ such that $(M, a_0, ..., a_n) \Rightarrow_\exists (M', b_0, ..., b_n)$, where $M' = cl(A')$.

(b) A - $(\nabla_1, \nabla_2) - cl - \Sigma^*$-nice-set in theory T if the condition in (a) holds with '$\Rightarrow_\exists$' replaced both places it occurs by '$\equiv_\exists$' and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$-$\Sigma^*$-nice model of theory T.

(c) A - $(\nabla_1, \nabla_2) - cl - \Delta$-nice set in theory T if the condition in (a) holds with '$\Rightarrow_\Delta$' replaced both places it occurs by '$\equiv_\Delta$', where $\Delta \subseteq L$, $\Delta = \forall \cap \exists$.
and obtained model M is said to be $(\nabla_1, \nabla_2) - cl - \Delta$-nice model of theory T.

Principle of «rheostat»

Let two countable models A_1, A_2 of some Jonsson theory T be given. Moreover, A_1 is an atomic model in the sense of [13], and X is $(\nabla_1, \nabla_2) - cl$ -algebraically prime set of theory T and $cl(X) = A_2$. Since $\nabla_1 = \nabla_2 = L$, then $A_1 \cong A_2$.

By the definition of (∇_1, ∇_2) - algebraic primeness of the set X, the model A_2 is both existentially closed and algebraically prime. Thus, the model A_2 is isomorphically embedded in the model A_1. Since by condition the model A_1 is countably atomic, then according to the Vaught's theorem, A_1 is prime, i.e. it is elementarily embedded in the model A_2. Thus, the models A_1, A_2 differ from each other only by the interior of the set X. This follows from the fact that any element of $a \in A_2 \backslash X$ implements some main type, since $a \in cl(X)$. That is, all countable atomic models in the sense of [13] are isomorphic to each other, then by increasing X we find more elements that do not realize the main type and, accordingly, $cl(X)$ is not an atomic model in the sense of [13]. Thus, the principle of rheostat is that, by increasing the power of the set X, we move away from the notion of atomicity in the sense of [13] and on the contrary, decreasing the power of the set X we move away from the notion of atomicity in the sense of [14].

Let $APC \in \{$atomic, algebraically prime, core$\}$. Thus, by specifying the set X as $(\nabla_1, \nabla_2) - cl - APC$, (where APC is a semantic property), we can also specify atomicity in the sense [14] in relation to atomicity in the sense of [13]. And accordingly, according to the principle of "rheostat" after the APC property is defined, we obtain the corresponding concepts of atomic models, the role of which is played A_2 from the principle of "rheostat".

One of the new directions in the study of the Jonsson spectrum is the study of model-theoretical properties of hybrids of the Jonsson classes of the spectrum under consideration. This problem is interesting in many respects, one of which is the existential model compatibility of a fragment of the algebraic construction of semantic models of these classes with the primordial theory.

The following definition 35 is taken from [15].

Let us define the essence of the operation of the symbol $\boxdot$ for algebraic construction of models, which will be play important role in the definition of hybrids. Let $\boxdot \in \{\cup, \cap, \times, +, \oplus, \prod_F, \prod_U\}$, where

$\cup$-union, $\cap$-intersection, $\times$-Cartesian product, $+$-sum and $\oplus$-direct sum, $\prod_F$-filtered product and $\prod_U$-ultraproduct.

Definition 35. A hybrid of classes $[T]_1, [T]_2$ is the class $[T]_i \in JSp(\mathcal{A})/_{\bowtie}$ if $Th_{\forall\exists}(C_1 \boxdot C_2) \in [T]_i$, we denote such hybrid as $H([T]_1, [T]_2)$.

Note the following fact:

Fact 1. For the theory $H([T]_1, [T]_2)$ in order to be Jonsson enough to be that $(C_1 \boxdot C_2) \in E_{[T]_i}$, where $[T]_i \in JSp(\mathcal{A})/_{\bowtie}$.

References

1 Ешкеев А.Р. JSp-косемантичность и JSB-свойство абелевых групп / А.Р. Ешкеев, О.И. Ульбрихт // Siberian Electronic Mathematical Reports [Электронный ресурс]. — 2016. — Вып. 13. — С. 861-874. — Режим доступа: http://semr.math.nsc.ru.

2 Ешкеев А.Р. JSp-косемантичность R модулей / А.Р. Ешкеев, О.И. Ульбрихт // Siberian electronics mathematical reports [Электронный ресурс]. — 2019. — Вып. 16. — С. 1233-1244. — Режим доступа: http://semr.math.nsc.ru.

3 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во КарГУ, 2016. — 370 с.

4 Yeshkeyev A.R. Strongly minimal Jonsson sets and their properties / A.R. Yeshkeyev // Bulletin of the Karaganda University - Mathematics. — 2015. — Vol. 80. — No. 4. — P. 47-51.

5 Yeshkeyev A.R. Properties of lattices of the existential formulas of Jonsson fragments / A.R. Yeshkeyev, M.T. Kasymetova // Bulletin of the Karaganda University. Series Mathematics. — 2015. — Vol. 79. — No. 3. — P. 25-32.

6 Yeshkeyev A.R. The Properties of Similarity for Jonsson's Theories and Their Models / A.R. Yeshkeyev // Bulletin of the Karaganda University. Series Mathematics. — 2015. — Vol. 80. — No. 4. — P. 52-59.

7 Yeshkeyev A.R. Δ-cl-atomic and prime sets / A.R. Yeshkeyev, A.K. Issayeva // Bulletin of the Karaganda University. Series Mathematics. — 2019. — Vol. 93. — No. 1. — P. 88-94.

8 Yeshkeyev A.R. Companions of the fragments in the Jonsson enrichment / A.R. Yeshkeyev // Bulletin of the Karaganda University. Series Mathematics. — 2017. — Vol. 85. — No. 1. — P. 41-45.

9 Yeshkeyev A.R. The properties of central types with respect to enrichment by Jonsson set / A.R. Yeshkeyev // Bulletin of the Karaganda University. Series Mathematics. — 2017. — Vol. 85. — No. 1. — P. 36-40.

10 Yeshkeyev A.R. Criterion for the cosemanticness of the Abelian groups in the enriched signature / A.R. Yeshkeyev, M.T. Kassymetova, O.I. Ulbrikht // Bulletin of the Karaganda University. Series Mathematics. — 2018. — Vol. 89. — No. 1. — P. 49-60.

11 Yeshkeyev A.R. The J-minimal sets in the hereditary theories / A.R. Yeshkeyev, M.T. Omarova, G.E. Zhumabekova // Bulletin of the Karaganda University. Series Mathematics. — 2019. — Vol. 94. — No. 2. — P. 92-98.

12 Yeshkeyev A.R. The atomic definable subsets of semantic model / A.R. Yeshkeyev, N.M. Mussina, A.K. Issayeva // Bulletin of the Karaganda University. Series Mathematics. — 2019. — Vol. 94. — No. 2. — P. 84-91.

13 Vaught R. Denumerable models of complete theories in Infinitistic Methode / R. Vaught // Pergamon. — London, 1961. — P. 303-321.

14 Baldwin J.T. Algebraically prime models / J.T. Baldwin, D.W. Kueker // Ann. Math. Logic. — 1981. — 20. — P. 289-330.

15 Yeshkeyev A.R. Hybrids of classes from Jonsson spectrum / A.R. Yeshkeyev, N.M. Mussina // The Bulletin of Symbolic Logic. — 2019. — Vol. 25. — No. 4. — P. 523.

А.Р. Ешкеев

Йонсондық спектрдің модельді-теоретикалық сұрақтары

Макалада йонсондық спектрді зерттеу аясында жаңа ұғымдар анықталған. Индуктивті, жалпы айтқанда, толық емес теориялар класындағы элементарлық эквиваленттіліктің жалпылауы болып табылатын косемантикалық ұғымына қатысты спектр қарастырылған. Сондай-ақ, йонсондық спектрдің көмегімен йонсондық теориялар мен олардың модельдерінің кластарын зерттеудің нақты бағыттары анықталды, атап айтқанда индуктивтік теориялардың жеткілікті кең ішкі классын анықтайтын және толықтық, модельді және модельді компаньон сияқты модельдер теориясының классикалық сұрақтарын зерттеу болып табылады. Сондықтан, йонсондық спектрдің модельді-теоретикалық қасиеттерін зерттегенде, біз элементарлық эквиваленттік тұжырымдамасынан косемантикалық тұжырымдамасына, сонымен қатар теориялар мен модельдерге көшкенде табиғи түрде пайда болатын сол ұғымдардың анықтамасын нақтылауымыз қажет. Сонымен бірге, йонсондық спектрдің кейбір модельді-теоретикалық қасиеттері қарастырылған. Йонсондық спектрге қатысты барлық берілген есептердің мағынасы бар және олардың шешімдерін шығару пайдалы болуы мүмкін, өйткені бұл мәселе йонсондық теориялар саласында белсенді түрде зерттелген.

Кілт сөздер: йонсондық теория, йонсондық спектр, косеманттылық, толықтық, ұқсастылық.

А.Р. Ешкеев

Теоретико-модельные вопросы йонсоновского спектра

В статье определены новые понятия в рамках изучения йонсоновских спектров. Рассмотрен спектр относительно понятия косемантичности, который является обобщением элементарной эквивалентности в классе индуктивных, вообще говоря, неполных теорий. Также с помощью йонсоновских спектров изучены актуальные направления йонсоновских теорий и их классов моделей, а именно классических вопросов теории моделей, таких как полнота модели, компаньон модели в рамках упомянутых выше условий, которые определяют довольно широкий подкласс индуктивных теорий и какие называются йонсоновскими теориями. По этой причине при изучении теоретико-модельных свойств йонсоновских спектров мы нуждаемся в уточнении определений тех понятий, которые естественным образом возникают при переходе от понятия элементарной эквивалентности к понятию косемантичности, причем, как теорий, так и моделей. Рассмотрены некоторые теоретико-модельные свойства йонсоновских спектров. В случае рассмотрения йонсоновского спектра все задачи, которые заданы в данной работе, имеют смысл, и их решение может оказаться полезным для решения смежных задач, потому что данная проблематика является активно изучаемой в области йонсоновских теорий.

Ключевые слова: йонсоновская теория, йононовский спектр, косемантичность, полнота, подобие.

References

1 Yeshkeyev, A.R. & Ulbrikht, O.I. (2016). JSp-kosemantichnost i JSB-svoistvo abelevykh hrupp [JSp-cosemanticnes and JSB property of abelian groups]. Siberian Electronic Mathematical Reports. Retrieved from http://semr.math.nsc.ru. [in Russian].

2 Yeshkeyev, A.R. & Ulbrikht, O.I. (2019). JSp-kosemantichnost R-modulei [JSp-cosemanticness of R-modules]. Siberian Electronic Mathematical Reports. Retrieved from http://semr.math.nsc.ru. [in Russian].

3 Yeshkeyev, A.R. & Kassymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models].* Karaganda: Izdatelstvo Karahandinskoho hosudarstvennoho universiteta [in Russian].

4 Yeshkeyev, A.R. (2015). Strongly minimal Jonsson sets and their properties. *Bulletin of the Karaganda University. Series Mathematics, Vol. 80, 4,* 47-51.

5 Yeshkeyev, A.R. & Kasymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University. Series Mathematics, Vol. 79, 3,* 25-32.

6 Yeshkeyev, A.R. (2015). The Properties of Similarity for Jonsson's Theories and Their Models. *Bulletin of the Karaganda University. Series Mathematics, Vol. 80, 4,* 52-59.

7 Yeshkeyev, A.R. & Issayeva, A.K. (2019). Δ-cl-atomic and prime sets. *Bulletin of the Karaganda University. Series Mathematics, Vol. 93, 1,* 88-94.

8 Yeshkeyev, A.R. (2017). Companions of the fragments in the Jonsson enrichment. *Bulletin of the Karaganda University. Series Mathematics, Vol. 85, 1,* 41-45.

9 Yeshkeyev, A.R. (2017). The properties of central types with respect to enrichment by Jonsson set. *Bulletin of the Karaganda University. Series Mathematics, Vol. 85, 1,* 36-40.

10 Yeshkeyev, A.R., Kasymetova, M.T. & Ulbrikht, O.I. (2018). Criterion for the cosemanticness of the Abelian groups in the enriched signature. *Bulletin of the Karaganda University. Series - Mathematics, Vol. 89, 1,* 49-60.

11 Yeshkeyev, A.R., Omarova M.T. & Zhumabekova G.E. (2019). The J-minimal sets in the hereditary theories. *Bulletin of the Karaganda University. Series Mathematics, Vol. 94, 2,* 92-98.

12 Yeshkeyev, A.R., Issayeva, A.K., & Mussina, N.M. (2019). The atomic definable subsets of semantic model. *Bulletin of the Karaganda University. Series Mathematics, Vol. 94, 2,* 84-91.

13 Vaught, R. (1961). *Denumerable models of complete theories in Infinitistic Methode.* Pergamon. London.

14 Baldwin, J.T. & Kueker, D.W. (1981). Algebraically prime models. *Ann. Math. Logic., 20,* 289-330.

15 Yeshkeyev, A.R., & Mussina, N.M. (2019). Hybrids of classes from Jonsson spectrum. *The Bulletin of Symbolic Logic, — Vol. 25, 4,* 523.

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The hybrids of the $\Delta - PJ$ theories

When studying Jonsson theories, which are a wide subclass of inductive theories, it becomes necessary to study the so-called Jonsson sets. Similar problems are considered both in model theory and in universal algebra. This topic is related to the study of model-theoretical properties of positive fragments. These fragments are a definable closure of special subsets of the semantic model of a fixed Jonsson theory. In this article are considered model-theoretical properties of a new class of theories, namely $\Delta - PJ$ theories of countable first-order language. These are theories that are obtained from $\Delta - PJ$ theories by replacing in the definition of $\Delta - PJ$ theories of morphisms (Δ-continuities) with morphisms (Δ-immersions). A number of results were obtained, $\Delta - PJ$ fragments, $\Delta - PJ$ sets, hybrids of $\Delta - PJ$ theories. All questions considered in this article are relevant in the study of Jonsson theories and their model classes.

Keywords: Jonsson theory, $\Delta - PJ$ theory, $\Delta - PJ$ fragment, semantic model, hybrid of the $\Delta - PJ$ theories.

In this article we want to define the concepts of a hybrid for a special positive case of Jonsson theories. Prior to this, we have defined the concepts of a hybrid of Jonsson theories, which are closely related to some fixed Jonsson theory. To familiarize ourselves with this material, we refer the reader to the following sources [1–5]. On the other hand, it is well known that the concept of Jonsson theory can be considered in a more general context, namely in the framework of the study of positive model theory. The main sources in this direction we would like to mention the following works: [6–8].

Further, as part of the study of the positive model theory, the study of Jonsson theories was begun [9]. In this paper, we consider a special case of a positive model theory and note that this particularity is related to the form of the formulas that are preserved during immersions; immersions, in turn, are a special case of homomorphisms.

About Jonsson theories, more detailed information can be extracted in the monograph [10] and in the works [11–15].

Let L be a first-order language. At is the set of atomic formulas of this language. $B^+(At)$ is a closed set of relatively positive Boolean combinations (conjunction and disjunction) of all atomic formulas, their subformulas, and variable substitutions. $Q(B^+(At))$ is the set of prenex normal formulas obtained by applying quantifiers ($\forall$ and $\exists$) to $B^+(At)$. A formula is called positive formula if it belongs to the set $Q(B^+(At)) = L^+$. A theory is called positively axiomatizable if its axioms are positive. $B(L^+)$ is an arbitrary Boolean combination of formulas from L^+. It is easy to see that $\Pi(\Delta) \subseteq B(L^+)$ for $\Delta = B^+(At)$, where $\Pi = \Pi(\Delta) = \{\forall y \neg\varphi(x, y) : \varphi \in \Delta\} = = \{\neg\psi : \psi \in \Delta\}$.

Following [6, 7] we define Δ - morphisms between structures. Let M and N be language structures, $\Delta \subseteq B(L^+)$. A mapping $h : M \to N$ is called a Δ - homomorphism (symbolically $h : M \to N$), if for any $\varphi(\overline{x}) \in \Delta$, $\forall \overline{a} \in M$ from the fact that $M \models \varphi(\overline{a})$, follows that $N \models \varphi(h(\overline{a}))$.

Following [6, 7], the M model is called the beginning in N and we say that M extends to N, and $h(M)$ is called the continuation of M. If the map h is injective, then it is said that the map h immerses M in N (symbolically $h : M \underset{\Delta}{\leftrightarrow} N$). In the future, we will use the term Δ-continuation and Δ-immersion. In the framework of this definition (Δ homomorphism), it is easy to notice that isomorphic embedding and elementary embedding are Δ-immersions when $\Delta = B(At)$ and $\Delta = L$, respectively.

Consider the following necessary definitions.

Definition 1. If K is a class of L-structures, then we say that an element M of K is Δ positively existentially closed in K, if every Δ is a homomorphism from M to any element of K is Δ-immersion. The class of all Δ - positively existentially closed models is denoted by $(E_K^\Delta)^+$; if $K = ModT$ for some theory T, then by E_T, $(E_T^\Delta)^+$ we mean, respectively, the class of existentially closed and Δ-positively existentially closed models of this theory.

Definition 2. We will say that the theory of T admits $\Delta - JEP$, if for any two $A, B \in ModT$ there is $C \in ModT$ and Δ-continuation $h_1 : A \underset{\Delta}{\to} C$, $h_2 : B \underset{\Delta}{\to} C$.

Definition 3. We will say that the theory T admits $\Delta - AP$, if for any $A, B, C \in ModT$ such that $h_1 : A \underset{\Delta}{\to} C$, $g_1 : A \underset{\Delta}{\to} B$, where h_1, g_1 is a Δ-continuation, there is $D \in ModT$ and $h_2 : C \underset{\Delta}{\to} D$, $g_2 : B \underset{\Delta}{\to} D$, where h_2, g_2 – Δ-continuation such that $h_2 \circ h_1 = g_2 \circ g_1$.

Definition 4. A theory T is called a Δ-positive Jonsson ($\Delta - PJ$) theory if it satisfies the following conditions:

1) T has an infinite model;

2) is positively $\forall\exists$-axiomatizable;

3) admits $\Delta - JEP$;

4) admits $\Delta - AP$.

Let C be a semantic model of some fixed Jonsson theory T.

Definition 5. Let $cl : P(C) \to P(C)$ be an operator on the power set of C. We say that (C, cl) is a Jonsson pregeometry if the following conditions are satisfied.

If $A \subseteq C$, then $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$.

If $A \subseteq B \subseteq C$, then $cl(A) \subseteq cl(B)$.

(Exchange). $A \subseteq C$, $a, b \in C$, and $a \in cl(A \cup \{B\})$, then $a \in cl(A), b \in cl(A \cup \{a\})$.

(Finite character). If $A \subseteq C \bigvee$ and $a \in cl(A)$, then there is a finite $A_0 \subseteq A$ such that a $a \in cl(A_0)$.

We say that $A \subseteq C$ is closed if $cl(A) = A$.

Definition 6. If (C, cl) is a Jonsson pregeometry, we say that A is Jonsson independent (J-independent) subset in C, if $a \notin cl(A \setminus \{a\})$ for all $a \in A$ and B is J-basis for Y, $Y \subseteq C$, if B-J-independent and $Y \subseteq acl(B)$.

Lemma 1. If (C, cl) is a J-pregeometry, $Y \subseteq C$, $B_1, B_2 \subseteq Y$ and each B_i is a J-basis for Y, then $\mid B_1 \mid = \mid B_2 \mid$.

We call $\mid B_i \mid$ the J-dimension of Y and write $Jdim(Y) = \mid B_i \mid$.

If$A \subseteq C$, we also consider the localization $cl_A(B) = cl(A \cup B)$.

Lemma 2. If (C, cl) is a J-pregeometry, then (C, cl_A) is a J-pregeometry.

If (C, cl) is a J-pregeometry, we say that $Y \subseteq C$ is J-independent over A if Y is J-independent in (C, cl_A). We let $Jdim(Y/A)$ be the J-dimension of Y in the localization (C, cl_A). We call $Jdim(Y/A)$ the J-dimension of Y over A.

Definition 7. We say that (C, cl) is a modular pregeometry if, for any finite-dimensional closed $A, B \subseteq C$ the following is true

$$dim(A \cup B) = dimA + dimB - dim(A \cap B).$$

Definition 8. If (C, cl) is modular, then the Jonsson theory T s called modular, where C is a semantic model of theory T.

Definition 9. Let $X \subseteq C$. We will say that a set X is Δ-positive Jonsson subset of C, if X satisfies the following conditions:

1) X is Δ-definable set (this means that there is a formula from Δ, the solution of which in the C is the set X, where $\Delta \subseteq B(L^+)$, that is Δ is a view of formula, for example $\exists^+, \forall^+, \forall\exists^+$ и тд.);

2) $dcl(X) = M$, $M \in (E_T^\Delta)^+$, where dcl is a definable closure operator and cl is an operator defining pregeometry over C.

All morphisms which we are considering below will be Δ-immersions.

Lemma 3. Let T be a $\Delta - PJ$-positive perfect Jonsson theory, $(E_T^{\Delta})^+$ is class of its existentially closed models. Then for any model $A \in (E_T^{\Delta})^+$ theory $Th_{\forall\exists^+}(A)$ is a $\Delta - PJ$ theory.

Definition 10. The inductive theory T is called the existentially prime if: it has a algebraically prime model, the class of its AP (algebraically prime models) denote by AP_T; class E_T non trivial intersects with class AP_T, i.e. $AP_T \bigcap E_T \neq 0$.

Definition 11. The theory T is called convex if for any its model A and for any family $\{B_i \mid i \in I\}$ of substructures of A, which are models of the theory T, the intersection $\bigcap_{i \in I} B_i$ is a model of T.

Let T be a $\Delta - PJ$ theory and C is a semantic model of $\Delta - PJ$ theory. Let X_1, X_2 be a $\Delta - PJ$ subsets of C.

$Fr(X_1), Fr(X_2)$ are $\Delta - PJ$ fragments.

Let $M_1 = dcl(X_1)$, $M_2 = dcl(X_2)$, where $M_1, M_2 \in (E_T^{\Delta})^+$.

$Th_{\forall\exists^+}(M_1) = T_1$, $Th_{\forall\exists^+}(M_2) = T_2$,

C_1 is the semantic model of $\Delta - PJ$ theory of T_1, C_2 is the semantic model of $\Delta - PJ$ theory of T_2.

$T_1 = Th_{\forall\exists^+}(M_1) = Fr^+(X_1)$, $T_2 = Th_{\forall\exists^+}(M_2) = Fr^+(X_2)$,

We define the essence of the operation of an algebraic construction.

Let $\boxdot \in \{\cup, \cap, \times, +, \oplus, \prod\limits_F, \prod\limits_U\}$, where $\cup$-union, $\cap$-intersection, $\times$-Cartesian product, $+$-sum and $\oplus$-direct sum, $\prod\limits_F$-filtered and $\prod\limits_U$-ultra-production.

The following definition gives a hybrid of two $\Delta - PJ$ fragments of the same signature.

Definition 12. A hybrid $H(Fr^+(X_1), Fr^+(X_2))$ of $\Delta - PJ$ fragments $Fr^+(X_1)$, $Fr^+(X_2)$ is called the theory $Th_{\forall\exists^+}(C_1 \boxdot C_2)$, if it is $\Delta - PJ$ theory, where C_i are the semantic models of $Fr^+(X_i)$, $i = 1, 2$.

Note the following fact:

Fact. For the theory $H(Fr^+(X_1), Fr^+(X_2))$ to be a $\Delta - PJ$ theory enough to $(C_1 \boxdot C_2) \in (E_T^{\Delta})^+$.

The following examples will be examples of hybrids of $\Delta - PJ$ theories:

Let $\Delta = B^+(At)$.

1) Let T be a $\Delta - PJ$ theory, C be a semantic model of $\Delta - PJ$ theory of T. A, B are the $\Delta - PJ$ subsets, $A, B \subseteq C$. $dcl(A) = M_1$, $dcl(B) = M_2$, where $M_1, M_2 \in (E_T^{\Delta})^+$. Then $Th_{\forall\exists^+}(M_1 \times M_2)$ will be a hybrid of $\Delta - PJ$ theories.

2) Let T_1, T_2 be the $\Delta - PJ$ theories of Abelian groups, C_1, C_2 be the semantic models of $\Delta - PJ$ theories of T_1, T_2, respectively. Then $Th_{\forall\exists^+}(C_1 \times C_2) = H(T_1, T_2)$ will be a hybrid of $\Delta - PJ$ theories.

3) Let V be a linear space, V_1, V_2 be the linear subspaces,$V_1, V_2 \subseteq V$. Then $Th_{\forall\exists^+}(V_1 \oplus V_2)$ will be a hybrid of $\Delta - PJ$ theories.

And also, there are a number of tasks that will be examples of hybrids of $\Delta - PJ$ theories.

1) Let G be a group, $T = Th(G)$, H_1, H_2 are normal divisors of the group G. $X_1, X_2 \subseteq C$, C is the semantic model of $\Delta - PJ$ theory of T. Let $H_1 = cl(X_1)$, $H_2 = cl(X_2)$, where $H_1, H_2 \in (E_T^{\Delta})^+$. $Th_{\forall\exists^+}(H_1) = T_1$, $Th_{\forall\exists^+}(H_2) = T_2$, T_1, T_2 are a $\Delta - PJ$ theory. Then their hybrid will be $H(T_1, T_2) = Th_{\forall\exists^+}(C_1 \boxdot C_2)$, where C_1 is the semantic model of $\Delta - PJ$ theory of T_1, C_2 is the semantic model of $\Delta - PJ$ theory of T_2,respectively. Then is there such a theory T_3, $T_3 = H(T_1, T_2) = Th_{\forall\exists^+}(C_1 \boxdot C_2)$ and if there is a theory of T_3, then which $H(T_1, T_2)$ satisfy these conditions? Here for the place of algebraic construction will be direct sum: $\boxdot = \oplus$.

2) Let T_1, T_2 be a $\Delta - PJ$ theory and T_3, T_4 be a $\Delta - PJ$ theory. Then C_1, C_2 are the semantic models of $\Delta - PJ$ theory of T_1, T_2, C_3, C_4 are the semantic models of $\Delta - PJ$ theory of T_3, T_4, respectively.

If $C_1 \equiv C_2$, $C_3 \equiv C_4$, то $C_1 \times C_3 \equiv C_2 \times C_4$, then are there such theories $\exists T_5 : T_5 = H(T_1, T_3) = Th_{\forall\exists^+}(C_1 \times C_3)$, $\exists T_6 : T_6 = H(T_2, T_4) = Th_{\forall\exists^+}(C_2 \times C_4)$, which will be hybrids of $\Delta - PJ$ theories?

In the study of this class of theories, we obtained the following results:

Let $\Delta = B^+(At)$.

Theorem 1. Let $Fr^+(X)$ be perfect convex existentially prime complete for $\forall\exists^+$-sentences a $\Delta - PJ$ fragment. X_1, X_2 are the $\Delta - PJ$-sets of the theory $Th_{\forall\exists^+}(C)$, where M_i=$dcl(X_i) \in E_{Fr(Th_{\forall\exists^+}(C))}$, $Fr^+(X_i) = Th_{\forall\exists^+}(M_i)$ are also perfect convex existentially prime complete for $\forall\exists^+$-sentences a $\Delta - PJ$ fragments. C_1, C_2 are their semantic models, respectively. Then, if their hybrid $H(Fr^+(X_1),\ \ Fr^+(X_2))$ is a model consistent with $Fr^+(X_i)$, then $H(Fr^+(X_1), Fr^+(X_2))$ is a perfect $\Delta - PJ$ theories for $i = 1, 2$.

Proof. Suppose the contrary. Then, since the hybrid $H(Fr^+(X_1), Fr^+(X_2))$ is a $\Delta - PJ$ theories and has a semantic model M, by the assumption not perfectness of this hybrid $H(Fr^+(X_1), Fr^+(X_2))$, the considered semantic model M will not be saturated in its power. And this means that there is such $X \subseteq M$ and such type $p \in S_1(X)$, which is not realized in M, more precisely in $(M, m)_{m \in X}$. By virtue of the consistency of type p, this type is realized in some elementary extension $M' \succ M$. y virtue of the Jonssonness of hybrid $\Delta - PJ$ fragments $H(Fr^+(X_1), Fr^+(X_2))$ and model consistency with $Fr^+(X_i)$, $i = 1, 2$ there is a model $A_i \in ModFr^+(X_i)$, $i = 1, 2$ such that M' is a submodel of A. A in turn, is embedded in the semantic model C_i, $i \in 1, 2$, but C_i is a saturated model of the theory $Fr^+(X_i)$, $i \in 1, 2$. By virtue of the Δ-immersion, suppose h from M' in A, $h(X) \subseteq A$ and since the type of p is realized in M' it will be realized in C_i. But $C_i \in E_{Fr^+(X_i)}$ and since $Fr^+(X_i)$ are existentially prime convex theories, there exists a countable model $N_i \in E_{Fr^+(X_i)}$, in which the type p will be realized. By virtue of convexity, the model N_i will be a nuclear model, i.e. it is algebraically prime embedded in other models from $Mod(Fr^+(X_i))$ exactly one time. But by virtue of the model consistency of $Fr^+(X_i)$ with the hybrid $H(Fr^+(X_i))$, N_i Δ-immerse oneself in some model from $ModH(Fr^+(X_i))$. Since $Fr^+(X_i)$ are perfect theories, their center is model-complete, i.e. any monomorphism is elementary between the models of this center. And such, by virtue of perfection, are all the models from $E_{Fr^+(X_i)}$. Then the above Δ-immersion will be elementary, i.e. type p is realized in a countable submodel of model M. We got a contradiction with the assumption of imperfection.

Theorem 2. Let $Fr^+(X), Fr^+(X_1), Fr^+(X_2)$ satisfy the conditions of Theorem 1 and $Fr^+(X_1)$, $Fr^+(X_2)$ be ω-categorical $\Delta - PJ$ fragments. Then their hybrid $H(Fr^+(X_1), Fr^+(X_2))$ is also a ω-categorical $\Delta - PJ$ theory.

Proof. We note that, by virtue of the above Theorem 1, the hybrid $H(Fr^+(X_1), Fr^+(X_2))$ will be a perfect $\Delta - PJ$ theory. Suppose the contrary, i.e. the hybrid $H(Fr(X_i))$ is not a ω-categorical Jonsson theory. Let A and B be two countable models from $ModH(Fr^+(X_i))$. Then there are A' and B' countable models from $E_{H(Fr^+(X_i))}$ such that A is isomorphically embedded into A', and B is isomorphically embedded into B'. This follows from the fact that in any inductive theory any model is isomorphically embedded in some existentially closed model of this theory. But fragments of $Fr^+(X_i)$ are mutually model consistent with $H(Fr^+(X_i))$ by virtue of the condition of the theorem. Then A' and B' are Δ-immerse oneself in some countable model $D \in E_{Fr^+(X_i)}$, но but as $Fr^+(X_i)$ are convex fragments, then the image of A' and the image of B' in the model D intersects non-empty. Let this intersection be a model R. y virtue of the above existential primeness and countable categoricity of $Fr^+(X_i)$, since $R \in E_{T_i}$ it follows that in $R \models \varphi(x) \wedge \neg\varphi(x)$, where in $A' \models \varphi(x)$, and in $B' \models \neg\varphi(x)$. But this is not true, as T_i are ω-categorical by condition. Consequently, we obtain a contradiction with the assumption of non-ω-categoricity $H(Fr^+(X_i))$.

References

1 Yeshkeyev A.R. Properties of hybrids of Jonsson theories / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University - Mathematics. — 2018. — Vol. 92. — No. 4. — P. 99-104.

2 Yeshkeyev A.R. The atomic definable subsets of semantic model /A.R. Yeshkeyev, N.M. Mussina, A.K. Issayeva // Bulletin of the Karaganda University - Mathematics. — 2019. — Vol. 94. — No. 2. — P. 84-91.

3 Yeshkeyev A.R. Small models of hybrids for special subclasses of Jonsson theories / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University - Mathematics. — 2019. — Vol. 95. — No. 3. — P. 68-73.

4 Yeshkeyev A.R. Hybrids of classes from Jonsson spectrum / A.R. Yeshkeyev, N.M. Mussina // The Bulletin of Symbolic Logic. — 2019. — Vol. 25. — No. 4. — P. 523.

5 Yeshkeyev A.R. Hybrid's core models of Jonsson theories / A.R. Yeshkeyev, N.M. Mussina // The Bulletin of Symbolic Logic.—2019. — Vol. 25. — No. 4. — P. 473-474.

6 Itay Ben-Yaacov. Positive model theory and compact abstract theories /I. Ben-Yaacov // Journal of Mathematical Logic. — 2003. — Vol. 3. — No. 1. — P. 85-118.

7 Itay Ben-Yaacov. Compactness and independence in non first order frameworks /I. Ben-Yaacov // Bulletin of Symbolic logic. — 2005. — Vol. 11. — No. 1. — P. 28-50.

8 Itay Ben-Yaacov. Fondements de la Logique positive /B. Poizat, I. Ben-Yaacov // Journal of Symbolic Logic. — 2007. — Vol. 72. — No. 4. — P. 1141-1162.

9 Poizat B. Positive Jonsson Theories / B. Poizat, A.R. Yeshkeyev // Logica Universalis. — 2018. — Vol. 12. — No. 1-2. — P. 101-127.

10 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во КарГУ, 2016. — 370 с.

11 Yeshkeyev A.R. Strongly minimal Jonsson sets and their properties / A.R. Yeshkeyev // Bulletin of the Karaganda University - Mathematics. — 2015. — Vol. 80. — No. 4. — P. 47-51.

12 Yeshkeyev A.R. Properties of lattices of the existential formulas of Jonsson fragments / A.R. Yeshkeyev, M.T. Kasymetova // Bulletin of the Karaganda University - Mathematics. — 2015. — Vol. 79. — No. 3. — P. 25-32.

13 Yeshkeyev A.R. On lattice of existential formulas for fragment of Jonsson set / A.R. Yeshkeyev, O.I. Ulbrikht // Bulletin of the Karaganda University - Mathematics. — 2015. — Vol. 79. — No. 3. — P. 33-39.

14 Yeshkeyev A.R. The Properties of Similarity for Jonsson's Theories and Their Models / A.R. Yeshkeyev // Bulletin of the Karaganda University - Mathematics. — 2015. — Vol. 80. — No. 4. — P. 52-59.

15 Yeshkeyev A.R. JSp-cosemanticness of R-modules / A.R. Yeshkeyev, O.I. Ulbrikht // Siberian electronics mathematical reports. — 2019. — Vol. 16. — P. 1233–1244.

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Δ-PJ теориялардың гибридтері

Индуктивті теориялардың кең класы болып табылатын йонсондық теорияларды зерттегенде, йонсондық ішкі жиындарды зерттеу қажет болады. Осыған ұқсас мәселелер модельдер теориясында да, әмбебап алгебрада да қарастырылған. Бұл тақырып позитивті фрагменттердің модельді-теоретикалық қасиеттерін зерттеумен байланысты, яғни фрагменттер белгілі бір йонсондық теорияның семантикалық моделінің арнайы жиындарының тұйықтамасы болып табылады. Мақалада теориялардың жаңа класының модельді-теоретикалық қасиеттері, атап айтқанда бірінші реттегі тілдегі Δ-PJ теориялары қарастырылған. Бұл Δ-PJ теориясынан алынған Δ-PJ морфизмдер теориясын (Δ-жалғасы) морфизмдермен (Δ - бату) алмастыру арқылы алынған теориялар. Сонымен қатар берілген жұмыста бірқатар нәтижелер алынды, олар: Δ-PJ фрагменттер, Δ-PJ жиындар, Δ-PJ теориялардың гибридтері. Мақалада қарастырылған барлық сұрақтар йонсондық теорияларды және олардың модельдер кластарын зерттеуде өзекті болып табылады.

Кілт сөздер: йонсондық теория, Δ-PJ теория, Δ-PJ фрагмент, семантикалық модель, Δ-PJ-теориялардың гибриді.

А.Р. Ешкеев, Н.М. Мусина

Гибриды Δ-PJ-теорий

При изучении йонсоновских теорий, которые являются широким подклассом индуктивных теорий, возникает необходимость изучения так называемых йонсоновских множеств. Подобные задачи рассматриваются как в теории моделей, так и в универсальной алгебре. Данная тематика связана с изучением теоретико-модельных свойств позитивных фрагментов, которые являются определимым замыканием специальных подмножеств семантической модели фиксированной йонсоновской теории. В статье рассмотрены теоретико-модельные свойства нового класса теорий, а именно Δ-PJ-теорий счетного языка первого порядка. Это теории, которые получаются из Δ-PJ-теорий заменой в определении Δ-PJ-теорий морфизмов (Δ-продолжений) на морфизмы (Δ-погружения). При этом получен ряд результатов, Δ-PJ-фрагменты, Δ-PJ-множества, гибриды Δ-PJ-теорий. Следует заметить, что основные синтаксические и семантические атрибуты этих новых классов являются новыми понятиями, и они появились при изучении позитивных йонсоновских классов теорий. Все вопросы, рассматриваемые в статье, являются актуальными в области изучения йонсоновских теорий и их классов моделей.

Ключевые слова: йонсоновская теория, Δ-PJ-теория, Δ-PJ-фрагмент, семантическая модель, гибрид Δ-PJ-теорий.

References

1 Yeshkeyev, A.R., & Mussina, N.M. (2018). Properties of hybrids of Jonsson theories. *Bulletin of the Karaganda University - Mathematics, Vol. 92, 4,* 99-104.

2 Yeshkeyev, A.R., Issayeva, A.K., & Mussina, N.M. (2019). The atomic definable subsets of semantic model. *Bulletin of the Karaganda University - Mathematics, Vol. 94, 2,* 84-91.

3 Yeshkeyev, A.R., & Mussina, N.M. (2019). Small models of hybrids for special subclasses of Jonsson theories. *Bulletin of the Karaganda University - Mathematics, Vol. 95, 3,* 68-73.

4 Yeshkeyev, A.R., & Mussina, N.M. (2019). Hybrids of classes from Jonsson spectrum. *The Bulletin of Symbolic Logic, Vol. 25, 4,* 523.

5 Yeshkeyev, A.R., & Mussina, N.M. (2019). Hybrid's core models of Jonsson theories. *The Bulletin of Symbolic Logic, Vol. 25, 4,* 473-474.

6 Itay, Ben-Yaacov. (2003). Positive model theory and compact abstract theories. *Journal of Mathematical Logic, Vol. 3, 1,* 85–118.

7 Itay Ben-Yaacov. (2005). Compactness and independence in non first order frameworks. *Bulletin of Symbolic logic, 1,* 28–50.

8 Itay Ben-Yaacov & Bruno Poizat, (2007). Fondements de la Logique positive. *Journal of Symbolic Logic, 72, 4,* 1141-1162.

9 Poizat, B., & Yeshkeyev, A.R. (2018). Positive Jonsson Theories *Logica Universalis, Vol. 12, 1-2,* 101-127.

10 Yeshkeyev, A.R., & Kasymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models].* Karaganda: Izdatelstvo Karahandinsoho universiteta [in Russian].

11 Yeshkeyev, A.R. (2015). Strongly minimal Jonsson sets and their properties. *Bulletin of the Karaganda University - Mathematics, Vol. 80, 4,* 47-51.

12 Yeshkeyev, A.R., & Kasymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University - Mathematics, Vol. 79, 3,* 25-32.

13 Yeshkeyev, A.R., & Ulbrikht, O.I. (2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University - Mathematics, Vol. 79, 3,* 33-39.

14 Yeshkeyev, A.R. (2015). The Properties of Similarity for Jonsson's Theories and Their Models. *Bulletin of the Karaganda University - Mathematics, Vol. 80, 4,* 52-59.

15 Yeshkeyev, A.R. & Ulbrikht, O.I. (2019). JSp-kosemantichnost R-modulei [JSpcosemanticness of R-modules]. Siberian Electronic Mathematical Reports. Retrieved from http://semr.math.nsc.ru. [in Russian].

МЕХАНИКА
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Wave field in a strip with symmetric located holes

In the linear formulation, the problem of the propagation of unsteady stress waves in an elastic body with symmetrically located rectangular holes is considered. Formulated in terms of stresses and velocities, the mixed problem is modeled numerically using an explicit difference scheme of the end-to-end calculation based on the method of spatial characteristics. The wave process is caused by applying an external dynamic load on the front border of the rectangular region, and the side boundaries of the region are free of stresses. The lower boundary of the rectangular region is rigidly fixed. The contour of symmetrically arranged rectangular openings is free of stress. Based on the numerical technique developed in this work, the calculated finite - difference relations of dynamic problems are obtained at the corner points of a rectangular hole, where the "smoothness"of functions "familiar"to dynamic problems is violated. At these corner points, the first and second derivatives of the desired functions suffer a discontinuity of the first kind. The isoline presents the results of changes in wave fields in an elastic body with symmetrically located rectangular holes. The concentration of dynamic stresses in the vicinity of the corner points of a rectangular hole is investigated. By numerical implementation, the stability of computational algorithms for a sufficiently large time is established.

Keywords: elastic, wave process, stress, speed, plane deformation, numerical solution.

Introduction

In recent years, the problem of developing scientifically sound and effective numerical methods for analyzing the health of structures with cuts, holes, foreign inclusions and other characteristic features has become increasingly relevant. These features make it necessary to develop new and improve traditional numerical methods for calculating and designing structures. This will make it possible to take advantage of the enormous advantages of mathematical modeling — to combine a physical experiment with a more economically viable numerical experiment and to provide answers to questions of interest to engineers with the least expenditure of funds and effort. To realize this possibility, it is necessary to solve a range of issues related to the construction of mathematical models of environments that take into account the complex features of the environment. On the other hand, the second necessary component of this approach is the creation of reliable and economical methods for the numerical calculation of the corresponding dynamics problems. Prediction of the dynamic behavior of structural elements taking into account a number of weakening factors (holes, cavities, cutouts, etc.) has not only theoretical, but also applied value, determined by the demands of engineering practice [1–17]. The methodological apparatus of scientific research is based on finite-difference methods based on the use of characteristic surfaces and compatibility relations on them. The research methodology is confirmed by scientific and theoretical justification, the correctness and rigor of the mathematical formulation of the investigated problems.

Formulation of the problem. Let the rectangular cross section of the strip, weakened by two equal rectangular holes symmetrically spaced relative to the axis $x_2 = 0$ (Figure 1), be in an undisturbed state at $t \leq 0$, i.e.

$$v_1(x_1, x_2, 0) = v_2(x_1, x_2, 0) = p(x_1, x_2, 0) = q(x_1, x_2, 0) = \tau(x_1, x_2, 0) \tag{1}$$

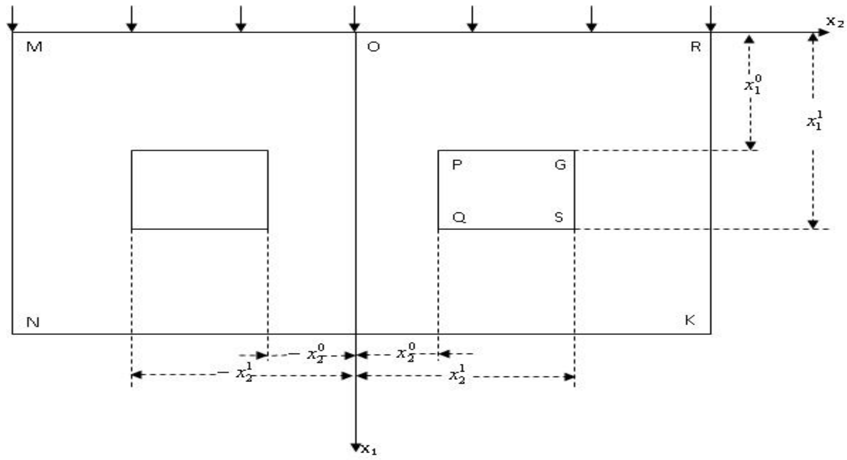

Figure 1. The study area

Wave processes in a strip with symmetrical holes are described by a system of differential equations of hyperbolic type, containing dimensionless stresses p, q, τ, displacement velocities v_1, v_2 as unknowns [1–2]:

$$\begin{gathered} v_{1,t} - p_{,1} - q_{,1} - \tau_{,2} = 0; \quad v_{2,t} - p_{,2} + q_{,2} - \tau_{,1} = 0, \\ \gamma^2(\gamma^2 - 1)^{-1} p_{,t} - v_{1,1} - v_{2,2} = 0; \quad \gamma^2 q_{,t} - v_{1,1} + v_{2,2} = 0, \\ \gamma^2 \tau_{,t} - v_{1,2} - v_{2,1} = 0. \end{gathered} \tag{2}$$

Dimensionless variables are introduced by the formulas [1–2]:

$$\begin{gathered} \overrightarrow{t} = \frac{tc_1}{b}; \quad \overrightarrow{x_i} = \frac{x_i}{b}; \quad v_i = \frac{1}{c_1}\frac{\partial u_i}{\partial t}; \quad (i = 1, 2) \\ p = \frac{\sigma_{11} + \sigma_{22}}{2\rho c_1^2}; \quad q = \frac{\sigma_{11} - \sigma_{22}}{2\rho c_1^2}, \\ \tau = \frac{\sigma_{12}}{\rho c_1^2}; \quad \gamma = \frac{c_1}{c_2}, \end{gathered} \tag{3}$$

where b is the characteristic length, ρ is the density of material, c_1, c_2 are the velocities of waves of expansion and shear, $\sigma_{11}, \sigma_{22}, \sigma_{12}$ are the components of the stress tensor, γ is a constant parameter. In the future, the bar over dimensionless parameters is omitted.

The solution of the system of equations (2) with respect to the desired quantities v_1, v_2, p, q, τ constructed for zero initial (1) and the following boundary conditions ($t \geq 0$):

$$v_1 = f(t), \quad v_2 = 0 \quad when \quad x_1 = 0, \quad -L \leq x_2 \leq L. \tag{4}$$

$$p - q = 0, \quad \tau = 0 \quad when \ |x_2| = L, \quad 0 \leq x_1 \leq \ell, \tag{5}$$

$$v_1 = v_2 = 0 \quad when \ |x_1| = \ell, \quad |x_2| \leq L, \tag{6}$$

$$p+q=0,\quad \tau=0 \quad when \quad x_1=x_1^0,\quad x_2^0\le|x_2|\le x_2^1 \quad and \quad x_1=x_1^1,\ \ x_2^0\le|x_2|\le x_2^1x_1^1, \quad (7)$$

$$p-q=0,\quad \tau=0 \quad when \ \ |x_2|=x_2^0 \quad x_1^0\le x_1\le x_1^1 \quad and \quad |x_2|=x_2^1,\ \ x_1^0\le x_1\le x_1^1. \quad (8)$$

Analysis of the calculation results. Numerical results are given for the rectangular region $0\le x_1\le 100\cdot h_1$, $|x2|\le 100h_2$. Moreover, the accepted values of the steps along the coordinate are the same $h_1=h_2=h=0.05$. The body material has the following characteristics: elastic modulus $E=200GPa$, Poisson's ratio $\nu=0.3$, density $\rho=7.9\cdot10^3kg/m^3$, $c_1=5817m/sec$, $c_2=3109m/sec$, $\gamma=1.87$. The dimensions of the holes are taken as follows: $x_1^0=25\cdot h$, $x_1^1=75\cdot h$, $|x_2^0|=25\cdot h$, $|x_2^1|=75\cdot h$. The wave field parameters were obtained with the following initial data

$$f(t)=A\cdot t\cdot e^{-st}, A=1, cs=0.2, k=0.025, h=0.05.$$

Here A is a constant factor, the parameter s characterizes the rate of change of the external load. Since the body under study has free boundaries $x_2=\pm100\cdot h$ and contains rectangular holes inside itself, then over time reflection waves (diffracted) superimposed on each other determine the complex nature of the manifestation of displacement, strain, and stress velocities in it. The corner points of the rectangular region and the corner points of rectangular holes are sources of disturbance, causing both longitudinal and transverse waves.

The study of the stability showed that a grid ratio k/h of 0.5 provides the stable results for a sufficiently large period of time, with multiple reflections and diffraction waves. In fact, the calculation was performed up to $t=1000\cdot k$. In the calculations at any time t, all boundary conditions are exactly satisfied both at the corner points of the strip and at the corner points of rectangular holes. This circumstance, unlike many approximate methods, ensures the reliability of the obtained solutions and the corresponding results.

The solution of the system of equations (2) under initial (1) and boundary (4)–(8) conditions is found by the method of spatial characteristics at the nodal points into which the entire studied area is divided [1–2]. A feature of the body under consideration is that at the corner points (P,G,Q,S) of the rectangular hole, the "smoothness" of functions that is "familiar" to dynamic problems is violated. It was precisely such features that were not extended, or in general, as we know, there was no method for solving such problems. In addition to the known relations [1–2], the calculated relations are obtained at the internal corner points (P,G,Q,S) of the rectangular hole [6].

The calculation results are presented by graphs in figures 2–3. Due to the symmetry of the location of the rectangular holes and the nature of the loading, the desired parameters v_1,p,q, are even, and v_2,τ are the odd functions relative to the axis $x_2=0$ of the strip.

The construction of dynamic stress fields generated from the interaction of symmetrically located holes will allow, using various strength criteria, to judge the shape and possible size of the fracture zones. In Figs. 2–3, in the plane $x_1/h\cdot x_2/h$ for the time instant $t=400\cdot k$, the contours of normal stresses $p+q=const$, $p-q=const$ are shown, respectively. The stress state due to the symmetry of the problem relative to the axis $x_2=0$ of the strip is shown only for positive values $x_2(x_2\ge0)$ near a symmetrically located hole.

The presented graphs clearly illustrate the strong interaction of wave fields on the stress distribution in the vicinity of the holes, which leads to the formation of a number of local extrema. The location of the latter changes in time due to repeatedly reflected and diffracted waves from the boundaries of the holes and strip.

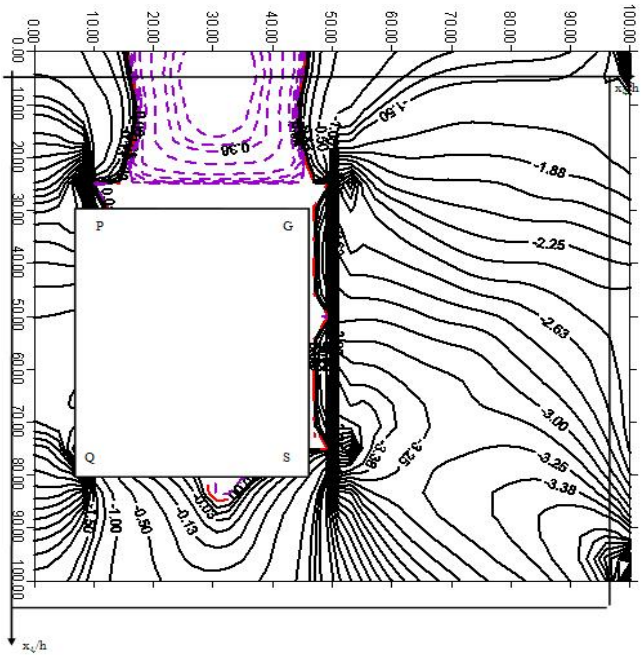

Figure 2. Isolines of normal stresses $p + q = const$ at time $t = 400 \cdot k$

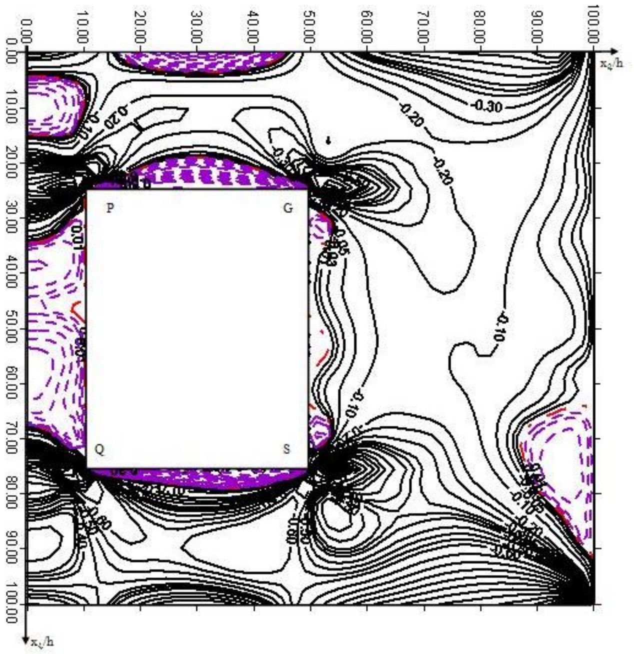

Figure 3. Isolines of normal stresses $p - q = const$ at time $t = 400 \cdot k$

It should be noted (Figure 2) the appearance of an extensive zone of tensile stresses $p + q = const$ in front of the hole. In the region behind the hole, the stress level is extremely low. Although the maximum values of tensile stresses $p + q = const$ turned out to be lower by an order of magnitude than the maximum absolute values of compressive stresses, nevertheless, for a number of materials (such as soils, bulk material, etc.) they can pose a serious danger. Therefore, a detailed study of the appearance and development of extension zones is an important applied problem. In the vicinity of the front corner points (P, G) of the hole relative to the loading surface, the concentration of compressive stresses $p + q = const$ at a given time is maximum. In the vicinity of the lower corner points (Q, S), where the stress amplitude is less, they remain compressive throughout the entire considered time.

Slightly smaller stress gradients than in the vicinity of the corner points are observed near the lateral free edges (PQ, GS) of the hole. The maximum concentration of compressive stresses $p + q = const$ is formed near the corner points (R, K) of the strip, in the vicinity of which the stress fields have the greatest gradients.

Analysis of normal stress isolines $p - q$ (Figure 3) shows that the resulting zone of tensile stresses on the lower free face (QS) of the hole is less extensive than on the front free face (PG). This is due to the fact that a perturbation is specified on the surface $x_1 = 0$, and the strip surface $x_1 = 100 \cdot h$ is pinched.

A small stretch zone is localized near the lateral face (PQ) of symmetrically located holes, as well as near the front (MR) and lateral (RK) faces of a rectangular strip. Moreover, the zone of tensile stresses is bordered by an isoline of zero stresses. The appearance of a large number of local maximums of compressive stresses in the vicinity of the corner points of the symmetric hole is due to the complex interaction of reflected, diffraction, and interference phenomena, which is characteristic of the stress state. As a result of the reflection of the waves, a region of compressive stresses is formed near the upper (MR), lower (NK), and lateral (RK) surfaces of the strip. From an engineering point of view, when calculating and constructing structural elements with several holes, it must be borne in mind that under dynamic loads, the stress concentration does not change monotonically when the holes approach each other, as is the case in static, but is determined by a more complex dependence and interaction of holes, corner points. In some cases, the stress concentration can even decrease as the holes approach each other, which leads to the appearance of a class of problems for optimizing the design with holes. When carrying out the calculations, it is necessary to take into account in advance the possible frequency ranges and disturbances in which the structure will subsequently operate.

References

1 Clifton R.J. A difference method for plane problems in dynamic elasticity / R.J. Clifton // Quart. Appl. Math. — 1967. — 25. — No. 1. — P. 97–116.

2 Ержанов Ж.С. Двумерные волны напряжений в однородных и структурно-неоднородных средах / Ж.С.Ержанов, Т.Д.Каримбаев, Т.Б.Байтелиев. — Алма-Ата: Наука, 1983. — 171 с.

3 Кукуджанов В.Н. Деформирование, повреждаемость и разрушение сред и материалов с дефектами и со структурами дефектов / В.Н. Кукуджанов // Вестн. Нижегород. ун-та им. Н.И. Лобачевского. — 2011. — № 4 (4). — С. 1557–1558.

4 Alexeeva L.A. Mathematical model of massive dynamics in the neighborhood of disturbance focus / L.A. Alexeeva, B.T. Sarsenov // AIP Conference Proceedings 1676, 020067 (2015); http://dx.doi.org /10.1063/1.4930481.

5 Ashirbayev N. Modeling and solving the two-dimensional non-stationary problem in an elastic body with a rectangular hole / N. Ashirbayev, Zh. Ashirbayeva, T.Sultanbek, R.Bekmoldayeva // AIP Conference Proceedings 1759, 020078 (2016); http://dx.doi.org/10. 1063/1.4959692.

6 Ashirbayev N. Features of the Wave Field in a Finite Body with a Noncentral Hole / N. Ashirbayev, Zh. Ashirbayeva, M. Shomanbayeva // AIP Conference Proceedings 1880, 060013 (2017); http://doi.org /10.1063/1.5000667.

7 Ashirbayev N. Investigation of Wave Processes in a Rectangular Region with Discontinuous Boundary Conditions / N. Ashirbayev, Zh. Ashirbayeva, G. Sarsenova, R. Bekmoldayeva, Zh. Nurmaganbetova, P.Duisebayeva // ISSN 1392–1207. Mechanika. — 2018. — 24. — No. 5. — P. 680–684.

8 Мамаев Ш.М. Численное исследование распространения динамических возмущений в составных стержнях сеточно-характеристическим методом / Ш.М. Мамаев, Н.Т. Борашева // Изв. НАН РК. — 2015. — № 3(301). — С. 145–154.

9 Seitmuratov A.Z. Equations of vibration of a two-dimensionally layered plate strictly based on the decision of various boundary-value problems / A.Z. Seitmuratov, B.M. Nurlanova, N. Medeubaev // Bulletin of the Karaganda university. Series mathematics. — 2017. — 87. — No. 3. — P. 109–116.

10 Seitmuratov A.Z. Integro-Differential equations with regular kernels / A.Z. Seitmuratov, A.Z. Madelkhanova, M.Z. Parmenova, K. Kanibaikyzy // News of the national academy of sciences of the republic of Kazakhstan. Series physic-mathematical. — 2018. — 2. — No. 318. P. 37–45.

11 Gavrilov S.N. Wave propagation in a semi-infinite heteromodular elastic bar suljected to a harmonic loading / S.N. Gavrilov, G.C. Herman // J. Sound and Vibr. — 2012. — 331. — No. 20. — P. 4464–4480.

12 Kochaev A.L. Mathematical modeling of elastic wave propagation in crystals: 3d-wave surfaces / A.L. Kochaev, R.A. Brazhe // Acta mechanika. — 2011. — 222. — No. 1, 2. — P. 193–198.

13 Parvanova S. Elastic wave fields in a half-plane with free-surface relief, tunnels and multiple buried inclusions / S. Parvanova, P.Dineva, G.D.Manolis // Acta Mechanica. — 2014. — 225. — No. 7 — P. 1843–1865.

14 Lys E.V. Interaction of seismic waves with zones of concentration of initial stresses / E.V. Lys, E.I. Romenski, V.A. Cheverda, M.I. Epov // Doklady Earth Sciences. — 2013. — 449. — No. 2. — P. 402–405.

15 Lesan D. On the deformation of or thotropic Cosserat elastic cylinders / D. Lesan, A. Scalia // Mathematics and Mechanics of Solids. — 2011. — 16. — No. 2. — P. 177–199.

16 Israilov M.Sh. Reduction of Boundary Value Problems of Dynamic Elasticity to Scalar Problems for Wave Potentials in Curvilinear Coordinates / M.Sh. Israilov // Mechanics of Solids. — 2014. — 46. — No. 1. — P. 104–108.

17 Yi-chao Chen. On the dynamic behavior of a class of Cauchy elastic materials / Chen Yi-chao // Mathematics and Mechanics of Solids. — 2016. — 16. — No. 5. — P. 563–569.

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Симметриялы орналасқан тесіктері бар денедегі толқындық өріс

Жұмыста сызықты жағдайда берілген симметриялы орналасқан тіктөртбұрышты тесіктері бар серпімді денедегі стационар емес толқындық процестің таралу есебі қарастырылған. Кернеулер мен жылдамдықтар терминінде қойылған аралас есеп айқын айырымдық схема, атап айтқанда сандық кеңістіктік сипаттамалар әдісімен шешілген. Толқындық процесс тіктөртбұрышты дененің беттік шекаралық нүктелерінде сырттай динамикалық күштің берілуінен пайда болады, ал дененің бүйір қабырғаларында кернеулік нөлге тең. Тіктөртбұрышты дененің төменгі шекаралық нүктелері қатаң бекітілген. Симметриялы орналасқан тіктөртбұрышты тесіктердің контурында кернеуліктер нөлге тең. Осы жұмыста әзірленген сандық техникалық негізінде динамикалық есептердің ақырғы-айырымдық қатынастары динамикалық есептерге «әдеттегі» шекті функциялардың бұзылған тіктөртбұрышты тесіктің бұрыштарында алынған. Бұл бұрыштық нүктелерде ізделінді, функциялардың бірінші және екінші ретті туындылары бірінші текті үзілісті. Симметриялы орналасқан тіктөртбұрышты тесіктері бар серпімді денедегі толқындық өріс өзгерісінің нәтижелері изосызық түрінде келтірілген. Тік бұрышты тесіктің бұрыштық нүктелерінің маңайында кернеуліктің динамикалық концентрациясы зерттелген. Сандық әдісті қолдану нәтижесінде есептеу алгоритмдерінің жеткілікті үлкен уақытқа тұрақтылығы анықталған.

Кілт сөздер: серпімді, толқындық процесс, кернеу, жылдамдық, жазықтық деформациясы, сандық шешім.

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Волновое поле в полосе с симметрично-расположенными отверстиями

В линейной постановке рассмотрена задача о распространении нестационарных волн напряжений в упругом теле с симметрично-расположенными прямоугольными отверстиями. Сформулированная в терминах напряжений и скоростей смешанная задача моделируется численно с помощью явной разностной схемы сквозного счета, основанной на методе пространственных характеристик. Волновой процесс вызывается прикладыванием внешней динамической нагрузки на лицевой границе прямоугольной области, а боковые границы области свободны от напряжений. Нижняя граница прямоугольной области жестко закреплена. Контур симметрично-расположенных прямоугольных отверстий свободен от напряжений. На основе разработанной в работе численной методики получены расчетные конечно-разностные соотношения динамических задач в угловых точках прямоугольного отверстия, где нарушается «привычная» для динамических задач гладкость функций. В этих угловых точках первые и вторые производные искомых функций терпят разрыв первого рода. В виде изолинии представлены результаты изменения волновых полей в упругом теле с симметрично-расположенными прямоугольными отверстиями. Исследована концентрация динамических напряжений в окрестности угловых точек прямоугольного отверстия. Путем численной реализации установлена устойчивость расчетных алгоритмов для достаточно большого времени.

Ключевые слова: упругость, волновой процесс, напряжение, скорость, плоская деформация, численное решение.

References

1 Clifton, R.J. (1967). A difference method for plane problems in dynamic elasticity. *Quart. Appl. Math., 25(1),* 97–116.

2 Erzhanov, Zh.S., Karimbaev, T.D., & Baiteliev, T.B. (1983). *Dvumernye volny napriazhenii v odnorodnykh i strukturno-neodnorodnykh sredakh [Two-dimensional stress waves in homogeneous and structurally inhomogeneous media].* Alma-Ata: Nauka [in Russian].

3 Kukudzhanov, V.N. (2011). Deformirovanie, povrezhdaemost i razrushenie sred i materialov s defektami i so strukturami defektov [Deformation, damage and destruction of media and materials with defects and with structures of defects]. *Vestnik Nizhehorodskoho universiteta imeni N.I. Lobachevskoho, 4(4),* 1557–1558 [in Russian].

4 Alexeeva, L.A., & Sarsenov, B.T. (2015). Mathematical model of massive dynamics in the neighbor-hood of disturbance focus. *AIP Conference Proceedings, 1676*, 020067. *dx.doi.org.* Retrieved from http://dx.doi.org /10.1063/1.4930481.

5 Ashirbayev, N., Ashirbayeva, Zh., Sultanbek, T., & Bekmoldayeva, R. (2016). Modeling and solving the two-dimensional non-stationary problem in an elastic body with a rectangular hole. *AIP Conference Proceedings, 1759*, 020078. Retrieved from http://dx.doi.org/10.1063/1.4959692.

6 Ashirbayev, N., Ashirbayeva, Zh., & Shomanbayeva, M. (2017). Features of the Wave Field in a Finite Body with a Noncentral Hole. *AIP Conference Proceedings, 1880*, 060013. Retrieved from http://doi.org /10.1063/1.5000667.

7 Ashirbayev, N., Ashirbayeva, Zh., Sarsenova, G., Bekmoldayeva, R., Nurmaganbetova, Zh., & Duisebayeva, P. (2018). Investigation of Wave Processes in a Rectangular Region with Discontinuous Boundary Conditions. *ISSN 1392-1207. Mechanika, 24(5),* 680–684.

8 Mamaev, Sh.M., & Borasheva, N.T. (2015). Chislennoe issledovanie rasprostraneniia dinamicheskikh vozmushchenii v sostavnykh sterzhniakh setochno-kharakteristicheskim metodom [A numerical study of the propagation of dynamic disturbances in composite rods by the grid-characteristic method]. *Izvestiia NAN RK, 3(301),* 145–154 [in Russian].

9 Seitmuratov, A.Z., Nurlanova, B.M., & Medeubaev, N. (2017). Equations of vibration of a two-dimensionally layered plate strictly based on the decision of various boundary-value problems. *Bulletin of the Karaganda university. Series mathematics, 87(3),* 109–116.

10 Seitmuratov, A.Z., Madelkhanova, A.Z., Parmenova, M.Z., & Kanibaikyzy, K. (2018). Integro-Differential equations with regular kernels. *News of the national academy of sciences of the republic of Kazakhstan. Series physic-mathematical, 2(318),* 37–45.

11 Gavrilov, S.N., & Herman, G.C. (2012). Wave propagation in a semi-infinite heteromodular elastic bar suljected to a harmonic loading. *J. Sound and Vibr.,* 331(20), 4464–4480.

12 Kochaev, A.L., & Brazhe, R.A. (2011). Mathematical modeling of elastic wave propagation in crystals: 3d-wave surfaces. *Acta Mechanika, 222(1–2),* 193–198.

13 Parvanova, S., Dineva, P., & Manolis, G. (2014). Elastic wave fields in a half-plane with free-surface relief, tunnels and multiple buried inclusions. *Acta Mechanica, 225 (7),* 1843–1865.

14 Lys, E.V., Romenski, E.I., Cheverda, V.A., & Epov, M.I. (2013). Interaction of seismic waves with zones of concentration of initial stresses. *Doklady Earth Sciences, 449 (2),* 402–405.

15 Lesan, D., & Scalia, A. (2011). On the deformation of or thotropic Cosserat elastic cylinders. *Mathematics and Mechanics of Solids, 16(2),* 177–199.

16 Israilov, M.Sh. (2014). Reduction of Boundary Value Problems of Dynamic Elasticity to Scalar Problems for Wave Potentials in Curvilinear Coordinates. *Mechanics of Solids, 46 (1),* 104–108.

17 Yi-chao Chen. (2016). On the dynamic behavior of a class of Cauchy elastic materials. *Mathematics and Mechanics of Solids, 16(5),* 563–569.

ИНФОРМАТИКА
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Fuzzy Multi-criteria Selection of Alternatives by Quasi-best Case as the Basis for Choosing Robotic Machine-Assembling Technologies

The process of choosing the robotic machine-assembling technologies (RMAT) is implemented as the fuzzy multi-criteria selection of alternatives by the suggested previousely method of quasi-best case. The basic concept feature of the given method is the developed specific correlations which are based on the corresponding comparisons to better alternative P°nd to the most important criterion. All the mentioned above determines the practical and scientific value of this paper. The results of strict ranking of the elements of local criteria discrete set (LCDS) are input dP°ta. It is performed by the method of expert survey and at the same time demonstrates RMAT phenomenon. The idea of selections the process of ordering constituents of initially unordered LCDS elements which finally form ordered set. The selection is performed within the set of these elements. The obtained ordering of RMAT phenomenon as a result of selection is recommended to be analyzed in the process of choosing. The base of solving the task of RMAT selection is its formalized description and on its base the generalized content formalisms of quasi-best case method are determined. The performance of the presented the theoretical issues is demonstrated step-by-step with the real example of the automated RMAT selection.

Keywords: alternative, automation, selection, local criterion, fuzziness, optimization, robotic machine-assembling technology, quasi-best case.

1 Introduction

Industrial robots (IR) are widely used in modern automated machine - assembling enterprises of machine and instrument engineering which implement robotic machine-assembling technologies (RMAT). The International Federation of Robotics (IFR) reports that annual increment in release and introduction of such universal and expensive means of industrial automation, which IR of various design and technology performance are [1], is about (14–16)% [2, 3] for recent years. The conduct of different by content and formulation researches is important and topical in order to improve the effectiveness and further development of robotic machine-assembling enterprises. It is recommended to develop either latest or using and modifying known approaches, methods and techniques by adapting them to specifics of formulation and content of the tasks solved.

One of the problems which occur here is the problem of proper RMAT selection taking into account their final set that is generated before [4]. It implies the preliminary determination and analysis of the ordered sequence of the selection local criteria (see further).

Every RMAT is presented with the set of phenomena which are local criteria discrete set (LCDS) $S = (S_j | j = \overline{1, m})$. Its components are the following [5]: Gm — geometric; Kn — kinematic; Dn — dynamic; Ct — control; En — energy; Tr — trajectory; $\tau(Q)$ — time (productivity); Rl — reliability; Ec — economy; Ac — accuracy; Fc — force; $Fopt$ — the component which is determined by other types of optimization criteria (e.g. technical and economic [6] etc.):

$$S = (Gm, Kn, Dn, Ct, En, Tr, \tau(Q), Rl, Ec, Ac, Fc, Fopt). \quad (1)$$

The complexity and content feature of RMAT selection tasks can be explained: by the necessity to take into account the desired multi-criteria of the extremity of every LCDS criterion; by the obvious ambivalence related to the ordering of LCDS elements consideration; by the necessity of decision-making within the set with the obtained alternatives. It can be possible provided that there are various criteria with different physical nature and different measurement scales (see above). The presented task is a task of multi-objective optimization in terms of prior ambivalence. The methods of finding solutions to such tasks are featured by variability and multiplicity [7].

The idea of many approaches to solving such tasks is in using information obtained from the experts as a result of sampling by survey method [8]. Here, the calculated value of Kendall concordance coefficient W [8] is the correlation of expert opinions. The ordered sequence (list) of local criteria is the desired solution. This sequence is formed at the correlation of expert opinions $W \approx 1$ (ideally $W = 1$). The using of rank correlation method is possible in other case [8]. The criterion for decision-making here is also the value of W with its indicated interpretation. If W is $\ll 1$, using of other approaches is possible. The examples of such approaches can be the method of pair-wise comparison of alternatives, which is based on the ideas of Bellman-Zade [9, 10], and Saaty hierarchies [11] and also fuzzy multi-criteria selection of alternatives by worst-case suggested by Rotstein [12]. The first ones among them (methods of Bellman-Zade and Saaty) are time-consuming and it is due to the performance of total alternative enumerating at the pair-wise comparison and the long processing of matrix information with the further computation of membership function as to every single expert as well as to every single alternative. The latter approach (Rotstein method) does not require time-consuming matrix formation of pair-wise comparison and further processing of this Information. Relatively simple computation a ratios correlations are used instead. It is profoundly compared to the worst alternative and the least important criterion [12].

The scale of corresponding rates in the given task is 12-score one (by the number of elements within LCDS, see the expression (1). It is the scale used by every among 10 experts ($n = 10$) to estimate each local criterion of LCDS without repeating estimations for various local criteria, i.e. the strict ranking [8] of LCDS is performed. Here the least important criterion obtained score 1 and the most important one got 12.

The result analysis of expert survey (see Table 1) as to consistency of experts by Kendall concordance coefficient for the primary processing ($W = 0,204$) and by rank correlation ($W = 0,271$) demonstrates the discordance of the experts' opinions. Such discordance does not contradict the possibility of using other methods of fuzzy multi-criteria selection of alternatives, e.g. by the method of the worst-case [12], that has been already used while selecting RMAT fuzzy multi-criteria [13].

The main idea of the method of quasi-best case [14] applied here is the answer to the regular question: why would not make another principle as the basis of the process of fuzzy multi-criteria selection of alternatives, if the optimal solution used by any approaches at fuzzy multi-criteria alternative selection is either de-jure or de-facto unknown. The principle of comparison of each from local criterion of LCDS to other local criterion, for example, the best one, can serve as a solution. The method is named as "quasi-best" due to the relativity of the term "best".

Table 1

Matrix $M_c[n \times m]$ of expert survey results $({}_i c_j)$

	$S_1 = Gm$	$S_2 = Kn$	$S_3 = Dn$	$S_4 = Ct$	$S_5 = En$	$S_6 = Tr$	$S_7 = \tau(Q)$	$S_8 = Rl$	$S_9 = Ec$	$S_{10} = Ac$	$S_{11} = Fc$	$S_{12} = Fopt$
E_1	11	12	3	10	4	9	8	5	6	7	2	1
E_2	5	4	2	3	7	6	10	9	11	8	1	12
E_3	9	10	8	7	5	12	6	4	1	11	3	2
E_4	5	9	10	12	6	4	7	3	8	1	11	2
E_5	12	11	4	5	6	7	2	8	3	9	10	1
E_6	3	8	4	2	12	5	11	7	9	10	6	1
E_7	9	8	7	10	5	4	2	12	1	11	6	3
E_8	7	10	6	5	12	11	9	3	2	8	1	4
E_9	4	5	7	8	9	1	6	12	10	11	2	3
E_{10}	9	12	3	5	7	11	10	2	6	8	4	1
$\sum S_j$	74	89	54	67	73	70	71	65	57	84	46	30

In general, the substantive *features* of the method of quasi-best case [14] used is the decrease of subjectivity level at the process of forming the ordered set of LCDS criteria and using the corresponding comparisons to the best alternative as well as to the most important criterion. It is the most important while solving the given task.

Therefore, the adapted to the content of quasi-best case method, the content of the task for RMAT multi-criteria selection becomes the fuzzy forming the ordered set of local criteria from LCDS for their further analysis. It is performed within the set of alternatives (expertsвЂ™ opinions) and their content is determined by the results of strict ranking of LCDS elements. Taking into account all the mentioned above, the purpose of the given paper is to increase feasibility and to decrease time-consuming factor of decision-making at selecting RMAT. It is recommended to use scientific and methodological set ordering of discrete local criteria of RMAT phenomenon by applying fuzzy multi-criteria selection of alternatives with the method of quasi-best case.

2 Formalized task statement

In its general form the essence of fuzzy multi-criteria RMAT selection [14] comes to forming LCDS element set $S = (S_j | j = \overline{1,m})$ from initially unordered one into finally ordered set $S_{<>} = S_j | j = \overline{1,m}$ as a result of execution of some certain computation procedures $\varphi = (\varphi_k | k = \overline{1,l})$ with total number l. In the given case $l = 7$ equals the number of methodically determined steps ***Рљ*** (see further). Every φ_k-th computation procedure, as well as its corresponding step, implements the obtaining of intermediate and final results. The latter ones are calculated within the information sets of input data, namely, within the set of experts $E = (E_i | i = \overline{1,n})$ and RMAT $S = (S_j | j = \overline{1,m})$. Therefore, the simplified form is:

$$\varphi : (E \times S) \to \langle S_{(j)max} \rangle \quad (2)$$

Here $\to$ is the symbol of suractive reflection of input data with united by Cartesian product (symbol $\times$) to corresponding computation data [15], which are implemented by the mentioned above set of computation procedures φ from [14]: $\varphi = (\varphi_c,\ \varphi_w,\ \varphi_\alpha,\ \varphi_{Ew},\ \varphi_{Ew^a},\ \varphi_{(j)max},\ \varphi_{<>})$.

Expression (2) obtains the form:

$$\begin{gathered}\left(\varphi = \left(\varphi_k | k = \overline{1,l}\right)\right) : ((((((((E = (E_i | i = \overline{1,n})) \times (S = (S_j | j = \overline{1,m})) \xrightarrow{K1} \\ ((_i c_j) \subset M_c)) \xrightarrow{K2} ((_i w_j) \subset M_w)) \xrightarrow{K3} ((_i \alpha_j) \subset M_\alpha)) \xrightarrow{K4} ((_E w_j) \subset M_{E^W})) \xrightarrow{K5} \\ \left(\left({}_i w_j^{(_i\alpha_j)max}\right) \subset M_{Ew^\alpha}\right)\right) \xrightarrow{K6} \left({}_i w_j^{\alpha\,\max}\right) \subset \left(S_{(j)\max}\right)\,) \xrightarrow{K7} \langle S_{(j)\max} \rangle | \forall i = \overline{1,n}; \forall j = \overline{1,m}.\end{gathered} \quad (3)$$

Here ***Рљ1***, ... , ***Рљ7*** are the designations of methodically resulted steps that correspond the implementation of the corresponding procedure set $\varphi = (\varphi_c,\ \varphi_w,\ \varphi_\alpha,\ \varphi_{Ew},\ \varphi_{Ew^a},\ \varphi_{(j)_{\max}},\ \varphi_{<>})$ with [14], namely:

– ***Рљ1*** is in correspondence with the procedure φ_c, that forms matrix M_c of final results of strict expert ranking, matrix M_c elements $(_i c_j)$ are integral natural numbers. The value and significance of every number are determined by ranking conditions;

– ***Рљ2*** corresponds the implementation of procedure φ_w which determines elements $(_i w_j)$ of matrix M_w as the weight of all alternatives by relation of ranks of all E_i-th alternatives to the rank of the best alternative $S_{j\,\max}$;

– step ***Рљ3*** (procedure φ_α) is used to form elements $(_i \alpha_j)$ of matrix M_α as fuzzy set taking into account the significance of every S_j-th criterion due to its weight α_j within the set of alternatives E;

– the content of step ***Рљ4*** is the implementation of procedure φ_{Ew}, that determines the importance of estimation of every E_i-th expert from the set E by determining the weights of alternatives related to S_j-th criterion, i.e. finding elements $(_E w_j)$ of matrix M_{Ew};

– ***Рљ5*** implements procedure φ_{Ew^a} of determination of significance of alternative (of every E_i-th expert) by the weight $_E\alpha$ of every of them within the set of criteria S forming elements $\left({}_i w_j^{(_i\alpha_j)\max}\right)$

of matrix M_{Ew^a} as fuzzy set; the very content of step ***K5*** determines the significant peculiarities of quasi-best case method;

– by implementing ***PЉ6***, elements $\left({}_iw_j^{({}_i\alpha_j)\max}\right)$ max of set $\left(S_{(j)\max}|j=\overline{1,m}\right)$ of fuzzy estimations for every local criterion $S_{j\max}$ are formed. It means that unordered set of membership functions is formed within the set of their highest values and this is the content of procedure $\varphi_{(j)\max}$ execution;

– ***PЉ7*** is used to implement procedure $\varphi_{<>}$ that orders elements of unordered set $\left(S_{(j)\max}|j=\overline{1,m}\right)$ obtained within ***K6*** into the ordered one $S_{<>}=\langle S_{(j)\max}|j=\overline{1,m}\rangle$ by solving *max-max* task. This is the final solution to the task of fuzzy multi-criteria selection of alternatives using the method of quasi-best case.

All mentioned above matrixes and namelCŕ M_c, M_w, M_α, M_{Ew^α} and M_{Ew^a} have the dimension $[n\times m]$ and sets $\left(S_{(j)\max}|j=\overline{1,m}\right)$ and $\langle S_{(j)\max}|j=\overline{1,m}\rangle$ have dimension $[1\times m]$.

3 Task solution

The process of fuzzy multi-criteria of RMAT selection is presented step by step ***PЉ1***, ... , ***PЉ7*** based on the results of actual strict expert sampling. It is performed by using the proposed method in accordance with its content [14] and formalized statement of the given task (see expression (3)).

PЉ1. Forming matrix $M_c[n\times m]$ of the results of expert sampling. Every element ${}_ic_j$ of this matrix is estimation (rank) of 12-score scale, which was used by every E_i-th expert to estimate every S_j-th criterion. Matrix $M_c[n\times m]$, as well as other matrixes, has the form of a table in the given case Table 1.

K2. The calculation of weights ${}_iw_j$ of alternatives by ratio of ranks ${}_ir_j$ of all E_i-th alternatives (Table 1) to the rank of the best alternative ${}_iw_j$ max. The latter one for every S_j-th criterion is determined as following:

$$ {}_iw_j\max=\frac{{}_ir_j\max}{\sum_{i=1}^{n}{}_ir_j}\mid\forall j=\overline{1,m}. \qquad (4) $$

Here the nominator is actually the total of ranks (estimations) given by the experts of set E for every S_j-th criterion. For matrix $M_c[n\times m]$ (Table 1) this is the total of elements from its every column. For example, the highest rank (score) 12 was given by expert E_5 to $S_1=Gm$. It means that this rating is the highest ${}_5r_{Gm}\max=12$ by the given criterion within the set of all experts (alternatives). Here and further the pre subscript specifies the reference number of expert (in the given case i=5) by Table 1. The weight of the best alternative Gm_5w_{Gm} max within the set of estimations of all experts (alternatives) taking into account (4) is determined with following expression:

$$ \sum_{i=1}^{n}{}_ir_j=\frac{{}_1r_{Gm}}{{}_5r_{Gm}\max}+\frac{{}_2r_{Gm}}{{}_5r_{Gm}\max}+\frac{{}_3r_{Gm}}{{}_5r_{Gm}\max}+\frac{{}_4r_{Gm}}{{}_5r_{Gm}\max}+\frac{{}_5r_{Gm}}{{}_5r_{Gm}\max}+\frac{{}_6r_{Gm}}{{}_5r_{Gm}\max}+ $$

$$ +\frac{{}_7r_{Gm}}{{}_5r_{Gm}\max}+\frac{{}_8r_{Gm}}{{}_5r_{Gm}\max}+\frac{{}_9r_{Gm}}{{}_5r_{Gm}\max}+\frac{{}_{10}r_{Gm}}{{}_5r_{Gm}\max}= $$

$$ =\frac{{}_1r_{Gm}+_2r_{Gm}+_3r_{Gm}+_4r_{Gm}+_5r_{Gm}+_6r_{Gm}+_7r_{Gm}+_8r_{Gm}+_9r_{Gm}+_{10}r_{Gm}}{{}_5r_{Gm}\max}; $$

$$ {}_5w_{Gm}\max=\frac{1}{\frac{{}_1r_{Gm}+{}_2r_{Gm}+{}_3r_{Gm}+{}_4r_{Gm}+{}_5r_{Gm}+{}_6r_{Gm}+{}_7r_{Gm}+{}_8r_{Gm}+{}_9r_{Gm}+{}_{10}r_{Gm}}{{}_5r_{Gm}\max}}= $$

$$ =\frac{1}{\frac{11}{12}+\frac{5}{12}+\frac{9}{12}+\frac{5}{12}+\frac{12}{12}+\frac{3}{12}+\frac{9}{12}+\frac{7}{12}+\frac{4}{12}+\frac{9}{12}}=\frac{12}{74}=0,1622. $$

Obtained similar to (4) weights for all other criteria of set S practically form matrix $M_w[n\times m]$ and are added to Table 2.

Table 2

Matrix $M_w[n \times m]$ of alternative weights $({}_iw_j)$ for various criteria as fuzzy set

	Gm	Kn	Dn	C	E	T	$\tau(Q)$	Rl	Ec	Ac	Fc	$Fopt$
E_1	.1487	**.1348**	.0556	.1493	.0548	.1286	.1127	.0769	.1053	.0833	.0435	.0333
E_2	.0676	.0449	.0370	.0448	.0969	.0857	.1409	.1385	**.1930**	.0952	.0217	**.4000**
E_3	.1216	.1124	.1482	.1045	.0685	.1714	.0845	.0615	.0175	**.1310**	.0652	.0667
E_4	.0676	.1011	**.1852**	**.1791**	.0822	.0571	.0986	.0462	.1404	.0119	**.2391**	.0667
E_5	**.1622**	.1236	.0741	.0746	.0822	.1000	.0282	.1231	.0526	.1071	.2174	.0333
E_6	.0406	.0899	.0741	.0299	**.1644**	.0714	**.1549**	.1077	.1579	.1191	.1304	.0333
E_7	.1216	.0899	.1296	.1493	.0685	.0571	.0282	**.1846**	.0175	**.1310**	.1304	.1000
E_8	.0946	.1124	.1111	.0746	**.1644**	.1571	.1268	.0462	.0351	.0952	.0217	.1333
E_9	.0541	.0562	.1296	.1194	.1233	.0143	.0845	**.1846**	.1754	**.1310**	.0435	.1000
E_{10}	.1216	**.1348**	.0556	.0746	.0959	.1571	.1409	.0308	. 1053	.0952	.0870	.0333

Here, the following condition is met for every S_j-th criterion (column of the table 2):

$$\left(\sum_{i=1}^{n} {}_iw_j | \forall j = \overline{1,m} \right) = 1 \tag{5}$$

Obtained elements of matrix $M_w[n \times m]$ (Table 2) are quantitative estimation of membership degree for every S_j-th criterion from LCDS to fuzzy sets. They can be explanted as weights which are included into fuzzy sets (4).

Obtained weights of alternatives for various criteria (see Table 2) allow to present criteria as fuzzy sets that are given within the universal sets of alternatives. It enables every S_j-th criterion forming set D_w by selecting maximum element (underlined in Table 2). Here and further some data in brackets]...[is not calculated but it is of informative character. Finally we obtain:

$$D_w = \left(\frac{{}_iw_1 \max}{]S_1[}; \ldots; \frac{{}_iw_j \max}{]S_j[}; \ldots; \frac{{}_iw_m \max}{]S_m[} \right) = \left(\frac{{}_iw_j \max}{]S_j[} | i = \overline{1,n}; \forall j = \overline{1,m} \right).$$

For the task of RMAT selection that is being solved in the given presentation for every S_j-th criterion, we obtain:

$$D_w = \left(\frac{0,1622}{]Gm[}; \frac{0,1348}{]Kn[}; \frac{0,1852}{]Dn[}; \frac{0,1741}{]Ct[}; \frac{0,1644}{]En[}; \frac{0,1714}{]Tr[}; \frac{0,1549}{]\tau(Q)[}; \frac{0,1846}{]Rl[}; \frac{0,1930}{]Ec[}; \frac{0,1310}{]Ac[}; \frac{0,2391}{]Fc[}; \frac{0,4000}{]Fopt[} \right).$$

The elements (i.e. the numerators) ${}_iw_j \max$ of obtained set D_w can be explanted as a set of potentially good solutions [12].

The obtained set D_w is a fuzzy one and it contains alternative membership in relation to optimal in some sense solution. By analyzing elements of set D_w the ranking of its elements from *max*to *min* was performed and ordered set $D_{\langle w \rangle}$ of local criteria form set S was formed. *Max-max* task is applied to demonstrate the elements of matrix $M_w[n \times m]$ (Table 2):

$$D_{\langle w \rangle} = \left\langle \frac{0,4000}{]Fopt[}; \frac{0,2391}{]Fc[}; \frac{0,1930}{]Ec[}; \frac{0,1846}{]Rl[}; \frac{0,1852}{]Dn[}; \frac{0,1741}{]Ct[}; \frac{0,1714}{]Tr[}; \frac{0,1644}{]En[}; \frac{0,1622}{]Gm[}; \frac{0,1549}{]\tau(Q)[}; \frac{0,1348}{]Kn[}; \frac{0,1310}{]Ac[} \right\rangle.$$

Рљ3. Determination of significance of every single criterion within set S. Let α_j be the weight for criterion $S_j \subset S$ that characterizes its significance. Taking into account the weights of criteria from Table 2 the fuzzy set of solutions is formed as following [9]:

$$D = ({}_iw_1)^{\alpha_1} \cap ({}_iw_2)^{\alpha_2} \cap \ldots \cap ({}_iw_j)^{\alpha_j} \cap \ldots \cap ({}_iw_m)^{\alpha_m} \, |\forall j = \overline{1,m} = \cap_i^n \, ({}_iw_j)^{\alpha_j} \, |\forall j = \overline{1,m}.$$

Criteria are compared only to the most significant one (the best) among them at the next stage. Here it is accepted that the more significant the weight α_j for S_j-th criterion is, the higher is its range R_j [12]:

$$\frac{\alpha_1}{R_1} = \frac{\alpha_2}{R_2} = \ldots = \frac{\alpha_j}{R_j} = \ldots = \frac{\alpha_m}{R_m}.$$

Let α_j max and R_j max be weight and rank correspondingly for the most significant criterion S_j. If the requirement (5) is met regarding parameter ${}_i\alpha_j$, i.e. ${}_i\alpha_j \left(\sum_{i=1}^n {}_i\alpha_j \, |\forall j = \overline{1,m}\right) = 1$ by the similar way to ${}_iw_j$ (see expression (4)), the weights of criteria are distributed in accordance with the ranks as following:

$$ {}_i\alpha_{jmax} = \frac{1}{\frac{R_1}{R_{jmax}} + \frac{R_2}{R_{jmax}} + \ldots + \frac{R_j}{R_{jmax}} + \ldots + \frac{R_m}{R_{jmax}}} = \frac{1}{\sum_{j=1}^m \frac{R_j}{R_{jmax}}} = \frac{R_{jmax}}{\sum_{j=1}^m R_j}; \qquad (6)$$

$$ {}_i\alpha_1 = {}_i\,\alpha_{jmax}\frac{R_1}{R_{jmax}}; \ldots; \ {}_i\alpha_j = {}_i\,\alpha_{jmax}\frac{R_j}{R_{jmax}}; \ldots; \ {}_i\alpha_m = {}_i\,\alpha_{jmax}\frac{R_m}{R_{jmax}}. \qquad (7)$$

As one can see in Table 1, the total of elements in column 2 equals 89 and is the biggest one in relation to the totals of elements of other columns that characterize the ranks of other local criteria as LCDS elements. It means that criterion Kn is the most significant one resulting from the expert sampling being analyzed:

$$ {}_i\alpha_{jmax} = {}_i\,\alpha_2 = {}_i\,\alpha_{Kn}; \ R_{2max} = \sum_{i=1}^n {}_iS_2 = 89.$$

Weights of all other local criteria of set S are calculated by using (6) and (7):

$$ {}_i\alpha_{Kn} = {}_i\,\alpha_{2max} = \frac{1}{\frac{74+89+54+67+73+70+71+65+57+84+46+30}{89}} = \frac{89}{780} = 0,1141;$$

$$ {}_i\alpha_{Kn} = {}_i\,\alpha_{2max} = {}_i\,\alpha_{Kn} * \frac{74}{89} = 0,1141 * \frac{74}{89} = 0,0949; \ldots;$$

$$ {}_i\alpha_{Opt} = {}_i\,\alpha_{Kn} * \frac{30}{89} = 0,1141 * \frac{30}{89} = 0,0385.$$

The obtained values of weights ${}_i\alpha_j$ for every S_j-th criterion within the set of alternatives (experts) E allow finding every fuzzy criterion within the set of alternatives as following:

$$S_j = \left(\frac{{}_iw_j^{({}_i\alpha_j)}}{]E_i[} | i = \overline{1,n}; \forall j = \overline{1,m} \right),$$

where ${}_iw_j$ are the elements of matrix $M_w[n \times m]$ (see Table 2); ${}_i\alpha_j$ is the power to which all corresponding elements of matrix $M_w[n \times m]$ are raised and it is expressed with (6) and (7).

Therefore:

$$Gm = \left(\frac{{}_iw_{Gm}^{({}_i\alpha_{Gm})}}{]E_1[} | i = \overline{1,n}; \forall j = \overline{1,m} \right) =$$

$$= \Big(\frac{0,1487^{0.0949}}{]E_1[}; \ \frac{0,0676^{0.0949}}{]E_2[}; \ \frac{0,1216^{0.0949}}{]E_3[}; \ \frac{0,0666^{0.0949}}{]E_4[}; \ \frac{0,1622^{0.0949}}{]E_5[}; \ \frac{0,0405^{0.0949}}{]E_6[};$$

$$\frac{0,1216^{0.0949}}{]E_7[}; \ \frac{0,0946^{0.0949}}{]E_8[}; \ \frac{0,0541^{0.0949}}{]E_9[}; \ \frac{0,1216^{0.0949}}{]E_{10}[} \Big),$$

otherwise we obtain:

$$Gm = \left(\frac{0,8346}{]E_1[}; \frac{0,7744}{]E_2[}; \frac{0,8188}{]E_3[}; \frac{0,7744}{]E_4[}; \frac{0,8415}{]E_5[}; \frac{0,7378}{]E_6[}; \frac{0,8188}{]E_7[}; \frac{0,7995}{]E_8[}; \frac{0,7582}{]E_9[}; \frac{0,8188}{]E_{10}[} \right).$$

The calculations for all the other local criteria within LCDS, i.e. within set S, are performed similarly and they are transferred to Table 3. Matrix $M_\alpha[n \times m]$ is formed and its elements are significance of every S_j-th criterion within LCDS due to its weight α_j within the set of alternatives E.

Table 3

Matrix $M_\alpha[n \times m]$ of significance of every of its criteria due to its weight α_j within the set of alternatives E as a fuzzy set

	Gm	Kn	Dn	C	E	T	$\tau(Q)$	Rl	Ec	Ac	Fc	$Fopt$
E_1	.8346	.7956	.8187	.8493	.7620	.8319	.8198	.8076	.8483	.7652	.8312	**.8374**
E_2	.7744	.7019	.7960	.7658	.2030	.8021	.8366	.8481	.8867	.7763	.7979	**.9654**
E_3	.8188	.7792	.8762	.8236	.7781	.8536	.7986	.7927	.7442	.8034	.8513	**.0667**
E_4	.7744	.7699	.8898	.8627	.7915	.7735	.8099	.7739	.8663	.6205	**.9191**	.9011
E_5	.8415	.7878	.8351	.8002	.7915	.8133	.7226	.8398	.8064	.7862	**.9139**	.8774
E_6	.7379	.7597	.8351	.7396	.8445	.7891	.8439	.8305	.8738	.7952	**.8868**	.8774
E_7	.8188	.7597	.8681	.8493	.7781	.7735	.7226	.8687	.7442	.8034	.8868	**.9153**
E_8	.7995	.7792	.8589	.8002	.8445	.8470	.8286	.7739	.7829	.7763	.7979	**.9254**
E_9	.7582	.7200	.8681	.8331	.8221	.6830	.7986	.8687	.8806	.8034	.8312	**.9153**
E_{10}	.8188	.7956	.8187	.8002	.8030	.8470	.8357	.7482	.8483	.7763	.8659	**.8774**

Values of matrix $M_\alpha[n \times m]$ elements (see Table 3) give the possibility to form set D_I. Its elements indicate the degree of membership as to optimal solution. It means that they contain maximum elements of matrix $M_\alpha[n \times m]$ and they are underlined for every E_i-th expert (see denominator of every element in brackets $]E_i[$ of set D_I):

$$D_I = \left(\frac{0,8774}{]E_1[}; \frac{0,9654}{]E_2[}; \frac{0,9011}{]E_3[}; \frac{0,9191}{]E_4[}; \frac{0,9139}{]E_5[}; \frac{0,8868}{]E_6[}; \frac{0,9153}{]E_7[}; \frac{0,9254}{]E_8[}; \frac{0,9153}{]E_9[}; \frac{0,8774}{]E_{10}[} \right).$$

Ordering of elements of set D_I from *max* to *min* gives ordered set $D_{\langle I \rangle}$ that characterizes ordered significance of experts by the degree of membership of alternatives to optimal solution:

$$D_{\langle I \rangle} = \left\langle \frac{0,9654}{]E_2[}; \frac{0,9254}{]E_8[}; \frac{0,9191}{]E_4[}; \left(\frac{0,9153}{]E_7[}; \frac{0,9153}{]E_9[} \right); \frac{0,9139}{]E_5[}; \frac{0,9011}{]E_3[}; \frac{0,8868}{]E_6[}; \left(\frac{0,8774}{]E_1[}; \frac{0,8774}{]E_{10}[} \right) \right\rangle.$$

K4. Determination of assessment significance $({}_iw_j)$ of every E_i-th expert within their set $E = \left(E_i | i = \overline{1,n}\right)$, i.e. calculation of alternative weights regarding every S_j-th criterion within LCDS. The content of this step is similar to ***K3*** execution taking into account the essence of step ***K2*** and data of Table 1, but it relates every E_i-th expert.

Table 1 is used to determine the biggest total of elements of columns for all local criteria within LCDS. It means that elements of set S are taken to determine rank ${}_ir_j$ of corresponding S_j-th criterion. Obtained data is used for further calculation while performing step ***K4***.

Therefore, local criterion Kn ${}_1r_{j\,\max} = {}_1 r_{Kn} = 12$ obtains the highest estimation for E_i-th expert (lower left index in ${}_ir_j$ and in ${}_iw_j$). It makes the following calculations determine weights of alternatives for E_1 possible:

$${}_1w_{Kn} = \frac{1}{\frac{r_{Gm}}{r_{Kn}} + \frac{r_{Kn}}{r_{Kn}} + \frac{r_{Dn}}{r_{Kn}} + \frac{r_{Ct}}{r_{Kn}} + \frac{r_{En}}{r_{Kn}} + \frac{r_{Tr}}{r_{Kn}} + \frac{r_{\tau(Q)}}{r_{Kn}} + \frac{r_{Rl}}{r_{Kn}} + \frac{r_{Ec}}{r_{Kn}} + \frac{r_{Ac}}{r_{Kn}} + \frac{r_{Fc}}{r_{Kn}} + \frac{r_{Fopt}}{r_{Kn}}} = $$

$$= \frac{r_{Kn}}{\sum_{j=1}^{m} {}_ir_j} | \forall i = \overline{1,n}.$$

In general case we get the following for every E_i-th expert:

$$ {}_iw_j = \frac{{}_ir_j \max}{\sum_{j=1}^{m} {}_ir_j} | \forall i = \overline{1,n}. \tag{8}$$

The calculations for determining alternative weights for expert E_1 for all local criteria, in other words criteria being analyzed within set S, are the following:

$$ {}_1w_{Kn} = \frac{12}{78} = 0,1539; \; {}_1w_{Gm} = {}_1 w_{Kn} * \frac{r_{Gm}}{r_{Kn}} = 0,1539 * \frac{11}{12} = 0,1410; \quad \text{and so on}$$

$$ {}_1w_{Fopt} = {}_1w_{Kn}, \dots, \; {}_1w_{Fopt} = {}_1w_{Kn} * \frac{r_{Fopt}}{r_{Kn}} = 0,1539 * \frac{1}{12} = 0,0128 \; .$$

Other elements ${}_iw_j$ are calculated similarly following expression (8) for all criteria of set S for every expert $E_i \subset E$ and are transferred to Table 4. Maximum values of elements in every line are underlined.

K5. Determination of alternative significance (expert opinion) by determination of alternative weights in relation to every criterion. The content of the given step is the actions similar to ***K3*** actions and they relate not every S_j-th criterion, but every E_i-th alternative. Here every element of each i-th line (fuzzy information from every E_i-th expert) of matrix $M_{E^w}[n \times m]$ (Table 4) is raised to power which is maximum element of corresponding line (E_i-th expert) of matrix $M_\alpha[n \times m]$, i.e. $({}_ia_j)\max$ is underlined (see Table 3). The following set is formed in such way

$$ {}_iw_j^{({}_i\alpha_j)\max} = \left(\frac{{}_iw_j^{({}_i\alpha_j)\max}}{]S_j[} | \forall j = \overline{1,m}; \; i = \overline{1,n} \right). \tag{9}$$

Table 4

Matrix $M_{E^w}[n \times m]$ of weights of expert opinion ${}_iw_j$ in relation to every criterion as fuzzy set

	Gm	Kn	Dn	C	E	T	$\tau(Q)$	Rl	Ec	Ac	Fc	$Fopt$
E_1	.1410	**.1539**	.0385	.1282	.0513	.1154	.1026	.0641	.0769	.0897	.0256	.0128
E_2	.0641	.0513	.0256	.0385	.0897	.0769	.1282	.1154	**.1410**	.1026	.0128	.1539
E_3	.1154	.1282	.1026	.0897	.0641	**.1539**	.0769	.0513	.0128	**.1410**	.0385	.0256
E_4	.0641	.1154	**.1282**	**.1539**	.0769	.0513	.0897	.0385	.1026	.0128	**.1410**	.0256
E_5	**.1539**	.1410	.0513	.0641	.0769	.0897	.0256	.1026	.0385	.1154	.1282	.0128
E_6	.0385	.1026	.0513	.0256	**.1539**	.0641	**.1410**	.0897	.1154	.1282	.0769	.0128
E_7	.1154	.1026	.0897	.1282	.0641	.0513	.0256	**.1539**	.0128	**.1410**	.0769	.0385
E_8	.0897	.1282	.0769	.0641	**.1539**	.1410	.1154	.0385	.0256	.1026	.0128	.0513
E_9	.0513	.0641	.0897	.1026	.1154	.0128	.0769	**.1539**	.1282	**.1410**	.0256	.0385
E_{10}	.1154	**.1539**	.0385	.0641	.0897	.1410	.1282	.0257	.0769	.1026	.9513	.0128

For example, each element in Table 4 of matrix $M_{E^w}[n \times m]$, i.e. 0,1410 for Gm (0,1539 for Kn and so on) is raised to power $({}_ia_j)\max = ({}_1a_{Gm})\max = 0,8774$ for every expert E_1. Therefore, we have the following for E_1:

$$ {}_1w_j^{({}_i\alpha_j)\max} = \Big(\frac{0,1410^{0,8774}}{]Gm[}; \; \frac{0,1539^{0,8774}}{]Kn[}; \; \frac{0,0385^{0,8774}}{]Dn[}; \; \frac{0,1282^{0,8774}}{]Ct[}; \; \frac{0,0513^{0,8774}}{]En[}; $$

$$ \frac{0,1154^{0,8774}}{]Tr[}; \; \frac{0,1026^{0,8774}}{]\tau(Q)[}; \; \frac{0,0641^{0,8774}}{]Rl[}; \; \frac{0,0769^{0,8774}}{]Ec[}; \; \frac{0,0897^{0,8774}}{]Ac[}; $$

$$ \frac{0,0256^{0,8774}}{]Fc[}; \; \frac{0,0128^{0,8774}}{]Fopt[} \Big), $$

otherwise, after calculations have been performed:

$$ {}_1w_j^{(_i\alpha_j)\max} = \left(\frac{0,1793}{]Gm[}; \frac{0,1935}{]Kn[}; \frac{0,0574}{]Dn[}; \frac{0,1649}{]Ct[}; \frac{0,0738}{]En[}; \frac{0,1504}{]Tr[}; \frac{0,1356}{]\tau(Q)[}; \frac{0,0878}{]Rl[}; \frac{0,1054}{]Ec[}; \frac{0,1206}{]Ac[}; \frac{0,0402}{]Fc[}; \frac{0,0219}{]Fopt[} \right). $$

Similar calculations are executed for other experts of set E by expression (9) and all data is transferred to Table 5. The elements of this table are $\left({}_iw_j^{(_i\alpha_j)\max}\right)$ and they form matrix $M_{Ew^\alpha}[n\times m]$ by implementing computation procedure of φ_{Ew^α} [14].

K6. Obtaining the fuzzy sets (solutions) of matrix $M_{Ew^\alpha}[n\times m]$ for every S_j-th criterion. Maximum value $\left({}_iw_j^{(_i\alpha_j)\max}\right)$ max is selected and underlined (see Table 5) for every S_j-th criterion of matrix $M_{Ew^\alpha}[n\times m]$. Procedure $\varphi_{(j)\max}$ is used to form ordered (in context of selection the maximum value $\left({}_iw_j^{(_i\alpha_j)\max}\right)$ max for every S_j-th criterion) set $\left(S_{(j)\max}|j=\overline{1,m}\right)$ of solutions:

$$ \left(S_{(j)\max}|j=\overline{1,m}\right) = \left(\frac{\left({}_iw_j^{(_i\alpha_j)\max}\right)\max}{]S_j[} | \forall j=\overline{1,m}; i=\overline{1,n} \right). $$

We obtain the following for the example considered in Table 5:

$$ \left(S_{(j)\max}|j=\overline{1,m}\right) = \left(\frac{0,1807}{]Gm[}; \frac{0,1935}{]Kn[}; \frac{0,1514}{]Dn[}; \frac{0,1790}{]Ct[}; \frac{0,1902}{]En[}; \frac{0,1851}{]Tr[}; \frac{0,1760}{]\tau(Q)[}; \frac{0,1803}{]Rl[}; \frac{0,1526}{]Ec[}; \frac{0,1712}{]Ac[}; \frac{0,1653}{]Fc[}; \frac{0,1642}{]Fopt[} \right). $$

Table 5

Matrix $M_{Ew^\alpha}[n\times m]$ of weights of expert opinion ${}_iw_j$ in relation to every criterion as fuzzy set Matrix $M_{Ew^\alpha}[n\times m]$ of significance of alternatives (expert opinions) by determining weighs of alternatives regarding every S_j-th criterion $\left({}_iw_j^{(_i\alpha_j)\max}\right)$ as fuzzy set

	Gm	Kn	Dn	C	E	T	$\tau(Q)$	Rl	Ec	Ac	Fc	$Fopt$
E_1	.1793	**.1935**	.0574	.1649	.0738	.1504	.1356	.0898	.1054	.1206	.0402	.0219
E_2	.0705	.0568	.0291	.0431	.0976	.0840	.1377	.1243	.1509	.1110	.0149	**.1642**
E_3	.1429	.1571	.1285	.1139	.0841	**.1851**	.0991	.0688	.0197	**.1712**	.0531	.0368
E_4	.0801	.1374	**.1514**	**.1790**	.0947	.0652	.1091	.0501	.1233	.0182	**.1653**	.0345
E_5	**.1807**	.1669	.0662	.0812	.0959	.1104	.0352	.1248	.0509	.1390	.1530	.0187
E_6	.0556	.1327	.0718	.0388	**.1902**	.0875	**.1760**	.1179	.1473	.1618	.1028	.0210
E_7	.1386	.1244	.1101	.1526	.0809	.0660	.0350	**.1803**	.0186	**.1665**	.0956	.05078
E_8	.1074	.1494	.0931	.0787	.1769	.1632	.1355	.0491	.0337	.1216	.0177	.0640
E_9	.0660	.0809	.1101	.1244	.1386	.0186	.0956	**.1830**	**.1526**	.1665	.0350	.0507
E_{10}	.1504	**.1935**	.0574	.0898	.1206	.1793	.1649	.0402	.1054	.1356	.0738	.0029

K7. Ordering of elements of set $\left(S_{(j)\max}|j=\overline{1,m}\right)$ by implementing procedure $\varphi_{<>}$ applying the rule from *max* to *min* in relation to fuzzy estimations (nominators of each from elements within set $\left(S_{(j)\max}|j=\overline{1,m}\right)$ of every S_j-th criterion of LCDS. It means that *max-min* task regarding elements of matrix $M_{Ew^\alpha}[n\times m]$ is solved by ordering elements of set $\left(S_{(j)\max}|j=\overline{1,m}\right)$ forming ordered set $\langle S_{(j)\max}|j=\overline{1,m}\rangle$:

$$ \langle S_{(j)\max}|j=\overline{1,m}\rangle = \langle \frac{{}_iw_j^{(_i\alpha_j)\max}\max}{]S_j[} | \forall j=\overline{1,m}; i=\overline{1,n}\rangle. $$

We have the following in the given case:

$$\begin{gathered}\langle S_{(j)\max}|j=\overline{1,m}\rangle=\Big\langle\frac{0,1935}{]Kn[};\frac{0,1902}{]En[};\frac{0,1851}{]Tr[};\frac{0,1807}{]Gm[};\frac{0,1803}{]Rl[};\frac{0,1790}{]Ct[};\\ \frac{0,1760}{]\tau(Q)[};\frac{0,1712}{]Ac[};\frac{0,1653}{]Fc[};\frac{0,1642}{]Fopt[};\frac{0,1526}{]Ec[};\frac{0,1514}{]Dn[}\Big\rangle.\end{gathered} \tag{10}$$

Ordered list of local criteria in brackets $]\ldots[$ is the solution to the task being solved, which is the task of multi-criteria of RMAT selection by the method of quasi-best case. It means that the final result of solution to the task being solved taking into account (10) is the following (index $QBMS$ demonstrates the result obtained with the method of quasi-best case):

$$\langle S_{(j)\max}|j=\overline{1,m}\rangle_{QBMS}=\langle Kn,En,Tr,Gm,Rl,Ct,\tau\left(Q\right),Ac,Fc,Fopt,Ec,Dn\rangle. \tag{11}$$

4 The results obtained and their discussion

The ordered set of LCDS elements is obtained by expression (11) after having solved the given task using the method of quasi-best case (see item 3).

As it can be seen, the mapping of elements of sets (S) and $\langle S_{(j)\max}|j=\overline{1,m}\rangle_{QBMS}$ by expressions (1) and (11) vary without coincidence in relation to places of local criteria within these mappings excluding criterion $\tau\left(Q\right)$. Obviously, it does not demonstrate infeasibility of the method used [14].

It makes sense to compare the obtained result $\langle S_{(j)\max}|j=\overline{1,m}\rangle_{QBMS}$ to the result of ordered set $\langle S_{(j)\max}|j=\overline{1,m}\rangle_{WMS}$, which was obtained while selecting RMAT using the method of the worst case [13] (to index WMS):

$$\langle S_{(j)\max}|j=\overline{1,m}\rangle_{WMS}=\langle Kn,En,Gm,Dn,\tau\left(Q\right),Ct,Ac,Rl,Tr,Ec,Fopt,Fc\rangle. \tag{12}$$

As it is seen, the places (order number) of only three criteria coincide within the sets $\langle S_{(j)\max}|j==\overline{1,m}\rangle_{QBMS}$ and $\langle S_{(j)\max}|j=\overline{1,m}\rangle_{WMS}$. These criteria are: $Kn, EnandCt$. This does not also demonstrate the infeasibility of quasi-best method used. The comparison of these sets in general demonstrates different final results. It proves feasibility of both methods of fuzzy multi-criteria of RMAT selection as well as the feasibility of quasi-best case method adapted to the particularity of topical area.

Generally, the comparison of the elements of sets $S=\left(S_j|j=\overline{1,m}\right)$ by (1), $\langle S_{(j)\max}|j=\overline{1,m}\rangle_{QBMS}$ by (11) and $\langle S_{(j)\max}|j=\overline{1,m}\rangle_{WMS}$ by (12) is not contradictory. The result of analysis of these sets first of all reproduces the features of solving the tasks of such content and formulation (see item 3).

Therefore, it is recommended to make decision for stated input data taking into account the obtained ordered set of LCDS elements by expression (11) at fuzzy multi-criteria selection of RMAT using the developed and applied method of quasi-best case. It means that local criteria (RMAT phenomena) have to be analyzed by the following ordered sequence: kinematics ($KC,$), energy components (En), geometrical parameters (Gm) and so on, finishing with dynamics components (Dn). It is obvious that the number of RMAT being analyzed by every local criterion may be significantly decreased due to the selection of in feasible RMAT by every criterion. This occurs while analyzing previously synthesized final set of RMAT [4] as a result of RMAT analysis by every local criterion in sequence of set [11] elements. All the mentioned above causes complexity decrease and makes decision-making rational even at non-automated solution of the task of fuzzy multi-criteria selection of RMAT using the method of quasi-best case.

The results obtained demonstrate the achieved purpose of the given paper (see item 2).

The following *directions of further researches* have been determined as results of conducted researches:

• the development of set of related methods of fuzzy multi-criteria alternative selection based on the results of strict and non-strict expert sampling and does not fundamentally contradict the possibility to automate fuzzy multi-criteria selection of alternatives;

• the known and the latest approaches related to fuzzy multi-criteria selection of alternatives are automatically implemented in the form of computer software.

5 Conclusions

1. The approach of fuzzy multi-criteria selection of alternatives by quasi-best case was chosen as theoretical and methodological basis to solve the task of RMAT selection taking into account its essence and formulation. This approach was used because it is invariant one in relation to the content of the task, to the origin and the number of discrete local criteria. The selection is performed within the finite sets of the mentioned above criteria. One more reason to choose the described method is its ability to provide the lower level of subjectivism and to increase the reasonableness of decisions made while ordering criteria from their LCDS.

2. The key points of the method used are adapted to solving tasks of fuzzy multi-criteria selection of RMAT. It was performed for the first time by applying the executed formalization of generalizations of content features of the task components. The mentioned above task, in its turn, can be further implemented in an automated way.

3. The performed formalization of RMAT selection task, which is being solved, is implemented with meaningfully grounded steps. The content of these steps is the methodological basis of fuzzy multi-criteria RMAT selection by the method of quasi-best case.

References

1 Industrial Robot Integration [Electronic resource]. — Access mode: https:// www.robots.com/ robots

2 Executive Summary World Robotics Industrial Robots [Electronic resource]. — Access mode: https:// ifr.org/ downloads/ press/ Executive_Summary_WR_2018_Industrial_Robots.pdf

3 International Federation of Robotics [Electronic resource]. — Access mode: https:// ifr.org/downloads /press2018/ WR_Presentation_Industry_and_Service_ Robots_ rev_ 5_ 12_18.pdf

4 Кирилович В.А. Автоматизований синтез роботизованих механоскладальних технологій: автореф. дис. ... д-ра. техн. наук: 05.02.08 – «Технологія машинобудування» [Электронный ресурс] / В.А. Кирилович. — Київ, 2015. — 48 с. — Режим доступа: http:// rada.kpi.ua/ files/ aref_Kyrylovych_V.A.pdf

5 Кирилович В.А. Система техніко-економічних критеріїв як основа умов критеріальної реалізованості при автоматизованому синтезі роботизованих механоскладальних технологій / В.А. Кирилович // Енергетика і автоматика. — 2015. — № 3. — С. 5–18.

6 Кирилович В.А. Умови функціональної вирішуваності при автоматизованому синтезі роботизованих механообробних технологій в гнучких виробничих комірках / В.А. Кирилович // Науковий журн. «Технологічні комплекси». — Луцьк, 2010. — № 1. — С. 136–145.

7 Steuer R.E. Multiple Criteria Optimization: Theory, Computations, and Application. — NewYork: John Wiley & Sons, Inc., 1986.

8 Макаров И.М. Теория выбора и принятия решений / И.М. Макаров, Т.М. Виноградская, А.А. Рубчинский. — М.: Наука, 1982. — 328 с.

9 Bellman, R.E., & Zadeh, L.A. (1970). Decision-Makingin Fuzzy Enviroment / R.E. Bellman, L.A. Zadeh // Management Science, Vol. 17, 4, 141-160.

10 Заде Л.А. Понятие лингвистической переменной и его применение к принятию решений / Л.А. Заде. — М.: Мир, 1976. — 167 с.

11 Саати Т. Принятие решений: Метод анализа иерархий / Т. Саати; пер. с англ. — М.: Радио и связь, 1993. — 278 с.

12 Ротштейн А.П. Нечёткий многокритериальный выбор альтернатив: метод наихудшего случая / А.П. Ротштейн // Изв. РАН. Теория и системы управления. — 2009. — № 3. — С. 51–55.

13 Kyrylovych V. Automation of fuzzy multi-criteria selection of robotic machine-assembly technologies using worst-case approach / V. Kyrylovych, L. Dimitrov, I. Kryzhanivska, P. Melnychuk // 6-th International BAPT Conference «Power Transmitions 2019», 19-22.06.2019, Varna, 2019. — Vol. III. — P. 380–385.

14 Кирилович В. Семантическая модель нечеткого многокритериального выбора альтернатив методом квазилучшего случая / В. Кирилович, Л. Димитров, П. Мельничук, Д. Бельский, В. Шадура // Теоретические и прикладные проблемы математики, механики и информатики: материалы Междунар. научн. конф., приуроч. к 70-летию д.ф.-м.н., проф. М.И. Рамазанова. — Караганда, 2019. — С. 140–141.

15 Сигорский В.П. Математический аппарат инженера / В.П. Сигорский. — Киев: Техника, 1975. — 538 с.

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Роботтандырылған механикалық құрастыру технологиясын таңдау негізінде баламаны квази-жақсы жағдай әдісімен анық емес көпкритерилі таңдау

Роботтандырылған механикалық құрастыру технологиясын (РМҚТ) таңдау процесі бұрын ұсынылған баламаны квази-жақсы жағдай әдісімен анық емес көпкритерилі таңдаумен жүзеге асты. Жұмыстың практикалық және ғылыми құндылығын анықтайтын әдістің негізгі мазмұнды ерекшелігі ол неғұрлым ерекше критериймен ең жақсы баламамен салыстыру арқылы негізделген арнайы әзірленген қатынастар болып табылады. Бастапқы деректер ретінде РМҚТ көріністері болатын, эксперттік сауалнама әдісімен орындалған локальды критерийлердің дискреттік жиынының (ЛКДЖ) элементтерін қатаң ранжирлеу нәтижелері болды. Таңдау мазмұны болып таңдау орындалатын жиында реттелген ақырлы жиынға ЛКДЖ ретсіз элементтерінің бастапқы құраушыларын реттеу процесі табылған. Алынған РМҚТ көрінісінің тізбегі таңдау процесінде таңдау нәтижесі ретінде саралауға ұсынылған. РМҚТ таңдау есебінің шешуінің негізі болып оның қалыптастырылған қойылымы және оның негізінде алғаш рет анықталған квази-жақсы жағдай әдісінің мазмұнының жалпылынған формализмдері болды. Келтірілген теориялық жағдайлардың қадам сайын жұмыс қабілеттілігі РМҚТ автоматтандырылған таңдаудың нақты мысалымен көрсетілген.

Кілт сөздер: балама, автоматтандыру, таңдау, локальды критерий, анық емес, тиімділеу, роботтандырылған механикалық құрастыру технологиясы, квази-жақсы таңдау.

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Нечёткий многокритериальный выбор альтернатив методом квазилучшего случая как основа выбора роботизированных механосборочных технологий

Процесс выбора роботизированных механосборочных технологий (РМСТ) реализован как нечеткий многокритериальный выбор альтернатив предложенным ранее методом квазилучшего случая. Основной содержательной особенностью данного метода, определяющей практическую и научную ценность данной работы, являются разработаные специальные соотношения, основанные на соответствующих сравнениях с лучшей альтернативой и с наиболее важным критерием. Входными данными являются результаты строгого ранжирования элементов дискретного множества локальных критериев (ДМЛК), являющиеся проявлениями РМСТ, выполненных методом экспертного опроса. Содержанием выбора является процесс упорядочения составляющих изначально неупорядоченных элементов ДМЛК, на множестве которых выполняется выбор, в конечное упорядоченное множество. Полученная последовательность проявлений РМСТ как результат выбора рекомендована к анализу в процессе выбора. Основой решения задачи выбора РМСТ является ее формализованная постановка и впервые определенные на ее основе обобщенные формализмы содержания метода квазилучшего случая. Работоспособность изложенных теоретических положений пошагово продемонстрирована реальным примером автоматизированного выбора РМСТ.

Ключевые слова: альтернатива, автоматизация, выбор, локальный критерий, нечеткость, оптимизация, роботизированная механосборочная технология, квазилучший выбор.

References

1 Industrial Robot Integration. *robots.com.* Retrived from https://www.robots.com/robots.

2 Executive Summary World Robotics Industrial Robots. *ifr.org.* Retrived from https: //ifr.org/ downloads/ press/ Executive_Summary_WR_2018_Industrial_Robots.pdf.

3 International Federation of Robotics. *ifr.org.* Retrived from https://ifr.org/ downloads/ press2018 /WR _Presentation_Industry_and_Service_Robots_rev_5_12_18.pdf

4 Kyrylovych, V.A. (2015). Avtomatizovanyi sintez robotyzovanykh mekhanoskladalmnykh tekhnolohii [Automated synthesis of robotized mechanical assembly technologies]. *Extended abstract of Doctors thesis.* Kyiv, KPI. Retrived from: http:// rada.kpi.ua/ files/ aref_Kyrylovych_V.A.pdf [in Ukrainian].

5 Kyrylovych, V.A. (2015). Systema tekhniko-ekonomichnykh kryteriiv yak osnova umov kriterialnoi realizovanosti pri avtomatizovanomu sintezi robotyzovanykh mekhanoskladalnykh tekhnolohii [The system of technical and economic criteria as a basis for the criteria of criterion realization in the automated synthesis of robotic mechanical assembly technologies]. *Enerhetika i avtomatika — Power engineering and automation, 3,* 5–18 [in Ukrainian].

6 Kyrylovych,V.A. (2010). Umovi fynktsionalnoi vyrishuvanosti pry avtomatizovanomu sintezi robotyzovanykh mekhanoobrobnikh tekhnolohii v hnuchkikh virobnichikh komirkakh. Naukovii zhurnal «Tekhnolohichni kompleksy» [Functional solvability conditions for automated synthesis of robotic machining technologies in flexible manufactoring cells]. *Naukovii zhurnal «Tekhnolohichni kompleksi» — Scientific journal «Technological complexes», 1,* 136–145 [in Ukrainian].

7 Steuer, R.E. (1986). *Multiple Criteria Optimization: Theory, Computations, and Application.* NewYork: JohnWiley & Sons, Inc.

8 Makarov, I.M., Vinogradskaya, T.M., & Rubchinskiy, A.A. (1982). *Teoriia vybora i priniatiia reshenii [Theory of Choice and Decision Making].* Moscow: Nauka [in Russian].

9 Bellman, R.E., & Zadeh, L.A. (1970). Decision-Makingin Fuzzy Enviroment. *Management Science, Vol. 17, 4, 141-160.*

10 Zade, L.A. (1976). *Poniatie linhvisticheskoi peremennoi i eho primenenie k priniatiiu reshenii [The concept of a linguistic variable and its application to decision making].* Moscow: Mir [in Russian].

11 Saaty, T. (1993). Priniatie reshenii: Metod analiza ierarkhii *[Decision Making: Analytic Hierarchy Process].* Moscow: Radio i sviaz [in Russian].

12 Rothstein, A.P. (2009). Nechetkii mnohokriterialnyi vybor alternativ: metod naikhudsheho sluchaia [Fuzzy multi-criteria choice of alternatives: worst case method]. *Izvestiia RAN. Teoriia i sistemy upravleniia – Proceedings of the RAS. Theory and control systems*, *3*, 51-55 [in Russian].

13 Kyrylovych, V., Dimitrov,L., Kryzhanivska, I., & Melnychuk, P. (2019). Automation of fuzzy multi-criteria selection of robotic machine-assembly technologies using worst-case approach. *6-th International BAPT Conference «Power Transmitions 2019», 19-22.06.2019, Varna. Vol. III*, 380–385.

14 Kyrylovych, V., Dimitrov, L., Melnychuk, P., Belskiy, D., & Shadura, V. (2019). Semanticheskaia model nechetkoho mnohokriterialnoho vybora alternativ metodom kvaziluchsheho sluchaia [Semantic model of fuzzy multi-criteria choice of alternatives by the quasi-best case method]. *Teoreticheskie i prikladnye voprosy matematiki, mekhaniki i informatiki — Theoretical and applied questions of mathematics, mechanics and computer science,* 140–141 [in Russian].

15 Sigorskij, V.P. (1975). *Matematicheskii apparat inzhenera [Math for Engineer].* Kiev: Tekhnika [in Russian].

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Mapes of secondary sources in the problem of ERT probing 2D medium: numerical method and analytical solutions

The paper considers a mathematical model of electrical tomography above the media with local inclusions. Numerical solutions of a system of integral equations for a medium with local inclusion are compared against a numerical implementation of the analytical solution of the problem for a case of a sphere in homogeneous space. The parameters of local inclusion and the depth of heterogeneity are varied. Maps of secondary sources in the ERT (Electrical Resistivity Tomography) probing problem are constructed: for local inclusion in the form of the ellipsoid, an ellipsoid in a homogeneous space (analytical solution of the problem) and for two-layer half-spaces as well. Numerical results are presented, and maps of secondary sources in the cases where the immersed heterogeneity is an insulator and a conductor are computed.

Keywords: map of secondary sources; analytical solution of the problem with immersed heterogeneity; ellipsoid in a homogeneous space; the integral equation method.

Introduction

Modeling the problems of electrical exploration is very relevant nowadays. The solution of direct and inverse problems of electrical tomography are the main subject of many works ([1–13] and references therein). In solving the problems of electrical tomography [14, 15], the finite element method is used; the novelty of our work is the application of the method of integral equations [16–22] to the solution of the problem specified below.

When modeling the electric field in complex media, it is important to take into account the geometric parameters of the desired objects (shape, number of elements, depth and dimensions) [23, 24]. The listed geometric parameters strongly affect the amplitude and shape of the electric field anomalies [23]. Theoretical calculations by analytical formulas and numerical algorithms should be performed for models that are found in the practice of geophysical research.

In addition to the size and the depth, the resistivity of the heterogeneity and the peculiarity of the medium such as angles of incidence of flat boundaries and the orientation of the buried object have a great influence on the results of the work [24]. In our work, we consider a model containing heterogeneity in the form of an ellipsoid located in a homogeneous half-space and full space (analytical solution) and a two-layer horizontally layered medium.

The numerical results are obtained for two types of modeling:

1. Tests of the numeral solutions have been performed using the method of integral equations against the analytical solutions by A. I. Zaborovsky [23].

2. The distribution of secondary sources on the earth's surface and the internal contact boundary are shown.

The study of the electrical field for such kind of media is important for isolating and tracing local objects, their depth and surface shape. Approximate solutions are known for a sphere, and for compressed and elongated ellipsoids in a homogeneous medium [23, 24].

A special case of an ellipsoid is a sphere in a homogeneous medium. Due to the complexity of calculations by explicit formulas, for mass calculations of field parameters in the software, approximate solutions of A.I. Zaborovsky are implemented [23, 25].

The electrical potential inside the medium with a sphere near the surface of a half-space is determined using the exact formulas [23]. Depending on the location of the source and receiving electrodes along the measuring profile, which does not necessarily passes above the center of the inclusion, there are four possible analytical formulas for the potential of the electrical field:

$$U_1 = \frac{I\rho_1}{2\pi}\left[\frac{1}{r} + \int_{n=0}^{\infty} \frac{npc^n}{d^{n+1}} P_n\left(\cos\theta\right)\right], d > a > r;$$

$$U_2 = \frac{I\rho_1}{2\pi}\left[\frac{1}{r} + \int_{n=0}^{\infty} \frac{npd^n}{c^{n+1}} P_n\left(\cos\theta\right)\right], c > a > d; \quad (1)$$

$$U_3 = \frac{I\rho_1}{2\pi}\left[\frac{1}{r} + \int_{n=0}^{\infty} \frac{(-1)pa^{2n+1}}{(cd)^{n+1}} P_n\left(\cos\theta\right)\right], a > r, d > a;$$

$$U_4 = \frac{I\rho_1}{2\pi}\left[\frac{1}{r} + \int_{n=0}^{\infty} \frac{(-1)p(n+1)(cd)^n}{c^{2n+1}} P_n\left(\cos\theta\right)\right], a > c, d < a;$$

where r is a distance from the source to the receiver; a is a radius of the half sphere; d, c are the distances between the supply and receiving electrodes and the center of the hemisphere, respectively; θ is the angle between directions c and d; $P_n\left(\cos\theta\right)$ is the Legendre polynomial of the first kind of the order n from cos θ and

$$\theta = \left\{ \begin{array}{c} 0 \\ \pi \end{array} \right. , \quad P_n\left(\cos\theta\right) = \left\{ \begin{array}{c} 1^n \\ (-1)^n \end{array} \right. .$$

Using these formulas (1), we can numerically solve the problem with submerged heterogeneity in the form of a ball and compare with the solution of the problem with immersed inhomogeneity obtained by the method of integral equations [16] - [22].

The second part of our testing was the construction of distributions of secondary field sources. Secondary sources of the anomalous field appear as a result of the excitation of the Earth's surface by a source electrode. Maps of secondary sources determine abnormal electric fields.

The authors constructed a map of secondary sources for a two-layer half-space, and for an immersed inhomogeneity by the method of integral equations and for the analytical solution for a ball in a homogeneous space according to A.I. Zaborovsky to show how secondary sources are distributed over the surface.

Numerical solutions

As mentioned above, the algorithm was tested in two ways: comparing the solution obtained by the method of integral equations with the analytical solution by A.I. Zaborovsky. For the best of our knowledge there is no analytical solutions of the problem for the heterogeneity placed inside a homogeneous half-space. On the other hand, we implement the method of integral equations for half-space, whereas the Zaborovsky solution is obtained for full-space; therefore, to make a reasonable comparison, the inclusion should be placed in such depth where the boundary of the half space does not significantly impact on the electric field.

Comparison with the analytical solution has been carried out for different parameters of the immersed heterogeneity, its size and depth. In the computations by the method of integral equations, when we placed the ball to the lower depths than $z = 1.5\ r$, the influence of the inhomogeneity on the anomalies of the resistivity curves became very small, so we have to reduce the depth of the inclusion. But in the analytical solution of A.I. Zaborovsky, the higher we lift up the ball from this depth, the more the difference in models appears, namely, the reflection from the boundary of upper half-space influences on the electric field.

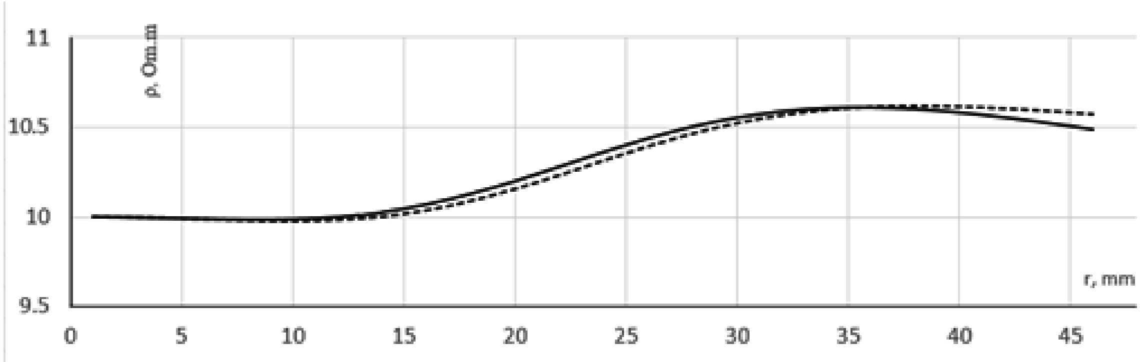

Figure 1. Comparison of a solution by the method of integral equations with an analytical solution by A.I. Zaborovsky, (-) - a solution by the method of integral equations, (- -) - an analytical solution by A.I. Zaborovsky

With the parameters of the sphere $a=1\ r$, $z = 1.5\ r$, it turned out that similar results are obtained. With these data, a comparison result about of 5% is obtained.

To construct maps of secondary sources for each case, media models are considered for inclusions that are an insulator or a conductor.

In Figure 2, the upper layer is flat, the parameters of local inclusion in taken the form of an ellipsoid at a depth of $z = 0.5\ r$ with parameters $ax = 0.21r$, $by = 0.2128\ r$, $cz = 0.21\ r$ with layer resistivities $\rho 1 = 10\ Ohm{\cdot}m$, $\rho 2 = 100\ Ohm{\cdot}m$. Although Matlab's state-of-the-art math package makes it possible to construct triangulations, for our purposes these triangulations turn to be unacceptable. This is due to the fact that the thickening of the grid should occur in the vicinity of the measuring line, and the source and measuring electrodes should be located at the vertices of the triangles, in nodes with the same geometry of triangulation. Therefore, we have had to construct our own algorithm of the triangulation. An example of a grid constructed for a case with spherical local inclusion is shown in Figure 2.

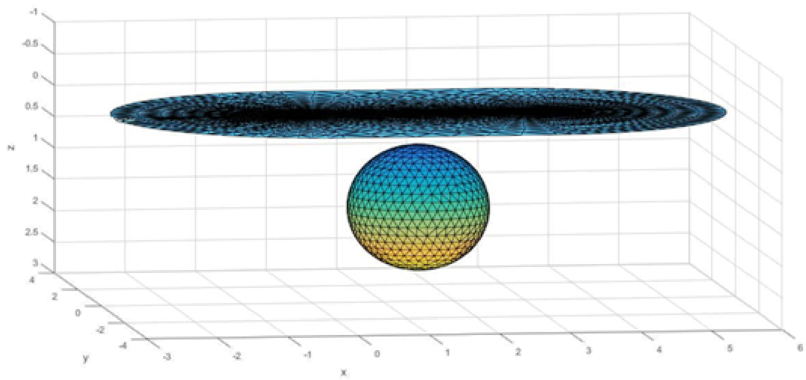

Figure 2. Triangulation for spherical inclusion under the flat surface

Figure 3 shows the secondary sources for resistivities of the surrounding medium $\rho 1 = 10\ Ohm{\cdot}m$ and an inclusion with resistivity $\rho 2 = 100\ Ohm{\cdot}m$.

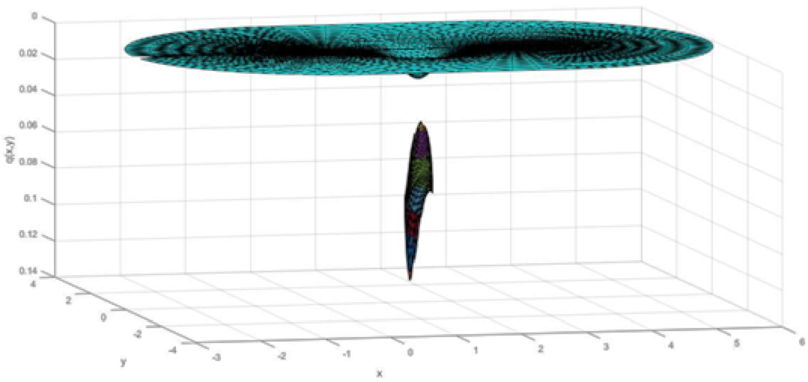

Figure 3. Map of secondary sources at $\rho 1$ = *10 Ohm·m* and $\rho 2$ = *100 Ohm·m*

Figure 4 shows a map of secondary sources for the case of resistivity of the surrounding medium $\rho 1$ = *100 Ohm·m* and there is an immersed conductor with resistivity $\rho 2$ = *10 Ohm·m*.

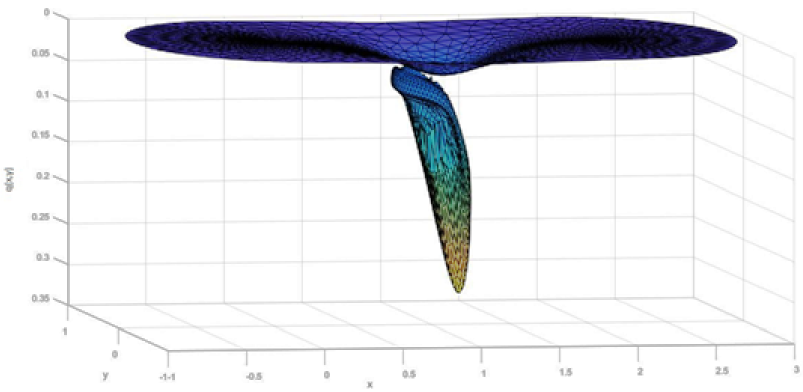

Figure 4. Map of secondary sources for $\rho 1$ = *100 Ohm·m and* $\rho 2$ = *10 Ohm·m*

Figure 5 shows a ball in homogeneous space, the parameters of the ball *a=1 r*. The triangulation constructed for this case is shown in Figure 5.

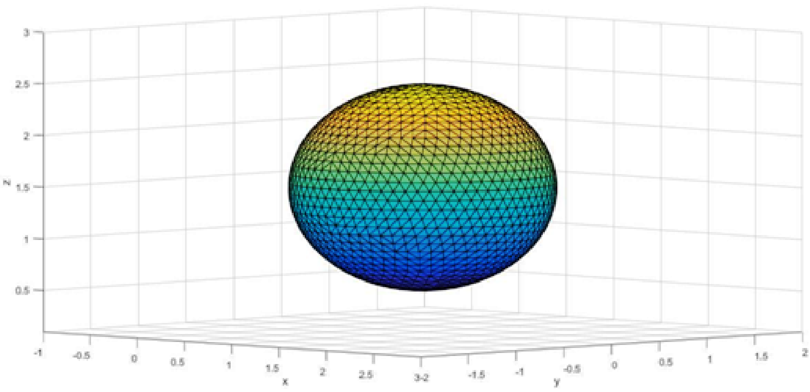

Figure 5. Triangulation constructed for a ball in a homogeneous space

Figure 6 shows a map of secondary sources projected on the plane (Oxy) for a ball in a homogeneous space, for resistivities $\rho 1$ = *10 Ohm·m*, $\rho 2$ = *100 Ohm·m*.

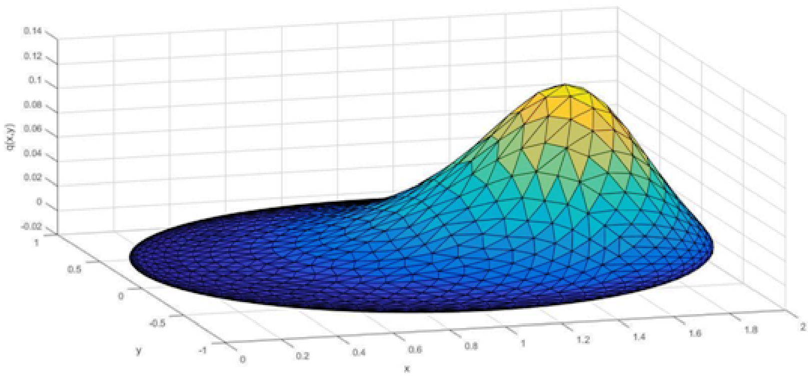

Figure 6. Map of secondary sources at *ρ1 = 10 Ohm·m and ρ2 = 100 Ohm·m*

Figure 7 shows a map of secondary sources projected on the plane (Oxy) for the resistivity of the containing medium *ρ1 = 100 Ohm·m* and the resistivity of the ball *ρ2 = 10 Ohm·m*.

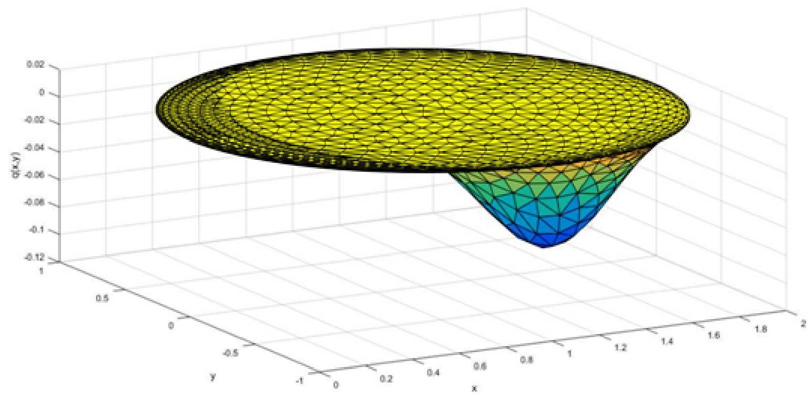

Figure 7. Map of secondary sources at *ρ1 = 100 Ohm·m* and *ρ2 = 10 Ohm·m*

Additional computations are performed for a two-layer medium in a half-space; both layers are supposed to be plane. The triangulation constructed for this case is shown in the Figure 8.

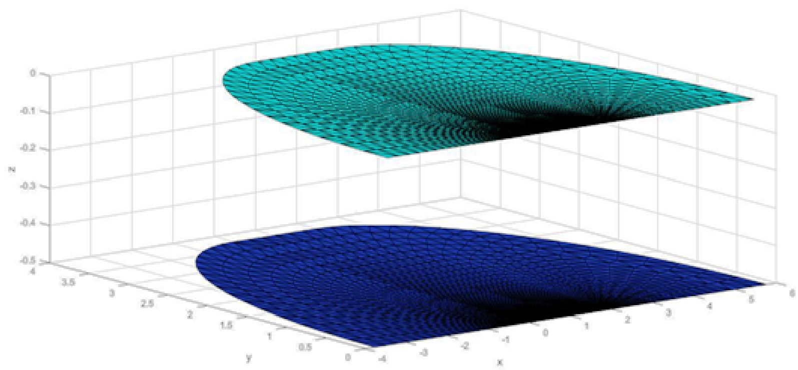

Figure 8. Triangulation built for a two-layer environment

Figure 9 shows a map of secondary sources for the case when the second insulating layer has a resistivity *ρ2 = 100 Ohm·m* and is contacting with the layer of *ρ1 = 10 Ohm·m*.

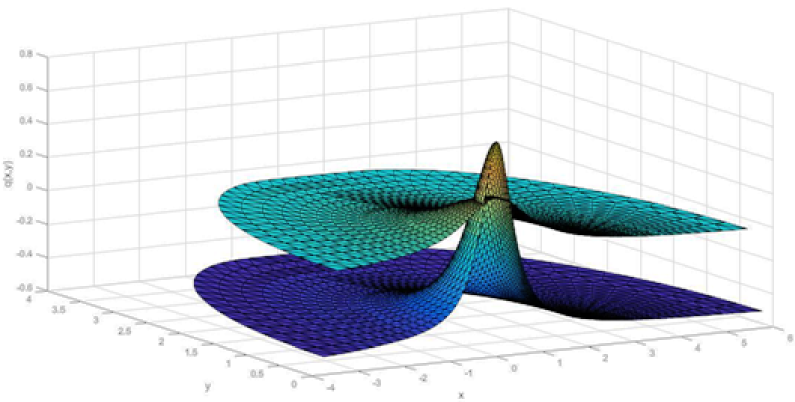

Figure 9. Map of secondary sources at *ρ1 = 10 Ohm·m and ρ2 = 100 Ohm·m*

Figure 10 shows a map of secondary sources for the resistivity of the upper layer *ρ1 = 100 Ohm·m*, and the lower layer with *ρ*2 = *10 Ohm·m*.

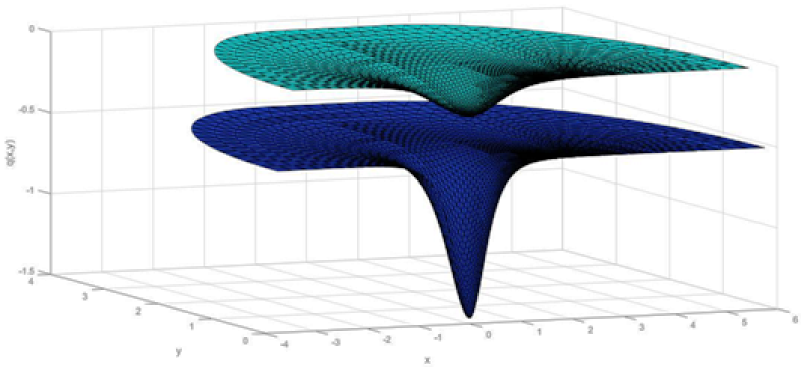

Figure 10. Map of secondary sources with *ρ1 = 100 Ohm·m and ρ2 = 10 Ohm·m*

Conclusion

Numerical solutions obtained by the method of integral equations are compared with analytical solutions of A.I. Zaborovsky. It turns out that even the models are different (half space and full space) the difference in apparent resistivity curves for the depth of inclusion $a = 1.5r$ are above 5%. Maps of secondary sources of the electric field for the following cases are constructed:

– buried inclusion by the method of integral equations,

– a sphere in homogeneous space according to A.I. Zaborovsky,

– a case of two-layer medium.

Calculations by the method of integral equations have shown that the distribution of secondary sources on the surface of inhomogeneities that determine the structure of an anomalous electric field is close to the solutions known in the theory of geophysics.

References

1 Dahlin T. On the Automation of 2D Resistivity Surveying for Engineering and Environmental Applications / T. Dahlin // PhD thesis, Lund University. — 1993. — P. 187.

2 Dahlin T. 2D resistivity surveying for environmental and engineering applications / T. Dahlin // First Break. — Vol. 14. — No. 7. — 1996. — P. 275–283.

3 Dey A. Resistivity modeling for arbitrary shaped two-dimensional structures / A. Dey, H.F. Morrison // Geophysical Prospecting. — Vol. 27. — No. 1. — 1979. — P. 106–136.

4 Alekseev A.S. Optimizational method for solving the inverse problem of geophysical prospecting by electric means under direct current for vertically-inhomogeneous media / A.S. Alekseev, V.A. Tcheverda, Sh. Niambaa // Inv. mod expl. geoph. — 1989. — P. 171–189.

5 Hasanov A. An inverse problem related to the determination of elastoplastic properties of a plate / A. Hasanov, A. Mamedov // Inverse problems. — Vol. 10. — 1994. — P. 601–615.

6 Peng Y.J. An inverse problem in petroleum exploitation / Y.J. Peng // Inverse problems. — Vol. 13. — 1997. — P. 1533–1546.

7 Orunkhanov M. Inverse resistivity problem: Geoelectric uncertainty principle and numerical reconstruction method / M. Orunkhanov // Mathematics and computers in simulation. — Vol. 80. — 2010. — P. 2091–2108.

8 Mukanova B. An inverse resistivity problem: 1. Lipschitz continuity of the gradient of the objective functional / B. Mukanova // Applicable analysis. — Vol. 5. — 2009. — P. 749–765.

9 Mukanova B. An inverse resistivity problem: 2. Unilateral convexity of the objective functional / B. Mukanova // Applicable analysis. — Vol. 5. — 2009. — P. 767–788.

10 Orunkhanov M. The integral equations method in problems of electrical sounding / M. Orunkhanov, B. Mukanova // Advances in high performance computing and computational sciences. — Vol. 93. — 2009. — P. 15.

11 Orunkhanov M. Convergence of the method of integral equations for quasi three-dimensional problem of electrical sounding / M. Orunkhanov, B. Mukanova // Compuutational science and high performance computing II. — Vol. 91. — 2016. — P. 175.

12 Borcea L. Electrical impedance tomography / L. Borcea // Inverse problems. — 2002. — Vol. 18. — P. R99–R136.

13 Geng Min. Numerical Simulation in Whole Space for Resistivity Logging through Casing under Approximate Conditions / Geng Min. Liang Huaqing. Yin Hongdong // 2012 International workshop on information and electronics engeneering. — Vol. 29. — 2012. — P. 3600–3607.

14 Klibanov M.V. Nanostructures imaging via numerical solution of a 3-D inverse scattering problem without the phase information / M.V. Klibanov, L.H. Nguyen, K.J. Pan // Applied numerical mathematics. — Vol. 110. — 2016. — P. 190–203.

15 Klibanov, M.V. Explicit solution of 3D phaseless inverse scattering problem for the Schrodinger eqation: the plane wave case / M. V. Klibanov, V. G. Romanov // Eurasian journal of mathematical and computer applications. — Vol. 3. — 2015. — P. 48–63.

16 Mukanova B. Modelling the Influence of Ground Surface Relief on Electric Sounding Curves Using the Integral Equations Method / B. Mukanova, T. Mirgalikyzy, D. Rakisheva // Mathematical Problems in Engineering. — 2017. — P. 217–228.

17 Mukanova B. A numerical solution to the well resistivity-sounding problem in the axisymmetric case / B. Mukanova // Invers problems in science and engineering. — Vol. 21(5). — 2012. — P. 767–780.

18 Ракишева Д.С. Аппроксимация поверхности рельефа дневной поверхности методом RBF / Д.С. Ракишева, Т. Миргаликызы, Б.Г. Муканова // Вестн. Национальной академии наук Республики Казахстан. — 2017. — Т. 1. — № 365. — С. 210–215.

19 Mukanova B. Modeling the impact of relief boundaries in solving the direct problem of direct current electrical sounding Communications in Computer and Information Science / B. Mukanova, T. Mirgalikyzy // Mathematical Modeling of Technological Processes: International Conference. CITech-2015. — Almaty: Proceedings. Springer. — 2015. — P. 117–132.

20 Mirgalikyzy T. Method of Integral Equations for the Problem of Electrical Tomography in a Medium with Ground Surface Relief / T. Mirgalikyzy, B. Mukanova, I. Modin // Journal of Applied Mathematics. — 2015.

21 Mukanova B. Modeling the Impact of Relief Boundaries in Solving the Direct Problem of Direct Current Electrical Sounding / B. Mukanova, T. Mirgalikyzy // In: Danaev N., Shokin Y., Darkhan AZ. (Eds) Mathematical Modeling of Technological Processes. Communications in Computer and Information Science, Springer. Cham. — Vol. 549.

22 Муканова Б.Г., Мирғалиқызы Т., Ракишева Д.С. Моделирование влияния рельефа земной поверхности на кривые электрического зондирования методом интегральных уравнений // Информатика и прикладная математика: II-я Междунар. науч. конф. — Ч. II. — Алматы, 2017. — С. 352–366.

23 Заборовский А.И. Электроразведка / А.И. Заборовский // Гос. науч.-техн. изд-во нефт. и гор.-топлив. лит. — М., 1963.

24 Якубовский Ю.В. Электроразведка / Ю.В. Якубовский, И.В. Ренард. — 3-е изд. — М.: Недра, 1991.

25 Mukanova B. The Boundary Element Method in Geophysical Survey / B. Mukanova, I. Modin // Springer. — T. 4. — 2017.

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2D орталарының ERT зондтау есебіндегі қайталама зарядтардың карталары: сандық әдіс және аналитикалық шешімдер

Мақалада локальді қосылуы бар аналитикалық шешім үшін электрлік томографияның математикалық моделі қарастырылған. Интегралдық теңдеулер жүйесіне негізделген эллипсоид түрінде қосылуы бар математикалық модель мен біртектес ортадағы шар аналитикалық шешімнің сандық есептелуі салыстырылды. Екі жағдай үшін локальді қосылудың параметрлерімен тереңдігін өзгертіп, оптималды тереңдік анықталды. Сонымен қатар, ERT барлау есебінде, эллипсойд және біртектес ортадағы шар түрінде локальді қосылуы бар және екі қабатты орта жағдайлары үшін қосалқы зарядтар картасы құрылған. Аналитикалық шешіммен салыстырудың сандық нәтижелерімен аталып кеткен үш жағдайдың өткізгіш пен изолятор орналасқан қосалқы зарядтар картасы келтірілген.

Кілт сөздер: қосалқы зарядтар картасы, қондырылған біртектесі бар аналитикалық шешім, біртектес ортадағы шар, интегралдық теңдеулер әдісі.

Д.С. Ракишева, И.Н. Модин, Б.Г. Муканова

Карты вторичных источников в задаче ERT зондирования 2D сред: численный метод и аналитические решения

В статье рассмотрена математическая модель электрической томографии для аналитического решения с локальным включением. Проведено сравнение с математической моделью, основанной на системе интегральных уравнений с локальным включением в виде эллипсоида, с численным решением аналитического решения задачи с шаром в однородном пространстве. Варьировались параметры локального включения и глубина залегания для определения оптимальной глубины для обоих случаев. Построены карты вторичных источников в задаче ERT зондирования: для локального включения в виде эллипсоида, для эллипсоида в однородном пространстве (аналитическое решение задачи) и двухслойных полупространств. Приведены численные результаты для сравнения с аналитическим решением и карты вторичных источников в случаях, когда погружен изолятор или проводник.

Ключевые слова: карта вторичных источников, аналитическое решение задачи с погруженной неоднородностью, эллипсоид в однородном пространстве, метод интегральных уравнений.

References

1 Dahlin, T. (1993). On the Automation of 2D Resistivity Surveying for Engineering and Environmental Applications, *PhD thesis,* Lund University.

2 Dahlin, T. (1996). 2D resistivity surveying for environmental and engineering applications, *First Break, 14(7),* 275–283.

3 Dey, A., & Morrison, H.F. (1979). Resistivity modeling for arbitrary shaped two-dimensional structures, *Geophysical Prospecting, 27(1),* 106–136.

4 Alekseev, A.S., Tcheverda, V.A. & Niambaa, Sh. (1989). Optimizational method for solving the inverse problem of geophysical prospecting by electric means under direct current for vertically-inhomogeneous media, *Inv. mod expl. Geoph.,* 171-189.

5 Hasanov, A, & Mamedov, A. (1994). An inverse problem related to the determination of elastoplastic properties of a plate, *Invers problems, 10,* 601-615.

6 Peng, Y.J. (1997). An inverse problem in petroleum exploitation, *Invers problems, 13,* 1533-1546.

7 Orunkhanov, M. (2010). Inverse resistivity problem: Geoelectric uncertainty principle and numerical reconstruction method, *Mathematics and computers in simulation, 80,* 2091-2108.

8 Mukanova, B. (2009). An inverse resistivity problem: 1. Lipschitz continuity of the gradient of the objective functional, *Applicable analysis, 5,* 749-765.

9 Mukanova, B. (2009). An inverse resistivity problem: 2. Unilateral convexity of the objective functional, Applicable analysis, 5, 767-788.

10 Orunkhanov, M., & Mukanova, B. (2009). The integral equations method in problems of electrical sounding, *Advances in high perfomans computing sciences, 93,* 15.

11 Orunkhanov, M., & Mukanova, B. (2016). Convergence of the method of integral equations for quasi three-dimensional problem of electrical sounding, *Compuutational science and high performance computing, 91(2),* 175.

12 Borcea, L. (2002). Electrical impedance tomography, Invers problems, 18, R99-R136.

13 Geng Min, Liang Huaqing & Yin Hongdong (2012). Numerical Simulation in Whole Space for Resistivity Logging through Casing under Approximate Conditions, *International workshop on information and electronics engineering, 29,* 3600-3607.

14 Klibanov, M.V., Nguyen, L.H. & Pan, K.J., (2016). Nanostructures imaging via numerical solution of a 3-D inverse scattering problem without the phase information, *Applied numerical mathematics, 110,* 190-203.

15 Klibanov, M.V., Romanov, V.G. (2015). Explicit solution of 3 D phaseless inverse scattering problem for the Schrodinger equations: the plane wave case, *Eurasian jornal of mathematical and computer applications, 3,* 48-63.

16 Mukanova, B, Mirgalikyzy, T. & Rakisheva, D. (2017). Modelling the Influence of Ground Surface Relief on Electric Sounding Curves Using the Integral Equations Method, *Mathematical Problems in Engineering,* 217–232.

17 Mukanova, B., (2012). A numerical solution to the well resistivity-sounding problem in the axisymmetric case, *Invers problems in science and engineering, 21(5),* 767–780.

18 Rakisheva, D.S., Mirgalikyzy, T., & Mukanova, B.G. (2017). Approksimatsiia poverkhnosti relefa dnevnoi poverkhnosti metodom RBF [Approximation of the surface relief of the surface by the RBF method]. *Vestnik Natsionalnoi akademii nauk Respubliki Kazakhstan, 1, 365,* 210–215 [in Russian].

19 Mukanova, B., & Mirgalikyzy, T. (2015). Modeling the impact of relief boundaries in solving the direct problem of direct current electrical sounding Communications in Computer and Information Science, *Mathematical Modeling of Technological Processes: International Conference, CITech-2015,* 210–215. Almaty: Proceedings, Springer.

20 Mirgalikyzy, T., Mukanova, B. & Modin, I. (2015). Method of Integral Equations for the Problem of Electrical Tomography in a Medium with Ground Surface Relief, *Journal of Applied Mathematics,* 117–132.

21 Mukanova, B., Mirgalikyzy, T. (2015). Modeling the Impact of Relief Boundaries in Solving the Direct Problem of Direct Current Electrical Sounding, In: Danaev N., Shokin Y., Darkhan A.Z. (Eds) Mathematical Modeling of Technological Processes. Communications in Computerand Information Science, Springer, 549.

22 Mukanova, B.G., Mirgalikyzy, T. & Rakisheva, D.S. (2017). Modelirovanie vliianiia relefa zemnoi poverkhnosti na krivye elektricheskoho zondirovaniia metodom intehralnykh uravnenii [Modeling the effect of the earth's relief on electric sounding curves by the method of integral equations]. Informatika i prikladnaia matematika: *II Mezhdunarodnaia nauchnaia konferentsiia — II International Scientific and Practical Conference, Part. II.* (pp. 352–366). Almaty [in Russian].

23 Zaborovskii, A.I. (1963). *Elektrorazvedka [Electrical intelligence].* Moscow: Hosudarstvennoe nauchno-tekhnicheskoe izdatelstvo neftianoi i horno-toplivnoi literatury [in Russian].

24 Iakubovskii, Yu.V., & Renard, I.V. (1991). *Elektrorazvedka [Electrical intelligence].* (3d ed.). Moscow Nedra [in Russian].

25 Mukanova, B., & Modin, I. (2017).The Boundary Element Method in Geophysical Survey. *Springer, 4.*

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Development of an algorithm for calculating the concentration of radiation defects during ion irradiation

The paper considers mathematical modeling of radiation defect formation processes in materials under ion irradiation. Algorithms for calculating cascade probability functions (CPF) taking into account energy, spectra of primary knocked out atoms (PKA) and the concentration of radiation defects during ion irradiation loss are presented. The defect concentration was calculated for various incident particles and targets of the periodic table. The regularities of the behavior of the concentration of defects depending on various physical parameters are revealed. The calculation algorithm is presented in the form of flowcharts. The work was performed as part of the cascade-probability method, the essence of which is to obtain and further use CPF. CPFs have the meaning of the probability that a particle generated at a certain depth h' will reach the detection depth h after the nth number of collisions.

Keywords: algorithm, calculation, probability, regularity, cascade probability function, concentration, radiation defects, ion, results area.

Introduction

In recent years, much attention has been paid to the problems of mathematical modeling of radiation-physical processes. The development of mathematical models, calculation algorithms, objects of research allows us to describe many phenomena [1]. We consider computer simulation of radiation defect formation processes in solids when they are irradiated with various charged particles, and computer simulation features of cascade probability functions and radiation defects for ions. The necessity for a such work is related to the problem of controlling the evolution of defects in a solid, in order to obtain, ultimately, materials with desired properties. For metals, irradiation with ions is an effective way to change properties such as metal strength, corrosion resistance, fatigue, deprecation, etc. Without research in this direction, solid state radiation physics would remain a fairly academic occupation that is not of interest for practical applications [1–11]. A lot of works have been devoted to the problems of the interaction of particles with matter and the generation of radiation defects upon irradiation of matter with ions [5–13]. Most of the work in this direction is carried out as part of the cascade-probability method [14–17].

During the interaction of charged particles with matter along the path of their movement, continuous energy losses occur. These losses lead to a strong dependence of both the energy spectra of the incident particles themselves and of the initially knocked out atoms on the penetration depth. The range of interaction for the formation of PKA substantially depends on energy, and therefore it became necessary to obtain physical and mathematical models that take into account the real dependence of various parameters of an elementary act on energy and depth. Previously, in most cases, in specific calculations, the simplest cascade probability function (CPF) was mainly used, this is not always justified, since the interaction path depends on the energy. It is necessary to study the behavior of the obtained CPFs, taking into account the energy losses for ions, to prove the properties that they must possess both from a physical and mathematical point of view, develop calculation algorithms and calculate the CPF depending on the number of interactions and the penetration depth of the particles, the primary knocked out atoms and concentration of radiation defects [18–20].

1 Experimental

Calculation of radiation defect concentration at ionic irradiation is made by formula [21, 22]:

$$C_k(E_0,h) = \int\limits_{E_c}^{E_{2\max}} W(E_0,E_2,h)dE_2, \tag{1}$$

$$E_{2\max} = \frac{4m_1c^2m_2c^2}{(m_1c^2+m_2c^2)^2}E_1;$$

m_2c^2 – rest energy of ion. $C_k(E_0,h)$ is determined taking into account of particle energy at depth h is $E_1(h)$. As $E_1(h) = E_0 - \Delta E(h)$, that power waste on ionization and agitation is assigned by ΔE, found corresponding depths of monitoring h from Bete-Blokh [1] formula or Komarov-Kumakhov table [22]. Spectrum of primary expelled atoms is determined by following ratio:

$$W\left(E_0,E_2,h\right) = \sum_{n=n_0}^{n_1}\int\limits_{h-k\lambda_2}^{h} \psi_n(h')\exp\left(-\frac{h-h'}{\lambda_2}\right)\frac{w\left(E_1,E_2,h'\right)dh'}{\lambda_1(h')\lambda_2}, \tag{2}$$

where n_0, n_1 – initial and finite value of interactions number from domain of cascade-probability function. Cascade-probability function $\psi_n(h')$, incoming to expression (2) is given by form:

$$\psi_n(h') = \frac{1}{n!\lambda_0^n}\left(\frac{E_0}{E_0-kh'}\right)^{\frac{1}{\lambda_0 ak}}\exp\left(\frac{h'}{\lambda_0}\right)\left(\frac{\ln\left(\frac{E_0}{E_0-kh'}\right)}{ak}-h'\right)^n, \tag{3}$$

$$\lambda_1(h') = \frac{1}{\sigma_0 n_0\left(\frac{1}{a\left(E_0-kh'\right)}-1\right)}\cdot 10^{24}\,(\text{cm}),\quad \lambda_2 = \frac{1}{\sigma_2 n_0}\cdot 10^{24}\,(\text{cm}).$$

Section of σ_2 calculated by Rutherford formula, z_1 – atomic number of flying particle, z_2 – atomic number of target. Spectrum of primary expelled atom (PEA) in elementary act is calculated by formula:

$$w(E_1,E_2) = \frac{d\sigma(E_1,E_2)/dE_2}{\sigma\left(E_1\right)}. \tag{4}$$

Substituting the expression (4) in formulae (1) and (2), we found:

$$C_k(E_0,h) = \frac{E_dE_{2\max}}{E_{2\max}-E_d}\int\limits_{E_c}^{E_{2\max}}\frac{dE_2}{E_2^2}\sum_{n=n_0}^{n_1}\int\limits_{h-k\lambda_2}^{h}\psi_n(h')\exp\left(-\frac{h-h'}{\lambda_2}\right)\frac{dh'}{\lambda_1(h')\lambda_2}.$$

operating transformations, we get to following ratio:

$$C_k\left(E_0,h\right) = \frac{E_d}{E_c}\frac{E_{2\max}-E_c}{E_{2\max}-E_d}\sum_{n=n_0}^{n_1}\int\limits_{h-k\lambda_2}^{h}\psi_n(h')\exp\left(-\frac{h-h'}{\lambda_2}\right)\frac{dh'}{\lambda_1(h')\lambda_2},$$

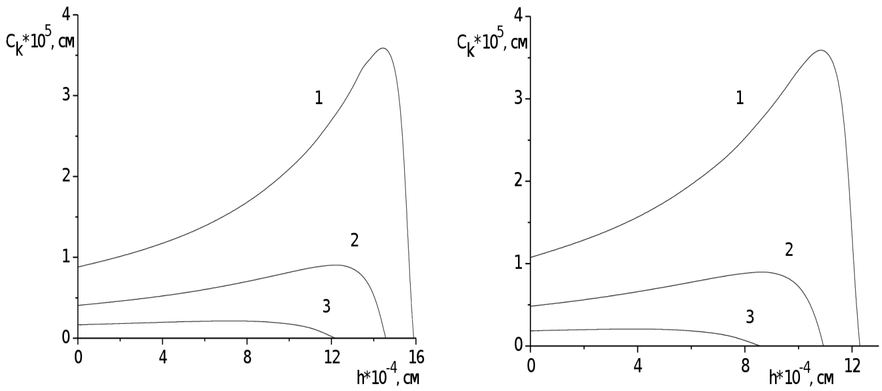
where E_d – average energy of displacement, E_0 – initial energy of particle, E_c – boundary energy, E_{2max} – maximum energy, transmitting to atom at head-on collision, $\psi n(h')$ – cascade-probability function.

Concentration of radiation defects is made by formula (1). It is impossible to put expression in the form of (3) instead of $\psi_n(h')$as a repletion appears in every term of cascade-probability function (CPF). Expression for $\psi_n(h')$ used in the form [20]:

$$\Psi_n(h',h,E_0) =$$
$$= \exp\left(-\ln n! - n\ln\lambda_0 - \frac{1}{\lambda_0 ak}\ln\left(\frac{E_0}{E_0-kh'}\right) + \frac{h'}{\lambda_0} + n\ln\left(\frac{\ln\left(\frac{E_0}{E_0-kh'}\right)}{ak} - h'\right)\right).$$

2 Results and Discussion

Results of calculations are given in 1–2 figures. Algorithm of calculations is given in the form of logic diagram (1, 2). As the calculations show (Figure 1, 2), with an increase in the threshold energy, the curves pass much lower, the transition through the maximum is realized smoother. The concentration of radiation defects depending on the depth increases, reaching a maximum, then decreases to 0.



(a) (b)

Figure 1. Dependence of the concentration of radiation defects on depth upon irradiation of silicon with silver ions at: (a) $E_0 = 1000\,\text{keV}$, $E_c = 50\,\text{keV}$ (1), $100\,\text{keV}$ (2), $200\,\text{keV}$ (3); (b) $E_0 = 800\,\text{keV}$, $E_c = 50\,\text{keV}$ (1), $100\,\text{keV}$ (2), $200\,\text{keV}$ (3)

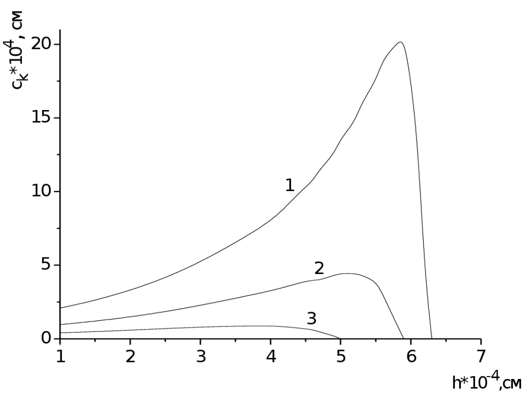


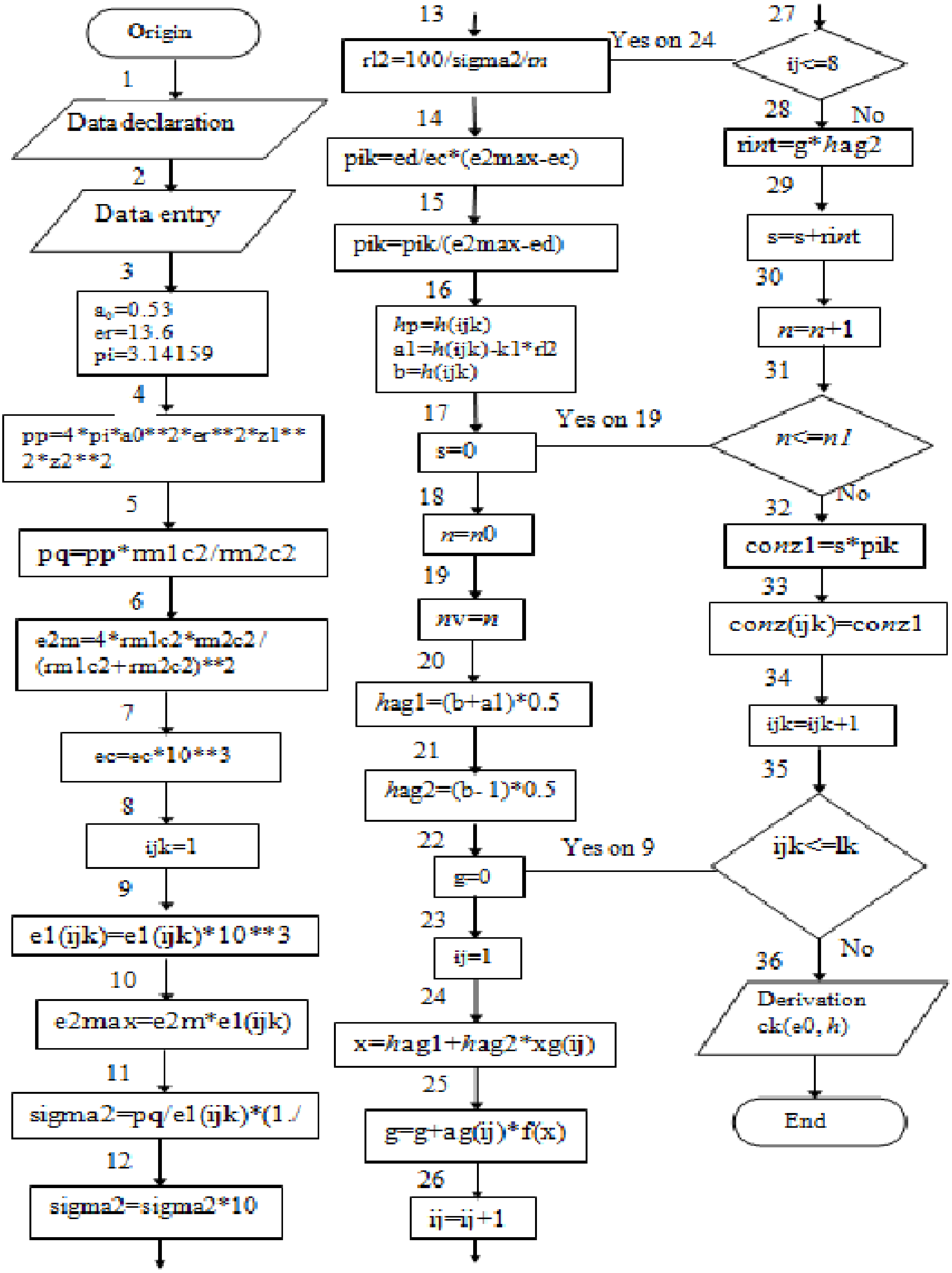
Figure 2. Dependence of the concentration of radiation defects on depth during ion irradiation for aluminum in iron at $E_0 = 1000\,\text{keV}$; $E_c = 50\,(1),\ 100\,(2),\ 200\,(3)\,\text{keV}$

The finding of area of result of radiation defect concentration at ionic irradiation has allowed revealing the following regularities:

1. The interval of area of result displaces to the right with reduction of initial energy of primary particle, values of radiation defect concentration increase.

2. Calculating time strongly increases and reaches tens of hours (table 1,2) at the great atomic weight of a flying particle and small atomic weight of target.

Algorithmic logic diagram of calculation of CK (E_0, h) :



Logic diagram of calculation function $f(h_1)$:

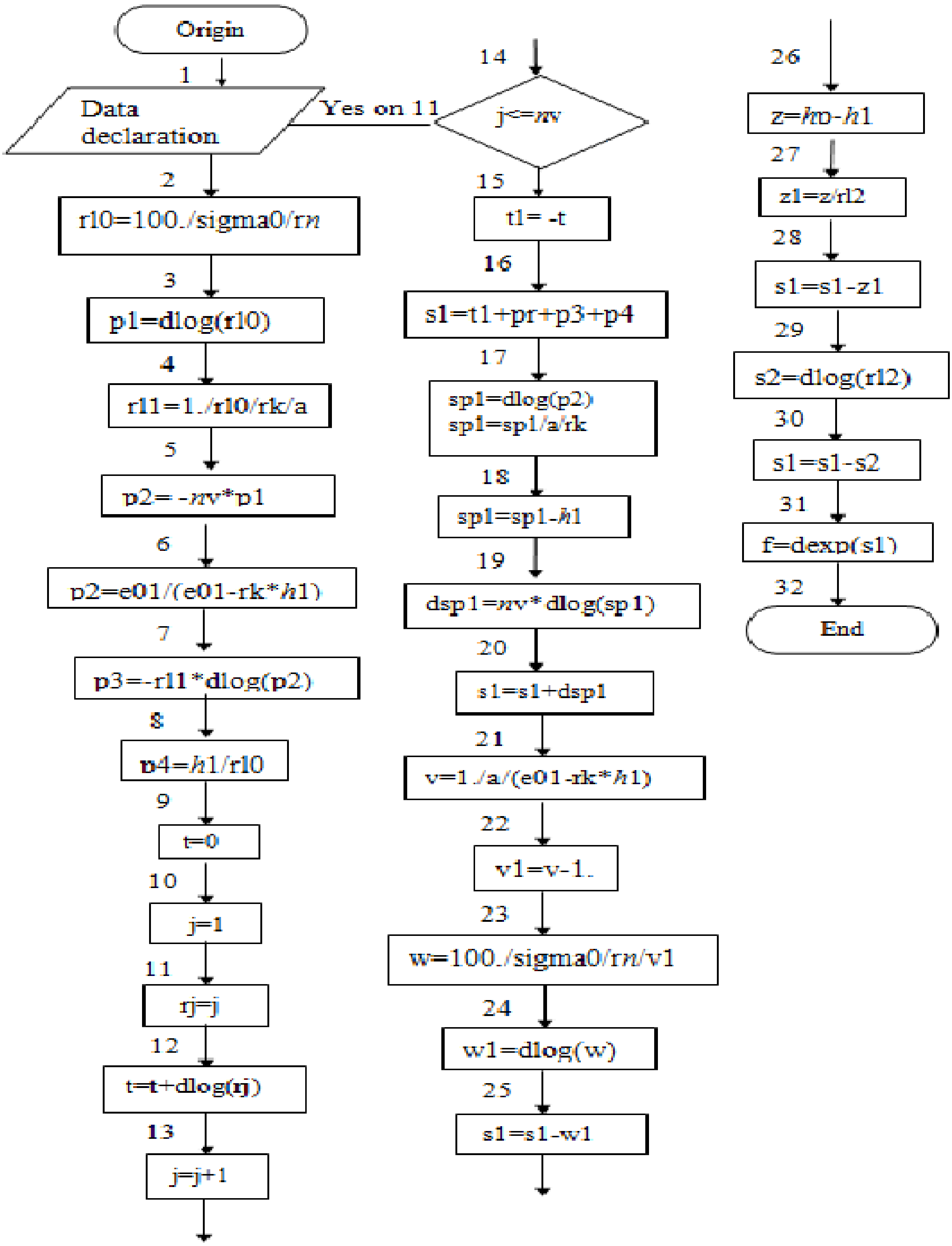


Table 1

Definitional domain boundaries of radiation defect concentration, numbers of interactions and calculating time for germanium in aluminium at $E_c = 50\,\text{keV}$ and $E_0 = 1000\,\text{keV}$

$h \cdot 10^4$, *cm*	C_k, cm^{-1}	E_0, keV	n_0	n_1	τ
0,1	31453	1000	342	663	$5''$ 80
2,0	34641	900	10319	11772	$1'$ $26''$
4,0	38790	800	22509	24628	$4'$ $10''$
6,0	44061	700	36526	39213	$7'$ $26''$
8,1	51426	600	53795	57102	$12'$ $57''$
10,1	61078	500	73611	77333	$19'$ $31''$
12,0	74085	400	96941	101215	$29'$ $03''$
13,0	83450	350	111802	116424	$36'$ $54''$
14,0	95401	300	129231	134213	$44'$ $35''$
14,3	99134	280	135104	140152	$47'$ $20''$
14,7	105238	260	143505	148774	$53'$ $36''$
15,1	112049	240	152665	158192	$59'$
15,5	119653	220	162734	168361	$1°$ $03'$
15,8	124508	200	171000	176816	$1°$ $09'$
16,2	133078	180	183184	189101	$1°$ $14'$
16,6	142065	160	197029	203203	$1°$ $23'$
16,9	144496	140	208809	215170	$1°$ $30'$
17,3	148840	120	226991	233671	$1°$ $44'$
17,6	135244	100	243135	250085	$1°$ $55'$
18,0	99266	80	269617	277085	$2°$ $17'$
18,1	48552	70	277471	284876	$2°$ $22'$
18,3	0	60	295266	302961	$2°$ $33'$

Table 2

Definitional domain boundaries of radiation defect concentration, numbers of interactions and calculating time for silver in silicon at $E_c = 200\,\text{keV}$ and $E_0 = 1000\,\text{keV}$

$h \cdot 10^5$, *cm*	C_k, cm^{-1}	E_0, keV	n_0	n_1	τ
0,01	16563	1000	174	417	$4''$ 35
1,81	17930	900	60299	63742	$4''$ 38
3,60	19347	800	130090	135256	$15'$
5,37	20642	700	210534	217106	$46'$
7,11	21479	600	304159	311851	$1°$ $33'$
8,84	20904	500	416958	426047	$2°$ $36'$
10,52	16167	400	553995	564515	$4°$ $11'$
11,35	10063	350	636131	647055	$7°$ $39'$
12,17	0	300	730486	742372	$9°$ $33'$

3. The interval of area of result greatly displaces to the right and increases with increase of atomic number of a flying particle, value of concentration in a point of maximum and values of concentration strongly increase. With an increase in the atomic number of the target for the same incident particle,

the value of the function at the maximum point increases slightly, the depths decrease, i.e. more vacancy clusters are formed in the heavier target, especially in the near-surface region.

4. Initial and final values of number of interactions increase depending on penetration depth, the interval of area of result (n_0, n_i) also increases and displaces to the right.

5. Values of concentration abruptly increase and calculating time increases at the great atomic weight of a flying particle and target, for the rest a behavior of area of result similar increases.

6. If the atomic weight of the incident particle is much less than the atomic number of the target, then the concentration of radiation defects becomes zero at energies significantly higher than the threshold energy. As a rule, the conversion to zero occurs at $E_1 = 50, 60\,\mathrm{keV}$ for a concentration calculated at $E_c = 50\,\mathrm{keV}$.

7. When increasing threshold energy at the same penetration depth, the values of the concentration of radiation defects decrease significantly, the boundaries of the result area do not change.

Conclusion

Thus, the work developed algorithms for calculating the CPF, PKA spectra, and the concentration of radiation defects. Calculations were made for silver in silicon, germanium in aluminum, aluminum in iron at various values of the initial and threshold energy. Regularities of the behavior and finding of the region of the result of the concentration of radiation defects are obtained depending on the initial ion energy, threshold energy, penetration depth, atomic number of the incident particle and target. It should be noted that earlier in the calculations, energy losses were not taken into account and the simplest CVF was used. In this work, we used the expressions for the CPF, PKA spectra, and defect concentrations taking into account energy losses. The obtained models and calculation algorithms can be used by specialists in the field of solid state radiation physics. Using the results obtained, they can be used for calculations of various incident particles and targets of the periodic table.

References

1 Boos, E.G., Kupchishin, A.A., Kupchishin, A.I., Shmygalev, E.V. & Shmygaleva, T.A. (2015). Cascade and probabilistic method, solution of radiation and physical tasks, Boltzmann's equations. *Communication with Markov's chains.* Almaty: KazNPU after Abay, Scientific Research Institute for New Chemical Technologies and Materials under al-Farabi KazNU.

2 Czarnacka, K., Komarov, F.F., Romanov, I.A., Zukowski, P. & Parkhomenko, I.N. (2017). C measurements and dielectric properties of nitrogen-rich silicon nitride thin films. *Proceedings of the 2017 IEEE 7th International Conference on Nanomaterials: Applications and Properties, 5, 86,* 657–669 [in Russian].

3 Komarov, F.F., Milchanin, O.V., Parfimovich, I.D., Tkachev, A.G. & Bychanok, D.S. (2017). Absorption and Reflectance Spectra of Microwave Radiation by an Epoxy Resin Composite with Multi-Walled Carbon Nanotubes. *Journal of Applied Spectroscopy, 84, 4,* 570–577.

4 Komarov, F.F. (2017). Nano- and microstructuring of solids by swift heavy ions. *Physics-Uspekhi, 187, 5,* 465–504.

5 Komarov, F.F., Romanov, I.A., Vlasukova, L.A., Mudryi, A.V. & Wendler, E. (2017). Optical and Structural Properties of Silicon with Ion-Beam Synthesized InSb Nanocrystals. *Spectroscopy, 120, 1,* 204–207.

6 Komarov, F.F., Konstantinov, S.V., Strel'nitskij, V.E.& Pilko, V.V. (2016). Effect of Helium ion irradiation on the structure, the phase stability, and the microhardness of TiN, TiAlN, and TiAlYN nanostructured coatings. *Technical Physics, 61, 5,* 696–702.

7 Al'zhanova, A., Dauletbekova, A., Komarov, F., Akilbekov, A. & Zdorovets, M. (2016). Peculiarities of latent track etching in SiO2/Si structures irradiated with Ar, Kr and Xe ions. *Nuclear*

Instruments and Methods in Physics Research, Section B: Beam Interactions with Materials and Atoms, 374, 1, 121–124.

8 Komarov, F.F., Zayats, G.M., Komarov, A.F.,Michailov, V.V. & Komsta, H. (2015). Simulation of radiation effects in SiO2/Si Structures. *Acta Physica Polonica A, 3, 1,* 854–860.

9 Komarov, F., Uglov, V., Vlasukova, L., Zlotski, S.& Yuvchenko, V. (2015). Drastic structure changes in pre-damaged GaAs crystals irradiated with swift heavy ions. *Nuclear Instruments and Methods in Physics Research, Section B: Beam Interactions with Materials and Atoms, 360, 4,* 150–155.

10 Kodanova, S.K., Ramazanov, T.S., Issanova, M.K., Golyatina, R.I. & Maiorov, S.A. (2018). Calculation of ion stopping in dense plasma by the Monte-Carlo method. *Journal of Physics: Conference Series, 946, 1,* 148–154.

11 Smyrnova, K.V., Pogrebnjak, A.D., Beresnev, V.M., Kravchenko, Y.O.& Bondar, O.V. (2018). Microstructure and Physical–Mechanical Properties of (TiAlSiY)N Nanostructured Coatings Under Different Energy Conditions. *Metals and Materials International, 24, 2,* 241–439.

12 Kozhamkulov, B.A., Kupchishin, A.I., Bitibaeva, Z.M.& Tamuzs, V.P. (2017). Radiation-Induced Defect Formation in Composite Materials and Their Destruction Under Electron Irradiation. *Mechanics of Composite Materials, 53, 1,* 59–64.

13 Kupchishin, A.I. (2017). On positrons irradiation in the defects of vacancy type. *IOP Conference Series: Materials Science and Engineering, 184, 1,* 234–248.

14 Kupchishin, A.I., Niyazov, M.N., Voronova, N.A., Kirdiashkin, V.I.& Abdukhairova, A.T. (2017). The effect of temperature, static load and electron beam irradiation on the deformation of linear polymers. *IOP Conference Series: Materials Science and Engineering, 168, 1,* 229–274.

15 Kupchishin, A.I., Kupchishin, A.A.& Voronova, N.A. (2016). Cascade-probabilistic methods and solution of Boltzman type of equations for particle flow. *IOP Conference Series: Materials Science and Engineering, 110, 1,* 123–129.

16 Kupchishin, A.I., Kupchishin, A.A., Voronova, N.A.& Kirdyashkin, V.I. (2016). Positron sensing of distribution of defects in depth materials. *IOP Conference Series: Materials Science and Engineering, 110, 4,* 248–258.

17 Kupchishin, A.I., Shmygalev, E.V., Shmygaleva, T.A.& Jorabayev, A.B. (2015). Relationship between markov chains and radiation defect formation processes by ion irradiation. *Modern Applied Science, 9, 3,* 59–70.

18 Kupchishin, A.I., Kupchishin, A.A., Shmygalev, E.V., Shmygaleva, T.A.& Tlebaev, K.B. (2014). Calculation of the spatial distribution of defects and cascade- Probability functions in the materials. *Journal of Physics: Conference Series, 552, 2,* 89–96.

19 Voronova, N. A., Kupchishin, A.I., Kupchishin, A.A., Kuatbayeva, A.A.& Shmygaleva, T.A.(2018). Effect of the nanoparticles with various structures and morphologies on the thermo-mechanical properties of epoxy-based carbon fiber-reinforced plastics, "Computer Modeling of Depth Distribution of Vacancy Nanoclusters in Ion-Irradiated Materials". *Key Engineering Materials, 769,* 358–363.

20 Kupchishin, A., Voronova, N., Shmygaleva, T.& Kupchishin, A.(2018). Computer Simulation of Vacancy Clusters Distribution by Depth in Molybdenum Irradiated by Alpha Particles. *Key Engineering Materials, 781,* 3–72.

21 Kupchishin, A.I., Kupchishin, A.A., Shmygaleva, T.A.& Shafii, C.A. (2019). Computer simulation of the energy spectra of PKA in materials irradiated by protons in the framework of the Cascade-Probabilistic method. *IOP Conference Series: Materials Science and Engineering, 510, 012024,* 1–6.

22 Komarov F.F.& Kumakhov M.A. (1980). Tables of spatial distribution parameters of ion-implanted impurities. *BGU named after Lenin V.I. publisher, Minsk,* 352.

Т.А. Шмыгалева, А.А. Қонысбаева

Иондық сәулелеудегі радиациялық ақаулардың концентрациясын есептеу алгоритмдерін құрастыру

Жұмыс каскадты-ықтималды әдіс аясында жасалынған, оның мәнісі – әртүрлі бөлшектерге арналған каскадты-ықтималды функциясын (КЫФ) алу және әрі қарай қолдануында болады. КЫФ ықтималдығының мәні – бірсыпыра "h" тереңдікте туындалған бөлшек "n" рет соқтығысуынан кейін белгілі бір "h" тереңдігіне жетуде. Мақалада есептеу сапасы мен уақытты қысқарту мақсатында әрекеттестіктер және бөлшектердің ену санына, иондық сәулелендіру кезіндегі радиациялық ақау концентрациясына байланысты каскадты-ықтималды функциясын (КЫФ) есептеуге арналған алгоритмдерді оңтайландыру ұсынылған. КЫФ және радиациялық ақаулар шоғырлануын есептеу үшін нәтиже аймағы, осы аймақ шекаралары мен есептеу қадамы табылады. Шекараларды іріктеу және есептеу қадамын автоматтандырылуы орындалған.

Кілт сөздер: алгоритм, есептеу, ықтималдық, ұдайы, каскадты-ықтималды функциясы, концентрация, радиациялық ақаулар, ион, нәтиже аймағы.

Т.А. Шмыгалева, А.А. Конысбаева

Разработка алгоритмов расчета концентрации радиационных дефектов при ионном облучении

Работа выполнена в рамках каскадно-вероятностного метода, суть которого заключается в получении и дальнейшем использовании каскадно-вероятностных функций (КВФ) для различных частиц. КВФ имеют смысл вероятности того, что частица, сгенерированная на некоторой глубине h, достигнет определенной глубины h после n-го числа соударений. Авторами предложена оптимизация алгоритмов для расчета каскадно-вероятностных функций (КВФ) в зависимости от числа взаимодействий и глубины проникновения частиц, концентрации радиационных дефектов при ионном облучении с целью уменьшения времени и качества расчета. Для расчета КВФ и концентрации радиационных дефектов находится область результата, границы этой области и шаг для расчета. Выполнена автоматизация подбора границ и шага для расчета.

Ключевые слова: алгоритм, расчет, вероятность, регулярность, каскадная функция вероятности, концентрация, радиационные дефекты, ион, область результатов.

АВТОРЛАР ТУРАЛЫ МӘЛIМЕТТЕР
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