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On strongly loaded heat equations

The article is devoted to the research of boundary value problems for the spectrum - loaded operator of heat conduction with the moving point of loading to the temporary axle in zero or on infinity. For strongly loaded parabolic 2k-order equations the adjoint boundary value problems, when order of loaded term is greater then one of differential part of equation, is studied. In this article we continue a investigation of the boundary value problems for spectrally loaded parabolic equations in unbounded domains.The boundary value problem for the spectral-loaded equation of thermal conductivity, which on the one hand is quite close to the problems with the load containing the second derivative of the spatial variable, and is of independent interest on the other hand in this work, is considered.

Keywords: loaded heat equation, class of essentially bounded functions, inverse Laplace transformation, residue.

1 Statement of the problem

We consider the first boundary value problem of heat conduction in the degenerating domain $Q = \{x \in (0,\infty),\ t \in (0,\infty)\}$ the cogeralized boundary value problems for a heavily loaded heat equation (which generally is called a heat equation order $2k$) in the domain :

$$L_\lambda u = f \Leftrightarrow \begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + \lambda \dfrac{\partial^{2k} u}{\partial x^{2k}}\ |_{x=a} = f, \\ u(x,0) = 0, u(0,t) = 0; \end{cases} \tag{1}$$

$$L_\lambda^* \upsilon = g \Leftrightarrow \begin{cases} -\dfrac{\partial \upsilon}{\partial t} - \frac{\partial^2 \upsilon}{\partial x^2} + \overline{\lambda}\delta^{(2k)}(x-a) \bigotimes \int_0^\infty \upsilon(\xi,t)d\xi = g, \\ u(x,\infty) = 0, \upsilon(0,t) = \upsilon(\infty,t) = 0, \end{cases} \tag{2}$$

where $a = const, a > 0, k \geq 2, \lambda = \lambda_1 + \iota\lambda_2 \in \mathcal{C}$ is the parameter

$$f, u \in L_1(Q), \frac{\partial^{2k} u}{\partial x^{2k}}\ |_{x=a} \in L_1(0,\infty); \qquad g, \upsilon, \int_0^\infty \upsilon(\xi,t)d\xi \in L_\infty(Q). \tag{3}$$

2 Reducing the problem to an integral equation

By inverting the differential part in the boundary value problem we obtain the following (1), we will have:

$$u(x,t) = -\lambda \int_0^t erf(\frac{x}{2\sqrt{t-\tau}}) * \frac{\partial^{2k} \upsilon(\eta,\tau)}{\partial \eta^{2k}} \mid_{\eta=a} d\tau +$$

$$+ \int_0^t \int_0^\infty G(x,\xi,t-\tau) f(\xi,\tau) d\xi d\tau, \tag{4}$$

where

$$G(x,\xi,t) = \frac{1}{2\sqrt{\Pi t}} exp(-\frac{(x-\xi)^2}{4t}) - exp(-\frac{(x+\xi)^2}{4t}) \tag{5}$$

Then differentiating (4) x by $2k$ times and assuming $x = a$, we contain the integral equation Volterra of the second kind

$$K_{\lambda\mu} \equiv \mu(t) - \lambda \int_0^t K_{2k}(t-\tau)\mu(\tau)d\tau = f_1(t) \tag{6}$$

where the following notation is used:

$$\mu(t) = \frac{\partial^{2k} u}{\partial x^{2k}} \mid_{x=a}, f_1(t) = \left(\frac{\partial^{2k}}{\partial x^{2k}} \int_0^t \int_0^\infty G(x,\xi,t-\tau) f(\xi,\tau) d\xi d\tau \right)_{|x=a}$$

$$K_{2k}(t-\tau) = \frac{d^{2k}}{dx^{2k}} \left\{ erf\left(\frac{x}{2\sqrt{t-\tau}} \right) \right\}_{|x=a}$$

or for the kernel $K_{2k}(\theta)$, you can use the ratio

$$K_{2k(\theta)} = \frac{1}{\sqrt{\pi\theta}} \cdot \frac{d^{2k-1}}{dx\sqrt{2k-1}} \left\{ exp\left(-\frac{x^2}{4\theta} \right) \right\} \mid_{x=a}$$

For example, we write the explicit form of kernels for $k = 2,3,4$:

$$K_{4(\theta)} = \frac{1}{4\sqrt{\pi}} \left[-\frac{a\sqrt{3}}{2\theta^{7/2}} + \frac{3a}{\theta^{5/2}} \right] exp\left(-\frac{a^2}{4\theta} \right),$$

$$K_{6(\theta)} = \frac{1}{8\sqrt{\pi}} \left[-\frac{a\sqrt{5}}{4\theta^{11/2}} + \frac{5a^3}{\theta^{9/2}} - \frac{15a}{\theta^{7/2}} \right] exp\left(-\frac{a^2}{4\theta} \right),$$

$$K_{8(\theta)} = \frac{1}{16\sqrt{\pi}} \left[-\frac{a\sqrt{7}}{8\theta^{15/2}} + \frac{21a^5}{4\theta^{13/2}} - \frac{105a^3}{2\theta^{11/2}} + \frac{105a}{\theta\theta^{9/2}} \right] exp\left(-\frac{a^2}{4\theta} \right),$$

Inverting the differential part to problem (2) in the same way as in problem (1), we will have:

$$\upsilon(x,t) = -\overline{\lambda} \int_t^\infty \int_0^\infty G(x,\xi,\tau-t) \delta^{(2k)}(\xi-\tau) \bigotimes \int_0^\infty \upsilon(\eta,\tau) d\eta d\xi d\tau +$$

$$+ \int_t^\infty \int_0^\infty G(x,\xi,\tau-t) g(\xi,\tau) d\xi d\tau. \tag{7}$$

Integrating the relation (7) over the variable x from 0 to ∞ and denoting

$$\upsilon(t) = \int_0^\infty \upsilon(\eta,t) d\eta, \tag{8}$$

we will obtain the integral equation

$$K_\lambda^* \upsilon \equiv \upsilon(t) - \overline{\lambda} \int_t^\infty K_{2k}(\tau-t)\upsilon(\tau)d\tau = g_1(t). \tag{9}$$

where

$$g_1(t) = \int_t^\infty \int_0^\infty erf\left(\frac{\xi}{2\sqrt{\tau-t}} \right) g(\xi,\tau) d\xi d\tau.$$

3 Laplace transformation. The partition plane of the spectral parameter

Tne equation (9) is an equation with a difference kernel, so you can use a transformation of Laplace to hear. In this case we use the following formulas [1–3]:

$$L\left\{\int_0^t K(t-\tau)\varphi(\tau)d\tau\right\} = \widehat{K}(p)\cdot\widehat{\varphi}L(p),$$

$$L\left\{\int_t^\infty K(t-\tau)\varphi(\tau)d\tau\right\} = \widehat{K}(-p)\cdot\widehat{\varphi}L(p),$$

where

$$\widehat{K}(-p) = \int_0^\infty K(t)e^{pt}dt,$$

we also use an easily checkable equality.

$$K_{2k}(t-\tau) = \frac{d^{2k}}{dx^{2k}}\left\{erf\left(dfracx2\sqrt{t-\hat{\tau}}\right)\right\}|_{x=a} = \frac{d^k}{dt^k}\left\{erf\left(\frac{x}{2\sqrt{t-\tau}}\right)\right\}|_{x=a}$$

Then if we apply the Laplace transform to the homogeneous equation (9), we obtain the following transcendental equation

$$\widehat{\varphi}L(p)\cdot[1-\overline{\lambda}\cdot\widehat{K}(-p)] = 0. \tag{10}$$

If we assume $\widehat{\varphi}L(p) \neq 0$ then the following equality must hold.

$$1-\overline{\lambda}\cdot\widehat{K}(-p) = 0. \tag{11}$$

Let equation (11) have one simple root -p_0 i.e.

$$1-\overline{\lambda}\cdot\widehat{K}(-p)] = (p-p_0)\cdot\psi(p),$$

where $\widehat{\psi}L(p) \neq 0$. Then equation (10) takes the form $\widehat{\varphi}L(p)\cdot(p-p_0) = 0$, therefore $\widehat{\varphi}L(p) = \delta(p-p_0)$, a, so $\varphi(t) = e^{p_0 t}$, $R_{ep_0} < 0$. From [2; 390, theorem 146] it follows that functions of this kind are the only solutions of the homogeneous equation (9) [4–6].
In our case

$$\widehat{K}(-p) = (-p)^{k-1}e^{-a\sqrt{-p}}.$$

Therefore, we need to find the roots of the transcendental equation

$$1-\overline{\lambda}\cdot(-p)^{k-1}e^{-a\sqrt{-p}} = 0, Re(-p) > 0. \tag{12}$$

In contrast to the previously considered case (spectral-loaded, k1), the roots of equations (12) can be found only approximately (for each numerically given X), in roots cannot be found clearly. To clarify the existence and number of roots of equation (12) for concretes – of the values of parameter A we rewrite it as follows:

$$\lambda = \frac{e^{a\sqrt{-p}}}{(-p)^{k-1}}. \tag{13}$$

Considering this equation as a function $\lambda = \lambda(p)$ whose domain is $Re(-p) > 0$, that, is as a conformal map, we find in what is displayed (on the complex plane λ) the domain of the variable p . By requirement $Re(-p) > 0$, $-\pi/2 < arg(-p) < \pi/2$,means $-\pi/4 < arg\sqrt{(-p)} < \pi/4$, if $z = \sqrt{(-p)} = x + \iota y$, means the boundary of the domain of definition the variable $-p$ is the lines $y = \pm x$. According to the law of correspondence of boundaries it is enough to find the images this line [7–9].

We have

$$|\lambda| = \frac{|e^{az}|}{|z|^{2(k-1)}} = \frac{e^{ax}}{(x^2+y^2)^{k-1}}, \tag{14}$$

$$\arg\lambda = \arg e^{ax+i(ay+2\pi n)} - 2(k-1)\arg z = ay + 2\pi n - 2(k-1)\,\mathrm{arctg}\,\frac{y}{x}.$$

Considering $y = \pm x$, we have

$$\arg\lambda = ax + 2\pi n - 2(k-1)\frac{\pi}{4} = ax + (2n - \frac{k-1}{2})\pi, n \in Z,$$

$$ax = \arg\lambda - (2n - \frac{k-1}{2})\pi, n \in Z. \tag{15}$$

Thus, from equation (14) (talking into account (15) and the fact that $y = \pm x$) we obtain that the lines defined by the equation

$$|\lambda| = \frac{(a/\sqrt{2})^{2k-2}}{|\arg\lambda + \left(2n + \frac{k-1}{2}\right)\pi|^{2k-2}} \cdot exp|\arg\lambda + \left(2n + \frac{k-1}{2}\right)\pi|,$$

where $n = 0, 1, 2, ...$ divide the complex $\lambda-$ plane into disjoint domains $D_m, m = 0, 1, 2, ...$

Comment.

Note that in addition to the areal domain D_0, which has only the outer boundary of $\Gamma_0 = \partial D_0$ each of the domains D_m has a boundary∂D_m consisting of Γ_m an external Γ_{m-1} part:

$$\partial D_m = \Gamma_{m-1} \bigcup \Gamma_m, where, \Gamma_{m-1} \bigcap \Gamma_m = (-1)^m exp\{m\pi\},$$

those the external Γ_m and internal Γ_{m-1} parts of the boundary ∂D_m of the domain D_m have one common point, lying on the real axis of the complex plane of parameter λ. D_0 is the area into which that part of the plane of the complex variable p I displayed, for which $-\pi/4 < argp < 7\pi/4$,, those are the exterior of the angle lying between the lines $y = -x$ and $y = x$ This just means that if $\lambda \in D_0$, then equation (12) does not have the roots we need, i.e. such for which $Re(-p) > 0$.

Obviously, for $k = 1$ we get our previously established picture of the partition of the complex plane λ [10].

Difference is that for each domain D_m, that is, when $\lambda \in D_m$ equation (12) will have exactly $2m$ roots, (this is easy to trace, for example, for real values λ, p) [11–13].

4 Solution of integral equations

With $\forall\lambda \in D_m,\ m = 1, 2, ...,$ homogeneous equation (9) has a General solution of the species.

$$\upsilon(t) = \Sigma_{k=1}^{2m} c_k \cdot e^{p_k t}, \tag{16}$$

where c_k - is an arbitrary constants, p_k- is the corresponding roots of equation (12).
We find a proprietary solution of the in homogeneous equation (9). Applying Laplace's transformation to it we get

$$\widehat{\upsilon}(p) \cdot \left[1 - \overline{\lambda}(-p)^{k-1} e^{-a\sqrt{-p}}\right] = \widehat{g_1}(p), \text{ with } Rep \le 0, \tag{17}$$

where $\widehat{\upsilon}(p), \widehat{g_1}(p)$ - is the Laplace transformation, corresponding to the $\upsilon(t)$ and $g_1(t)$ functions; Since the function

$$\widehat{A}(p, \overline{\lambda}) \equiv 1 - \overline{\lambda}(-p)^{k-1} e^{-a\sqrt{-p}}.$$

It is determined only at $Rep \le 0$, then we will continue it analytically on the whole complex plane with a cut along the positive real axis. Suppose that the Laplace transform of functions $g(t)$ is analytic in the band $-\varepsilon < Rep < \varepsilon$. Then from equality (17) at $\forall\lambda\Gamma_m$,
$(m = 0, 1, 2, ...,)$ we get

$$\widehat{\upsilon}(p) = \widehat{g_1}(p) + \overline{\lambda}\frac{(-p)^{k-1} exp(-a\sqrt{-p})}{1 - \overline{\lambda}(-p)^{k-1} exp(-a\sqrt{-p})} \cdot \widehat{g_1}(p).$$

Passing in this balance to the originals we get

$$\upsilon(t) = g_1(t) + \overline{\lambda}\int_t^\infty r_\lambda - (t - \tau) g_1(\tau) d\tau, \tag{18}$$

where

$$r_\lambda - (y) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{(-p)^{k-1} exp(-a\sqrt{-p})}{1 - \overline{\lambda}(-p)^{k-1} exp(-a\sqrt{-p})} exp(yp) dp. \tag{19}$$

If the roots of the equation
$$1-\overline{\lambda}(-p)^{k-1}exp(-a\sqrt{-p})=0.$$
lie on the imaginary axis integration will perform along the contour, bypassing these points on the left. At the sometime integral should be understood in the sense of main value for Cauchy. Thus, the General solution of equation (9) at $\lambda \in D_m$ has the form

$$\upsilon(t)=g_1(t)+\overline{\lambda}\int_t^{\infty} r_{\lambda}-(t-\tau)g_1(\tau)d\tau+\Sigma_{k=1}^{2m}c_k\cdot e^{p_kt}. \tag{20}$$

We formulate the results in the form of the following lemmas.

Lemma 1. The $\lambda \in D_0$ values are regular numbers of the $\mathbf{K}_{\lambda^*}$ (9) operator.

Lemma 2. The set $C\setminus D_0$ consists of the characteristic numbers of the $\mathbf{K}_{\lambda^*}$ operator (9). And if $\lambda \in D_m\cup\Gamma_{m-1}\setminus\{(-1)^m e^{m\pi}\},\ m=1,2,...,$ then $Ker(\mathbf{K}_{\lambda^*})=2m$ and the corresponding own functions have the form:
$$\upsilon_{\lambda k}(t)=exp(p_kt),\quad k=1,...,2m.$$
where p_k- is the corresponding roots of equation (12).

Now consider the integral equation (6), which is usually called the *recovery equation* [14, 15]. *This name is explained by the fact that such equations arise in the theory of restoration - a section of probability theory, which describes a wide range of phenomena related to with the Failure and recovery of elements of the any system. The recovery equation is also of great importance in the study of problems of both applied and theoretical nature in reliability, queueing theory, in the theory of stocks, in the theory of branching processes and so on.* [16–18].

Applying the Laplace transform to equation (6)and using the convolution theorem we obtain

$$\widehat{\mu}(p)=\widehat{f}(p)+\frac{\lambda}{p}^{k-1}e^{-a\sqrt{p}}1-\lambda p^{k-1}e^{-a\sqrt{p}}\cdot\widehat{f_1}(p),\quad p=s+i\sigma,\quad Rep=s>0,$$

Using the inverse Laplace transform we have

$$\mu(p)=f_1(t)+\lambda\int_0^t r_{\lambda}+(t-\tau)f_1(\tau)dt, \tag{21}$$

here the resolvent $r_{\lambda+}(\theta)$ is defined by the formula

$$r_{\lambda}+(\theta)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\lambda p^{k-1}e^{-a\sqrt{p}}}{1-\lambda p^{k-1}e^{-a\sqrt{p}}}\cdot e^{p\theta}dp,\quad p=s+i\sigma. \tag{22}$$

Where the integration path is parallel to the imaginary axis of the complex plane, to the Right of all singular points of the integrand, that is, to the right of all zeros of the function
$$\widehat{A}(p,\lambda)=1-\lambda\cdot p^{k-1}e^{-a\sqrt{p}}.$$

In order for the $\mu(t)$function defined by equality (21) to be substantially limited, it is necessary and sufficient that the conditions are satisfied

$$\int_0^{\infty} f_1(t)\cdot exp(-p_kt)dt=0,\quad 1\le k\le 2m, \tag{23}$$

where p_k the roots of the function $\widehat{A}(p,\lambda)$ for which $Rep_k>0$ and they coincide with the roots of equation (12) with the opposite sign. Thus, the right side of equation (6) must be orthogonal to the eigenfunctions (16) of the conjugate integral equation (9). Thus, the fair

Lemma 3. If $\lambda\in D_0$, then the inhomogeneous equation (6) is definitely torn; if $\lambda\in C\setminus D_0,\lambda\in D_m$, then for the unambiguous solvability of equation (6), it is necessary and sufficient fulfillment of $m-$ the conditions of solvability (23). The conditions (23) mean that the free member of the integral equation (6) must be orthogonal to the solutions of the homogeneous conjugate integral (9).

The validity of these statements, as well as conditions (23), can be shown as follows.

The image of the solution of the integral equation (6) is determined by the equality

$$\widehat{\mu}(p) = \frac{\widehat{f}_1(p)}{1 - \lambda p^{k-1} e^{-a\sqrt{p}}}. \qquad (24)$$

The following options are possible.

1. The $\widehat{A}(p, \lambda) = 1 - \lambda \cdot p^{k-1} e^{-a\sqrt{p}}$. function doesn't have zeros in the right half-plane (this means that the $\lambda \in D_0$). In this case, the equation for any right part of $f(t)$ has the only solution, which is expressed through the $r_{\lambda+}(\theta)$ resolvent defined by the formula (22)

$$\mu(t) = f(t) + \lambda \int_0^t r_\lambda + (t-\tau) f_1(\tau) d\tau, \quad t \in \Re_+. \qquad (25)$$

2. Function $\widehat{A}(p, \lambda) = 1 - \lambda \cdot p^{k-1} e^{-a\sqrt{p}}$. has a $p_k, (k = 1, 2, ...2m)$ zeroes in the right half plane (it means $\lambda \in D_m,\ m = 1, 2, ...,)$. The $\widehat{f}_1(t)$ function must then be zero at these points p_k. In this Case, the function (24) again will not have pluses in the *Re* $p > 0$ area, so the equation (6) also has the only solution of the species (25). Condition $\widehat{f}_1(t) = 0$, on the conversion of the $\widehat{f}_1(t)$ function to zero at points $p = p_k$ just is exactly the some as the condition:

$$\int_0^\infty f(t) \cdot e^{-p_k t} dt = 0, \quad k = 1, 2, ..., 2m. \qquad (26)$$

So we proved the following statement.

Lemma 4. On the complex plane C there are no characteristic numbers of the operator $\mathbf{K}_{\lambda^*}$ (6).

5 Main result

Directly from lemmas and integral representations (4)–(7) follows

Theorem. Boundary value problems (1)–(2) are Noetherian if $\lambda \in \{C \setminus D_0\}$ in Addition, if $\lambda \in D_m$, then $dimKer(L_{\lambda^*} = 2mm)$ and $dimKer(L_\lambda) = -2m$.

References

1 Krasnov, M.L. (1975). *Integral Equations.* Moscow: Nauka.

2 Titchmarsh, E. (1986). *Introduction to the theory of Fourier integrals.* (3d ed.) Chelsea Pub Co.

3 Koks, D.V., & Smith, E. (1967). *Recovery theory.* Moscow.

4 Akhmanova, D.M., Dzhenaliev, M.T., & Ramazanov, M.I. (2011). On a particular second kind Volterra integral equation with a spectral parameter. *Siberian Mathematical Journal, 52, 1,* 1–10. DOI: 10.1134 /S0037446606010010

5 Amangaliyeva, M.M., Akhmanova, D.M., Dzhenaliev, M.T., & Ramazanov, M.I. (2011). Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity. *Differential Equations, 47, 2,* 231–243. DOI: 10.1134/S0012266111020091

6 Kosmakova, M.T. (2016). On an integral equation of the Dirichlet problem for the heat equation in the degenerating domain. *Bulletin of the Karaganda University. Mathematic sriies, 1 (81),* 62–67.

7 Amangaliyeva, M.M., Dzhenaliev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). On one homogeneous problem for the heat equation in an infinite angular domain. *Siberian Mathematical Journal, Vol. 56, No. 6,* 982–995. DOI: 10.1134/S0037446615060038

8 Kosmakova, M.T., Akhmanova, D.M., Iskakov, S.A., Tuleutaeva, Zh.M., & Kasymova, L.Zh. (2019). Solving one pseudo-Volterra integral equation. Bulletin of the Karaganda University. Mathematics seriies, 1 (93), 72–77. DOI 10.31489/2019M1/72-77.

9 Dzhenaliev, M.T., & Ramazanov, M.I. (2006). On the boundary value problem for the spectrally loaded heat conduction operator. *Siberian Mathematical Journal, 47, 3,* 433–451. DOI: 10.1007/s11202-006-0056-z

10 Dzhenaliev, M.T., & Ramazanov, M.I. (2010). *Loaded equations as perturbations of differential equations.* Almaty: Nauka.

11 Dzhenaliev, M.T., & Ramazanov, M.I. (2007). On a boundary value problem for a spectrally loaded heat operator: I. *Differential Equations, 43, 4,* 513–524. DOI: 10.1134/S0012266107040106

12 Dzhenaliev, M.T., & Ramazanov, M.I. (2007). On a boundary value problem for a spectrally loaded heat operator: II. *Differential Equations, 43, 6,* 806–812. DOI: 10.1134/S0012266107060079

13 Ramazanov, M.I., Kosmakova, M.T., Romanovsky, V.G., Zhanbusinova, B.H., & Tuleutaeva, Z.M. (2018). Boundary value problems for essentially-loaded parabolic equation. *Bulletin of the Karaganda University. Mathematics seriies, 4 (92),* 79–86. DOI: 10.31489/2018M4/79-86

14 Kosmakova, M.T., Orumbayeva, N,T., Medeubaev, N.K., & Tuleutaeva, Zh.M. (2018). Problems of Heat Conduction with Different Boundary Conditions in Noncylindrical Domains. AIP Conference Proceedings. 1997. UNSP 020071-1. DOI: 10.1063/1.5049065

15 Akhmanova D.M., Ramazanov M.I., & Yergaliyev M.G. (2018). On an integral equation of the problem of heat conduction with domain boundary moving by law of t = x(2). *Bulletin of the Karaganda University-Mathematics, 1 (89),* 15–19. DOI: 10.31489/2018M1/15-19

16 Kosmakova, M.T., Ramazanov, M.I., Tokesheva, A.S., & Khairkulova, A.A. (2016). On the non-uniqueness of solution to the homogeneous boundary value problem for the heat conduction equation in an angular domain. *Bulletin of the Karaganda University-Mathematics, 4 (84),* 80–87. DOI: 10.31489/2016M4/80-87

17 Jenaliyev, M.T., Iskakov, S.A., & Ramazanov, M.I. (2017). On a parabolic problem in an infinite corner domain. *Bulletin of the Karaganda University-Mathematics, 1 (85),* 28–35. DOI: 10.31489/2017M1/28-35

18 Jenaliyev, M.T., Iskakov, S.A., Ramazanov, M.I., & Tuleutaeva, Z.M. (2018). On the solvability of the first boundary value problem for the loaded equation of heat conduction. *Bulletin of the Karaganda University. Mathematics series, 1 (89),* 33-41.

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Жылуөткізгіштіктің әлді жүктелген теңдеуі туралы

Нөлге немесе шексіздікте жүктеме сызығының уақыт осіне жуықталумен берілген спектралды — жүктелген параболалық операторлар үшін шекаралық есептерді зерттеуге арналған. $2k$ ретті қатты жүктелген параболалық теңдеу үшін жүктелген қосындылардың тәртібі теңдеудің дифференциалды бөлігінің тәртібінен асып кеткен жағдайда ұштасқан шекаралық есептер зерттелді. Мақалада шексіз аймақтардағы спектрлік жүктелген параболалық теңдеулер үшін шеттік есептерді зерттеу жалғастырылған. Жылуөткізгіштіктің спектралды-жүктелген теңдеуі үшін шеттік есеп қарастырылды, ол бірінші жағынан екінші ретті туындыны қамтитын жүктемелерге жеткілікті жақын кеңістіктік айнымалы бойынша анықталса, екінші жағынан өз бетінше қызығушылықты білдірді.

Кілт сөздер: жүктелген теңдеу, маңызды шектелген функциялар классы, Лапластың кері түрлендіруі, шегерім.

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О сильно-нагруженных уравнениях теплопроводности

Даны постановки граничных задач для спектрально-нагруженных параболических уравнений в четверти плоскости, когда порядок производной в нагруженном слагаемом равен порядку дифференциальной части уравнения с движущейся пространственной точкой нагрузки по степенному закону. Для сильно нагруженного параболического уравнения порядка 2k исследованы сопряженные граничные задачи в случае, когда порядок нагруженного слагаемого превышает порядок дифференциальной части уравнения. В статье продолжено исследование краевых задач для спектрально-нагруженных параболических уравнений в неограниченных областях. Рассмотрена краевая задача для спектрально-нагруженного уравнения теплопроводности, которая, с одной стороны, достаточно близка к задачам с нагрузкой, содержащей вторую производную по пространственной переменной, и представляющая самостоятельный интерес — с другой.

Ключевые слова: нагруженное уравнение, класс существенно ограниченных функций, обратное преобразование Лапласа, вычет.

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On a characteristic problem for a loaded hyperbolic equation

The paper studies a loaded hyperbolic equation with one-dimensional wave equation in its main part. In the loaded components there are two points, which are distributed respectively along a pair of intersecting characteristics at a constant speed. For such an equation we study the Cauchy problem with characteristics of the one dimensional wave equation belonging to any of the pair of intersecting straight lines. If to impose certain conditions at the boundary points on function presenting Cauchy data and the derivatives of the first, second and third orders we can prove the existence and uniqueness of the problem. The proof of the existence and uniqueness of the solution follows directly from the method of its production. We consider also issues concerning domains of dependence, influence, and definitions for Cauchy data that are specified on one of the characteristic curves. The above verifies once more the thesis of load influencing on posing of various initial - boundary value problem for partial differential equations.

Keywords: Cauchy problem, loaded equation, wave equation, characteristics, domain of influence, domain of dependence.

Introduction

A.M. Nakhushev first made the most general definition for a loaded equation in [1]. In [2] he had introduced concepts and a detailed classification for various loaded equations: loaded differential, loaded integral, loaded integrodifferential, loaded functional equations. Besides fundamental work on the study of loaded integral and loaded ordinary differential equations, we wish to acknowledge some of the proceedings [3–6]. Together with A.M. Nakhushev and his successors [7–16] a systematic researches and significant contribution to the boundary value problems for loaded differential equations had been made by the Kazakh mathematicians M.T. Dzhenaliev and M.I.Ramazanov, and their students [17, 18]. They investigated a wide class of homogeneous and inhomogeneous boundary value problems for essentially loaded parabolic and hyperbolic-elliptic equations, as well as the spectral issues on the corresponding homogeneous problems, when the load is specified with respect to the space variable and the load point moves at constant and variable speeds. Works [19–21] are devoted to uniqueness classes for solution of the Cauchy problem and non-trivial solutions of the homogeneous Cauchy problem for some classes of loaded differential equations and linear loaded systems of the first order.

The Cauchy problem and the Cauchy-Dirichlet problem for a spectrally loaded parabolic equation with a load at a fixed time variable are investigated in [22, 23].

The effect of the load on the convergence of spectral expansions for operators is considered in [24].

In [25–27] the Goursat problem for second-order strictly and weakly hyperbolic equations with two independent variables is investigated, where it is shown that the load allows eliminating the inequality between characteristics that are data in the Goursat problem.

This paper considers the following loaded equation

$$u_{xx} - u_{yy} = \lambda u(x+y,\, 0) + \mu u(x-y,\, 0), \tag{1}$$

where λ, μ are arbitrary real constants.

Using characteristic variables $\xi = x - y$, $\eta = x + y$ equation (1) has the form

$$v_{\xi\eta} = \frac{\lambda}{4} v(\eta,\, \eta) + \frac{\mu}{4} v(\xi,\, \xi), \tag{2}$$

where $v(\xi,\, \eta) = u\left(\frac{\xi+\eta}{2},\, \frac{\eta-\xi}{2}\right)$.

By (2) it follows that

$$v(\xi,\,\eta)=f(\xi)+g(\eta)+\frac{\lambda}{4}\,\xi\int_0^{\eta}v(t,t)dt+\frac{\mu}{4}\,\eta\int_0^{\xi}v(t,t)dt, \tag{3}$$

where $f(\xi)$, $g(\eta)$ are arbitrary smooth enough functions.

As $\eta=\xi$ by (3) obtain

$$\frac{d}{d\xi}\left[e^{-\frac{\lambda+\mu}{8}\xi^2}\int_0^{\xi}v(t,t)dt\right]=e^{-\frac{\lambda+\mu}{8}\xi^2}\,[f(\xi)+g(\xi)].$$

Hence

$$\int_0^{\xi}v(t,t)dt=\int_0^{\xi}e^{\frac{\lambda+\mu}{8}(\xi^2-t^2)}\,[f(t)+g(t)]\,dt. \tag{4}$$

Substituting into (3) instead of integrals values obtained using formula (4) and moving to x, y, coordinates we get

$$u(x,y)=f(x-y)+g(x+y)+\frac{\lambda}{4}(x-y)\int_0^{x+y}e^{\frac{\lambda+\mu}{8}\left[(x+y)^2-t^2\right]}\,[f(t)+g(t)]\,dt+$$
$$+\frac{\mu}{4}(x+y)\int_0^{x-y}e^{\frac{\lambda+\mu}{8}\left[(x-y)^2-t^2\right]}\,[f(t)+g(t)]\,dt. \tag{5}$$

Formula (5) is an analogue of d'Alembert formula for equation (1) and obviously as $\lambda=\mu=0$ coincides with the d'Alembert formula for equation (1) as $\lambda=\mu=0$.

Assume $\Omega=\{(x,y):\,0<x+y<1,\ 0<x-y<1\}$ is a characteristic quadrilateral.

Cauchy problem. In the domain $\bar{\Omega}$ find a regular solution u(x, y) of equation (1) continuous in $\bar{\Omega}$ and satisfying the conditions

$$u\left(\frac{x}{2},\,\frac{x}{2}\right)=\varphi(x),\quad 0\le x\le 1, \tag{6}$$

$$u_y\left(\frac{x}{2},\,\frac{x}{2}\right)=\psi(x),\quad 0\le x\le 1, \tag{7}$$

where $\varphi(x)$, $\psi(x)$ – are specified functions.

It is well known that as $\lambda=\mu=0$ this problem is not well posed. The necessary and sufficient condition for its solvability is

$$\varphi'(x)-\psi(x)=\varphi'(0)-\psi(0),\quad 0\le x\le 1.$$

If the above condition is satisfied, the function $u(x,y)$ is a solution to problem (6), (7) for equation (1) and has the form

$$u(x,y)=f(x-y)-f(0)+\varphi(x+y),$$

where f is an arbitrary twice continuously differentiable function.

The following theorem is valid.

Theorem. Assume $\lambda\neq 0$, $\varphi\in C^3(\bar{J})\cap C^4(J)$, $\psi\in C^2(\bar{J})\cap C^3(J)$ and the coordination conditions

$$\varphi(0)=\psi(0)=\varphi'(0)=0, \tag{8}$$

$$\varphi''(0)-\psi'(0)=0, \tag{9}$$

$$\varphi'''(0)-\psi''(0)=0. \tag{10}$$

Then the solution for the Cauchy problem exists, is unique and representable in the form of

$$u(x,y)=\frac{4}{\lambda}\,[\varphi''(x-y)-\psi'(x-y)]-\left(x-\frac{\lambda+2\mu}{\lambda}y\right)[\varphi'(x-y)-\psi(x-y)]+$$
$$+(x-y)\,[\varphi'(x+y)-\psi(x+y)]+\varphi(x+y)-\varphi(x-y). \tag{11}$$

Indeed the function $u(x,y)$ is a solution to the Cauchy problem if and only if it is representable as (5). Therefore, by substituting (5) into (6) and (7), we can get

$$f(0)+g(x)=\varphi(x), \tag{12}$$

$$-f'(0)+g'(x)-\frac{\lambda}{4}\int_0^{x}e^{\frac{\lambda+\mu}{8}(x^2-t^2)}\,[f(t)+g(t)]\,dt-$$

$$-\frac{\mu}{4}x\left[f(0)+g(0)\right]=\psi(x). \tag{13}$$

Taking into account (8) from (12) and (13) it is clear that

$$f(0)+g(0)=0,\;\; f'(0)=0.$$

Hence, by (12) and (13) we have

$$g(x)=\varphi(x)-f(0), \tag{14}$$

$$\int_0^x e^{\frac{\lambda+\mu}{8}(x^2-t^2)}\left[f(t)+g(t)\right]dt=\frac{4}{\lambda}\left[\varphi'(x)-\psi(x)\right]. \tag{15}$$

By differentiation of (15) with respect to x and subtracting identity (15), previously multiplied by $\frac{\lambda+\mu}{4}x$ we can get

$$f(x)+g(x)=\frac{4}{\lambda}\left[\varphi''(x)-\psi'(x)\right]-\frac{\lambda+\mu}{\lambda}x\left[\varphi'(x)-\psi(x)\right]$$

or

$$f(x)=f(0)-\varphi(x)+\frac{4}{\lambda}\left[\varphi''(x)-\psi'(x)\right]-\frac{\lambda+\mu}{\lambda}x\left[\varphi'(x)-\psi(x)\right].$$

Substituting the obtained values into $f(x)$ and $g(x)$ in (5) and taking into account (15) when calculating the last two terms of formula (5), and after simple conversions we arrive at formula (11).

Substitution into equation (1) ensures us that the function $u(x,y)$ calculated by formula (11) is the solution to (1). It is easy to check that under conditions (8)–(10) the function $u(x,y)$ satisfies conditions (6), (7).

Note that the Cauchy problem as $\lambda=0$ is not a well posed one. Indeed by (12), (13) the condition

$$\varphi'(x)-\psi(x)=\varphi'(0)-\psi(0)+\frac{\mu}{4}\varphi(0)$$

is necessary and sufficient for the solvability of the problem. In case this condition is satisfied the solution to the Cauchy problem is

$$u(x,y)=f(x-y)-f(0)+\varphi(x+y)+\frac{\mu}{4}(x+y)\int_0^{x-y}\left[f(t)-f(0)+\varphi(t)\right]e^{\frac{\mu}{4}\left[(x-y)^2-t^2\right]}dt,$$

where f — is an arbitrary twice continuously differentiable function.

If we make a replacement $u(x,y)=v(x,-y)$ in equation (1) then problem

$$u\left(\frac{x}{2},-\frac{x}{2}\right)=\varphi(x),\quad 0\le x\le 1, \tag{16}$$

$$u_y\left(\frac{x}{2},-\frac{x}{2}\right)=\psi(x)\quad 0\le x\le 1, \tag{17}$$

becomes the problem

$$v\left(\frac{x}{2},\,\frac{x}{2}\right)=\varphi(x),\quad v_y\left(\frac{x}{2},\,\frac{x}{2}\right)=-\psi(x)$$

for the equation $v_{xx}-v_{yy}=\mu v(x+y,\,0)+\lambda\, v(x-y,\,0)$.

Therefore, assuming that $\mu\neq 0$ and

$$\varphi(0)=\psi(0)=\varphi'(0)=0;\qquad \varphi''(0)+\psi'(0)=0;\qquad \varphi'''(0)+\psi''(0)=0,$$

the solution to the problem is representable as

$$v(x,y)=\frac{\mu}{4}\left[\varphi''(x-y)+\psi'(x-y)\right]-\left(x-\frac{\mu+2\lambda}{\mu}y\right)\left[\varphi'(x-y)+\psi(x-y)\right]+$$
$$+(x-y)\left[\varphi'(x+y)+\psi(x+y)\right]+\varphi(x+y)-\varphi(x-y).$$

This implies the solution to problem (16), (17) for equation (1) casts into the form

$$u(x,y)=\frac{\mu}{4}\left[\varphi''(x+y)+\psi'(x+y)\right]-\left(x+\frac{\mu+2\lambda}{\mu}y\right)\left[\varphi'(x+y)+\psi(x+y)\right]+$$
$$+(x+y)\left[\varphi'(x-y)+\psi(x-y)\right]+\varphi(x-y)-\varphi(x+y). \tag{18}$$

It is known [28] that in case with three spatial variables corresponding to the Cauchy problem a wave is completely determined by the Cauchy data on a sphere. This fact in the theory of sound is called Huygens principle. We also know that with two spatial variables in wave processes the Huygens principle does not hold since to determine the wave the Cauchy data must be specified not only on the circle but also at all points of the corresponding circle. In the case of one variable, to determine the value of the oscillation at the point (x, y), one of the components of the Cauchy data must be set on the segment boundary $[x - y, x + y]$, and the second at all points of this segment.

The idea is that to determine $u(x, y)$ at the point (x, y) in formulas (11), (18) you need to know the Cauchy data only on the segment boundary $[x - y, x + y]$. That is, we can say that there is a one-dimensional version for the Huygens principle.

References

1 Нахушев А.М. О задаче Дарбу для одного вырождающегося нагруженного интегро-дифференциального уравнения второго порядка / А.М. Нахушев // Дифференциальные уравнения. — 1976. — Т. 12, № 1. — С. 103–108.

2 Нахушев А.М. Уравнения математической биологии / А.М. Нахушев. — М.: Высш. шк., 1995. — 301 с.

3 Knezer A. Rendiconti del Cirolo Matematico di Palermo / A. Knezer. — 1914. — V. 37. — P. 169–197.

4 Крейн М.Г. О нагруженных интегральных уравнениях, функции распределения которых не монотонны / Д.А. Траве // Сб. памяти акад. — М.; Л., 1940. — С. 88–103.

5 Krall A.M. Differential - boundary operators: Trans. Amer. Math. Soc. — 1971. — 154 p.

6 Phillips R.S. Dissipativ operators and hyperbolic systems of partional differential equations / R.S. Phillips // Trons. Amer. Math. Sos. — 1990. — P. 193–254.

7 Нахушев А.М. Нагруженные уравнения и их приложения / А.М. Нахушев. — М.: Наука, 2012. — 232 с.

8 Нахушев А.М. О нелокальных краевых задачах со смещением и их связи с нагруженными уравнениями / А.М. Нахушев // Дифференциальные уравнения. — 1985. — Т. 21, № 1. — С. 92–101.

9 Нахушев А.М. Нелокальные задачи и задача Гурса для нагруженного уравнения гиперболического типа и их приложения к прогнозу почвенной влаги / А.М. Нахушев // Докл. АН СССР. — 1978. — Т. 242, № 5. — С. 1008–1011.

10 Нахушев А.М. Краевые задачи для нагруженных интегро-дифференциальных уравнений гиперболического типа и некоторых их приложениях к прогнозу почвенной влаги / А.М. Нахушев // Дифференциальные уравнения. — 1979. — Т. 15, № 1. — С. 96–105.

11 Нахушева В.А. Дифференциальные уравнения математических моделей нелокальных процессов / А.М. Нахушев. — М.: Наука, 2006. — 174 с.

12 Елеев В.А. О некоторых краевых задачах для смешанных нагруженных уравнений второго и третьего порядка / В.А. Елеев // Дифференциальные уравнения. — 1994. — Т. 30, № 2. — С. 230–237.

13 Псху А.В. Уравнения в частных производных дробного порядка / А.В. Псху. — М.: Наука, 2005. — 199 с.

14 Казиев В.М. Задача Гурса для одного нагруженного интегро-дифференциального уравнения / В.М. Казиев // Дифференциальные уравнения. — 1981. — Т. 17, № 2. — С. 313–319.

15 Хубиев К.У. Аналог задачи Трикоми для характеристически нагруженного уравнения гиперболо-параболического типа с переменными коэффициентами / К.У. Хубиев // Уфим. математ. журн. — 2017. — Т. 9, № 2. — С. 94–103.

16 Хубиев К.У. Задачи со смещением для нагруженного уравнения гиперболо-параболического типа с оператором дробной диффузии / К.У. Хубиев // Вестн. Удмурт. ун-та. Сер. Математика. Механика. Компьютерные науки. — 2018. — Т. 28, № 1. — С. 82–90.

17 Дженалиев М.Т. К теории линейных краевых задач для нагруженных дифференциальных уравнений / М.Т. Дженалиев. — Алматы: Компьютерный центр ТИПМ, 1995. — 270 с.

18 Дженалиев М.Т. Нагруженные уравнения как возмущения дифференциальных уравнений / М.Т. Дженалиев, М.И. Рамазанов. — Алматы: Ғылым, 2010. — 334 с.

19 Огородников Е.Н. Некоторые характеристические задачи для систем нагруженных дифференциальных уравнений и их связь с нелокальными краевыми задачами / Е.Н. Огородников // Вестн. Самар. гос. техн. ун-та. Сер. Физ.-мат. науки. — 2003. — Т. 19. — С. 22–28.

20 Борок В.М. О существовании и единственности решения задачи Коши для нагруженных уравнений / В.М. Борок // Изв. вузов. Сер. Матем. — 1986. — № 4. — С. 8–13.

21 Борок В.М. О единственности решения задачи Коши для систем линейных нагруженных уравнений / В.М. Борок // Украин. математ. журн. — 1985. — Т. 37, № 1. — С. 108–109.

22 Дженалиев М.Т. О задаче Коши для существенно нагруженного параболического уравнения / М.Т. Дженалиев, М.И. Рамазанов // Математ. журн. — 2004. — Т. 4, № 3(13). — С. 22–26.

23 Дженалиев М.Т. О гранично-начальной задаче для существенно нагруженного параболического уравнения / М.Т. Дженалиев, М.И. Рамазанов // Изв. НАН РК, Сер. физ-мат. — 2005. — № 5. — С. 36–43.

24 Ломов И.С. Нагруженные дифференциальные операторы: сходимость спектральных разложений / И.С. Ломов // Дифференциальные уравнения. — 1983. — Т. 19, №1. — С. 86–94.

25 Аттаев А.Х. Задача Гурса для нагруженного вырождающегося гиперболического уравнения второго порядка с оператором Геллерстедта в главной части / А.Х. Аттаев // Вестн. Самар. гос. техн. ун-та. Сер. Физ.-мат. науки. — 2016. — №1 (20). — С. 7–22.

26 Аттаев А.Х. Задача Гурса для нагруженного гиперболического уравнения / А.Х. Аттаев // Докл. Адыгской (Черкесской) МАН. — 2014. — Т. 16, № 3. — С. 9-12.

27 Аттаев А.Х. Задача с данными на параллельных характеристиках для нагруженного волнового уравнения / А.Х. Аттаев // Докл. Адыгской (Черкесской) МАН. — 2013. — Т. 15. — №2. — С. 25–28.

28 Бицадзе А.В. Уравнения математической физики / А.В. Бицадзе. — М.: Наука, 1982. — 336 с.

А.Х. Аттаев

Жүктелген гиперболалық теңдеу үшін бір сипаттамалық есеп жайлы

Бұл жұмыста басты бөлігі шектің тербелісінің бірөлшемді теңдеуі болатын жүктелген гиперболалық теңдеу объект болып табылады. Жүктелген қосылғыштарда тұрақты жылдамдықпен қиылысатын сипаттаушы жұптарының бойына сәйкес таралатын жүктеменің екі нүктесі бар. Осындай теңдеу үшін бірөлшемді шектің тербелісінің теңдеуінің сипаттаушылары болып табылатын қиылысатын түзулердің кез келген жұбының деректерімен Коши есебін зерттеу жүргізілді. Коши деректерін беретін функцияға нүктелік сипаттағы белгілі шарттарда және олардың бірінші, екінші, үшінші туындылары қойылған есептің бар болуы және жалғыздығы дәләлденді, ал шешуі жайлы түсінік анық жазылды. Шешудің бар болуы және жалғыздығын дәлелдеу оны алу әдісінің өзінен шығады. Сонымен қоса бір сипаттаушыда берілетін, Коши деректерінің тәуелділік, әсер ету және анықталу облыстарымен байланысты сұрақтар қарастырылды. Осылайша дербес туындылы жүктемелі дифференциалдық теңдеулер үшін бастапқы-шектік есептердің қойылымына жүктеменің әсерінің тиімділігі жайлы тезис кезекті рет нақтыланды.

Кілт сөздер: Коши есебі, жүктемелі теңдеу, шектің тербеліс теңдеуі, сипаттамалар, әсер ету облысы, тәуелсіздік облысы.

А.Х. Аттаев

Об одной характеристической задаче для нагруженного гиперболического уравнения

Объектом исследования статьи является нагруженное гиперболическое уравнение, главная часть которого представляет собой одномерное уравнение колебания струны. В нагруженных слагаемых присутствуют две точки нагрузки, которые распространяются соответственно вдоль пары пересекающихся характеристик с постоянной скоростью. Для такого уравнения проводится исследование задачи Коши с данными на любой из пары пересекающихся прямых, являющихся характеристиками уравнения колебания одномерной струны. При определенных условиях точечного характера на функции, задающие данные Коши и производные первого, второго и третьего порядков от них, доказывается существование и единственность поставленной задачи, а представление самого решения выписывается в явном виде. Доказательство существования и единственности решения непосредственно следует из самого способа его получения. Также затрагиваются вопросы, связанные с областями зависимости, влияния и определения данных Коши, задаваемых на одной из характеристик. Этим самым в очередной раз подтверждается тезис об эффекте влияния нагрузки на постановку тех или иных начально-краевых задач для нагруженных дифференциальных уравнений с частными производными.

Ключевые слова: задача Коши, нагруженное уравнение, уравнение колебания струны, характеристики, область влияния, область зависимости.

References

1 Nakhushev, A.M. (1976). O zadache Darbu dlia odnoho vyrozhdaiushchehosia nahruzhennoho intehro-differentsialnoho uravneniia vtoroho poriadka [On the Darboux problem for a single degenerate loaded second-order integro-differential equation]. *Differentsialnye uravneniia – Differential Equations, Vol. 12, No. 1,* 103–108 [in Russian].

2 Nakhushev, A.M. (1995). *Uravneniia matematicheskoi biolohii [Equations of mathematical biology].* Moscow: Vysshaia shkola [in Russian].

3 Knezer, A. (1914). *Rendiconti del Cirolo Matematico di Palermo, Vol. 37,* 169–197.

4 Kerin, M.G. (1940). O nahruzhennykh intehralnykh uravneniiakh, funktsii raspredeleniia kotorykh ne monotonny [On loaded integral equations with non-monotonous distribution functions]. Sb. Memory Acad. D.A. Trave. Moscow; Leningrad, 88–103 [in Russian].

5 Krall, A.M. (1971). Differential - boundary operators: Trans. Amer. Math. Soc.

6 Phillips, R.S. (1990). Dissipativ operators and hyperbolic systems of partional differential equations. *Tpons. Amer. Math. Sos.,* 193–254.

7 Nakhushev, A.M. (2012). *Nahruzhennye uravneniia i ikh prilozheniia [Loaded equations and their applications].* Moscow: Nauka [in Russian].

8 Nakhushev, A.M. (1985). O nelokalnykh kraevykh zadachakh so smeshcheniem i ikh sviazi s nahruzhennymi uravneniiami [On nonlocal boundary value problems with displacement and their relationship with loaded equations]. *Differentsialnye uravneniia – Differential Equations, Vol. 21, no. 1,* 92–101 [in Russian].

9 Nakhushev, A.M. (1978). Nelokalnye zadachi i zadacha Hursa dlia nahruzhennoho uravneniia hiperbolicheskoho tipa i ikh prilozheniia k prohnozu pochvennoi vlahi [Nonlocal problems and the Goursat problem for a loaded hyperbolic type equation and their application to the soil moisture forecast]. *Doklady AN SSSR – Doklady Academy of Sciences of the USSR, Vol. 242, No. 5,* 1008–1011 [in Russian].

10 Nakhushev, A.M. (1979). Kraevye zadachi dlia nahruzhennykh intehro-differentsialnykh uravnenii hiperbolicheskoho tipa i nekotorye ikh prilozheniiakh k prohnozu pochvennoi vlahi [Boundary value problems for loaded integro-differential equations of hyperbolic type and some of their applications to the soil moisture prediction]. *Differentsialnye uravneniia – Differential Equations, Vol. 15, No. 1,* 96–105 [in Russian].

11 Nakhusheva, V.A. (2006). *Differentsialnye uravneniia matematicheskikh modelei nelokalnykh protsessov [Differential equations of mathematical models of non-local processes].* Moscow: Nauka [in Russian].

12 Eleev, V.A. (1994). O nekotorykh kraevykh zadachakh dlia smeshannykh nahruzhennykh uravnenii vtoroho i treteho poriadka [On some boundary value problems for mixed loaded equations of the second and third orders]. *Differentsialnye uravneniia – Differential Equations, Vol. 30, No. 2,* 230–237 [in Russian].

13 Psku, A.V. (2005). *Uravneniia v chastnykh proizvodnykh drobnoho poriadka [Differential equations with partial derivatives].* Moscow: Nauka [in Russian].

14 Kaziev, V.M. (1981). Zadacha Hursa dlia odnoho nahruzhennoho intehro-differentsialnoho uravneniia [The Goursat problem for a single loaded integro-differential equation]. *Differentsialnye uravneniia – Differential Equations, Vol. 17, No. 2,* 313–319 [in Russian].

15 Khubiev, K.U. (2017). Analoh zadachi Trikomi dlia kharakteristicheski nahruzhennoho uravneniia hiperbolo-parabolicheskoho tipa s peremennymi koeffitsientami [Analog of the Tricomi problem for a characteristic loaded hyperbolic-parabolic equation with variable coefficients]. *Ufimskii matematicheskii zhurnal – Ufa Mathematical Journal, Vol. 9, No. 2,* 94–103 [in Russian].

16 Khubiev, K.U. (2018). Zadachi so smeshcheniem dlia nahruzhennoho uravneniia hiperbolo-parabolicheskoho tipa s operatorom drobnoi diffuzii [Tasks with an offset for a loaded equation of a hyperbalo-parabolic type with a fractional diffusion operator]. *Vestnik Udmurtskoho universiteta. Seriia Matematika. Mekhsnika. Kompiuternye nauki. – Bulletin of Udmurt University. Math. Mechanic. Computer Sciences, Vol. 28, No. 1,* 82–90 [in Russian].

17 Dzhenaliev, M.T. (1995). *K teorii lineinykh kraevykh zadach dlia nahruzhennykh differentsialnykh uravnenii [On the theory of linear boundary value problems for loaded differential equations].* Almaty: Kompiuternyi tsentr TIPM [in Russian].

18 Dzhenaliev, M.T. & Ramazanov, M.I. (2010). *Nahruzhennye uravneniia kak vozmushcheniia differentsialnykh uravnenii [Loaded equations as perturbations of differential equations].* Almaty: Hylym [in Russian].

19 Ogorodnikov, E.N. (2003). Nekotorye kharakteristicheskie zadachi dlia sistem nahruzhennykh differentsialnykh uravnenii i ikh sviaz s nelokalnymi kraevymi zadachami [Some characteristic problems for systems of loaded differential equations and their relations with nonlocal boundary value problems]. *Vestnik Samarskoho hosudarstvennoho universiteta. Seriia phys.-math. nauk – Bulletin Samara State Tech. University. Series phys.-math. sciences, Vol. 19,* 22–28 [in Russian].

20 Borok, V.M. (1986). O sushchestvovanii i edinstvennosti resheniia zadachi Koshi dlia nahruzhennykh uravnenii [On existence and uniqueness of the solution to the Cauchy problem for loaded equations]. *Izvestie vuzov. Seriia Matematika – Izv. universities. Math., No. 4,* 8—13 [in Russian].

21 Borok, V.M. (1985). O edinstvennosti resheniia zadachi Koshi dlia sistem lineinykh nahruzhennykh uravnenii [On the uniqueness of the solution to the Cauchy problem for systems of linear loaded equations]. *Ukrainskii matematicheskii zhurnal – Ukrainian Mathematical Journal, Vol. 37, No. 1,* 108, 109 [in Russian].

22 Dzhenaliev, M.T. & Ramazanov, M.I. (2004). O zadache Koshi dlia sushchestvenno nahruzhennoho parabolicheskoho uravneniia [On the Cauchy problem for a substantially loaded parabolic equation]. *Matematicheskii zhurnal – Mathematical Journal, Vol. 4, No. 3(13),* 22–26 [in Russian].

23 Dzhenaliev, M.T. & Ramazanov, M.I. (2005). O hranichno-nachalnoi zadache dlia sushchestvenno nahruzhennoho parabolicheskoho uravneniia [On boundary - initial problem for a substantially loaded parabolic equation]. *Izvestiia NAS RK, Seriia Phys. math., No. 5,* 36–43 [in Russian].

24 Lomov, I.S. (1983). Nahruzhennye differentsialnye operatory: skhodimost spektralnykh razlozhenii [Loaded Differential Operators: Convergence of Spectral Expansions]. *Differentsialnye uravneniia – Differential Equations, Vol. 19, No. 1,* 86–94 [in Russian].

25 Attayev, A.Kh. (2016). Zadacha Hursa dlia nahruzhennoho vyrozhdaiushchehosia hiperbolicheskoho uravneniia vtoroho poriadka s operatorom Hellerstedta v hlavnoi chasti [The Goursat problem for a loaded degenerate second-order hyperbolic equation with the Gellerstedt operator in the principal part]. *Vestnik Samarskoho hosudarstvennoho tekhnicheskoho universiteta. Seriia phys.-mat. nauk, No. 1(20),* 7–22 [in Russian].

26 Attayev, A.Kh. (2014). Zadacha Hursa dlia nahruzhennoho hiperbolicheskoho uravneniia [Goursat problem for a loaded hyperbolic equation]. *Reports of the Adygskaya (Circassian) International Academy of Sciences, Vol. 16, No. 3,* 9–12 [in Russian].

27 Attayev, A.Kh. (2013). Zadacha s dannymi na parallelnykh kharakteristikakh dlia nahruzhennoho volnovoho uravneniia [A problem with data on parallel characteristics for a loaded wave equation]. *Reports of the Adygha (Circassian) International Academy of Sciences, Vol. 15, No. 2,* 25–28 [in Russian].

28 Bitsadze, A.V. (1982). *Uravneniia matematicheskoi fiziki [Equations of mathematical physics].* Moscow: Nauka [in Russian].

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Singularly perturbed control problems in the case of the stability of the spectrum of the matrix of an optimal system

The paper considers a singularly perturbed control problem with a quadratic quality functional. Such problems in their standard formulation under known spectrum restrictions (the points of the spectrum of the optimal system are not purely imaginary and are located symmetrically with respect to the imaginary axis) were previously considered using the Vasilyeva - Butuzov method of boundary functions. If at least one of the points of the spectrum for some values of the independent variable falls on the imaginary axis, the boundary functions method does not work. It is precisely this situation with the assumption of purely imaginary points of the spectrum that is investigated in this paper. In this case, you have to develop a different approach based on the ideas of the regularization method S.A. Lomov. It should also be noted that in the control problems considered earlier, the cost functional either did not depend on a small parameter at all, or allowed a smooth dependence on the parameter. In this paper, an irregular dependence on a small parameter is allowed, in particular, the presence in them of a rapidly changing damping function in the form of an exponential factor under the integral sign. In this case, the spectrum behavior of the optimal system depends on the damping coefficient, which (under certain conditions) can shift the spectrum in one direction or another in the complex plane. In this case, a situation may arise when some points of the spectrum for individual values (or even on a certain continuum set) of an independent variable can become purely imaginary. This situation is not amenable to investigation by the previously mentioned Vasilieva - Butuzov method of boundary functions. However, it can be fully studied using the regularization method S.A. Lomov, the algorithm of which is applied to the considered control problem in the present paper. The presentation of this method begins with a brief description of the maximum principle of L.S. Pontryagin for the classical optimal control problem, which then, along with other ideas, is used to justify the results in the considered control problem.

Keywords: singularly perturbed, Pontryagin's maximum principle, regularization, asymptotic convergence.

Introduction

The presentation of the regularization method of S.A. Lomov [1] begins with a brief description of the Pontryagin's maximum principle for the classical optimal control problem, which is then applied to a linear singularly perturbed control problem with a quadratic quality functional (cost functional) in the case of a stable spectrum of the matrix of an optimal system.

Consider a linear control system

$$\frac{dx}{dt} = A(t)x + B(t)u + h(t),\ x(0) = x^0; \tag{1}$$

$$J(u) = \frac{1}{2}\int_0^T (x^*Q(t)x + u^*R(t)u)dt \to \min, \tag{2}$$

where $x(t)$, $h(t)$ are n — dimensional; $u(t)$ is m — dimensional vector functions, x^0 is a constant n-dimensional vector; $A(t)$ is $(n \times n)-$matrix; $B(t)$ is $(n \times m)-$ matrix; $Q\,(t)$ is a symmetric non-negative definite $(n \times n)-$ matrix, $R\,(t)$ is a symmetric positive definite $(m \times m)-$ matrix, * is a transposition sign. It is required to transfer the system (1) from a given initial position $x(0) = x^0$ to a position $x(T)$ in a fixed time $T < +\infty$ ($x(T)$ is not fixed) so that the functional $J(u)$ takes the minimum value. A similar problem was considered in many sources devoted to the theory of optimal control. Our presentation follows the monograph [2]. We introduce a variable

$x_0 = x_0(t)$, satisfying the equation $\dot{x}_0 = f_0(x,u,t)$, $x(t_0) = 0$ (where $f_0(x,u,t) \equiv \frac{1}{2}(x^*Q(t)x + u^*R(t)u)$). Then the problem (1)–(2) will be rewritten as

$$\begin{array}{l} \frac{dx_0}{dt} = f_0(x,u,t), x(0) = 0; \\ \frac{dx}{dt} = A(t)x + B(t)u + h(t), x(0) = x^0; \\ x_0(T) \to \min,\ \ 0 \le t \le T. \end{array} \tag{3}$$

Denote $f(x,u,t) = \{f_1, ..., f_n\} \equiv A(t)x + B(t)u + h(t)$ and make Hamiltonian

$$\tilde{H}(\psi,x,u,t) = \psi_0 f_0 + \sum_{j=1}^{n} \psi_j f_j \equiv \psi_0 f + (\psi_1, ..., \psi_n) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} =$$
$$= \tfrac{1}{2}\psi_0(x^*Q(t)x) + \tfrac{1}{2}\psi_0(u^*R(t)u) + \psi^*A(t)x + \psi^*B(t)u + \psi^*h(t),$$

where $\psi^* \equiv (\psi_1, \ldots, \psi_n)$. According to the maximum principle should be $\psi_0(t) = \text{const} \le 0$. In the problem with a fixed time T and with a free end of the trajectory are equalities

$$\psi_0(T) = -1, \psi_1(T) = \psi_2(T) = \ldots = \psi_n(T) = 0$$

(transversality conditions; see [2; 260]), therefore $\psi_0(t) \equiv -1$, and the function $\tilde{H}$ takes the form

$$\begin{array}{c} \tilde{H}(\psi,x,u,t) = -[\frac{1}{2}(x^*Q(t)x) + \frac{1}{2}(u^*R(t)u) - \\ -\psi^*A(t)x - \psi^*B(t)u - \psi^*h(t)] \equiv -[\frac{1}{2}(x^*Q(t)x) + \\ +\frac{1}{2}(u^*R(t)u) + p^*A(t)x + p^*B(t)u + p^*h(t)], \end{array}$$

where it is indicated: $p^* \equiv -\psi^*$. The function $\tilde{H}$ must reach a maximum on the optimal control $u = u(t)$, which means, that the function $\hat{H}(p^*,x,u,t) \equiv \frac{1}{2}(x^*Q(t)x) + \frac{1}{2}(u^*R(t)u) + p^*A(t)x + p^*B(t)u + p^*h(t)$ should reach to $u = u(t)$ the minimum. We note now that $\frac{dp_j}{dt} = -\frac{d\psi_j}{dt} (j = \overline{1,n})$, therefore, the auxiliary functions $p_j \equiv -\psi_j(t)$ satisfy the system of differential equations $\frac{dp_j}{dt} = \frac{\partial \tilde{H}}{\partial x_j}, j = \overline{1,n}$ (the equation $\dot{p}_0 = \frac{\partial \tilde{H}}{\partial x_0}$ is not written out, since it is trivially satisfied, because $\psi_0(t) \equiv -1$, $\frac{\partial \tilde{H}}{\partial x_0} \equiv 0$). In terms of the function $\hat{H}$, this system can be written in the form

$$\frac{dp_j}{dt} = -\frac{\partial \hat{H}}{\partial x_j},\ j = \overline{1,n}. \tag{4}$$

Calculate $\frac{\partial \hat{H}}{\partial x_j}$. We have (consider that $Q(t)$ is a symmetric matrix)

$$\begin{array}{c} \frac{\partial \hat{H}}{\partial x_j} = \frac{\partial}{\partial x_j}(\frac{1}{2}x^*Q(t)x + p^*A(t)x) = \frac{1}{2}e_j^*Q(t)x + \\ +\frac{1}{2}x^*Q(t)e_j + p^*A(t)e_j = \frac{1}{2}(Q(t)x, e_j) + \\ +\frac{1}{2}(Q(t)e_j, x) + (A(t)e_j, p) = \\ = \frac{1}{2}(x, Q(t)e_j) + \frac{1}{2}(Q(t)e_j, x) + (A(t)e_j, p) = (e_j, Q(t)x) + \\ +(e_j, A^*(t)p) = (e_j, Q(t)x + A^*(t)p), \end{array}$$

where $e_j = \left\{0, ..., 0, \underset{(j)}{1}, 0, ..., 0\right\}, j = \overline{1,n}$. Therefore, the system (4) has the form

$$\begin{array}{c} \frac{dp_j}{dt} = -(e_j, Q(t)x + A^*(t)p), j = \overline{1,n} \Leftrightarrow \\ \Leftrightarrow \frac{dp}{dt} = -Q(t)x - A^*(t)p. \end{array} \tag{5}$$

On the other hand, to search for the minimum points of the function $\hat{H}(p^*, x_0, x, u, t)$ relative to u, we must find its critical points (note that $R(t)$ is a symmetric matrix):

$$\begin{gathered}\frac{\partial \hat{H}}{\partial u_j}=0, j=\overline{1,n} \Leftrightarrow \\ \Leftrightarrow \frac{1}{2}e_j^* R(t)u+\frac{1}{2}u^* R(t)e_j+p^* B(t)e_j=0 \Leftrightarrow \\ \Leftrightarrow (e_j, R(t)u)=-(B(t)e_j, p) \Leftrightarrow \\ \Leftrightarrow (e_j, R(t)u)=-(e_j, B^*(t)p), j=\overline{1,n} \\ \Leftrightarrow R(t)u=-B^*(t)p \Leftrightarrow u=-R^{-1}(t)B(t)^*p.\end{gathered}$$

So, the only one critical point $u=-R^{-1}(t)B^*(t)p$ is obtained. In order to check whether it will realize the minimum of the function $\hat{H}$ for u, consider the matrix of the second derivatives:

$$\begin{gathered}\frac{\partial^2 \hat{H}}{\partial u_j \partial u_k}=\frac{\partial}{\partial u_k}[(e_j, R(t)u)+(e_j, B^*(t)p]= \\ =\frac{\partial}{\partial u_k}(e_j, R(t)u)=(e_j, R(t)e_k)=R_{jk}(t), j,k=\overline{1,n},\end{gathered}$$

where R_{jk} are the elements of the matrix $R(t)$. This shows that the matrix $(\frac{\partial^2 \hat{H}}{\partial u_j \partial u_k})$ does not depend on u. Since $R(t)$ is a positive definite matrix, then the point $u=-R^{-1}(t)B^*(t)p$ is indeed the minimum point of the function $\hat{H}$ for u. So, if the optimal control of problem (1)–(2) exists, then it necessarily has the form $u=-R^{-1}(t)B^*(t)p$, where $p=p(t)$ satisfies system (5) and $x=x(t)$ is calculated from the system (3). In other words, the optimal system has the form

$$\begin{gathered}\frac{dx}{dt}=A(t)x-B(t)R^{-1}(t)B^*(t)p+h(t), x(0)=x^0; \\ \frac{dp}{dt}=-Q(t)x-A^*(t)p, p(T)=0.\end{gathered} \tag{6}$$

It follows from [3] that the control $u=-R^{-1}(t)B^*(t)p$, where $p=p(t)$ satisfies system (6), is optimal, and the corresponding trajectory $x=x(t)$is the optimal trajectory. The boundary-value problem (6) with continuous matrices $A(t), B(t), R(t), Q(t)$ and with continuous function $h(t)$ on a segment $[0,T]$ can be ambiguously solvable, and then the optimal control $u=-R^{-1}(t)B^*(t)p$ is calculated ambiguously. For the unique solvability of problem (6), it is necessary to require that the corresponding homogeneous boundary value problem

$$\begin{gathered}\frac{dx}{dt}=A(t)x-B(t)R^{-1}(t)B^*(t)p, x(0)=0; \\ \frac{dp}{dt}=-Q(t)x-A^*(t)p, p(T)=0.\end{gathered} \tag{6_0}$$

Has only a trivial solution $(x(t), p(t))\equiv 0$. This will take place, for example, in the case when $h(t)\equiv 0$ and $p(t)$ is linear depends on phase coordinates: $p(t)=K(t)x(t)$. Indeed, in this case (as shown in [3; 318, 319]), the $(n\times n)-$ matrix $K(t)\not\equiv 0$ is symmetric and satisfies the nonlinear Riccati matrix differential equation:

$$\begin{gathered}\dot{K}=-K\cdot A(t)-A^*(t)\cdot K+ \\ +K\cdot B(t)R^{-1}(t)\cdot B^*(t)\cdot K-Q(t), K(T)=0 (t\in[0,T]),\end{gathered}$$

and homogeneous problem (6_0) takes the form

$$\begin{gathered}\frac{dx}{dt}=\left[A(t)-B(t)R^{-1}(t)B^*(t)K(t)\right]x,\ x(0)=0; \\ p(t)=K(t)x(t).\end{gathered}$$

This problem has only one solution $(x(t), p(t))\equiv 0$.

2 Singularly perturbed control problems. Building an optimal system

We now consider a singularly perturbed control system

$$\begin{gathered}\varepsilon\dot{x}=A(t)x(t,\varepsilon)+B(t)u(t,\varepsilon)+f(t), x(0,\varepsilon)=x^0, 0\le t\le T; \\ J_\varepsilon(u)=\frac{1}{2}\int_0^T (x^*Q(t)x+u^*R(t)u)\, exp\left(\frac{1}{\varepsilon}\int_T^t \mu(\theta)d\theta\right)dt,\end{gathered} \tag{7}$$

where $\varepsilon>0$ is a small parameter, $x(t,\varepsilon)$, $f(t)$ are n — dimensional, $u(t,\varepsilon)$ is m — dimensional vector functions, x^0 is a constant n-dimensional vector; $A(t)$ is $(n\times n)-$matrix; $B(t)$ is $(n\times m)-$matrix; $Q(t)$ is a symmetric non-negative definite $(n \times n)-$ matrix; $R(t)$ is a symmetric positive definite $(m \times m)-$ matrix, $\mu(t)$ is a scalar function, $t \in [0,T]$, $*$ is a transposition sign. It is required to transfer the system (7) from a given initial position $x(0) = x^0$ to a position $x(T)$ in a fixed time $T < +\infty$ ($x(T)$ is not fixed) so that the functional $J(u)$takes the minimum value. In order to obtain asymptotic representations for $x(t,\varepsilon)$ and $u(t,\varepsilon)$ in the form of series in powers of ε, we require the following conditions to be satisfied:

1^0. The elements of the matrices $A(t), B(t), Q(t)$ and $R(t)$, as well as the components of the vector $f(t)$ and the scalar function $\mu(t)$ belong to $C^\infty([0,T],\mathrm{R})$.

Applying the Pontryagin's maximum principle (see system (1) in the previous section) and taking into account that the role of $A(t)$ is played by the matrix $\frac{1}{\varepsilon}A(t)$, role $B(t)-$ matrix $\frac{1}{\varepsilon}B(t)$, role $Q(t)-$ matrix $Q(t)\,exp\left(\frac{1}{\varepsilon}\int\limits_T^t \mu(\theta)\,d\theta\right)$; role $R(t)$ – matrix $R(t)\,exp\left(\frac{1}{\varepsilon}\int\limits_T^t \mu(\theta)\,d\theta\right)$; role $h(t)-$ function $\frac{1}{\varepsilon}f(t)$, we get the following optimal system:

$$\varepsilon^2\dot{x} = \varepsilon A(t)x - B(t)R^{-1}(t)B^*(t)exp\left(-\frac{1}{\varepsilon}\int\limits_T^t \mu(\theta)d\theta\right)p + \varepsilon f(t), x(0,\varepsilon) = x^0,$$
$$\varepsilon\dot{p} = -\varepsilon Q(t)exp\left(\frac{1}{\varepsilon}\int\limits_T^t \mu(\theta)d\theta\right)x - A^*(t)p, p(T,\varepsilon) = 0.$$

Doing successive replacements

$$\varepsilon x = y, \quad exp\left(-\frac{1}{\varepsilon}\int\limits_T^t \mu(\theta)d\theta\right)p = q, \quad \{y,q\} = z, \tag{8}$$

and taking into account that

$$\varepsilon \cdot \frac{d}{dt}p(t) = \left(\varepsilon \cdot \frac{d}{dt}q(t)\right)e^{\frac{1}{\varepsilon}\int_T^t \mu(\theta)\,d\theta} + q(t)\,\mu(t)\,e^{\frac{1}{\varepsilon}\int_T^t \mu(\theta)\,d\theta},$$

we arrive at the following singularly perturbed boundary value problem with weak inhomogeneity:

$$\varepsilon\frac{d}{d\,t}\,y(t) = A(t)\,y(t) - B(t)\cdot R^{-1}(t)\cdot B^*(t)\,q(t) + \varepsilon f(t);$$
$$\varepsilon\frac{d}{dt}q(t) = -q(t)\,\mu(t) - Q(t)\,y(t) - A^*(t)\cdot q(t),$$

or

$$\begin{array}{c} \varepsilon\dot{z} = S(t)z + \varepsilon h(t), \quad 0 \le t \le T; \\ Gz \equiv Mz(0,\varepsilon) + Nz(T,\varepsilon) = \varepsilon\alpha, \end{array} \tag{9}$$

where $S(t)$ is $(2n \times 2n)-$matrix with elements from the class $C^\infty[0,T]$:

$$S(t) = \begin{pmatrix} A & -BR^{-1}B^* \\ -Q & -(A^* + \mu I) \end{pmatrix},$$

where $M = diag(1,...,1,0,...,0)$, $N = diag(0,...,0,1,...,1)$,$h(t) = f(t),0,...0$, $\alpha = x^0,0,...,0$ are $2n-$ dimensional vectors. In this case, the optimal control (see 1) will be

$$u(t) = -\frac{1}{\varepsilon}R^{-1}(t)B^*(t)q(t). \tag{10}$$

3 Regularization of the optimal system. Construction of solutions of iterative problems

Without detracting from the community, we may assume that $T = 1$. Let $b_j(t)$, $d_j(t)$ $\left(j = \overline{1,2n}\right)$ are eigenvectors of matrices $S(t)$ and $S^*(t)$ corresponding to eigenvalues $\lambda_j(t)$ and $\overline{\lambda}_j(t)$, respectively, and, moreover, $(b_j(t), d_j(t)) = \delta_{ij}$ is Kronecker's symbol. Suppose that besides the condition 1^0, two more conditions are performed:

2^0. The spectrum $\{\lambda_j(t)\}$ of the matrix $S(t)$ has the properties:

a) $\lambda_j(t) \ne 0, j = \overline{1,2n}, \forall t \in [0,1]$;

b) $\lambda_i(t) \ne \lambda_j(t), i \ne j, i,j = \overline{1,2n}, \forall t \in [0,1]$;

c) Re $\lambda_j(t) < 0, j = \overline{1,n}, \mathrm{Re}\,\lambda_j(t) \ge 0, j = \overline{n+1,2n}$, $\forall t \in [0,1]$;

d) $\mathrm{Re}\lambda_1(t) \le \mathrm{Re}\lambda_2(t) \le ... \le \mathrm{Re}\lambda_{2n}(t)$.

3^0. $\det(b_{ij}(0))_{i,j=1}^{n} \cdot \det(b_{ij}(1))_{i,j=n+1}^{2n} \neq 0$.

Here $b_j = \mathrm{colon}(b_{ij},...,b_{2nj}), j = \overline{1,2n}$; the conditions 3^0 mean that in the matrix $\mathfrak{B}(t) = (b_{i,j}(t))_{i,j=\overline{1,2n}}$ the left angle minor $\det(b_{ij}(t))_{i,j=1}^{n}$ of order n does not vanish at the point $t=0$, and the right angle minor $\det(b_{ij}(t))_{i,j=n+1}^{2n}$ of order n does not vanish at the point $t=1$.

The listed conditions $1^0 - 3^0$ are realizable. To verify this, consider the following scalar problem:

$$\varepsilon\left(\tfrac{d}{dt}x(t)\right) = -x(t) + e^t u(t) + f(t), x(0) = x^0 \ (0 \le t \le 1);$$
$$J_\varepsilon(u) = \tfrac{1}{2}\int_0^1 \left(x(t)^2 + u(t)^2\right) e^{\frac{t-1}{\varepsilon}}\, dt \to \min.$$

By completing the replacement (8), we obtain the optimal system

$$\varepsilon\begin{pmatrix}\dot y\\ \dot z\end{pmatrix} = \begin{bmatrix}-1 & -e^{2t}\\ -1 & 2\end{bmatrix}\begin{pmatrix}y\\ z\end{pmatrix} + \varepsilon\begin{pmatrix}f(t);\\ 0\end{pmatrix} \ (0 \le t \le 1),$$
$$y(0) = \varepsilon x^0, \quad q(1) = 0.$$

The eigenvalues and eigenvectors of the matrix $S(t)$ are follows:

$$\lambda_1(t) = \frac{1}{2} - \frac{1}{2}\sqrt{9+4e^{2t}} \leftrightarrow b_1(t) = \begin{pmatrix}\frac{2e^{2t}}{-3+\sqrt{9+4e^{2t}}}\\ 1\end{pmatrix},$$

$$\lambda_2(t) = \frac{1}{2} + \frac{1}{2}\sqrt{9+4e^{2t}} \leftrightarrow b_2(t) = \begin{pmatrix}\frac{-2e^{2t}}{3+\sqrt{9+4e^{2t}}}\\ 1\end{pmatrix}.$$

Obviously, all the conditions $1^0 - 3^0$ for them are realized.

We draw attention to the fact that in the considerate problem may arise a situation when some points of the spectrum on separate values (or even on some continual set) of independent variable can become purely imaginary. This situation is not amenable to the study of the known methods of asymptotic integration (for example, using the Vasilyeva-Butuzov method of boundary functions [4–7]. However, it can be fully studied using the Lomov's regularization method [1, 8–10]. Note, that the regularization method [1, 8] allows us to investigate a wide class of problems in the theory of singular perturbations [11–25].

For the regularization of problem (9), we apply the algorithm of the regularization method, developed for singularly perturbed problems in [1]. We introduce regularizing variables

$$\tau_k = \tfrac{1}{\varepsilon}\int_{t_k}^{t}\lambda_k(s)ds \equiv \varphi_k\left(t, \tfrac{1}{\varepsilon}\right), k = \overline{1,2n};$$
$$t_1 = ... = t_n = 0, \quad t_{n+1} = ... = t_{2n} = 1$$

and consider a new function $\tilde z = \tilde z(t,\tau,\varepsilon)$, for which we set the following problem:

$$\begin{gathered}\varepsilon\tfrac{\partial\tilde z}{\partial t} + \sum_{j=1}^{2n}\lambda_j(t)\tfrac{\partial\tilde z}{\partial\tau_j} - S(t)\tilde z = \varepsilon h(t);\\ G\tilde z(t,\tau,\varepsilon) \equiv M\tilde z(M_0,\varepsilon) + N\tilde z(M_1,\varepsilon) = \varepsilon\alpha,\end{gathered} \tag{11}$$

where

$$M_0\left(0,\varphi\left(0,\frac{1}{\varepsilon}\right)\right), M_1\left(1,\varphi\left(1,\frac{1}{\varepsilon}\right)\right), \varphi\left(t,\frac{1}{\varepsilon}\right) \equiv \left(\varphi_1\left(t,\frac{1}{\varepsilon}\right),...,\varphi_{2n}\left(t,\frac{1}{\varepsilon}\right)\right).$$

If $\tilde z = \tilde z(t,\tau,\sigma,\varepsilon)$ is a solution to problem (11), then the restriction $z(t,\varepsilon) \equiv \tilde z(t,\varphi\left(t,\frac{1}{\varepsilon}\right),\varepsilon)$ will obviously be a solution to problem (9). The resulting problem (11) is regular in ε, and therefore we are looking for its solution in the form of a series

$$\tilde z = \sum_{i=0}^{\infty}\varepsilon^i z_i(t,\tau) \tag{12}$$

by non-negative powers of the parameter ε. To determine the coefficients of this series, we obtain the following iteration systems:

$$L_0 z_0 \equiv \sum_{j=1}^{2n}\lambda_j(t)\frac{\partial z_0}{\partial\tau_j} - S(t)z_0 = 0, \quad Gz_0(t,\tau) \equiv Mz_0(M_0) + Nz_0(M_1) = 0, \tag{13_0}$$

$$L_0 z_1 = h\left(t\right) - \frac{\partial z_0}{\partial t}, Gz_1 = \alpha; \qquad (13_1)$$

$$L_0 z_i = -\frac{\partial z_{i-1}}{\partial t},\ Gz_i = 0,\ \ i = 2, 3, \ldots\ . \qquad (13_i)$$

In order to study the solvability of problems (13_0), (13_1), (13_i), consider the general iterative system

$$L_0 \xi = H(t, \tau, \sigma). \qquad (14)$$

We will find a solution to system (14) in the space of functions

$$U = \left\{ \xi(t,\tau) : \xi = \sum_{k=1}^{2n} \xi_k(t) e^{\tau_k} + \xi_0(t)\, \xi_k\left(t\right) \in C^\infty([0,1]), \mathbb{C}^{2n}), k = \overline{0, 2n} \right\},$$

in which is the following scalar (for each $t \in [0,1]$) product is introduced

$$< \xi\left(t,\tau\right), \zeta\left(t,\tau\right) >= \sum_{k=0}^{2n} \left(\xi_k\left(t\right), \zeta_k\left(t\right)\right), \quad \xi = \xi(t,\tau),\ \zeta\left(t,\tau\right) \in U,$$

where $(,)$ is the usual scalar product in the complex space $\mathbb{C}^{2n}$. Suppose that the right-hand side of system (14) belongs to U, that is $H(t,\tau) = \sum\limits_{k=1}^{2n} h_k(t) e^{\tau_k} + H_0(t) \in U$. The following assertion holds true, which is proved in the same way as the analogous assertion in [1].

Theorem 1. If conditions 1^0,2^0*(a, b, c) are satisfied and* $H(t,\tau) \in U$*, then for the solvability of system (14) in space* U*, it is necessary and sufficient that*

$$< H(t, \tau,), d_k\left(t\right) e^{\tau_k} >\equiv 0, \forall t \in [0,1],\ \ k = \overline{1, 2n}. \qquad (15)$$

Remark 1. Under the conditions of Theorem 1 the system (14) has the following solution in the space U:

$$\xi(t,\tau) = \sum_{k=1}^{2n} \left(\alpha_k(t) b_k(t) + \sum_{s=1, s\neq k}^{2n} H_{ks}(t) b_s(t) \right) e^{\tau_k} + \sum_{k=1}^{n} H_{0k}(t) b_k(t); \qquad (16)$$

where $\alpha_k(t) \in C^\infty([0,1], \mathbb{C}^1)$are arbitrary functions, $H_{ks}(t) \equiv (\lambda_k(t) - \lambda_s(t))^{-1} (H_k(t), d_s(t))$;

$$H_{0k}(t) = -(H_0(t), d_k(t))/\lambda_k(t),\ \ k, s = \overline{1, 2n}.$$

Theorem 2. Let the conditions 1^0*–*3^0 *be satisfied,* $H(t,\tau) \in U$*satisfies the requirements (15). Then the system (14) under boundary conditions* $G\xi = \xi^0$ *and additional conditions*

$$< -\frac{\partial \xi}{\partial t} + g(t),\ d_k\left(t\right) e^{\tau_k} >\equiv 0, \overline{1, 2n}, \forall t \in [0,1], \qquad (17)$$

where $g(t) \in C^\infty([0,1], \mathbb{C}^{2n})$ *is a known function, is uniquely solvable in* U *for* $\varepsilon \in (0, \varepsilon_0]$, where $\varepsilon_0 > 0$ *is sufficiently small.*

Proof. We use the conditions of orthogonality (17) and obtain equations for the unknown functions $\alpha_k(t)$ in the representation (16):

$$\dot{\alpha}_k(t) + (\dot{b}_k\left(t\right), d_k\left(t\right)) \alpha_k = -\sum_{s=1, s\neq k}^{2n} H_{ks} (\dot{b}_s\left(t\right), d_k\left(t\right)), k = \overline{0, 2n}. \qquad (18)$$

From equations (18) we find

$$\alpha_k(t) = e^{-\int\limits_0^t (\dot{b}_k, d_k) d\theta} [\alpha_k(0) - \sum_{s=1, s\neq k}^{2n} \int\limits_0^t e^{\int\limits_0^x (\dot{b}_k, d_k) d\theta} H_{ks}\left(x\right) (\dot{b}_s\left(x\right), d_k\left(x\right)) dx], \qquad (19)$$

where $\alpha_k(0)$, $k=\overline{1,2n}$, are while unknown numbers. Subordinate the vector function (16) to the boundary condition $G\xi=\xi^0 \Leftrightarrow M\xi\left(0,\varphi\left(0,\frac{1}{\varepsilon}\right)\right)+N\xi\left(1,\varphi\left(1,\frac{1}{\varepsilon}\right)\right)=\xi^0$. To simplify the calculations, we write the solution (16) in the form

$$\xi(t,\tau)=\Phi\left(t\right)\mathrm{diag}\left(e^{\tau_1},...,e^{\tau_{2n}}\right)\alpha\left(t\right)+\hat{\xi}(t,\tau),$$

where $\alpha\left(t\right)=\left\{\alpha_1\left(t\right),...,\alpha_{2n}\left(t\right)\right\}$, $\Phi\left(t\right)\equiv\left(b_1(t),...,b_{2n}(t)\right)$ is the matrix of the eigenvectors of the matrix $T(t)$, and by $\hat{\xi}(t,\tau)$ is denoted the particular solution of system (14):

$$\hat{\xi}(t,\tau)=\sum_{k=1}^{2n}\left(\sum_{s=1,s\neq k}^{2n}H_{ks}(t)b_s(t)\right)e^{\tau_k}+\sum_{k=1}^{n}H_{0k}(t)b_k(t).$$

The condition $G\xi=\xi^0$ gives:

$$\begin{gathered}M\Phi\left(0\right)\mathrm{diag}\left(1,...,1,e^{\varphi_{n+1}\left(0,\frac{1}{\varepsilon}\right)},...,e^{\varphi_{2n}\left(0,\frac{1}{\varepsilon}\right)}\right)\alpha\left(0\right)+\\ +N\Phi\left(1\right)\mathrm{diag}\left(e^{\varphi_1\left(1,\frac{1}{\varepsilon}\right)},...,e^{\varphi_n\left(1,\frac{1}{\varepsilon}\right)},1,...,1,\right)\alpha\left(1\right)=\\ =l\left(\varepsilon\right),\ l\left(\varepsilon\right)\equiv\xi^0-M\hat{\xi}\left(0,\varphi\left(0,\tfrac{1}{\varepsilon}\right)\right)+N\hat{\xi}\left(1,\varphi\left(1,\tfrac{1}{\varepsilon}\right)\right).\end{gathered}\tag{20}$$

From equality (19) we find that

$$\alpha_k(1)=e^{-\int\limits_0^1(\dot{b}_k,d_k)d\theta}[\alpha_k(0)-\sum_{s=1,s\neq k}^{2n}\int\limits_0^1 e^{\int\limits_0^x(\dot{b}_k,d_k)d\theta}H_{ks}\left(x\right)(\dot{b}_s\left(x\right),d_k\left(x\right))dx],\ k=\overline{1,2n},$$

so

$$\alpha(1)=\mathrm{diag}(e^{-\int\limits_0^1(\dot{b}_1,d_1)d\theta},...,e^{-\int\limits_0^1(\dot{b}_{2n},d_{2n})d\theta})\alpha(0)+\beta,$$

where β is a constant vector having the form

$$\begin{gathered}\beta=-\{\sum_{s=1,s\neq 1}^{2n}\int\limits_0^1 e^{\int\limits_1^x(\dot{b}_1,d_1)d\theta}H_{1s}\left(x\right)\left(\dot{b}_s\left(x\right),d_1\left(x\right)\right)dx,...;\\ \sum_{s=1,s\neq 1}^{2n}\int\limits_0^1 e^{\int\limits_1^x(\dot{b}_{2n},d_{2n})d\theta}H_{1s}\left(x\right)\left(\dot{b}_s\left(x\right),d_{2n}\left(x\right)\right)dx\}.\end{gathered}$$

Substituting $\alpha\left(1\right)$ in (20), we will have

$$\begin{gathered}[M\Phi\left(0\right)\mathrm{diag}\left(1,...,1,e^{\varphi_{n+1}\left(0,\frac{1}{\varepsilon}\right)},...,e^{\varphi_{2n}\left(0,\frac{1}{\varepsilon}\right)}\right)+\\ +N\Phi\left(1\right)\mathrm{diag}\left(e^{\varphi_1\left(1,\frac{1}{\varepsilon}\right)-\int\limits_0^1(\dot{b}_1,d_1)d\theta},...,e^{\varphi_n\left(1,\frac{1}{\varepsilon}\right)-\int\limits_0^1(\dot{b}_n,d_n)d\theta},e^{-\int\limits_0^1(\dot{b}_{n+1},d_{n+1})d\theta},...,e^{-\int\limits_0^1(\dot{b}_{2n},d_{2n})d\theta}\right)]\times\\ \times\alpha\left(0\right)=l\left(\varepsilon\right)-N\beta.\end{gathered}\tag{21}$$

We divide the matrix $\Phi\left(t\right)$ into blocks of size $n\times n:\Phi\left(t\right)=\begin{pmatrix}\Phi_{11}\left(t\right)&\Phi_{12}\left(t\right)\\ \Phi_{21}\left(t\right)&\Phi_{22}\left(t\right)\end{pmatrix}$ and we introduce the notation:

$$\begin{gathered}\Lambda_1\left(t\right)=\mathrm{diag}\left(\lambda_1\left(t\right),...,\lambda_n\left(t\right)\right),\ \Lambda_2\left(t\right)=\mathrm{diag}\left(\lambda_{n+1}\left(t\right),...,\lambda_{2n}\left(t\right)\right);\\ G_1=\mathrm{diag}\left(-\int\limits_0^1(\dot{b}_1,d_1)d\theta,...,-\int\limits_0^1(\dot{b}_n,d_n)d\theta\right);\\ G_2=\mathrm{diag}\left(-\int\limits_0^1(\dot{b}_{n+1},d_{n+1})d\theta,...,-\int\limits_0^1(\dot{b}_{2n},d_{2n})d\theta\right).\end{gathered}$$

Then the determinant $\Delta\left(\varepsilon\right)$ of the system (21) can be written as

$$\Delta\left(\varepsilon\right)=\begin{vmatrix}\Phi_{11}\left(0\right)&\Phi_{12}\left(0\right)e^{\frac{1}{\varepsilon}\int_1^0\Lambda_2(\theta)d\theta}\\ \Phi_{21}\left(1\right)e^{\frac{1}{\varepsilon}\int_0^1\Lambda_1(\theta)d\theta+G_1}&\Phi_{22}\left(1\right)e^{G_2}\end{vmatrix}.$$

To calculate it, we apply the block Gauss method (see [26; 44, 45]). Multiply the first «column» of this determinant by $-\Phi_{11}^{-1}(0)\,\Phi_{12}(0)\,e^{\frac{1}{\varepsilon}\int_1^0 \Lambda_2(\theta)d\theta}$ the matrix and adding the result to the second «column», we get that

$$\Delta(\varepsilon) = \begin{vmatrix} \Phi_{11}(0) & 0 \\ \Phi_{21}(1)\,e^{\frac{1}{\varepsilon}\int_0^1 \Lambda_1(\theta)d\theta + G_1} & K \end{vmatrix} = \det \Phi_{11}(0)\cdot \det K,$$

where $K = \Phi_{22}(1)\,e^{G_2} - \Phi_{11}^{-1}(0)\,\Phi_{12}(0)\,e^{\frac{1}{\varepsilon}\int_1^0 \Lambda_2(\theta)d\theta}\Phi_{21}(1)\,e^{\frac{1}{\varepsilon}\int_0^1 \Lambda_1(\theta)d\theta + G_1}$. Considering conditions 2^0c), we conclude that $\Delta(\varepsilon) \to \Delta(0) = \det \Phi_{11}(0)\cdot \det[\Phi_{22}(1)\,e^{G_2}\ (\varepsilon \to +0)$. By virtue of the condition 3^0 the determinant $\Delta(0) \neq 0$, and therefore $\Delta(\varepsilon) \neq 0$ for $\varepsilon \in (0, \varepsilon_0]$ and $\varepsilon_0 > 0$ is sufficiently small. This means that system (21) has a unique solution $\alpha(0) = \alpha^0(\varepsilon)$. Substituting this solution into (19), we find uniquely the functions $\alpha_k(t) = \alpha_k(t,\varepsilon)$, and, therefore, we calculate the solution (16) of system (14) in the space U in a unique way. The theorem is proved.

4 Correct solvability of a singularly perturbed two-point boundary value problem and estimation of the remainder term

Applying Theorems 1, 2 to iterative problems (13_i), we find in a unique way solutions of these problems in space U and construct series (12). Arguing by analogy with the monograph [1; 107–126], we prove the following result.

Lemma 1. Let the conditions $1^0 - 3^0$ *be fulfilled. Then the function* $z_{\varepsilon N}(t) = S_N(t,\ \varphi\left(t,\ \frac{1}{\varepsilon}\right),\ \varepsilon) \equiv \sum\limits_{i=0}^{N} \varepsilon^i z_i\left(t, \frac{\varphi(t)}{\varepsilon}\right)$ *satisfies the problem*

$$\varepsilon\frac{dz_{\varepsilon N}}{dt} = S(t)z_{\varepsilon N}(t) + \varepsilon h(t) + \varepsilon^{N+1}R_N(t,\varepsilon),\ \ Gz_{\varepsilon N}(t) = \varepsilon\alpha,$$

where $||R_N(t,\varepsilon)||_{C[0,1]} \le \bar{R}_N = const$ *for* $\varepsilon \in (0,\varepsilon_0]$ *(*$\varepsilon_0 > 0$ *is sufficiently small).*

To substantiate the asymptotic convergence of the formal solution $z_{\varepsilon N}(t) = S_N(t, \varphi\left(t, \frac{1}{\varepsilon}\right), \varepsilon)$ of the problem (9) to its exact solution $z(t,\varepsilon)$, we must prove the solvability of an arbitrary boundary value problem

$$\varepsilon\frac{dz}{dt} = S(t)z + h(t),\ Gz(t,\varepsilon) = z^0,\ t \in [0,T], \tag{22}$$

under the above conditions $1^0 - 3^0$ and give an estimate of its solution through $||z^0||$ and $||h(t)||$. Under conditions of stability of the spectrum $\{\lambda_j(t)\}$ of an operator $S(t)$, such an estimate was carried out using the Green function (see, for example, [1]), the construction of which essentially uses a rather strict condition 2^0d). Here we propose another procedure based on the study of a special integral equation equivalent to system (22). The implementation of this procedure (its main ideas are presented in [27]) makes it possible to avoid constructing the Green function, and also to get rid of condition 2^0d). We formulate some of the statements from [27], which will be used later.

Consider a more general, than (22), boundary value problem

$$\frac{dw}{dt} = S(t)w + h(t),\ lw \equiv M_1 w(0) + N_1 w(T) = w^0,\ t \in [0,T], \tag{23}$$

not containing a parameter. We assume that M_1 and N_1 are arbitrary $(m\times m)$-matrices, $w = \{w_1,\ldots,w_m\}$, $S(t)$ is a well-known $(m\times m)$-matrix, $h(t) = \{h_1,\ldots,h_m\}$ is a well-known vector function, $w^0 = \{w_1^0,\ldots,w_m^0\}$ is a well-known constant vector. We formulate conditions under which the boundary value problem (23) is solvable.

Let $\Phi(t)$ be the fundamental matrix of solutions of the corresponding homogeneous system $\dot{w} = S(t)w$ with columns $c_j(t)$, i.e. $\Phi(t) = (c_1(t),\ldots,c_m(t))$. Form the matrices

$$\Phi_1(t) = (c_1(t),\ldots,c_k(t);0,\ldots,0);$$
$$\Phi_2(t) = (0,\ldots,0;c_{k+1}(t),\ldots,c_m(t)),$$

where $k \in \{1,\ldots,m\}$ is an arbitrary number. The following properties of these matrices are obvious:

a) $\Phi_1(t) + \Phi_2(t) = \Phi(t)$; b) $\dot{\Phi}_j(t) \equiv S(t)\Phi_j(t), j = 1,2$; c) $\dot{\Phi}_1(t) + \dot{\Phi}_2(t) = \dot{\Phi}(t)$.

Lemma 2 [27]. *Let the matrix* $S(t) \in C^{\infty}([0,T], \mathbb{C}^{m\times m}), h(t) \in C^{\infty}([0,T], \mathbb{C}^{m})$.*Then the general solution of the system* $\dot{w} = S(t)w + h(t)$ *on the segment* $[0,T]$ *can be written as*

$$w(t,c_0) = \Phi(t)c_0 + \Phi_1(t)\int\limits_0^t \Phi^{-1}(\theta)h(\theta)d\theta + \Phi_2(t)\int\limits_T^t \Phi^{-1}(\theta)h(\theta)d\theta, \tag{24}$$

where $c_0 \in C^m$ *is an arbitrary constant vector.*

Proof. Since $\Phi(t)c_0$ it is a general solution of the corresponding homogeneous system, it is necessary to show that the vector function

$$\tilde{w}(t) = \Phi_1(t)\int\limits_0^t \Phi^{-1}(\theta)h(\theta)d\theta + \Phi_2(t)\int\limits_T^t \Phi^{-1}(\theta)h(\theta)d\theta \tag{25}$$

is a particular solution to a non-homogeneous system of equations $\dot{w} = S(t)w + h(t)$. Using the properties a) – c) of the matrices $\Phi_j(t)$, we will have

$$\frac{d\tilde{w}}{dt} = \dot{\Phi}_1(t)\int\limits_0^t \Phi^{-1}(\theta)h(\theta)d\theta + \Phi_1(t)\Phi^{-1}(t)h(t) +$$

$$+\dot{\Phi}_2(t)\int\limits_T^t \Phi^{-1}(\theta)h(\theta)d\theta + \Phi_2(t)\Phi^{-1}(t)h(t) =$$

$$= S(t)\Phi_1(t)\int\limits_0^t \Phi^{-1}(\theta)h(\theta)d\theta + S(t)\Phi_2(t)\int\limits_T^t \Phi^{-1}(\theta)h(\theta)d\theta +$$

$$+[\Phi_1(t) + \Phi_2(t)]\Phi^{-1}h(t) = S(t)[\Phi_1(t)\int\limits_0^t \Phi^{-1}(t)(\theta)h(\theta)d\theta +$$

$$+\Phi_2(t)\int\limits_T^t \Phi^{-1}(\theta)h(\theta)d\theta] + h(t) = A(t)\tilde{w}(t) + h(t),$$

that is, function (25) is indeed a solution of an non-homogeneous system $\dot{w} = S(t)w + h(t)$. The lemma is proved.

Lemma 3 [27]. *Let the conditions of Lemma 2 be fulfilled. Then for the unique solvability of the boundary value problem (23) it is necessary and sufficient that*

$$\det[M_1\Phi(0) + N_1\Phi(T)] \neq 0. \tag{26}$$

If condition (26) is satisfied, then the solution of the problem (23) is given by formula (24), where the vector c_0*has the form*

$$\begin{gathered} c_0 = [M_1\Phi(0)c_0 + M_1\Phi(T)]^{-1}(w^0 - M_1\Phi_2(0)\int\limits_T^0 \Phi^{-1}(\theta)h(\theta)d\theta - \\ -N_1\Phi_1(T)\int\limits_0^T \Phi^{-1}(\theta)h(\theta)d\theta). \end{gathered} \tag{27}$$

Proof. Subject (24) to the boundary condition $lw = w^0$. We will have

$$M_1\Phi(0)c_0 + M_1\Phi_2(0)\int\limits_T^0 \Phi^{-1}(\theta)h(\theta)d\theta + N_1\Phi(T)c_0 +$$

$$+N_1\Phi_1(T)\int\limits_0^T \Phi^{-1}(\theta)h(\theta)d\theta = w^0;$$

or

$$[M_1\Phi(0)+M_2\Phi(T)]c_0 = w^0 - M_1\Phi_2(0)\int\limits_T^0 \Phi^{-1}(\theta)h(\theta)d\theta)-$$

$$-N_1\Phi_1(t)\int\limits_0^T \Phi^{-1}(\theta)h(\theta)d\theta.$$

For the unique solvability of this system, it is necessary and sufficient to satisfy condition (26). At the same time, c_0 has the form (27). The lemma is proved.

We now turn to the study of the solvability of the boundary value problem (23). According to [1; 109, 110], under the assumption that $S(t)\in C^1([0,T],\mathbb{C}^{2n\times 2n})$ transformation

$$z=[B(t)(I+\varepsilon B_1(t))]\eta(t,\varepsilon),$$

where $B^{-1}(t)T(t)B(t)=\Lambda(t)=\operatorname{diag}(\lambda_1(t),\dots,\lambda_{2n}(t))$;

$$(B_1(t))_{ij}=\begin{cases} 0, & i=j,\\ \frac{1}{\lambda_i(t)-\lambda_j(t)}(B^{-1}(t)B^{'}(t))_{ij}, & i\neq j \quad (i,j=\overline{1,2n}),\end{cases}$$

leads system (22) to the system

$$\begin{gathered}\varepsilon\tfrac{d\eta}{dt}=\Lambda_0(t,\varepsilon)\eta+\varepsilon^2C(t,\varepsilon)\eta+H_1(t,\varepsilon);\\ P_1(\varepsilon)\eta(0,\varepsilon)+P_2(\varepsilon)\eta(T,\varepsilon)=z^0,\end{gathered} \tag{28}$$

where indicated:

$$P_1(\varepsilon)=M[B(0)(I+\varepsilon B_1(0)],\quad P_2(\varepsilon)=N[B(T)(I+\varepsilon B_1(T)];$$
$$\Lambda_0(t,\varepsilon)=\operatorname{diag}(\lambda_1(t)+\varepsilon\mu_1(t),\dots,\lambda_{2n}(t)+\varepsilon\mu_{2n}(t));$$
$$H_1(t)=[B(t)(I+\varepsilon B_1(t)]^{-1}h(t)$$

and $C(t,\varepsilon)$ is a known matrix. Let $\Phi(t,\varepsilon)$ be the fundamental solution system of a shortened system

$$\varepsilon\frac{dz}{dt}=\Lambda_0(t,\varepsilon)z. \tag{29}$$

Obviously, it can be taken in the following form:

$$\Phi(t,\varepsilon)=\begin{pmatrix} e^{\frac{1}{\varepsilon}\int\limits_0^t\overline{\Lambda}(\theta,\varepsilon)d\theta} & 0\\ 0 & e^{\frac{1}{\varepsilon}\int\limits_T^t\overline{\overline{\Lambda}}(\theta,\varepsilon)d\theta}\end{pmatrix}, \tag{30}$$

where

$$\overline{\Lambda}(t)=\operatorname{diag}(\lambda_1(t)+\varepsilon\mu_1(t),\dots,\lambda_n(t)+\varepsilon\mu_n(t));$$
$$\overline{\overline{\Lambda}}(t)=\operatorname{diag}(\lambda_{n+1}(t)+\varepsilon\mu_{n+1}(t),\dots,\lambda_{2n}(t)+\varepsilon\mu_{2n}(t)).$$

Using (30) and Lemma 3, we reverse the system (28) and obtain an equivalent system of integral equations

$$\begin{gathered}\eta(t,\varepsilon)=\Phi(t,\varepsilon)c_0(\eta)+\\ +\Phi_1(t,\varepsilon)\int\limits_0^t\varepsilon\Phi^{-1}(s,\varepsilon)C(s,\varepsilon)\eta(s,\varepsilon)ds+\\ +\tfrac{\Phi_1(t,\varepsilon)}{\varepsilon}\int\limits_0^t\Phi^{-1}(s,\varepsilon)H_1(s,\varepsilon)ds+\\ +\Phi_2(t,\varepsilon)\int\limits_T^t\varepsilon\Phi^{-1}(s,\varepsilon)C(s,\varepsilon)\eta(s,\varepsilon)ds+\\ +\tfrac{\Phi_2(t,\varepsilon)}{\varepsilon}\int\limits_T^t\Phi^{-1}(s,\varepsilon)H_1(s,\varepsilon)ds,\end{gathered} \tag{31}$$

where

$$c_0(\eta) = [P_1(\varepsilon)\Phi(0,\varepsilon) + P_2(\varepsilon)\Phi(T,\varepsilon)]^{-1}\times$$
$$\times\left[z^0 - P_1(\varepsilon)\Phi_2(0,\varepsilon)\int_T^0 \varepsilon\Phi^{-1}(s,\varepsilon)C(s,\varepsilon)\eta(s,\varepsilon)ds-\right.$$
$$-P_2(\varepsilon)\Phi_1(T,\varepsilon)\int_0^T \varepsilon\Phi^{-1}(s,\varepsilon)C(s,\varepsilon)\eta(s,\varepsilon)ds-$$
$$-P_1(\varepsilon)\frac{\Phi_2(0,\varepsilon)}{\varepsilon}\int_T^0 \Phi^{-1}(s,\varepsilon)H_1(s,\varepsilon)ds- \tag{32}$$
$$\left.-P_2(\varepsilon)\frac{\Phi_1(T,\varepsilon)}{\varepsilon}\int_0^T \Phi^{-1}(s,\varepsilon)H_1(s,\varepsilon)ds\right]$$

and by $\Phi_j(t,\varepsilon)$ we denoted $(2n\times 2n)$-matrices

$$\Phi_1(t,\varepsilon) = \mathrm{diag}(e^{\frac{1}{\varepsilon}\int_0^t \overline{\Lambda}(\theta,\varepsilon)\theta},0); \qquad \Phi_2(t,\varepsilon) = \mathrm{diag}(0,\ e^{\frac{1}{\varepsilon}\int_T^t \overline{\overline{\Lambda}}(\theta,\varepsilon)\theta}).$$

Substituting (32) into (31), we write the integral system in operator form:

$$\eta = A\eta. \tag{33}$$

Obviously, the operator A acts from space $C\left([0,T],\mathbb{C}^{2n}\right)$ to itself. In assessing the norm $\|A\eta_1 - A\eta_2\|$, we use the fact that the product $\Phi_1(t,\varepsilon)\Phi^{-1}(s,\varepsilon)$ is uniformly bounded of ε (for sufficiently small $\varepsilon>0$) for s and t satisfying the inequalities $0\le s\le t\le T$, and the product $\Phi_2(t,\varepsilon)\Phi^{-1}(s,\varepsilon)$ is uniformly bounded (for sufficiently small $\varepsilon>0$) for s and t satisfying the inequalities $0\le t\le s\le T$. Let us show this. We have at $0\le s\le t\le T$:

$$||\Phi_1(t,\varepsilon)\Phi^{-1}(s,\varepsilon)|| \equiv$$
$$\equiv ||exp\{\tfrac{1}{\varepsilon}\int_s^t \overline{\Lambda}\,(\theta)\,d\theta\}|| = ||exp\{\tfrac{1}{\varepsilon}\int_s^t Re\overline{\Lambda}\,(\theta)\,d\theta\}|| \le$$
$$\le ||diag(exp\{\frac{1}{\varepsilon}\int_s^t Re\lambda_1 d\theta\},\ldots,exp\{\frac{1}{\varepsilon}\int_s^t Re\lambda_n d\theta\}||\times$$
$$\times||diag(exp\{\int_s^t Re\mu_1 d\theta\},\ldots,exp\{\int_s^t Re\mu_n d\theta\}|| \le \nu_1 = const;$$

because $Re\lambda_i(\theta)\le 0$ when $0\le s\le\theta\le t\le T, i=\overline{1,n}$. When $0\le t\le s\le T$ we have

$$||\Phi_2(t,\varepsilon)\Phi^{-1}(s,\varepsilon)|| \equiv$$
$$\equiv ||exp\{\tfrac{1}{\varepsilon}\int_s^t \overline{\overline{\Lambda}}d\theta\}|| = ||exp\{-\tfrac{1}{\varepsilon}\int_t^s Re\overline{\overline{\Lambda}}d\theta\}|| \le$$
$$\le ||diag(exp\{-\frac{1}{\varepsilon}\int_t^s Re\lambda_{n+1}d\theta\},\ldots,(exp\{-\frac{1}{\varepsilon}\int_t^s Re\lambda_{2n}d\theta\}||\times$$
$$\times||diag(exp\{\int_s^t Re\mu_{n+1}d\theta\},\ldots,exp\{\int_s^t Re\mu_{2n}d\theta\})|| \le$$
$$\le \nu_2 = const,$$

since $Re\lambda_j(\theta)\ge 0$ when $0\le t\le\theta\le s\le T, j=\overline{n+1,2n}.$ In this case, constants ν_1 and ν_2 does not depend on ε when $\varepsilon\in(0,\varepsilon_0]$, where $\varepsilon_0>0$ is sufficiently small.

We now turn to the estimate $||A\eta_1 - A\eta_2|| \equiv ||A\eta_1 - A\eta_2||_{C[0,T]}$. Using the boundedness of matrices $\Phi_1(t,\varepsilon) \times \Phi^{-1}(t,\varepsilon)$ and $\Phi_2(t,\varepsilon) \cdot \Phi^{-1}(t,\varepsilon)$, we will have

$$||A\eta_1 - A\eta_2|| \leq ||\Phi(t,\varepsilon)(c_0(\eta_1) - c_0(\eta_2))||+$$
$$+\varepsilon||\Phi_1(t,\varepsilon)\int_0^t \Phi^{-1}(s,\varepsilon)C(s,\varepsilon)(\eta_1(s,\varepsilon) - \eta_2(s,\varepsilon)ds||+$$
$$+\varepsilon||\Phi_2(t,\varepsilon)\int_T^t \Phi^{-1}(s,\varepsilon)C(s,\varepsilon)(\eta_1(s,\varepsilon) - \eta_2(s,\varepsilon)ds|| \leq$$
$$\leq \nu_0||c_0(\eta_1) - c_0(\eta_2)|| + \varepsilon\nu_3||\eta_1 - \eta_2|| + \varepsilon\nu_4||\eta_1 - \eta_2||.$$

On the other hand,

$$||c_0(\eta_1) - c_0(\eta_2)|| \leq ||(P_1(\varepsilon)\Phi(0,\varepsilon) + P_2(\varepsilon)\Phi(T,\varepsilon))^{-1}||\times$$
$$\times(||\varepsilon P_1(\varepsilon)\Phi_2(0,\varepsilon)\int_T^0 \Phi^{-1}(s,\varepsilon)C(s,\varepsilon)(\eta_1(s,\varepsilon) - \eta_2(s,\varepsilon)ds||+$$
$$+||\varepsilon P_2(\varepsilon)\Phi_1(T,\varepsilon)\int_0^T \Phi^{-1}(s,\varepsilon)C(s,\varepsilon)(\eta_1(s,\varepsilon) - \eta_2(s,\varepsilon)ds||) \leq$$
$$\leq \varepsilon\nu_5||\eta_1 - \eta_2||.$$

Substituting this into the previous inequality, we get $||A\eta_1 - A\eta_2|| \leq \varepsilon\nu_6||\eta_1 - \eta_2||$, where ν_6 do not depend on ε at $\varepsilon \in (0,\varepsilon_0]$. From here it follows that the operator A is contraction operator in space $C\left([0,T],\mathbb{C}^{2n}\right)$, and therefore, the equation (33) is uniquely solvable in $C\left([0,T],\mathbb{C}^{2n}\right)$.

From (31) and (32), passing to the norms in $C\left([0,T],\mathbb{C}^{2n}\right)$,we get

$$||\eta(t,\varepsilon)|| \leq \nu_0||c_0(\eta)|| + \varepsilon\nu_7||\eta|| + \frac{\nu_8}{\varepsilon}||H_1||;$$

$$||c_0(\eta)|| \leq \nu_9.||z^0|| + \varepsilon\nu_{10}||\eta|| + \frac{\nu_{11}}{\varepsilon}||H_1||,$$

which means, that

$$||\eta|| \leq \frac{1}{1-\varepsilon(\nu_7 + \nu_{10}\nu_0)}(\nu_0\nu_9||z^0|| + \frac{\nu_8 + \nu_0\nu_{11}}{\varepsilon}||H_1||).$$

Take $\varepsilon > 0$ such that

$$1 - \varepsilon(\nu_7 + \nu_{10}\nu_0) \geq \frac{1}{2}.$$

Then

$$||\eta|| \leq 2(\nu_0\nu_9||z^0|| + \frac{\nu_8 + \nu_0\nu_{11}}{\varepsilon}||H_1||).$$

Since $z = [B(t)I + \varepsilon B_1(t)]\eta(t,\varepsilon)$, then the initial boundary problem (22) is uniquely solvable in $C^1\left([0,T],\mathbb{C}^{2n}\right)$ and for its solution $z(t,\varepsilon)$ we have the estimate

$$||z(t,\varepsilon)||_{C[0,T]} \leq K_1||z^0|| + \frac{K_2}{\varepsilon}||h||_{C[0,T]}. \tag{34}$$

However, all our calculations are correct only when

$$|\det[P_1(\varepsilon)\Phi(0,\varepsilon) + P_2(\varepsilon)\Phi(T,\varepsilon)]| \geq \delta_1 = const > 0 \tag{35}$$

at $\varepsilon \in (0,\varepsilon_0]$. Insofar as $P_1(\varepsilon) = MB(0) + \varepsilon MB_1(0), P_2(\varepsilon) = NB(T) + \varepsilon NB_1(T)$, then inequality (35) for $\varepsilon \in (0,\varepsilon_0]$ follows from the fact that

$$|\det[MB(0)\Phi(0,\varepsilon) + NB(T)\Phi(T,\varepsilon)]| \geq \delta_2 = const > 0$$

for $\varepsilon \in (0,\varepsilon_0]$. This fact follows from the conditions $3^0a) - 3^0c)$, what can be proved in the same way as the inequality $|\Delta(\varepsilon)| \geq \delta_0 = const$ (see system (21) and the following calculations).

The following result is proved.

Theorem 3. Suppose that $S(t) \in C^1([0,T],\mathbb{C}^{2n\times 2n}), h(t) \in C([0,T],\mathbb{C}^{2n})$ *and the conditions* $1^0, 2^0\text{a}) - 2^0c), 3^0$ *are satisfied. Then for sufficiently small* $\varepsilon \in (0,\varepsilon_0]$ *the boundary value problem (22) has a unique solution* $z(t,\varepsilon)$ *and estimate (34) is valid for it.*

Using the estimate (34) for the remainder term $r_N(t,\varepsilon) = z(t,\varepsilon) - z_{\varepsilon N}(t)$, as well as Lemma 1, we can easily prove the following statement.

Theorem 4. Let the conditions $1^0, 2^0\text{a}) - 2^0c), 3^0$ *are fulfilled. Then the boundary value problem (9) for sufficiently small* $\varepsilon > 0 (0 < \varepsilon \le \varepsilon_0)$ *is uniquely solvable in the class* $C^1([0,T],\mathbb{C}^{2n})$ *and the estimate*

$$||z(t,\varepsilon) - z_{\varepsilon N}(t)||_{C[0,T]} \le C_N \varepsilon^{N+1}$$

holds true (here $C_N > 0$ *is a constant independent of* $\varepsilon \in (0,\varepsilon_0]$, $z_{\varepsilon N}(t) = S_N(t,\varphi\left(t,\frac{1}{\varepsilon}\right),\varepsilon)$ *the formal asymptotic solution constructed above.*

So, we have obtained the asymptotic solution of the problem (9) in the form of a series (12), taken at the restriction $\tau = \varphi(t,\frac{1}{\varepsilon})$. Using equations (8) and (10), we construct the asymptotics of the optimal control $u(t,\varepsilon)$ and the optimal trajectory $x(t,\varepsilon)$.

Remark 2. Due to the uniqueness of the solution of the system (13_0) in space U, it will have only a trivial solution $z_0 = z_0(t,\tau) \equiv 0$, therefore the asymptotic series for the optimal control (10) and the optimal trajectory $x(t,\varepsilon) = \frac{1}{\varepsilon} y(t,\varepsilon)$ will not contain coefficients with a negative degree of the parameter ε.

Remark 3. Condition 2^0d) is related to the construction of the Green function for the boundary value problem (9) and its application in estimating the remainder term (see [1; 108–126]). Using our technique, one can do without constructing the Green function and then condition 2^0d) can be removed.

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References

1 Ломов С.А. Введение в общую теорию сингулярных возмущений / С.А. Ломов. — М.: Наука, 1981.

2 Ройтенберг Я.Н. Автоматическое управление / Я.Н. Ройтенберг. — М.: Наука, 1971.

3 Болтянский И.Г. Достаточные условия оптимальности и обоснования метода динамического программирования / И.Г. Болтянский // Изв. АН СССР. Сер. Матем. — Т. 28, № 3. — 1964. — С. 481–514.

4 Глизер В.Я. Метод пограничного слоя при решении некоторых задач аналитического проектирования регулятора / В.Я. Глизер, В.Я. Глизер, М.Г. Дмитриев // Тез. докл. Всесоюз. конф. по асимп. методам. — Фрунзе: Илим, 1975. — С. 112–115.

5 Васильева А.Б. Об асимптотике решения типа контрастной структуры / А.Б. Васильева, В.Ф. Бутузов // Матем. заметки. — 1987. — Т. — 42. — №. 6. — С. 831–841.

6 Бутузов В.Ф. Асимптотическая теория контрастных структур / В.Ф. Бутузов, А.Б. Васильева, Н.Н. Нефедов // Автомат. и телемех. — 1977. — 7. — С. 3–32.

7 Васильева А.Б. Контрастные структуры в сингулярно возмущенных задачах / А.Б. Васильева, В.Ф. Бутузов, Н.Н. Нефедов // Фундамент. и приклад. матем. — 1998. — Т. 4(3). — С. 799–851.

8 Ломов С.А. Основы математической теории пограничного слоя / С.А. Ломов, И.С. Ломов. — М.: Изд-во МГУ, 2011.

9 Ломов С.А. Теория сингулярных возмущений в случае спектральных особенностей предельного оператора / С.А. Ломов, А.Г. Елисеев // Матем. сб. — 1986. — Т. 131 (173), № 4 (26). — С. 544–557.

10 Бободжанов А.А. Регуляризация оптимальных сингулярно возмущенных систем с быстро меняющейся демпфирующей функцией / А.А. Бободжанов, Н.А. Рашепкина, В.Ф. Сафонов // Вестн. МЭИ. — 1998. — № 6. — С. 18–26.

11 Иманбаев Н.С. Алгоритм метода регуляризации для сингулярно возмущенной задачи с нестабильным значением ядра интегрального оператора / Н.С. Иманбаев, Б.Т. Калимбетов, М.А. Темирбеков // Вестн. Караганд. ун-та. Сер. математика. — 2013. — № 4(72). — С. 64–70.

12 Калимбетов Б.Т. Математическое описание внутреннего пограничного слоя для нелинейной интегро-дифференциальной системы / Б.Т. Калимбетов, Б.И. Ескараева, М.А. Темирбеков // Вестн. Караганд. ун-та. Сер. математика. — 2014. — № 3(75). — С. 77–87.

13 Ескараева Б.И. Дискретный пограничный слой в случае нулевых точек спектра для систем интегро-дифференциальных уравнений / Б.И. Ескараева, Б.Т. Калимбетов, М.А. Темирбеков // Вестн. Караганд. ун-та. Сер. математика. — № 3(75). — 2014. — С. 88–95.

14 Kalimbetov B.T. Internal boundary layer for integral-differential equations with zero spectrum of the limit operator and rapidly changing kernel / B.T. Kalimbetov, B.I. Yeskarayeva, A.S. Tolep // Applied Mathematical Sciences. — 9. — 141–144. — 2015. — P. 7149–7165.

15 Kalimbetov B.T. Asymptotic solution of singular perturbed problems with an instable spectrum of the limiting operator / B.T. Kalimbetov, M.A. Temirbekov, Zh.O. Habibullaev // Abstract and Applied Analysis. — 2012. Article ID 120192.

16 Kalimbetov B.T. Regularized asymptotical solutions of integro-differential systems with spectral singularites / B.T. Kalimbetov, N.S. Imanbaev, D.A. Sapakov, L.T. Tashimov // Advances in Difference Equations. — 2013. — Vol. 109. — DOI: 10.1186/1687-1847-2013-109.

17 Safonov V.F. A regularization method for systems with instable spectral of the kernel of the integral operator / V.F. Safonov, B.T. Kalimbetov // Differential Equations. — 31. — 4. — 1995. — P. 647–656.

18 Safonov V.F. Regularized asymptotic solutions of the initial problem of systems of integro-partial differential equations / V.F. Safonov, A.A. Bobodzhanov // Mathematical Notes. — 102. — 1. — 2017. — P. 22–30.

19 Safonov V.F. Regularized asymptotics of solutions to integro-differential partial differential equations with rapidly varying kernels / V.F. Safonov, A.A. Bobodzhanov // Ufimsk. Mat. Zh. — 10. — 2. — 2018. — P. 3–12.

20 Bobodzhanov A.A. A generalization of the regularization method to the singularly perturbed integro-differential equations with partial derivatives / A.A. Bobodzhanov, V.F. Safonov // Russian Math. (Iz. vuz). — 62. — 3. — 2018. — P. 6–17.

21 Bobodzhanov A.A. A problem with inverse time for a singularly perturbed integro-differential equation with diagonal degeneration of the kernel of high order / A.A. Bobodzhanov, V.F. Safonov // Izv. Math. — 80. — 2. — 2016. — P. 285–298.

22 Bobodzhanov A.A. Asymptotic solutions of Fredholm integro-differential equations with rapidly changing kernels and irreversible limit operator / A.A. Bobodzhanov, V.F. Safonov // Russian Math. (Iz. VUZ). — 59. — 10. — 2015. — P. 1–15.

23 Bobodzhanov A.A. The method of normal forms for singularly perturbed systems of Fredholm integro-differential equations with rapidly varying kernels / A.A. Bobodzhanov, V.F. Safonov // Sb. Math. — 204. — 7. — 2013. — P. 979–1002.

24 Safonov V.F. Volterra integral equations with rapidly varying kernels and their asymptotic integration / V.F. Safonov, A.A. Bobodzhanov // Sb. Math. — 192. — 8. — 2001. — P. 1139–1164.

25 Safonov V.F. «Splashes» in Fredholm Integro-Differential Equations with Rapidly Varying Kernels / V.F. Safonov, A.A. Bobodzhanov // Math. Notes. — 85. — 2. — 2009. — P. 153–167.

26 Gantmakher F.R. Theory of matrices / F.R. Gantmakher. — Moscow: GITTL, 1953.

27 Konyaev Yu.A. On a method of studying some problems of the perturbation theory / Yu.A. Konyaev // Mathem. Coll. — 1993. — No. 12. — P. 133–144.

А.А. Бободжанов, Б.Т. Калимбетов, В.Ф. Сафонов

Оңтайлы жүйенің матрицасының спектрі ұдайы болған жағдайдағы басқарудың сингуляр ауытқымалы есебі

Мақалада сапаның квадрат функционалы бар сингуляр ауытқымалы есебі қарастырылды. Мұндай есептер бұрын спектрдің белгілі шектеулерінде (оңтайлы жүйенің спектр нүктелері таза жорамал емес және жорамал оське қатысты симметриялы орналасса) Васильева-Бутузовтың шекаралық функциялар әдісі арқылы қарастырылған. Егер спектрдің нүктелерінің ең болмағанда біреуі тәуелсіз айнымалының кейбір мәндерінде жорамал оське түссе, шекаралық функциялар әдісі жұмыс істемейді.

Жұмыста спектрдің таза жорамал нүктелері бар жағдай зерттелді. Бұл жағдайда С.А. Ломовтың регуляризациялау әдісінің идеяларына негізделген басқа әдісті дамытуға тура келеді. Сондай-ақ бұрын қаралған басқару есептерінде шығындар функционалы кіші параметрге мүлдем тәуелді емес немесе кіші параметрге тегіс тәуелді болатыны байқалған. Бұл мақалада шығынның кіші параметрге регуляр емес тәуелділігі, яғни интеграл таңбасы астында экспоненциалды көбейткіш түрінде жылдам өзгеретін демпфирлеу функциясының бар болуы жағдайы қарастырылған. Осы жағдайда оңтайлы жүйе спектрінің беталысы демпфирлеу коэффициентіне байланысты болады, ол (белгілі бір жағдайларда) спектрді комплекс жазықтықтың қандай да бір жағына жылжытуы мүмкін. Онда спектрдің кейбір нүктелері жекеленген мәндерде (немесе тіпті кейбір континуалды жиындарда) тәуелсіз айнымалы таза жорамал болған жағдай туындауы мүмкін. Мұндай жағдайды жоғарыда аталған Васильева-Бутузовтың шекаралық функциялары әдісімен зерттеуге болмайды. Алайда қарастырылып отырған басқару есебін зерттеуге қатысты С.А. Ломовтың регуляризациялау әдісінің алгоритмін қолдану осы жұмыста толық қарастырылған. Бұл әдісті баяндау оңтайлы басқарудың классикалық есебі үшін Л.С. Понтрягиннің максимум принципін қысқаша сипаттаудан басталып, одан соң басқа идеялармен қатар, қарастырылып отырған басқару есебінің нәтижелерін негіздеу үшін қолданылды.

Кілт сөздер: сингуляр ауытқу, Понтрягинның максимум принципі, регуляризация, асимптотикалық жинақтылық.

А.А. Бободжанов, Б.Т. Калимбетов, В.Ф. Сафонов

Сингулярно возмущенные задачи управления в случае стабильности спектра матрицы оптимальной системы

В статье рассмотрена сингулярно возмущенная задача управления с квадратичным функционалом качества. Такие задачи в их стандартной постановке при известных ограничениях на спектр (точки спектра оптимальной системы не являются чисто мнимыми и расположены симметрично относительно мнимой оси) были рассмотрены ранее с помощью метода пограничных функций Васильевой-Бутузова. Если же хотя бы одна из точек спектра при некоторых значениях независимой переменной попадает на мнимую ось, метод погранфункций не работает. Именно такая ситуация с допущением чисто мнимых точек спектра исследована в настоящей работе. В этом случае приходится развивать другой подход, основанный на идеях метода регуляризации С.А. Ломова. Следует заметить также, что в рассмотренных ранее задачах управления функционал затрат либо вообще не зависел от малого параметра, либо допускал гладкую зависимость от параметра. В данной работе допущена нерегулярная зависимость от малого параметра, в частности, наличие в них быстро изменяющейся функции демпфирования в виде экспоненциально множителя под знаком интеграла. В этом случае поведение спектра оптимальной системы зависит от коэффициента демпфирования, который (при определенных условиях) может смещать спектр в ту или иную сторону в комплексной плоскости. При этом может возникнуть ситуация, когда некоторые точки спектра при отдельных значениях (или даже на некотором континуальном множестве) независимой переменной могут становиться чисто мнимыми. Эта ситуация не поддается исследованию упомянутым ранее методом пограничных функций Васильевой-Бутузова. Однако его можно полностью изучить с помощью метода регуляризации С.А. Ломова, алгоритм которого применительно к рассматриваемой задаче управления развивается в настоящей работе. Изложение этого метода начинается с краткого описания принципа максимума Л.С. Понтрягина для классической задачи оптимального управления, который затем, наряду с другими идеями, применяется для обоснования результатов в рассматриваемой задаче управления.

Ключевые слова: сингулярное возмущение, принцип максимума Понтрягина, регуляризация, асимптотическая сходимость.

References

1 Lomov, S.A. (1981). *Vvedenie v obshchuiu teoriiu sinhuliarnykh vozmushchenii [Introduction to the general theory of singular perturbations]*. Moscow: Nauka [in Russian].

2 Reutenberg, J.N. (1971). *Avtomaticheskoe upravlenie [Automatic control]*. Moscow: Nauka [in Russian].

3 Boltyansky, I.G. (1964). Dostatochnye usloviia optimalnosti i obosnovaniia metoda dinamicheskoho prohrammirovaniia [Sufficient conditions for optimality and justification of the dynamic programming method]. *Izvestiia Akademii nauk SSSR. Seriia Matematika – News of Academy of Sciences of the USSR, Mathematical series, 28, 3,* 481–514 [in Russian].

4 Glizer, V.Ya., & Dmitriev, M.G. (1975). Metod pohranichnoho sloia pri reshenii nekotoryikh zadach analiticheskoho proektirovaniia rehulyatora [The boundary layer method in solving some problems of analytical design of the regulator]. *Tezisy dokladov Vsesoiuznoi konferentsii po asimptoticheskim metodam – Abstracts of the All-Union conf. by asymp. methods, Frunze: Ilim,* 112–115 [in Russian].

5 Vasilyeva, A.B., & Butuzov, V.F. (1987). Ob asimptotike resheniia tipa kontrastnoi struktury [On the asymptotics of a solution of the type of contrast structure]. *Mathematicheskie zametki – Mathem. notes, 42, 6,* 831–841 [in Russian].

6 Butuzov, V.F., Vasilyeva, A.B., & Nefedov, N.N. (1977). Asimptoticheskaia teoriia kontrastnykh struktur [Asymptotic theory of contrast structures]. *Avtomatika i Telemekhanika – Avtomat. and Telemech., 7,* 3–32 [in Russian].

7 Vasilyeva, A.B., Butuzov, V.F., & Nefedov, N.N. (1998). Kontrastnye struktury v sinhuliarno vozmushchennykh zadachakh [Contrast structures in singularly perturbed problems]. *Fundamentalnaiia i prikladaiia mathematika – Fundament. and Appl. Math., 4(3),* 799–851 [in Russian].

8 Lomov, S.A., & Lomov, I.S. (2011). *Osnovy matematicheskoi teorii pohranichnoho sloia [Foundations of mathemathical theory of boundary layer].* Moscow: Izdatelstvo Moskovskoho hosudarstvennoho universiteta [in Russian].

9 Lomov, S.A., & Eliseev, A.G. (1986). Teoriia sinhuliarnykh vozmushchenii v sluchae spektralnykh osobennostei predelnoho operatora [The theory of singular perturbations in the case of spectral singularities of the limit operator]. *Mathematicheskii sbornik – Math. collect., 131(173), 4(26), 544-557* [in Russian].

10 Bobodzhanov, A.A., Rashepkina, N.A., & Safonov, V.F. (1998). Rehuliarizatsiia optimalnykh sinhuliarno vozmushchennykh sistem s bystro meniaiushcheisia dempfiruiushchei funktsiiei [Regularization of optimal singularly perturbed systems with a rapidly changing damping function]. *Vestnik MEI – Vestnik MPEI, 6, 18-26* [in Russian].

11 Imanbaev, N.S., Kalimbetov, B.T., & Temirbekov, M.A. (2013). Alhoritm metoda rehuliarizatsii dlia sinhuliarno vozmushchennoi zadachi s nestabilnym znacheniem yadra intehralnoho operatora [Algorithm the regularization method for a singularly perturbed problem with unstable value of kernel of the integral operator]. *Vestnik Karahandinskoho universiteta. Matematicheskaia seriia – Vestnik of Karaganda University, Mathematics series, 4(72)* , 64–70 [in Russian].

12 Kalimbetov, B.T., Yeskaraeva, B.I., & Temirbekov, M.A. (2014). Matematicheskoe opisanie vnutrenneho pohranichnoho sloia dlia nelineinoi intehro-differentsialnoi sistemy [Mathematical description of the inner boundary layer for a nonlinear integro-differential system]. *Vestnik Karahandinskoho universiteta. Seriia matematika – Vestnik of Karaganda University. Mathematics series, 3(75),* 77–87 [in Russian].

13 Yeskaraeva, B.I., Kalimbetov, B.T., & Temirbekov, M.A. (2014). Diskretnyi pohranichnyi sloi v sluchae nulevykh tochek spektra dlia sistem intehro-differentsialnykh uravnenii [Discrete boundary layer in the case of zero points of the spectrum for systems of integro-differential equations]. *Vestnik Karahandinskoho universiteta. Seriia matematika – Vestnik of Karaganda University, Mathematics series, 75, 3, 88–95* [in Russian].

14 Kalimbetov, B.T., Yeskarayeva, B.I., & Tolep A.S. (2015). Internal boundary layer for integral-differential equations with zero spectrum of the limit operator and rapidly changing kernel. *Journal Applied Mathematical Sciences, 9, 141-144, 7149-7165.*

15 Kalimbetov, B.T., Temirbekov, M.A., & Habibullaev, Zh.O. (2012). Asymptotic solution of singular perturbed problems with an instable spectrum of the limiting operator. *Journal Abstract and Applied Analysis, ID 120192.*

16 Kalimbetov, B.T., Imanbaev, N.S., Sapakov, D.A., & Tashimov, L.T. (2013). Regularized asymptotical solutions of integro-differential systems with spectral singularites. *Journal Advances in Difference Equations, 109, doi: 10.1186/1687-1847-2013-109.*

17 Safonov, V.F., & Kalimbetov, B.T. (1995). A regularization method for systems with unstable spectral value of the kernel of the integral operator. *Journal Differential Equations, 31(4), 647-656.*

18 Safonov, V.F., & Bobodzhanov, A.A. (2017). Regularized asymptotic solutions of the initial problem of systems of integro-partial differential equations. *Mathematical Notes, 102, 1, 22-30.*

19 Safonov, V.F., & Bobodzhanov, A.A. (2018). Regularized asymptotics of solutions to integro-differential partial differential equations with rapidly varying kernels. *Ufimsk. Mat. Zh., 10, 2. 3-12.*

20 Bobodzhanov, A.A., & Safonov, V.F. (2018). A generalization of the regularization method to the singularly perturbed integro-differential equations with partial derivatives. *Journal Russian Math. (Izvest. VUZ), 62, 3, 6–17.*

21 Bobodzhanov, A.A., & Safonov, V.F. (2016). A problem with inverse time for a singularly perturbed integro-differential equation with diagonal - of the kernel of high order. *Journal Izvest. Math, 80, 2, 285–298.*

22 Bobodzhanov, A.A., & Safonov, V.F. (2015). Asymptotic solutions of Fredholm integro-differential equations with rapidly changing kernels and irreversible limit operator. *Journal Russian Math. (Izvest. vuz), 59, 10, 1–15.*

23 Bobodzhanov, A.A., & Safonov, V.F. (2013). The method of normal forms for singularly perturbed systems of Fredholm integro-differential equations with rapidly varying kernels. *Sibir. Math J., 204, 7, 979–1002.*

24 Safonov, V.F., & Bobodzhanov, A.A. (2001). Volterra integral equations with rapidly varying kernels and their asymptotic integration. *Sibir. Math. J., 192, 8, 1139–1164.*

25 Safonov, V.F., & Bobodzhanov, A.A. (2009). «Splashes» in Fredholm integro-differential equations with rapidly varying kernels. *Math. Notes, 85, 2. 153–167.*

26 Gantmakher, F.R. (1953). *Teoriia matrits [Theory of matrices].* Moscow: HITTL [in Russian].

27 Konyaev, Yu.A. (1993). On a method of studying some problems of the perturbation theory. *Mathematicheskii sbornik – Mathem. collect., 12,* 133–144 [in Russian].

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Modeling of a Static Incompressible Medium

In the article mathematical modeling of the solution of the static Stokes problem is considered by solving the static problem of the theory of elasticity, with Hooke's law. We obtain estimates for the proximity of solutions of these problems under asymptotics, namely, when $\lambda \to \infty$, in this case the Poisson coefficient tends to $0,5$, which corresponds to an incompressible medium. Further, it was proved that the estimate of the proximity of the solutions of the problems considered as unimprovable in order of $\frac{1}{\lambda}$. This allows us to use previously differential schemes for the static problem of the theory of elasticity, as an approximate, it is obtained a method for solving the static Stokes problem on a sequence of grids.

Keywords: Incompressible medium, Hooke's law, stresses, deformations, displacements, Lame coefficients, the task of the Stokes, theory of elasticity, equilibrium equation, single-linked area.

In a bounded simply connected domain $D \subset R^3$ with a sufficiently smooth boundary γ, we will make a solution to the problem of the theory of elasticity for an incompressible medium that satisfies the equilibrium equation:

$$\mu \triangle \overline{u} - \nabla p + \overline{f} = 0, \ \ x \in D, \tag{1}$$

medium incompressibility condition

$$\operatorname{div} \overline{u} = 0, x \in D, \tag{2}$$

displacement-strain relations

$$2\varepsilon_{ik} = \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right), \ \ i,k = 1,2,3, \tag{3}$$

equations of state of the environment

$$\sigma_{ik} = -\delta_{ik} p + 2\mu \varepsilon_{ik}, \tag{4}$$

where $u(x)$ is the displacement vector, $p(x)$ is the pressure function, $f(x)$ is the vector of volume forces, $\varepsilon_{ik}(x)$ is the components of the strain tensor, $\sigma_{ik}(x)$ are the components of the stress tensor and the boundary condition

$$\sum_{k=1}^{3} \sigma_{ik} n_k = 0, \ \ x \in \gamma, \tag{5}$$

where n_k is guides of cosines by norms to the boundary γ.

We also consider in the domain D the formulation of the static problem of the theory of elasticity for a compressible medium and use following equations.

Equilibrium equations

$$\mu \triangle \overline{u}^{\lambda} + (\lambda + \mu) \nabla \operatorname{div} \overline{u}^{\lambda} + \overline{f} = 0, x \in D, \tag{6}$$

displacement-strain relations

$$2\varepsilon_{ik}^{\lambda} = \left(\frac{\partial u_i^{\lambda}}{\partial x_k} + \frac{\partial u_k^{\lambda}}{\partial x_i} \right), \ \ i,k = 1,2,3, \tag{7}$$

equations of state, Hooke's law

$$\sigma_{ik} = \lambda \delta_{ik} \theta^{\lambda} + 2\mu \varepsilon_{ik}^{\lambda}, \tag{8}$$

In the formula (8) $\theta^{\lambda} = \sum\limits_{j=1}^{3} \varepsilon_{jj}^{\lambda}$, λ, $\mu > 0$ is Lame coefficients.

Border conditions

$$\sum_{k=1}^{3} \sigma_{ik}^{\lambda} n_k = 0, \ \ x \in \gamma, \tag{9}$$

for the solvability of problems (6) - (9) the following conditions are necessary [1,2]

$$\int_D \overline{f} dx = 0, \ \int_D \overline{r} \times \overline{f} dx = 0, \ \int_D \overline{u} dx = 0, \ \int_D \operatorname{rot} \overline{u} dx = 0, \tag{10}$$

where $\overline{r} = r(x)$ is the radius vector of the point $x \in D$.

Earlier in the work, it was obtained an estimate for the proximity of the solution of problem (6)–(10) to the solution of problem (1)–(5) as $\lambda \to \infty$. Here we get this assessment in another way.

Let us denote $\overline{u} = \overline{u}_0 + \frac{1}{\lambda}\overline{\mathrm{w}}$ where $\overline{u}_0$ is the solution of problem (1)–(5). By virtue of (6)–(10) and (1)–(5) we have:

$$E(\overline{u}_0, \overline{\mathrm{v}}) + \frac{1}{\lambda} E(\overline{\mathrm{w}}, \overline{\mathrm{v}}) + \int_D \operatorname{div} \overline{u}_0 \cdot \operatorname{div} \overline{\mathrm{v}} dx + \int_D \operatorname{div} \overline{\mathrm{w}} \cdot \operatorname{div} \overline{\mathrm{v}} dx = \int_D \overline{f} \cdot \overline{\mathrm{v}} dx, \tag{11}$$

for an arbitrary function $\overline{\mathrm{v}} \in W_2^1(D)$

$$E(\overline{u}, \overline{\mathrm{v}}) = \frac{1}{2}\mu \int_D \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial \mathrm{v}_i}{\partial x_j} + \frac{\partial \mathrm{v}_j}{\partial x_i}\right) dx.$$

In (11) we add and subtract the expressions $\int_D p \operatorname{div} \overline{\mathrm{v}} dx$, as a result we get

$$\frac{1}{\lambda} E(\overline{\mathrm{w}}, \overline{\mathrm{v}}) + \int_D \operatorname{div} \overline{\mathrm{w}} \cdot \operatorname{div} \overline{\mathrm{v}} dx = -\int_D p \operatorname{div} \overline{\mathrm{v}} dx. \tag{12}$$

Assuming in (12) $\overline{\mathrm{v}} = \overline{\mathrm{w}}$, the estimate follows

$$\frac{1}{\lambda} E(\overline{\mathrm{w}}, \overline{\mathrm{w}}) + \| \operatorname{div} \overline{\mathrm{w}}\|^2 \le \|p\| \cdot \| \operatorname{div} \overline{\mathrm{w}}\| \le \frac{1}{2}\|p\|^2_{L_2(D)} + \frac{1}{2}\| \operatorname{div} \overline{\mathrm{w}}\|^2.$$

By virtue of the fact that

$$\|p\|_{L_2(D)} \le \|f\|_{W_2^{-1}(D)}.$$

then inequality holds true

$$\frac{1}{\lambda}\|\overline{\mathrm{w}}\|^2_{W_2^1(D)} + \frac{1}{2}\| \operatorname{div} \overline{\mathrm{w}}\|^2 \le \frac{1}{2}\|f\|^2_{W_2^{-1}(D)}. \tag{13}$$

After that we will finally get the next formula:

$$\lambda \| \operatorname{div} \overline{u}\| \le c\overline{f}_{W_2^{-1}(D)} \tag{14}$$

Next, we estimate the higher derivatives of the solution of problem (6)–(10).

Theorem 1. Let $f(x) \in L_2(D)$, $\gamma \in C^2$. Then, to solve the problem (6)–(10), the inequality is

$$\|u\|_{W_2^2(D)} + \frac{1}{\varepsilon}\|\nabla \operatorname{div} \overline{u}\|_{L_2(D)} \le c\|f\|_{L_2(D)} \tag{15}$$

Evidence. If we follow to the work [3], from the smoothness condition for the boundary γ, it implies that there is a local change of variables $y = \psi(x)$, where $x \in \gamma$, let ω be a neighborhood of the point x such that the piece of the boundary $\omega \bigcap \gamma$ becomes a piece of the hyperplane. Let denote through ω_1 the small neighborhood of this point, which is at a positive distance from the boundary to the neighborhood ω. Let $z(x)$ be a non-negative in $\overline{\omega} = \omega \bigcup \partial\omega$ a smooth function such that $\operatorname{supp} z \subset \omega$ and $z(x) = 1$ if $x \in \omega$. Finally, let us assume that the y_k axis, $k = 1, 2, \ldots, n-1$ lies in the indicated hyperplane, and the y_n axis is directed along the norms drawn to it. Solutions of problem (6) - (10) satisfy a neutral identity:

$$E(\overline{\mathrm{u}}, \overline{\mathrm{v}}) + \lambda \int_D \operatorname{div} \overline{u} \cdot \operatorname{div} \overline{\mathrm{v}} dx = \int_D \overline{f} \cdot \overline{\mathrm{v}} dx \tag{16}$$

for any $\overline{\mathrm{v}}(x) \in W_2^1(D)$. Put in (16) $\mathrm{v} = z(x)\frac{\partial}{\partial y_k}(z(x)u_{y_k})$, after integration by parts, evaluating the terms in the same way as in [4] we will have $\|u_{y_k}\|^2_{W_2^1(w)} \le \|u\|_{W_2^2(D)} \cdot \|u\|_{W_2^1(\omega)}$, by virtue of (6)–(10), multiplying by $z(x)u_{y_k}$ integrating we get

$$\lambda \left\| \frac{\partial}{\partial y_k} \nabla \operatorname{div} \overline{u} \right\|_{\overset{0}{W}{}_2^{-1}(D)} \le M \left[\|f\| + \|u\|^{1/2}_{W_2^2(D)} \cdot \|f\|^{1/2}_{L_2(D)} \right]$$

to estimate $\frac{\partial}{\partial y_k} \operatorname{div} \overline{u}$, we use the identity $\nabla \operatorname{div} \overline{u} = \operatorname{div}\left(\frac{\overline{f}}{\lambda + 2\mu}\right)$, which is obtained from (6), if we take the div by going to the variables , we express $\frac{\partial}{\partial y_k} \operatorname{div} \overline{u}$ through the remaining derivatives of the first and second order, which are already estimated. Then we get

$$\left\| \nabla \frac{\partial}{\partial y_k} \operatorname{div} \overline{u} \right\|_{\overset{0}{W}{}_2^{-1}(\omega)} \le c \left[\|f\| + \|u\|^{1/2}_{W_2^2(D)} \cdot \|f\|^{1/2}_{L_2(D)} \right].$$

Let us suppose that the set ω is a finite cover $\overline{D}$, then in the last inequality on the left we can take the norm over the whole domain D. So we have

$$\|u\|_{W_2^2(D)} \le c \left(\|f\|_{L_2(D)} + \|f\|^{1/2}_{L_2(D)} \cdot \|u\|^{1/2}_{W_2^2(D)} \right).$$

The theorem is proved. Similarly we prove

Theorem 2. Let $f \in L_2(D)$, a domain D is bounded, simply connected, with a sufficiently smooth boundary γ, then the estimate is valid.

$$\|\overline{u}^\lambda - \overline{u}_0\|_{W_2^2(D)} + \|\lambda \operatorname{div} \overline{u}^\lambda - p\|_{W_2^1(D)} \le c \cdot \lambda^{-1}. \tag{17}$$

In the work [5], it was obtained an estimate for the proximity estimate for the solutions of problem (6)–(10) to the solution of problem (1)–(5).

$$\|\overline{u}^\lambda - \overline{u}\|^2_{W_2^1(D)} + \|\lambda \operatorname{div} \overline{u}^\lambda - p\|^2_{L_2(D)} \le c \cdot \lambda^{-2}. \tag{18}$$

Let us show that this estimate is best possible in the order of the parameter $1/\lambda$.

Theorem 3. An estimate of the proximity of the solutions of problems (6)–(10) and (1)–(5) is best possible for λ.

Evidence. Suppose the contrary, i.e.

$$\|\overline{u}^\lambda - \overline{u}\|^2_{W_2^1(D)} + \|\lambda \operatorname{div} \overline{u}^\lambda - p\|^2_{L_2(D)} \le c \cdot \lambda^{-(2+\alpha)}.$$

where $\alpha > 0$ is a constant, perhaps small enough, therefore we have:

$$\|\lambda \operatorname{div} \overline{u}^\lambda - p\| \ge \|p\| - \lambda \| \operatorname{div} \overline{u}^\lambda \|$$

$$\|p\| \le \lambda \| \operatorname{div} \overline{u}^\lambda \| + \|\lambda \operatorname{div} \overline{u}^\lambda - p\| \le c \left(\|\pi\| + \frac{\lambda}{\lambda^{1+\alpha/2}} \right) \tag{19}$$

Here $\pi = p - \lambda \operatorname{div} \overline{u}^\lambda$. In inequality (19), we wil pass to the limit as $\lambda \to \infty$; we can obtain it by virtue of (18). $\|p\|_{L_2(D)} = 0$, i.e. $p = 0$. Thus, for $\overline{u}$ and p we get the following problem

$$\mu \triangle \overline{u} - \nabla p + \overline{f} = 0,\ x \in D,\ \operatorname{div} \overline{u} = 0,\ \ p = 0;$$

$$\sum_{k=1}^{3} \sigma_{ik} n_k = 0, \ \ x \in \gamma. \tag{20}$$

As we see this contradicts our assumption, since the latter problem (21) is unsolvable (the initial problem (1)–(5)) is correct and it was required.

References

1 Кобельков Г.М. Численное решение некоторых задач теории упругости: дис. ... канд. физ.-мат. наук: 01.01.07 – «Вычислительная математика» / Г.М. Кобельков. — М.: Изд-во МГУ, 1975. — 104 с.

2 Фикера Г. Теоремы существования в теории упругости / Г. Фикера. — М.: Мир, 1974. — 159 с.

3 Ладыженская О.А. О некоторых задачах векторного анализа и обобщенных постановках краевых задач для уравнения Навье-Стокса / О.А. Ладыженская, В.А. Солонников // Записки научного семинара ЛОМИ, 1976. — Т. 59. — С. 81–116.

4 Ладыженская О.А. Линейные и квазилинейные уравнения эллиптического типа / О.А. Ладыженская, Н.Н. Уральцева. — М.: Наука, 1973. — С. 120.

5 Степаненко В.М. Аппроксимация решения задачи теории упругости для несжимаемого материала / В.М. Степаненко // Сборник численных методов механики сплошной среды. — 1978. — Т. 9, № 6. — С. 136–145.

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Статикалық қысылмайтын ортаны модельдеу

Мақалада Гук заңымен серпімділік теориясының статикалық есебін шешу арқылы Стокстың статикалық есебінің шешімін математикалық моделдеу қарастырылған. Асимптотика кезінде осы міндеттерді шешу жақындығының бағасы алынды, атап айтқанда $\lambda \to \infty$, бұл жағдайда Пуассон коэффициенті $0,5$-ке ұмтылады, бұл қысылмайтын ортаға сәйкес келеді. Одан әрі жұмыста қарастырылған есептер шешімдерінің жақындығын бағалау $1/\lambda$ ретімен жақсартылмайтын болып табылатыны дәлелденді. Бұл серпімділік теориясының статикалық есебі үшін бұрын айырымдық схемаларды қолдануға мүмкіндік береді, жақындатылған тордың жүйелілігіне Стокстың статикалық есебін шешу әдісі алынды.

Кілт сөздер: қысылмайтын орта, Гук заңы, кернеу, деформация, орын ауыстыру, Ламе коэффициенттері, Стокс есебі, серпімділік теориясы, тепе-теңдік теңдеуі, бір байланысты облыс.

М.М. Букенов, Д.К. Койкелова

Моделирование статической несжимаемой среды

В статье рассмотрено математическое моделирование решения статической задачи Стокса с помощью решения статической задачи теории упругости с законом Гука. Получены оценки близости решений этих задач при асимптотике, а именно когда $\lambda \to \infty$, в этом случае коэффициент Пуассона стремится к $0,5$, что соответствует несжимаемой среде. Далее в работе доказано, что оценка близости решений рассмотренных задач является неулучшаемой по порядку $1/\lambda$. Это позволяет использовать ранее неисследованные разностные схемы для статической задачи теории упругости, как приближенный получен метод решения статической задачи Стокса на последовательности сеток.

Ключевые слова: несжимаемая среда, закон Гука, напряжения, деформации, перемещения, коэффициенты Ламе, задача Стокса, теория упругости, уравнение равновесия, односвязная область.

References

1 Kobelkov, G.M. (1975). Chislennoe reshenie nekotorykh zadach teorii upruhosti [Numerical solution of some problems of the theory of elasticity]. *Candidate's thesis.* Moscow: Izdatelstvo MHU [in Russian].

2 Fickera, G. (1974). *Teoremy sushchestvovaniia v teorii upruhosti* [*The existence theorems in the theory of elasticity*]. Moscow: Mir [in Russian].

3 Ladyzhenskaya, O.A, & Solonnikov, V.A. (1976). O nekotorykh zadachakh vektornoho analiza i obobshchennykh postanovkakh kraevykh zadach dlia uravneniia Nave-Stoksa [On some problems of vector analysis and generalized formulations of boundary value problems for the Navier-Stokes equation]. *Zapiski nauchnoho seminara LOMI – Notes of the LOMI scientific seminar, Vol. 59,* 81–116 [in Russian].

4 Ladyzhenskaya, O.A., & Uraltseva, N.N. (1973). *Lineinye kvazilineinye uravneniia ellipticheskoho tipa* [*Linear and quasilinear equations of elliptic type*]. Moscow: Nauka [in Russian].

5 Stepanenko, V.M. (1978). Approksimatsiia resheniia zadachi teorii upruhosti dlia neszhimaiemoho materiala [Approximation of the solution of the problem of the theory of elasticity for incompressible material]. *Sbornik chislennykh metodov mekhaniki sploshnoi sredy – Collection of Numerical Methods of continuum mechanics, Vol. 9, 6,* 136–145 [in Russian].

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On eigenvalues of third order composite type equations with regular boundary value conditions

In the paper the question about distribution of eigenvalues of third-order composite type equations with regular, more precisely, with periodic boundary value conditions is studied. After, applying the Fourier method, the original problem splits into two problems on eigenvalues of third-order ordinary differential operators with periodic boundary value conditions in $L_2(0,1)$. Characteristic determinants are calculated and zeros of entire analytic functions are found, and their location on the complex plane is determined. Existence of an infinite number of eigenvalues of a third order composite type operator is proved. Distance between the neighboring eigenvalues of the third order composite type operator of each series, which lie on rays, perpendicular to sides of a conjugate indicator diagram, that is, a regular hexagon on the complex plane, is determined. Moreover, it is determined that zero is not an eigenvalue of a third order composite type operator, in other words, zero is a regular point of the operator that belongs to resolvent set of the original operator. Adjoint operator with periodic boundary value conditions is constructed.

Keywords: composite type equations, regular, periodic boundary value conditions, rectangular domain, Fourier method, characteristic determinant, entire analytic functions, eigenvalue, zeros of entire functions.

Introduction and Formulation of the problem

Series of spectral boundary value problems for composite type equations

$$\frac{\partial}{\partial x}\left(u_{xx}+u_{yy}\right)+\lambda u=0$$

has been investigated in [1, 2]. Solution of initial-boundary value problems for partial differential equations by the Fourier method is almost always reduced to the problem on determining eigenvalues and eigen-functions of some differential operators [3–6].

This paper is devoted to finding the eigenvalues of one boundary value problem for the equation

$$Lu \equiv u_{xxx}+u_{yyy}+\lambda u=0, \tag{1}$$

where λ is a spectral parameter and complex number, which is also composite type equation [1].

In a rectangular domain D we consider the problem on eigenvalues of the equation (1), satisfying the following boundary conditions:

$$u|_{\partial D}=0, \quad u_x\left(0,y\right)=u_x\left(1,y\right), \quad u_y\left(x,0\right)=u_y\left(x,1\right), \tag{2}$$

where $D=\{x,y: \ \ 0<x<1, \ \ 0<y<1\}$.

Solution of the problem

Looking for a solution of the problem (1), (2) by the Fourier method as follows:

$$u\left(x,y\right)=X\left(x\right)\cdot Y\left(y\right),$$

we come to the following spectral problems in the space $L_2\left(0,1\right)$ for ordinary differential operators

$$L_0X \equiv X'''+\mu X=0, \quad X\left(0\right)=X\left(1\right)=0, \quad X'\left(0\right)=X'\left(1\right), \tag{3}$$

$$L_1Y \equiv Y''' + \nu Y = 0, \quad Y(0) = Y(1) = 0, \quad Y'(0) = Y'(1), \tag{4}$$

moreover, $\lambda = \mu + \nu$. Boundary value problems in (3),(4) are regular by G.D. Birkhoff [7]. In the monograph of M.A. Naimark [8; 67] a subclass of regular boundary conditions is distinguished, and it is noted that for an odd order of the equation, all strongly regular conditions are regular.

We solve the problem (3) (problem (4) is solved analogically). General solution of the equation (3) has the form

$$X(x) = C_1 e^{2ax} + \left(C_2 \cdot \cos\sqrt{3}ax + C_3 \cdot \sin\sqrt{3}ax\right) \cdot e^{-ax}, \tag{5}$$

where $C_1,\ C_2,\ C_3$ are arbitrary constants,

$$a = \frac{\sqrt[3]{-\mu}}{2} \neq 0. \tag{6}$$

Putting (5) into the boundary value condition (3), we will have the linear system concerning to the coefficients C_j:

$$\begin{cases} C_1 + C_2 = 0, \\ C_1 \cdot e^{2a} + C_2 \cdot e^{-a} \cdot \cos\sqrt{3}a + C_3 \cdot e^{-a} \cdot \sin\sqrt{3}a = 0, \\ C_1 \cdot \left(2a - 2a \cdot e^{2a}\right) + C_2 \cdot \left(-a + \sqrt{3}a \cdot e^{-a} \cdot \sin\sqrt{3}a + a \cdot e^{-a} \cdot \cos\sqrt{3}a\right) + \\ +C_3 \cdot \left(\sqrt{3}a - \sqrt{3}a \cdot e^{-a} \cdot \cos\sqrt{3}a + a \cdot e^{-a} \cdot \sin\sqrt{3}a\right) = 0. \end{cases}$$

Its determinant will be a characteristic determinant for the problem (3):

$$\Delta(a) = \begin{vmatrix} 1 & 1 & 0 \\ e^{2a} & e^{-a}\cos\sqrt{3}a & e^{-a}\sin\sqrt{3}a \\ 2a - 2ae^{2a} & ae^{-a}\left(\sqrt{3}\sin\sqrt{3}a + \cos\sqrt{3}a\right) - a & \sqrt{3}a - ae^{-a}\left(\sqrt{3}\cos\sqrt{3}a + \sin\sqrt{3}a\right) \end{vmatrix}. \tag{7}$$

From where by standard calculations and transformations the determinant (7) is reduced to the form:

$$\Delta(a) = \left(\sqrt{3} + 3i\right) e^{\left(1 + i\sqrt{3}\right)a} + \left(\sqrt{3} - 3i\right) e^{\left(1 - i\sqrt{3}\right)a} + \left(\sqrt{3} + 3i\right) e^{-\left(1 + i\sqrt{3}\right)a} +$$

$$+ \left(\sqrt{3} - 3i\right) e^{-\left(1 - i\sqrt{3}\right)a} - 2\sqrt{3}e^{2a} - 2\sqrt{3}e^{-2a}. \tag{8}$$

We formulate the obtained result as the following theorem.

Theorem 1. Characteristic determinant of the spectral problem (3) is represented as a form of quasi-polynomial (7) and is the entire analytic function of the variable a.

Connection of quasi-polynomials zeros with spectral problems is reflected in [9–13].

Sometimes entire analytical functions coincide with quasi-polynomials, zeros of which are investigated in [8, 14–18].

The papers [19, 20] are devoted to study of zeros of entire functions with an integral representation, related to spectral problems of a third-order differential operator with nonlocal boundary value conditions.

In [21, 22] the characteristic determinant of spectral problem for the Sturm-Liouville operator with perturbed regular boundary value conditions, which is an entire analytic function of the spectral parameter, is calculated. Also in this paper, they study stability problems of basis property of root functions systems of the original operator.

Zeros of the entire analytical function $\Delta(a)$ in (8) are eigenvalues of the operator L_0. Therefore, further we consider the question about distribution of eigenvalues of the entire function $\Delta(a)$ on the complex plane a.

Taking into account results of the monographs [9, 10, 16], the conjugate indicator diagram of the function $\Delta(a)$ will be a regular hexagon on the complex plane a. Sides of the hexagon consist of the following segments:

$$\left[\overline{1 - i\sqrt{3}};\ \overline{-1 - i\sqrt{3}}\right],\ \left[\overline{-1 + i\sqrt{3}};\ \overline{1 + i\sqrt{3}}\right],\ \left[\overline{-2};\ \overline{-1 - i\sqrt{3}}\right],$$

$$\left[\overline{-1 + i\sqrt{3}};\ \overline{-2}\right],\ \left[\overline{1 + i\sqrt{3}};\ \overline{2}\right],\ \left[\overline{2};\ \overline{1 - i\sqrt{3}}\right],$$

where lines mean complex conjugation, and they are commensurable numbers, that is, the length of each segment is equal to $d = 2$, and therefore they form the regular hexagon. From the origin we draw rays that are perpendicular to the sides of the regular hexagon.

Rays, which are perpendicular to the indicator diagram, are called critical. According to [10] the critical rays on the plane a are exactly six, that is

$$\arg\sqrt[3]{a} = \frac{\pi}{6} + \frac{\pi n}{3}, \quad n = 0,\ 1,\ 2,\ 3,\ 4,\ 5.$$

Along the ray, perpendicular to the segment passing through the points $\overline{2};\ \overline{1-i\sqrt{3}}$, there are zeros of the quasi-polynomial $\left(\sqrt{3}+3i\right)\cdot e^{\left(1+i\sqrt{3}\right)a} - 2\sqrt{3}\cdot e^{2a}$ from (8), which are majorizing exponents. Moreover, along this ray other exponents from (8) do not contribute.

We find zeros of the quasi-polynomial:

$$\left(\sqrt{3}+3i\right)\cdot e^{\left(1+i\sqrt{3}\right)a} - 2\sqrt{3}\cdot e^{2a} = 0,$$

$$\left(\sqrt{3}+3i\right)\cdot e^{\left(1+i\sqrt{3}\right)a} = 2\sqrt{3}\cdot e^{2ik\pi},$$

$$a_{k1} = \frac{2ik\pi}{-1+i\sqrt{3}} + \frac{\ln\left|\frac{2\sqrt{3}}{\sqrt{3}+3i}\right| + iArg\left(\frac{2\sqrt{3}}{\sqrt{3}+3i}\right)}{-1+i\sqrt{3}}, \quad k = 1,\ 2,\ 3,\ ...,$$

which are zeroes of the first series, where $\ln\left|\frac{2\sqrt{3}}{\sqrt{3}+3i}\right| + iArg\left(\frac{2\sqrt{3}}{\sqrt{3}+3i}\right) = const.$

Otherwise, from (7) if follows that eigenvalues of the first series of the operator L_0 will be

$$\mu_{k1} = -\left(\frac{4ik\pi}{-1+i\sqrt{3}} + \frac{const}{-1+i\sqrt{3}}\right)^3, \quad k = 1,\ 2,\ 3,\ ....$$

A similar procedure is carried out on the other sides of the hexagon, and along the other perpendicular rays we have the corresponding series of quasi-polynomials zeros from (8):

– segment $\left[\overline{-1-i\sqrt{3}};\ \overline{1-i\sqrt{3}}\right]$, 2-nd series of zeros

$$a_{k2} = \frac{ik\pi}{1+i\sqrt{3}} + \frac{const}{2\left(1+i\sqrt{3}\right)}, \quad k = 1,\ 2,\ 3,\ ...$$

$$\mu_{k2} = -\left(\frac{2ik\pi}{1+i\sqrt{3}} + \frac{const}{1+i\sqrt{3}}\right)^3, \quad k = 1,\ 2,\ 3,\ ...;$$

– segment $\left[\overline{-1+i\sqrt{3}};\ \overline{1+i\sqrt{3}}\right]$, 3-rd series of zeros

$$a_{k3} = ik\pi + const, \quad k = 1,\ 2,\ 3,\ ...,$$

$$\mu_{k3} = -(2ik\pi + const)^3, \quad k = 1,\ 2,\ 3,\ ....;$$

– segment $\left[\overline{-2};\ \overline{-1-i\sqrt{3}}\right]$, 4-th series of zeros

$$a_{k4} = \frac{2ik\pi}{1+i\sqrt{3}} + \frac{const}{1+i\sqrt{3}}, \quad k = 1,\ 2,\ 3,\ ...,$$

$$\mu_{k4} = -\left(\frac{2ik\pi}{1+i\sqrt{3}} + \frac{const}{1+i\sqrt{3}}\right)^3, \quad k = 1,\ 2,\ 3,\ ....;$$

– segment $\left[\overline{2};\ \overline{1+i\sqrt{3}}\right]$, 5-th series of zeros

$$a_{k5} = -\frac{2ik\pi}{1+i\sqrt{3}} - \frac{const}{1+i\sqrt{3}}, \quad k = 1,\ 2,\ 3,\ ...,$$

$$\mu_{k5} = \left(\frac{2ik\pi}{1+i\sqrt{3}} + \frac{const}{1+i\sqrt{3}}\right)^3, \quad k = 1,\ 2,\ 3,\ ....$$

– segment $\left[\overline{-1+i\sqrt{3}};\ \overline{-2}\right]$, 6-th series of zeros

$$a_{k6}=\frac{2ik\pi}{1-i\sqrt{3}}+\frac{const}{1-i\sqrt{3}},\quad k=1,\ 2,\ 3,\ ...,$$

$$\mu_{k6}=-\left(\frac{4ik\pi}{1-i\sqrt{3}}+\frac{const}{1-i\sqrt{3}}\right)^3,\quad k=1,\ 2,\ 3,\ ....$$

Thus,

Proposition 1.

1. There exists an infinite number of operator eigenvalue of the operator L_0.

2. Distance between neighboring eigenvalues of the operator L_0 of the each series is equal to $\frac{2\pi}{|d|}$.

3. Eigenvalues of each series of the operator L_0 lie on the rays, perpendicular to the segment containing the numbers

$$\left(\overline{1-i\sqrt{3}};\ \overline{-1-i\sqrt{3}}\right),\ \left(\overline{-1+i\sqrt{3}};\ \overline{1+i\sqrt{3}}\right),\ \left(\overline{-2};\ \overline{-1-i\sqrt{3}}\right),\ \left(\overline{-1+i\sqrt{3}};\ \overline{-2}\right),$$
$$\left(\overline{1+i\sqrt{3}};\ \overline{2}\right),\ \left(\overline{2};\ \overline{1-i\sqrt{3}}\right).$$

Similarly, by repeating the whole process of researching the problem (3), we solve the problem (4), and we obtain eigenvalues of the operator L_1:

$$\nu_{l1}=-\left(\frac{4il\pi}{-1+i\sqrt{3}}+\frac{const}{-1+i\sqrt{3}}\right)^3,\quad l=1,\ 2,\ 3,\ ....$$

$$\nu_{l2}=-\left(\frac{2il\pi}{1+i\sqrt{3}}+\frac{const}{1+i\sqrt{3}}\right)^3,\quad l=1,\ 2,\ 3,\ ....$$

$$\nu_{l3}=-(2il\pi+const)^3,\quad l=1,\ 2,\ 3,\ ....$$

$$\nu_{l4}=-\left(\frac{2il\pi}{1+i\sqrt{3}}+\frac{const}{1+i\sqrt{3}}\right)^3,\quad l=1,\ 2,\ 3,\ ....$$

$$\nu_{l5}=\left(\frac{2il\pi}{1+i\sqrt{3}}+\frac{const}{1+i\sqrt{3}}\right)^3,\quad l=1,\ 2,\ 3,\ ....$$

$$\nu_{l6}=-\left(\frac{4il\pi}{1-i\sqrt{3}}+\frac{const}{1-i\sqrt{3}}\right)^3,\quad l=1,\ 2,\ 3,\ ....$$

Analogically, all points of Proposition 1 are true for the operator L_1.

So, we have proved the following:

Theorem 2. Suppose that all conditions of Theorem 1 and Proposition 1 hold for the operators L_0 and L_1. Then eigenvalues of the operator L are $\lambda_{klj}=\pm(\mu_{kj}+\nu_{lj})$, where $k=1,2,3,...,\ l=1,2,3,...,\ j=\overline{(1-6)}$ mean the each series.

Remark. In the case $a=\frac{\sqrt[3]{-\mu}}{2}=0$, representing general solution (3) as $X(x)=ax^2+bx+c$ and satisfying the boundary value conditions in (3), we have $X(x)=0$, that is, $\mu_0=0$ is not eigenvalue of the operator L_0. Similarly, $\nu_0=0$ is a regular point of the operator L_1. Thus, $\lambda_0=0$ is not eigenvalue of the operator L.

Conjugate problems

$L_0X\equiv l_0(X)=X'''(x)$. Applying the method of integration by parts, we get the Lagrange formula:

$$\int_0^1 l_0(X)\overline{\upsilon(x)}dx+\int_0^1 X(x)\overline{l_0^*(\upsilon)}dx=X''(1)\overline{\upsilon(1)}-X''(0)\overline{\upsilon(0)}-$$
$$-\left[\overline{\upsilon'(0)}-\overline{\upsilon'(1)}\right]\cdot X'(0)+X(1)\cdot\overline{\upsilon''(1)}-X(0)\cdot\overline{\upsilon''(0)}.$$

Here $l_0^*(\upsilon)$ is the conjugate differential expression:

$$l_0^*(\upsilon)=-\upsilon'''(x),\quad 0<x<1. \tag{9}$$

Consequently, the operator оператор L_0^*, conjugate to the operator L_0, is given by the differential expression (9) and the boundary value conditions

$$\upsilon(1) = \upsilon(0) = 0, \quad \upsilon'(0) - \upsilon'(1) = 0. \tag{10}$$

Analogically, for the operator L_1 conjugate operator is

$$L_1 Y \equiv l_1(Y) = Y'''(y), \quad L_1^* : l_1^*(\upsilon) = -\upsilon'''(y), \quad 0 < y < 1$$

with the boundary value conditions (10). From this, it follows that in the domain D conjugate problem to the problem (1), (2) will be

$$L^* V = V_{xxx} + V_{yyy} - \lambda V = 0,$$

satisfying the boundary value conditions

$$V|_{\partial D} = 0, \quad V_x(1, y) = V_x(0, y), \quad V_y(x, 0) = V_y(x, 1).$$

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References

1 Джураев Т.Д. О классификации и приведении к каноническому виду уравнений с частными производными третьего порядка / Т.Д. Джураев, Я. Попелек // Дифференциальные уравнения. — 1991. — Т. 27, № 10. — С. 1734–1745.

2 Джураев Т.Д. Вычисление собственных значений и собственных функций некоторых дифференциальных операторов третьего и четвертого порядков / Т.Д. Джураев, Б.В. Логинов, И.А. Малюгина // Дифференциальные уравнения математической физики и их приложения. — Ташкент: Фан, 1989. — С. 24–36.

3 Ильин В.А. Спектральная теория дифференциальных операторов. Разложение по фундаментальным функциям самосопряженных эллиптических операторов / В.А. Ильин. — М.: Наука, 1991. — 167 с.

4 Моисеев Е.И. Уравнения смешанного типа со спектральным параметром / Е.И. Моисеев. — М.: Изд-во МГУ, 1988. — 150 с.

5 Кальменов Т.Ш. О спектре задачи Трикоми для уравнения Лаврентьева — Бицадзе / Т.Ш. Кальменов // Дифференциальные уравнения. — 1977. — Т. 13, — № 8. — С.1418–1425.

6 Садыбеков М.А. Спектральные свойства одного класса краевых задач для параболо-гиперболического уравнения / М.А. Садыбеков, Г.Д. Тойжанова // Дифференциальные уравнения. — 1992. — Т. 28, № 1. — С. 176–179.

7 Birkhoff G.D. On the asymptotic chapacter of the certain linear differential equations containing a parameter / G.D. Birkhoff // Transactions of the American Mathematical Society. — 1908. — Vol. 9, No. 2. — P. 219–231.

8 Наймарк М.А. Линейные дифференциальные операторы / М.А. Наймарк. — М.: Наука, 1969. — 496 с.

9 Беллман Р. Дифференциально-разностные уравнения / Р. Беллман, К. Кук. — М.: Наука, 1967. — 548 с.

10 Леонтьев А.Ф. Целые функции и ряды экспонент / А.Ф. Леонтьев. — М.: Наука, 1983. — 176 с.

11 Hald O.H. Discontinuous Inverse Eigen Value Problems / O.H. Hald // Communications on Pure Applied Mathematics. — 1984. — Vol. 37. — P. 539–577.

12 Левин Б.Я. Распределение корней целых функций / Б.Я. Левин. — М.: Наука, 1956. — 632 с.

13 Malamud M.M. On the completeness of root vectors of first order systems/ M.M. Malamud, L.L. Oridoroga // Doklady Mathematics. — 2010. — Vol. 82, No. 3. — P. 809–905.

14 Садовничий В.А. О нулях целых функций одного класса / В.А. Садовничий, В.А. Любишкин, Ю. Белаббаси // Тр. семинара им. И.Г. Петровского. — 1982. — 8. — С. 211–217.

15 Шкаликов А.А. О базисности собственных функций обыкновенных дифференциальных операторов с интегральными краевыми условиями / А.А. Шкаликов // Вест. Москов. ун-та. Сер. 1. Математика. Механика. — 1982. — № 6. — С. 12–21.

16 Кангужин Б.Е. Дифференциальные операторы на отрезке. Распределение собственных значений / Б.Е. Кангужин, М.А. Садыбеков. — Шымкент: Ғылым, 1996. — 270 с.

17 Imanbaev N.S. Distribution of Eigen values of a Third-Order Differential Operator with Strongly Regular non local Boundary Conditions / N.S. Imanbaev // AIP Conference Proceedings. — 2018. — Vol. 1997. — P. 020027-1 – 020027-5.

18 Imanbaev N.S. On zeros of a Quasi-Polynomial of Exponential Type Connected with a Regular Third order Differential Operator / N.S. Imanbaev // Математ. журн. — 2018. — Т. 18, № 2. — С. 124–132.

19 Иманбаев Н.С. О нулях целых функций, имеющих интегральное представление / Н.С. Иманбаев, Б.Е. Кангужин // Изв. НАН РК. Сер. физ.-мат. — 1995. — № 3. — С. 47–52.

20 Imanbaev N.S. On zeros the characteristic determinant of the spectral problem for a third-order differential operator on a segment with nonlocal boundary conditions / N.S. Imanbaev, B.E. Kanguzhin, B.T. Kalimbetov // Advances in Difference Equations. — 2013. — Doi:10.1186/ 1687-1847-20113-110.

21 Sadybekov M.A. Characteristic determinant of a boundary value problem, which does not have the basis property / M.A. Sadybekov, N.S. Imanbaev // Eurasian Mathematical Journal. — 2017. — 8, 2. — P. 40–46.

22 Иманбаев Н.С. Характеристический определитель спектральной задачи для оператора Штурма-Лиувилля с интегральным возмущением краевого условия антипериодического типа / Н.С. Иманбаев // Вестн. Караганд. ун-та. Сер. Математика. — 2016. — № 2(82). — С. 68–72.

Н.С. Иманбаев, М.Н. Оспанов

Регулярлық шеттік шарттармен берілген үшінші ретті құрамдас типті теңдеудің меншікті мәндері жайлы

Мақалада регулярлық, дәлірек айтқанда, периодтық шеттік шарттармен берілген үшінші ретті құрамдас типті теңдеудің меншікті мәндерінің орналасуы туралы мәселе зерттелді. Бастапқы есепті Фурье әдісімен шешуді қолданғаннан кейін, $L_2(0,1)$ кеңістігінде периодтық шарттармен берілген үшінші ретті жай дифференциалдық теңдеудің меншікті мәндерін зерттеуге арналған екі есепке тармақталған. Осы есептердің сипаттамалық анықтауыштарының бүтін аналитикалық функциялар болатындығы дәлелденіп, олардың нөлдері табылып, кешенді жазықтықтағы орындары анықталған. Бастапқы оператордың меншікті мәндерінің саналымды, шексіз екендігі көрсетілген. Түйіндес индикаторлық диаграммасы құрылып, әр сериядағы меншікті мәндердің перпендикуляр сәулелердің бойында арақашықтықтары анықталған. Спектралдық параметрдің нөлдік мәні оператордың меншікті мәні болмайтындығы көрсетілген. Түйіндес операторы құрылған.

Кілт сөздер: құрамдас типті теңдеу, регулярлық периодтық шеттік шарттар, төртбұрыш аймақ, Фурье әдісі, характеристикалық анықтауыш, бүтін аналитикалық функция, меншікті мәндер, бүтін функцияның нөлдері.

Н.С. Иманбаев, М.Н. Оспанов

О собственных значениях уравнений третьего порядка составного типа с регулярными краевыми условиями

В статье исследован вопрос распределения собственных значений уравнений третьего порядка составного типа с регулярными, точнее, с периодическими краевыми условиями. После применения метода Фурье исходная задача распадается на две задачи, т.е. на собственные значения обыкновенных дифференциальных операторов третьего порядка с периодическими краевыми условиями в $L_2(0,1)$. Вычислены характеристические опеределители и найдены нули целых аналитических функций и определено их расположение на комплексной плоскости. Доказано существование бесконечного числа собственных значений оператора третьего порядка составного типа. Определено расстояние между соседними собственными значениями оператора третьего порядка составного типа каждой серии, которое лежит на лучах перпендикулярно сторонам сопряженной индикаторной диаграммы, то есть правильного шестиугольника на комплексной плоскости. Доказано, что нуль не является собственным значением оператора третьего порядка составного типа, иначе говоря, нуль является регулярной точкой оператора, которая принадлежит резольвентному множеству исходного оператора. Построен сопряженный оператор с периодическими краевыми условиями.

Ключевые слова: уравнения составного типа, регулярные, периодические краевые условия, прямоугольная область, метод Фурье, характеристический определитель, целые аналитические функции, собственные значения, нули целых функций.

References

1 Djuraev, T.D. & Popelek, Ya. (1991). O klassifikatsii i privedenii k kanonicheskomu vidu uravnenii s chastnymi proizvodnymi tretieho poriadka [On classification and reduction to the canonical form of third-order partial differential equations]. *Differentsialnye uravnenia – Differential Equations, 27, 10,* 1734–1745 [in Russian].

2 Djuraev, T.D., Loginov, B.V. & Malugina, I.A. (1989). Vychislenie sobstvennykh znachenii i sobstvennykh funktsii nekotorykh differentsialnykh operatorov tretieho i chetvertoho poriadkov [Calculation of eigenvalues and eigen-functions of some differential operators of the third and fourth orders]. *Differentsialnye uravnenia matematicheskoi fiziki i ikh prilohenia – Differential equations of mathematical physics and their applications,* 24–36. Tashkent: Fan [in Pussian].

3 Iliin, V.A. (1991). *Spektralnaia teoria differentsialnykh operatorov. Razlozhenie po fundamentalnym funktsiam samosopriazhennykh ellipticheskikh operatorov [Spectral theory of differential operators. Expansion by fundamental functions of self-adjoint elliptic operators].* Moscow: Nauka [in Russian].

4 Moiseev, E.I. (1988). *Uravnenia smeshannoho tipa so spektralnym parametram [Mixed type equations with spectral parameters].* Moscow: Isdatelstvo MHU [in Russian].

5 Kalmenov, T.Sh. (1977). O spektre zadachi Trikomi dlia uravnenia Lavrentieva-Bitsadze [On spectrum of the Tricomi problem for the Lavrent'ev-Bitsadze equation]. *Differentsialnye uravnenia – Differential Equations, 13, 8,* 1418–1425 [in Russian].

6 Sadybekov, M.A. & Toizhanova, G.D. (1992). Spektralnye svoistva odnoho klassa kraevykh zadach dlia parabolo-hiperbolicheskoho uravnenia [Spectral properties of a class of boundary value problems for a parabolic-hyperbolic equation]. *Differentsialnye uravnenia – Differential Equations, 28, 1,* 176–179 [in Russian].

7 Birkhoff, G.D. (1908). On asymptotic character of the certain linear differential equations containing a parameter. *Transactions of the American Mathematical Society, 9, 2,* 219–231.

8 Naimark, M.A. (1969). *Lineinye differentsialnye operatory [Linear differential operators].* Moscow: Nauka [in Russian].

9 Bellman, R. & Kuk, K. (1967). *Differentsialno-rrznostnye uravnenia [Differential-difference Equations].* Moscow: Nauka [in Russian].

10 Leontiev, A.F. (1983). *Tselye funktsii i riady eksponent [Entire Functions and Exponential Series].* Moscow: Nauka [in Russian].

11 Hald, O.H. (1984). Discontinuous Inverse Eigen Value Problems. *Communications on Pure Applied Mathematics, 37,* 539–577.

12 Levin, B.Ia. (1956). *Raspredelenie kornei tselykh funktsii [Distribution of roots of entire functions].* Moscow: Nauka [in Russian].

13 Malamud, M.M. & Oridoroga, L.L. (2010). On completeness of root vectors of first order systems. *Doklady Akademii nauk. Matematika. Doklady Mathematics, 82, 3,* 809–905.

14 Sadovnichii, V.A., Liubishkin, V.A. & Belabbasi, U. (1982). O nuliakh tselykh funktsii odnoho klassa [On zeros of entire functions of one class]. *Trudy seminara imeni I.G. Petrovskoho – Proceedings of the I.G. Petrovsky seminar, 8,* 211–217 [in Russian].

15 Shkalikov, A.A. (1982). O bazisnosti sobstvennykh funktsii obyknovennykh differetsialnykh operatorov s intehralnymi kraevymi usloviami [On basis property of eigen-functions of ordinary differential operators with integral boundary value conditions]. *Vestnik Moskovskoho universiteta. Seria 1. Matematika. Mekhanika — Moscow University Mathematics. Mechanics Bulletin, 6,* 12–21 [in Russian].

16 Kanguzhin, B.E. & Sadybekov, M.A. (1996). *Differentsialnye operatory na otrezke. Raspredelenie sobstvennykh znachenii. [Differential operators on a segment. Distribution of eigenvalues].* Shymkent: Hylym [in Russian].

17 Imanbaev, N.S. (2018). Distribution of eigenvalues of a third-order differential operator with strongly regular non local boundary conditions. AIP Conference Proceedings, 1997: 020027-1 – 020027-5.

18 Imanbaev, N.S. (2018). On zeros of an exponential type quasi-polynomial connected with a regular third order differential operator. *Matematicheskii zhurnal – Mathematical Journal, 18, 2,* 124–132.

19 Imanbaev, N.S. & Kanguzhin, B.E. (1995). O nuliakh tselykh funktsii, imeiushikh intehralnoe predstavlenie [On zeros of entire functions, having an integral representation]. *Izvestia NAN RK. Seria fiziko-matematicheskaia – News of NAS RK. Physics. Mathematics Series, 3,* 47–52 [in Russian].

20 Imanbaev, N.S., Kanguzhin, B.E. & Kalimbetov, B.T. (2013). On zeros of characteristic determinant of spectral problem for a third-order differential operator on a segment with nonlocal boundary conditions. Advances in Difference Equations, doi:10.1186/1687-1847-20113-110.

21 Sadybekov, M.A. & Imanbaev, N.S. (2017). Characteristic determinant of a boundary value problem, which does not have basis property. *Eurasian Mathematical Journal, 8, 2,* 40–46.

22 Imanbaev, N.S. (2016). Harakteristicheskii opredelitel spektralnoi zadachi dlia operatora Shturma-Liuvillia s intehralnym vozmusheniem kraevoho uslovia antiperiodicheskoho tipa [Characteristic determinant of the spectral problem for the Sturm-Liouville operator with the perturbed boundary value conditions]. *Vestnik Karahandinskoho universiteta. Seria Matematika – Bulletin of the Karaganda University. Mathematics, 2 (82),* 68–72 [in Russian].

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Solving a nonhomogeneous integral equation with the variable lower limit

An nonhomogeneous integral equation with a singular kernel is considered. A feature of the equation under study is the incompressibility of the integral operator. In the study of the equation, an auxiliary simpler equation is used with the right-hand side equal to 1. The incompressibility of the integral operator for the equation under study is shown. Using the relations for an independent variable, the equation is equivalently reduced to a certain simplified equation. With the help of replacements for independent variables, the equation is reduced to an integral equation with a difference kernel. By applying the Laplace transform, the obtained equation is reduced to an ordinary first-order differential equation (linear). Its solution is found. By using the inverse Laplace transform, a solution of the auxiliary integral equation is obtained in the form of a convergent series in some domain. The solution of the initial equation with an arbitrary right-hand side is written through the solution of the auxiliary equation.

Keywords: nonhomogeneous singular integral equation, auxiliary equation, Laplace transform, convergent series.

Introduction

The most complex and interesting objects of study of linear integral equations are irregular situations when the phenomenon of uniqueness of a solution is violated. One of these equations is an equation of the form:

$$\varphi(t) - \frac{1}{2\,a\sqrt{\pi}} \int_t^{\infty} \left[\frac{\tau + t}{(\tau - t)^{\frac{3}{2}}} \ \exp\left\{ -\frac{(\tau + t)^2}{4a^2(\tau - t)} \right\} + \right.$$

$$\left. + \frac{1}{(\tau - t)^{\frac{1}{2}}} \ \exp\left\{ -\frac{\tau - t}{4a^2} \right\} \right] \varphi(\tau)\, d\tau = f(t), \qquad (t > 0). \tag{1}$$

For the kernel of equation (1):

$$K(\tau, t) = \frac{1}{2\,a\sqrt{\pi}} \left[\frac{\tau + t}{(\tau - t)^{\frac{3}{2}}} \ \exp\left\{ -\frac{(\tau + t)^2}{4a^2(\tau - t)} \right\} + \frac{1}{(\tau - t)^{\frac{1}{2}}} \ \exp\left\{ -\frac{\tau - t}{4a^2} \right\} \right], \tag{2}$$

we have [1]:

$$\lim_{t \to \infty} \int_t^{\infty} K(\tau, t)\, d\tau = \lim_{t \to \infty} \left(2e^{-\frac{2t}{a^2}} + 1 \right) = 1_{+0}.$$

Hence, the characteristic part of equation (1) is the second term of the kernel (2).

Than we have [1].

Theorem 1. For the singular integral Volterra equation (1) with the kernel (2) the norm of an integral operator acting in classes of continuous functions is equal to 3.

1 An auxiliary equation and reducing the integral equation to an equation with a difference kernel

Using relations:

$$\tau + t = 2\tau - (\tau - t), \quad \frac{(\tau+t)^2}{4a^2\,(\tau-t)} = \frac{\tau\, t}{a^2\,(\tau-t)} + \frac{\tau - t}{4a^2},$$

equation (1) will be rewritten as:

$$\varphi(t) - \int_t^{\infty} \frac{1}{2a\sqrt{\pi}} \left\{ \frac{2\tau}{(\tau-t)^{3/2}} \exp\Big\{ -\frac{\tau t}{a^2(\tau-t)} \Big\} + \right.$$

$$\left. + \frac{1}{\sqrt{\tau-t}} \Big(1 - \exp\Big\{ -\frac{\tau t}{a^2(\tau-t)} \Big\} \Big) \right\} \cdot \exp\left\{ -\frac{\tau-t}{4a^2} \right\} \varphi(\tau)\, d\tau = f(t).$$

It is enough to find a solution to the «simplified» equation [2; 215]:

$$\psi(t) - \int_t^{\infty} k^*(t,\tau)\, \psi(\tau)\, d\tau \; = g(t), \tag{3}$$

where

$$k^*(t,\tau) = \frac{1}{2a\sqrt{\pi}} \left\{ \frac{2\tau}{(\tau-t)^{3/2}} \exp\Big\{ -\frac{\tau t}{a^2(\tau-t)} \Big\} + \frac{1}{\sqrt{\tau-t}} \Big(1 - \exp\Big\{ -\frac{\tau t}{a^2(\tau-t)} \Big\} \Big) \right\};$$

$$g(t) = \exp\left\{ -\frac{t}{4a^2} \right\} \cdot f(t), \qquad \psi(t) = \exp\left\{ -\frac{t}{4a^2} \right\} \cdot \varphi(t).$$

We consider an auxiliary equation with $g(t) = 1$ in the (3):

$$\psi(t) - \frac{1}{2a\sqrt{\pi}} \int_t^{\infty} \left\{ \frac{2\tau}{(\tau-t)^{3/2}} \exp\Big\{ -\frac{\tau t}{a^2(\tau-t)} \Big\} + \right.$$

$$\left. + \frac{1}{\sqrt{\tau-t}} \Big(1 - \exp\Big\{ -\frac{\tau t}{a^2(\tau-t)} \Big\} \Big) \right\} \psi(\tau)\, d\tau = 1. \tag{4}$$

Integral equation (4) is reduced to an equation with a difference kernel by means of replacements:

$$t = \frac{1}{t_1}, \qquad \tau = \frac{1}{\tau_1}.$$

We have

$$\psi\left(\frac{1}{t_1}\right) - \frac{1}{2a\sqrt{\pi}} \int_0^{t_1} \frac{\sqrt{t_1}}{\tau_1^{3/2}\sqrt{t_1-\tau_1}} \left(1 - \exp\left\{ -\frac{1}{a^2\,(t_1-\tau_1)} \right\} \right) \psi\left(\frac{1}{\tau_1}\right) d\tau_1 -$$

$$-\frac{1}{2a\sqrt{\pi}} \int_0^{t_1} \frac{2\, t_1^{3/2}}{\tau_1^{3/2}} \,(t_1-\tau_1)^{3/2} \exp\left\{ -\frac{1}{a^2\,(t_1-\tau_1)} \right\} \psi\left(\frac{1}{\tau_1}\right) d\tau_1 = 1.$$

After that dividing both sides of the last equality by $t_1^{3/2}$, we introduce the following notation:

$$y\,(t_1) = \frac{1}{t_1^{3/2}} \cdot \psi\left(\frac{1}{t_1}\right).$$

As a result, we obtain the equation:

$$t_1 \cdot y_1(t_1) - \frac{1}{2a\sqrt{\pi}} \int\limits_0^{t_1} \frac{1}{(t_1-\tau_1)^{1/2}} \Big(1 - \exp\Big\{ -\frac{1}{a^2(t_1-\tau_1)} \Big\} \Big) y(\tau_1)\, d\tau_1 -$$

$$-t_1 \cdot \frac{1}{2a\sqrt{\pi}} \int\limits_0^{t_1} \frac{2}{(t_1-\tau_1)^{3/2}} \exp\Big\{ -\frac{1}{a^2(t_1-\tau_1)} \Big\} y(\tau_1)\, d\tau_1 = \frac{1}{\sqrt{t_1}}. \tag{5}$$

2 Solving the equation with a difference kernel

Applying the Laplace transform to the equation (5) we obtain the operator equation:

$$-\overline{y}'(p)-\frac{1}{2a\sqrt{p}}\left(1-\exp\left(-\frac{2\sqrt{p}}{a}\right)\right)\overline{y}(p)+\left\{\exp\left(-\frac{2\sqrt{p}}{a}\right)\overline{y}(p)\right\}'_p=\frac{\sqrt{\pi}}{\sqrt{p}}.$$

After simple transformations we finally get

$$\overline{y}'(p)+\frac{1}{2a\sqrt{p}}\frac{ch\frac{\sqrt{p}}{a}}{sh\,\frac{\sqrt{p}}{a}}\overline{y}(p)=-\frac{\sqrt{\pi}}{\sqrt{p}\left(1-\exp\left(-\frac{2\sqrt{p}}{0}\right)\right)}. \tag{6}$$

The solution of the differential equation (6) is the following function:

$$\overline{y}(p)=\frac{C}{sh\frac{\sqrt{p}}{a}}-a\sqrt{\pi}\frac{\exp\left(\frac{\sqrt{p}}{a}\right)}{sh\frac{\sqrt{p}}{a}}. \tag{7}$$

We rewrite (7) in the form:

$$\overline{y}(p)=\frac{2C}{\exp\left(\frac{\sqrt{p}}{a}\right)-\exp\left(-\frac{\sqrt{p}}{a}\right)}-\frac{2a\sqrt{\pi}}{1-\exp\left(-\frac{2\sqrt{p}}{a}\right)},$$

or in the form

$$\overline{y}(p)=\frac{2C}{\exp\left(\frac{\sqrt{p}}{a}\right)-\exp\left(-\frac{\sqrt{p}}{a}\right)}-2a\sqrt{\pi}\sum_{n=0}^{\infty}\exp\left(-\frac{2n\sqrt{p}}{a}\right). \tag{8}$$

To (8) we apply the inverse Laplace transform [2] we get the solution to equation (5):

$$y(t_1)=-C\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\left(\frac{\nu}{2};a^2t_1\right)\right]_{\nu=0}-2\sum_{n=1}^{\infty}\frac{n}{t_1^{\frac{3}{2}}}\exp\left(-\frac{n^2}{a^2t_1}\right),$$

where

$$\widehat{\theta}_0(\nu;t)=\frac{1}{\sqrt{\pi x}}\left\{\sum_{n=0}^{\infty}\exp\left(-\frac{1}{x}\left(\nu+n+\frac{1}{2}\right)^2\right)-\sum_{n=-1}^{-\infty}n\cdot\exp\left(-\frac{1}{x}\left(\nu+n+\frac{1}{2}\right)^2\right)\right\}$$

is the modified theta function.

3 Solving the «simplified» equation

Returning to the original variables, we get

$$\psi(t)=-\frac{C}{t^{\frac{3}{2}}}\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\left(\frac{\nu}{2};\frac{a^2}{t}\right)\right]_{\nu=0}-2\sum_{n=1}^{\infty}n\cdot\exp\left(-\frac{n^2}{a^2}t\right). \tag{9}$$

(9) is the solution of the auxiliary equation (4) with the right-hand side $g(t)=1$.

We denote (9) by

$$\omega(t)=-\frac{C}{t^{\frac{3}{2}}}\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\left(\frac{\nu}{2};\frac{a^2}{t}\right)\right]_{\nu=0}-2\sum_{n=1}^{\infty}n\cdot\exp\left(-\frac{n^2}{a^2}t\right). \tag{10}$$

Then [1; 546] the solution of the «simplified»- equation (3) with an arbitrary right-hand side $g(t)$ is expressed in terms of $\omega(t)$ using the formula

$$\psi(t)=g(0)\,\omega(t)+\int_t^{\infty}\omega(t-\tau)\,g'(\tau)\,d\tau.$$

In view of notation (10), we obtain a solution to the equation (3)

$$\psi(t) = -g(0)\left(\frac{C}{t^{\frac{3}{2}}}\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\left(\frac{\nu}{2};\frac{a^2}{t}\right)\right]_{\nu=0} + 2\sum_{n=1}^{\infty} n\cdot\exp\left(-\frac{n^2}{a^2}t\right)\right) -$$

$$-\int_t^{\infty}\left(\frac{C}{(t-\tau)^{\frac{3}{2}}}\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\left(\frac{\nu}{2};\frac{a^2}{t-\tau}\right)\right]_{\nu=0} + 2\sum_{n=1}^{\infty} n\cdot\exp\left(-\frac{n^2}{a^2}(t-\tau)\right)\right) g'(\tau)\,d\tau. \qquad (11)$$

As

$$-\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\left(\frac{\nu}{2};x\right)\right]_{\nu=0} =$$

$$= \frac{1}{2\sqrt{\pi}x^{\frac{3}{2}}}\left\{\sum_{n=0}^{+\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4x}\right) - \sum_{n=-1}^{-\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4x}\right)\right\} =$$

$$= \frac{1}{\sqrt{\pi}x^{\frac{3}{2}}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4x}\right),$$

then equality (11) transforms to the form

$$\psi(t) = g(0)\left(\frac{C}{a^3\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4a^2}t\right) - 2\sum_{n=1}^{\infty} n\cdot\exp\left(-\frac{n^2}{a^2}t\right)\right) +$$

$$+\int_t^{\infty}\left(\frac{C}{a^3\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4a^2}(t-\tau)\right) - 2\sum_{n=1}^{\infty} n\cdot\exp\left(-\frac{n^2}{a^2}(t-\tau)\right)\right) g'(\tau)\,d\tau. \qquad (12)$$

The following theorem is proved:

Theorem 2. The integral equation (3) in the class of essentially bounded functions at $g(t)\in L_\infty(0<t<+\infty)$ has the solution defined by the formula (12).

4 Main result

The solution of the integral equation (1) taking into account the obtained expression (12) and [2; 215] has the explicit form:

$$\varphi(t) = f(0)\left(\frac{C}{a^3\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{n^2+n}{a^2}t\right) - 2\sum_{n=1}^{\infty} n\cdot\exp\left(-\frac{4n^2-1}{4a^2}t\right)\right) +$$

$$+\int_t^{\infty}\left[\frac{C}{a^3\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{n^2+n}{a^2}(t-\tau)\right) -\right.$$

$$\left.-2\sum_{n=1}^{\infty} n\cdot\exp\left(-\frac{4n^2-1}{4a^2}(t-\tau)\right)\right]\left(f'(\tau) - \frac{1}{4a^2}f(\tau)\right)d\tau. \qquad (13)$$

5 Main result

Theorem 3. The solution of the integral equation (1) with the singular kernel (2) in the class of essentially bounded functions at $t\ge t_0>0$ has an explicit form defined by the formula (13).

Remark. Singular homogeneous integral equations with kernels of Volterra type were considered in works [3–5]. Their kernels were also «incompressible». The weight classes of the solution existence were found. We also note that boundary value problems for a spectrally loaded parabolic equation reduce to this kind of singular integral equations, when the load line moves according to the law $x=t$ [6–11] and problems for essentially loaded equation of heat conduction [12–16].

In works [17, 18] it is shown that the homogeneous Volterra integral equation of the second kind, to which the homogeneous boundary value problem of heat conduction in the degenerating domain is reduced, has a nonzero solution.

References

1 Kosmakova, M.T., Romanovski, V.G., Orumbayeva, N.T., Tuleutaeva, Zh.M., & Kasymova, L.Zh. (2019). On the integral equation of an adjoint boundary value problem of heat conduction. *Bulletin of the Karaganda University. Series Mathematics, 3 (95),* 33–43. doi: 10.31489/2019M2/33-43

2 Polyanin, A.D., & Manzhirov, A.V. (2008). *Handbooks of Integral Equations* Chapman and Hall / CRC (2 ed.).

3 Jenaliyev, M., Amangaliyeva, M., Kosmakova, M., & Ramazanov, M. (2014). About Dirichlet boundary value problem for the heat equation in the infinite angular domain. *Boundary Value Problems, 213,* 1–21. DOI: 10.1186/s13661-014-0213-4.

4 Amangaliyeva, M.M., Dzhenaliev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). On one homogeneous problem for the heat equation in an infinite angular domain. *Siberian Mathematical Journal, Vol. 56, No. 6*, 982–995. DOI: 10.1134/S0037446615060038.

5 Jenaliyev, M., Amangaliyeva, M., Kosmakova, M., & Ramazanov, M. (2015). On a Volterra equation of the second kind with 'incompressible' kernel. *Advances in Difference Equations, 71,* 1–14. doi: 10.1186/s13662-015-0418-6.

6 Akhmanova, D.M., Dzhenaliev, M.T., & Ramazanov, M.I. (2011). On a particular second kind Volterra integral equation with a spectral parameter. *Siberian Mathematical Journal, 52, 1,* 1–10. DOI: 10.1134 /S0037446606010010.

7 Amangaliyeva, M.M., Akhmanova, D.M., Dzhenaliev, M.T., & Ramazanov, M.I. (2011). Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity. *Differential Equations, 47, 2,* 231–243. DOI: 10.1134/S0012266111020091.

8 Kosmakova, M.T. (2016). On an integral equation of the Dirichlet problem for the heat equation in the degenerating domain. *Bulletin of the Karaganda University. Series Mathematics, 1 (81),* 62–67.

9 Dzhenaliev, M.T., & Ramazanov, M.I. (2006). On the boundary value problem for the spectrally loaded heat conduction operator. *Siberian Mathematical Journal, 47, 3,* 433–451. DOI: 10.1007 /s11202-006-0056-z

10 Dzhenaliev, M.T., & Ramazanov, M.I. (2007). On a boundary value problem for a spectrally loaded heat operator: I *Differential Equations, 43, 4,* 513-524. DOI: 10.1134/S0012266107040106.

11 Dzhenaliev, M.T., & Ramazanov, M.I. (2007). On a boundary value problem for a spectrally loaded heat operator: II *Differential Equations, 43, 6,* 806-812. DOI: 10.1134/S0012266107060079.

12 Ramazanov, M.I., Kosmakova, M.T., Romanovsky, V.G., Zhanbusinova, B.H., & Tuleutaeva, Zh.M. (2018). Boundary value problems for essentially-loaded parabolic equation. *Bulletin of the Karaganda University. Mathematics, 4 (92),* 79-86. DOI: 10.31489/2018M4/79-86.

13 Jenaliyev, M., & Ramazanov, M. (2016). On a homogeneous parabolic problem in an infinite corner domain. *AIP Conference Proceedings, 1759,* 020085. DOI: 10.1063/1.4959699.

14 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). Uniqueness and non-uniqueness of solutions of the boundary value problems of the heat equation. *AIP Conference Proceedings, 1676,* 020028. DOI: 10.1063/1.4930454.

15 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2014). On the spectrum of Volterra integral equation with the «incompressible» kernel. *AIP Conference Proceedings, 1611,* 127–132. DOI: 10.1063/1.4893816.

16 Kosmakova, M.T., Orumbayeva, N.T., Medeubaev, N.K., & Tuleutaeva, Zh.M. (2018). Problems of Heat Conduction with Different Boundary Conditions in Noncylindrical Domains. *AIP Conference Proceedings, 1997,* UNSP 020071-1. DOI: 10.1063/1.5049065.

17 Akhmanova, D.M., Ramazanov, M.I., & Yergaliyev. M.G. (2018). On an integral equation of the problem of heat conduction with domain boundary moving by law of t = x(2). *Bulletin of the Karaganda University. Mathematics series, 1 (89),* 15–19. DOI: 10.31489/2018M1/15-19.

18 Kosmakova, M.T., Ramazanov, M.I., Tokesheva, A.S., & Khairkulova, A.A. (2016). On the non-uniqueness of solution to the homogeneous boundary value problem for the heat conduction equation in an angular domain. *Bulletin of the Karaganda University. Mathematics series, 4 (84),* 80–87. DOI: 10.31489/2016M4 /80-87.

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Бір біртекті емес айнымалы төменгі шекті интегралдық теңдеудің шешілуі

Сингулярлы ядролы біртекті емес интегралдық теңдеу қарастырылған. Зерттелетін теңдеудің ерекшелігі интегралдық оператордың сығылмайтындығы болып табылады. Теңдеуді зерттеу кезінде оң жағы 1-ге тең қарапайым қосалқы теңдеу қолданылды. Тәуелсіз айнымалы үшін қатынастар пайдаланып, теңдеу эквивалентті қандайда бір ықшам теңдеуге келтірілді. Тәуелсіз айнымалылар үшін ауыстырулар қолданылып, теңдеу айырымдық ядролы интегралдық теңдеуге сәйкестендірілді. Алынған теңдеу Лаплас түрлендіруін қолдану арқылы бірінші ретті кәдімгі дифференциалдық (сызықтық) теңдеуге келтірілді. Оның шешуі табылды. Лапластың кері түрлендіруі көмегімен қосымша интегралдық теңдеудің жинақты қатар түріндегі қандайда бір облыстағы шешуі алынды. Кез келген оң жағымен берілген бастапқы теңдеудің шешуі көмекші теңдеудің шешуі арқылы жазылды.

Кілт сөздер: біртекті емес сингулярлы интегралдық теңдеу, қосалқы теңдеу, Лаплас түрлендіруі, жинақты қатар.

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Решение одного неоднородного интегрального уравнения с переменным нижним пределом

Рассмотрено неоднородное интегральное уравнение с сингулярным ядром. Особенностью исследуемого уравнения является несжимаемость интегрального оператора. При исследовании уравнения использовано вспомогательное более простое уравнение с правой частью, равной 1. Используя соотношения для независимой переменной, уравнение эквивалентно сводится к некоторому упрощенному уравнению. С помощью замен для независимых переменных уравнение сводится к интегральному уравнению с разностным ядром. Применением преобразования Лапласа полученное уравнение сведено к обыкновенному дифференциальному уравнению первого порядка (линейному). Найдено его решение. С помощью обратного преобразования Лапласа получено решение вспомогательного интегрального уравнения в виде сходящегося ряда в некоторой области. Выписано решение исходного уравнения с произвольной правой частью через решение вспомогательного уравнения.

Ключевые слова: неоднородное сингулярное интегральное уравнение, вспомогательное уравнение, преобразование Лапласа, сходящийся ряд.

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Extension method for a class of loaded differential equations with nonlocal integral boundary conditions

In this paper we investigate a class of loaded ordinary differential equations with nonlocal integral boundary conditions in terms of an abstract operator equation

$$Bu = A^2u - q\Psi(u) = f, \quad f \in Y, \tag{1}$$
$$D(B) = \{u \in D(A^2) : \Phi(u) = NF(Au), \quad \Phi(Au) = PF(Au)\}.$$

A loaded part and nonlocal integral boundary conditions of these equations are described using functional vectors $\Psi(u)$ and $F(Au)$, respectively. Such equations follow from Extension Theory of linear operators. The necessary and sufficient solvability conditions of these equations are given by the determinant of some matrix. In the case when this determinant is nonzero, a direct method for exact solution of this class of loaded differential equations is proposed. If some problem can be reduced to the type of equation under consideration, then it can be easily solved using the extension method. This method, for $q = \vec{0}$, also gives the necessary and sufficient solvability conditions and the exact solution of a class of ordinary differential equations with nonlocal integral boundary conditions in terms of an abstract operator equation $Bu = A^2u = f, \quad D(B) = \{u \in D(A^2) : \Phi(u) = NF(Au), \quad \Phi(Au) = PF(Au)\}, \quad f \in Y.$

Keywords: loaded ordinary differential equations, differential equations, nonlocal integral boundary conditions, injective and correct operators, exact solutions.

Introduction

In recent years the theory of loaded functional and differential equations (or equations with aftereffect) has been advanced. These equations describe the problems of optimal control, such as: longstanding prediction method and regulation of the layer of soil water and ground moisture, the problems of underground fluid and gas dynamics, mathematical biology, ecology and economics [1–5]. The stationary one-speed transport equation is an important example of loaded differential equations [6]. A methodical study on the theory of boundary value problems for loaded functional and differential equations and their applications has been conducted by Nakhushev [4]. An equation is called loaded if it contains the solution function on a manifold with dimension less than the dimension of domain of this function [4]. For example an ordinary loaded differential equation is represented by

$$dy/dx = f(x, y) + h(y(x_j)), \quad x \in [0, 1], x_j \in [0, 1],$$

where x_j are fixed points. Boundary value problems consisting of general boundary conditions and the so-called differential boundary equations (loaded differential equations) have been investigated by many researchers in the last century, see, for example, the survey paper by Krall [7] and the references cited in it. Big interest represent the theory of loaded equations in Pure and Applied Mathematics [8–15]. Usually boundary value problems for the loaded ordinary differential equations with integral boundary conditions are investigated by numerical methods [16–19]. Here, the necessary and sufficient solvability conditions of the abstract operator equations (1) and their exact solutions by Extension Method are obtained in closed form. This formalism is applied to solve the ordinary loaded differential equations with integral boundary conditions, when q, Φ, Ψ, F are vectors, N, P matrices, A is an ordinary differential operator, $\Psi(u)$ is a loaded part of equations and $F(Au)$ defines the integral boundary conditions. Such problems arise naturally from Oinarov extensions of linear operators [20, 21] in Banach space which are not restrictions of a maximal operator, unlike the classical M. Krein, J. Von. Neuman extensions [22, 23] in Hilbert space and Otelbaev, Kokebaev, Shynybekov extensions [24] in Banach space. Extension Method uses a symple correct restiction $\widehat{A}$ of a maximal operator A defined by $\widehat{A}u = Au, \quad D(\widehat{A}) = \{u \in D(A) : \Phi(u) = \mathbf{0}\}$, to solve complex problems of type (1). This method was applied in [25] to simpler abstract operator equations $Bu = Au - q\Psi(u) = f, \quad D(B) = \{u \in D(A) : \Phi(u) = NF(Au)\}, \quad f \in Y$ for the study of ordinary loaded differential and loaded integro-differential equations with integral boundary conditions and their exact solutions in closed form. There are many problems of type (1). The loaded differential operator [8]

$$Lu(x) = u''(x) + a(x)u\left(\frac{1}{4}\right) + b(x)u\left(\frac{1}{2}\right) + u(1); \tag{2}$$

$$u(0) = \int_0^1 \gamma(x)u(x)dx, \quad u'(1) = \int_0^1 \nu(x)u(x)dx,$$

is an operator of type (1) if we take $Au = u'(x)$, $q_1 = -a(x)$, $q_2 = -b(x)$, $q_3 = -1$, $\Psi_1(u) = u(1/4)$, $\Psi_2(u) = u(1/2)$, $\Psi_3(u) = u(1)$, $\Phi_1(u) = u(1)$, $\Phi_2(u) = u'(1)$, $\int_0^1 \gamma(x)dx = \int_0^1 \nu(x)dx = 0$, $\widehat{A}u = Au$, $D(\widehat{A}) = \{u(x) \in D(A) : u(1) = 0\}$. Then $\widehat{A}^{-1}f(x) = \int_1^x f(x)dx$ and the boundary conditions of (2) are represented as

$$u(1) = \int_0^1 \left[1 - \int_0^x \gamma(s)ds\right] u'(x)dx, \quad u'(1) = -\int_0^1 \int_0^x \nu(s)dsu'(x)dx.$$

So $N = 1$,
$F_1(Au) = \int_0^1 \left[1 - \int_0^x \gamma(s)ds\right] u'(x)dx$, $P = -1$, $F_2(Au) = \int_0^1 \int_0^x \nu(s)dsu'(x)dx$ and Extension Method can be applied.
The loaded differential operator [8]

$$Lv(x) = v''(x) + \gamma(x)v'(0) + \nu(x)v(1); \tag{3}$$

$$v(0) = 0, \quad v'(1) = \int_0^1 v(x)dx,$$

is also an operator of type (1) if we take the operators $A, \widehat{A}$, the vectors Φ_1, Φ_2 as in (2) and $q_1 = -\gamma(x)$, $q_2 = -\nu(x)$, $\Psi_1(v) = v'(0)$, $\Psi_2(v) = v(1)$. Note that the boundary conditions of (3) are represented as $v(1) = \int_0^1 v'(x)dx$, $v'(1) = \int_0^1 (1-x)v'(x)dx$. So $N = P = 1$, $F_1(Av) = \int_0^1 v'(x)dx$, $F_2(Av) = \int_0^1 (1 - -x)v'(x)dx$, and Extension Method can be applied. The technique of this method is simple to use and can be easily incorporated to any Computer Algebra System (CAS). The paper is organized as follows. First we recall some basic terminology and notation about operators. Then we prove the main general results and give an example of boundary value problem with ordinary loaded differential equation and nonlocal integral boundary conditions which show the usefulness of our results.

Terminology and notation

Let X, Y be a complex Banach spaces and X^* is the adjoint space of X, i.e. the set of all complex-valued linear and bounded functionals on X. We denote by $f(x)$ the value of functional $f \in X^*$ on $x \in X$. We write $D(A)$ and $R(A)$ for the domain and the range of the operator A, respectively. An operator A_2 is said to be an *extension* of an operator A_1, or A_1 is said to be a *restriction* of A_2, in symbol $A_1 \subset A_2$, if $D(A_2) \supseteq D(A_1)$ and $A_1x = A_2x$, for all $x \in D(A_1)$. An operator A is called *maximal* if $R(A) = Y$ and $\ker A \neq \{0\}$. An operator $\widehat{A} : X \to Y$ is said to be *correct* if $R(\widehat{A}) = Y$ and the inverse $\widehat{A}^{-1}$ exists and is continuous on Y. An operator $\widehat{A}$ is called a *correct restriction* of the maximal operator A if it is a correct operator and $\widehat{A} \subset A$. An operator $A : X \to Y$ is said to be *injective* if for all $u_1, u_2 \in D(A)$ such that $Au_1 = Au_2$, follows that $u_1 = u_2$. Remind that a linear operator A is injective if and only if $\ker A = \{0\}$. If $\Psi_i \in X^*, i = 1, \ldots, n,$ then we denote by $\Psi =$col$(\Psi_1, \ldots, \Psi_n)$ and $\Psi(x) =$col$(\Psi_1(x), \ldots, \Psi_n(x))$. Let $g = (g_1, \ldots, g_n)$ be a vector of X^n. We will denote by $\Psi(g)$ the $n \times n$ matrix whose i, j-th entry $\Psi_i(g_j)$ is the value of functional Ψ_i on element g_j. Note that $\Psi(gC) = \Psi(g)C$, where C is a $n \times k$ constant matrix. We will also denote by $\mathbf{0}_{ln}$ the zero $l \times n$ matrix and by I_n the identity $n \times n$ matrix. By $\vec{0}$ we will denote the zero column vector.

Let $A : X \overset{on}{\to} Y$ be an ordinary m order differential operator

$$Au(x) = \alpha_0 u^{(m)}(x) + \alpha_1 u^{(m-1)}(x) + ... + \alpha_m u(x), \quad \alpha_i \in \mathbb{R}, \alpha_0 \neq 0 \tag{4}$$

and X, Y be the Banach spaces. Usually $X = Y = C[a, b]$ or $X = Y = L_p(a, b)$, $p \geq 1$. Everywhere below we denote by

$$X_A^m = \left(D(A), ||\cdot||_{X_A^m}\right)$$

the Banach space m times differentiable functions with norm

$$||u(x)||_{X_A^m} = \sum_{i=0}^{m} ||u^{(i)}(x)||_X.$$

It is a well-known fact that the operator $\widehat{A}: C[a,b] \overset{on}{\to} C[a,b]$ defined by

$$\widehat{A}u(x) = Au(x) = f, \tag{5}$$

$$D(\widehat{A}) = \{u(x) \in C^m[a,b] : u(x_0) = u'(x_0) = ... = u^{(m-1)}(x_0) = 0\}, \quad x_0 \in [a,b],$$

is a correct restriction of A and the unique solution of (5) for $\alpha_0 = 1, \alpha_1 = ... = \alpha_m = 0$ is

$$u(x) = \widehat{A}^{-1}f(x) = \frac{1}{(m-1)!}\int_{x_0}^{x}(x-t)^{m-1}f(t)dt, \qquad f(x) \in C[a,b]. \tag{6}$$

Lemma 1. Let X, Y be complex Banach spaces, $A : X \overset{on}{\to} Y$ an operator defined by (4) with finite dimensional kernel $\mathbf{z} = (z_1, ..., z_m)$ which is a basis of $\ker A$. Suppose also that the components of a functional vector $\Phi = (\Phi_1, ..., \Phi_m)$ belong to $[X_A^m]^*$, a set $\Phi_1, ..., \Phi_m$ is biorthogonal to $z_1, ..., z_m$, i.e. $\Phi(\mathbf{z}) = I_m$ and $\widehat{A}$ is a correct restriction of A defined by

$$\widehat{A} \subset A, \quad D(\widehat{A}) = \{u \in D(A) : \Phi(u) = 0\}. \tag{7}$$

Then the operator $\widehat{A}^2$ is correct and defined by

$$\widehat{A}^2 \subset A^2, \quad D(\widehat{A}^2) = \{u \in D(A^2) : \Phi(u) = 0, \quad \Phi(Au) = 0\}.$$

Proof. By definition since (7) we get

$$D(\widehat{A}^2) = \{u \in D(\widehat{A}) : \widehat{A}u \in D(\widehat{A})\} =$$

$$= \{u \in D(A) : \Phi(u) = 0, \widehat{A}u \in D(A), \Phi(\widehat{A}u) = 0\} =$$

$$= \{u \in D(A) : \Phi(u) = 0, Au \in D(A), \Phi(Au) = 0\} =$$

$$= \{u \in D(A^2) : \Phi(u) = 0, \quad \Phi(Au) = 0\}.$$

Then $\widehat{A}^2 \subset A^2$. Finally the operator $\widehat{A}^2$ is correct as superposition of two correct operators.

Remark 2. If the operator $\widehat{A}$ is defined by (5) then

$$\widehat{A}^2u(x) = \alpha_0 u^{(2m)}(x) + \alpha_1 u^{(2m-1)}(x) + ... + \alpha_{2m}u(x);$$

$$D(\widehat{A}^2) = \{u(x) \in C^{2m}[a,b] : u(x_0) = u'(x_0) = ... = u^{(2m-1)}(x_0) = 0\}$$

Theorem 3. Let the spaces X, Y, the operators $A, \widehat{A}$ and a vector Φ be defined as in Lemma 1, and the components of the functional vectors $\Psi = col(\Psi_1, \dots, \Psi_l)$ and $F = col(F_1, \dots, F_n)$ belong to $[X_A^m]^*$ and Y^*, respectively. Suppose also that the components of the vector $q = (q_1, \dots, q_l)$ are linearly independent on Y and N, P are the $m \times n$ constant matrices. Then:

(i) The operator B defined by

$$Bu = A^2u - q\Psi(u) = f, \quad f \in Y, \tag{8}$$

$$D(B) = \{u \in D(A^2) : \Phi(u) = NF(Au), \quad \Phi(Au) = PF(Au)\},$$

is injective if and only if

$$\det L = \det \begin{pmatrix} I_l - \Psi(\widehat{A}^{-2}q) & -\Psi(\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N) \\ -F(\widehat{A}^{-1}q) & I_n - F(\mathbf{z})P \end{pmatrix} \neq 0. \tag{9}$$

(ii) If B is injective, then B is correct and for all $f \in Y$ the unique solution of (8) is given by

$$u = B^{-1}f = \widehat{A}^{-2}f + (\widehat{A}^{-2}q, \widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N)L^{-1}\begin{pmatrix} \Psi(\widehat{A}^{-2}f) \\ F(\widehat{A}^{-1}f) \end{pmatrix}. \tag{10}$$

Proof (i). From boundary conditions (8), since $\Phi(\mathbf{z}) = I_m$, we obtain

$$\Phi(u - \mathbf{z}NF(Au)) = 0, \quad \Phi(Au - \mathbf{z}PF(Au)) = 0. \tag{11}$$

From (11), taking into account (7), we get $u - \mathbf{z}NF(Au) \in D(\widehat{A})$, $Au - \mathbf{z}PF(Au) \in D(\widehat{A})$. Further, using these relations, the correctness of $\widehat{A}$ and $\mathbf{z} \in [\ker A]^m$, for every $u \in D(B)$ from (8) we obtain

$$Bu = A\left(Au - \mathbf{z}PF(Au)\right) - q\Psi(u) = f, \quad f \in Y,$$

$$Bu = \widehat{A}\left(Au - \mathbf{z}PF(Au)\right) - q\Psi(u) = f,$$

$$Au - zPF(Au) - \widehat{A}^{-1}q\Psi(u) = \widehat{A}^{-1}f, \tag{12}$$

$$F(Au) - F(\mathbf{z})PF(Au) - F(\widehat{A}^{-1}q)\Psi(u) = F(\widehat{A}^{-1}f),$$

$$[I_n - F(\mathbf{z})P]F(Au) - F(\widehat{A}^{-1}q)\Psi(u) = F(\widehat{A}^{-1}f). \tag{13}$$

From (12), since $u - \mathbf{z}NF(Au) \in D(\widehat{A})$ and $\widehat{A}$ is correct, we get

$$A(u - \mathbf{z}NF(Au)) - \mathbf{z}PF(Au) - \widehat{A}^{-1}q\Psi(u) = \widehat{A}^{-1}f,$$

$$\widehat{A}(u - \mathbf{z}NF(Au)) - \mathbf{z}PF(Au) - \widehat{A}^{-1}q\Psi(u) = \widehat{A}^{-1}f,$$

$$u - \mathbf{z}NF(Au) - \widehat{A}^{-1}\mathbf{z}PF(Au) - \widehat{A}^{-2}q\Psi(u) = \widehat{A}^{-2}f,$$

$$u - \left(\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N\right)F(Au) - \widehat{A}^{-2}q\Psi(u) = \widehat{A}^{-2}f, \tag{14}$$

$$\Psi(u) - \Psi\left(\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N\right)F(Au) - \Psi(\widehat{A}^{-2}q)\Psi(u) = \Psi(\widehat{A}^{-2}f),$$

$$[I_l - \Psi(\widehat{A}^{-2}q)]\Psi(u) - \Psi\left(\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N\right)F(Au) = \Psi(\widehat{A}^{-2}f), \tag{15}$$

From (13), (15) we have

$$\begin{pmatrix} I_l - \Psi(\widehat{A}^{-2}q) & -\Psi(\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N) \\ -F(\widehat{A}^{-1}q) & I_n - F(\mathbf{z})P \end{pmatrix} \begin{pmatrix} \Psi(u) \\ F(Au) \end{pmatrix} = \begin{pmatrix} \Psi(\widehat{A}^{-2}f) \\ F(\widehat{A}^{-1}f) \end{pmatrix}. \tag{16}$$

Let $det\, L \neq 0$ and $u \in \ker B$. Then in (8) $f = 0$ and

$$Bu = A^2u - g\Psi(u) = 0, \quad \Phi(u) = NF(Au), \quad \Phi(Au) = PF(Au). \tag{17}$$

By the similar way as above is proved the type (16) for $f = 0$, viz.

$$\begin{pmatrix} I_l - \Psi(\widehat{A}^{-2}q) & -\Psi(\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N) \\ -F(\widehat{A}^{-1}q) & I_n - F(\mathbf{z})P \end{pmatrix} \begin{pmatrix} \Psi(u) \\ F(Au) \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix},$$

which, since $\det L \neq 0$ implies $\Psi(u) = \vec{0}$, $F(Au) = \vec{0}$. Substituting these values into (17), we obtain $Bu = A^2u = 0$, $\Phi(u) = \Phi(Au) = \vec{0}$. Then $u \in D(\widehat{A}^2)$, $Bu = \widehat{A}^2u = 0$. The last implies $u = 0$, since $\widehat{A}$ is correct. This proves that $\ker B = \{0\}$. So B is injective.
Conversely. We will prove that if B is injective, then $det\, L \neq 0$, or equivalently if $det\, L = 0$, then B is not injective. Let $det\, L = 0$. Then there exists a vector $\vec{c} = col(\vec{c}_1, \vec{c}_2) \neq \vec{0}$,
$\vec{c}_1 = col(c_{11}, ..., c_{1l})$, $\vec{c}_2 = col(c_{21}, ..., c_{2m})$ such that $L\vec{c} = 0$ or

$$\begin{pmatrix} I_l - \Psi(\widehat{A}^{-2}q) & -\Psi(\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N) \\ -F(\widehat{A}^{-1}q) & I_n - F(\mathbf{z})P \end{pmatrix} \begin{pmatrix} \vec{c}_1 \\ \vec{c}_2 \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}. \tag{18}$$

Consider the element $u_0 = \widehat{A}^{-2}q\vec{c}_1 + (\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N)\vec{c}_2$. It easy to verify that $u_0 \neq 0$, otherwise since the components of q are linearly independent, we get $\vec{c}_1 = \vec{0}$, $(\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N)\vec{c}_2 = \vec{0}$, $\mathbf{z}P\vec{c}_2 = \vec{0}$, $F(\mathbf{z})P\vec{c}_2 = \vec{0}$ and from (18) it follows that $\vec{c}_2 = \vec{0}$. Thus $\vec{c} = \vec{0}$, which contradicts the hypothesis that $\vec{c} \neq \vec{0}$. Note that $u_0 \in D(B)$, since $\Phi(u_0) = N\vec{c}_2$, $\Phi(Au_0) = P\vec{c}_2$, $F(Au_0) = F(\widehat{A}^{-1}q)\vec{c}_1 + F(\mathbf{z})P\vec{c}_2$ and

$$\Phi(u_0) - NF(Au_0) = N\vec{c}_2 - NF(\widehat{A}^{-1}q)\vec{c}_1 - NF(\mathbf{z})P\vec{c}_2 =$$

$$= N\left(-F(\widehat{A}^{-1}q), I_n - F(\mathbf{z})P\right) col(\vec{c}_1, \vec{c}_2) = \vec{0}, \tag{19}$$

$$\Phi(Au_0) - PF(Au_0) = P\vec{c}_2 - PF(\widehat{A}^{-1}q)\vec{c}_1 - PF(\mathbf{z})P\vec{c}_2,$$

$$= P\left(-F(\widehat{A}^{-1}q), I_n - F(\mathbf{z})P\right) col(\vec{c}_1, \vec{c}_2) = \vec{0}. \tag{20}$$

The last equations in (19), (20) follow since (18). So $u_0 \in D(B)$. We will show now that $u_0 \in \ker B$.

$$Bu_0 = A^2u_0 - q\Psi(u_0) = q\vec{c}_1 - q\Psi(\widehat{A}^{-2}q)\vec{c}_1 - q\Psi(\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N)\vec{c}_2$$

$$= q\left(I_l - \Psi(\widehat{A}^{-2}q), -\Psi(\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N)\right) col(\vec{c}_1, \vec{c}_2) = q\vec{0} = \vec{0},$$

since (18). So we obtain $u_0 \neq \mathbf{0}$ and $u_0 \in \ker B$. Hence $\ker B \neq \{0\}$ and B is not injective. The statement (i) holds.
(ii) Let $det\, L \neq 0$ and $Bu = f$. From (8) as in the proof (i) we get (13), (14) and (15). Then

$$\begin{pmatrix} \Psi(u) \\ F(Au) \end{pmatrix} = \begin{pmatrix} I_l - \Psi(\widehat{A}^{-2}q) & -\Psi(\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N) \\ -F(\widehat{A}^{-1}q) & I_n - F(\mathbf{z})P \end{pmatrix}^{-1} \begin{pmatrix} \Psi(\widehat{A}^{-2}f) \\ F(\widehat{A}^{-1}f) \end{pmatrix}.$$

Substituting these values into (14) we obtain the solution (10) of the problem (8) for every $f \in Y$. Because f in (10) is arbitrary, we obtain $R(B) = Y$. Since the operator $\widehat{A}^{-1}$ and the functionals $\Phi_1, ..., \Phi_m$, $\Psi_1, ..., \Psi_l$ are bounded, from (10) follows the boundedness of B^{-1}. Hence, the operator B is correct if and only if (9) holds and the unique solution of (8) is given by (10). The theorem is proved.

From the previous theorem for $q = \vec{0}$ follows the next corollary which is useful for solving differential equations with integral boundary conditions.

Corollary 4. Let the spaces X, Y, the operators $A, \widehat{A}$, the vector $\mathbf{z} = (z_1, ..., z_m)$, functional vectors Φ, F and the matrices N, P be defined as in Theorem 3. Then:
(i) The operator B defined by

$$Bu = A^2u = f, \quad f \in Y, \tag{21}$$

$$D(B) = \{u \in D(A^2) : \Phi(u) = NF(Au), \quad \Phi(Au) = PF(Au)\}$$

is injective if and only if

$$\det\, V = \det[I_n - F(\mathbf{z})P] \neq 0.$$

(ii) If B is injective, then B is correct and for all $f \in Y$ the unique solution of (21) is given by

$$u = B^{-1}f = \widehat{A}^{-2}f + (\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N)V^{-1}F(\widehat{A}^{-1}f). \tag{22}$$

Proof (i). For $q = 0$ from (9) follows that

$$\det\, L = \det\begin{pmatrix} I_l & -\Psi(\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N) \\ \mathbf{0}_{nl} & I_n - F(\mathbf{z})P \end{pmatrix} = \det[I_n - F(\mathbf{z})P] = \det V \neq 0.$$

It is easy to verify that

$$L^{-1} = \begin{pmatrix} I_l & \Psi(\widehat{A}^{-1}\mathbf{z}P + \mathbf{z}N)V^{-1} \\ \mathbf{0}_{nl} & V^{-1} \end{pmatrix}.$$

Then from (10) for $q = \vec{0}$ follows (22).

Example. The next problem with loaded differential equation and nonlocal integral boundary conditions on $C[0,1]$

$$u''(t) - 4t\, u(1/2) - (2t+1)\, u(1) = 1 - 5t, \tag{23}$$

$$u(0) = -6\int_0^1 x^2u'(x)dx + 15\int_0^1 x^4u'(x)dx,$$

$$u'(0) = 6\int_0^1 x^2u'(x)dx - 15\int_0^1 x^4u'(x)dx,$$

is correct and the unique solution of (23) is given by

$$u(t) = t^2 - t + 1. \tag{24}$$

Proof. First we rewrite the boundary conditions (24) in the form

$$u(0) = (-6, 15)col\left(\int_0^1 x^2u'(x)dx, \int_0^1 x^4u'(x)dx\right), \tag{25}$$

$$u'(0) = (6, -15)col\left(\int_0^1 x^2u'(x)dx, \int_0^1 x^4u'(x)dx\right).$$

If we compare (23), (25) with (8), it is natural to take $X = Y = C[0, 1]$, $m = 1$, $n = l = 2$, $Au(t) = u'(t)$, $D(A) = \{u(t) \in C^1[0, 1]\}$, $X_A^1 = C^1[0, 1]$. $A^2u = u''(t)$, $D(A^2) = \{u(t) \in C^2[0, 1]\}$. It is evident that $z = 1$ constitute a basis of $\ker A$. As the operator $\widehat{A}$ it is natural to take $\widehat{A}u(t) = Au(t) = u'(t)$, $D(\widehat{A}) = \{u(t) \in \in D(A) : u(0) = 0\}$, $\widehat{A}^2u(t) = A^2u(t) = u''(t)$, $D(\widehat{A}^2) = \{u(t) \in D(A^2) : u(0) = u'(0) = 0\}$, The initial problem $\widehat{A}u(t) = f(t)$ is correct and has the unique solution $\widehat{A}^{-1}f(t) = \int_0^t f(x)dx$. Then, since (6), $\widehat{A}^{-2}f(t) = \int_0^t (t-x)f(x)dx$. By comparing (23), (25) with (8), it is natural to take $q_1 = 4t$, $q_2 = 2t+1$, $q = (q_1, q_2) = (4t, 2t + 1)$, $f = 1 - 5t$, $M = (-6, 15)$, $P = (6, -15)$, $\Phi(u) = u(0)$,$\Phi(Au) = u'(0)$,

$$\Psi(u) = \begin{pmatrix}\Psi_1(u)\\ \Psi_2(u)\end{pmatrix} = \begin{pmatrix}u(1/2)\\ u(1)\end{pmatrix}, \quad F(Au) = \begin{pmatrix}F_1(Au)\\ F_2(Au)\end{pmatrix} = \begin{pmatrix}\int_0^1 x^2u'(x)dx\\ \int_0^1 x^4u'(x)dx\end{pmatrix}.$$

Then $F(f) = \begin{pmatrix}F_1(f)\\ F_2(f)\end{pmatrix} = \begin{pmatrix}\int_0^1 x^2f(x)dx\\ \int_0^1 x^4f(x)dx\end{pmatrix}$.
It is evident that $\Phi(z) = 1$ for $z = 1$ and that $|F_1(f)| \le ||f||_C$, $|F_2(f)| \le ||f||_C$ for all $f \in C[0, 1]$. So $F_1, F_2 \in C^*[0, 1] = Y^*$. Because of $|\Psi(u)| = |u(t_0)| \le ||u(t)||_C \le ||u(t)||_C + ||u'(t)||_C = ||u(t)||_{C^1} = ||u(t)||_{X_A^1}$, we conclude that $\Psi_1, \Psi_2 \in [X_A^1]^*$. In the same way is proved that $\Phi \in [X_A^1]^*$. So we can apply Theorem 3. We calculate $\widehat{A}^{-1}z = \int_0^t 1dx = t$, $\widehat{A}^{-1}zP + zN = t(6, -15) + (-6, 15) = (6t - 6, -15t + 15) = (v_1, v_2)$,

$$\Psi(\widehat{A}^{-1}zP + zN) = \begin{pmatrix}\Psi_1(v_1) & \Psi_1(v_2)\\ \Psi_2(v_1) & \Psi_2(v_2)\end{pmatrix} = \begin{pmatrix}v_1(1/2) & v_2(1/2)\\ v_1(1) & v_2(1)\end{pmatrix} = \begin{pmatrix}-3 & 15/2\\ 0 & 0\end{pmatrix},$$

$$\widehat{A}^{-1}q_1(t) = \int_0^t 4xdx = 2t^2, \quad \widehat{A}^{-1}q_2(t) = \int_0^t (2x+1)dx = t^2 + t,$$

$$\widehat{A}^{-1}q = \left(2t^2,\, t^2 + t\right), \quad \widehat{A}^{-2}q_1(t) = \int_0^t (t-x)4xdx = \frac{2}{3}t^3,$$

$$\widehat{A}^{-2}q_2(t) = \int_0^t (t-x)(2x+1)dx = \frac{1}{3}t^3 + \frac{1}{2}t^2,$$

$$\widehat{A}^{-2}q = (\widehat{A}^{-2}q_1, \widehat{A}^{-2}q_2) = \left(\frac{2}{3}t^3, \frac{1}{3}t^3 + \frac{1}{2}t^2\right),$$

$$\Psi(\widehat{A}^{-2}q) = \begin{pmatrix}\Psi_1(\widehat{A}^{-2}q_1) & \Psi_1(\widehat{A}^{-2}q_2)\\ \Psi_2(\widehat{A}^{-2}q_1) & \Psi_2(\widehat{A}^{-2}q_2)\end{pmatrix} = \begin{pmatrix}(\widehat{A}^{-2}q_1)(1/2) & (\widehat{A}^{-2}q_2)(1/2)\\ (\widehat{A}^{-2}q_1)(1) & (\widehat{A}^{-2}q_2)(1)\end{pmatrix} =$$

$$= \begin{pmatrix}1/12 & 1/6\\ 2/3 & 5/6\end{pmatrix},$$

$$F(\widehat{A}^{-1}q) = \begin{pmatrix}F_1(\widehat{A}^{-1}q_1) & F_1(\widehat{A}^{-1}q_2)\\ F_2(\widehat{A}^{-1}q_1) & F_2(\widehat{A}^{-1}q_2)\end{pmatrix} = \begin{pmatrix}2/5 & 9/20\\ 2/7 & 13/42\end{pmatrix},$$

$I_l - \Psi(\widehat{A}^{-2}q) = \begin{pmatrix}11/12 & -1/6\\ -2/3 & 1/6\end{pmatrix}$, $F(z) = \begin{pmatrix}F_1(z)\\ F_2(z)\end{pmatrix} = \begin{pmatrix}1/3\\ 1/5\end{pmatrix}$,
$F(z)P = \begin{pmatrix}1/3\\ 1/5\end{pmatrix}(6, -15) = \begin{pmatrix}2 & -5\\ 6/5 & -3\end{pmatrix}$, $I_n - F(z)P = \begin{pmatrix}-1 & 5\\ -6/5 & 4\end{pmatrix}$.

Then by using (9) we find

$$L = \begin{pmatrix} 11/12 & -1/6 & 3 & -15/2 \\ -2/3 & 1/6 & 0 & 0 \\ -2/5 & -9/20 & -1 & 5 \\ -2/7 & -13/42 & -6/5 & 4 \end{pmatrix}.$$

Since $\det L \neq 0$, by Theorem 3, the problem (23), is correct. We find the inverse matrix

$$L^{-1} = \begin{pmatrix} -140/303 & -548/303 & 70/101 & 175/101 \\ -560/303 & -374/303 & 280/101 & -700/101 \\ -248/909 & -565/1818 & 730/303 & -2135/606 \\ -1172/4545 & -289/909 & 299/303 & -889/606 \end{pmatrix}.$$

For $f(t) = 1 - 5t$ we compute
$\widehat{A}^{-1}f(t) = \int_0^t (1-5x)dx = t - \frac{5}{2}t^2$, $\widehat{A}^{-2}f(t) = \int_0^t (t-x)(1-5x)dx = \frac{1}{2}t^2 - \frac{5}{6}t^3$,

$$F_1(\widehat{A}^{-1}f) = \int_0^1 x^2 \left(x - \tfrac{5}{2}x^2\right) dx = -1/4, \quad F_2(\widehat{A}^{-1}f) = \int_0^1 x^4 \left(x - \tfrac{5}{2}x^2\right) dx = -4/21,$$

$$F(\widehat{A}^{-1}f) = \begin{pmatrix} -1/4 \\ -4/21 \end{pmatrix}, \quad \Psi(\widehat{A}^{-2}f) = \begin{pmatrix} \Psi_1(\widehat{A}^{-2}f) \\ \Psi_2(\widehat{A}^{-2}f) \end{pmatrix} = \begin{pmatrix} 1/48 \\ -1/3 \end{pmatrix}.$$

By substituting these values into (10) we obtain the unique solution of (23)

$$u(t) = \frac{1}{2}t^2 - \frac{5}{6}t^3 + \left(\frac{2}{3}t^3, \frac{1}{3}t^3 + \frac{1}{2}t^2, 6t-6, -15t+15\right) L^{-1} \begin{pmatrix} 1/48 \\ -1/3 \\ -1/4 \\ -4/21 \end{pmatrix},$$

which yields (24).

References

1 Нахушев А.М. Краевые задачи для нагруженных параболических уравнений и их применение к прогнозу уровня грунтовых вод / А.М. Нахушев, В.Н. Борисов // Дифференциальные уравнения. — 1977. — Вып. — 13. — № 1. — С. 105–110.

2 Нахушев А.М. Нагруженные уравнения и их приложения / А.М. Нахушев. — М.: Наука, 2012.

3 Нахушев А.М. Уравнения математической биологии / А.М. Нахушев. — М.: Высш. шк., 1995. — 205 с.

4 Нахушев А.М. Нагруженные уравнения и их приложения / А.М. Нахушев // Дифференциальные уравнения. — 1983. — Вып. 19 (1). — С. 86–94.

5 Дженалиев М.Т. Нагруженные уравнения как возмущения дифференциальных уравнений / М.Т. Дженалиев, М.И. Рамазанов. — Алматы: Ғылым, 2010. — 334 с.

6 Владимиров В.С. Математические задачи теории переноса односкоростной теории переноса частиц / В.С. Владимиров // Тр. Мат. ин-та им. В.А. Стеклова АН СССР. — 1961. — 61. — 158 с.

7 Krall A.M. The development of general differential and general differential-boundary systems / A.M. Krall // Rocky Mountain J. Math. — 1975. — 5. — No. 4. — P. 493–542. https: // doi.org /10.1216 /RMJ-1975-5-4-493.

8 Lomov I.S. Loaded differential operators: Convergence of spectral expansions / I.S. Lomov // Diff. Equat. — 2014. — 50. — No. 8. — P. 1070–1079 https: // doi.org /10.1134 /S0012266114080060.

9 Lomov I.S. Study of Spectral Properties of a Loaded Second-Order Differential Operator / I.S. Lomov, V.V. Chernov // Diff. Equat. — 2015. — Vol. 51. — No. 7. — P. 857–861.

10 Дженалиев М.Т. О граничной задаче для спектрально нагруженного оператора теплопроводности / М.Т. Дженалиев, М.И. Рамазанов // Сиб. мат. журн. — 2006. — Вып. 47. — № 3. — С. 527–547.

11 Дженалиев М.Т. Оптимальное управление линейными нагруженными параболическими уравнениями / М.Т. Дженалиев // Дифференциальные уравнения. — 1989. — 25. — No. 4. — С. 641–651.

12 Ramazanov M.I. Boundary value problems for essentialy loaded parabolic equation / M.I. Ramazanov, M.T. Kosmakova, V.G. Romanovsky, B.H. Zhanbusinova, Zh.M. Tuleutaeva // Bulletin of the Karaganda University. Mathematics Series. — 2018. — Vol. 4. — No. 92. — P. 79–86.

13 Imanbaev N. To the eigenvalue problems of a special-loaded first-order differential operator / N. Imanbaev, B. Kalimbetov, Z. Khabibullayev // Int. J. Math. Anal. (Ruse). — 2014. — Vol. 8. — No. 45. — P. 2247–2254. http: // dx.doi.org /10.12988 /ijma.2014.48263.

14 Кожанов А.И. Нелинейные нагруженные уравнения и обратные задачи / А.И. Кожанов // Журн. вычислительной математики и мат физики. — 2004. — 44. — № 4. — С. 657–675.

15 Parasidis I.N. An exact solution method for a class of nonlinear loaded difference equa-tions with multipoint boundary conditions / I.N. Parasidis, E. Providas // Journal of Difference Equations and Applications, Taylor — Francis groop, Published on line. — 2018. — No. 05. — P. 1–15.

16 Dzhumabaev D. Computational methods for solving the boundary value problems for the loaded differential and Fredholm integro-differential equations / D. Dzhumabaev / Math. Meth. Appl. Sci. — 2017. — P. 1–24. Available at https: // doi.org /10.1002 /mma.4674.

17 Abdullaev V.M. Numerical method of solution to loaded nonlocal boundary value problems for ordinary differential equations / V.M. Abdullaev, K.R. Aida-Zade // Journal of Computational Mathematics and Mathematical Physics, Springer. — 2014. — Vol. 54. — No. 7. — P. 1096–1109.

18 Aida-Zade K.R. On the numerical solution of loaded system of ordinary differential equations with nonseparated multipoint and integral conditions / K.R. Aida-Zade, V.M. Abdullaev // Numerical Analysis and Applications, Pleiades Publishing, Ltd., 2014. — Vol. 7. — No. 1. — P. 1–14.

19 Alikhanov A.A. Boundary value problems for certain classes of loaded differential equa-tions and solving them by finite difference methods / A.A. Alikhanov, A.M. Berezgov, M.X. Shkhanukov-Lafishev // Comput. Math. and Math. Phys. Pleiades Publishing, Ltd., 2008. — 48. — No. 9. — P. 1581–1590. https: // doi.org /10.1134 /S096554250809008X.

20 Ойнаров Р.О. Корректно разрешимые расширения операторов с конечными дефектами в банаховом пространстве / Р.О. Ойнаров, И.Н. Парасиди // Изв. АН КазССР. — 1988. — No. 5. — С. 42–46.

21 Ойнаров Р.О. Гладкие расширения минимального оператора с конечным дефектом в банаховом пространстве / Р.О. Ойнаров, С.С. Сагинтаева // Изв. НАН РК. Сер. физ.-мат. — 1994. — № 5. — С. 43–48.

22 Крейн М.Г. Теория самосопряженных расширений полуограниченных операторов Эрмита и их приложения / М.Г. Крейн // Мат. сб. — 1947. — 20. — No. 3. — С. 431–495.

23 Neumann J.Von. Allgemeine Eigenwerttheorie Hermitescher Functional operatoren / J.Von Neumann / Math. Ann., Bd. — 1929–1930. — 102. — P. 49–131.

24 Кокебаев Б.К. О сужениях и расширениях операторов / Б.К. Кокебаев, М.О. Отелбаев, А.Н. Шыныбеков // Докл. АН СССР. — 1983. — Вып. 271. — № 6. — С. 1307–1310.

25 Parasidis A.I. Loaded Differential and Fredholm Integro-Differential Equations with nonlocal integral boundary conditions [Electronic resource] / I.N. Parasidis, E. Providas, V. Dafopoulos // Applied Math. and control science. — 2018. — No.3. — P. 31–50. — Access mode: vestnik.pstu.ru.

И.Н. Парасидис

Локальды емес шеттік интегралдық шарттарымен жүктелген дифференциалдық теңдеулер класстары үшін кеңейту әдісі

Мақалада локальды емес шеттік интегралдық шарттарымен жүктелген кәдімгі дифференциалдық теңдеулер класы абстрактілі операторлық теңдеу терминінде зерттелді

$$Bu = A^2u - q\Psi(u) = f, \quad f \in Y, \tag{1}$$

$$D(B) = \{u \in D(A^2) : \Phi(u) = NF(Au), \quad \Phi(Au) = PF(Au)\}.$$

Бұл теңдеулердің жүктелген бөлігі және локальды емес шеттік интегралдық шарттары сәйкесінше $\Psi(u)$ және $F(Au)$, функционалдық векторларының көмегімен сипатталды. Мұндай теңдеулер сызықтық операторларды кеңейту теориясынан шығады. Қарастырылатын теңдеулердің шешілгіштігінің қажетті және жеткілікті шарттары қандай да бір матрицаның анықтауышы көмегімен өрнектеледі. Жүктелген теңдеулердің қарастырылатын класстарының нақты шешуін табу үшін, анықтауыш нольге тең емес жағдайында, дайын формула ұсынылады. Егер қандай да бір есеп (1) түріне келтірілетін болса, онда оны ұсынылып отырған кеңейту әдісімен оңай шешуге болады. Сонымен қоса, $q = \vec{0}$, үшін бұл әдіс шешілгіштіктің қажетті және жеткілікті шарттарын және локальды емес шеттік интегралдық шарттарымен жүктелген кәдімгі дифференциалдық теңдеулер кластарын абстрактілі операторлық теңдеу терминінде нақты шешу үшін формула береді.

Кілт сөздер: жүктелген кәдімгі дифференциалдық теңдеулер, дифференциалдық теңдеулер, локальды емес интегралдық шеттік шарттар, инъективті және дұрыс операторлар, нақты шешулер.

И.Н. Парасидис

Метод расширения для класса нагруженных дифференциальных уравнений с нелокальными граничными интегральными условиями

В статье исследован класс нагруженных обыкновенных дифференциальных уравнений с нелокальными граничными интегральными условиями в терминах абстрактного операторного уравнения

$$Bu = A^2u - q\Psi(u) = f, \quad f \in Y; \tag{1}$$

$$D(B) = \{u \in D(A^2) : \Phi(u) = NF(Au), \quad \Phi(Au) = PF(Au)\}.$$

Нагруженная часть и нелокальные граничные интегральные условия этих уравнений описываются с помощью функциональных векторов $\Psi(u)$ и $F(Au)$ соответственно. Такие уравнения следуют из теории расширений линейных операторов. Необходимые и достаточные условия разрешимости рассматриваемых уравнений выражаются с помощью определителя некоторой матрицы. В случае когда этот определитель ненулевой, предлагается готовая формула для нахождения точного решения рассматриваемого класса нагруженных уравнений. Если некоторая задача может быть приведена к виду (1), то ее можно легко решить предлагаемым методом расширения. Данный метод для $q = \vec{0}$ также дает необходимые и достаточные условия разрешимости и формулу для точного решения для класса обыкновенных дифференциальных уравнений с нелокальными граничными интегральными условиями в терминах абстрактного операторного уравнения

$$Bu = A^2u = f; \quad D(B) = \{u \in D(A^2) : \Phi(u) = NF(Au); \quad \Phi(Au) = PF(Au)\}; \quad f \in Y.$$

Ключевые слова: нагруженные обыкновенные дифференциальные уравнения, дифференциальные уравнения, нелокальные интегральные граничные условия, инъективные и корректные операторы, точные решения.

References

1 Nakhushev, A.M., & Borisov, V.N. (1977). Kraevye zadachi dlia nahruzhennykh parabolicheskikh uravnenii i ikh primenenie k prohnozu urovnia hruntovykh vod [Boundary value problems for loaded parabolic equations and their application to forecast of groundwater level]. *Differentsialnye uravneniia — Differential equations, 13, 1*, 105–110 [in Russian].

2 Nakhushev, A.M. (2012). *Nahruzhennye uravneniia i ikh prilozheniia [Loaded equations and their applications].* Moscow: Nauka [in Russian].

3 Nakhushev, A.M. (1995). *Uravneniia matematicheskoi biolohii [Equations of Mathematical Biology].* Moscow: Vysshaia shkola [in Russian]

4 Nakhushev, A.M. (1983). Nahruzhennye uravneniia i ikh prilozheniia [Loaded equations and their applications]. *Differentsialnye uravneniia — Differential equations, 19, 1,* 86–94 [in Russian].

5 Dzhenaliev, M.T., & Ramazanov, M.I. (2010). *Nahruzhennye uravneniia kak vozmushcheniia differentsialnykh uravnenii [Loaded equations as perturbations of differential Equations].* Almaty: Hylym [in Russian].

6 Vladimirov, V.S. (1961). Matematicheskie zadachi odnoskorostnoi teorii perenosa chastits [Mathematical problems in one-speed theory of particle transport]. *Trudy Matematicheskoho instituta imeni V.A. Steklova Akademii nauk SSSR — Proceeding of V.A. Steklov Math. Inst., 61,* 3–158 [in Russian].

7 Krall, A.M. (1975). The development of general differential and general differential-boundary systems *Rocky Mountain J. Math., 5, 4,* 493–542. Retrieved from https://doi.org/10.1216/RMJ-1975- 5-4-493.

8 Lomov, I.S. (2014). Loaded differential operators: Convergence of spectral expansions. *Differential equations, 50, 8,* 1070–1079. Retrieved from https://doi.org/10.1134/S0012266114080060.

9 Lomov, I.S., & Chernov, V.V. (2015). Study of Spectral Properties of a Loaded Second-Order Differential Operator. *Differential equations, 51, 7,* 857–861.

10 Dzenaliev, M.T., & Ramazanov, M.I. (2006). O hranichnoi zadache dlia spektralno-nahruzhennoho operatora teploprovodnosti [On the boundary value problem for the spectrally loaded heat condition operator]. *Sibirskii matematicheskii zhurnal — Sibirian Math. Journal, 47, 3,* 527–547 [in Russian].

11 Dzhenaliev, M.T. (1989). Optimalnoe upravlenie lineinymi nahhruzhennymi parabolicheskimi uravneniiami [Optimal control of linear loaded parabolic equations]. *Differentsialnye uravneniia — Differential equations, 25, 4,* 641–651 [in Russian].

12 Ramazanov, M.I., Kosmakova, M.T., Romanovsky, V.G., Zhanbusinova, B.H., & Tuleutaeva, Zh.M. (2018). Boundary value problems for essentialy loaded parabolic equation. *Bulletin of the Karaganda University. Mathematics Series, 4, 92,* 79–86.

13 Imanbaev, N., Kalimbetov, B., & Khabibullayev, Z. (2014). To the eigenvalue problems of a special-loaded first-order differential operator. *Int. J. Math. Anal. (Ruse), 8, 45,* 2247–2254. Retrieved from http://dx.doi.org/10.12988/ijma.2014.48263.

14 Kozhanov, A.I. (2004). Nelineinye nahruzhennye uravneniia i obratnye zadachi [Nonlinear Loaded Equations and Inverse Problems]. *Zhurnal vychislitelnoi matematiki i matematicheskoi fiziki — J. Comput. Math. Math. Phys, 44. 4,* 657–675 [in Russian].

15 Parasidis, I.N., & Providas, E. (2018). An exact solution method for a class of nonlinear loaded difference equations with multipoint boundary conditions. *Journal of Difference Equations and Applications, Taylor — Francis groop, Published on line, 05,* 1–15.

16 Dzhumabaev, D. (2017). Computational methods for solving the boundary value problems for the loaded differential and Fredholm integro-differential equations. *Math. Meth. Appl. Sci.*, 1–24. Retrieved from https://doi.org/10.1002/mma.4674.

17 Abdullaev, V.M., Aida-Zade, K.R. (2014). Numerical method of solution to loaded nonlocal boundary value problems for ordinary differential equations. *Journal of Computational Mathematics and Mathematical Physics, Springer, 54, 7,* 1096–1109.

18 Aida-Zade, K.R., & Abdullaev, V.M. (2014). On the numerical solution of loaded system of ordinary differential equations with nonseparated multipoint and integral conditions. *Numerical Analysis and Applications, Pleiades Publishing, LTD., 7, 1,* 1–14.

19 Alikhanov, A.A., Berezgov, A.M., & Shkhanukov-Lafishev, M.X. (2008). Boundary value problems for certain classes of loaded differential equations and solving them by finite difference methods. *Comput. Math. and Math. Phys., Pleiades Publishing, LTD., 48, 9,* 1581–1590. https://doi.org/10.1134 /S09655425 0809008X.

20 Oinarov, R.O., & Parasidi, I.N. (1988). Korrektno razreshimye rasshireniia operatorov s konechnymi defektami v banakhovom prostranstve [Correct extensions of operators with finite defect in Banach spases]. *Izvestiia AN KazSSR, 5,* 42–46 [in Russian].

21 Oinarov, R.O., & Sagintaeva, S.S. (1994). Hladkie rasshireniia minimalnoho operatora s konechnym defektom v banakhovom prostranstve [Smooth extensions of a minimal operator with a finite defect in a Banach space]. *Izvestiia NAN RK. Seriia Fiz.-mat., 5,* 43–48 [in Russian].

22 Krein, M.G. (1947). Teoriia samosopriazhennykh rasshirenii poluohranichennykh operatorov Ermita i ikh prilozheniia [The theory of self-adjoint extensions of semi-bounded Hermitian operators and its aplications] *Matematicheskii sbornik — Math. Notes, 20, 3,* 431–495 [in Russian].

23 J.von Neumann. (1929–1930). Allgemeine Eigenwerttheorie Hermitescher Functional operatoren. *Math. Ann., Bd., 102,* 49–131.

24 Kokebaev, B.K., Otelbaev, M., & Shynybekov, A.N. (1983). O suzheniiakh i rasshireniiakh operatorov [About Restrictions and Extensions of operators]. *Doklady AN SSSR — Reports of the USSR Academy of Sciences, 271, 6,* 1307–1310 [in Russian].

25 Parasidis, I.N., Providas, E., & Dafopoulos, V. (2018). Loaded Differential and Fredholm Integro-Differential Equations with nonlocal integral boundary conditions. *Applied Math. and control science, vestnik.pstu.ru., 3,* 31–50.

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Chains of existentially closed models of positive (n_1, n_2)-Jonsson theories

In this article are considered model-theoretical properties of chains of positive (n_1, n_2)-Jonsson theories. Herewith considered theories is perfect in the sense of the existence of appropriate model companion. The main obtained results are as follows: introduced new concepts n_2-elimination of quantifier for positive theory, (n_1, n_2)-Jonsson theory, n_1-Jonsson chain; indicated a feature of perfect (n_1, n_2)-positive Jonsson theory.

Keywords: (n_1, n_2)-Jonsson theory, positive (n_1, n_2)-Jonsson theory, n_2-elimination of quantifier, n_1-Jonsson chain.

This article is devoted to the study of positive properties of (n_1, n_2)-Jonsson theories. At the study of positive Jonsson theories in the framework of [1], were considered various aspects of positivity which appear at translating of properties of Jonsson theories into the language of positivity [2].

In this paper, we continue to study (n_1, n_2)-Jonsson theories with the condition of positive axiomatizability, and instead of morphisms we consider immersions, special cases of homomorphisms [1]. The indices n_1, n_2 respectively determine n_1-model completeness, n_2-elimination of quantifiers. These indices are related to the concepts of chains of theories under consideration.

Many classical examples from algebra satisfy the natural axioms that define Jonsson theories. On Jonsson theories, more detailed information can be extracted in the monograph [3] and in the works [4–8]. We consider groups, Abelian groups, Boolean algebras, fields of fixed characteristic, rings, modules, polygons. All of these types of algebras are examples of such algebras whose theories are Jonsson. Abelian groups and Boolean algebras are good because their invariants are known, that is, some constants that allow us to describe these types of algebras up to elementary equivalence. In Jonsson theories, the concepts of elementary equivalence are replaced by cosemanticness. In the works [9, 10] are presented results that describe Abelian groups and modules up to cosemanticness. In the case of Boolean algebras the work [11] is known were defined generalized Jonsson theories (α-Jonsson) were defined and a description of α-Jonson theories of Boolean algebras was developed on the basis of the apparatus developed in the framework of the definitions of this article. Regardless of the study of Jonsson theories, in [12] the authors considered various generalizations of the concepts of model completeness and elimination of quantifiers. Also, within the framework of these generalizations, n-chains were considered in [12] which naturally appear as a consequence of the n-inductance of the theories under consideration. In fact, all of the above concepts definitely are related to the central concept of Robinson model theory, namely, the concept of a model companion. Well known the achievements of A. Robinson and his followers related to the application of theorems on the properties of model companions of the considered theories for classical algebraic objects, such as fields of fixed characteristic, Boolean algebras, Abelian groups, rings, modules, etc. Unfortunately, not all inductive theories have a model companion. It is well known from [13] that a theory has a model companion iff the class of its existentially closed models is elementary. In particular, the Jonsson theories that have a model companion are reasonably well arranged in the sense that their semantic invariant is a saturated model. Also in [14], was considered a generalization of the concept of a model companion, but somewhat from a different perspective. Therefore, in view of the foregoing, it would be interesting to redefine some of the concepts from works [11, 12] and [14], in the framework of the study of positive Johnson theory, where all the above properties of companions of Jonsson theories will be considered in the framework of positivity, which will be defined in this work.

Now we want to define the notion of (n_1, n_2)-positive Jonsson's theories.

Let L be a first-order language. At is a set of atomic formulas of the language. $B^+(At)$ is a closed set with respect to the positive Boolean combinations (conjunction and disjunction) of all atomic formulas and their subformulas and substitution variables. $Q_n(B^+(At))$ is a set of formulas in prenix normal form obtained by the use of quantifiers ($\forall$ and $\exists$) to $B^{+}(At)$, where n is a number of quantifier changes in prenix. We call the formula positive if it belongs to $Q_n(B^{+}(At))$. The theory is positively axiomatizable if its axioms are positive.

Following [15] we define the Δ-morphisms between structures.

Let M and N, the structure of language, $\Delta \in Q_n(B^{+}(At))$. The mapping $h : M \to N$ is called Δ-homomorphism (in symbols $h : M \underset{\Delta}{\longrightarrow} N$) if for any $\varphi(\overline{x}) \in \Delta$, $\forall \overline{a} \in M$ from the fact that $M \models \varphi(\overline{a})$, it follows that $N \models \varphi(h(\overline{a}))$.

Following [2], the model M is said to begin in N and we say that M continues to N, with $h(M)$ is a continuation of M. If the map h is injective, we say that h immersion from M into N (symbolically h : $M \underset{\Delta}{\longleftrightarrow} N$). In the following we will use the terms Δ-continuation and Δ-immersion. In frames of this definition (Δ-homomorphism), it is easy to see that an isomorphic embedding, and an elementary embedding are Δ-immersions, when $\Delta = B(At)$ and $\Delta = L$, respectively.

Definition 1 [15]. If C - class of L-structures, then we say that an element M of C Δ-positive existential closed in C if every Δ-homomorphism from M to any element of C is a Δ-immersion. The class of all existential positive Δ-closed models will be denoted by $(E_C^{\Delta})^{+}$; if $C = ModT$ for some theory T, then by E_T, $(E_T^{\Delta})^{+}$ we mean, respectively, the class of existential closed and Δ-positive existential of the closed models of theory T.

Definition 2 [15]. We say that theory T admits a $\Delta - JEP$, if for any two $A, B \in ModT$ exists $C \in ModT$ and Δ-homomorphisms $h_1 : A \underset{\Delta}{\longrightarrow} C$, $h_2 : B \underset{\Delta}{\longrightarrow} C$.

Definition 3 [15]. We say that theory T admits a of $\Delta - AP$, if for any $A, B, C \in ModT$ such that $h_1 : A \underset{\Delta}{\longrightarrow} C$, $g_1 : A \underset{\Delta}{\longrightarrow} B$, where h_1, g_1 – Δ-homomorphisms, there exists a $D \in ModT$ and $h_2 : C \underset{\Delta}{\longrightarrow} D$, $g_2 : B \underset{\Delta}{\longrightarrow} D$, where h_2, g_2 – Δ-homomorphisms such that $h_2 \circ h_1 = g_2 \circ g_1$.

Definition 4 [15]. The theory T is called Δ-positive Jonsson's ($\Delta - PJ$)-theory if it satisfies the following conditions:

1) T has an infinite model;
2) T positive $\forall\exists$-axiomatizable;
3) T admits $\Delta - JEP$;
4) T admits $\Delta - AP$.

When $\Delta = B(At)$ we obtain the usual Jonsson's theory, the only difference that it has only positive $\forall\exists$-axiom.

In future, by (n_1, n_2)-positive Jonsson theory we understand some Δ-positive Jonsson theory ($\Delta - PJ$) which n_1-positive model complete and n_2-positive elimination of quantifiers (E.Q.). Positive model completeness is model completeness of theory without using a symbol $\neg$. Positive E.Q. is a redaction to positive quantifiers-free formulas.

We give the necessary definitions of concepts which we will use.

A concept of model completeness introduced by A. Robinson is played large role in the study of model companions of various types of classical algebras.

Definition 5 [12]. A theory T is model complete if for any $B, D \in ModT$ and B is a submodel of D, then $B \prec D$.

Definition 6 [13]. A theory T is called model companion of T if:

1) T and T^* are mutually model consistent;
2) T^* is model complete.

The following fact is well known, connecting the theory with model companion with respect to E.Q.

Lemma 1 [13]. 1) Let T^* be model companion of theory T, where T is universal theory. In this case T^* is model completion of T iff T^* admit quantifier elimination.

2) Let T^* be model companion of theory T. In this case T^* is model completion of T iff T^* has amalgam property.

In the framework of concept of n-embedding, following work [11], we have n-Jonsson theory that differs from concept of generalized α-Jonsson theory only a semantic model [16]. Continuing to work within the framework of n-Jonsson theory we can fully define the concept n-model completeness, n-model companion and correspondingly on this base to prove the theorem which is a criterion of perfectness of generalized α-Jonsson theory [from 11] in following view.

Proposition 1 [11]. Let T be arbitrary n-Jonsson theory, then the following conditions are equivalent:

1) T is perfect;
2) T^* is n-model companion of T.

In work [12] are given following definitions generalizing a concepts of model completeness and E.Q.

Definition 7 [12]. $B \subseteq_n D$ means that $B \models \psi(\bar{b})$ if $D \models \psi(\bar{b})$ for all $\bar{b} \in B$ and all Σ_n (or Π_n) formulas ψ.

Definition 8 [12]. 1. A theory T is n-model complete if for all models B, D of T, $B \subseteq_n D$ implies $B \prec D$.

2. T is nearly n-model complete if any formula is equivalent (mod T) to Boolean combination of Σ_{n+1} (or Π_{n+1}) formulas.

3. A chain $\{B_n\}_{n\in\omega}$ under $\subseteq_n$ is called a n-chain.

Definition 9. We will call the theory n_2-E.Q. if any formula of this theory is presented in the view of Boolean combination of formulas from $Q_{n_2}(B(At))$.

Combining these concepts in the framework of study of Jonsson theories we can distinguish the following class of Jonsson theories:

Definition 10. Theory T is called (n_1, n_2)-Jonsson theory if it is n_1-model complete and n_2-E.Q.

In work [12] was determined the next kind of chain of models for some theory and statements with respect to this concept as 1-model completeness.

Definition 11 [12]. A chain $\{B_k\}_{k\in\omega}$ is eventually elementary if for all $\bar{b} \in \cup_{k\in\omega} B_k$ and all formulas $\psi(\bar{x})$ exists some $k_0 \in \omega$ such that either $B_k \models \psi(\bar{b})$ for every $k \geq k_0$ or $B_k \models \neg\psi(\bar{b})$ for every $k \geq k_0$.

Proposition 2 [12]. A theory T is 1-model complete iff for any formulas $\psi(\bar{x})$ exists a formula $\varphi(\bar{x})$ which is a $\forall_2 \cap \exists_2$-formulas such that $T \models \forall\bar{x}[\psi \leftrightarrow \varphi]$.

If the chain $\{B_k\}_{k\in\omega}$ is not eventually elementary, then there are $\psi(\bar{x})$ and $\bar{b}$ such that $B_k \models \psi(\bar{b})$ and $B_m \models \neg\psi(\bar{b})$ both hold for arbitrarily large $k, m \in \omega$. Thus, such a chain can be refined to be an alternating chain for $\psi(\bar{x})$ as in the following definition.

Definition 12 [12]. A chain $\{B_k\}_{k\in\omega}$ is an alternating chain for $\psi(\bar{x})$ if there is some $\bar{b} \in B_0$ such that $B_{2k} \models \psi(\bar{b})$ and $B_{2k+1} \models \neg\psi(\bar{b})$ for all $k \in \omega$.

Proposition 3. A theory T is 1-E.Q. iff for any formulas $\psi(\bar{x})$ exists a formula $\varphi(\bar{x})$ which is Boolean combination of $\forall_1 \cap \exists_1$-formulas such that $T \models \forall\bar{x}[\psi \leftrightarrow \varphi]$.

Recall the basic definitions and statements from work [11], in which are determined all basic concepts related with α-Jonsson theories. In our case, $\alpha = n_1$.

Definition 13 [11]. Let $\Gamma \subset L$. Then

1) map $f : A \to B$ is called a Γ-embedding if for any $\bar{a} \in A$ and $\varphi(\bar{x}) \in \Gamma$ and $A \models \varphi(\bar{a})$ follows $B \models \varphi(f(\bar{a}))$;

2) if $B \subseteq \mid A \mid$, then $Th_\Gamma(A, B)$ denote a set of all Γ-sentence of language L_B which are true in A;

3) if $A \subseteq B$, then notation $A \subseteq_\Gamma B$ denote that $Th_\Gamma(A, |A|) \subseteq Th_\Gamma(B, |B|)$;

4) sequence of models A_i, $i < \beta$, is called a Γ-chain if $A_i \subseteq_\Gamma A_j$ at $i < j < \beta$.

Lemma 2 [11]. map $f : A \to B$ is a Π_α-embedding iff it is a $\Sigma_{\alpha+1}$-embedding.

Definition 14 [11]. Theory T preserved with respect to the union of Π_α-chains (or α-inductive) if an union of any Π_α-chain of model of T again is a model of T.

Definition 15 [11]. A model $M \models T$ is called $\Sigma_{\alpha+1}$-saturated model if for any subset $E \subseteq \mid M \mid$ less power than M, for any model $B \models T$ such that $M \subseteq_{\Pi_\alpha} B$, and any element $b \in B$ exists element $m \in M$ satisfying an inclusion $Th_{\Sigma_{\alpha+1}}(M, E \cup m) \supseteq Th_{\Sigma_{\alpha+1}}(B, E \cup b)$.

Theorem 1 [11]. Let T be α-Jonsson theory, $M \models T$. Then the following conditions are equivalent:

1) M is $T - \alpha$-universal $T - \alpha$-homogeneous model;

2) M is $\Sigma_{\alpha+1}$-saturated model.

Proposition 4 [11]. For α-Jonsson theory T the following conditions are equivalent:

1) T is perfect;

2) T^* is α-model completion (i.e. $D(T^*) - \alpha$-model completion) of T.

Definition 16 [11]. A model $A \models T$ is called $\Sigma_{\alpha+1}$-closed if for any model $B \models T$ and any formula $\varphi(\bar{x}) \in \Sigma_{\alpha+1}$ with constants from A is performed $A \models \exists\bar{x}\varphi(\bar{x})$ provided that $A \subseteq_{\Pi_\alpha} B$ and $B \models \exists\bar{x}\varphi(\bar{x})$.

In future, a set $\Sigma_{\alpha+1}$-closed models of theory T denote by $\Sigma_{n_1}(T)$.

Definition 17. A chain $\{B_k\}_{k\in\omega}$ is called n_1-Jonsson if it will consist only from models $\Sigma_{n_1}(T)$.

The following facts are well known.

Lemma 3 [17]. Let A_β, $\beta < \alpha$, be some Σ_n-chain of models and $A = \cup_{\beta<\alpha} A_\beta$. Then

1) a model A is a Σ_n-extension of every model A_β;

2) any Π_{n+1}-sentence which is true in all models A_β, is true in A.

Theorem 2 [17]. The following statements are equivalent (where $n > 0$):

1) sentence φ equivalent to both some Σ_{n+1} and some Π_{n+1};

2) sentence φ equivalent to some Boolean combination Σ_n.

The result on the stability of theory is well known with respect to arbitrary chain of her models. Such theory must be $\forall\exists$-axiomatizable. The following definition is considered a generalization of concept of induction, i.e. stability with respect to a chain of models.

Definition 18 [17]. A theory T is stable with respect to a union Π_α-chains (or α-inductive) if a union of any Π_α-chain of models T is again a model of T.

The following result is known in the connection with the above definition.

Proposition 5 [17]. The following conditions are equivalent:

1) $T \in \Pi_{\alpha+2} C_\Delta$;

2) a theory T is α-inductive.

A class of models of her center in the connection with perfectness α-Jonsson theory coincides with a class of her $\Sigma_{\alpha+1}(T)$-models.

Lemma 4 [17]. Let T be perfect α-Jonsson theory, T^* be her center, $A \models T$. Then $A \in \Sigma_{\alpha+1}(T) \Leftrightarrow A \models T^*$.

In future, we will work in the case when the theory n_1-model complete and so that to use the result (Theorem 2) in the definition of $\Delta - PJ$-theory we suppose that Δ is contained in the $Q_{n_1}(B^+(At))$, where $Q_{n_1} = \Sigma_{n_1} \cap \Pi_{n_1}$.

The following theorem is positive generalization of theorem (Theorems 2.6 and 2.7 from [12]) in the framework of study of (n_1, n_2)-positive Jonsson theories.

Theorem 3. Let T be perfect (n_1, n_2)-positive Jonsson theory, where $n_1 = n_2 + 1; n_1, n_2 > 0$. Then the following conditions are equivalent:

1) T is n_2-E.Q.;

2) any n_2-chain of models which belong to $\Sigma_{n_2}(T)$ is Jonsson eventually elementary;

3) T is n_1-model complete;

4) any n_2-chain of models, which belong to $\Sigma_{n_2}(T)$, whose union is also a model of $\Sigma_{n_2}(T)$, is Jonsson eventually elementary.

It is easy to see that since all homomorphisms are immersions, we can use all the necessary results from work [11] without loss of generality for positive (n_1, n_2)-Jonsson theory.

Essential point in the proof of following result is a perfectness of Jonsson theory in the sense of work [11].

Proof. A equivalence of the aforecited statements follows from following three equivalences:

a) (2) equivalent to (4) follows from Proposition 5 and Lemma 4;

b) from (1) in (3) proof is obvious;

c) from (3) in (1) follows from Theorem 2;

d) (1) equivalent to (2) follows from Theorem 2.6. [12].

References

1 Poizat B. Positive Jonsson theories / B. Poizat, A.R. Yeshkeyev // Logica universalis. — 2018. — Vol. 12. — No. 1, 2. — P. 101–127.

2 Yaacov I.B. Foundations of positive logic / I.B. Yaacov, B. Poizat // Journal of Symbolic Logic. – 2007. — Vol. 72. — No. 4. — P. 1141–1162.

3 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд–во КарГУ, 2016. — 346 с.

4 Yeshkeyev A.R. Strongly minimal Jonsson sets and their properties / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 80. — No. 4. — P. 47–51.

5 Yeshkeyev A.R. Properties of of the existential formulas of Jonsson fragments / A.R. Yeshkeyev, M.T. Kasymetova // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79. — No. 3. — P. 25–32.

6 Yeshkeyev A.R. On lattice of existential formulas for fragment of Jonsson set / A.R. Yeshkeyev, O.I. Ulbrikht // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79. — No. 3. — P. 33–39.

7 Yeshkeyev A.R. dcl-cl-atomic and prime sets / A.R. Yeshkeyev, A.K. Issayeva // Bulletin of the Karaganda University. Mathematics series. — 2019. — Vol. 93. — No. 1. — P. 88–94.

8 Yeshkeyev A.R. Properties of hybrids of Jonsson theories / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University. Mathematics series. — 2018. — Vol. 92. — No. 4. — P. 99–104.

9 Ешкеев А.Р. JSp-косемантичность и JSB-свойство абелевых групп / А.Р. Ешкеев, О.И. Ульбрихт // Сибирские электронные математические известия. — 2016. — Т. 13. — С. 861–874.

10 Ешкеев А.Р. JSp-косемантичность R-модулей / А.Р. Ешкеев, О.И. Ульбрихт // Сибирские электронные математические известия. — 2019. — Т. 16. — С. 1233–1244.

11 Мустафин Т.Г. Обобщенные условия Йонсона и описание обобщенно-йонсоновских теорий булевых алгебр / Т.Г. Мустафин // Матем. тр. — 1998. — № 2. — С. 135–197.

12 Kueker D.W. Nearly model complete theories / D.W. Kueker, P. Turnquist // Mathematical Logic Quarterly. — 1999. — No 3. — P. 291–298.

13 Теория моделей: справочная книга по математической логике: [в 4–х ч.]. — Ч.1. / под ред. Дж. Барвайса. — М.: Наука, 1982. — 392 с.

14 Weispfenning V. Model theory and lattices of formulas. Patras logic symposion / V. Weispfenning. — North–Holland Publishing Company, 1982. — P. 261–295.

15 Yeshkeyev A.R. The structure of lattices of positive existential formulae of $\Delta - PJ$-theories / A.R. Yeshkeyev // Science Asia. — 2013. — 39s. — P. 19–24.

16 Mustafin Ye.T. Quelques properties des theories de Jonsson / Ye.T. Mustafin // The Journal of Symbolic logic. — 2002. — No. 2. — P. 528–536.

17 Кейслер Г. Теория моделей: учеб. пос. / Г. Кейслер, Ч.Ч. Чэн; под ред. Ю.Л. Ершова, А.Д. Тайманова. — М.: Мир, 1977. — 614 с.

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Позитивті (n_1, n_2)-йонсондық теориялардың экзистенциалды-тұйық модельдерінің тізбелері

Мақалада (n_1, n_2)-йонсондық теориялар тізбелерінің модельді-теоретикалық қасиеттері қарастырылған. Сонымен қатар қарастырылып отырған теориялар модельді компаньонның мағынасына сәйкес келеді. Негізгі алынған нәтижелер мынадай: жаңа ұғымдар енгізілді; n_2 позитивті теориялар үшін n_2-кванторлар элиминациясы, (n_1, n_2)-йонсондық теориясы; n_1-йонсондық тізбе; (n_1, n_2)-позитивті йонсондық теориясының ерекшелігі көрсетілген.

Кілт сөздер: (n_1, n_2)-йонсондық теория, позитивті (n_1, n_2)-йонсондық теория, n_2-кванторлар элиминациясы, n_1-йонсондық тізбе.

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Цепи экзистенциально-замкнутых моделей позитивных (n_1, n_2)-йонсоновских теорий

В статье рассмотрены теоретико-модельные свойства цепей позитивных (n_1, n_2)-йонсоновских теорий. При этом рассматриваемые теории являются совершенными в смысле существования соответственного модельного компаньона. Основные полученные результаты следующие: введены новые понятия; n_2-элиминация кванторов для позитивной теории, (n_1, n_2)-йонсоновская теория, n_1-йонсоновская цепь; указана особенность совершенной (n_1, n_2)-позитивной йонсоновской теории.

Ключевые слова: (n_1, n_2)-йонсоновская теория, позитивная (n_1, n_2)-йонсоновская теория, n_2-элиминация кванторов, n_1-йонсоновская цепь.

References

1 Poizat, B., & Yeshkeyev, A.R. (2018). Positive Jonsson theories. *Logica universalis, Vol. 12, 1–2,* 101–127.

2 Yaacov, I.B., & Poizat, B. (2007). Foundations of positive logic. *Journal of Symbolic Logic, Vol. 72, 4,* 1141–1162.

3 Yeshkeyev, A.R., & Kassymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models].* Karaganda: Izdatelstvo Karahandinskoho hosudarstvennoho universiteta [in Russian].

4 Yeshkeyev, A.R. (2015). Strongly minimal Jonsson sets and their properties. *Bulletin of the Karaganda University. Mathematics series, Vol. 80, 4,* 47–51.

5 Yeshkeyev, A.R., & Kasymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3,* 25–32.

6 Yeshkeyev, A.R., & Ulbrikht, O.I. (2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3,* 33–39.

7 Yeshkeyev, A.R., & Issayeva, A.K. (2019). dcl-cl-atomic and prime sets. *Bulletin of the Karaganda University. Mathematics series, Vol. 93, 1,* 88–94.

8 Yeshkeyev, A.R., & Mussina, N.M. (2018). Properties of hybrids of Jonsson theories. *Bulletin of the Karaganda University. Mathematics series, Vol. 92, 4,* 99–104.

9 Yeshkeyev, A.R., & Ulbrikht, O.I. (2016). JSp-kosemantichnost i JSB svoistvo abelevykh hrupp [JSp-cosemantic and JSB property of abelian groups]. *Sibirskie elektronnye matematicheskie izvestiia - Siberian Electronic Mathematical Reports, Vol. 13,* 861–874 [in Russian].

10 Yeshkeyev, A.R., & Ulbrikht, O.I. (2019). JSp-kosemantichnost R-modulei [JSp-cosemanticness of R-modules]. *Sibirskie elektronnye matematicheskie izvestiia – Siberian Electronic Mathematical Reports, Vol. 16,* 1233–1244 [in Russian].

11 Mustafin, T.G. (1998). Obobshchennye usloviia Ionsona i opisanie obobshchenno-ionsonovskikh teorii bulevykh alhebr [Generalized conditions of Jonsson and description of generalized Jonsson theories of Boolean algebras]. *Matematicheskie trudy – Mathematical works, 2,* 135–197 [in Russian].

12 Kueker, D.W. & Turnquist, P. (1999). Nearly model complete theories. *Mathematical Logic Quarterly, 3,* 291–298.

13 Barwise, Ed.J. (1982). *Teoriia modelei [Model theory].* Ed.J. Barwise, Yu.L. Yershov (Ed.). (Parts 1–4). Moscow: Nauka [in Russian].

14 Weispfenning, V. (1982). Model theory and lattices of formulas. *Patras logic symposion,* 261–295. North-Holland Publishing Company.

15 Yeshkeyev, A.R. (2013). The structure of lattices of positive existential formulae of $\Delta - PJ$-theories. *ScienceAsia, 39s.,* 19–24.

16 Mustafin, Ye.T. (2002). Quelques properties des theories de Jonsson. *The Journal of Symbolic logic, 67:2,* 528–536.

17 Keisler, G., & Chang, C.C. (1977). *Teoriia modelei [Model theory].* Moscow: Mir [in Russian].

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Companions of (n_1, n_2)-Jonsson theory

In given work are considered model-theoretical properties of companions of (n_1, n_2)-Jonsson theory. Also were considered a communications between center and (n_1, n_2)-Jonsson theory. Herewith considered theories is perfect in the sense of the existence of appropriate model companion. In given article introduced new concepts: (n_1, n_2)-Jonsson theory, $D-\alpha$-model companion. New results are shown with respect to model companions of α-Jonsson theory and 1-perfect 1-Jonsson theory.

Keywords: Jonsson theory, model complete, n-model complete, nearly n-model complete, $D-\alpha$-model companion, model completion, (n_1, n_2)-Jonsson theory.

In the work [1] were determined generalized Jonsson theories and in the language of the introduced definitions were given a descriptions of all generalized Jonsson theories of Boolean algebras. In [2] were defined the concepts of 1-model completeness and near model completeness, and in this work, it was pointed out about the possibility of transferring these two concepts to arbitrary n. In fact, there is a direct connection between these two works are related to model completeness, thereby determining a utensils of essence of given works to one of classical directions in the model theory, determined by the studies of Abraham Robinson. In general, the development of this subjects after the beginning of the 70 was naturally suspended due to the fact that the main trends of development of model theory were based on technology and concepts, related to the study of complete theories. On the other hand, the main ideas of Robinson's directions relate to the study of inductive theories, which generally are not complete. The special subclass of inductive theories is the class of Jonsson theories. Toward this class can be attributed basic algebraic examples of theories, which are play important role in the various modern sections of mathematics. For example, theories of groups, theories of Abelian groups, theories of fields of fixed characteristic, theories of Boolean algebras, theories of polygons, etc. The given examples of theories show the relevance of studying model-theoretical properties of Jonsson theories.

In the classic textbook, in the form of reference book [3] can find the definition of Jonsson theory, later on, the study of Jonsson theories was developed in the following list of works: [4–6].

In [1], the concept of a semantic model is used in a substantial way, and this concept by its existence is connected with an additional axiom about the existence of a strongly unattainable cardinal to the existing system of axioms of Zermelo-Frenkel set theory. In the work [7] a new definition of the semantic model was given, in the framework of which developed and develops the further research of Jonsson theories [8–10]. In this definition of the semantic model in work [7] is no requirement about the existence of a strongly unattainable cardinal. The greatest success at the study of Jonsson theories can be achieved in case of saturation of semantic model. As it was determined, in [11] such theories, by analogy with [1], will be called perfect Jonsson theories. These theories have many good semantic properties related to the theory. For example, the theory of algebraically closed fields of a fixed characteristic is model companion of theory of fields of the same characteristic. A concept of model companion was defined by A. Robinson and this concept is closely related with the concept of model completeness.

In connection with the above, we want to consider the properties of Jonsson theories, which use the concept of n-model completeness from work [2] for some n and the corresponding concept of a model companion, using the results from [1], but within the framework of a new approach to Jonsson theories [11–14] using the new definition of semantic model and companions in study of model completeness from [2].

We give the necessary definitions of concepts which we will use.

We will start with definition of Jonsson theory.

Definition 1 [11]. A theory T is called Jonsson if the following conditions are satisfied:

1) T has infinite models;
2) T is inductive;
3) T has the joint embedding property (JEP);

4) T has the amalgam property (AP).

Following definitions (1-5) and facts (theorems 1-2) allows the reader to get acquainted with the inner structure of semantic model as part of the definition from [7].

Definition 2 [7]. Let $\kappa \geq \omega$. A model $\mathfrak{M}$ of theory T is called κ-universal for T if every model T of strictly less power κ isomorphically embedded in $\mathfrak{M}$.

Definition 3 [7]. Let $\kappa \geq \omega$. A model $\mathfrak{M}$ of theory T is called κ-homogeneous for T if for any two models $\mathfrak{A}$ and $\mathfrak{A}_1$ of theory T, which are submodels of $\mathfrak{M}$, power strictly less, than κ, and isomorphic $f : \mathfrak{A} \to \mathfrak{A}_1$, for every extension $\mathfrak{B}$ of model $\mathfrak{A}$, which is submodel of $\mathfrak{M}$ and model of T of strictly less power κ exists extension $\mathfrak{B}_1$ of model $\mathfrak{A}_1$, which is submodel of $\mathfrak{M}$, and isomorphic $g : \mathfrak{B} \to \mathfrak{B}_1$, continuing f.

Definition 4 [7]. κ-homogeneous-universal model for theory T of power κ, where $\kappa \geq \omega$, is called homogeneous-universal model for T.

Theorem 1 [7]. Every Jonsson theory T has κ^+-homogeneous-universal model of power 2^κ. Inversely, if T is inductive, has infinite model and has ω^+-homogeneous-universal model, then a theory T is Jonsson theory.

Theorem 2 [7]. Let T be a Jonsson theory. Two models $\mathfrak{M}$ and $\mathfrak{M}_1$ κ-homogeneous-universal for T are elementary equivalent.

Definition 5 [7]. ω^+-homogeneous-universal model of theory T is called semantic model C_T of Jonsson theory T.

For any Jonsson theory a semantic model always exists, therefore it plays an important role as a semantic invariant.

From definition of semantic model follows that:

Proposition 1 [15]. Any two semantic models of Jonsson theory T is elementary equivalent between themselves.

Lemma 1 [15]. Semantic model C_T of Jonsson theory T is T-existential closed.

Definition 6 [15]. Elementary theory of semantic model C of Jonsson theory T is called semantic completion (center) T^* of this T, i.e. $T^* = Th(C_T)$.

As we have already noticed, greatest progress in learning of Jonsson theories, as a rule, can be achieved provided that of perfection of Jonsson theory.

Definition 7 [15]. A Jonsson theory T is called perfect if every semantic model of T is saturated model of T^*.

It is well know, that only at work with perfect Jonsson theories a class of existential closed models of considered theory is elementary.

Theorem 3 [15]. Let E_T be a class of all existential closed models of theory T. If a Jonsson theory T is perfect, then $E_T = \mathrm{Mod}\, T^*$, where $T^* = Th(C_T)$.

A concept of model completeness introduced by A. Robinson is played large role in the study of model companions of various types of classical algebras.

Definition 8 [2]. A theory T is model complete if for any $B, D \in ModT$ and B is a submodel of D, then $B \prec D$.

Definition 9 [2]. $B \subseteq_1 D$ satisfied if B is a submodel of D, for every $\forall$-formulas (equivalently, $\exists$-formulas) $\psi(\overline{x})$ and for every $\overline{b} \in B$ will performed $B \models \psi(\overline{b})$ provided that $D \models \psi(\overline{b})$.

Generalization of definition 8 of model completeness, namely, definition 10, was consider in [2] by the authors using the concept (definition 9).

Definition 10 [2]. A theory T is 1-model complete if for any $B, D \in ModT$ and $B \subseteq_1 D$, then $B \prec D$.

One of the interesting properties of classical model theory is a property of quantifier eliminable which is also associated with a special case of model companion. In [2] was determined generalization of concept of quantifier eliminable, namely, definition 11.

Definition 11 [2]. A theory T is nearly model complete if for any formulas $\psi(\overline{x})$ exists a formula $\varphi(\overline{x})$ which is Boolean combination of $\forall$-formulas such that $T \models \forall \overline{x}[\psi \leftrightarrow \varphi]$.

Moreover in work [2] criterion was obtained (proposition 2.).

Proposition 2 [2]. A theory T is 1-model complete iff for any formulas $\psi(\overline{x})$ exists a formula $\varphi(\overline{x})$ which is a $\forall$-formulas such that $T \models \forall \overline{x}[\psi \leftrightarrow \varphi]$.

On the other hand, in work [1] was considered a generalization of Jonsson theory and the main tool of this generalization was a concept of Γ-embedding which is generalizated a concept of isomorphic embedding with respect to considered formulas. Instead of boolean combination of atomic formulas is considered a formulas with quantifier prenix of length α. In place of Boolean combination of atomic formulas we consider a formulas with quantifier prenix of length α. Under Γ we understand a kind of formulas, for example, $\Gamma = \Pi_\alpha$.

A set of all formulas (is a view of formula $\forall\exists\ldots\psi$) denote by Π_n, $\Sigma_n = \{\psi |\ \neg\psi \in \Pi_n\}$.

Definition 12 [1]. A map $f : A \to B$ is called a Γ-embedding if for any $\overline{a} \in A$ and $\psi(\overline{x}) \in \Gamma$ from $A \models \psi(\overline{a})$ implies $B \models \psi(f(\overline{a}))$.

A concept of model companion was determined by A. Robinson, and it is played important role in the study of various types of algebras, theories of which has model companion [1] (chapter 4).

Definition 13 [3]. A theory T is called model companion of T if:

1) T and T^* are mutually model consistent;

2) T^* is model complete.

Using next theorem we understand a value of concept of model companion for any Jonsson theory, semantic model of which is saturated.

Theorem 4 [15]. Let T be arbitrary Jonsson theory, then the following conditions are equivalent:

1) T is perfect;

2) T^* is model companion of T.

Using concept of finite diagram from work [16], T.G. Mustafin is determined a concept of model completion for generalized Jonsson theory. In the future on throughout of all paper in the results concerning work [1], as a semantic model we use a model as part of the definition 5.

Definition 14 [1]. 1. A set $D(\mathfrak{B}) = \cup_{n<\infty}\{Th(\mathfrak{B}, \overline{b})| \; \overline{b} \in |\mathfrak{B}|^n\}$ is called finite diagram of system $\mathfrak{B}$.

2. Algebraic system $\mathfrak{A}$ is called a $D(\mathfrak{B})$-system if satisfied $Th(\mathfrak{A}) = Th(\mathfrak{B})$ and $D(\mathfrak{A}) \subseteq D(\mathfrak{B})$.

3. If T is arbitrary theory, then any this model is called a $D(T)$-model.

In the future we will consider that $D = D(T)$ or $D = D(\mathfrak{B})$ for some model $\mathfrak{B}$ of theory T.

Using Γ-embeddings at work [1] was determined special case of α-model companion, namely, of concept of α-model completion which can be obtained from definitions 15 and 16 from work [1].

Definition 15 [1]. We say that a theory T is $D - \alpha$-model complete if a theory $T \cup Th_{\Pi_\alpha}(B, |B|)$ is complete with respect to D for any model $B \models T$.

Definition 16 [1]. Let T_1, T_2 be arbitrary theories of one language. A theory T_2 is called $D - \alpha$-model completion of T_1 if:

1) any model of T_1 is Π_α-embeddable in some D-model of T_2, and conversely, every D-model of T_2 is Π_α-embeddable in suitable (or some) model of T_1;

2) T_2 is $D - \alpha$-model complete;

3) a theory $T_2 \cup Th_{\Pi_\alpha}(B, |B|)$ is complete with respect to D for any model B of T_1.

In the future we will say that if satisfied condition (1) from definition 16, then considered theories are $D - \Pi_\alpha$-mutually model consistent, where $D = D(T)$ or $D = D(\mathfrak{B})$ for some model $\mathfrak{B}$ of theory T.

This theorem is α-Jonsson generalization of criterion of perfectness of Jonsson theory (theorem 4).

Proposition 3 [1]. Let T be arbitrary α-Jonsson theory, then the following conditions are equivalent:

1) T is perfect;

2) T^* is α-model completion of T.

Proceed to the main result of given paper. For this we must define a concept of (n_1, n_2)-Jonsson theory. Let n_1, n_2 be arbitrary natural numbers.

Definition 17. Theory T is called (n_1, n_2)-Jonsson theory if it is n_1-model complete and nearly n_2-model complete theory.

It is clear from definition 17 that $n_1 \geq n_2$. If $n_2 = 0$, then aл center of Jonsson theory T^* admit elimination of quantifiers. If n_1=0, then Jonsson theory T^* is model complete theory. Note that $(0, n)$-Jonsson theory is perfect for any natural n.

In other words, n_1-model completeness denote by the index n_1, but near model completeness denote by the index n_2. It is clear that it 2 various indices, and they may be dependentes, i.e. considered theories can be with one index or simultaneously with two indices.

We determine a concept of α-model companion of α-Jonsson theory.

Definition 18. Let T_1, T_2 be α-Jonsson theories of one language. A theory T_2 is called $D-\alpha$-model companion of T_1 if:

1) any model of T_1 is Π_α-embeddable in some D-model of T_2, and conversely, every D-model of T_2 is Π_α-embeddable in suitable (or some) model of T_1;

2) T_2 is $D - \alpha$-model complete.

Theorem 5. Every α-Jonsson theory T has no more than one α-model companion.

Proof. Let's say on the contrary, i.e. α-Jonsson theory has as minimum two various α-model companion, or example, T_1, T_2. Hence, theory T is perfect. Then by definition a theories T_1 and T_2 are mutually model consistent. By criterion of perfectness of α-Jonsson theory we can conclude that theories T_1 and T_2 are mutually model consistent with T^*, where T^* is center of theory T. And this means that T_1 and T_2 are cosemantic among themselves, and means they are equal.

By criterion of perfectness (proposition 3.) we can be conclude that α-Jonsson theory (when α=1) is 1-perfect if the theory has 1-model companion.

Next theorem allows you to get a description of 1-perfect 1-Jonsson theory in the sense of work [1].

Theorem 6. 1-Jonsson theory is 1-perfect iff the following conditions are equivalence:

1) a theory T has 1-model companion T^m in sense of work [1];

2) a theory $T^m = T^c$, where T^c is a center of theory T;

3) a theories $T^m = T^c$ and T are $D - \Pi_1$-mutually model consistent, where $D = D(C)$, is semantic model of theory T. A theory T^m is 1-model complete in sense of work [2].

Proof. From (1) to (2) and from (2) to (1) follows from proposition 3 at α=1.

We prove from (1) to (3). Since a theory T has 1-model companion T^m, then by definition 18, when α=1, we have that T^m is $D-1$-model complete, where $D = D(C)$, and C is semantic model of theory T. A theory T^m is $D - 1$-model complete if a theory $T \cup Th_{\Pi_1}(C, |C|)$ is complete. Since any model of theory T isomorphically embedded in model C, then easy to notice that by universality of formulas Π_1 and $D - 1$-model completeness all models of theory T with respect to Π_1-sentences are elementary equivalent. We need to show that from what $A \subseteq_1 B$ follows that $A \preceq B$ for any $A, B \in ModT$. Suppose the contrary. This means that there are such that $A, B \in ModT$, but it is not true that $A \npreceq B$. This is equivalent to that $B \notin ModD(A)$, but it is not true, since $D(A) \subseteq D(C)$ and B is a model $D(C)$ by $D - 1$-model completeness of theory T.

We prove from (3) to (1). Suppose the contrary. This means that a theory T is not 1-perfect, and this means that it is non-perfect in sense of work [1] (proposition 3.), i.e. she does not have $D - 1$-model completion. But by (3) a theories T^m and T are $D - \Pi_1$-mutually model consistent, where $D = D(C)$, C is semantic model of theory T, moreover T^m is 1-model complete in sense of work [2]. But then we can use a criterion (proposition 2.), which say that: a Jonsson theory T is 1-model complete iff for any formulas $\psi(\overline{x})$ exists a formula $\varphi(\overline{x})$ which is a $\forall$-formulas such that $T \models \forall\overline{x}[\psi \leftrightarrow \varphi]$. 1-nonperfectness is means that there is such type p, consisting from universal formulas with constants from some subset X of model C, which is not implemented in C. Without loss of generality, we can assume that X is contained in some model A of theory T^m. Since a type p is consistent set, then there is elementary extension A' of model A, which is implemented a type p. But since p is consisted from $\forall$-formulas, then p is implemented and in A. By $D - \Pi_1$-model compatibility A is invested in C, and this means, that a type p is implemented and in C. And from that one we can conclude about avalibility of the contradiction.

References

1 Мустафин Т.Г. Обобщенные условия Йонсона и описание обобщенно-йонсоновских теорий булевых алгебр / Т.Г. Мустафин // Матем. тр. — 1998. — № 2. — С. 135–197.

2 Kueker D.W. Nearly model complete theories. / D.W. Kueker, P. Turnquist // Mathematical Logic Quarterly. — 1999. — No. 3. — P. 291–298.

3 Теория моделей: справочная книга по математической логике: [В 4-х ч.]. Ч.1. / Под ред. Дж. Барвайса. — М.: Наука, 1982. — 392 с.

4 Yeshkeyev A.R. Properties of lattices of the existential formulas of Jonsson fragments / A.R. Yeshkeyev, M.T. Kasymetova // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79. — No. 3. — P. 25–32.

5 Yeshkeyev A.R. On lattice of existential formulas for fragment of Jonsson set / A.R. Yeshkeyev, O.I. Ulbrikht // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79. — No. 3. — P. 33–39.

6 Yeshkeyev A.R. Strongly minimal Jonsson sets and their properties / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 80. — No. 4. — P. 47–51.

7 Mustafin Y.T. Quelques properties des theories de Jonsson. / Y.T. Mustafin // The Journal of Symbolic logic. — 2002. — No. 2. — P. 528-536.

8 Ешкеев А.Р. JSp-косемантичность и JSB-свойство абелевых групп / А.Р. Ешкеев, О.И. Ульбрихт // Сибирские электронные математические известия. — 2016. – Т. 13. – С. 861–874.

9 Yeshkeyev A.R. The Properties of Similarity for Jonsson's Theories and Their Models / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 80. — No. 4. — P. 52–59.

10 Yeshkeyev A.R. Companions of the fragments in the Jonsson set / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2017. — Vol. 85. — No. 1. — P. 41–45.

11 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во КарГУ, 2016. – 346 с.

12 Yeshkeyev A.R. dcl-cl-atomic and prime sets / A.R. Yeshkeyev, A.K. Issayeva // Bulletin of the Karaganda University. Mathematics series. — 2019. — Vol. 93. — No. 1. — P. 88–94.

13 Yeshkeyev A.R. Properties of hybrids of Jonsson theories / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University. Mathematics series. — 2018. — Vol. 92. — No. 4. — P. 99–104.

14 Yeshkeyev A.R. Model-theoretic properties of the #-companion of a Jonsson set / A.R. Yeshkeyev, M.T. Kassymetova, N.K. Shamataeva // Eurasian mathematical journal. — 2018. — Vol. 9. — No. 2. — P. 68–81.

15 Ешкеев А.Р. Йонсоновские теории: учеб. пос. / А.Р. Ешкеев. — Караганда: Изд-во КарГУ, 2009. — 249 с.

16 Shelah S. Finite diagrams stable in power / S. Shelah // Ann. Math. Logic. — 1971. — No. 3. — P. 271–362.

А.Р. Ешкеев, М.Т. Омарова

(n_1, n_2)-Йонсон теориясының компаньондары

Мақалада (n_1, n_2)-йонсон теориясы компаньондерінің модельдік-теоретикалық қасиеттері қарастырылған. Сонымен бірге (n_1, n_2)-йонсон теориясымен орталық арасындағы байланыс зерттелген. Бұл ретте қарастырылып отырған теориялар модельді компаньонның мағынасына сәйкес келеді. Авторлар жаңа ұғымдар енгізді, атап айтқанда: (n_1, n_2)-йонсон теориясы; $D - \alpha$-модельді компаньон. α-йонсон және 1-кемел 1-йонсон теориясының модельдік-компаньондарына қатысты жаңа нәтижелер көрсетілген.

Кілт сөздер: йонсондық теория, модельді толық, n-модельді толық, дерлік n-модельді толық, $D - \alpha$-модельді компаньон, модельді толықтыру, (n_1, n_2)-йонсондық теория.

А.Р. Ешкеев, М.Т. Омарова

Компаньоны (n_1, n_2)-йонсоновских теорий

В статье рассмотрены теоретико-модельные свойства компаньонов (n_1, n_2)-йонсоновской теории. Также изучены связи между центром и (n_1, n_2)-йонсоновской теорией. При этом рассматриваемые теории являются совершенными в смысле существования соответственного модельного компаньона. Авторами введены новые понятия, а именно: (n_1, n_2)-йонсоновская теория; $D - \alpha$-модельный компаньон. Показаны новые результаты относительно модельных компаньонов α-йонсоновской и 1-совершенной 1-йонсоновской теорий.

Ключевые слова: йонсоновская теория, модельно полная, n-модельно полная, почти n-модельно полная, $D - \alpha$-модельный компаньон, модельное пополнение, (n_1, n_2)-йонсоновская теория.

References

1 Mustafin, T.G. (1998). Obobshchennye usloviia Ionsona i opisanie obobshchenno-ionsonovskikh teorii bulevykh alhebr [Generalized conditions of Jonsson and description of generalized Jonsson theories of Boolean algebras]. *Matematicheskie trudy — Mathematical works, 2,* 135–197 [in Russian].

2 Kueker, D.W. & Turnquist, P. (1999). Nearly model complete theories. *Mathematical Logic Quarterly, 3,* 291–298.

3 Barwise, Ed.J. (1982). *Teoriia modelei [Model theory].* Ed.J. Barwise, Yu.L. Yershov (Ed.). (Parts 1-4). Moscow: Nauka [in Russian].

4 Yeshkeyev, A.R. & Kasymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3,* 25–32.

5 Yeshkeyev, A.R., & Ulbrikht, O.I. (2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3,* 33–39.

6 Yeshkeyev, A.R. (2015). Strongly minimal Jonsson sets and their properties. *Bulletin of the Karaganda University. Mathematics series, Vol. 80, 4,* 47–51.

7 Mustafin, Ye.T. (2002). Quelques properties des theories de Jonsson. *The Journal of Symbolic logic, 2,* 528–536.

8 Yeshkeyev, A.R., & Ulbrikht, O.I. (2016). JSp-kosemantichnost i JSB svoistvo abelevykh hrupp [JSp-cosemantic and JSB-property of abelian groups]. *Sibirskie elektronnye matematicheskie izvestiia – Siberian Electronic Mathematical Reports, Vol. 13,* 861–874 [in Russian].

9 Yeshkeyev, A.R. (2015). The Properties of Similarity for Jonsson's Theories and Their Models. *Bulletin of the Karaganda University. Mathematics series, Vol. 80, 4,* 52–59.

10 Yeshkeyev, A.R. (2017). Companions of the fragments in the Jonsson set. *Bulletin of the Karaganda University. Mathematics series, Vol. 85, 1,* 41–45.

11 Yeshkeyev, A.R., & Kassymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models].* Karaganda: Izdatelstvo Karahandinskoho hosudarstvennoho universiteta [in Russian].

12 Yeshkeyev, A.R., & Issayeva, A.K. (2019). dcl-cl-atomic and prime sets. *Bulletin of the Karaganda University. Mathematics series, Vol. 93, 1,* 88–94.

13 Yeshkeyev, A.R., & Mussina, N.M. (2018). Properties of hybrids of Jonsson theories. *Bulletin of the Karaganda University. Mathematics series, Vol. 92, 4,* 99–104.

14 Yeshkeyev, A.R., Kassymetova, M.T., & Shamataeva, N.K. (2018). Model-theoretic properties of the #-companion of a Jonsson set. *Eurasian mathematical journal, Vol. 9, 2,* 68–81.

15 Yeshkeyev, A.R. (2009). *Ionsonovskie teorii [Jonsson theories].* Karaganda: Izdatelstvo Karahandinskoho hosudarstvennoho universiteta [in Russian].

16 Shelah, S. (1971). Finite diagrams stable in power. *Ann. Math. Logic, 3,* 271–362.

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Mathematical model of a root harvester after-cleaning system

A mathematical model of root crops after-cleaning and their movement pattern in a technological run for loading into means of transportation has been presented. The impact of both controlled and uncontrolled factors on the efficient continuous transportation of root crops has been determined on the basis of the above-mentioned model. The use of the suggested mathematical model will enable to narrow the search of the most efficient design, kinematic and dynamic parameters in conducting some experimental research on efficient operation of after-cleaning systems of harvesters with their possible adjustment according to the natural and climatic conditions of harvesting.

Keywords: mathematical model of root crops after-cleaning, root crops movement pattern, transporting-separating system, design, kinematic and dynamic parameters of tools.

Introduction

To provide the quality performance of root crops harvesters, especially concerning sugar beets root crops cleaning of soil and plants remains one needs some complex approach combining both theoretical and practical investigation.

Quality mathematical modeling of some processes of root crops after-cleaning resulted in the recommended limits of design and kinematic parameters will enable to reduce the number of experiments significantly. In this way we will save some time and costs to achieve the target goal.

Auger mechanisms in layout schemes of root crops harvesters are widely used. Theory of root crops movements and other similar products have been described in numerous scientific articles. The problems dealing with this subject are highlighted namely in the papers [1–5].

Minimal damage of the harvested crops during their simultaneous transportation and cleaning is also paid a great attention to in the investigation. These issues have been presented in the papers [6–9].

Some dynamic models of agricultural products transportation providing the quality of technological process performance by necessary tools and their reliable operation have been described in the articles [10–14].

Moreover, minimum power consumption is also important in materials transportation and other technological operations and this problem was discussed in the papers [15, 16].

The investigation results dealing with the sugar beets root crops after-cleaning by augers are given in the articles [17–19].

The made analysis of the known research has proved that the problem of root crops quality cleaning in machine harvesting has not been solved completely.

The purpose of the theoretical research is to find the most effective design, kinematic and dynamic parameters of the auger taking the root crops aside to provide the quality cleaning of root crops.

Material and method

Let's consider the process of an auger-root crop interaction whilst its transportation (Fig. 1). As we have seen in the paper [20], the case when the assumed central axis of a root crop is perpendicular to the belt motion direction is the most unfavorable in terms of sugar beets passing through the clearance between the belt surface 1 and auger rotation surface 2.

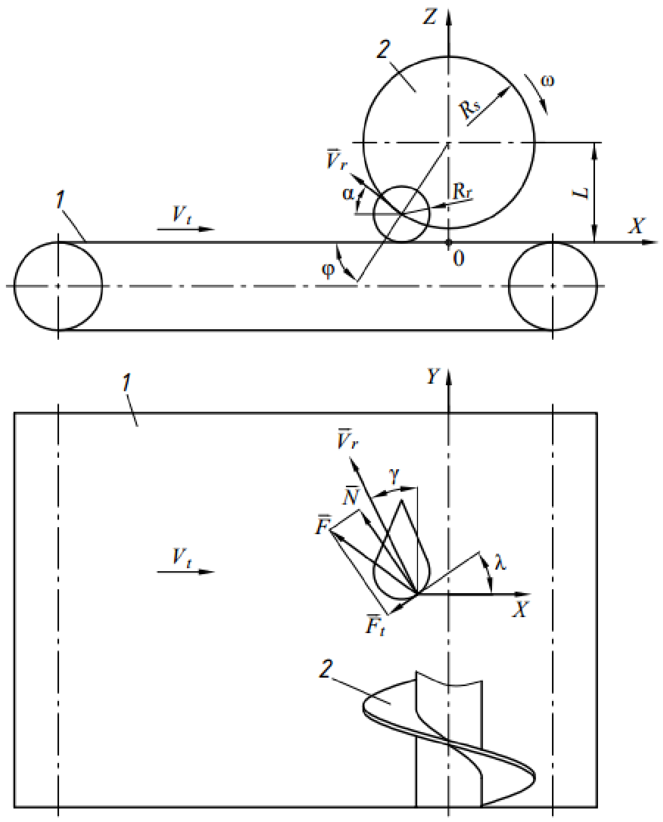

Figure 1. Auger (taking root crops aside)-root crop interaction pattern

In case when a root crop, in plane projection OXZ, is overlapped with the auger rotational surface less than the crown radius value, it will be thrown back with some turning and will be a little risen over the belt surface. It can be explained by the fact, that during the interaction the root crop is acted by the moment whose regularized force will be passing through the beet's center and the arm will be close to the beet crown radius located on the belt surface.

A case when root crop overlapping with the auger rotational surface is more than the crown radius value is scarcely probable for big and medium root crops as the most efficient structural and kinematic parameters of augers are chosen for relatively small, standard root crops (crown diameter is 40 *mm*).

Thus, when doing some calculations, we'll accept an option when normal reaction force of the auger to the root crop will be passing through the central axis of the latter, and the contact will take place with the beet crown.

So, while doing some calculations, we'll make the following assumptions: a beet crown is of perfect semi-sphere shape; longitudinal and lateral oscillations of the belt are neglected, its surface is considered to be perfectly smooth; screw pitch and height, angle of ascent of auger screw line are the same along the whole length; transporting conveyor linear velocity and angular velocity of auger rotation are constant.

We assume that a rot crop is travelling in horizontal plane on the transporting conveyor surface with the velocity V_t, and the auger, which is perpendicular to the motion direction, is rotating with angular velocity ω and is throwing the root beet aside and back. The root crop and auger contact is taking place when the center of imaginary sphere circling the root crop coincides with the auger edge.

Angle of ascent of screw surface on the given radius in the place of contact

$$\lambda = \text{arctg}\left(\frac{T}{2\pi R}\right),$$

where T — auger pitch; R — radius of imaginary circle in auger-root crop contact point ($R \leq R_s$); R_s — auger radius.

The angle between the horizontal line and the line connecting the centers of projections of the root crop sphere and the auger axis is found by

$$\varphi = \arcsin\left(\frac{L - R_r\,(1 - \sin\lambda)}{R}\right),$$

where L — height of auger axis location over the transporting conveyor; R_r — a root crop radius.

Reaction N and friction force $F_{fr} = Nf$ are acting on a root crop from the auger's side, which are directed, respectively, flatwise to the screw surface and at a tangent to it towards screw rotation direction (f — friction coefficient).

Velocity V_r of a root crop takeoff consists of two parts — velocity V_1, which is equal to the projection of linear velocity of the auger contact point on the direction of vectors sum of forces, and velocity V_2 of the root crop thrown back of the surface of stationary by convention auger due to initial velocity of transportation V_t taking into account throwback coefficient K_V.

Vector $\vec{V}_1' = \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$ is found by the direction of the auger resultant of resistant force

$$\vec{F} = \vec{N} + \vec{F}_{fr} = \begin{pmatrix} X_F \\ Y_F \\ Z_F \end{pmatrix}.$$

To find the vectors direction we use rotation matrixes about axis X and Y, respectively, on angles λ and φ

$$M_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda & -\sin\lambda \\ 0 & \sin\lambda & \cos\lambda \end{pmatrix};$$

$$M_\varphi = \begin{pmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix}.$$

Then the correspondent vectors are found by rotation matrixes.

Vector of the auger resultant of resistant force is equal to

$$\vec{F} = M_\varphi M_\lambda \begin{pmatrix} 0 \\ N \\ Nf \end{pmatrix}.$$

Velocity vector of auger-root crop contact point

$$\vec{V}_s = \begin{pmatrix} X_s \\ Y_s \\ Z_s \end{pmatrix} = M_\varphi \begin{pmatrix} 0 \\ 0 \\ \omega R \end{pmatrix}.$$

Modulus of throwing velocity

$$V_1' = V_s \cos\beta,$$

where β — angle between vectors $\vec{V}_1'$ and $\vec{V}_s$, which is equal to the angle between vectors $\vec{F}$ and $\vec{V}_s$, which is found from the known condition of angle cosine between two vectors

$$\cos\beta = \frac{X_F X_s + Y_F Y_s + Z_F Z_s}{\sqrt{X_F^2 + Y_F^2 + Z_F^2}\sqrt{X_s^2 + Y_s^2 + Z_s^2}}.$$

To find takeoff velocity vector $\vec{V}_1$ we find directing vector of force $\vec{F}$ of unit length

$$\vec{\mu} = \frac{\vec{F}}{N\sqrt{1+f^2}},$$

where vector modulus $\vec{F}$ is written in the denominator.

Then the vector value of takeoff speed $\vec{V}_1$ is found as a product of unit directing vector $\vec{\mu}$ and modulus of velocity value of contact point of auger and a root crop in its projection on this vector axis

$$\vec{V}_1 = \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = \vec{\mu}\omega R_s \cos\beta = \frac{\vec{F}\omega R_s \cos\beta}{N\sqrt{1+f^2}} = \frac{\omega R_s \cos\beta}{N\sqrt{1+f^2}} M_\varphi M_\lambda \begin{pmatrix} 0 \\ N \\ Nf \end{pmatrix}.$$

A root crop velocity due to «mirror» reflection of the auger surface $\vec{V}_2 = \begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix}$ is determined from the condition of equality of angles of descent and reflection and the coplanarity condition of these vectors with normal vector to the auger surface in the place of contact.

The coplanarity condition of the above-mentioned vectors are determined by their mixed product equality to zero (correspondent determinant)

$$\begin{vmatrix} X_2 & Y_2 & Z_2 \\ X_N & Y_N & Z_N \\ V_t & 0 & 0 \end{vmatrix} = 0,$$

where normal vector to the auger surface in the contact point is found by the dependence

$$\begin{pmatrix} X_N \\ Y_N \\ Z_N \end{pmatrix} = M_\varphi M_\lambda \begin{pmatrix} 0 \\ N \\ 0 \end{pmatrix}.$$

The condition of equality of angles of descent and reflection after some necessary transformations is written in the following way

$$\frac{X_2 X_N + Y_2 Y_N + Z_2 Z_N}{\sqrt{X_2^2 + Y_2^2 + Z_2^2}} = -X_N.$$

Being thrown back of the auger surface taking into account the reflection coefficient the modulus of velocity $\vec{V}_2$ is taking the following value

$$V_2 = K_V V_t = \sqrt{X_2^2 + Y_2^2 + Z_2^2}.$$

Sum velocity of root crop takeoff is found as vector sum of two velocities

$$\vec{V}_r = \begin{pmatrix} X_r \\ Y_r \\ Z_r \end{pmatrix} = \vec{V}_1 + \vec{V}_2 = \begin{pmatrix} X_1 + X_2 \\ Y_1 + Y_2 \\ Z_1 + Z_2 \end{pmatrix}.$$

On the basis of velocity vector $\vec{V}_r$ we calculate vertical α and horizontal γ angles of takeoff and projected throw path length S_L of a root crop

$$\alpha = arctg\left(\frac{Z_r}{\sqrt{X_r^2 + Y_r^2}}\right);$$

$$\gamma = arctg\left(\frac{X_r}{Y_r}\right);$$

$$S_L = \frac{V_r^2 \sin 2\alpha}{g} = \frac{\left(X_r^2 + Y_r^2 + Z_r^2\right)\sin 2\alpha}{g}.$$

A program of calculating velocities, takeoff angles and throw path length for different kinematic and geometrical parameters of the system under consideration has been developed using the above-mentioned formulae to analyze and improve the transport auger design.

Results

Figures 2, 3, 4, 5 show the correspondent calculating curves of dependencies of vertical and horizontal angles α and γ of the root crop takeoff, root crop takeoff velocities V_r and range of throw S_L on the change of basic design and physical parameters of transporting-separating system, namely: auger pitch T, angular velocity of its rotation ω, radius R_s of the auger peripheral surface, linear velocity of the root crop transportation till its contact with the auger rib V_t, friction coefficient of a root crop on the auger surface f, beet crown radius R_r and the coefficient of its throwback of the auger edge K_V.

Analysis of curves shown on Figure 2 proves that radius value R_s of peripheral surface of the auger taking root crops aside and its pitch T are controlled dominant factors which influence the vertical angle α of the root crop takeoff.

Other controlled factors, namely angular velocity of rotation of the auger taking root crops aside ω and linear velocity V_t of the root crop transportation till the moment of its contact with the auger do not make substantial impact on the vertical angle α of the root crop throw by the auger taking root crops aside.

Here it should be noticed that uncontrolled factors, namely the beet crown radius R_r and friction coefficient of a root crop on the auger surface f make substantial impact on vertical angle α of the root crop throw by the auger taking root crops aside, their value increase results in increased value of angle α.

The coefficient of root crop throw away of the auger rib K_V does not make great impact on the vertical angle α.

Analyzing the curves shown on Figure 3, we can state that the dominant controlled factors making impact on horizontal angle γ of a root crop takeoff are the auger pitch T and linear velocity of horizontal transport conveyor V_t. Increased value T results in decreased value of angle γ. Conversely, increased linear velocity of the root crop transportation V_t causes the increased value of angle γ.

Negative sign at angle γ means that a root crop is thrown away in opposite to the axis OX direction.

Compared to the previous factors, which in fact are characterized by linear dependencies of influence on angle γ, the angular velocity of rotation of the auger taking root crops aside ω, while speeding up from 10 to 25 rad/s, results in sharp increase of angle γ absolute value but its further influence on the angle γ can be neglected. Increased value of peripheral surface radius of auger taking root crops aside R_s leads to a small increase of angle γ modulus.

As for the uncontrollable factors, it should be said that high soil humidity and as a result lower value of friction coefficient f makes a great impact on angle γ modulus increase. At the same time, the increased radius of beet crown R_r and the coefficient of its beating off the auger edge K_V on the contrary results in decreased modulus of angle γ.

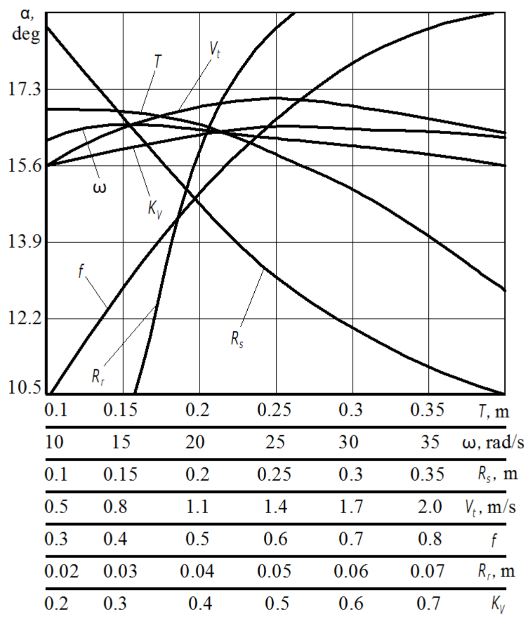

Figure 2. Dependencies of vertical angle α of root crop takeoff on parameters T, ω, R_s,V_t, f, R_r and K_V

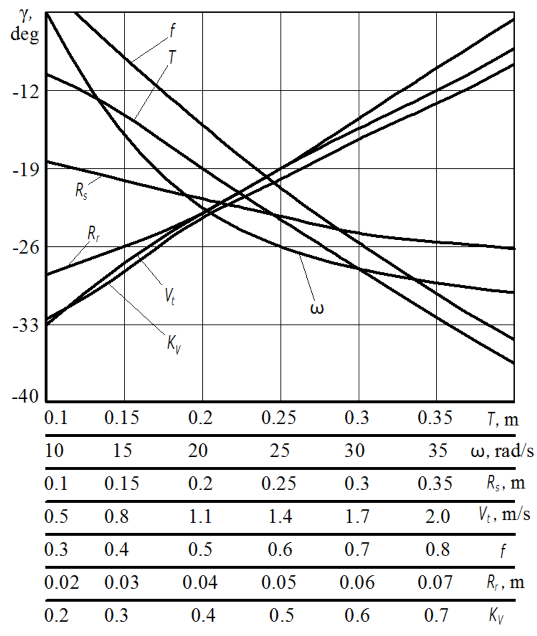

Figure 3. Dependencies of horizontal angle γ of root crop takeoff on parameters $T, \omega, R_s, V_t, f, R_r$ and K_V

The analysis of curves (Fig. 4) describing the impact of different factors on the root crop takeoff speed V_r has proved that only increase of radius value of peripheral surface of the auger taking root crops aside R_s and angular velocity ω of its rotation cause the increase of value V_r, while other factors do not make a substantial impact on value V_r change.

Similar to the previous case, only angular velocity ω of rotation of the auger taking root crops aside (considerable impact) and its peripheral surface radius R_s have a significant influence on root crop range of throw S_L (Fig. 5).

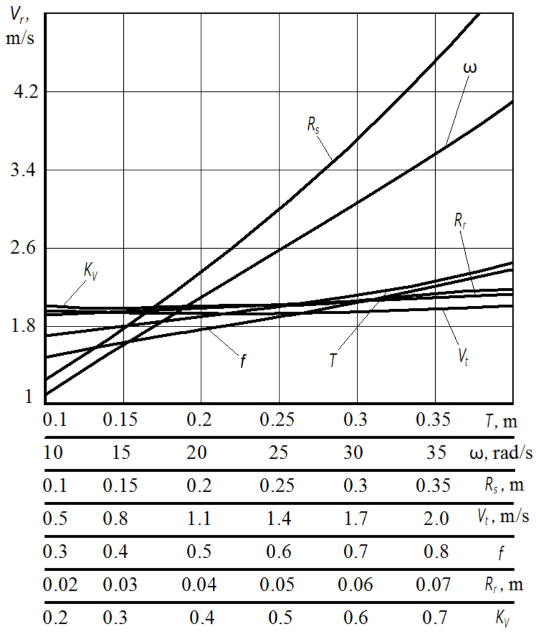

Figure 4. Dependencies of a root crop takeoff speed V_r on parameters T, ω, R_s, V_t, f, R_r and K_V

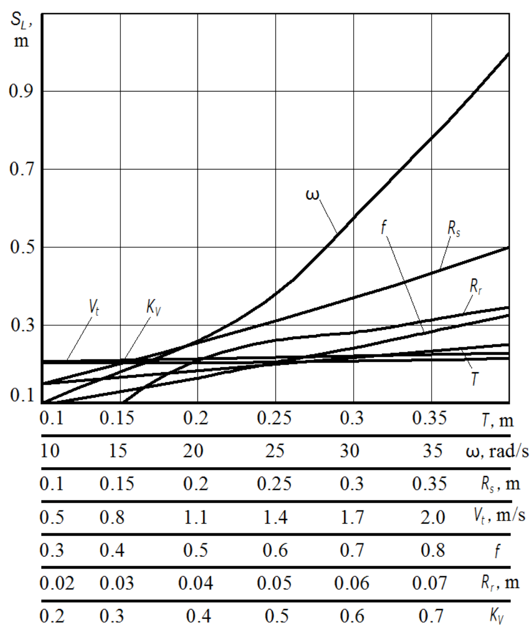

Figure 5. Dependencies of a root crop range of throw S_L on parameters T, ω, R_s, V_t, f, R_r and K_V

Conclusions

On the basis of results analysis of conducted theoretical investigation on root crops movement behavior after their contact with the auger taking root crops aside it has been found out that increased auger pitch from 0.1 to 0.4 m causes decreased angle α (from 17° to 12°) and increased angle γ (from $-10°$ to $-37°$). At the same time within the given range of auger pitch increase T both the velocity of root crop takeoff V_r and, therefore, the range of its throw S_L is getting increased. Root crops range of throw must be within the limits $0.2 \ldots 0.6$ m. At $S_L < 0.2$ m root crops separation is not efficient due to little shaking effect after their interaction with the belt. New contacts with the auger are also getting more possible which cause additional damage of the root crops. At $S_L > 0.6$ m the root crops will be thrown towards shields and frames structures of the machine hopper, and this also results in their bigger damage. Thus, auger pitch T can't be smaller than 0.15 m.

The bigger the friction coefficient f is, the bigger the range of throw of a root crop is. Though the maximum values $f = 0.8 \ldots 0.9$ (use of special friction materials for an auger spiral rib making) do not cause the S_L increase for more than 0.5 m.

A beet crown radius R_r increase has caused its range of throw S_L increase, though small root crops ($R_r = 0.02 \ldots 0.03$ m) will be thrown not far from (up to 0.15 m).

Coefficient of throwback K_V change can be neglected regarding the range of throw of root crops ($S_L = 0.2 \ldots 0.215$ m).

Thus, the variable parameters are recommended to be chosen within certain boundaries: $T = 0.2 \ldots 0.3$ m; $\omega = 18 \ldots 30$ rad/s; $R_s = 0.15 \ldots 0.25$ m; $V_t = 1.2 \ldots 1.3$ m/s.

References

1 Baranovsky V.M. Theoretical analysis of the technological feed of lifter root crops / V.M. Baranovsky, M.V. Potapenko // INMATEH: Agricultural engineering. Bucharest, Romania. — 2017. — Vol. 51. — No. 1. — P. 29–38.

2 Baranovsky V.M. Justification of rational parameters of a pneumoconveyor screw feeder / V.M. Baranovsky, R.B. Hevko, V.O. Dzyura, O.M. Klendii, M.B. Klendii, R.M. Romanovsky // INMATEH: Agricultural engineering. Bucharest, Romania. — 2018. — Vol. 54. — No. 1. — P. 15–24.

3 Bulgakov V. Study on the interaction of feeler and roots within the topping process of sugar beet / V. Bulgakov // Bulletin of the Transilvania University of Braєov. Romania. — 2002. — Vol. 9. — Series A. — P. 79–84.

4 Bratucu Gh. Establishing the optimum operating mode of sugar beet head cutting equipment using a cylindrical palpator / Gh. Bratucu , D.D. Paunescu // Bulletin of the Transilvania University of Braєov. Romania. — 2015. — Vol. 8. — No. 1. — Series II. — P. 51–56.

5 Hevko B. Dynamics of auger working body of a multifunctional conveyor / B. Hevko, O. Lyashuk, M. Sokil, O. Skyba, O. Marunych, D. Shmatko // Bulletin of Karaganda State University. — 2018. — № 1(89). — P. 105–112.

6 Hevko R.B. Development and investigation of reciprocating screw with flexible helical surface / R.B. Hevko, S.Z. Zalutskyi, I.G. Tkachenko, O.M. Klendiy // INMATEH: Agricultural engineering. Bucharest, Romania. — 2015. — Vol. 46. — No. 2. — P. 133–138.

7 Hevko R.B. Development of a pneumatic screw conveyor design and substantiation of its parameters / R.B. Hevko, O.M. Strishenets, O.L. Lyashuk, I.G. Tkachenko, O.M. Klendii, V.O. Dzyura // INMATEH: Agricultural engineering. Bucharest, Romania. — 2018. — Vol. 54. — No. 1. — P. 153–160.

8 Liebe S. Impact of root rot pathogens on storage of sugar beets and control measures / S. Liebe, M. Varrelmann // Zuckerindustrie. Sugar industry. Berlin, Germany. — 2014. — Vol. 139(7). — P. 443–452.

9 Hevko R.B. Determination of interaction parameters and grain material flow motion on screw conveyor elastic section surface / R.B. Hevko, S.Z. Zalutskyi, Y.B. Hladyo, I.G. Tkachenko, O.L. Lyashuk, O.M. Pavlova et al. // INMATEH: Agricultural engineering. Bucharest, Romania. — 2019. — Vol. 57. — No. 1. — P. 123–134.

10 Lutsiv I.V. Investigation of dynamical impact loads in screw conveyer drives with safety clutches / I.V. Lutsiv, I.B. Hevko, O.L. Lyashuk, T.S. Dubynyak // INMATEH: Agricultural engineering. Bucharest, Romania. — 2017. — Vol. 51. — No. 1. — P. 69–76.

11 Lyashuk O.L. Modelling of the vertical screw conveyer loading / O.L. Lyashuk, O.R. Rogatynska, D.L. Serilko // INMATEH: Agricultural engineering. Bucharest, Romania. — 2015. — Vol. 45. — No. 1. — P. 87–94.

12 Lyashuk O.L. Mathematical model of bending vibrations of a horizontal feeder-mixer along the flow of grain mixture / O.L. Lyashuk, M.B. Sokil, V.M. Klendiy, O.P. Skyba , O.L. Tretiakov, L.M. Slobodian et al. // INMATEH: Agricultural engineering. Bucharest, Romania. — 2018. — Vol. 55. — No. 2. — P. 35–44.

13 Roberts A.W. Bulk Solids: Optimizing Screw Conveyors / A.W. Roberts // Chemical engineering. New York, USA. — 2015. — Vol. 122(2). — P. 62–67.

14 Rohatynskyi R.M. The research of the torsional vibrations of the screw in terms of impulsive force impacts / R.M. Rohatynskyi, Iv.B. Hevko , A.Ye. Dyachun // Науковий вісник Національного гірничого університету. — Дніпропетровськ: НГУ, 2015. — № 5. — С. 64–68.

15 Tripathi N. Energy loss at bends in the pneumatic conveying of fly ash / N. Tripathi, A. Sharma, S.S. Mallick, P.W. Wypych // Particuology.— 2015. — Vol. 21. — P. 65–73.

16 Yao Y. Overall performance evaluation of tubular scraper conveyors using a TOPSIS-based multiattribute decision-making method / Y. Yao, Z. Kou, W. Meng, G. Han // The Scientific World Journal. New York, USA, 2014.

17 Hevko R.B. Development of design and investigation of operation processes of small-sclale root crop and potato harvesters / R.B. Hevko, I.G. Tkachenko, S.V. Synii, I.V. Flonts // INMATEH: Agricultural engineering. Bucharest, Romania. — 2016. — Vol. 49. — No. 2. — P. 53–60.

18 Hevko R. Advances in methods of cleaning root crops / R. Hevko, R. Brukhanskyi, I. Flonts, S. Synii, O. Klendii // Bulletin of the Transilvania University of Braєov. — 2018. — Vol. 11. — No. 1. — Series II. — P. 127–138.

19 Синій С.В. Підвищення ефективності очищення коренеплодів / С.В. Синій ., Р.Б. Гевко, І.В. Фльонц, О.М. Клендій // Сільськогосподарські машини. Зб. наук. ст. — Луцьк: ЛНТУ. — 2018. — Випуск 40. — С. 89–100.

20 Гандзюк М.О. Кінематичний аналіз доочисного пристрою коренезбиральної машини / М.О. Гандзюк, Ю.Б. Гладьо, Р.Б. Гевко // Сільськогосподарські машини. Зб. наук. ст. — Луцьк: ЛДТУ. — 2001. — Випуск 8 — С. 44–53.

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Түбірлік өсімдіктерді жинау машинасының тазалау жүйесінің математикалық моделі

Мақалада түбірлік дақылдарды тазарту үрдісінің математикалық моделі және олардың қозғалысын көлік құралдарына түсірудің технологиялық табиғаты ұсынылған. Осы модельдің негізінде, бақыланатын және бақыланбайтын факторлардың түбірлік дақылдардың үздіксіз қозғалысын сапалы қамтамасыз етуге әсері анықталды. Ұсынылған математикалық модельді пайдалану түбір жинау машиналары үшін тазарту жүйелерінің тиімді жұмыс істеуіне, жинаудың климаттық жағдайларына сәйкес оларды реттеу мүмкіндігімен, эксперименттік зерттеулер жүргізген кезде қажетті оңтайлы дизайн, кинематикалық және динамикалық параметрлердің өрісін қысқартуға мүмкіндік береді.

Кілт сөздер: түбірлік дақылдарды кейіннен емдеу процесінің математикалық моделі, түбірлік дақылдардың қозғалысы, көлік және бөлу жүйесі, жұмыс органдарының конструктивті, кинематикалық және динамикалық параметрлері.

Р.Б. Гевко, И.Г.Ткаченко, Ю.Б. Гладьо, С.В. Синий, Н.А. Гандзюк, О.Н. Троханяк

Математическая модель системы доочистки корнеуборочной машины

Представлена математическая модель процесса доочистки корнеплодов и характера их движения в технологическое русло для выгрузки в транспортные средства. На основе данной модели было установлено влияние как управляемых, так и неуправляемых факторов на качественное обеспечение непрерывного перемещения корнеплодов. Использование предложенной математической модели позволит сузить поле искомых оптимальных конструктивных, кинематических и динамических параметров при проведении экспериментальных исследований для эффективной работы систем доочистки корнеуборочных машин с возможностью их регулирования в соответствии с природно-климатическими условиями уборки.

Ключевые слова: математическая модель процесса доочистки корнеплодов, характер движения корнеплодов, транспортно-сепарирующая система, конструктивные, кинематические и динамические параметры рабочих органов.

References

1 Baranovsky, V.M., & Potapenko, M.V. (2017). Theoretical analysis of the technological feed of lifter root crops. *INMATEH: Agricultural Engineering, Bucharest, Romania. Vol. 51, 1,* 29–38.

2 Baranovsky, V.M., Hevko, R.B., Dzyura, V.O., Klendii, O.M., Klendii, M.B., & Romanovsky, R.M. (2018). Justification of rational parameters of a pneumoconveyor screw feeder. *INMATEH: Agricultural Engineering, Bucharest, Romania. Vol. 54, 1*, 15–24.

3 Bulgakov, V. (2002). Study on the interaction of feeler and roots within the topping process of sugar beet. *Bulletin of the Transilvania University of Braeov, Romania. Vol. 9,* Series A, 79–84.

4 Bratucu, Gh., & Paunescu, D.D. (2015). Establishing the optimum operating mode of sugar beet head cutting equipment using a cylindrical palpator. *Bulletin of the Transilvania University of Braeov, Romania. Vol. 8, 1,* Series II, 51–56.

5 Hevko, B., Lyashuk, O., Sokil, M., Skyba, O., Marunych, O., & Shmatko, D. (2018). Dynamics of auger working body of a multifunctional conveyor. *Vestnik Karahandinskoho universiteta – Bulletin of the Karaganda University, 1(89),* 105-112.

6 Hevko, R.B., Zalutskyi, S.Z., Tkachenko, I.G., & Klendiy, O.M. (2015). Development and investigation of reciprocating screw with flexible helical surface. *INMATEH: Agricultural Engineering, Bucharest, Romania. Vol. 46,* 2, 133–138.

7 Hevko, R.B., Strishenets, O.M., Lyashuk, O.L., Tkachenko, I.G.; Klendii, O.M., & Dzyura, V.O. (2018). Development of a pneumatic screw conveyor design and substantiation of its parameters. *INMATEH: Agricultural Engineering, Bucharest, Romania. Vol. 54,* 1, 153–160.

8 Liebe, S., & Varrelmann, M. (2014). Impact of root rot pathogens on storage of sugar beets and control measures. *Zuckerindustrie. Sugar industry, Berlin, Germany. Vol. 139(7),* 443–452.

9 Hevko, R.B., Zalutskyi, S.Z., Hladyo, Y.B., Tkachenko, I.G., Lyashuk, O.L., Pavlova, O.M., & et al. (2019). Determination of interaction parameters and grain material flow motion on screw conveyor elastic section surface. *INMATEH: Agricultural Engineering, Bucharest, Romania. Vol. 57,* 1, 123–134.

10 Lutsiv, I.V., Hevko, I.B., Lyashuk, O.L., & Dubynyak, T.S. (2017). Investigation of dynamical impact loads in screw conveyer drives with safety clutches. *INMATEH: Agricultural Engineering, Bucharest, Romania. Vol. 51,* 1, 69–76.

11 Lyashuk, O.L., Rogatynska, O.R., & Serilko, D.L. (2015). Modelling of the vertical screw conveyer loading. *INMATEH: Agricultural Engineering, Bucharest, Romania. Vol. 45,* 1, 87–94.

12 Lyashuk, O.L., Sokil, M.B., Klendiy, V.M., Skyba, O.P., Tretiakov, O.L., Slobodian, L.M., & et al. (2018). Mathematical model of bending vibrations of a horizontal feeder-mixer along the flow of grain mixture. *INMATEH: Agricultural Engineering, Bucharest, Romania. Vol. 55,* 2, 35–44.

13 Roberts, A.W. (2015). Bulk Solids: Optimizing Screw Conveyors. *Chemical engineering, New York, USA. Vol. 122(2),* 62–67.

14 Rohatynskyi, R.M., Hevko, Iv.B. & Dyachun, A.Ye. (2015). The research of the torsional vibrations of the screw in terms of impulsive force impacts. *Naukovyi visnyk Natsionalnoho hirnychoho universyteta – Scientific Bulletin of National Mining University, Dnipropetrovsk, NMU. 5,* 64–68.

15 Tripathi, N., Sharma, A., Mallick, S.S., & Wypych, P.W. (2015). Energy loss at bends in the pneumatic conveying of fly ash. *Particuology, Vol. 21,* 65–73.

16 Yao, Y., Kou, Z., Meng, W., & Han, G. (2014). Overall performance evaluation of tubular scraper conveyors using a TOPSIS-based multiattribute decision-making method. *The Scientific World Journal, New York, USA.*

17 Hevko, R.B., Tkachenko, I.G., Synii, S.V., & Flonts, I.V. (2016). Development of design and investigation of operation processes of small-sclale root crop and potato harvesters. *INMATEH: Agricultural Engineering, Bucharest, Romania. Vol. 49, 2,* 53–60.

18 Hevko, R., Brukhanskyi, R., Flonts, I., Synii, S. &, Klendii, O. (2018). Advances in methods of cleaning root crops. *Bulletin of the Transilvania University of Braeov, Romania. Vol. 11, 1,* Series II, 127-138.

19 Synii, S.V., Hevko, R.B., Flonts, I.V., &, Klendii, O.M. (2018). Pidvyshchennia efektyvnosti ochyshchennia koreneplodiv [Improvement of the root crop cleaning efficiency]. *Silskohospodarski mashyny. Zbirnyk naukovykh statei – Agricultural machines. Collection of scientific articles, Lutsk, LNTU. Vol. 40*, 89–100 [in Ukrainian].

20 Gandziuk, M.O., Hlado Y.B., & Hevko, R.B. (2001). Kinematychnyi analiz doochysnoho prystroiu korenezbyralnoi mashyny [Kinematic analysis of the root harvest post-cleaning device]. *Silskohospodarski mashyny. Zbirnyk naukovykh statei – Agricultural machines. Collection of scientific articles, Lutsk, LSTU. Vol. 8*, 44–53 [in Ukrainian].

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Determining stress intensity factor in bending reinforced Concrete beams

There has been analytically solved the problem of determining the stress state in the cross section of bending reinforced concrete beams with a crack in a linear formulation. For this, the beam is cut along the crack line and the equilibrium condition of the cut-off part of the beam determines the height of the compression zone and the tensile stress at the crack tip. The remaining parameters of the stress state are expressed in terms of these values. The value of the bending moment is determined, above which there takes place increasing the initial length of the crack. For this case, the length of the operational crack is determined. The solution is valid for beams of arbitrary section shape. Determining the stress intensity factor (SIF) is based on the assumptions that the longitudinal forces at the tip of the crack are equal with and without taking into account stress concentration. The size of the stress concentration zone is determined from the condition that the local stress is equal to the nominal stress. On this basis a formula has been obtained for determining the stress intensity factor in rectangular beams. The paper analyzes the dependence of the stress intensity factor on the crack length, the moment and other geometric and operational factors. The obtained results allow estimating the bearing capacity of beams with a crack, as well as their crack resistance by a stress intensity factor.

Keywords: reinforced concrete, beam, bending, compressed zone, crack, reinforcement, stress state, bearing capacity, stress intensity factor, crack resistance.

A very common type of defects in reinforced concrete structures are cracks. They appear both at the manufacturing stage and at the operation stage [1–3]. The appearance of cracks in bending elements does not mean exhaustion of its bearing capacity. It leads to increasing the efforts in the sections with a crack, which reduces the strength of the element. Due to the opening of the crack width corrosion of reinforcement increases that reduces the durability of structures. The calculation of the stress state of reinforced concrete structures with cracks is the focus of many books and articles [1–9]. The crack opening width is one of the criteria for the ultimate state of reinforced concrete structures with cracks. Its definition is dealt with a lot of works [10–14]. The issues of crack formation are considered in works [15–17].

The above problems are dealt with in a series of experimental studies [8, 18, 19]. A lot of studies have been carried out by numerical methods [6, 7, 11, 20–22].

Under certain conditions unstable crack development is possible, which is estimated through the parameters of fracture mechanics [1, 23]. The stress intensity factor (SIF) is the main parameter of linear fracture mechanics. Some works dealing with determining the SIF in reinforced concrete beams have appeared only recently [24, 25]. It should be noted that these calculations were carried out in a linear formulation. However, the relationship between stress and strain in concrete is not linear. In this paper an approximate analytical method is proposed for determining the SIF in reinforced concrete beams.

We will first define the nominal stress in a beam with a crack of the known length l in a linear formulation. Let's consider an I-section with the vertical axis of symmetry (Fig. 1, *a*). We will introduce the notations:

A_{ct}, A_c is the area of the shelves overhang in the tension and compression zones;

h_t, h_c is the thickness of the shelves in the tension and compression zones;

a, a' is the thickness of the protective layer of concrete in the tension and compression zones;

A_s, A'_s is the area of reinforcement in the tension and compression zones;

N_a, N'_a are internal forces in the reinforcement in the tension and compression zones.

Let's cut the beam by the section passing through the crack and show the curve of the stress distribution in it (Fig. 1, *b*).

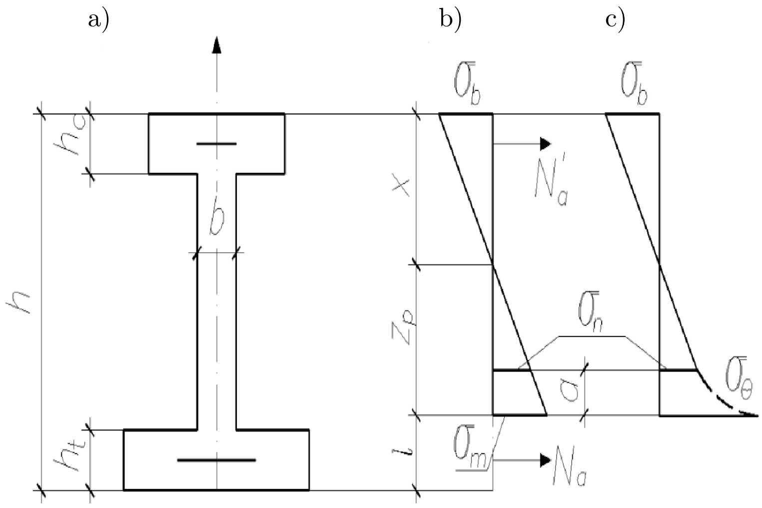

Figure 1. Towards determining stresses and SIF

The main unknown tasks will be tensile stress at the tip of the crack σ_m and the height of the compressed zone x. Given the linearity of the stress profile, we will determine the maximum compressive stress in concrete

$$\sigma_b = \sigma_m E/z_p, \tag{1}$$

where z_p is the height of the tensile zone.

In the zone of the crack the adhesion of the reinforcement to the concrete is broken, and the tensile forces are mainly perceived by the reinforcement. In sections between cracks tensile forces are also perceived by reinforcement and concrete. The deformations and stresses in the reinforcement and concrete, as well as the height of the compressed zone between the cracks vary. This non-uniformity in the calculations is taken into account by introducing special coefficients that are equal to the ratios of the average values in the section between the cracks and the values in the section with a crack [2]

$$\psi_s = \varepsilon_{sm}/\varepsilon_s, \quad \psi_b = \varepsilon_{bm}/\varepsilon_b.$$

Taking this into account, we will determine the deformations at the reinforcement level as:

$$\varepsilon_s = \varepsilon_m \frac{\psi_b}{\psi_s} \cdot \frac{h-a-x}{z_p} = \varepsilon_m \psi_{bs}(h-a-x)/z_p, \quad \varepsilon'_s = \varepsilon_m \psi_{bs}(x-a')/z_p.$$

The stress in the reinforcement will be

$$\sigma_s = E_s \varepsilon_s = \alpha \sigma_m \psi_{bs}(h-a-x)/z_p \quad \sigma'_s = \sigma'_{sp} - E_s \varepsilon'_s = \sigma'_{sp} - \alpha \sigma_m \psi_{bs}(x-a')/z_p, \tag{2}$$

where α is the ratio of elasticity modulus of the reinforcement and concrete; σ'_{sp} is pre-stress in the reinforcement of the compression zone.

The condition of longitudinal forces equilibrium in the section

$$\sigma_m b z_p/2 + \sigma_s A_s - \sigma_b b x/2 + \sigma'_s A'_s - \sigma_b A_c (x - h_c/2)/x = 0.$$

The second equilibrium equation in the form of the sum of all the forces moments relative to the zero line has the form

$$\sigma_b b x^2/3 + \sigma_m b z_p^2/3 + \sigma_s A_s (h-a-x) + \sigma_b A_c (x-h_c/2)^2/x - \sigma'_s A'_s (x-a') = M_{bn},$$

where the external moment M_{bn} is equal to the bending moment in the section with a crack M for the reinforcement without pre-stress. For the pre-stressed reinforcement of the tension zone the σ s stress occurs after the moment of external forces M exceeds the moment of pre-stress force. Then in the equation the external moment will be

$$M_{bn} = M - \sigma_{sp} A_s (h-a-x).$$

The equilibrium equations are common for all reinforced concrete elements with and without pre-stress, with different cross-sectional shapes: I-shaped, T-shaped, rectangular beams. For a T-section A_c or A_{ct} is equal to zero. For a rectangular cross section, the area of both overhangs is zero. In order to obtain explicit calculation ratios, we will perform a further solution for the most common case of a rectangular cross section with reinforcement in the tension zone. In this case the equilibrium equations will take the form:

$$\sigma_m b z_p/2 + \sigma_s A_s - \sigma_b bx/2 = 0;$$

$$\sigma_b bx^2/3 + \sigma_m b z_p^2/3 + \sigma_s A_s(h - a - x) = M_{bn}.$$

Let's introduce the following dimensionless parameters:

$$\xi = x/h, \quad z = l/h, \quad \lambda = z_p/h = 1 - z - \xi, \quad \overline{h} = (h-a)/h, \quad \mu = A_s/bh.$$

Taking into account the expressions for σ_b and σ_s, the equilibrium equations will transform into the form:

$$\lambda^2 - \xi^2 + \alpha\mu\psi_{bs}(\overline{h}_0 - \xi) = 0;$$

$$\sigma_m \left[\xi^3/\lambda + \lambda^2 + 3\alpha\mu\psi_{bs}(\overline{h} - \xi)^2/\lambda\right] = 3M_{bn}/bh^2.$$

From the first equation we will determine the height of the compression zone, and then that of the tension zone

$$\xi = \frac{(1-z)^2 + \alpha\mu\psi_{bs}\overline{h}}{2(1-z) + \alpha\mu\psi_{bs}}, \quad \lambda = 1 - z - \xi. \tag{3}$$

From the second equation of equilibrium we will determine the nominal stress at the crack tip

$$\sigma_m = LM_{bn}/W, \tag{4}$$

where

$$W = bh^2/6, \quad L = 0.5\lambda/\left[\xi^3 + \lambda^3 + 3\alpha\mu\psi_{bs}(\overline{h} - \xi)^2\right].$$

The $\lambda(\xi)$, L parameters do not depend on the section dimensions and the acting loads. Therefore, it is expedient to build in advance the graph of these parameters dependence on the crack length. Figure 2 shows such a graph for

$$\overline{h} = 0.907, \quad \nu = \alpha\mu\psi_{bs} = (200/24) \cdot 0.015(0.6/0.5) = 0.15.$$

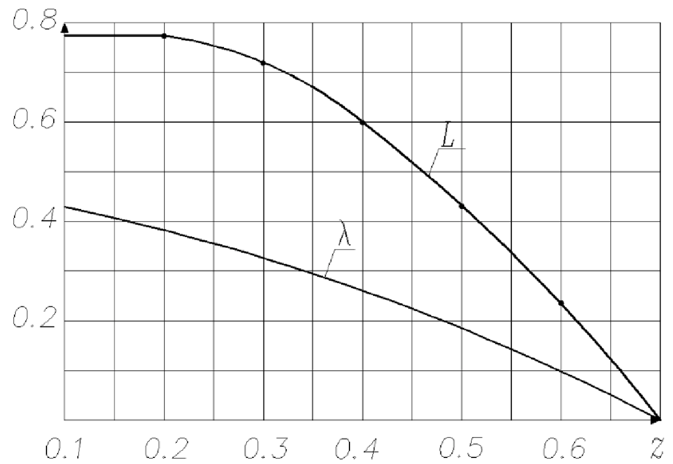

Figure 2. Stress state parameters dependence on the crack length

This solution is valid as long as the nominal stress is less than or equal to the concrete tensile strength R_{bt}. Otherwise, the initial crack length will increase. The value of the moment at which this happens is equal to

$$M_m = R_{bt}W/L.$$

Using the graph in Figure 2 and this formula, we can find the moment value for crack growth M_m. The second line of Table 1 shows the values of this moment in kNm for the beam section (15×30) cm, where the destructive moment $M_p = 53.6kNm$.

If the nominal stress at the crack tip σ_m exceeds the tensile strength of the concrete $R_{bt}(M_{bn} > M_m)$, then the crack length will increase. To determine a new crack length in (4), we assume $L = R_{bt}W/M$. Then, from the system of nonlinear equations (3) and (4) we also determine ξ and z. To solve this system, we can use the graphs in Figure 2.

Let's determine the SIF at the crack tip K_I. The tip of the crack is a strong stress concentrator. According to the linear theory of elasticity, the stress there is determined by the formula

$$\sigma_\theta = K_I/\sqrt{2\pi r}, \tag{5}$$

where r is the distance from the crack tip to the point considered.

Determining the SIF is based on the equality of the longitudinal forces at the crack tip with and without stress concentration. Local stresses are determined by formula (5), and nominal stresses by formulas (3) and (4).

Let us determine the length of the stress concentration zone "a" from the condition of equality of the nominal and local stresses at the end of this zone (Fig. 1, c). The rated stresses at the distance «a» from the crack tip

$$\sigma_n = \sigma_m(1 - a/z_p) = \sigma_m(1 - t).$$

Equating the local stress to this value, we will obtain

$$a = K_I^2/2\pi\sigma_n^2.$$

The longitudinal forces per unit width of the beam in the zone of concentration are

$$I_1 = \int_0^a \sigma_\theta dr = K_I\sqrt{2a/\pi} = K_I^2/\pi\sigma_n.$$

The longitudinal forces in the same zone without considering concentration are equal

$$I_2 = \frac{\sigma_m + \sigma_n}{2}a = \frac{\sigma_m + \sigma_n}{2}\frac{K_I^2}{2\pi\sigma_n^2}.$$

Equating these forces, we will obtain

$$\sigma_m = 3\sigma_n.$$

From here it follows that $t = 2/3$.

Now we will determine the SIF

$$K_I = \sqrt{2\pi a}\sigma_n = 0.683\sigma_m\sqrt{z_p}.$$

Substituting here (4), we will finally obtain

$$K_I = 0.683L\sqrt{\lambda h}M_{bn}/W. \tag{6}$$

If the nominal stress at the crack tip σm exceeds the tensile strength of the concrete $R_{bt}(M_{bn} > M_m)$, then it is necessary to find a new operational crack length, and the SIF is calculated using the formula

$$K_I = 0.683\sqrt{\lambda h}R_{bt}.$$

Let's analyze the SIF changing dependent on the crack length. In third line of Table 1 there is shown the SIF changing ($N/m^{3/2}$) in the beam of the (15×30) cm section at the acting moment $M = 4.5kNm$ $(M < M_m)$.

Table 1

SIF change depending on the crack length

Z	0.1	0.2	0.3	0.4	0.5	0.6	0.7
M_m	4.61	4.63	5.0	6.0	8.28	15.5	$> M_p$
K_I	0.378	0.347	0.293	0.217	0.131	0.051	0

We see that with increasing the crack length, the SIF decreases to zero. This is explained by the fact that in bending there necessarily exists a minimum zone of compressive stresses. With increasing the length of the crack, this reduces the length of the zone of tensile stresses to zero. At $z_p = 0$ tensile forces are perceived only by the reinforcement, the crack does not develop. In this case, the SIF becomes equal to zero and the beam is destroyed due to the achievement of the limit of the yield strength in the reinforcement or crushing the concrete of the compression zone.

If, with a known crack length and a given moment, the SIF will be smaller than the maximum SIF value $_{IC}$ for this material, then there is a steady development of the crack. If $K_I \geq K_{IC}$, the crack is unstable and a rapid crack growth is possible with a slight increase of the moment. The growth of the crack length continues until the SIF drops to the $_{IC}$ value. This SIF value corresponds to a certain value of the crack length, which is determined from expression (6). This value is easier to find graphically by plotting the $_I$ dependence on z using the graphs in Figure 2.

The assessment of the bearing capacity of the beam is made by the magnitude of the stress in the reinforcement (2) and the maximum compressive stress in the concrete (1). The parameters σ_m, x, z_p, found through the initial ($M < M_m$) or operational ($M > M_m$) crack length are substituted into these formulas.

Let's analyze the SIF changing dependent on the crack length at $M = 6kHm$. From Table 1 we conclude that in this case $M > M_m$ for $z_0 = 0.1 \div 0.3$, $M = M_m$ at $z_0 = 0.4$ and $M < M_m$ at $z_0 = 0.5$ and 0.6. At $z_0 \leq 0.4$

$$L = 1.6 \cdot 10^3 \cdot 2.25 \cdot 10^{-3}/6 = 0.6.$$

From the graph in Figure 2 we will find $\lambda = 0,233$ and then by the formula for the SIF we will obtain

$$K_I = 0.683 \cdot 1.6\sqrt{0.233 \cdot 0.3} = 0.29MN/m^{3/2}.$$

At $z_0 = 0.5$

$$\sigma = M/W = 6 \cdot 10^3/2.25 \cdot 10^{-3} = 2.667MPa.$$

By the graph in Figure 2 we will find $L = 0.435$, $\lambda = 0.164$. Then by formula (6) we will obtain $K_I = 0.175MN/m^{3/2}$. Similarly we determine the SIF at $z_0 = 0.6 : K_I = 0.069MN/m^{3/2}$.

From here we see that at $M > M_m$ the SIF does not depend on the length of the initial crack. It depends only on the length of the operational crack that depends on the acting moment.

Let's analyze the SIF changing with the external moment changing. To do this, we will calculate the SIF at $z_0 = 0.2$ ($M_m = 4.63kNm$). The results are shown in Table 2, where the moment is in kNm, the SIF is in $MN/m^{3/2}$.

Table 2

SIF changing dependent on the moment

M	2	4	4.63	6	8	12	16	20
K_I	0.154	0.308	0.357	0.29	0.255	0.203	0.178	0.159

The Table shows that if $M < M_m$, then the SIF increases in direct proportion to the moment, reaching its maximum at $M = M_m$. At $M > M_m$ the SIF decreases due to increasing the length of the operational crack.

The obtained results allow estimating the bearing capacity of beams with a crack, as well as their crack resistance by the stress intensity factor.

References

1 Зайцев Ю.В. Механика разрушения для строителей / Ю.В Зайцев. — М.: Высш. шк., 1991. — 288 с.

2 Карпенко Н.И. Общие модели механики железобетона / Н.И. Карпенко. — М.: Стройиздат, 1996. — 416 с.

3 Нугужинов Ж.С. Теория расчета железобетонных конструкций с повреждениями / Ж.С. Нугужинов. — Караганда: Изд-во КарГТУ, 2012. — 203 с.

4 Кодыш Э.Н. Расчет железобетонных конструкций из тяжелого бетона по прочности, трещиностойкости и деформациям / Э.Н. Кодыш, И.К. Никитин, Н.Н. Трекин. — М.: Изд-во АВС, 2010. — 352 с.

5 Беглов А.Д. Теория расчета железобетонных конструкций на прочность и устойчивость. Современные нормы и евростандарты / А.Д. Беглов, Р.С. Санжаровский. — М.: Изд-во АВС, 2006. — 221 с.

6 Тамразян А.А. Конечно-элементное исследование напряженно-деформированного состояния железобетонных балок с нормальными трещинами / А.А. Тамразян, М.М. Орлова // Научное обозрение. — 2016. — № 6. — С. 8–11.

7 Chernin L. Prediction of corrosion-induced cover cracking in reinforced concrete structures / L. Chernin, D.V. Val // Construction and Building Materials. — 2014. — 25. — No. 4. — P. 1854–1869.

8 Тамразян А.А. Экспериментальные исследования напряженно-деформированного состояния железобетонных изгибаемых элементов с трещинами / А.А. Тамразян, М.М. Орлова // Вестн. ТГСАУ. — 2015. — № 6(53). — С. 98–105.

9 Kolchunov V.I. The modeling method of discrete cracks in reinforced concrete under the torsion with bending / V.I. Kolchunov, A.I. Dem'yanov // Magazine of Civil Engineering. — 2018. — No. 5. — P. 160–173.

10 Kwan A.K.H. Crack width analysis of reinforced concrete under direct tension by finite element method and crack queuing algorithm / A.K.H. Kwan, F.J. Ma // Engineering Structures. — 2016. — 126. — P. 618–627.

11 Ma F.J. Crack width analysis of reinforced concrete members under flexure by finite element method and crack queuing algorithm / F.J. Ma, A.K.H. Kwan // Engineering Structures. — 2015. — 105. — P. 209–219.

12 Oliver-Leblond C. Modelling of three-dimensional crack patterns in deep reinforced concrete structures / C. Oliver-Leblond, A. Delaplace, F. Ragueneau // Engineering Structures. — 2015. — 83. — P. 176–186.

13 Tan R. Experimental and theoretical investigation of crack width calculation methods for RC ties / R. Tan, K. Eileraas, O. Whispers, G. Zirgulis, M. Hendriks, M. Geiker, Dan-Evert Brekke, T. Kanstad // Structural Concrete. — 2018. — 19. — 5. — P. 1436–1447.

14 Amin A. Instantaneous Crack Width Calculation for Steel Fiber-Reinforced Concrete Flexural Members / A. Amin, R.I. Gilbert // ACI Structural Journal. — 2018. — 115. — 2. — P. 535–543.

15 Hua Zhu. Crack formation of steel reinforced concrete structure under stress in construction period / Zhu Hua // Frattura ed Integrita' Strutturale. — 2016. — No. 36. — P. 191–200.

16 Salnikov A. The computational model of spatial formation of cracks in reinforced concrete constructions in torsion with bending / A. Salnikov, V. Kolchunov, I. Yakovenko // Applied Mechanics and Materials. — 2015. — P. 784–789.

17 Sheng J. Experimental and theoretical investigations on crack spacing and stiffness of textile-reinforced concrete-strengthened reinforced concrete beams / J. Sheng, S.P. Yin, S.L. Xu, Z.Y. Jin // Advances in Structural Engineering. — 2018. — 21. — 11. — P. 1696–1707.

18 Fantilli A.P. Golden Ratio in the Crack Pattern of Reinforced Concrete Structures / A.P. Fantilli, B. Chiaia // Journal of Engineering Mechanics. — 2013. — 139. — P. 1178–1184.

19 Bykov A.A. Shock wave method for monitoring crack repair processes in reinforced concrete structures / A.A. Bykov, V.P. Matveenko, I.N. Shardakov, A.P. Shestakov // Mechanics of Solids. — 2017. — 52. — P. 378–383.

20 Ma F.J. Finite element analysis of concrete shrinkage cracks / F.J. Ma, A.K.H. Kwan // Advances in Structural Engineering. — 2018. — 21. — No. 10. — P. 1454–1468.

21 Rossi P. Analysis of cracking in steel fibre-reinforced concrete (SFRC) structures in bending using probabilistic modeling / P. Rossi, D. Daviau-Desnoyers, J.L. Tailhan // Structural Concrete. — 2015. — 16. — P. 381–388.

22 Cohen M. Finite element analysis of intermediate crack debonding in fibre reinforced polymer strengthened reinforced concrete beams / M. Cohen, A. Monteleone, S. Potapenko // Canadian Journal of Civil Engineering. — 2018. — 45. — 10. — P. 840–851.

23 Черепанов Г.И. Механика разрушения / Г.И. Черепанов. — Ижевск: ИКИ, 2012. — 872 с.

24 Garcia-Alvarez V.O. Analysis of mixed-mode fracture in concrete using interface elements and a cohesive crack model / V.O. Garcia-Alvarez, R. Gettu, I. Carol // SADHANA-ACADEMY PRO-CEEDINGS IN ENGINEERING SCIENCES. — 2012. — 37. — 1. — P. 187–205.

25 Клюева Н.В. Проблемные задачи развития гипотез механики разрушения применительно к расчету железобетонных конструкций / Н.В. Клюева, В.И. Колчунов, Н.А. Яковенко // Изв. Казан. гос. архитек.-строит. ун-та. — 2014. — № 3. — С. 41–45.

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Иілетін темірбетонды арқалықтардағы кернеудің қарқындылық коэффициентін анықтау

Есептің аналитикалық шешімі жарығы бар иілетін темірбетонды арқалықтардың көлденең қимасындағы кернеуді анықтайды. Ол үшін арқалық жарық бойымен кесіледі және арқалықтың бөлінген бөлігінің тепе-теңдік жағдайынан сығылған аймақтық биіктігі мен жарықтың ұшындағы созылатын кернеу анықталды. Кернеудің қалған параметрлері осы өлшемдер арқылы көрсетілді. Жарықтың бастапқы ұзындығының ұлғаюына байланысты иілу моменті анықталды. Ол үшін жарықтың пайдалану ұзындығы анықталған. Бұл қиманың еркін формасындағы арқалықтар үшін әділ шешім. Кернеудің қарқындылық коэффициентін анықтау кернеудің шоғырлануын елемей және жарық ұшындағы бойлық күштердің теңдігіне негізделген. Кернеудің шоғырлану аймағының өлшемі жергілікті кернеудің номиналдық кернеудегі теңдік шарттарынан анықталды. Осы негізде арқалықтардың тікбұрышты қимасындағы кернеудің қарқындылық коэффициентін анықтайтын формула алынған. Мақалада кернеудің қарқындылық коэффициенті жарықтың ұзындығына, моментіне, басқа да геометриялық және пайдалану факторларына тәуелділігі талданған. Алынған нәтижелер жарығы бар арқалықтардың көтергіш қабілетін, сондай-ақ кернеудің қарқындылық коэффициенті бойынша олардың жарыққа төзімділігін бағалауға мүмкіндік береді.

Кілт сөздер: темірбетон, арқалық, иілу, сығылған аймақ, жарық, арматура, кернеулі жағдай, салмақ көтеру қабілеті, кернеудің қарқындылық коэффициенті, жарыққа төзімділік.

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Определение коэффициента интенсивности напряжений в изгибаемых железобетонных балках

Аналитически решена задача об определении напряженного состояния в сечении изгибаемых железобетонных балок с трещиной в линейной постановке. Для этого балка разрезается по линии трещины и из условий равновесия отсеченной части балки определяются высота сжатой зоны и растягивающее напряжение у вершины трещины. Остальные параметры напряженного состояния выражаются через эти величины. Определено значение изгибающего момента, при превышении которого происходит увеличение первоначальной длины трещины. Для этого случая определена длина эксплуатационной трещины. Решение справедливо для балок произвольной формы сечения. Определение коэффициента интенсивности напряжений основано на предположении о равенстве продольных сил у вершины трещины с учетом и без учета концентрации напряжений. Размер зоны концентрации напряжений определяется из условия равенства местного напряжения номинальному напряжению. На этой основе получена формула для определения коэффициента интенсивности напряжений в балках прямоугольного сечения. В работе проанализирована зависимость коэффициента интенсивности напряжений от длины трещины, момента и других геометрических и эксплуатационных факторов. Полученные результаты позволяют оценить несущую способность балок с трещиной, а также их трещиностойкость по коэффициенту интенсивности напряжения.

Ключевые слова: железобетон, балка, изгиб, сжатая зона, трещина, арматура, напряженное состояние, несущая способность, коэффициент интенсивности напряжений, трещиностойкость.

References

1 Zaitsev, Yu.V. (1991). *Mekhanika razrusheniia dlia stroitelei [Mechanics of destruction for builders].* Moscow: Vysshaia shkola [in Russian].

2 Karpenko, N.I. (1996). *Obshchie modeli mekhaniki zhelezobetona [General models of reinforced concrete mechanics].* Moscow: Stroiizdat [in Russian].

3 Nuguzhinov, Zh.S. (2012). *Teoriia rascheta zhelezobetonnykh konstruktsii s povrezhdeniiami [Theory of designing reinforced concrete structures with damages].* Karaganda: Izdatelstvo Karahandinskoho hosudarstvennoho universiteta [in Russian].

4 Kodysh, E.N., Nikitin, I.K., & Trekin, N.N. (2010). *Raschet zhelezobetonnykh konstruktsii iz tiazheloho betona po prochnosti, treshchinostoikosti i deformatsiiam [Designing reinforced heavy con-crete structures for strength, crack resistance and strains].* Moscow: Izdatelstvo AVS [in Russian].

5 Beglov, A.D., & Sanzharovskii, R.S. (2006). *Teoriia rascheta zhelezobetonnykh konstruktsii na prochnost i ustoichivost. Sovremennye normy i evrostandarty [Theory of designing strength and stability of reinforced concrete structures. Present day norms and Euro-standards].* Moscow: Izdatelstvo AVS [in Russian].

6 Tamrazian, A.A., & Orlova, M.M. (2016). Konechno-elementnoe issledovanie napriazhenno-deformirovannoho sostoianiia zhelezobetonnykh balok s normalnymi treshchinami [Finite element studying strained state of reinforced concrete beams with normal cracks]. *Nauchnoe obozrenie – Scientific Review, 6,* 8–11 [in Russian].

7 Chernin, L., & Val, D.V. (2014). Prediction of corrosion-induced cover cracking in reinforced con-crete structures. *Construction and Building Materials, 25, 4,* 1854–1869.

8 Tamrazian, A.A., & Orlova, M.M. (2015). Eksperimentalnye issledovaniia napriazhenno-deformirovannoho sostoianiia zhelezobetonnykh izhibaemykh elementov s treshchinami [Experimental studying stress-strain state of reinforced bending elements with cracks]. *Vestnik TGASU – Journal of Construction and Architecture, 6,* 98–105 [in Russian].

9 Kolchunov, V.I., & Dem'yanov, A.I. (2018). The modeling method of discrete cracks in reinforced concrete under the torsion with bending. *Magazine of Civil Engineering, 5,* 160–173.

10 Kwan, A.K.H., & Ma, F.J. (2016). Crack width analysis of reinforced concrete under direct tension by finite element method and crack queuing algorithm. *Engineering Structures, 126,* 618–627.

11 Ma, F.J., & Kwan, A.K.H. (2015). Crack width analysis of reinforced concrete members under flexure by finite element method and crack queuing algorithm. *Engineering Structures, 105,* 209–219.

12 Oliver-Leblond, C., Delaplace, A., & Ragueneau, F. (2015). Modelling of three-dimensional crack patterns in deep reinforced concrete structures. *Engineering Structures, 83,* 176–186.

13 Tan, R., Eileraas, K., Whispers, O., Zirgulis, G., Hendriks, M., Geiker, M., Brekke, Dan-Evert, & Kanstad, T. et.al. (2018). Experimental and theoretical investigation of crack width calculation methods for RC ties. *Structural Concrete, 19, 5,* 1436–1447.

14 Amin, A., & Gilbert, R.I. (2018). Instantaneous Crack Width Calculation for Steel Fiber-Reinforced Concrete Flexural Members. *ACI Structural Journal, 115, 2,* 535–543.

15 Hua, Zhu. (2016). Crack formation of steel reinforced concrete structure under stress in construction period. *Frattura ed Integrita' Strutturale, 36,* 191–200.

16 Salnikov, A., Kolchunov, V. & Yakovenko, I. (2015). The computational model of spatial formation of cracks in reinforced concrete constructions in torsion with bending. *Applied Mechanics and Materials,* 784–789.

17 Sheng, J., Yin, S.P., Xu, S.L., & Jin, Z.Y. (2018). Experimental and theoretical investigations on crack spacing and stiffness of textile-reinforced concrete-strengthened reinforced concrete beams. *Advances in Structural Engineering, 21, 11,* 1696–1707.

18 Fantilli, A.P., & Chiaia, B. (2013). Golden Ratio in the Crack Pattern of Reinforced Concrete Structures. *Journal of Engineering Mechanics, 139,* 1178–1184.

19 Bykov, A.A., Matveenko, V.P., Shardakov, I.N., & Shestakov, A.P. (2017). Shock wave method for monitoring crack repair processes in reinforced concrete structures. *Mechanics of Solids, 52,* 378–383.

20 Ma, F.J., & Kwan A.K.H. (2018). Finite element analysis of concrete shrinkage cracks. *Advances in Structural Engineering, 21, 10,* 1454–1468.

21 Rossi, P., Daviau-Desnoyers, D., &. Tailhan, J.L. (2015). Analysis of cracking in steel fibre-reinforced concrete (SFRC) structures in bending using probabilistic modeling. *Structural Concrete, 16,* 381–388.

22 Cohen, M., Monteleone, A., & Potapenko, S. (2018). Finite element analysis of intermediate crack debonding in fibre reinforced polymer strengthened reinforced concrete beams. *Canadian Journal of Civil Engineering, 45, 10,* 840–851.

23 Cherepanov, G.I. (2012). *Mekhanika razrusheniia [Fracture mechanics].* Izhevsk: IKI [in Russian].

24 Garcia-Alvarez, V.O., Gettu, R., & Carol, I. (2012). Analysis of mixed-mode fracture in concrete using interface elements and a cohesive crack model. *SADHANA–ACADEMY PROCEEDINGS IN ENGINEERING SCIENCES, 37, 1,* 187–205.

25 Kliueva, N.V., Kolchunov, V.I., & Iakovenko, N.A. (2014). Problemnye zadachi razvitiia hipotez mekhaniki razrusheniia primenitelno k raschetu zhelezobetonnykh konstruktsii [Problem tasks of de-veloping hypotheses of mechanical destruction relating to designing reinforced concrete structures]. *Izvestiia Kazanskoho hosudarstvennoho arkhitekturno-stroitelnoho universiteta – News of the Kazan State University of Architecture and Civil Engineering, 3,* 41–45 [in Russian].

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The dynamic simulation model of apples contact interaction

The mathematical model of apples shock interaction with each other and with working surfaces of technological equipment during their gathering, transportation and technological processing has been presented. In the model using the Hertz theory, strain and other running parameters of bodies shock interaction have been determined. To set changes to these parameters in time has been a peculiarity of the model, which allows simulate the process in the mode of the calculated experiment. The graphic change dependences of time contact forces, as well as examples of apples kinematics changes at initial angular and tangential velocities have been given.

Keywords: shock, contact interaction, apple, Hertz contact strain, mathematical model, calculated experiment, recovery coefficient.

Introduction

One of the most important requirements for apples is to ensure a long shelf life without losing conditional qualities. For this purpose, during their harvesting and processing, damages and excessive shock loads must be avoided because that may violate the integrity and structure of apples pulp. Therefore, when preparing the apple handling process and designing the appropriate technological equipment, it is necessary to follow the requirements that will ensure safe modes of harvesting, transportation and processing of apples for storage. According to the results of well-known researches [1–5] and everyday practical experience, depending on the varieties of apples, they have different dimensional, physical and technical characteristics. Since, as a rule, apples of smaller sizes are used mainly for the manufacture of juices, then in storage are laid mainly apples of larger sizes with a diameter of 60–80 mm. Such apples, due to their greater weight, receive greater dynamic damage during transportation. In the shock theory, the assumption of fruits momentary impact, where the main parameters of interaction correspond to the law of saving the amount of movement, have been assumed. This assumption leads to a significant simplification of the interaction calculation model and it is inadmissible for the simulation of objects interaction in time and the simultaneous interaction of several fruits, their movement in the flow (container), etc., where the assumption of the momentality of the contact is meaningless. In studies [6–17], it has been shown that with a small impact load for simulation of shock interaction one can use a rheological model of an elastic body, and to determine the force of impact - the Hertz formula. However, the Hertz formula makes it impossible to establish the time for elastic contact of bodies and changes in contact forces, magnitude, and speed of convergence over time. Also, kinematics of bodies, mainly apples, at the time of and after contact, when they have initial angular or tangential velocities, have not been investigated.

Material and method

Long-term storage of apples is ensured by following high requirements that prevent their damage, that is, not exceeding the allowable contact strain when they interact with the technological surfaces and with each other. One of the most suitable ways of solving problems of apples contact interaction with each other and working surfaces is to construct a model based on the elastic contact interaction of apples on impact. This model has a real physical content at low kinetic energies of fruits. According to Hertz contact problem, the connection between the force of contact interaction P_{12}, the radius of the contact platform a_{12}, and the magnitude of the close convergence of the spheres (apples) u_{12} are as follows [5, 7, 8, 10]:

$$a_{12} = \sqrt[3]{\frac{3\pi P_{12}(q_1+q_2)R_1R_2}{4(R_1+R_2)}}; \quad u_{12} = \sqrt[3]{\frac{9\pi^2 P_{12}^2(q_1+q_2)^2(R_1+R_2)}{16R_1R_2}}, \tag{1}$$

where R_1 and R_2 — radii of the spheres, q_1 and q_2 — constants characterizing elastic properties of interaction bodies and are determined through the Lame constants λ_i and μ_i, or, relayively, Poisson coefficients ν_i and shear modules $\mu_i = G_i$ and Jung (elasticity)E_i.

$$q_i = \frac{\lambda_i + 2\mu_i}{4\pi\mu_i(\lambda_i + \mu_i)}; \quad \lambda_i = \frac{2\nu_i G_i}{1 - 2\nu_i}; \quad \mu_i = G_i = \frac{E_i}{2(1+\nu_i)}. \tag{2}$$

In accordance, the force of contact interaction P_{12} of apples between each other or apple with platform of magnitude of hard convergence u_{12} connected by dependence

$$P_{12} = \frac{k_{12} u_{12}^{3/2}}{\sqrt{K_1 + K_2}}, \tag{3}$$

where k_{12} — a parameter that takes into account the elastic properties of the contact body in the Hertz contact problem; K_1 and K_2 – the average curvature of the surfaces of the interaction bodies at the point of contact (for curved surfaces with a minus sign). For an apple in the form of a sphere in radius R_i average curvature of the surface (the sphere) $K_i = R_i$, in case of an apple impact with a flat platform, the average curvature of such is $K_j = 0$.

The model parameter of two bodies interaction k_{12} takes into account the mechanical properties of the interaction bodies,

$$k_{12} = \frac{4}{3\pi} \Big/ \left(\frac{1-\mu_2^2}{E_2} + \frac{1-\mu_1^2}{E_1} \right). \tag{4}$$

Consider the impact interaction of apples with spherical form radius R_1 with a hard flat platform. Implicit in the Cartesian coordinate system $Oxyz$ apple platform surface, the center of its gravity has running coordinates $C[x_0(t); y_0(t); z_0(t)]$, is described by the dependence [11],

$$f_S(x, y, x, t) = [(x - x_0(t))^2 + (y - y_0(t))^2 + (z - z_0(t))^2 - R_1^2]^{0,5} = 0. \tag{5}$$

The potential of a geometric field, which is described by a function (5) at an arbitrary point $A(x_A; y_A; z_A)$ will be equal to the distance to the surface of the fruit (p. E) at time t,. $p_A = f_S(x_A, y_A, z_A, t) = l_{AE}(t)$.

Accordingly, the surface of the platform describe the normal plane equation that describes a similar field. For the case of a stationary platform

$$f_P(x, y, x,) = \cos\alpha \cdot x + \cos\beta \cdot y + \cos\gamma \cdot z - d = 0, \tag{6}$$

where α, β, and γ — directional angles of the platform normali vector; d — the distance from the platform to the coordinates beginning.

At the point of contact of the apple with the plane $E(x_E; y_E; z_E)$ both functions (5) and (6) takes the value $f_S(x_E, y_E, z_E, t) = f_P(x_E, y_E, z_E) = 0$, but normali to the surface of the apple $\bar{n}_S$ and sufaces of the platform $\bar{n}_P$ will be $\bar{n}_S = -\bar{n}_P$.

Here $\bar{n}_S = grad f_S$; $\bar{n}_P = grad f_P = \cos\alpha \cdot \bar{i} + \cos\beta \cdot \bar{j} + \cos\gamma \cdot \bar{k}$.

So, the force vector of impact interaction $\bar{N}_P$, which influences on an apple from the side of the platform will be $\bar{N}_P = -\bar{P}_{21} = P_{21} \cdot \bar{n}_P$.

The apple movement speed at any given time t will be determined by the vector

$$\vec{\nu}_S(t) = \frac{\partial x_0(t)}{\partial t} \cdot \bar{i} + \frac{\partial y_0(t)}{\partial t} \cdot \bar{j} + \frac{\partial z_0(t)}{\partial t} \cdot \bar{k}. \tag{7}$$

The value of the hard convergence of an apple by radius R_1 with platform, which is described by the dependence (5), at the point of contact will be

$$u_{12}(t) = R_1 - f_{Pl}(x_0, y_0, x_0,) = R_1 - d + \cos\alpha \cdot x_0 + \cos\beta \cdot y_0 + \cos\gamma \cdot z_0 = 0. \tag{8}$$

In general case i is such apple by weight m_i can contact with other apples or work surfaces and have j contact points E_{ij}. On the selected movable i apple in general influence such forces as: earthly gravity $\bar{G}_i$; inertia — $m_i \bar{a}_i$, directed opposite to the acceleration vector $\bar{a}_i = d\bar{\nu}_S/dt$; normal and tangential forces $\bar{P}_{ij}$ and $\bar{F}_{ij}$ in each E_{ij} — contact zone from interaction with j body. Accordingly, moments also influence into an apple $\bar{M}_{Pij}$ and $\bar{M}_{Fij}$ from forces $\bar{P}_{ij}$ and $\bar{F}_{ij}$, the moments of twisting from the rotating motion of the fruit at the point of contact $\bar{T}_{ij}$ and inertial moments of forces. At the same time, equation of equilibrium of all forces, applied to the apple it is expedient to put in a fixed, basic coordinate system $Oxyz$, and equation of equilibrium of forces moments – in its own coordinate system of the apple, $O^\wedge x^\wedge y^\wedge z^\wedge$ the center of which $O^\wedge$ located in the center of the apple $C[x_0(t); y_0(t); z_0(t)]$, and the axes are tightly bound with the body of the apple [6–8, 11]

$$\sum_{j=1}^{k}\left(\bar{P}_{ij}+\bar{F}_{ij}\right)-m_i\bar{a}_i+\bar{G}_i=0; \quad \sum_{j=1}^{k}[\left(\bar{r}^\wedge_{ij}+\bar{\delta}^\wedge_{ij}\right)\times\left(\bar{P}^\wedge_{ij}+\bar{F}^\wedge_{ij}\right)+\sum_{j=1}^{k}T^\wedge_{ij}-\bar{L}^{(e)}_{oi}=0, \tag{9}$$

where $\bar{P}_{ij}$ and $\bar{P}^\wedge_{ij}$ — vectors of normal forces of elastic interaction on the Hertz model are given in the general and proper coordinate systems; $\bar{F}_{ij}$ and $\bar{F}^\wedge_{ij}$ — the corresponding vectors of tangential forces; $\bar{r}_{ij}$ and $\bar{r}^\wedge_{ij}$ — the corresponding radii-vectors of ij zone; $\bar{\delta}^\wedge_{ij}=\bar{F}_{ij}\nu_i/(4a_{ij}G_i)$ — the tangential displacement of contact platform from force $\bar{F}^\wedge_{ij}$, [6,7]; $\bar{L}^{(e)}_{oi}$ — vector amount of forces moments.

In the coordinate system of a moving body $\bar{L}^{(e)}_{0i}=\frac{d\bar{K}^\wedge_{0i}}{dt}+\left(\left[\bar{\omega}^\wedge_0\times\bar{K}^\wedge_{0i}\right]\right)$, where $\bar{K}^\wedge_{0i}$ — apple kinetic moment. The twisting moments form a small fate of power factors, that act on a particle (less than 2–3 %), that's why they are not taken into account in the simplified model. Friction forces have been determined by Amonton-Coulomb law $\bar{F}_{ij}=\mu\Delta\bar{\nu}^\wedge_{eij}/|\Delta\bar{\nu}^\wedge_{eij}|$, where μ — the coefficient of dry friction, $\bar{\nu}^\wedge_{eij}$ — speed of the point on the apple surface relative to the platform E_{ij} in contact zone.

The transition from the coordinate system $Oxyz$ to $O^\wedge x^\wedge y^\wedge z^\wedge$ have been conducted in a homogeneous base $\zeta_x\zeta_y\zeta_z\zeta$ and own apple $\zeta^\wedge_x\zeta^\wedge_y\zeta^\wedge_z\zeta$ coordinate systems, where $\zeta=\zeta^\wedge$ — a scale multiplier [11, 12]. For the objects of unchanging volume $\zeta=\zeta^\wedge=1$. The relation between coordinate systems $Oxyz$ and $O^\wedge x^\wedge y^\wedge z^\wedge$ with homogeneous coordinates will be the following

$$x=\zeta_x/\zeta; \quad y=\zeta_y/\zeta; \quad z=\zeta_z/\zeta; \quad x^\wedge=\zeta^\wedge_x/\zeta^\wedge; \quad y^\wedge=\zeta^\wedge_y/\zeta^\wedge; \quad z^\wedge=\zeta^\wedge_z/\zeta^\wedge.$$

The matrix record of the transition from its own coordinate system of the apple to the base will look like [12]:

$$M(R)=\begin{vmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} & x_0(t)\\ \alpha_{21} & \alpha_{22} & \alpha_{23} & y_0(t)\\ \alpha_{31} & \alpha_{32} & \alpha_{33} & z_0(t)\\ 0 & 0 & 0 & 1\end{vmatrix}M(R^\wedge)=\Pi_\nu\Pi_\omega(\alpha)M(R^\wedge), \tag{10}$$

where α_{ij} — guiding cosines between the axes of the base and their own coordinate systems; $M(R)=|xyz1|^T$ — a matrix that specifies the coordinates of an arbitrary point of the body in the general coordinate system $Oxyz$ and corresponds to the vector $r(t)=x(t)i+y(t)j+z(t)k$; $M(R^\wedge)=|x^\wedge y^\wedge z^\wedge 1|^T$ — matrix corresponding to the vector $r(t)=x^\wedge(t)i+y^\wedge(t)j+z^\wedge(t)k$ and specifies the coordinates of the same point in its own coordinate system of the object $O^\wedge x^\wedge y^\wedge z^\wedge$; $r_0(t)=x_0(t)i+y_0(t)j+z_0(t)k$ — radius vector that connects the beginning of the base coordinate system with the start of its own aperture coordinate system, which coincides with its gravity center; Π_ν and Π_ω — matrices of linear displacements and turns of its own coordinate system $O^\wedge x^\wedge y^\wedge z^\wedge$ in general $Oxyz$:

Similarly, the matrices of the inverse transform from the inertial coordinate system to the own coordinate system of the object jave been written.

In an expanded form, the system of equations (9) takes the form:

$$\sum_{i=1}^{k}P_{ij}\left[grad(f_i)-\frac{\mu\Delta\bar{\nu}_{eij}}{|\Delta\bar{\nu}_{eij}|}\right]-m_i\bar{a}_i+\bar{G}_i=0; \tag{11}$$

$$\sum_{i=1}^{m}P_{ij}\left\{(\bar{r}^\wedge_{ij}+\bar{\delta}^\wedge_{ij})\times\left[grad(f^\wedge_i)-\frac{\mu\Delta\bar{\nu}^\wedge_{eij}}{|\Delta\bar{\nu}^\wedge_{eij}|}\right]\right\}-\bar{L}^{(e)}_{0i}=0.$$

Here the first equation of the system (11) is written in the base (fixed) coordinate system, and the second - in the own (moving) coordinate system of the apple.

The indicated dependences are the basis for constructing an imitation model for the interaction of apples with each other and with technological surfaces. In the model, the description of bodies interaction has been given by the normal equations of their surfaces, whose potential geometric fields determined their mutual placement in space, their convergence and availability of contact points. In case of contact, the interval Δt was set for the procedure of numerical differentiation. Step by step the amount of hard convergence u_{12} and contact forces $\bar{P}_{ij}$ and $\bar{F}_{ij}$, that occur at such convergence and moments from these forces have been determined. The next step was to determine the linear and angular accelerations from the forces found and the new values of the velocities and displacements of each body (apple) interaction and their linear and angular velocities, which defined new coordinates of the bodies through the period of time Δt and new levels of hard convergence u_{ij} and contact forces $\bar{P}_{ij}$ and $\bar{F}_{ij}$.

In order to take into account energy losses in determining the contact interaction forces, equation (3) was represented as [6, 7]

$$P_{12} = \frac{k_{12} u_{12}^{3/2} [1 + k_e + (1 - k_e) th(-\lambda du_{ij}/dt)]}{2\sqrt{K_1 + K_2}}, \tag{12}$$

where k_e — coefficient, which depends on the coefficient of recovery at impact, $k_e = 0.8 - 0.95$; λ — parameter of the model smoothing the load curve at the point of maximum force value P_{12}, $\lambda > 10$; Parameters of the model k_e and λ are specified experimentally from the condition while providing a given recovery factor e.

Results

The constructed model allowed to conduct a computational experiment and set the time for contact interaction t_k and running forces of contact interaction, depending on the physical and mechanical characteristics of the interaction bodies, as well as kinematics of bodies (apples) during contact interaction, depending on the initial conditions.

The study of the shock interaction of one apple with a hard platform has a significant practical value, because it is one of the most common case of fruits bruising. Implementation of the models for different initial conditions is presented in Figure 1-5.

In particular, in Figure 1 it has been shown a graph of time variation of shock contact interaction $P = P_{12}$ for an apple with a diameter of 80 mm, the modulus shift of which is $\mu_1 = G_1 = 1.1\ MPa$, Poisson's coefficient $\nu_1 = 0.18$ with a rigid steel surface ($G_2 = 8.1 \times 10^{10}\ Pa$, $\nu_2 = 0.28$) for cases where the initial apple speed at the moment of contact varies from $\nu_0 = 0.2\ m/s$ till $\nu_0 = 1.4\ m/s$. In this case, the force of interaction increased from $P_{0.2} = 11.5\ N$ at the time of interaction $t_{0.2} = 0.0088\ s$, till 97.0 N at $t_{1.4} = 0.0055\ s$.

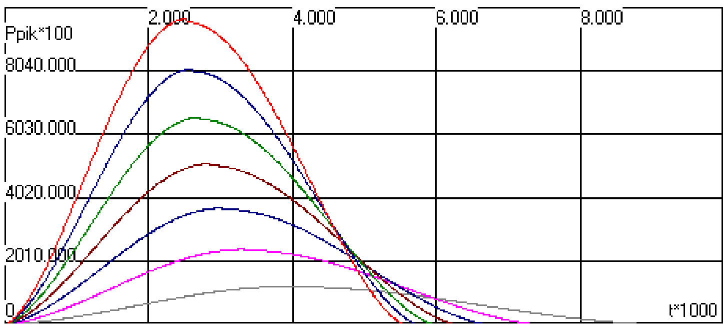

Figure 1. The change of the running force of the apple interaction contact ($G_1 = 1\ MPa$, $\nu_1 = 0.18$) with steel platform ($G_2 = 8.1 \times 10^{10}\ Pa$, $\nu_2 = 0.28$) at different initial velocities of interaction $\nu_0 \in [0.2; 0.4; 0.6; 0.8; 1.0; 1.2; 1.4;]\ m/s$

From the graph, it is clear that with the increase of the initial rate of contact, the force of impact interaction significantly increases with a slight decrease in contact interaction time.

Figure 2 shows a graph of changes in the time of the force of impact contact interaction $P = P_{12}$ for an apple of the same physical and mechanical characteristics for different diameters (6; 8 and 10 mm) with a steel platform at initial contact speeds $\nu_0 = 0.5\ m/s$ (bottom graph) and $\nu_0 = 1.0\ m/s$.

The analysis of Figure 2 shows that with increased apple size, the force and time of the contact interaction increases, and at the lower collision rates, the interaction time is greater. Figure 3 shows the graphical dependences of the time variation of the impact contact interaction force $_{12}$ with a steel platform for apples of different hardness, that is, different values of the displacement module, at the initial collision speed $\nu_0 = 1.0\ m/s$.

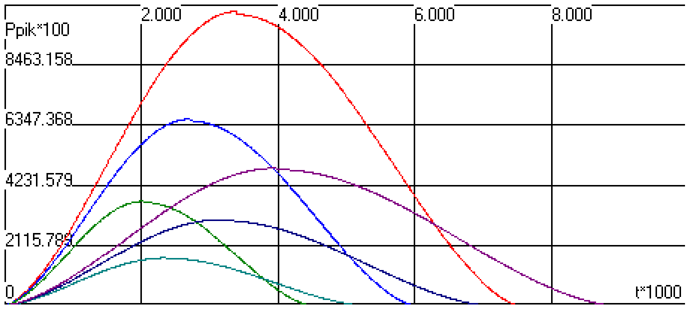

Figure 2. The change of the running force of the apple interaction contact ($D = 80\ mm, \nu_1 = 0.18$) with different diameters $D \in [60; 80; 100]\ mm$ with a steel platform ($G_2 = 8.1 \times 10^{10}\ Pa, \nu_2 = 0.28$) at interaction speed $\nu_0 = 0.5\ m/s$ and $\nu_0 = 1.0\ m/s$

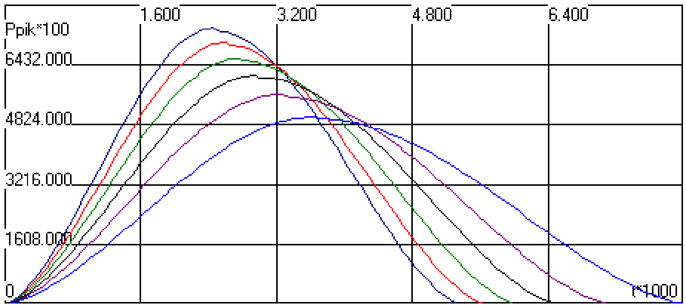

Figure 3. The change of the running force of the apple interaction contact at ($D = 80\ mm, \nu_1 = 0.18$) of different hardness $G_1 \in [0.6; 0.8; 1.0; 1.2; 1.4; 1.6]\ MPa$ with a steel platform ($G_2 = 8.1 \times 10^{10}\ Pa, \nu_2 = 0.28$) at a speed collision $\nu_0 = 1.0\ m/s$

From the graphs it has been seen that for harder apples, the impact forces are larger with less contact time interaction.

Figure 4 shows the interaction of apples with platforms of different materials.

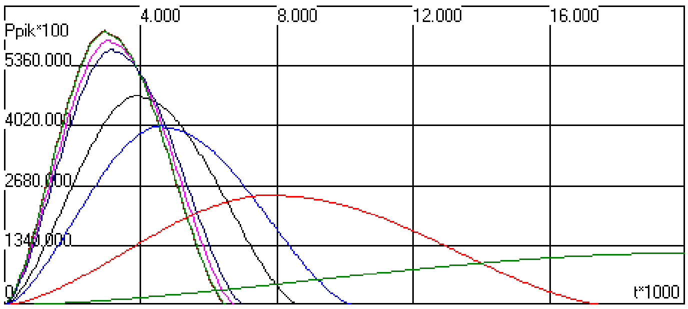

Figure 4. The change of the running force of the apple interaction contact ($D = 80\ mm, G_1 = 1\ MPa, \nu_1 = 0.18$) with an platform with different values of the shift modulus $G_2 \in [1 \cdot 10^4; 1 \cdot 10^5; 5 \cdot 10^5; 1 \cdot 10^6; 5 \cdot 10^6; 1 \cdot 10^7; 1 \cdot 10^8]$

It follows from the graphs (Fig. 4) that the impact strength can be significantly reduced only when the platform is made of a material for which the shift modulus (or the Young module) is smaller or at least equal to the apple displacement modulus. The shiftness platform change in the rate $G_2 > 1 \cdot 10^8$ practically does not affect on the change in the force of impact P_{12}. So, the replacement of the steel platform on the wooden ($G_2 = 4 \cdot 10^9\ Pa$) practically does not reduce the forces of contact interaction.

In the case of falling packets with apples placed in several layers into the platform, the most injured are apples placed in the lower layer, which are in contact with the rigid bottom and perceive loading from the upper layers.

Figure 5 shows graphs of force time variation of apples contact interaction in the presence of upper layers.

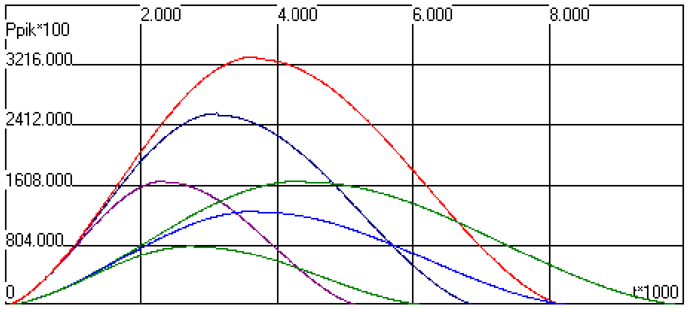

Figure 5. The change of the running force of the lower layers the apple interaction contact ($D = 80\ mm, G_1 = 1\ MPa, \nu_1 = 0.18$) in a container with a plastic platform ($G_2 = 1 \times 10^9\ Pa, \nu_2 = 0.3$) with the placement of apples in 1, 2 and 3 layers with the speed of interaction with the container to the platform $\nu_0 = 0.25\ m/s$ and $\nu_0 = 0.5\ m/s$

The approximation of dependencies of force impact change in time for a case of contact interaction with sufficient accuracy for practical use, it is advisable to make dependencies of the

$$P_{ij}(t) = P_{ij\,\max} \left(\frac{t}{\tau \cdot t_k} \right)^{\varepsilon} \left[\frac{t_k - t}{t_k(1-\tau)} \right]^{\varepsilon(1-\tau)/\tau}, \tag{13}$$

where ε– the shape coefficient of the curve, for elastic impact $\varepsilon \approx 2$, for visco-elastic and elastic-plastic bodies $\varepsilon < 1.5$; t_k - a time of contact interaction; τ – the asymmetry parameters of the curve interaction, $\tau = t_{P\,\max}/t_k$, for a symmetric curve $\tau = 0.5$. The bigger values of the coefficient ε are taken for greater speeds of collision and normal impact without a tangential component.

If the mass element j significantly greater than mass element i, $m_j > m_i$ so, in accordance with the law of conservation of momentum, element with mass m_i on impact it changes the magnitude and direction of movement (reflected from the surface of the platform) and therefore the ratio is the following

$$m_i \nu_0 (1 + e) = \int_0^{t_k} P_{ij} dt = k_P P_{ij\,\max} t_k, \tag{14}$$

where e– recovery coefficient at impact, for apples $e = 0.25 - 0.35$; k_P - the parameter determining the shape effect of the curve on the magnitude of the force impulse.

The shape of the curve varies a liitle bit from the physical and mechanical characteristics of the apple and to a small extent depends on the speed of impact. For critical speed of apples collision with a hard platform $k_P = 0.53 - 0.54$.

Firthermore, taking into account (14) the dependence (13) takes the form

$$P_{ij}(t) = \frac{m_i \nu_0 (1+e)}{k_P t_k} \left(\frac{t}{\tau \cdot t_k} \right)^{\varepsilon} \left[\frac{t_k - t}{t_k(1-\tau)} \right]^{\varepsilon(1-\tau)/\tau}. \tag{15}$$

Since dependence (15) describes the change in the force of contact interaction at a given contact time t_k, which is easy to install experimentally, so it has a significant practical value.

The developed model also allows to simulate a change in the kinematic parameters of apples when they are transported on technological surfaces, taking into account their initial linear and angular displacements. In Figure 6 and 7 the kinematics of the transition from slipping to rolling the apples on a steel platform at different initial values of linear and angular velocities (without impact) has been shown.

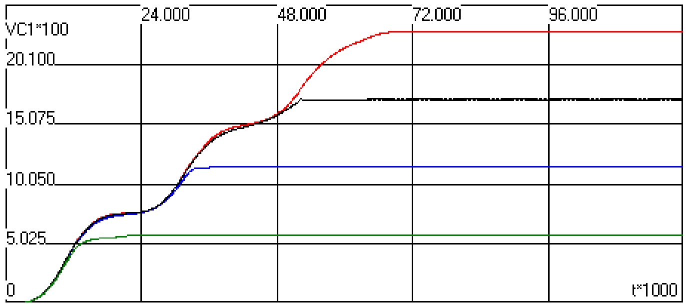

Figure 6. The change of apples linear speed ($D = 80\ mm, G_1 = 1\ MPa, \nu_1 = 0.18$) on steelsurface ($G_2 = 8.1 \times 10^{10}\ Pa, \nu_2 = 0.28$) in the transition from slipping to rolling at an initialspeed $\nu_x = 0$ and different angular velocities $\omega_y = 5; 10; 15; 20 s^{-1}$

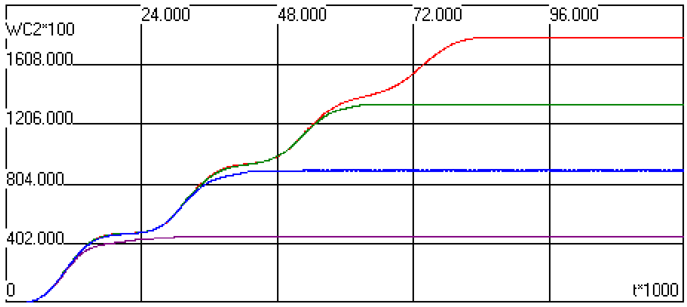

Figure 7. The change of apples linear speed ($D = 80\ mm, G_1 = 1\ MPa, \nu_1 = 0.18$) on steel surface ($G_2 = 8.1 \times 10^{10}\ Pa, \nu_2 = 0.28$) in the transition from slipping to rolling at an initial speed $\omega_y = 0$ and different angular velocities $\nu_x \in 0.25; 0.5; 0.75; 1.0\ m/s$

As follows from the figures, the transition from slipping to rolling of apples having an initial only linear or angular velocity occurs over a short period of time (less than $0.1s$).

Conclusions

The deduced dependencies are the basis of the simulation mathematical model, which allows to determine not only the parameters of contact interaction of apples between themselves or with technological surfaces, but also to simulate in the mode of the computational experiment the processes of transportation of apples, their movement during processing under the conditions of interaction with several objects simultaneously. According to the results of the study, it was found that the contact time of the apples between themselves and in hard containers does not exceed $0.001\ s$. It is grater for apples of larger size, of soft varieties and with layered apples placed in the container. To significantly reduce the force of contact interaction and increase the contact time (more than $0.01\ s$), it is necessary that the modulus of the elasticity of the platform was one-two times less than the modulus of apples elasticity. With the increase in the collision rate of apples with the platform or among themselves, the contact interaction strength increases significantly with a slight decrease in contact time. The critical values of the force of the impact contact interaction, in which the structure of the flesh of the apples for the solid varieties has not disturbed, is $P_{\max} = 80$, for soft — $35 - 40$.

References

1 Roudot A.-C.G. Modelling the Response of Apples to Loads / A.-C.G. Roudot, F. Duprat, C. Wenian // Journal of Agricultural Engineering Research. —1991. — 48(4). — 2497259.

2 Степанова Е.Г. Реологические свойства яблок при различных свойствах их обработки / Е.Г. Степанова, В.А. Причко // Известия вузов. Пищевая технология. — 1999 . — №5 – 6. — С.72.

3 Yuwana Y. Prediction of Apple Bruising Based on the Instantaneous Impact Shear Stress and Energy Absorbed / Y. Yuwana, F. Duprat // International Agrophysics. — 1998. — 12. — 1337140.

4 Rybczynski R. Mechanical. Resistance of Apple in Different Place of Fruit. / R. Rybczynski, B. Dobrzanski // International Agrophysics. – 1994. – 9(3). – 4557459.

5 Grotte M. Young's modulus, poisson's ratio, and lame's coefficients of golden delicious apple. / M. Grotte, F. Duprat, E. Piétri & D. Loonis // International Journal of Food Properties. — 2002. — 5(2). — P. 333–349. DOI: 10.1081/JFP-120005789.

6 Рогатинский Р.М. Силовое взаимодействие коренебульбоплодив с рабочими органами шнековых очистителей / Р.М. Рогатинский // Вестн. Нац. аграр. ун-та. — М.: НАУ, 1997. — Т. 1. — С. 98–103.

7 Рогатинский Р.М. Модель контактного взаимодействия частицы груза с рабочими поверхностями сельскохозяйственных машин / Р.М. Рогатинский и др. // Вестн. Харьков. гос. техн. ун-та сельского хозяйства «Механизация сельскохозяйственного производства». — Харьков, 2003. — Вип. 21. — С. 222–228.

8 Гупка Б.В. Модели контактного взаимодействия частиц технологической среды с рабочими поверхностями деталей машин / Б.В. Гупка, А.Р. Рогатинская // Науч. заметки: межвуз. сб. (по направлению «Инженерная механика»). — Вып. 11. — Луцк: Изд-во ЛГТУ, 2002. — С. 114–120.

9 Пик А.И. Динамическая модель взаимодействия частиц сыпучего груза между собой и с рабочими поверхностями машин / А.И. Пик, А.В. Дудин, А.Р. Рогатинская // Вестн. Харьков. гос. техн. ун-та сельского хозяйства. Механизация сельскохозяйственного производства. — Харьков, 2004. — Вып. 24. — С. 120–127.

10 Hanaor D.A.H. Contact mechanics of fractal surfaces by spline assisted discretisation / D.A.H. Hanaor, Y. Gan, I. Einav // International Journal of Solids and Structures. —2015. — 59. — P. 121–131. DOI:10. 1016/j.ijsolstr.2015.01.021.

11 Рогатинський Р.М. Моделювання процесів взаємодії шнекових робочих органів із коренебульбоплодами / Р.М. Рогатинський // Вісник Національного аграрного університету. — Киев: НАУ, 1997. — Т. 1. — С. 103–108.

12 Шенен П. Математика и САПР: [В 2-х кн.] Кн. 1. / П. Шенен, М. Коснар, И. Гардан и. др.; пер. с фр. — М.: Мир, 1988. — 204 с.

13 Гевко Р.Б. Направления совершенствования свеклоуборочной техники сникив / Р.Б. Гевко, И.Г. Ткаченко, С.В. Синий, В.М. Булгаков, Р.М. Рогатинский, О.Б. Павелчак. — Луцк: ЛГТУ, 1999. — 168 с.

14 Hevko R.B. Determination of interaction parameters and grain material flow motion on screw conveyor elastic section surface / R.B. Hevko, S.Z. Zalutskyi, Y.B. Hladyo, I.G. Tkachenko, O.L. Lyashuk, O.M. Pavlova, B.V. Pohrishchuk, O.M. Trokhaniak, N.V. Dobizha // INMATEH: Agricultural engineering. Bucharest, Romania. — 2019. — Vol. 57. — No. 1. — P. 123–134.

15 Hevko R.B. The influence of bluk material flow on technical and economical performance of a screw conveyor / R.B. Hevko, V.M. Baranovsky, O.L. Lyashuk, B.V. Pohrishchuk, Y.P. Gumeniuk, O.M. Klendii, N.V. Dobizha // INMATEH: Agricultural engineering. Bucharest, Romania. — 2018. — Vol. 56. — No. 3. — P. 175–184.

16 Lyashuk O.L. Modelling of the vertical screw conveyer loading / O.L. Lyashuk, O.R. Rogatynska, D.L. Serilko // INMATEH: Agricultural engineering. Bucharest, Romania. — 2015. — Vol. 45. — No. 1. — P. 87–94.

17 Lyashuk O.L. Mathematical model of bending vibrations of a horizontal feeder-mixer along the flow of grain mixture / O.L. Lyashuk, M.B. Sokil, V.M. Klendiy, O.P. Skyba, O.L. Tretiakov, L.M. Slobodian et al. // INMATEH: Agricultural engineering. Bucharest, Romania. — 2018. — Vol. 55. — No. 2. — P. 35–44.

Р.Рогатынский, Р.Гевко, Ю.Никеруй, О.Дмитров, Р.Розум

Алманың өзара жанасуы әрекетінің динамикалық имитациялық моделі

Алмалардың бір-бірімен, сондай-ақ технологиялық жабдықтардың жұмыс беттерімен оларды жинау, тасымалдау және технологиялық өңдеу кезіндегі соққымен өзара әсерлесуінің математикалық моделі ұсынылған. Герц теориясын пайдалана отырып модельде кернеуді, сондай-ақ дененің соққылық әсерінің басқа да ағымдағы параметрлерін анықтайды. Модельдің ерекшелігі уақыт параметрлерінің өзгеруін орнату болып табылады, бұл есептелген эксперимент режимінде процесті модельдеуге мүмкіндік береді. Сонымен қатар, бастапқы бұрыштық және тангенциялық жылдамдық кезіндегі алма кинематикасының өзгеру мысалдары келтірілген.

Кілт сөздер: соққы, өзара байланысу әрекеті, алма, Герцтің түйіспелі кернеуі, математикалық модель, есептелетін эксперимент, қалпына келтіру коэффициенті.

Р.Рогатынский, Р.Гевко, Ю.Никеруй, О.Дмитров, Р.Розум

Динамическая имитационная модель контактного взаимодействия яблок

Представлена математическая модель ударного взаимодействия яблок между собой, а также с рабочими поверхностями технологического оборудования при их уборке, транспортировке и технологической обработке. В модели с использованием теории Герца определяются напряжения, а также другие текущие параметры ударного взаимодействия тел. Особенность модели есть установление изменений параметров во времени, что позволяет моделировать процесс в режиме исчисляемого эксперимента. Приведены графические зависимости изменения контактных сил во времени, а также примеры изменения кинематики яблок при начальных угловых и тангенциальных скоростях.

Ключевые слова: удар, контактное взаимодействие, яблоко, контактные напряжения Герца, математическая модель, исчисляемый эксперимент, коэффициент восстановления.

References

1 Roudot, A.-C.G., Duprat, F., & Wenian, C. (1991). Modelling the Response of Apples to Loads. *Journal of Agricultural Engineering Research,* 48 (4), 2497259.

2 Stepanova, E.G., & Prichko, V.A. (1999). Reolohicheskie svoistva yablok pri razlichnyikh svoistvakh ikh obrabotki [The rheological properties of apples with different properties of their processing]. *Izvestiia vuzov. Pischevaia tehnolohiia — Izvestiia vuzov. Food technology,* No. 5–6, 72 [in Russian].

3 Yuwana, Y., & Duprat, F. (1998). Prediction of Apple Bruising Based on the Instantaneous Impact Shear Stress and Energy Absorbed. *International Agrophysics, 12,* 1337140.

4 Rybczynski, R., & Dobrzanski, B. (1994). Mechanical. Resistance of Apple in Different Place of Fruit. *International Agrophysics,* 9 (3), 4557459.

5 Grotte, M., Duprat, F., Piétri, E., & Loonis, D. (2002). Young's modulus, poisson's ratio, and lame's coefficients of golden delicious apple. *International Journal of Food Properties, 5:2,* 333–349, DOI: 10.1081 /JFP-120005789.

6 Rohatynskyi, R.M. (1997). Sylova vzaiemodiia korenebulboplodiv iz robochymy orhanamy shnekovykh ochysnykiv [The power interaction of root crops with working organs of screw cleaners]. *Visnyk Natsionalnoho ahrarnoho universiteta — Bulletin of the National Agrarian University, Vol. 1,* 98–103. Kyiv: NAU [in Ukrainian].

7 Rohatynskyi, R.M., & et al. (2003). Model kontaktnoi vzaiemodii chastynky vantazhu z robochymy poverkhniamy silskohospodarskykh mashyn [Model of contact interaction of a particle with the working surfaces of agricultural machines]. *Visnyk Kharkivskoho derzhavnoho tekhnichnoho universytetu silskoho hospodarstva «Mekhanizatsiia silskohospodarskoho vyrobnytstva» — Journal of the Kharkov State Technical University of Agriculture «Mechanization of agricultural production», Vol. 21,* 222–228. Kharkiv, LSTU [in Ukrainian].

8 Hupka, B.V., & Rohatynska, O.R. (2002). Modeli kontaktnoi vzaiemodii chastynok tekhnolohichnoho seredovyshcha z robochymy poverkhniamy detalei mashyn [Models of contact interaction of technological environment particles with working surfaces of machine parts]. *Naukovi notatky. Mizhvuzivskyi zb. (za napriamkom «Inzhenerna mekhanika») – Scientific notes. Interuniversity collection («Engineering Mechanics»), Lutsk, LDTU. Vol. 11,* 114–120 [in Ukrainian].

9 Pik, A.I., Dudin, O.V., & Rohatynska, O.R. (2004). Dynamichna model vzaiemodii chastynok sypkoho vantazhu mizh soboiu ta z robochymy poverkhniamy mashyn [The dynamic model of interaction of bulk cargo particles with each other and with working surfaces of machines]. *Visnyk Kharkivskoho derzhavnoho tekhnichnoho universytetu silskoho hospodarstva — Bulletin of Kharkiv State Technical University of Agriculture, Mechanization of agricultural production, Vol. 24,* 120–127 [in Ukrainian].

10 Hanaor, D. A. H., Gan, Y., & Einav, I. (2015). Contact mechanics of fractal surfaces by spline assisted discretisation *International Journal of Solids and Structures.* 59: 121–131. DOI:10.1016/j.ijsolstr.2015.01.021.

11 Rohatynskyi, R.M. (1997). Modeliuvannia protsesiv vzaiemodii shnekovykh robochykh orhaniv iz korenebulboplodamy [Modeling of the processes of interaction of screw working bodies with root tubes]. *Visnyk Natsionalnoho ahrarnoho universytetu — Bulletin of the National Agrarian University, Kyiv, NAU. Vol. X1,* 103–108. [in Ukrainian].

12 Shenen, P., Kosnar, M., & Gardan, I., et al. (1988). Matematika i SAPR: V dvuh kn. Kn. 1.Per. s frants. [Mathematics and CAD: In two books. Book. 1.Transl from Franch]. Moscow: Mir [in Russia].

13 Hevko, R.B., Tkachenko, I.H., Synii, S.V., Bulhakov, V.M., Rohatynskyi, R.M., & Pavelchak, O.B. (1999). *Napriamky vdoskonalennia buriakozbyralnoi tekhniky snykiv [Directions of improvement of beet harvesting technique of cutters].* Lutsk: LDTU [in Ukrainian].

14 Hevko, R.B., Zalutskyi, S.Z., Hladyo, Y.B., Tkachenko, I.G., Lyashuk, O.L., Pavlova, O.M., Pohrishchuk, B.V., Trokhaniak, O.M., & Dobizha, N.V. (2019). Determination of interaction parameters and grain material flow motion on screw conveyor elastic section surface. *INMATEH: Agricultural engineering. Bucharest, Romania, Vol. 57,* No. 1, 123–134.

15 Hevko, R.B., Baranovsky, V.M., Lyashuk, O.L., Pohrishchuk, B.V., Gumeniuk, Y.P., Klendii, O.M., & Dobizha, N.V. (2018). The influence of bluk material flow on technical and economical performance of a screw conveyor. *INMATEH: Agricultural engineering. Bucharest, Romania, Vol. 56,* No. 3, 175–184.

16 Lyashuk, O.L., Rogatynska, O.R., Serilko, D.L. (2015). Modelling of the vertical screw conveyer loading *INMATEH Agricultural Engineering, Bucharest. Romania. Vol. 45,* No. 1, 87–94,

17 Lyashuk, O.L., Sokil, M.B., Klendiy, V.M., Skyba, O.P., Tretiakov, O.L., Slobodian, L.M., & Slobodian, N.O. (2018). Mathematical model of bending vibrations of a horizontal feeder-mixer along the flow of grain mixture, *INMATEH Agricultural Engineering, Bucharest. Romania, Vol. 55,* No. 2, 35–44.

ИНФОРМАТИКА
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The development of the model of the intelligent system on the basis of fuzzy sets for microclimate control of building

Nowadays, much attention is paid to the creation of favorable conditions for the health and work of people and the microclimate in the design and further operation of buildings. It is necessary to investigate the processes of forming a microclimate in the room to assess the comfort of the microclimate, as well as to determine the required capacity of the equipment of engineering systems. Analytical research methods, methods of mathematical and computer modeling are used as research methods. The methods of analysis and synthesis of automatic control systems, mathematical apparatus of theory of fuzzy logic, the Matlab software environment, the system of visual modeling Simulink, and the system for designing fuzzy systems Fuzzy Logic Toolbox are used for research. The paper presents a model of the room, taking into account the heat loss through the enclosing structures, the model of the air conditioner, the structure of the fuzzy control system, the algorithm of its functioning, input and output variables of the fuzzy controller, the composition of their terms, membership functions, the formed complete database of rules. On the basis of research methods, the actual scientific and practical problem of developing an intelligent control system for the formation of the comfort microclimate of the building is solved.

Keywords: intelligent system, building microclimate, simulation, fuzzy inference system, fuzzy logic, linguistic rules, air conditioning system.

Introduction

The favorable microclimate is created by the building's engineering systems, such as heating, ventilation and air conditioning.

Using of computer simulation and the choice of an adequate microclimate model reduce labor costs in the design of the above systems.

At present, in the domestic and foreign literature, a large number of articles are devoted to research models and methods of microclimate control in buildings [1–15].

At the same time, there are no publications of scientists where research and development of a control system would be conducted taking into account the characteristics of the room and all disturbing influences.

All microclimate control models are divided into three classes: white, black and gray box models [16; 225].

Analysis of the works of foreign authors showed that the most used models are models based on the comfort index PMV/PDD (23 %), models based on fuzzy logic (22 %) and learning models (20 %).

The advantages and disadvantages of models of three classes are considered in the work [16; 226, 227].

Despite the shortcomings, the considered models have the ability to maintain thermal comfort in the building. But the most effective for solving this problem is the use of fuzzy systems [17–19].

Fuzzy systems are suitable for maintaining microclimate parameters in both industrial and office premises, for a number of reasons:

– the ability to control nonlinear systems with dynamically changing parameters, even in the absence of a complete priori information about the control object;

– the use of expert knowledge in a particular subject area and their representation in the form of linguistic variables that are close to human perception;

– tangible improvements in performance of control processes in the case of application of fuzzy controllers.

The purpose of this article is to research and develop the building microclimate control system based on fuzzy logic (fuzzy logic is a branch of mathematics that is a generalization of classical logic and set theory, based on the concept of fuzzy set), taking into account all disturbing influences and characteristics of room, as well as its approbation (checking the adequacy of the system by conducting a simulation).

1 Development of Matlab model for fuzzy temperature control system in the room

Temperature control is one of the major factors influencing microclimate in both industrial and residential buildings. Therefore, the modeling of the control system based on the fuzzy controller is tested on the basis of the room temperature control loop.

This control loop consists of the following blocks (Fig. 1):

– Block «Set temperature» establishes the desired temperature in the room.

– Block «Fuzzy controller» represents a model of the fuzzy controller, which is formed using Fuzzy Logic Toolbox package, designed specifically for constructing the fuzzy expert and / or control systems.

– Using «Room model» block, heat losses through constructions enclosing room are considered.

– Block «Conditioner Model». In buildings, the conditioner is installed, which can operate in two modes: a heating or cooling.

The system operates in the following way: the difference between the set and the current room temperature is provided to the input of the controller. Based on the generated base of rules, the controller provides an output signal to the conditioner, which includes the heating or the cooling mode with the appropriate productivity based on the error value.

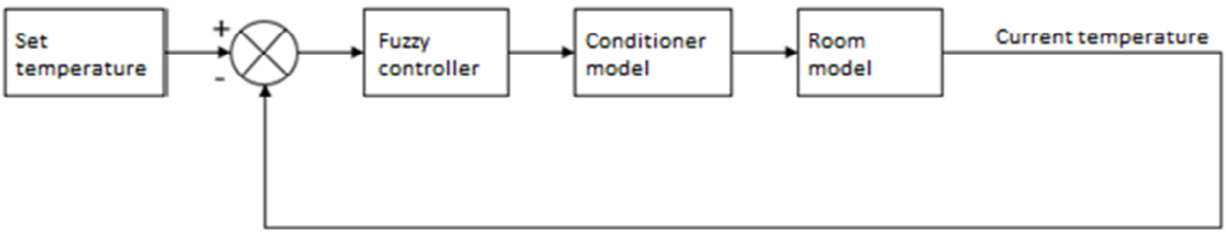

Figure 1. Temperature control loop block diagram

2 Determination and calculation of heat loss through the enclosing surfaces

Heat losses occur in the room through the walls, windows, ceiling and floor.

The following is the general formula for the calculation of heat loss:

$$Q_{heatloss} = Q_{walls} + Q_{windows} + Q_{ceiling} + Q_{floor},$$

where, $Q_{walls}, Q_{windows}, Q_{ceiling}, Q_{floor}$ — heat loss through exterior walls, windows, ceiling and floor, respectively.

The premise with the following characteristics is taken as an example for the calculation of heat losses.

The premise is located on the first floor of the business center.

The dimensions of the premises:

– Length - 8 m;

– Width - 5 m;

– Ceiling height - 3 m.

The premise consists of two external walls, one of which has two windows with the following dimensions (WxH): 1.45 x1.5 m and two internal walls, one of which has a doorway.

Characteristics of exterior walls.

Exterior wall construction:

– brickwork with a thickness of 640 mm (thermal conductivity is 0,7 kcal/m·hour·°C);

– internal plaster with a thickness of 15 mm (thermal conductivity is 0,6 kcal/m·hour·°C).

Heat transfer resistance R_{in} for internal surfaces of walls, floors, as well as ceilings with a smooth surface is 0.133 m^2·hour·°C/kcal. The heat transfer resistance for R_{ex} outer surfaces which are contacted with the outside air (exterior walls) is 0.5 m^2·hour·°C/kcal.

Window construction: plastic profile frame (heat transfer coefficient of plastic profile 0.2 W/m^2·°C), double glazing (heat transfer resistance coefficient 0.51 m^2·°C/W, heat transfer coefficient 1.96 W/m^2·°C).

Heat losses through the wall are determined according to the expression [20]:

$$Q_{walls} = k_{walls} \cdot S_{walls} \cdot (t_{in} - t_{out}),$$

where, Q_{walls} — heat loss through exterior walls, kcal/h; k_{walls} — heat transfer coefficient of the wall, kcal/m^2·hour·°C; S_{walls} — an area of the wall, m^2; t_{in}- internal temperature of the room, °C; t_{out} — outside temperature, °C.

Heat transfer coefficient of the wall k_{walls} is calculated by formula [21; 55]:

$$k_{walls} = 1/R_c,$$

where, R_c — heat resistance of enclosure construction m^2·hour·°C / kcal.

Since the wall is a multilayered enclosure, the value R_c is determined by the formula [21; 55]:

$$R_c = R_{in} + R_1 + R_2 + ...R_{ex},$$

where, R_{in} — heat resistance at the inner surface of the enclosure; R_1, R_2 — thermal resistance of the individual layers of enclosure; R_{ex} — heat resistance at the outer surface of the enclosure.

Thermal resistance of homogeneous enclosure or layer constituting the multilayer enclosures is calculated by the formula [21; 32]:

$$R = \delta/\lambda$$

where, δ — layer thickness, m; λ — thermal conductivity of the material, kcal / m·hour·°C.

Applying formulas 3 and 4, the heat transfer resistance of external walls is calculated:

$$R_c = 0.133 + 0.015/0.6 + 0.64/0.7 + 0.05 = 1.12m^2 \cdot hour \cdot {}^\circ C/kcal.$$

Heat transfer coefficient of exterior walls:

$$k_{walls} = 1/1.12 = 0.89kcal/m^2 \cdot hour \cdot {}^\circ C.$$

1 kcal = 4.19 J, then k_{walls}=3730 J/m^2·hour·°C.

Heat transfer coefficient of the ceiling and floor are calculated similarly heat transfer coefficient of exterior walls.

Heat loss through the window openings are determined according to the expression [20]:

$$Q_{windows} = k_{windows} \cdot S_{windows} \cdot (t_{in} - t_{out}),$$

where $Q_{windows}$ — heat loss through windows, W; $k_{windows}$ — heat transfer coefficient of windows (W/m^2·°C); $S_{windows}$ — the area of windows, m^2.

The heat transfer coefficient of windows $k_{windows}$ calculated by the following formula [22]:

$$k_{windows} = \frac{k_{gl}S_{gl} + k_f S_f}{S_t},$$

where k_{gl} – heat transfer coefficient of the glazing unit, W/(m^2·°C); S_{gl} — glazing area, m^2; k_f — heat transfer coefficient of frame (plastic profile), W/(m^2·°C); S_f — frame area, m^2; S_t — window area, m^2.

The heat transfer coefficient of windows $k_{windows}$:

$$k_{windows} = \frac{1.96 \cdot 1.5525 + 0.2 \cdot 06225}{2.175} = 1.46W/m^2 \cdot {}^\circ C.$$

The heat transfer coefficient of two windows $k_{windows}$= 2.92 W/(m^2·°C);

1 W = 3600 J/hour, then $k_{windows}$ = 10512 J/m^2·hour·°C.

3 Development of fuzzy controller for temperature control loop in the room

Input variable of fuzzy controller is the difference between set and current temperatures. The output parameter is the corresponding signal to the actuator of conditioner. Terms and data for constructing membership functions of the input and output variables are presented in Tables 1, 2.

The structure of the model of the fuzzy controller is shown in Figure 2.

Table 1

Terms of variable «Temperature difference»

No.	Name of the term	Designation of the term	Range, °C
1	Negative Big	NB	from -22 to -9
2	Negative Middle	NM	from -14 to -4
3	Negative Small	NS	from -9 to 0
4	Zero	Z	from -4 to 4
5	Positive Small	PS	from 0 to 8
6	Positive Middle	PM	from 3 to 13
7	Positive Big	PB	from 8 to 17

Table 2

Terms of variable «Heater (Cooler)»

No.	Name of the term	Designation of the term	Range, %
1	Stop	ST	0
2	Low Power	LP	0-30
3	Average Power	AP	30-60
4	High Power	HP	60-100

Table 3 shows the formed base of rules.

Table 3

The rule base of fuzzy temperature control system in the room

No.	If the temperature difference is important	Then the conditioner is operating in the mode of	
		cooling	heating
1	Negative Big	At full power	Off
2	Negative Middle	At average power	Off
3	Negative Small	At low power	Off
4	Zero	Off	Off
5	Positive Small	Off	At low power
6	Positive Middle	Off	At average power
7	Positive Big	Off	At full power

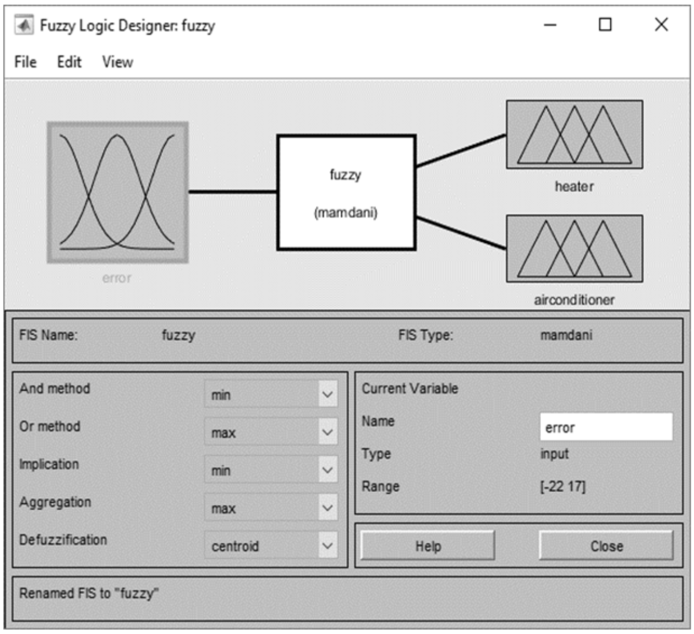

Figure 2. The structure of the model of the fuzzy controller

At the next stage, the verification of the adequacy of the developed control system based on a fuzzy controller for temperature control loop have produced in Simulink package in Matlab software environment (Fig. 3).

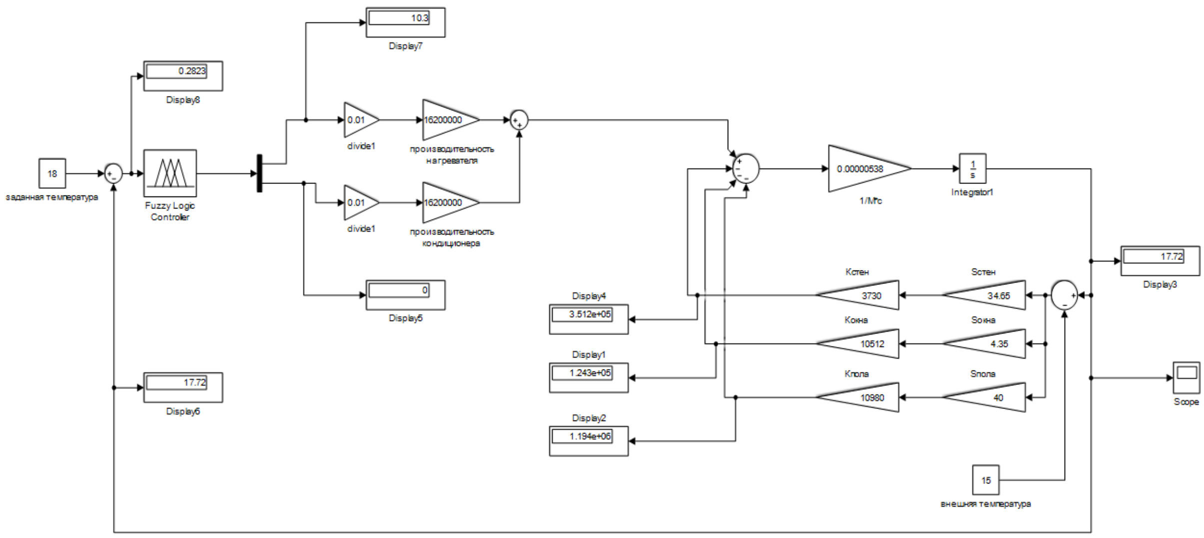

Figure 3. The fuzzy microclimate system for temperature control loop

Set temperature of Tset = 18 °C is defined for modeling. According to the simulation results, the temperature difference is equal to 0.2823 °C. In response to this difference, the controller initiates a control signal to the conditioner for turning on the heating mode with performance of 10.3% (Fig. 4). This scenario indicates that rule № 5 is implemented, when the error has a positive small value (it is slightly cold in the premise), it is required to turn off the cooling and turn on the heating at low power. The modeling results show that the fuzzy controller is suitable for maintaining microclimate parameters at the required level (Fig. 5).

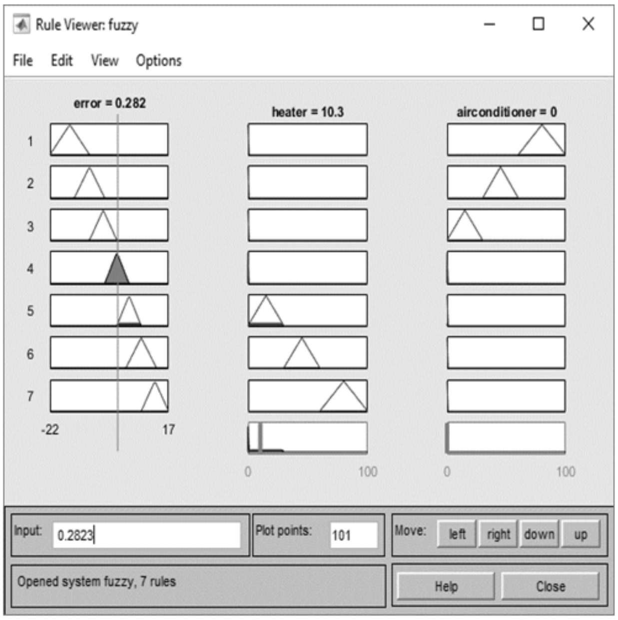

Figure 4. The value of the control signal for 0.2823°C temperature difference

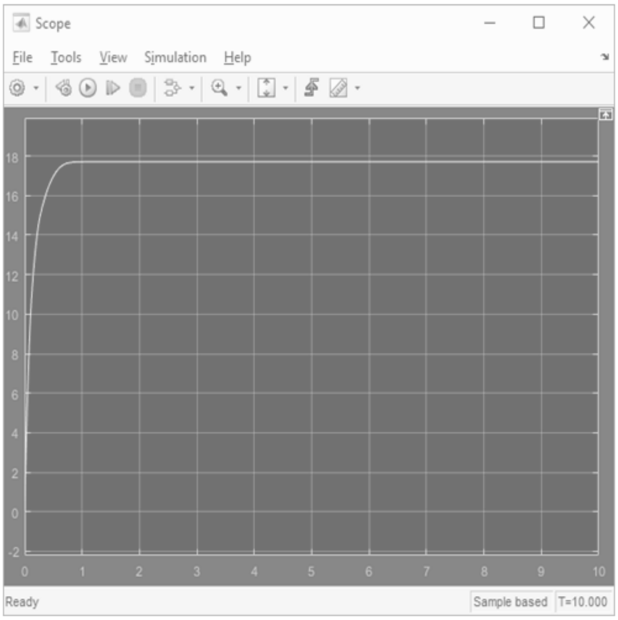

Figure 5. Plot of current temperature based on simulation results

Main result

The research has produced the following results, which have a scientific and practical importance:

1. The analysis of the microclimate control models has performed.

2. Heat loss through the enclosing surfaces (walls, windows, ceiling and floor) of the room have calculated in the article.

3. The model of fuzzy controller has developed using Fuzzy Logic Toolbox package.

4. The simulation of developed control system taking into account the characteristics of the premise has conducted in Simulink package of Matlab software environment.

5. The research results have demonstrated that the developed intelligent microclimate control system of buildings based on the fuzzy logic ensures the maintenance of microclimate parameters (temperature) at the required level.

The further researches in this direction refer to the exploring the possibility of using a fuzzy controller to optimize the performance of various climatic equipment (heater, humidifier, fan plants, etc.).

References

1 Сансызбай Л.Ж. Интеллектуальная система управления процессом формирования микроклимата в помещении / Л.Ж. Сансызбай, Б.Б. Оразбаев // Вест. ЕНУ им. Л.Н. Гумилева. Сер. естеств.-техн. наук. — 2017. — №4(119). — С. 161–169.

2 Бобриков Д.А. Многоуровневая автоматизированная система управления микроклиматом на основе нечеткой логики: дис. ... канд. техн. наук: 05.13.06 – «Автоматизация и управление технологическими процессами и производствами (по отраслям)» / Д.А. Бобриков. — М., 2017. — 154 с.

3 Пешко М.С. Адаптивная система управления параметрами микроклимата процессов производства и хранения пищевых продуктов: дис. ... канд. техн. наук: 05.13.06 – «Автоматизация и управление технологическими процессами и производствами (по отраслям)» / М.С. Пешко. — М., 2015. — 200 с.

4 Савосин С.И. Интеллектуальная система контроля влажности и температуры воздуха в теплице: дис. ... канд. техн. наук: 05.13.06 – «Автоматизация и управление технологическими процессами и производствами (по отраслям)» / С.И. Савосин. — М., 2009. — 132 с.

5 Hasim N. Intelligent room temperature controller system using MATLAB fuzzy logic toolbox / N. Hasim, M.S.M. Aras // International Journal of Science and Research (IJSR). — 2014. — Vol. 3. — P. 1748–1753.

6 Das T.K. Design of A Room Temperature and Humidity Controller Using Fuzzy Logic / T.K. Das, Y. Das // American Journal of Engineering Research (AJER). — 2013. — Vol. 2. — P. 86–97.

7 Alcala R. Fuzzy Rule Reduction and Tuning of Fuzzy Logic Controllers for a HVAC System / R. Alcala, J. Alcala–Fdez, M.J. Gacto, F. Herrera // Journal Fuzzy Applications in Industrial Engineering — 2006. — Vol. 201. — P. 89–117.

8 Calvino F. The control of indoor thermal comfort conditions: introducing a fuzzy adaptive controller / F. Calvino, M.L. Gennusa, G. Rizzo, G. Scaccianoce // Journal Energy and Buildings. — 2004. — Vol. 36. — P. 97–102.

9 Eftekhari M. Application of fuzzy control in naturally ventilated buildings for summer conditions / M. Eftekhari, L. Marjanovic // Journal Energy and Buildings. — 2003. — Vol. 35. — P. 645–655.

10 Kolokotsa D. Comparison of the performance of fuzzy controllers for the management of the indoor environment / D. Kolokotsa // Journal Building and Environment. — 2003. — Vol. 38. — P. 1439–1450.

11 Mirinejad H. A review of intelligent control techniques in HVAC systems / H. Mirinejad, K.C. Welch, L. Spicer // Proc. of the IEEE Energytech. — 2012. — P. 1–5.

12 Molina–Solana M. Unifying Fuzzy controller for Indoor Environment Quality / M. Molina–Solana, M. Ros, M. Delgado // Proc.Joint of IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS). — 2013. — P. 1080–1085.

13 Xu X.N. Intellectual controlling and energy–saving management for terminal equipment of air–conditioning system / X.N. Xu, Y.F. Ding // Building Science. — 2008. — Vol. 24(8). — P. 43–46.

14 Soyguder S. Design and simulation of selftuning pid–type fuzzy adaptive control for an expert hvac system / S. Soyguder, M. Karakose, H. Alli // Expert Systems with Applications. — 2009. — Vol. 36. — P. 4566–4573.

15 Dounis A. Advanced control systems engineering for energy and comfort management in a building environment–a review / A. Dounis, C. Caraiscos // Renewable and Sustainable Energy Reviews. — 2009. — Vol. 13. — P. 1246–1261.

16 Карпенко А.В. Модели управления микроклиматом в помещении / А.В. Карпенко, И.Ю. Петрова // Фундаментальные исследования. — 2016. — № 7(2). — С. 224–229.

17 Zadeh L. Fuzzy sets and systems / L. Zadeh // International Journal of General Systems. — 1990. — Vol. 17. — P. 129–138.

18 Takagi T. Fuzzy identification of systems and its applications to modeling and control / T. Takagi, M. Sugeno // IEEE Transactions on Systems, Man and Cybernetics. — 1985. — Vol. 15(1). — P. 116–132.

19 Mamdani E. Application of fuzzy logic to approximate reasoning using linguistic synthesis / E. Mamdani // IEEE Transactions on Computers. — 1977. — Vol. 26(12). — P. 1182–1191.

20 Пешко М.С. Раскрытая математическая модель микроклимата грибной теплицы / М.С. Пешко // Молодой ученый. — 2011. — № 9. — С. 42–48.

21 Малявина Е.Г. Теплопотери здания / Е.Г. Малявина. — М.: АВОК–ПРЕСС, 2007. — 146 с.

22 СНиП РК 2.04–03–2002. Строительная теплотехника. [Введен в действие от 2003–03–01]. — Астана: КАЗГОР, 2002. — 54 с.

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Ғимараттың микроклиматын интеллектуалды басқару жүйесінің моделін айқын емес жиындар теориясы негізінде құру

Қазіргі таңда ғимаратты жобалау және одан әрі пайдалану кезінде адамдардың денсаулығы мен жұмысы үшін қолайлы жағдай жасауға, микроклиматқа үлкен көңіл бөлінеді. Микроклиматтың жайлылығын бағалау, сондай-ақ инженерлік жүйелер жабдықтарының қажетті жұмыс қуатын анықтау үшін үй-жайлардың микроклиматының қалыптасу процестерін зерттеу қажет. Зерттеу әдістері ретінде аналитикалық, математикалық және компьютерлік модельдеу әдістері қолданылды. Зерттеу жүргізу үшін автоматты басқару жүйелерін талдау және синтездеу әдістері, анық емес логика теориясының математикалық аппараты, Matlab қолданбалы бағдарламалар пакеті, Simulink визуальды модельдеудің графикалық жүйесі, Fuzzy Logic Toolbox анық емес жүйелерін әзірлеуге арналған жүйесі қолданылды. Жұмыс барысында қоршау конструкциялары арқылы жылу шығынын ескеретін үй-жай моделі, кондиционер моделі, басқару жүйесінің құрылымы, оның жұмыс істеу алгоритмі, анық емес реттегіштің кіріс және шығыс айнымалылары, олардың термаларының құрамы, тиістілік функциялары анықталған, ережелердің толық базасы қалыптасқан. Теориялық зерттеу, математикалық және компьютерлік моделдеу жүргізу нәтижесінде үй-жайлардың микроклиматын интеллектуалды басқару жүйесі айқын емес логика базасында құрылып, қазіргі таңдағы өзекті ғылыми-практикалық мәселелердің бірі шешілді.

Кілт сөздер: интеллектуалдық бақылау жүйесі, ғимараттың микроклиматы, айқын емес қорытындылау жүйесі, айқын емес логика, лингвистикалық ережелер, кондиционерлеу жүйесі.

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Разработка модели интеллектуальной системы управления микроклиматом здания на основе теории нечетких множеств

На сегодняшний день при проектировании и дальнейшей эксплуатации зданий большое внимание уделяется созданию благоприятных условий для здоровья и работы людей, микроклимата. Для оценки комфортности микроклимата, а также определения требуемой мощности работы оборудования инженерных систем необходимо исследование процессов формирования микроклимата в помещении. В качестве методов исследования применены аналитические методы исследования, методы математического и компьютерного моделирования. Для проведения исследований использованы методы анализа и синтеза систем автоматического управления, математический аппарат теории нечеткой логики, пакет прикладных программ Matlab, система визуального моделирования Simulink, система для разработки нечетких систем Fuzzy Logic Toolbox. В работе представлена модель помещения, учитывающая теплопотери через ограждающие конструкции, модель кондиционера, определены структура нечеткой системы управления, алгоритм ее функционирования, входные и выходные переменные нечеткого регулятора, состав их термов, функций принадлежности, сформирована полная база правил. На основании проведения теоретического исследования, математического и компьютерного моделирований решена актуальная научно-практическая задача разработки интеллектуальной системы управления процессами формирования микроклимата помещений, реализуемой на базе теории нечеткой логики.

Ключевые слова: интеллектуальная система, микроклимат здания, система нечеткого вывода, нечеткая логика, лингвистические правила, система кондиционирования.

References

1 Sansyzbay, L.Zh. & Orazbayev, B.B. (2017). Intellektualnaia sistema upravleniia protsessom formirovaniia mikroklimata v pomeshchenii [Intelligent control systems indoor climate]. *Vestnik ENU imeni L.N. Gumileva. Seriia estestvenno-tekhnicheskikh nauk — Bulletin of the L.N.Gumilyov Eurasian National University. Natural and technical Series, 4(119),* 161–169 [in Russian].

2 Bobrikov, D.A. (2017). Mnohourovnevaia avtomatizirovannaia sistema upravleniia mikroklimatom na osnove nechetkoi lohiki [Multi-level automated climate control system based on fuzzy logic]. *Candidate's thesis.* Moscow [in Russian].

3 Peshko, M.S. (2015). Adaptivnaia sistema upravleniia parametrami mikroklimata protsessov proizvodstva i khraneniia pishchevykh produktov [Adaptive control system of microclimate parameters of food production and storage processes]. *Candidate's thesis.* Moscow [in Russian].

4 Savosin, S.I. (2009). Intellektualnaia sistema kontrolia vlazhnosti i temperatury vozdukha v teplitse [Intellectual control system of humidity and air temperature in the greenhouse]. *Candidate's thesis.* Moscow [in Russian].

5 Hasim, N., & Aras, M.S.M. (2014). Intelligent room temperature controller system using MATLAB fuzzy logic toolbox. *International Journal of Science and Research (IJSR), 3,* 1748–1753.

6 Das, T.K., & Das, Y. (2013). Design of A Room Temperature And Humidity Controller Using Fuzzy Logic. *American Journal of Engineering Research (AJER), 2,* 86–97.

7 Alcala, R., Alcala-Fdez, J., Gacto, M.J., & Herrera, F. (2006). Fuzzy Rule Reduction and Tuning of Fuzzy Logic Controllers for a HVAC System. *Journal Fuzzy Applications in Industrial Engineering, 201,* 89–117.

8 Calvino, F., Gennusa, M.L., Rizzo, G., & Scaccianoce, G. (2004). The control of indoor thermal comfort conditions: introducing a fuzzy adaptive controller. *Journal Energy and Buildings, 36,* 97–102.

9 Eftekhari, M., & Marjanovic, L. (2003). Application of fuzzy control in naturally ventilated buildings for summer conditions. *Journal Energy and Buildings, 35,* 645–655.

10 Kolokotsa, D. (2003). Comparison of the performance of fuzzy controllers for the management of the indoor environment. *Journal Building and Environment, 38*, 1439–1450.

11 Mirinejad, H., Welch, K.C., & Spicer, L. (2012). A review of intelligent control techniques in HVAC systems. *Proceedings of the IEEE Energytech,* 1–5.

12 Molina-Solana, M., Ros, M., & Delgado, M. (2013). Unifying Fuzzy controller for Indoor Environment Quality. *Proceedings of Joint of IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS),* 1080–1085.

13 Xu, X.N., & Ding, Y.F. (2008). Intellectual controlling and energy-saving management for terminal equipment of air-conditioning system. *Building Science, 24(8)*, 43–46.

14 Soyguder, S., Karakose, M., & Alli, H. (2009). Design and simulation of selftuning pid-type fuzzy adaptive control for an expert hvac system. *Expert Systems with Applications, 36(3),* 4566–4573.

15 Dounis, A., &, Caraiscos, C. (2009). Advanced control systems engineering for energy and comfort management in a building environment-a review. *Renewable and Sustainable Energy Reviews,13*, 1246-1261.

16 Karpenko, A.V., & Petrova, I.Iu. (2016). Modeli upravleniia mikroklimatom v pomeshchenii [Microclimate control models]. *Fundamentalnye issledovaniia– Fundamental research, 7(2),* 224–229 [in Russian].

17 Zadeh, L. (1990). Fuzzy sets and systems. *International Journal of General Systems, 17,* 129–138.

18 Takagi, T., & Sugeno, M. (1985). Fuzzy identification of systems and its applications to modeling and control. *IEEE Transactions on Systems, Man and Cybernetics, 15(1),* 116–132.

19 Mamdani, E. (1977). Application of fuzzy logic to approximate reasoning using linguistic synthesis. *IEEE Transactions on Computers, 26(12),* 1182–1191.

20 Peshko, M.S. (2011). Raskrytaia matematicheskaia model mikroklimata hribnoi teplitsy [The disclosed mathematical model of the microclimate in the mushroom greenhouses]. *Molodoi uchenyi — Young scientist, 9,* 42–48 [in Russian].

21 Maliavina, E. G. (2007). *Teplopoteri zdaniia [The heat losses of the building].* Moscow: AVOK-PRESS [in Russian].

22 SNiP RK 2.04-03-2002. Stroitelnaia teplotekhnika [Construction heat engineering] *from 1st March of 2003.* Astana: KAZGOR [in Russian].

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Development of neural network models for the analysis of infocommunication traffic

This article discusses the problems of today's infocommunication networks, the basis of which are multiservice networks serving all types of traffic, presented as a set of IP packets. The characteristic features of this traffic are analyzed, each of which is oriented to a certain class of services. The knowledge gained as a result of ongoing traffic research is an essential factor for increasing the effectiveness of decisions made in various fields of the telecommunications industry. The need for knowledge of the nature of traffic circulating in the network and the laws of its behavior is revealed and substantiated. Without this, it is impossible to effectively manage networks, develop solutions for their development, ensure network security and maintain the required level of quality. Despite the large number of works about building multi-service networks, a number of issues require further study. Analysis of traffic studies of modern converged, multiservice networks showed the lack of knowledge about its nature and laws of behavior, given the high variability of its characteristics. Thus, it can be argued that the parameters of the studied traffic are statistical, probabilistic in nature, can vary randomly over time and, accordingly, based on the study, the author proposes a study using statistical analysis methods. To study traffic, you should use the tools of probability theory and mathematical statistics.

Keywords: infocommunication networks, neural networks, multiservice traffic, probabilistic traffic characteristics, mathematical traffic models, multimedia services, multiservice traffic, neural network technologies, studied distributions, traffic classifications, information objects.

Introduction

Artificial neural networks being, although very primitive, but still, being an analog of the principles of the human brain are used to solve those problems that are unknown how to solve. To solve these problems, there is no finite or approximate solution algorithm that can be expressed by an equation or a block diagram. These, in particular, include recognition and recognition of speech and images, as well as complex forecasting tasks carried out with a limited set of known input data, analysis of big data, i.e. such tasks that are solved by a person «unconsciously». Such tasks include the tasks of studying telecommunications traffic.

Previous studies in [1–3] confirmed the earlier hypothesis about the stability of the distributions of the probability characteristics of traffic generated by typical information objects, such as student campuses, office centers, sports and entertainment facilities, etc.

Since the traffic of these objects is, in the probabilistic sense, stable in nature, its characteristic features can be described by certain models that can be used in the construction and development of communication networks focused on specific typical traffic sources.

Thus, there is reason to believe that network traffic generated by various information objects will have distinctive features of the probability distributions of the corresponding traffic characteristics. Accordingly, as one of the directions of the study of infocommunication traffic, one should single out the classification of the probability distributions of its characteristics.

The use of neural networks for the analysis of infocommunication traffic

An analytical review of modern scientific and technical literature in the field of infocommunications showed that the most widespread in works devoted to traffic approximation, approximation of observed distributions, as well as the construction of mathematical traffic models, were such types of distributions as: Poisson distribution, normal distribution, Weibull distribution, Pareto distribution and hyperexpo-nential distribution.

In multiservice networks, mathematical models of traffic flows from the simplest Poisson to the complex model of fractal processes can be used.

The advent of pervasive sensor networks, the Internet of things, NGN networks, and other technologies complicates the dynamic processes that occur in information and communication networks. Multimedia services (video streaming, interactive games, etc.) are gaining more and more popularity as compared to classical communication services.

Three main types of traffic can be distinguished [4]:

1. Homogeneous traffic in monoservice networks. Representatives of this type are telephone networks that provide only voice communication services; in most cases, this traffic corresponds to an elementary Poisson stream model. This model has been widely used in information distribution systems such as switched telephone networks. The description of flow models in classical telephone networks was mainly carried out on the basis of an ordinary flow without aftereffect, obeying the family of Poisson distributions [5]. The widespread use of the Poisson stream is due, in particular, to one of its essential properties - the additivity property - the sum of Poisson flows is also a Poisson stream.

2. Heterogeneous traffic in multiservice networks. The nature of this traffic is determined by the wide range of services provided and the integrated characteristics of the multiservice network. This type of traffic is characterized by increased unevenness.

3. Multiservice traffic in packet networks. Traffic of this type is heterogeneous and even more different from the Poisson stream. A single network is used to transmit streams of various services, but traffic sources differ in data transfer speed, type of traffic generation and transmission. Such flows are characterized by «burstiness», which causes even greater traffic unevenness.

Thus, packet switched multiservice networks traffic is already significantly different from Poisson stream. Multiservice traffic has non-stationary properties and is described by heavy-tailed distributions. Such type of distributions, including the phenomenon of long-term dependence, are signs a large number of popular IP applications correspond to slowly decaying distributions, for example, some studies show that VoIP is characterized by Pareto distribution, FTP / TCP traffic is characterized by Weibull distribution or lognormal distribution, and HTTP / TCP is characterized by lognormal distribution or Pareto distribution [6].

The Weibull and Pareto distributions are examples of slowly decreasing dependencies. Due to the large dispersion value, these distributions are characterized by certain computational difficulties, because of the need to consider large amounts of static information [7].

As you know, classification issues are effectively addressed using neural network technologies. Consider the possibility of using neural networks to classify the main typical probability distributions.

The analysis of neural network models determined the choice for further studies of a multilayer network with direct signal propagation.

There are various software tools for implementing neural networks: Matlab, Python, RStudio, C ++, etc. At the previous stage of the study, the Matlab software package was chosen as a tool for working with neural networks [8–10]. It provides a convenient graphical interface and contains most of the necessary built-in functions, however, experience with the package showed its lack of speed with an increase in the amount of input training effects. Therefore, the programming language Python was chosen as a platform for implementing a neural network.

Python provides the ability to select a neural network model, flexible parameter settings, as well as various implementation options for activation functions and learning algorithms. The disadvantage of this toolkit is the lack of a graphical interface. Another advantage of the Python programming language is the availability of various libraries for modeling neural networks [11, 12].

In this work, the PyBrain library is selected as the Python library for performing operations with neural networks. PyBrian is a modular library designed to implement various machine learning algorithms in Python.

To initialize a neural network in PyBrain, the net = buildNetwork (layers, options) function is used. This function creates a network consisting of the number of layers and neurons in them required by the developer. The layers argument indicates the number of neurons in the layer, and enumerating multiple values with a comma specifies the number of layers. The options argument is a variety of options that you can configure when creating a neural network. For example, you can set your own activation function for the output and for each of the hidden layers.

The training set for a neural network is set by the function ds = SupervisedDataSet (inp, target). The inp parameter is responsible for the data that will be input to the neural network. The target parameter is the target neural network training vector, i.e., the vector of those values that the neural network should produce for a given set of input values. The advantage of the PyBrain library is the ability to create data for training a neural network using both the Python programming language and load previously created values from a file.

The neural network learning algorithm is defined by the function corresponding to the name of the algorithm. In this paper, we study the training of a neural network using the error back propagation algorithm, and it is specified in this environment by the function trainer = BackpropTrainer (net, ds). The net argument is a link to the network in the form of the previously declared network name, which must be trained using this algorithm.

The neural network training is initialized using the trainer.train function, where the trainer is a link to the previously specified training algorithm. Calling the train method performs one iteration (era) of training and returns the value of the quadratic error. In order not to organize a cycle for passing through each era, there is a training network function for convergence - trainer.train Until Convergence (ds, maxEpochs). The ds argument indicates which data to use as the training set. The maxEpochs argument allows you to limit the number of iterations during which the neural network will be trained. Regulation of this argument reduces the training time of the neural network. If continuing training using initial vectors will worsen the approach to verification vectors, then network training will stop earlier. The functional toolkit of the PyBrain library also allows you to set the value of the target learning error and control the learning speed.

After the training of the neural network is completed, you can begin to test it. To test a neural network, you must use the previously described function SupervisedDataSet (inp, target) to specify the data that must be supplied to the input of the neural network, as well as the values of the target vector that should be obtained at the output. If the target vector for the test data is not known, it must be filled with zeros by default. Similar to the training sequence, you can create data for testing a neural network using the Python programming language or load previously created values from a file.

Neural network testing is initialized using the trainer.testOnClassData (ds) function. The ds parameter in this function indicates the data set on which to test. If this parameter is not specified, then by default testing is performed on the data used in training the neural network.

Building a neural network architecture

The problem of traffic classification that we are solving consists in assigning the investigated real traffic to one of the known probability distributions. To do this, you need to create a neural network that can identify the correspondence between the parameters of the input data and the known probability distributions, i.e. with a given certainty will ensure the formation of the output signal corresponding to a certain distribution law.

Let us turn to the traffic statistics of one of the typical information objects, on the basis of which there was a selection of parameters for study [10]. His analysis showed a significant predominance of TCP traffic. Therefore, at this stage, for further research, the following parameter was selected that characterizes TCP traffic - the duration of TCP sessions. It is also interesting because it to some extent represents an analogue of the duration of connections of traditional telephone networks, which provides the basis for future comparative analysis. The statistics used in this work contained information about TCP sessions, the overwhelming number of which did not exceed 3000s in duration.

It is known that to select the architecture of a neural network, it is necessary to determine the number of hidden layers and the number of neurons in each layer, the size of the input and output vectors, and in addition, select the activation function. A vector, in accordance with the terminology adopted in neural networks, is understood as a set of data supplied to or received from the input of a neural network.

The studied traffic mainly contained TCP sessions, the duration interval of which ranged from 0 to 3000 s. At this stage of the study, this interval [0: 3000] was divided into uniform segments with a step of 60, which allowed the formation of samples containing 51 values (one from each segment). As a result, it was assumed that the dimension of the input layer of the neural network contains 51 neurons.

An assumption was made that all considered samples have the same values of mathematical expectation and variance, corresponding to the nature of the TCP sessions of the analyzed traffic approximated by ideal distributions. The values of each distribution determine the number of TCP sessions from 0 to 3000 s in length, provided that they are approximated by these distributions. Therefore, 51 values of each of the studied distributions, which will form the input vector, should be fed to the input of the neural network. In fig. Figure 1 shows graphs of the studied distributions.

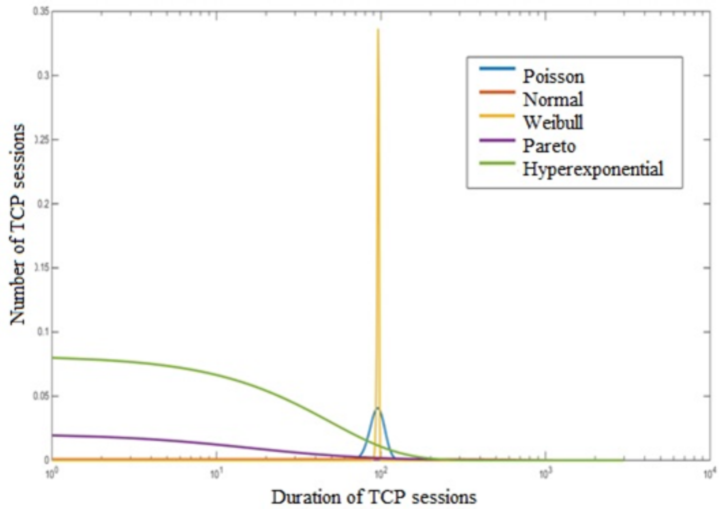

Figure 1. Graphs of the studied distributions

The output of a neural network determines which of the five commonly used distributions the input data have the most correspondence, based on which five neurons were involved in the output layer. Each of the neurons of the last layer is associated with an output that «encodes» the distribution. The assignment of the input sample to the corresponding distribution law is performed with the initially specified reliability. To verify the effectiveness of the proposed neural network architecture and search for an acceptable design option, the number of hidden layers and the number of neurons in them were changed [13, 14].

During testing, 100 experiments were conducted for each of the distributions. By one experiment is meant the input of a trained neural network (with a certain number of neurons and hidden layers, as well as with a given volume of the training sample) of the input sample corresponding to one of the known distributions and not included in the training sample. The result of each experiment was the value of the output vector, with a given reliability, showing the distribution law, which corresponded to the sample submitted to the input of the neural network. According to the results of all experiments, the proportion of successfully (correctly) recognized distribution laws was determined according to the success criterion (Tables 1 and 2). The proportion of correctly recognized distribution laws was estimated by assigning the resulting sum of successes to the total number of experiments.

Table 1

The number of input samples successfully recognized by the neural network, %

		Number of hidden layers						
		1				2	3	4
		The number of neurons in the hidden layer						
1	2	3	4	5	6	7	8	9
	The number of training samples	10	20	30	40	20, 10	30, 20, 10	40, 30, 20, 10
Poisson distribution	1 000	0	0	0	0	0	0	0
	10 000	11	3	3	3	16	26	28
	20 000	15	16	19	15	23	30	33
	30 000	20	19	14	19	33	29	32
Normal distribution	1 000	0	0	0	0	0	0	0
	10 000	0	0	0	0	0	0	0
	20 000	33	20	53	32	98	99	97
	30 000	60	68	84	83	97	97	97

1	2	3	4	5	6	7	8	9
Weibull distribution	1 000	0	0	0	0	0	0	0
	10 000	5	5	5	5	6	5	8
	20 000	5	5	5	5	6	5	8
	30 000	6	6	5	6	5	6	7
Pareto distribution	1 000	0	0	0	0	0	0	0
	10 000	0	0	0	0	0	0	0
	20 000	15	9	20	10	15	49	41
	30 000	37	28	44	37	29	22	43

The neurons in the layers were interconnected according to the principle of «each with each». All connections of two neurons were assigned a weight coefficient, which was then corrected by the network during training. In hidden layers, the sigmoidal activation function was used, and in the output layer, the softmax activation function was used. The training sequence was formed from 30 thousand vectors of each of the distributions. The values for them were calculated on an argument taken randomly from the interval [0: 3000].

Table 2

The number of input samples successfully recognized by the neural network, %

		Number of hidden layers						
		1				2	3	4
		The number of neurons in the hidden layer						
	The number of training samples	10	20	30	40	20, 10	30, 20, 10	40, 30, 20, 10
Hyperexpo-nential distribution	1 000	56	55	55	55	55	58	56
	10 000	54	58	53	53	57	63	62
	20 000	65	62	62	41	52	62	60
	30 000	53	65	57	60	63	60	63

During the study, it was found that an increase in the hidden layers or neurons in them leads to an increase in the recognition quality of not all distributions [15]. However, an increase in the training sample significantly increases the likelihood of correct recognition of some distributions supplied to the input of the neural network.

Based on the above experiments, the following neural network architecture was chosen: 51 neurons in the input layer, two hidden layers with 20 and 10 neurons, one output layer with five neurons (Fig. 2).

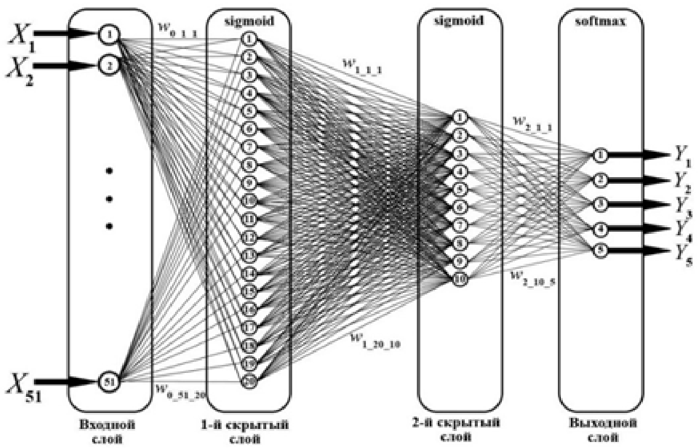

Figure 2. Architecture of the studied neural network

Training Sample Adjustment

It was found that a neural network constructed and trained in the presented way cannot accurately recognize all types of distributions. The values of many of the studied distributions are mainly contained in the interval [0: 300] and have a long «tail» of significantly lower values. With a uniform choice of x from the domain of definition of values characteristic of each distribution, it turned out insufficiently, as a result of which the neural network could not reveal the distinctive features of all [16]. As a result, the training sample was formed in such a way that a larger number of x values were concentrated on the interval [0: 300], near the mode of the corresponding distribution, and the remaining one was evenly distributed outside it randomly.

Table 3 presents the results of testing a neural network when vectors of constructed distributions that were not included in the training set were fed to it. For each distribution, 100 experiments were performed. In each experiment, the correspondence of the value of the output vector was determined, which determined the distribution law recognized by the network and the given distribution law of the input sample applied to the input of the neural network. According to the results of all experiments, the proportion of successful tests was determined [17].

Table 3

The number of input samples successfully recognized by the neural network, %

Distribution	Number of training sequences			
	30 000	50 000	70 000	100 000
Poisson distribution	76	84	78	81
Normal distribution	100	100	100	99
Weibull distribution	94	92	93	93
Pareto distribution	99	99	99	99
Hyperexponential distribution	100	100	100	100

From Table 3 it follows that the network recognizes the normal distribution, Pareto distribution and hyperexponential distribution as follows. The probability of successful recognition for the normal distribution is about 0.97, for the Pareto distribution – 0.99 and for the hyperexponential distribution – 0.99.

It is important that, compared with the results shown in Tables 1 and 2, the share of recognized Poisson and Weibull distributions significantly increased. Consequently, the hypothesis of the necessity (in the formation of training samples) of the prevailing use of argument values gravitating to the mode value of the corresponding distributions is confirmed.

Conclusion

The obtained results prove the possibility of creating a neural network that can solve the problem of assigning the distributions of the values of the parameters of real traffic to one of the known probabilistic distributions

Thus, it was found that a network with 2 hidden layers, 20 neurons in the first hidden layer and 10 neurons in the second hidden layer will be sufficient architecture for recognizing the given distributions. In hidden layers, the sigmoidal activation function was used, and in the output layer, the Softmax activation function was used.

It was also revealed that an increase in the hidden layers or neurons in them leads to an increase in the recognition quality of not all distributions. However, an increase in the training sample significantly increases the likelihood of correct recognition of the distributions supplied to the input of the neural network.

When testing a neural network, it was noted that the quality of recognition of distributions by a neural network is affected not only by the size of the training sample, but also by its composition. In the course of the experiment, the hypothesis was confirmed that with a uniform choice of x from the domain of definition of values characteristic of each distribution, the neural network could not reveal the distinctive features of all. Correction of the training sample so that a larger number of x values are concentrated near the mode of the corresponding distribution, and the remaining one is evenly distributed outside it randomly, thereby increasing the stability of correct recognition.

Based on the results of the analysis of models of neural networks, and also taking into account the fact that the data processed during the analysis of teletraffic, firstly, do not have a very high dimension, secondly, the input vector fully describes the process under study, and thirdly, the main predicted task of the proposed analysis is a classification - it should be concluded that the preferred use in further studies of multilayer neural networks with direct signal propagation.

References

1 Гойхман В.Ю. Исследование вероятностных характеристик трафика при помощи аппарата нейронных сетей / В.Ю. Гойхман, А.И. Лапий. — М.: Электросвязь, 2018. — № 4. — С. 64–67.

2 Авраменко М.В. Выбор дисциплины передачи данных сенсорного узла / М.В. Авраменко, В.Ю. Гойхман. — М.: Первая миля, 2017. — № 2. — С. 34–39.

3 Гойхман В.Ю. Решение задачи классификации трафика с использованием нейронной сети / В.Ю. Гойхман, А.И. Лапий // Тр. учебных заведений связи. — 2016. — Т. 2, № 1. — С. 43–47.

4 Поздняк И.С. Исследование сетевого трафика на степень самоподобия / И.С. Поздняк, М.А. Буранова. — Самара: ИУНЛ ПГУТИ, 2013. — 8 с.

5 Степанов С.Н. Основы телетрафика мультисервисных сетей / С.Н. Степанов. — М.: Эко-Трендз, 2010. — 392 с.

6 Ложковский А.Г. Модель трафика в мультисервисных сетях с коммутацией пакетов / А.Г. Ложковский. — Наукові праці ОНАЗ ім. О.С. Попова. — 2010. — 40 с.

7 Авраменко М.В. Выбор дисциплины передачи данных сенсорного узла / М.В. Авраменко, В.Ю. Гойхман // Первая миля. — № 2. — 2017. — С. 34–39.

8 Гойхман В.Ю. Анализ трафика студенческого общежития / В.Ю. Гойхман, Т.Ю. Лушникова, Ю.А. Зиненко // Вестн. кибернетики. — 2016. — № 1 (21).

9 Вардзинский Р.Н. Справочник по вероятностным распределениям / Р.Н. Вардзинский. — СПб.: Наука, 2001. — 295 с.

10 Шелухин О.И. Самоподобие и фракталы: телекоммуникационные приложения / О.И. Шелухин, А.В. Осин, С.М. Смольский. — М.: Физматлит, 2008. — 368 с.

11 Goykhman V. Research of Typical Information Objects Traffic / V. Goykhman, L. Korganbaeva, A. Ermakov, M. Nikolaeva // Advances in Science and Technology Research Journal. — 2019. — Vol. 13, Issue 2. — P. 51–55.

12 Goikhman V. Evaluating the packet traffic parameter measurements / V. Goikhman, Sh. Seilov, N. Sokolov, L. Korganbayeva // Telecommunications and Radio Engineering. — 2019. — No. 78(6). — P. 489–499.

13 Christopher M. Bishop. Neural Networks for Pattern Recognition / Christopher M. Bishop. — New York: Oxford University Press, 1995. — 498 p.

14 Ciresan D. Multicolumn Deep Neural Network for Traffic Sign Classification / D. Ciresan, U. Meier, J. Masci, J.Schmidhuber // Neural Networks. — August 2012. — Vol. 34. — P. 333–338.

15 Hinton G. Deep Neural Networks for Acoustic Modeling in Speech Recognition / G. Hinton, L. Deng, D. Yu, G. Dahl, A. Mohamed, N. Jaitly, A. Senior, V. Vanhoucke, P. Nguyen, T. Sainath, B. Kingsbury // IEEE Signal Processing Magazine. — 2012. — Vol. 29, No. 6. — P. 82–97.

16 Каллан Р. Основные концепции нейронных сетей / Р. Каллан. — М.: Изд. дом «Вильямс», 2001. — 287 с.

17 Башарин Г.П. Управление качеством и вероятностные модели функционирования сетей связи следующего поколения / Г.П. Башарин, Ю.В. Гайдамака, К.Е. Самуйлов, Н.В. Яркина. — М.: РУДН, 2008. — 157 с.

Ш. Сеилов, В. Гойхман, М. Касенова, А. Сейлов, Д. Шингисов

Инфокоммуникациялық трафикті талдау үшін нейрондық желінің модельдерін жасау

Мақалада IP пакеттер жиынтығы ретінде ұсынылған, трафиктің барлық түріне қызмет көрсететін мультисервистік желілер болып табылатын қазіргі инфокоммуникациялық желілердің мәселелері қарастырылды. Телекоммуникациялық желілер жиынтығы үшін осы трафиктің сипаттамалары талданды, олардың әрқайсысы белгілі бір қызмет түріне бағытталған. Трафикті зерттеу нәтижесінде алынған білім телекоммуникация саласының әртүрлі салаларында қабылданатын шешімдердің тиімділігін арттырудың маңызды факторы болып табылады. Желідегі трафиктің сипаты мен оның заңдылықтарын білу қажеттілігі анықталды және дәлелденді. Онсыз желілерді тиімді басқару, оларды дамыту үшін шешімдер әзірлеу, желінің қауіпсіздігін қамтамасыз ету және қажетті сапа деңгейін ұстап тұру мүмкін емес. Мультисервистік желілерді салу бойынша жұмыстардың көптігіне қарамастан, бірқатар мәселелер қосымша зерттеуді қажет етеді. Заманауи конвергентті трафикті, мультисервистік желілерді зерттеу нәтижесінде олардың сипаттамаларының жоғары өзгергіштігін ескерсек, оның табиғаты туралы білмейтінімізді көрсетті. Қазіргі желілердің трафигіне кездейсоқ факторлардың көптігі әсер етеді, бұл оның сипаттамаларын детерминант ретінде талдау мүмкіндігін жоққа шығарады. Осылайша, зерттелетін трафиктің параметрлері статистикалық, ықтималды сипатта болатынын айтуға болады және уақыт өте келе кездейсоқ өзгеруі мүмкін, сондықтан зерттеу негізінде автор статистикалық талдау әдістерін қолдана отырып зерттеу жүргізуді ұсынады. Трафикті зерттеу үшін ықтималдықтар теориясы мен математикалық статистика құралдарын пайдалану керек.

Кілт сөздер: инфокоммуникациялық желілер, нейрондық желілер, мультисервистік трафик, трафиктің ықтималды сипаттамалары, трафиктің математикалық модельдері, мультимедиялық қызметтер, мультисервистік трафик, нейрондық технологиялар, зерттелген таратулар, трафиктің жіктелімдері, ақпараттық объектілер.

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Разработка моделей нейронных сетей для анализа инфокоммуникационного трафика

В статье рассмотрены проблемы сегодняшних инфокоммуникационных сетей, основу которых составляют мультисервисные сети, обслуживающие трафик всех видов, представленный в виде совокупности IP-пакетов. Проанализированы характерные особенности этого трафика для набора сетей телекоммуникаций, каждая из которых ориентирована на определенный класс услуг. Знания, полученные в результате проводимых исследований трафика, являются существенным фактором для повышения действенности принимаемых решений в самых различных областях телекоммуникационной отрасли. Выявлена и обоснована необходимость знания природы циркулирующего в сети трафика и законов его поведения. Без этого невозможно эффективное управление сетями, выработка решений по их развитию, обеспечение сетевой безопасности и поддержка необходимого уровня качества. Несмотря на большое число работ о построении мультисервисных сетей, ряд вопросов требует дальнейшего изучения. Анализ исследований трафика современных конвергентных, мультисервисных сетей показал недостаточность имеющихся знаний о его природе и законах поведения, учитывая высокую изменчивость его характеристик. Трафик современных сетей подвержен влиянию большого количества случайных факторов, что исключает возможность анализа его характеристик как детерминированных. Таким образом, можно утверждать, что параметры исследуемого трафика носят статистический, вероятностный характер, могут изменяться в течение времени случайным образом, и, соответственно, на основе проведенного исследования авторами предложено исследование с использованием статистических методов анализа. Для изучения трафика следует использовать инструменты теории вероятностей и математической статистики.

Ключевые слова: инфокоммуникационные сети, нейронные сети, мультисервисный трафик, вероятностые характеристики трафика, математические модели трафика, мультимедийные услуги, мультисервисный трафик, нейросетевые технологии, исследуемые распределения, классификации трафика, информационные объекты.

References

1 Goikhman, V.Yu. (2018). *Issledovanie veroiatnostnykh kharakteristik trafika pri pomoshchi apparata neironnykh setei [Study of probabilistic characteristics of traffic using the apparatus of neural networks].* Moscow: Elektrosviaz [in Russian].

2 Avramenko, M.V., & Goikhman, V.Yu. (2017). Vybor distsipliny peredachi dannykh sensornoho uzla [Select the data transfer discipline of the sensor node]. *Pervaia milia — First mile, 2,* 34–39 [in Russian].

3 Goikhman, V.Yu. (2016). Reshenie zadachi klassifikatsii trafika s ispolzovaniem neironnoi seti [Solution of the problem of traffic classification using a neural network]. *Trudy uchebnykh zavedenii sviazi, Vol. 2, 1,* 43–47 [in Russian].

4 Pozdniak, I.S. (2013). *Issledovanie setevoho trafika na stepen samopodobiia [Study of network traffic on the degree of self-similarity].* Samara: IUNL PGUTI [in Russian].

5 Stepanov, S.N. (2010). *Osnovy teletrafika multiservisnykh setei [Fundamentals of teletraphy of multiservice networks].* Moscow: Eko-Trendz [in Russian].

6 Lozhkovskii, A.G. (2010). Model trafika v multiservisnykh setiakh s kommutatsiei paketov [Traffic model in multiservice networks with packet switching]. *Naukovi pratsi ONAZ imeni O.S. Popova* [in Russian].

7 Avramenko, M.V., & Goikhman, V.Yu (2017). Vybor distsipliny peredachi dannykh sensornoho uzla [The choice of discipline of data transmission of the sensor node]. *Pervaia milia — First mile, 2,* 34–39 [in Russian].

8 Goikhman, V.Yu., Lushnikova, T. Yu., & Zinenko, Yu.A. (2016). Analiz trafika studencheskoho obshchezhitiia [Traffic Analysis of student hostel]. *Vestnik kibernetiki, 1 (21)* [in Russian].

9 Vardzinskii, R.N. (2001). *Spravochnik po veroiatnostnym raspredeleniiam [The reference probability distributions].* Saint Petersburg: Nauka [in Russian].

10 Shelukhin, O.I. (2008). *Samopodobie i fraktaly: telekommunikatsionnye prilozheniia [Self-Similarity and fractals: telecommunication applications].* Moscow: Fizmatlit [in Russian].

11 Goykhman, V., Korganbaeva, L., Ermakov, A., & Nikolaeva, M. (2019). Research of Typical Information Objects Traffic. *Advances in Science and Technology Research Journal, Vol. 13, Issue 2,* 51–55.

12 Goikhman, V., Seilov, Sh., Sokolov, N., & Korganbayeva, L. (2019). Evaluating the packet traffic parameter measurements. *Telecommunications and Radio Engineering, 78 (6),* 489–499.

13 Christopher, M. Bishop. (1995). *Neural Networks for Pattern Recognition.* New York: Oxford University Press.

14 Ciresan, D., Meier, U., Masci, J. & Schmidhuber, J. (2012). Multicolumn Deep Neural Network for Traffic Sign Classification. *Neural Networks, Vol. 34,* 333–338.

15 Hinton, G., Deng, L., Yu, D., Dahl, G., Mohamed, A., & Jaitly, N. et al. (2012). Deep Neural Networks for Acoustic Modeling in Speech Recognition. *IEEE Signal Processing Magazine, Vol. 29, 6,* 82–97.

16 Kallan, R. (2001). *Osnovnye kontseptsii neironnykh setei [Basic concepts of neural networks].* Moscow: Izdatelskii dom «Wiliams» [in Russian].

17 Basharin, G.P. (2008) *Upravlenie kachestvom i veroiatnostnye modeli funktsionirovaniia setei sviazi slediushcheho pokoleniia [Quality management and probabilistic models of functioning of communication networks of the next generation].* Moscow: RUDN [in Russian].

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Mathematical analysis of trend indicators of Internet security resources cyber-diagrams dynamics in the Republic of Kazakhstan

In the research the development of a mathematical model for predicting threats of information security of Internet resources is carried out, which allows, based on the minimum amount of input data, to indicate the dynamics of the development of possible threats in the life cycle of information systems, which may in the long run be a reason for shaping the costs of preventing threats to information security.

Keywords: mathematical model, predicting of threats, information security, life cycle, internet resource.

The global tendency characteristic of the last decades of introducing the achievements of information and communication technologies with a pace that is significantly ahead of the formation of a culture of their use, and the rooting of social and industrial relations characteristic of the «information society», primarily in matters of ensuring cybersecurity, also finds the confirmation.

New technologies, electronic services have become an integral part of our daily life. Given that, society is becoming increasingly dependent on information and communication technologies every day, the protection and availability of these technologies is becoming a critical moment and a very important topic for national interests.

Today, a necessary condition for the development of the information society is cybersecurity, which can be followed by a virtually endless list of security problems and their solutions, ranging from technical to legislative.

Traditionally, the concept of security is viewed through the prism of three approaches: the absence of threats, the security and stability of the system. It is obvious that the specific features of the development of the information space make the approach based on the understanding of security as the absence of threats irrelevant. In particular, in accordance with the interactive map of cyber threats of Kaspersky Lab, Kazakhstan is in the top 30 countries by the number of tested cyber attacks, most often taking place in the corridor between 18 and 27 places [1]. Thus, it makes sense to evaluate information security in contexts of relative security and the ability of the system to adequately respond to emerging challenges and threats, and minimize risks.

In world practice, as well as in the information segment of Kazakhstan, there is a steady trend of transferring information assets to Internet platforms and the provision of cloud services. The control of privileged users is one of the main tasks in relations with Internet service providers, software developers and technical support specialists. Now information assets of business and government structures can be placed anywhere, they are serviced by a large number of contractors scattered around the world, who, as a rule, are not responsible for violation of information security.

Thus, Internet service providers, while performing work in their segment of an information system, may accidentally or intentionally gain access to foreign systems, unauthorized launch applications or make configuration changes. Therefore, it is extremely difficult to limit their actions at the level of network access to applications by traditional means of delimiting and controlling access in information systems. In consequence of this, recently decisions on the control of Internet users will continue to be highly demanded in information systems, and this demand will increase every year.

In accordance with the international standard ISO / IE C 27001: 2005, information security incident management is an important element in ensuring the continuity of an organization's business processes [2]. Incident management is a process that is fed to the data received as a result of logging information about events related to information security, and the process output is informed about the reasons for the incident and measures that need to be taken to prevent the incident from happening again.

In general, incident management is a cyclical process, the main stages of which are represented by the PDCA model (Plan-Do-Check-Act, continuous improvement model of processes). According to ISO 27001, the classical model includes four management stages: information security incident identification, information security incident response, investigation, corrective and preventive measures.

It is during the response and investigation of incidents that the specific information system vulnerabilities are revealed, traces of attacks and intrusions are detected, the operation of the protective equipment, the quality of the information security system architecture and its management are checked. Also important is the existence of procedures for analyzing and eliminating the consequences of incidents and taking corrective measures to reduce the likelihood of such incidents recurring in the future.

Thus, there is an urgent problem of prompt response to emerging incidents. It is necessary to decide which strategy out of the set of specific ones can be applied, or to determine that there is no suitable strategy and it is necessary to work it out (Table 1).

Table 1

The number of cyber incidents registered in Kazakhstan that violate the information security of users of Internet resources for the period from 2015–2017

№	Incident Type	Quantity in 2015	Quantity in 2016	Quantity in 2017	Total
1	Unauthorized access and modification of the content of Internet resources (website defacement) / attacks on an Internet resource	588	1934	658	3180

Note. Source: State Technical Service https://lsm.kz/kakie-banki-podvergalis-kiberatakam-v-2017-godu.

One of the promising areas of research for solving the problem of protection against cyber attacks on IP is the creation of methods for predicting their intensity by means of mathematical methods of analysis [3–13]. Note that the intensity of cyber attacks is the total number of these attacks per unit of time. In the case of a forecast that the intensity of cyber attacks on IP exceeds a certain predetermined value, additional measures of protection can be taken, including, for example, a more in-depth intelligent analysis of traffic [14–20].

In recent years, an increasing interest of researchers in trend analysis and prediction of cyber attacks has been observed [21–24]. This can be explained by the fact that trend forecasts make it possible to receive not only forecasts of directly future events, but also characterizing their estimates.

An important method of stochastic predictions is the exponential smoothing method. This method consists in the fact that a number of dynamics is smoothed with the help of a moving average in which the weights obey the exponential law [25, 26].

A special feature of the exponential smoothing method is that the procedure for finding the smoothed level uses only the previous levels of the series, taken with a certain weight, and the weight decreases as it moves away from the point in time for which the smoothed value of the series level is determined. If for the initial time series y_1, y_2, y_3,..., y_n the corresponding smoothed values of the levels are denoted by S_t, $t = 1,2,...,n$, this exponential smoothing is carried out according to the formula: $S_t = (1\text{-}\alpha)yt + \alpha S_{t-1}$.

Some sources give a different formula: $S_t = \alpha yt+(1\text{-}\alpha)S_{t-1}$, where α — smoothing parameter ($0 < \alpha < 1$); magnitude (1-α) called the discount factor.

In practical tasks of processing economic time series, it is recommended (unreasonable) to choose the value of the smoothing parameter in the range from 0,1 to 0,3. There are no other precise recommendations for choosing the optimal value of the parameter α. In some cases, it is proposed to determine the value of α based on the length of the series being smoothed: $\alpha = 2/(n+1)$.

As for the initial parameter S_0, then in tasks it is taken or equal to the value of the first level of the series Γi_1, or equal to the arithmetic average of the first few terms of the series. If, at the approach to the right end of the time series, the values smoothed by this method with the selected parameter α begin to differ significantly from the corresponding values of the original series, it is necessary to switch to another smoothing parameter. The advantage of this method is that with smoothing, neither the initial nor the final levels of the smoothing time series are lost. As S_0 take the first value of the row, $S_0 = y_1 = 588$ (Table 2).

Table 2

Calculated parameters of the model equation

t	y	S_t	Formula	$(y - S_t)^2$
2015	588	588	(1 – 0,3)*588 + 0,3*588	0
2016	1934	1530,2	(1 – 0,3)*1934 + 0,3*588	163054,44
2017	658	919,66	(1 – 0,3)*658 + 0,3*1530,2	68465,956
				231520,396

Note. The table is based on the calculation.

Forecasting data using exponential smoothing:

The prediction methods called «smoothing» take into account the effects of the overshoot function much better than the methods using regression analysis.

The basic equation is as follows:

$S(t+1) = S(t)(1 - \alpha) + \alpha Y(t)$

S(t) – this is a forecast made at a time t;

S(t+1) reflects the forecast in the time period immediately following the point in time t :

$S(3+1) = 919,66(1 - -0,3) + 0,3 * 658 = 841,162.$

Standard error (error) is calculated by the formula:

$$e_t = \sqrt{\frac{\sum (y_i - S_{i-1})^2}{n-1}},$$

where i = (t - 2, t)

$$e_t = \sqrt{\frac{231520,396}{3-1}} = 340,236.$$

The linear trend equation has the form: $y = bt + a$.

1. Find the parameters of the equation by the method of least squares:

The OLS system of equations takes the form:

$an + b\sum t = \sum y$

$a\sum t + b\sum t^2 = \sum y * t$ (Table 3).

Table 3

Calculated parameters in tabular form

t	y	t^2	y^2	t y
1	588	1	345744	588
2	1934	4	3740356	3868
3	658	9	432964	1974
6	3180	14	4519064	6430
Average value	1060	4.667	1506354.667	2143.333

Note. The table is based on the calculation.

For our data, the system of equations is:

$3\ a + 6\ b = 3180$

$6\ a + 14\ b = 6430.$

From the first equation we express a and substitute in the second equation.

Get $a = 990$, $b = 35$.

Get the trend equation: $y = 35\ t + 990$.

Empirical trend coefficients a and b are only estimates of theoretical coefficients β_i, and the equation itself reflects only a general tendency in the behavior of the variables in question.

Trend ratio $b = 35$ shows the average change in the effective index (in units of y) with a change in the time period t per unit of measurement. In this example, with t by 1 unit, y change on average by 35.

Estimate the quality of the trend equation using the average relative approximation error.

$$\overline{A} = \frac{\sum |y_t - y_i| : y_i}{n} 100\ \%.$$

The approximation error within 5 %–7 % indicates a good selection of the trend equation to the source data.

$$\overline{A} = \frac{1,8592}{3} 100\% = 61,97\ \%.$$

To determine the size of the error or accuracy of the forecast indicator Y calculate the coefficient of disparity Teil formula:

$$K_T = \frac{\sqrt{\sum (y_i - \overline{y})^2}}{\sqrt{\sum y_t^2}};$$

$$K_T = \frac{1145814}{4519064} = 0,254.$$

This indicator varies from 0 to 1. The closer its value is to zero, the better the prediction results.

Average values:

$$\overline{t} = \frac{\sum t_i}{n} = \frac{6}{3} = 2;$$

$$\overline{y} = \frac{\sum y_i}{n} = \frac{3180}{3} = 1060;$$

$$\overline{t \cdot y} = \frac{\sum t_i y_i}{n} = \frac{6430}{3} = 2143,3333.$$

Dispersion:

$$D\left(t\right) = \frac{\sum t_i^2}{n} - \overline{t}^2 = \frac{14}{3} - 2^2 = 0,6667;$$

$$D\left(y\right) = \frac{\sum y_i^2}{n} - \overline{y}^2 = \frac{4519064}{3} - 1060^2 = 382754,6667.$$

Standard deviation:

$$\sigma\left(t\right) = \sqrt{D(t)} = \sqrt{0,6667} = 0,8165;$$

$$\sigma\left(y\right) = \sqrt{D(y)} = \sqrt{382754,6667} = 618,6717.$$

Calculate the coefficient of determination:

$$R^2 = 1 - \frac{\sum (y_i - y_t)^2}{\sum (y_i - \overline{y})^2};$$

$$R^2 = 1 - \frac{1145814}{1148264} = 0,00213,$$

where in 0.21 % of cases, t affects the change y. In other words — the accuracy of the selection of the trend equation is low.

To assess the quality of the parameters of the equation, we construct a calculation table (Table 4).

Table 4

Calculated parameters in tabular form

t	y	y(t)	$(y_i - y_{cp})^2$	$(y_i - y(t))^2$	$(y_i - y(t)) : y_i$
1	588	1025	222784	190969	0,743
2	1934	1060	763876	763876	0,452
3	658	1095	161604	190969	0,664
		3180	1148264	1145814	1,859

Note. The table is based on the calculation.

2. Analysis of the accuracy of determining estimates of the parameters of the trend equation:
Dispersion error equation:

$$S_y^2 = \frac{\sum (y_i - y_t)^2}{n-m-1},$$

where m = 1 — the number of influencing factors in the trend model.

$$S_y^2 = \frac{1145814}{1} = 1145814.$$

Standard equation error:

$$S_y = \sqrt{S_y^2} = \sqrt{1145814} = 1070,427;$$

$$S_b = S_y \cdot \frac{\sqrt{\sum t^2}}{n\sigma_t};$$

$$S_b = 1070,427 \cdot \frac{\sqrt{14}}{3 \cdot 0,8165} = 1635,104;$$

$$S_a = \frac{S_y}{\sqrt{n}\sigma_t} = \frac{1070,427}{0,8165\sqrt{3}} = 756,906.$$

Perform interval forecast and determine the standard error of the predicted indicator.
$Uy = y_{n+L} \pm K$,
where

$$K = t_a \cdot S_y \cdot \sqrt{1 + \frac{1}{n} + \frac{3(n+2L-1)^2}{n(n^2-1)}},$$

L — lead period; y_{n+L} — point forecast by model on $(n+L)$ moment of time; n — number of observations in the time series; Sy — standard prediction error; T_{tabl} — the tabular value of student's criterion for the level of significance α and for the number of degrees of freedom equal to *n-2*.

According to the student's table we find T_{tabl}
T_{tabl} (*n-m-1*;$\alpha/2$) = (;) = 12,706.
Spot forecast, $t = 4$: y(4) = 35*4 + 990 = 1130.

$$K_1 = 12,706 \cdot 1070,43 \cdot \sqrt{1 + \frac{1}{3} + \frac{3\,(3+2\cdot 1-1)^2}{3\,(3^2-1)}} = 24831,63.$$

1130 – 24831,63 = -23701,63; 1130 + 24831,63 = 25961,63.
Interval forecast:
$t = 4$: (-23701,63;25961,63)
Spot forecast, $t = 5$: y(5)= 35*5 + 990 = 1165.

$$K_2 = 12,706 \cdot 1070,43 \cdot \sqrt{1 + \frac{1}{3} + \frac{3\,(3+2\cdot 2-1)^2}{3\,(3^2-1)}} = 32849,16.$$

1165 – 32849,16 = -31684,16; 1165 + 32849,16 = 34014,16.
Interval forecast:
$t = 5$: (-31684,16;34014,16).
Spot forecast, $t = 6$: y(6)= 35*6 + 990 = 1200.

$$K_3 = 12,706 \cdot 1070,43 \cdot \sqrt{1 + \frac{1}{3} + \frac{3\,(3+2\cdot 3-1)^2}{3\,(3^2-1)}} = 41551,27.$$

1200 - 41551.27 = -40351,27; 1200 + 41551,27 = 42751,27.

Interval forecast:
t = 6: (-40351,27;42751,27)
Spot forecast, t = 7: y(7) = 35*7 + 990 = 1235.

$$K_4 = 12,706 \cdot 1070,43 \cdot \sqrt{1 + \frac{1}{3} + \frac{3\,(3 + 2 \cdot 4 - 1)^2}{3\,(3^2 - 1)}} = 50585,88.$$

1235 − 50585,88 = -49350,88; 1235 + 50585,88 = 51820,88.
Interval forecast:
t = 7: (-49350,88;51820,88)
3. Testing hypotheses regarding the coefficients of the linear trend equation:
1) t- statistics. Student criterion.
According to the student's table we find T_{tabl}
T_{tabl} (n-m-1;α/2) = (1;0,025) = 12,706.

$$t_a = \frac{a}{S_a};$$

$$t_a = \frac{35}{756,906} = 0,04624 < 12,706.$$

The statistical significance of the coefficient a is not confirmed

$$t_b = \frac{b}{S_b};$$

$$t_b = \frac{99035}{1635,104} = 0,6055 < 12,706.$$

2) F- statistics. Fisher Criterion.
Coefficient of determination:

$$F = \frac{R^2}{1 - R^2}\frac{n - m - 1}{m} = \frac{0,00213}{1 - 0,00213}\frac{3 - 1 - 1}{1} = 0,00214.$$

Find from the table Fkp(1;1;0.05) = 161,
where m — the number of factors in the trend equation (m=1).

Insofar as $F < Fkp$, then the coefficient of determination (and, in general, the trend equation) is not statistically significant.

As a result of the study, the time dependence was studied Y from time t. At the specification stage, a linear trend was chosen. Its parameters are estimated by the method of least squares. The statistical significance of the equation is verified using the coefficient of determination and the Fisher criterion. It was found that in the situation under study, 0.21 % of the total variability Y due to the change in the time parameter. It was also established that the parameters of the model are not statistically significant. Economic interpretation of model parameters is possible — with each time period t value Y on average, increases by 35 units.

Thus, when creating an IP protection system to counter cyber attacks, in addition to implementing the information risk management system, their information audit and analysis, it is necessary to pay attention to predicting the intensity of cyber attacks.

As follows from the results of this work, interval forecasting of the intensity of cyber attacks on informatization objects of critical infrastructures is an important practical task. Experimental studies of interval prediction of cyber attack intensity by means of trend extrapolation methods with dynamic updating of the smoothing parameter showed that the proposed approach has the best accuracy of interval prediction of a selected indicator of cyber attack intensity.

References

1 Сайт компании «Лаборатория Касперского» [Электронный ресурс]. – Режим доступа: https://cybermap.kaspersky.com/ru/.

2 Сайт «Международная организация по стандартизации» [Электронный ресурс]. – Режим доступа: https://www.iso.org/standard/42103.html

3 Петренко С.А. Концепция раннего распознавания и предупреждения компьютерного нападения / С.А. Петренко, А.С. Петренко // Информационные системы и технологии в моделировании и управлении: материалы Всеросс. науч.-практ. конф. — 2016. – С. 82–86.

4 Петренко С.А. Национальная система раннего предупреждения о компьютерном нападении / С.А. Петренко, Д.Д. Ступин. – Иннополис: Изд. дом «Афина», 2017. – 440 с.

5 Werner G. Time series forecasting of cyber-attack intensity / G. Werner, S. Yang, K. McConky // Proceedings of the 12th Annual Conference on Cyber and Information Security. – 2017. – P. 224–240.

6 Гамбаров Г.М. Статистическое моделирование и прогнозирование: учеб. пос. / Г.М. Гамбаров, Н.М. Журавель, Ю.Г. Королев и др.; под ред. А.Г. Гранберга. – М.: Финансы и статистика, 1990. – 383 с.

7 Малюк А.А. Защита информации: современные проблемы / А.А. Малюк // Безопасность информационных технологий. — 2010. — № 1. — С. 5–9.

8 Домарев В.В. Безопасность информационных технологий. Системный подход / В.В. Домарев. — Киев: Изд-во «Диасофт», 2004. — 992 с.

9 Малюк А.А. К вопросу об интенсификации процессов защиты информации / А.А. Малюк // Безопасность информационных технологий. — 2011. — № 1. — С. 6–10.

10 Малюк А.А. Информационная безопасность; концептуальные и методологические основы защиты информации: учеб. пос. для вузов / А.А. Малюк. — М.: Горячая линия-Телеком, 2004. — 280 с.

11 Герасименко В.А. Основы защиты информации / В.А. Герасименко, А.А. Малюк. — М.: МИФИ, 1997. — 537 с.

12 Ажмухамедов И.М. Принципы обеспечения комплексной безопасности информационных систем / И.М. Ажмухамедов // Вестн. АГТУ. Сер. Управление, вычислительная техника и информатика. — 2011. — № 1. — С. 7–11.

13 Скородумов Б.И. О понятийно-терминологическом аппарате информационной безопасности / Б.И. Скородумов // БИТ. — 2008. — № 4. — С. 43–45.

14 Шаньгин В.Ф. Информационная безопасность компьютерных систем и сетей: учеб. пос. / В.Ф. Шаньгин. — М.: ИД ФОРУМ, НИЦ ИНФРА-М, 2013. — 416 с.

15 Партыка Т.Л. Информационная безопасность: учеб. пос. / Т.Л. Партыка, И.И. Попов. — М.: Форум, 2012. — 432 с.

16 Борисов В.В. Компьютерная поддержка сложных организационно-технических систем / В.В. Борисов, И.А. Бычков, А.В. Дементьев, А.П. Соловьев, А.С. Федулов. – М.: Горячая линия – Телеком, 2002. – 154 с.

17 Ажмухамедов И.М. Математическая модель комплексной безопасности компьютерных систем и сетей на основе экспертных суждений / И.М. Ажмухамедов // Инфокоммуникационные технологии. — 2009. — 7. — № 4. — С. 103–107.

18 Войтик А.И. Экономика информационной безопасности: учеб. пос. / А.И. Войтик, В.Г. Прожерин. – СПб.: НИУИТМО, 2012. – 120 с.

19 Вестник УрФО. Безопасность в информационной сфере [Электронный ресурс]. — Режим доступа:http://www.info-secur.ru/

20 Ажмухамедов И.М. Моделирование на основе экспертных суждений процесса оценки информационной безопасности / И.М. Ажмухамедов // Вестн. АГТУ. Сер. Управление, вычислительная техника и информатика. — 2009. — 2. — С. 101–109.

21 Ларина И.Е. Экономика защиты информации: учеб. пос. / И.Е. Ларина. — М.: МГИУ, 2007. — 92 с.

22 Ажмухамедов И.М. Повышение безопасности компьютерных систем и сетей на основе анализа сетевого трафика / И.М.Ажмухамедов, А.Н. Марьенков // Инфокоммуникационные технологии. — 2010. — 8. — № 3. — С. 106–108.

23 Сайт компании «Лаборатория Касперского» [Электронный ресурс]. — Режим доступа: http://media.kaspersky.com/.

24 Aamodt A. Case-Based Reasoning: Foundational Issues, methodological Variations, and System Approaches / A. Aamodt, E. Plaza // AI Communications. — 1994. — 7. — No. 1. — P. 39–59.

25 Люгер Д.Ф. Искусственный интеллект: стратегии и методы решения сложных проблем / Д.Ф. Люгер. — М.: Вильямс, 2003. — 864 с.

26 Берман А.Ф. Концепция построения прецедентной экспертной системы / А.Ф. Берман, О.А. Николайчук, А.И. Павлов, А.Ю. Юрин // Междунар. конф. по вычислительной механике и современным прикладным программным системам. — 2003. — 2. — С. 110, 111.

Б.К. Шаяхметова, Ш.Е. Омарова, В.Г. Дрозд

Қазақстан Республикасындағы интернет-қауіпсіздік ресурстарының кибер-диаграммалары динамикасының үрдістік көрсеткіштерін математикалық талдау

Мақалада интернет-ресурстардың ақпараттық қауіпсіздігіне қауіп-қатерлерді алдын ала болжау үшін математикалық модель әзірленген, бұл енгізілген деректердің ең аз мөлшеріне сүйене отырып, ақпараттық жүйелердің өмірлік циклінде мүмкін қауіптердің даму динамикасын көрсетуге мүмкіндік береді, бұл болашақта ақпарат қауіпсіздігіне төнетін қатердің алдын-алуға, шығындардың қалыптасуына әкелуі мүмкін.

Кілт сөздер: математикалық модель, қауіптерді болжау, ақпараттық қауіпсіздік, өмірлік цикл, интернет-ресурстар.

Б.К. Шаяхметова, Ш.Е. Омарова, В.Г. Дрозд

Математический анализ трендовых показателей динамики кибер-диаграмм ресурсов интернет-безопасности в Республике Казахстан

В статье разработана математическая модель прогнозирования угроз информационной безопасности интернет-ресурсов, позволяющая на основе минимального объема исходных данных указать динамику развития возможных угроз в жизненном цикле информационных систем, что может явиться в перспективе основанием по формированию затрат на предотвращение угроз информационной безопасности.

Ключевые слова: математическая модель, прогнозирование угроз, информационная безопасность, жизненный цикл, интернет-ресурсы.

References

1 Sait kompanii «Laboratoriia Kasperskoho» [The site of the company «Kaspersky laboratory»]. *kaspersky.com.* Retrieved from https://cybermap.kaspersky.com/ru/ [in Russian].

2 Sait «Mezhdunarodnaia orhanizatsiia po standartizatsii» [Sait of «ISO»]. *iso.org.* Retrieved from https://www.iso.org/standard/42103.html.

3 Petrenko, S.A., & Petrenko, A.S. (2016). Kontseptsiia ranneho raspoznavaniia i preduprezhdeniia kompiuternoho napadeniia [The concept of early detection and prevention computer attacks]. *Materialy Vserossiiskoi nauchno-prakticheskoi konferentsii «Informatsionnye sistemy i tekhnolohii v modelirovanii i upravlenii»,* 82–86 [in Russian].

4 Petrenko, S.A., & Stupin, D.D. (2017). *Natsionalnaia sistema ranneho preduprezhdeniia o kompiuternom napadenii [National Computer Attack Early Warning System].* Innopolis: Izdatelskii Dom «Afina» [in Russian].

5 Werner, G. Yang, S., & McConky, K. (2017). Time analysis of cyber-attack intensity *Proceedings of the 12th Annual Conference on Cyber and Information Security*, 224–240.

6 Gambarov, G.M., Zhuravel, N.M., & Korolev, Yu.G., et. al. (1990) *Statisticheskoe modelirovanie i prohnozirovanie [Statistical modeling and forecasting].* Moscow: Finansy i statistika [in Russian].

7 Maliuk, A.A. (2010). Zashchita informatsii: sovremennye problemy [Information security: current problems]. *Bezopasnost informatsionnykh tekhnolohii, 1,* 5–9 [in Russian].

8 Domarev, V.V. (2004). *Bezopasnost informatsionnykh tekhnolohii. Sistemnyi podkhod [Security information technology. System approach].* Kiev: Izdtelstvo «Diasoft» [in Russian].

9 Maliuk, A.A. (2011). K voprosu ob intensifikatsii protsessov zashchity informatsii [To the question of the intensification of information security processes]. *Bezopasnost informatsionnykh tekhnolohii, 1*, 6–10 [in Russian].

10 Maliuk, A.A. (2004). *Informatsionnaia bezopasnost: kontseptualnye i metodolohicheskie osnovy zashchity informatsii [Information Security: conceptual and methodological foundations of information security].* Moscow: Hotline-Telecom [in Russian].

11 Gerasimenko, V.A., & Malyuk, A.A. (1997). *Osnovy zashchity informatsii [Basics of information security].* Moscow: MEPI [in Russian].

12 Azhmukhamedov, I.M. (2011). Printsipy obespecheniia kompleksnoi bezopasnosti informatsionnykh sistem [Principles of ensuring integrated security of information systems]. *Bulletin of ASTU. Series Management, Computer Engineering and Informatics, 1*, 7-11 [in Russian].

13 Skorodumov, B.I (2008). O poniatiino-terminolohicheskom apparate informatsionnoi bezopasnosti [On the conceptual and terminological apparatus of information security]. *BIT, 4*, 43-45 [in Russian].

14 Shangin, V.F. (2013). *Informatsionnaia bezopasnost kompiuternykh sistem i setei [Information security of computer systems and networks].* Moscow: ID FORUM; SIC INFRA-M [in Russian].

15 Partyka, T.L., & Popov, I.I. (2012). *Informatsionnaia bezopasnost [Information Security].* Moscow: Forum [in Russian].

16 Borisov, V.V., Bychkov, I.A., Dementev, A.V., Solovev, A.P., & Fedulov, A.S. (2002). *Kompiuternaia podderzhka slozhnykh orhanizatsionno-tekhnicheskikh sistem [Computer support for complex organizational and technical systems].* Moscow: Hotline - Telecom [in Russian].

17 Azhmukhamedov, I.M. (2009). Matematicheskaia model kompleksnoi bezopasnosti kompiuternykh sistem i setei na osnove ekspertnykh suzhdenii [Mathematical model of integrated security of computer systems and networks based on expert judgments]. *Infocommunication technologies Vol. 7, 4,* 103–107 [in Russian].

18 Voitik, A.I. (2012). *Ekonomika informatsionnoi bezopasnosti [The Economics of Information Security].* Saint Petersburg: NIUITMO [in Russian].

19 Sait zhurnala «Vestnik UrFO. Bezopasnost v informatsionnoi sfere» [Site of Journal «Bulletin of the Ural Federal District. Information Security information security»]. *info-secur.ru.* Retrieved from http://www.info-secur.ru/ [in Russian].

20 Azhmukhamedov, I.M. (2009). Modelirovanie na osnove ekspertnykh suzhdenii protsessa otsenki informatsionnoi bezopasnosti [Modeling on the basis of expert judgments of the information security assessment process]. *ASTU Bulletin. Series: Management, Computer Engineering and Informatics, 2*, 101–109 [in Russian].

21 Larina, I.Ye. (2007). *Ekonomika zashchity informatsii [The Economics of Information Security].* Moscow: MHIU [in Russian].

22 Azhmukhamedov, I.M., & Marienkov, A.N. (2010). Povyshenie bezopasnosti kompiuternykh sistem i setei na osnove analiza setevoho trafika [Improving the security of computer systems and networks based on network traffic analysis]. *Infocommunication Technologies, Vol. 8, 3,* 106–108 [in Russian].

23 Sait kompanii «Laboratoriia Kasperskoho» [The sait of company «Kaspersky Lab»]. *kaspersky.com.* Retrieved from http://media.kaspersky.com/ [in Russian].

24 Aamodt, A., & Plaza, E. (1994). Case-Based Reasoning: Foundational Issues, Methodological Variations, and System Approaches. *AI Communications, Vol. 7, 1,* 39–59.

25 Luger D.F. (2002). *Iskusstvennyi intellekt: stratehii i metody resheniia slozhnykh problem [Artificial Intelligence: Strategies and Methods for Solving Difficult Problems].* Moscow: Williams [in Russian].

26 Berman, A.F., Nikolaichuk, O.A., Pavlov, A.I., & Yurin, A.Yu. (2003). Kontseptsiia postroeniia pretsedentnoi ekspertnoi sistemy [The concept of building a precedent expert system]. *International Conference on Computational Mechanics and Modern Applied Software Systems, 2,* 110–111 [in Russian].

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