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Splitting method and the existence of a strong solution of the Navier-Stokes equations[1]

In the author's article from the previous issue of the journal from the properties of the ONS solutions the relation between pressure and module square of velocity vector is set. Based on which the uniqueness of the weak and existence of strong solutions of the problem for three-dimensional equations of Navier-Stokes as a whole over time are proved. The result is a contribution to a qualitative mathematical theory of the Navier-Stokes equations. However, one of the actual problems in the theory of equations of Navier-Stokes is the choice of the mathematical method for proofs of the existence of a theorem. In the work splitting method is chosen to solve the Navier-Stokes equations. The rationale of this method is given. The compactness of the solution sequence is showed, thus the existence of strong solutions of the problem for three-dimensional Navier-Stokes equations as a whole over time is proved.

Keywords: the Navier-Stokes equations, splitting method for the Navier-Stokes equations, compactness, the existence of strong solutions, determination algorithm of strong solutions.

0.1 Problem statement and splitting method

In [1, 2] the initial-boundary value problem for nonlinear equations of Navier-Stokes relatively to the velocity vector $\mathbf{U} = (U_1, U_2, U_3) \in \mathring{\mathbf{J}}(Q)$ and the pressure P in domain $Q = (0, T] \times \Omega$ is reduced to

$$\frac{\partial \mathbf{U}}{\partial t} - \mu\Delta\mathbf{U} + (\mathbf{U}, \nabla)\mathbf{U} - \nabla\big|\mathbf{U}\big|^2 = \mathbf{f}(t, \mathbf{x}), \qquad (1a)$$

$$\mathbf{U}(0, \mathbf{x}) = \mathbf{\Phi}(\mathbf{x}),\ \ \mathbf{U}(t, \mathbf{x})\big|_{\mathbf{x}\in\partial\Omega} = 0, \qquad (1b)$$

where $t \in (0, T], \forall T < \infty;\ \mathbf{x} \in \Omega,\ \Omega \subset R_3,\ \partial\Omega$ — is the boundary of Ω, $\mathbf{x} \in \Omega \subset R_3$; Ω is a convex domain $\mathring{\mathbf{J}}(\Omega)$ - space solenoidal vectors; $\mathbf{L}_\infty(Q)$ — is the subspace of $\mathbf{C}(\bar{Q})$. $W^k_{p,0}(\Omega)$ is Sobolev space functions equal to zero on $\partial\Omega$;

The input data $\mathbf{f}$ and $\mathbf{\Phi}$ of the problem (1) meet the requirements:

i) $\mathbf{f}(t, \mathbf{x}) \in \mathbf{L}_\infty(0, T; \mathbf{L}_p(\Omega)) \cap \mathring{\mathbf{J}}(Q)$; **ii)** $\mathbf{\Phi}(\mathbf{x}) \in \mathbf{L}_p(\Omega) \cap \mathbf{W}^1_{2,0}(\Omega) \cap \mathring{\mathbf{J}}(\Omega)$, $\forall p$.

[1]The work was done on the personal initiative of the author, since there was no financial support.

Further, we use the Holder inequalities

$$\left|\int\limits_{\Omega} UV\,\mathbf{dx}\right| \le \left(\int\limits_{\Omega} |U|^p\,\mathbf{dx}\right)^{\frac{1}{p}} \left(\int\limits_{\Omega} |V|^q\,\mathbf{dx}\right)^{\frac{1}{q}} \tag{2}$$

in addition, the integration by parts formula

$$\int\limits_{\Omega} V\Delta U\,\mathbf{dx} = -\int\limits_{\Omega} \nabla V\,\nabla U\,\mathbf{dx} + \int\limits_{\partial\Omega} V\frac{\partial U}{\partial \mathbf{n}}\,\mathbf{dx}. \tag{3}$$

To solve the problem (1) we use the splitting method. Let known vector-function be an approximation $\{\mathbf{u}^n\}$ in the moment $n\tau$, $0<\tau-$ step, then vector functions $\{\mathbf{u}^{n+1/2}\}$, $\{\mathbf{u}^{n+1}\}$, $n=0,1,\cdots,M;\ T=M\tau<\infty$ determined by the decisions of the following subsystems:

$$\frac{\mathbf{U}^{n+1/2}-\mathbf{U}^n}{\tau} + (\mathbf{U}^n,\nabla)\mathbf{U}^{n+1/2} - \nabla|\mathbf{U}^{n+1/2}|^2 = \mathbf{f}^n, \tag{4}$$

with initial

$$\mathbf{u}^0(\mathbf{x}) = \mathbf{\Phi}(\mathbf{x}) \wedge \mathbf{\Phi}(\mathbf{x})\big|_{\partial\Omega} = 0 \tag{5}$$

and

$$\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n+1/2}}{\tau} - \mu\Delta\mathbf{U}^{n+1} = 0, \tag{6}$$

boundary conditions

$$\mathbf{U}^{n+1}\big|_{\partial\Omega} = 0, \quad n=0,1,\cdots,M-1. \tag{7}$$

Lemma. For solving the splitting method scheme(4)-(7) fair estimates:

$$\|\mathbf{U}^{n+1/2}\|_{L_p(\Omega)} \le \|\mathbf{U}^n\|_{L_p(\Omega)} + \tau\|\mathbf{f}^n\|_{L_p(\Omega)}; \tag{8}$$

$$\max_{0\le n\le M-1}\|\mathbf{U}^{n+1}\|_{L_p(\Omega)} \le \|\mathbf{\Phi}\|_{L_p(\Omega)} + T\max_{0\le n<M}\|\mathbf{f}^n\|_{L_p(\Omega)},$$
$$\forall p=2k,\ k\in N,\ n=0,1,\cdots,M-1. \tag{9}$$

Proof. Multiply the scalar equation (4) on a function vector

$$p(E^{n+1/2})^{p-1}\mathbf{U}^{n+1/2},$$

we integrate work over the domain Ω and let us use identity $E^p=\frac{1}{2^p}|\mathbf{U}|^{2p}$, then similarly, both of works (see [1], inequalities (10)–(14)) find

$$2\int\limits_{\Omega}(E^{n+1/2})^p d\mathbf{x} + \tau\int\limits_{\Omega}\mathbf{U}^n\nabla(E^{n+1/2})^p d\mathbf{x} - 2\tau\int\limits_{\Omega}\mathbf{U}^{n+1/2}\nabla(E^{n+1/2})^p d\mathbf{x} =$$

$$= p\int\limits_{\Omega}(E^{n+1/2})^{p-1}\mathbf{U}^{n+1/2}\big(\mathbf{U}^n+\tau\mathbf{f}^n\big)d\mathbf{x}. \tag{10}$$

The second term from the left side (10) is transformed with integration in parts[2], then the right-hand side is estimated by the inequality Holder:

$$\tau\int\limits_{\Omega}\mathbf{U}^n\nabla\left(E^{n+1/2}\right)^p d\mathbf{x} = -\tau\int\limits_{\Omega} div\mathbf{U}^n\left(E^{n+1/2}\right)^p d\mathbf{x} + \tau\int\limits_{\partial\Omega}\mathbf{U}^n\mathbf{n}\left(E^{n+1/2}\right)^p d\mathbf{x} = 0; \tag{11}$$

$$-2\tau\int\limits_{\Omega}\mathbf{U}^{n+1/2}\nabla\left(E^{n+1/2}\right)^p d\mathbf{x} = 2\tau\int\limits_{\Omega} div\mathbf{U}^{n+1/2}\left(E^{n+1/2}\right)^p d\mathbf{x} - 2\tau\int\limits_{\partial\Omega}\mathbf{U}^{n+1/2}\mathbf{n}\left(E^{n+1/2}\right)^p d\mathbf{x} = 0; \tag{12}$$

[2] Of(6) follows $div\mathbf{U}^{n+1/2}=0$, consequently in equality (12) $\mathbf{U}^{n+1/2}\mathbf{n}=0$ [3; 46].

$$p\int\limits_{\Omega}\left(E^{n+1/2}\right)^{p-1}\mathbf{U}^{n+1/2}\mathbf{U}^{n}d\mathbf{x}\le\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1/2}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}; \quad (13)$$

$$p\tau\int\limits_{\Omega}\left(E^{n+1/2}\right)^{p-1}\mathbf{U}^{n+1/2}\mathbf{f}^{n}d\mathbf{x}\le\tau\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1/2}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{f}^{n}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}. \quad (14)$$

Now from the identity (10), using the relation (11)–(14), we arrive at the inequality

$$\frac{p}{2^{p-1}}\int\limits_{\Omega}|\mathbf{U}^{n+1/2}|^{2p}d\mathbf{x}\le$$

$$\le\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}|\mathbf{U}^{n+1/2}|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\left(\int\limits_{\Omega}|\mathbf{U}^{n}|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}+\tau\left(\int\limits_{\Omega}|\mathbf{f}^{n}|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}\right).$$

Where there is the estimate for the fractional step $n+1/2$ of the splitting method, that is (8) of the lemma 1.

$$\Big(\int\limits_{\Omega}|\mathbf{U}^{n+1/2}|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}}\le\Big(\int\limits_{\Omega}|\mathbf{U}^{n}|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}}+\tau\Big(\int\limits_{\Omega}\left|\mathbf{f}^{n}\right|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}}\quad\text{or}$$

$$\|\mathbf{U}^{n+1/2}\|_{L_p(\Omega)}\le\|\mathbf{U}^{n}\|_{L_p(\Omega)}+\tau\|\mathbf{f}^{n}\|_{L_p(\Omega)},\ \forall p=2k,\ k\in N.$$

To obtain the estimate (9) for the whole step, multiply the equation (6) by vector function $p(E^{n+1})^{p-1}\mathbf{U}^{n+1}$ and integrate the result over Ω

$$2\int\limits_{\Omega}\left(E^{n+1}\right)^{p}d\mathbf{x}-\tau p\mu\int\limits_{\Omega}\left(\Delta\mathbf{U}^{n+1},\mathbf{U}^{n+1}\right)\left(E^{n+1}\right)^{p-1}d\mathbf{x}-$$

$$-\tau p\int\limits_{\Omega}\left(\nabla|\mathbf{U}^{n+1}|^{2},\mathbf{U}^{n+1}\right)\left(E^{n+1}\right)^{p-1}d\mathbf{x}=p\int\limits_{\Omega}\mathbf{U}^{n+1}\mathbf{U}^{n+1/2}\left(E^{n+1}\right)^{p-1}d\mathbf{x}+$$

$$+\tau p\int\limits_{\Omega}\mathbf{U}^{n+1}\mathbf{f}^{n}(E^{n+1})^{p-1}d\mathbf{x}. \quad (15)$$

For the second term on the left side (15) we find

$$-\tau p\mu\int\limits_{\Omega}\left(\Delta\mathbf{U}^{n+1},\mathbf{U}^{n+1}\right)(E^{n+1})^{p-1}d\mathbf{x}=\tau p\mu\int\limits_{\Omega}(E^{n+1})^{p-1}\sum_{\alpha=1}^{3}(\nabla U_{\alpha}^{n+1})^{2}d\mathbf{x}+$$

$$+\tau p(p-1)\mu\int\limits_{\Omega}(E^{n+1})^{p-2}(\nabla E^{n+1})^{2}d\mathbf{x}\ge0, \quad (16)$$

Taking into account (16) from (15) we have

$$2\int\limits_{\Omega}(E^{n+1})^{p}d\mathbf{x}+\tau p\mu\int\limits_{\Omega}(E^{n+1})^{p-1}\sum_{\alpha=1}^{3}(\nabla U_{\alpha}^{n+1})^{2}d\mathbf{x}+$$

$$+\tau p(p-1)\mu\int\limits_{\Omega}(E^{n+1})^{p-2}(\nabla E^{n+1})^{2}d\mathbf{x}\le$$

$$\le p\int\limits_{\Omega}(E^{n+1})^{p-1}\mathbf{U}^{n+1}\mathbf{U}^{n+1/2}d\mathbf{x}+\tau p\int\limits_{\Omega}(E^{n+1})^{p-1}\mathbf{U}^{n+1}\mathbf{f}^{n}d\mathbf{x}. \quad (17)$$

The right-hand sides (17) are estimated using the Holder inequality, i.e.

$$p\int\limits_{\Omega}\left(E^{n+1}\right)^{p-1}\mathbf{U}^{n+1}\mathbf{U}^{n+1/2}d\mathbf{x}\leq\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1/2}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}.$$

Hence, taking into account the estimate (8) for the fractional step $n+1/2$, we write

$$p\int\limits_{\Omega}\left(E^{n+1}\right)^{p-1}\mathbf{U}^{n+1}\mathbf{U}^{n+1/2}d\mathbf{x}\leq\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}} \tag{18}$$

and

$$\tau p\int\limits_{\Omega}\left(E^{n+1}\right)^{p-1}\mathbf{U}^{n+1}\mathbf{f}^{n}d\mathbf{x}\leq\tau\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{f}^{n}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}. \tag{19}$$

Now using (16), (18), (19) from inequalities (17) we obtain

$$\frac{p}{2^{p-1}}\int\limits_{\Omega}\left(\mathbf{U}^{n+1}\right)^{2p}d\mathbf{x}\leq\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}+$$

$$\tau\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{f}^{n}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}.$$

Where, dividing both parts by a positive value $\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}$, write down

$$\left(\int\limits_{\Omega}\mid\mathbf{U}^{n+1}\mid^{p}d\mathbf{x}\right)^{\frac{1}{p}}\leq\left(\int\limits_{\Omega}\left|\mathbf{U}^{n}\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}}+\tau\left(\int\limits_{\Omega}\left|\mathbf{f}^{n}\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}},\ \forall p=2k,\ k\in N.$$

Which, summing over $n=0,1,\cdots,M-1$, we have

$$\left(\int\limits_{\Omega}\mid\mathbf{U}^{n+1}\mid^{p}d\mathbf{x}\right)^{\frac{1}{p}}\leq\left(\int\limits_{\Omega}\left|\mathbf{\Phi}\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}}+T\max_{n}\left(\int\limits_{\Omega}\left|\mathbf{f}^{n}\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}}.$$

Or equivalently the norm of the space $L_p(\Omega)$ estimate (9) in Lemma, Q. E. D.

Corollary. For solutions of the splitting method schemes (4)–(5), the following estimates are valid:

$$\max_{0\leq n\leq M-1}\|\mathbf{U}^{n+1}\|_{L_2(\Omega)}\leq\|\mathbf{\Phi}\|_{L_2(\Omega)}+T\max_{0\leq n<M}\|\mathbf{f}^{n}\|_{L_2(\Omega)}\equiv R_1, \tag{20}$$

$$\max_{0\leq n\leq M-1}\|\mathbf{U}^{n+1}\|_{L_4(\Omega)}\leq\|\mathbf{\Phi}\|_{L_4(\Omega)}+T\max_{0\leq n<M}\|\mathbf{f}^{n}\|_{L_4(\Omega)}\equiv R_2. \tag{21}$$

$$\sum_{n=0}^{M-1}\tau\sum_{\alpha=1}^{3}\left\|\nabla U_{\alpha}^{n+1}\right\|_{L_2(\Omega)}^{2}\leq\frac{1}{\mu}\Big(\left\|\mathbf{\Phi}\right\|_{L_2(\Omega)}^{2}+T/2(1+T)\left\|\mathbf{f}^{n}\right\|_{L_2(\Omega)}^{2}\Big)\equiv R_3, \tag{22}$$

$$\sum_{n=0}^{M-1}\tau\left\|\nabla E^{n+1}\right\|_{L_2(\Omega)}^{2}\leq\frac{1}{2\mu}\Big(\left\|\mathbf{\Phi}\right\|_{L_{4(\Omega)}}+TR_2^3\left\|\mathbf{f}^{n}\right\|_{L_4(\Omega)}\Big)\equiv R_4. \tag{23}$$

Proof. The estimates (20), (21) follow from (9) respectively at $p = 2$ and $p = 4$. To prove the estimate (22), write (17) when $p = 1$

$$\int_\Omega |\mathbf{U}^{n+1}|^2 d\mathbf{x} + \tau\mu \int_\Omega \sum_{\alpha=1}^{3} (\nabla U_\alpha^{n+1})^2 d\mathbf{x} \le \int_\Omega \mathbf{U}^{n+1}\mathbf{U}^{n+1/2} d\mathbf{x} + \tau \int_\Omega \mathbf{U}^{n+1}\mathbf{f}^n d\mathbf{x}. \tag{24}$$

Where to the right parts, using successive inequalities $2ab \le (a^2 + b^2)$, (8) and Cauchy-Bunyakovsky, we get

$$\tau\mu \int_\Omega \sum_{\alpha=1}^{3} \left(\nabla U_\alpha^{n+1}\right)^2 d\mathbf{x} \le$$

$$\le \frac{1}{2} \int_\Omega |\mathbf{U}^n|^2 d\mathbf{x} + \tau \parallel \mathbf{U}^{n+1} \parallel_{L_2(\Omega)} \parallel \mathbf{f}^n \parallel_{L_2(\Omega)} .$$

Here, summing up n, we have

$$\frac{1}{2} \int_\Omega |\mathbf{U}^{n+1}|^2 \, \mathbf{dx} + \mu \sum_{n=0}^{M-1} \tau \sum_{\alpha=1}^{3} \int_\Omega (\nabla U_\alpha^{n+1})^2 d\mathbf{x} \le$$

$$\le \frac{1}{2} \int_\Omega |\Phi|^2 \, \mathbf{dx} + \sum_{n=0}^{M-1} \tau \left(\int_\Omega \left|\mathbf{U}^{n+1}\right|^2 d\mathbf{x} \right)^{\frac{1}{2}} \left(\int_\Omega \left|\mathbf{f}^n\right|^2 d\mathbf{x} \right)^{\frac{1}{2}} .$$

Where by virtue of the nonnegativity of the first integral in the left part and estimates (9), we arrive at an inequality (22), i.e.

$$\mu \sum_{n=0}^{M-1} \tau \sum_{\alpha=1}^{3} \left\|\nabla U_\alpha^{n+1}\right\|^2_{L_2(\Omega)} \le \frac{1}{2}\left\|\mathbf{\Phi}\right\|^2_{L_2(\Omega)} + \max_n \left\|\mathbf{U}^{n+1}\right\|_{L_2(\Omega)} T \max_n \left\|\mathbf{f}^n\right\|_{L_2(\Omega)}.$$

To prove (23) the estimate (17) we write at $p = 2$, replacing n with m

$$\frac{1}{4} \int_\Omega |\mathbf{U}^{m+1}|^4 \, \mathbf{dx} + \mu\tau \int_\Omega E^{m+1} \sum_{\alpha=1}^{3} \left(\nabla U_\alpha^{m+1}\right)^2 d\mathbf{x} + 2\mu\tau \int_\Omega \left(\nabla E^{m+1}\right)^2 d\mathbf{x} \le$$

$$\le \frac{1}{4} \int_\Omega | \, \mathbf{U}^m \, |^4 \, d\mathbf{x} + \tau \left(\int_\Omega \left|\mathbf{U}^{m+1}\right|^4 d\mathbf{x} \right)^{\frac{3}{4}} \left(\int_\Omega \left|\mathbf{f}^m\right|^4 d\mathbf{x} \right)^{\frac{1}{4}} .$$

Which is the sum of m from 0 to n,

$$\frac{1}{4} \sum_{m=0}^{n} \int_\Omega |\mathbf{U}^{m+1}|^4 \, \mathbf{dx} + \mu \sum_{m=0}^{n} \tau \int_\Omega E^{m+1} \sum_{\alpha=1}^{3} (\nabla U_\alpha^{m+1})^2 d\mathbf{x} +$$

$$+ 2\mu \sum_{m=0}^{n} \tau \int_\Omega (\nabla E^{m+1})^2 d\mathbf{x} \le \frac{1}{4} \sum_{m=0}^{n} \int_\Omega | \, \mathbf{U}^m \, |^4 \, d\mathbf{x} +$$

$$+ \sum_{m=0}^{n} \tau \Big(\int_\Omega \left|\mathbf{U}^{m+1}\right|^4 d\mathbf{x} \Big)^{\frac{3}{4}} \Big(\int_\Omega \left|\mathbf{f}^m\right|^4 d\mathbf{x} \Big)^{\frac{1}{4}} .$$

Here, as in the previous case, we find

$$2\mu \sum_{m=0}^{n} \tau \int_\Omega (\nabla E^{m+1})^2 d\mathbf{x} \le \left\|\mathbf{\Phi}\right\|_{L_4(\Omega)} + T \, R_2^3 \max_{0 \le m \le M} \left\|\mathbf{f}^m\right\|_{L_4(\Omega)}.$$

Where follows (23).

Having excluded from the subsystem (4), (5) a vector function with a fractional exponent $\{n+1/2\}$ of the splitting method, we obtain a system of the whole step

$$\mathbf{U}_{\bar{t}}^m - \mu\Delta\mathbf{U}^{m+1} + (\mathbf{U}^m,\nabla)\mathbf{U}^{m+1} - 2\nabla E^{m+1} = \mathbf{f}^m, \quad \mathbf{U}_{\bar{t}}^m = \big(\mathbf{U}^{m+1} - \mathbf{U}^m\big)/\tau, \tag{25}$$

with initial boundary conditions

$$\mathbf{U}^0(\mathbf{x}) = \mathbf{\Phi}(\mathbf{x}), \quad \mathbf{U}^m\big|_{\partial\Omega} = 0, \ m = 0,1,\cdots,M-1. \tag{26}$$

Theorem. If the input data of the problem (1) satisfy the requirements **i**), **ii**) *and* $\partial\Omega \in C^2$, *then there is a strong generalized solution to the* **U** *problem (1) and have place of evaluation of spaces*

$$\mathbf{U}^m \in \mathbf{W}_{2,0}^{2,1}(\Omega) \cap \mathring{\mathbf{J}}_\infty(\Omega), \forall m \in N,$$

$$\big\|\mathbf{U}_{\bar{t}}^m\big\|_{\mathbf{L}_2(\Omega)}^2 \le \mu \sum_{\alpha=1}^{3} \big\|\nabla\Phi_\alpha\big\|_{\mathbf{L}_2(\Omega)}^2 + 3T\big\|\mathbf{f}^m\big\|_{\mathbf{L}_2(\Omega)}^2 \equiv R_6, \tag{27}$$

$$\big\|\Delta\mathbf{U}^m\big\|_{\mathbf{L}_2(\Omega)}^2 \le R_6/\mu^2 \equiv R_7, \tag{28}$$

$$\big\|\nabla U_\alpha^m\big\|_{\mathbf{L}_2(\Omega)}^2 \le R_6/\mu \equiv R_8, \ \alpha = \overline{1,3}, \tag{29}$$

$$\|\mathbf{U}^m\|_{\mathbf{W}_2^2(\Omega)} \le R_9\|\Delta\mathbf{U}^m\|_{\mathbf{L}_2(\Omega)}, \quad R_9 - const, \ \forall m \in N. \tag{30}$$

Proof. In order to establish inequalities (27)–(29) from the equation (25) we pass to the identity

$$\int_\Omega \big(\mathbf{U}_{\bar{t}}^m - \mu\Delta\mathbf{U}^{m+1}\big)^2 \mathbf{dx} = \int_\Omega \big(\mathbf{f}^m - (\mathbf{U}^m,\nabla)\mathbf{U}^{m+1} + 2\nabla E^{m+1}\big)^2 \mathbf{dx}.$$

We square the integrals and from which we turn to inequality

$$\sum_{m=0}^{n} \tau \int_\Omega \Big(\big(\mathbf{U}_{\bar{t}}^m\big)^2 + \big(\mu\Delta\mathbf{U}^{m+1}\big)^2 - 2\mu\mathbf{U}_{\bar{t}}^m\Delta\mathbf{U}^{m+1}\Big)\mathbf{dx} \le 3T\max_m \|\mathbf{f}^m\|_{L_2(\Omega)} +$$

$$+3\sum_{m=0}^{n} \tau \int_\Omega |(\mathbf{U}^m,\nabla)\mathbf{U}^{m+1}|^2 d\mathbf{x} + 12\sum_{m=0}^{n} \tau\|\nabla E^{m+1}\|_{L_2(\Omega)}^2. \tag{31}$$

Then pair work on the left side, converting with integration by parts, find the inequality

$$-2\mu\sum_{m=0}^{n} \tau \int_\Omega \mathbf{U}_{\bar{t}}^m\Delta\mathbf{U}^{m+1} d\mathbf{x} \ge \mu\sum_{\alpha=1}^{3} \int_\Omega |\nabla U_\alpha^{n+1}|^2 d\mathbf{x} - \mu\sum_{\alpha=1}^{3} \int_\Omega \big|\nabla\Phi_\alpha\big|^2 \mathbf{dx}. \tag{32}$$

In the right part of such force on Young's inequality at $\epsilon = 1$ and $p = 2$:

$$3\sum_{m=0}^{n} \tau \int_\Omega |(\mathbf{U}^m,\nabla)\mathbf{U}^{m+1}|^2 d\mathbf{x} \le 3\max_m \|\mathbf{U}^m\|_{L_2(\Omega)}^2 \sum_{m=0}^{n} \tau \sum_{\alpha=1}^{3} \|\nabla U_\alpha{}^{m+1}\|_{L_2(\Omega)}^2 = 3R_1R_3.$$

As a result, from (31), taking into account (32) and estimates from Lemmas we obtain the inequalities (27)–(29) for strong generalized solutions of the problem (1a)–(1b).

Since the boundary of the domain $\partial\Omega \in C^2$ find the estimate (30), using inequalities from [4], just for any functions $U(x) \in W_2^2(\Omega) \cap W_{2,0}^2(\Omega)$:

$$\|\mathbf{U}^m\|_{\mathbf{W}_2^2(\Omega)} \le R_9\|\Delta\mathbf{U}^m\|_{\mathbf{L}_2(\Omega)}, \ \forall m \in N, \quad R_9 - const.$$

Theorem is proved.

To show the compactness and existence of solutions, we denote a set of approximate solutions of systems with initial-boundary conditions (4)–(7) via $\{\mathbf{U}^\tau\}$, and the predicted values on the interval $[0,T]-$ through $\widetilde{\mathbf{U}}^\tau$.

Of the estimates (27)–(30) implies the uniform boundedness the norms of interpolating functions $\widetilde{\mathbf{U}}^\tau \in \mathbf{W}^{2,1}_{2,0}(\Omega) \cap \mathring{\mathbf{J}}_\infty(\Omega)$. Consequently, the set $\{\widetilde{\mathbf{U}}^\tau\}$ strongly compact in the space $W^1_2(Q)$. From it you can select convergent subsequence. It will converge strongly in $W^1_2(Q)$ to some elements $\mathbf{U}(t,x) \in W^1_2(Q)$.

The second derivatives and nonlinear terms, respectively, will have its weak limits in $L_2(Q)$.

Remark. In [4] for some difference schemes corresponding to three-dimensional system of non-linear Burgers equations, proved stability in the space $\ell_p, \forall p$.

References

1 Akysh (Akishev), A.Sh. (2018). On natural properties of the Navier-Stokes equations. *Sobolev readings abstracts: International School–Conference dedicated to the 110th anniversary of the birthday of Sergei L'vovich Sobolev.* Novosibirsk, December 10–16, 48.

2 Akysh (Akishev), A.Sh. (2018). The natural solvability of the Navier-Stokes equations. *Bulletin of the Karagandy University. Mathematics series, Vol. 4(92),* 8–14.

3 Ladyzenskaja, O.A. (1970). *Mathematical problems of dynamics of viscous incompressible fluids.* Moscow: Nauka.

4 Akysh (Akishev), A.Sh. (2005). The ℓ_p stability of some difference schemes for one system of nonlinear parabolic equations. *Siberian Journal of Numer. Mathematics, Vol. 8, 4,* 273–280.

Ә.Ш. Ақыш (Ақышев)

Навье-Стокс теңдеуінің ыдырату әдісі және әлді шешімнің табылатындығы

Журналдың өткен санындағы автордың мақаласында үшөлшемді Навье-Стокс теңдеулері (НСТ) шешімдерінің қасиеттерінен қысым мен жылдамдық векторы модулі квадратының арақатынас байланысы табылған. Осы нәтиже негізінде НСТ-ның шешілетіндігі көрсетілген. Зерттеушінің таңдаған кеңістігінде үшөлшемді НСТ-ға қойылған есептің әлсіз шешімінің жалқылығы мен әлді шешімінің ұзақ уақыт бойы табылатындығы дәлелденген. Бұл алынған нәтиже НСТ-ның математикалық сапа теориясын дамытуға әрі қарай да өз үлесін қоса бермек. Ал қазіргі шақтағы өзекті мәселелердің негізгісінің бірі — қойылған есептің шешімін табу үшін математикалық әдістердің ыңғайлы біреуін негіздеу. Мақалада НСТ-ның шешімін табуға қолайлы әдіс ретінде ыдырату әдісі таңдалған. Әдіс негізделіп, әлді шешімді табу алгоритмі ұсынылған.

Кілт сөздер: Навье-Стокс теңдеулері, Навье-Стокс теңдеулеріне ыдырату әдісі, Навье-Стокс теңдеулерінің шешімінің табылатындығы, әлді шешімді табу алгоритмі.

А.Ш. Акыш (Акишев)

Метод расщепления и существование сильного решения уравнений Навье-Стокса

В предыдущем номере данного журнала в статье автора установлено соотношение между давлением и квадратом модуля вектора скорости из свойств решений УНС, на основе чего доказаны единственность слабых и существование сильных решений задачи для трехмерных уравнений Навье-Стокса в целом по времени. Результат является вкладом в качественную математическую теорию уравнений Навье-Стокса. Однако одной из актуальных проблем в теории уравнения Навье-Стокса является выбор математического метода для доказательства теоремы существования. В настоящей работе выбран метод расщепления для решения уравнений Навье-Стокса. Дано обоснование этого метода. Показана компактность последовательности решений, тем самым доказано существование сильных решений задачи для трехмерных уравнений Навье-Стокса в целом по времени.

Ключевые слова: уравнения Навье-Стокса, метод расщепления для уравнений Навье-Стокса, компактность, существование сильных решений, алгоритм определения сильных решений.

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Some new integral inequalities for (s, m)-convex and (α, m)-convex functions

The paper considers several new integral inequalities for functions the second derivatives of which, with respect to the absolute value, are (s, m)-convex and (α, m)-convex functions. These results are related to well-known Hermite-Hadamard type integral inequality, Simpson type integral inequality, and Jensen type inequality. In other words, new upper bounds for these inequalities using the indicated classes of convex functions have been obtained. These estimates are obtained using a direct definition for a convex function, classical integral inequalities of Hölder and power mean types. Along with the new outcomes, the paper presents results confirming the existing in literature upper bound estimates for integral inequalities (in particular well known in literature results obtained by U. Kırmacı in [7] and M.Z. Sarıkaya and N. Aktan in [35]). The last section presents some applications of the obtained estimates for special computing facilities (arithmetic, logarithmic, generalized logarithmic average and harmonic average for various quantities).

Keywords: convex function, (s, m)-convex, (α, m)-convex, Hermite–Hadamard inequalitiy, Jensen inequality, Hölder inequality, power mean inequality.

Introduction

Convexity has become a very attractive topic for many authors over the past decades, since it has applications in many areas of pure and applied mathematics. The following basic two definitions are well known in the literature [1]:

Definition 1. The function $f : [a, b] \to \mathbb{R}$, is said to be convex, if we have

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

Definition 2. A function $f : [a, b] \to R$ is called either midconvex or convex in the Jensen sence, or J–convex on $[a, b]$ if for all points $x, y \in [a, b]$ the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} \tag{1}$$

is valid. Many important inequalities are established for the class of convex functions, but one of the most important is so called Hermite–Hadamard's inequality (or Hadamard's inequality). This double inequality is stated as follows in literature:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and let $a, b \in I$, with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int\limits_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \tag{2}$$

The above inequality is in the reversed direction if f is concave.

In [2] Toader defines the m-convexity:

Definition 3. Let real function f be defined on a nonempty interval I of real numbers $\mathbb{R}$. The function f is said to be m-convex on I if inequality

$$f(\lambda x + m(1 - \lambda) y) \leq \lambda f(x) + m(1 - \lambda) f(y)$$

holds for all $x, y \in I$ and $m, \lambda \in [0, 1]$.

In [3] Breckner defined a new class of functions that are s-convex in the second sense:

Definition 4. $f: [0,\infty) \to \mathbb{R}$ is said to be s-convex function in the second sense if

$$f(\lambda x + (1-\lambda) y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in [0,\infty)$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$. It is clear that the ordinary convexity of functions defined on $[0,\infty)$ for $s = 1$.

In [4] Miheşan introduced the following class of functions:

Definition 5. $f : [0,\infty) \to \mathbb{R}$ is said to be (α, m)-convex function if

$$f(\lambda x + (1-\lambda) y) \le \lambda^\alpha f(x) + m(1-\lambda^\alpha) f(y)$$

holds for all $x, y \in [0,\infty)$, $\lambda \in [0,1]$; and for some fixed $\alpha, m \in (0,1]$.

A series of works ([5–38] and references therein) devoted to (α, m)-convex and (s, m)-convex functions and established some Hermite-Hadamard, Ostrovski, Jensen et al. type inequalities (1) and (2).

The following theorem was proved by Dragomir and Pearce, in [5]:

Theorem 1. Let $f : [0,\infty) \to \mathbb{R}$ be a m-convex function with $m \in (0,1]$. If $0 \le a < b < \infty$ and $f \in L_1[a,b]$, then one has the inequality:

$$\frac{1}{b-a}\int_a^b f(x)dx \le min\left\{\frac{f(a)+mf(\frac{b}{m})}{2}, \frac{f(b)+mf(\frac{a}{m})}{2}\right\}. \tag{3}$$

Some generalizations of this result can be found in [36–38].

In [6] Özdemir et al. the following lemma is proved.

Lemma 1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° (I° is interior of I), where $a, b \in I$ and $m \in (0,1]$. If $f'' \in L[a,b]$, then the following equality holds

$$\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)dx =$$

$$= \frac{(mb-a)^2}{2}\int_0^1 (t-t^2) f''(ta + m(1-t)b)dt. \tag{4}$$

In [7], Kırmacı proved the following lemma

Lemma 2. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° with $f^{''} \in L[a,b]$. Then we have

$$\frac{(b-a)^2}{2}(I_1 + I_2) = \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right], \tag{5}$$

where

$$I_1 = \int_0^{1/2} t\,(t-0.5)\, f''(ta + (1-t)b)dt,\ I_2 = \int_{1/2}^1 (t-0.5)\,(t-1)\, f''(ta + (1-t)b)dt$$

and I° denotes the interior of I.

In [8] B. Bayraktar and M. Gürbüz the following lemma is proved.

Lemma 3. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°($I^\circ$is interior of I), where $a, b \in I$. If $f^{''} \in L\,[a,b]$, then we have

$$\frac{f(a)+f(mb)}{2} - f\left(\frac{a+mb}{2}\right) =$$

$$= \frac{(mb-a)^2}{2}\left[\int_0^{1/2} t f''(at + m(1-t)b)dt + \int_{1/2}^1 (1-t) f''(at + m(1-t)b)dt\right]. \tag{6}$$

In this paper we give some integral inequalities of Hadamard type and inequalities Jensen type for twice differentiable (s, m)-convex and (α, m)-convex functions and give some applications to the special means of real numbers.

1 Some new results for (s,m)-convex functions

We start with the definition [9] of a (s,m)-convex functions.

Definition 6. For some fixed $s \in (0,1]$ and $m \in [0,1]$ a mapping $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be (s,m)-convex in the second sense on I if

$$f(tx + m(1-t)y) \le t^s f(x) + m(1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

It should be noted that the following proposition is true:

Proposition 1. Any m-convex function is (s,m)-convex function.

Proof. Indeed, for m-convex functions we have

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y), \ \forall\ m,\ t \in [0,1].$$

Since $t \le t^s$ and $1-t \le (1-t)^s$ for all $s \in (0,1]$ then we can write

$$tf(x) + m(1-t)f(y) \le t^s f(x) + m(1-t)^s f(y)$$

and then

$$f(tx + m(1-t)y) \le t^s f(x) + m(1-t)^s f(y).$$

The proof is completed.

Theorem 2. Let $f : [0,\infty) \to \mathbb{R}$ be an (s,m)-convex function with $s,m \in (0,1]$. If $0 \le a < b < \infty$ and $f \in L[a,b]$, then has the inequality:

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{s+1}\left[\frac{f(a) + mf(\frac{b}{m})}{2} + \frac{f(b) + mf(\frac{a}{m})}{2}\right]. \tag{7}$$

Proof. It's obvious that

$$\int_0^1 f(ta + (1-t)b)dt = \int_0^1 f((1-t)a + tb)dt = \frac{1}{b-a}\int_a^b f(x)dx; \tag{8}$$

and

$$\int_0^1 \left[f(ta + (1-t)b) + f((1-t)a + tb)\right]dt = \frac{2}{b-a}\int_a^b f(x)dx. \tag{9}$$

Since the function f is (s,m)-convex functions for all $t \in [0,1]$

$$f(ta + (1-t)b) = f\left(ta + m(1-t)\frac{b}{m}\right) \le t^s f(a) + m(1-t)^s f\left(\frac{b}{m}\right)$$

and

$$f(tb + (1-t)a) \le t^s f(b) + m(1-t)^s f\left(\frac{a}{m}\right)$$

then

$$\int_0^1 \left[f(ta + (1-t)b) + f((1-t)a + tb)\right]dt \le$$

$$\le \int_0^1 \left[t^s f(a) + m(1-t)^s f\left(\frac{b}{m}\right)\right]dt + \int_0^1 \left[t^s f(b) + m(1-t)^s f\left(\frac{a}{m}\right)\right]dt =$$

$$= \frac{f(a) + mf(\frac{b}{m})}{s+1} + \frac{f(b) + mf(\frac{a}{m})}{s+1}.$$

Taking into account equality (9) completes the proof.

Remark 1. From (7) for $m = 1$ and $s = 1$ we have right hand inequality (2)

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a) + f(b)}{2}.$$

Corollary 1. It is obvious that for (s,m)-convex functions the inequalities

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{\frac{f(a)+mf(\frac{b}{m})}{s+1},\ \frac{f(b)+mf(\frac{a}{m})}{s+1}\right\}. \tag{10}$$

Proof. Since the

$$\int_0^1 f(ta+(1-t)b)dt \le \frac{f(a)+mf(\frac{b}{m})}{s+1}$$

and

$$\int_0^1 f(tb+(1-t)a)dt \le \frac{f(b)+mf(\frac{a}{m})}{s+1}$$

and taking into account equalities (8) we have (10).

Remark 2. If we choose $s=1$ from (10) we have (3).

Theorem 3. Let $f: I\subset[0,b^*]\to\mathbb{R}$ be a twice differentiable mapping on I°such that $f''\in L[a,b]$ where $a,b\in I$. If $|f''|^q$ is (s,m)-convex on $[a,b]$ for $s,m\in(0,1]$, $q\ge 1$, then the following inequality holds

$$\frac{f(a)+f(mb)}{2}\le \tag{11}$$

$$\le\frac{(mb-a)^2}{2}\frac{\left(|f''(a)|^q+m|f''(b)|^q\right)^{1/q}}{6^{1-\frac{1}{q}}(s+2)(s+3)}.$$

Proof. Suppose that $q=1$. From (4) and using the (s,m)-convexity of $|f''|$, we have

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|\le\frac{(mb-a)^2}{2}\int_0^1\left(t-t^2\right)|f''(ta+m(1-t)b)|\,dt\le$$

$$\le\frac{(mb-a)^2}{2}\int_0^1\left(t-t^2\right)\left[t^s|f''(a)|+m(1-t)^s|f''(b)|\right)]\,dt=$$

$$=\frac{(mb-a)^2}{2}\frac{1}{(s+2)(s+3)}\left(|f''(a)|+m\,|f''(b)|\right)$$

which completes the proof for $q=1$.

Suppose now that $q>1$. From (4) in Lemma 1 and using the Hölder's integral inequality for $q>1$, we have

$$\int_0^1\left(t-t^2\right)|f''(ta+m(1-t)b)|\,dt= \tag{12}$$

$$=\int_0^1 (t-t^2)^{\frac{1}{p}}(t-t^2)^{\frac{1}{q}}\,|f''(ta+m(1-t)b)|\,dt\le$$

$$\le\left[\int_0^1\left((t-t^2)^{\frac{1}{p}}\right)^p dt\right]^{\frac{1}{p}}\left(\int_0^1\left[(t-t^2)^{\frac{1}{q}}\,|f''(ta+m(1-t)b)|\right]^q dt\right)^{\frac{1}{q}},$$

where $\frac{1}{q}+\frac{1}{p}=1$.

Since $|f''|^q$ is (s,m)-convex on$[a,b]$, we know that for all $t\in[0,1]$,

$$|f''(ta+m(1-t)b)|^q\le t^s\,|f''(a)|^q+m(1-t)^s\,|f''(b)|^q\,. \tag{13}$$

From (12) and (13) we have

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|\le\frac{(mb-a)^2}{2}\left(\frac{1}{6}\right)^{\frac{1}{p}}\times$$

$$\times\left[|f''(a)|^q\int_0^1(t-t^2)t^s dt+m\,|f''(b)|^q\int_0^1(t-t^2)(1-t)^s dt\right],$$

here

$$\int_0^1 (t-t^2)t^s dt = \frac{1}{(s+2)(s+3)} \text{ and } \int_0^1 (t-t^2)(1-t)^s dt = \frac{1}{(s+2)(s+3)},$$

we have (11). The proof is completed.

Corollary 2. From (11) for $s = m = q = 1$, we have estimates obtained by Sarıkaya and Aktan (see [35], Proposition 2):

$$\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)dx\right| \leq \frac{(b-a)^2}{24}\left[|f''(a)| + |f''(b)|\right].$$

Theorem 4. Let $f : I \subset \mathbb{R} \to \mathbb{R},\ I \subset [0, \infty)$, be twice differentiable function on I° such that $f^{''} \in L[a, b]$ with $0 \leq a < b < \infty$. If $\left|f^{''}\right|$ is (s, m)-convex function with $s, m \in (0, 1]$ then we have

$$\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] \leq$$

$$\leq \frac{(b-a)^2}{2}\xi(1+\tau)\left[|f''(a)| + m\left|f''\left(\frac{b}{m}\right)\right|\right], \tag{14}$$

where

$$\xi = \frac{1}{(s+2)(s+3)2^{s+3}} \text{ and } \tau = \frac{2^{s+2}(s-1)+s+5}{s+1}$$

Proof. Since f'' is a (s, m)-convex function

$$f''(ta+(1-t)b) = f''\left(ta + m(1-t)\frac{b}{m}\right) \leq t^s f(a) + m(1-t)^s f\left(\frac{b}{m}\right), \forall t \in [0, 1]$$

From equality (5) and using the triangle inequality, we can write

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq$$

$$\leq |f''(a)|\int_0^{1/2} t^{s+1}(0.5-t)\,dt + m\left|f''\left(\frac{b}{m}\right)\right|\int_0^{1/2} t\,(0.5-t)\,(1-t)^s\,dt \quad +$$

$$+|f''(a)|\int_{1/2}^1 t^s\,(t-0.5)\,(1-t)dt + m\left|f''\left(\frac{b}{m}\right)\right|\int_{1/2}^1 (t-0.5)\,(1-t)^{s+1}\,dt. \tag{15}$$

Obviously, the first and third integrals are easy to calculate:

$$\int_0^{1/2} t^{s+1}(0.5-t)\,dt = \xi, \quad \int_{1/2}^1 t^s\,(t-0.5)\,(1-t)dt = \xi\tau.$$

If we do $1 - t = z$ transformations in second and fourth integrals, we get:

$$\int_{1/2}^1 (t-0.5)\,(1-t)^{s+1}\,dt = \xi \text{ and } \int_0^{1/2} t\,(0.5-t)\,(1-t)^s\,dt = \xi\tau.$$

Substituting the values of the integrals in inequality (15) and completing the grouping, we complete the proof.

Corollary 3. Let $f : I \to \mathbb{R},\ I \subset [0, \infty)$ be twice differentable function on I° such as $f'' \in L[a, b]$, $0 \leq a < b < \infty$. If $|f''|$ is m-convex with $m \in (0, 1]$ then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq$$

$$\leq \frac{(b-a)^2}{96}\left[|f''(a)| + m\left|f''\left(\frac{b}{m}\right)\right|\right]. \tag{16}$$

Proof. In inequality (14) if we choose $s = 1$ we have:

$$\xi = \frac{1}{(s+2)(s+3)2^{s+3}} = \frac{1}{3 \cdot 2^6}; \ \tau = \frac{2^{s+2}(s-1)+s+5}{s+1} = 3, \xi(1+\tau) = \frac{1}{48}$$

and from (14) we get (16). The proof is completed. This inequality were obtained by Kırmacı (see [7], Corollary 1).

Corollary 4. If $\|f''\|_\infty = \sup\limits_{x\in[a,b]} |f''(x)| < \infty$ and $m \in (0,1]$, we have

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{2} \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{(b-a)^2}{96}(1+m) \|f''\|_\infty .$$

Also putting $m = 1$ we get inequality

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{2} \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{(b-a)^2}{48} \|f''\|_\infty .$$

The same estimates were obtained by U. Kırmacı (see [7], Remark 1).

2 Some new results for (α, m)-convex functions

The following theorem gives an upper estimate the value of the inequality (1) for a (α, m)-convex function.

Theorem 5. Let $f : [0,\infty) \to \mathbb{R}$ be twice differentiable function on I° such as $f^{''} \in L[a,b]$ with $0 \le a < b < \infty$. If $\frac{b}{m} \in I^\circ$ and $\left|f^{''}\right|$ is (α, m)-convex function with $\alpha, m \in (0,1]$ then we have

$$\left| \frac{f(a)+f(mb)}{2} - f\left(\frac{a+mb}{2}\right)\right| \le \tag{17}$$

$$\le \frac{(mb-a)^2}{2} \left[(\zeta+\eta)|f''(a)| + m|f''(b)| \left|\frac{1}{4} - (\zeta+\eta)\right|\right],$$

where

$$\zeta = \frac{1}{(\alpha+2)\,2^{\alpha+2}} \quad \text{and} \quad \eta = \frac{2^{\alpha+2}-\alpha-3}{(\alpha+1)(\alpha+2)\,2^{\alpha+2}}.$$

Proof. Using the triangle inequality for the equality (6) in Lemma 3, we can write

$$\frac{f(a)+f(mb)}{2} - f\left(\frac{a+mb}{2}\right) \le \frac{(mb-a)^2}{2} \times$$

$$\times \left[\left| \int_0^{1/2} t f''(at+m(1-t)b)dt \right| + \left| \int_{1/2}^1 (1-t) f''(at+m(1-t)b)dt \right| \right] =$$

$$= \frac{(mb-a)^2}{2} (|I_1| + |I_2|) . \tag{18}$$

Since f'' is a (α, m)-convex function

$$|I_1| \le \int_0^{1/2} t|f''(at+m(1-t)b)|dt \le |f''(a)| \int_0^{1/2} t^{\alpha+1}dt + m|f''(b)| \int_0^{1/2} t\,(1-t^\alpha)\,dt =$$

$$= \frac{1}{(\alpha+2)2^{\alpha+2}}|f''(a)| + m|f''(b)| \left| \frac{1}{8} - \frac{1}{(\alpha+2t)2^{\alpha+2}} \right| =$$

$$= \zeta|f''(a)| + m|f''(b)| \left|\frac{1}{8} - \zeta\right| .$$

For the second integral $|I_2|$ we can write

$$|I_2| \le |f''(a)| \int_{1/2}^{1} t^{\alpha}(1-t)\,dt + m\,|f''(b)| \int_{1/2}^{1} (1-t)(1-t^{\alpha})\,dt =$$

$$= \frac{2^{\alpha+2}-\alpha-3}{(\alpha+1)(\alpha+2)\,2^{\alpha+2}}\,|f''(a)| + m\,|f''(b)| \left|\frac{1}{8} - \frac{2^{\alpha+2}-\alpha-3}{(\alpha+1)(\alpha+2)\,2^{\alpha+2}}\right| =$$

$$= \eta\,|f''(a)| + m\,|f''(b)| \left|\frac{1}{8}-\eta\right|.$$

Substituting these inequalities for $|I_1|$ and $|I_2|$ into inequality (18), we complete the proof.

Corollary 5. In inequality (17) if we choose $\alpha = 1$ and $m = 1$ we have:

$$\left|\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{16}\left[|f''(a)| + |f''(b)|\right].$$

The same estimates were obtained by Bayraktar and Gürbüz (see [8], Corollary 2.2).

Theorem 6. Let $f : I = [0,b^*] \to \mathbb{R}$ be a twice differentiable function on I° such as $f'' \in L[a,b]$ where $a,b \in I^{\circ}$. If $\frac{b}{m} \in I^{\circ}$ and $|f''|^q$ is (α,m)-convex on I, for $\alpha, m \in (0,1]$ and $q \ge 1$, the following inequality holds

$$\left|\frac{f(a)+f(mb)}{2} - f\left(\frac{a+mb}{2}\right)\right| \le \frac{(mb-a)^2}{2^{4-\frac{3}{q}}} \times F,$$

where

$$F = \left[\eta\,|f''(a)|^q + m\left|\frac{1}{8}-\eta\right|\,|f''(b)|^q\right]^{\frac{1}{q}} + \left[\zeta\,|f''(a)|^q + m\left|\frac{1}{8}-\zeta\right|\,|f''(b)|^q\right]^{\frac{1}{q}};$$

$$\eta = \frac{1}{(\alpha+2)\,2^{\alpha+2}} \text{ and } \zeta = \frac{2^{\alpha+2}-1}{(\alpha+1)(\alpha+2)\,2^{\alpha+2}}.$$

Proof. Using the triangle inequality for the equality (6) in Lemma , we can write

$$\left|\frac{f(a)+f(mb)}{2} - f\left(\frac{a+mb}{2}\right)\right| \le$$

$$\le \frac{(mb-a)^2}{2}\left[\int_0^{1/2} t\,|f''(ta+m(1-t)b)|\,dt + \int_{1/2}^{1} (1-t)\,|f''(ta+m(1-t)b)|\,dt\right] =$$

$$= \frac{(mb-a)^2}{2}\left(I_1 + I_2\right). \tag{19}$$

Using the power mean inequality and (α,m)-convexity of $|f''|^q$ on $[a,b]$ we get

$$I_1 \le \left(\int_0^{1/2} t dt\right)^{1-\frac{1}{q}} \left[\int_0^{1/2} t\,|f''(at+m(1-t)b)|^q\,dt\right]^{\frac{1}{q}} \le$$

$$\le 2^{\frac{3(1-q)}{q}} \left[|f''(a)|^q \int_0^{1/2} t^{\alpha+1}dt + m\,|f''(b)|^q \left|\int_0^{1/2} t(1-t^{\alpha})dt\right|\right]^{\frac{1}{q}}.$$

And calculating these integrals, we have

$$I_1 \le 2^{\frac{3(1-q)}{q}} \left[\eta\,|f''(a)|^q + \left|\frac{1}{8}-\eta\right| m\,|f''(b)|^q\right]^{\frac{1}{q}}. \tag{20}$$

Similarly for I_2 we can write

$$I_2 \le \left(\int_{1/2}^{1} (1-t)\,dt\right)^{1-\frac{1}{q}} \left[\int_{1/2}^{1} (1-t)\,|f''(at+m(1-t)b)|^q\,dt\right]^{\frac{1}{q}} \le$$

$$\leq 2^{\frac{3(1-q)}{q}}\left[|f''(a)|^q\left|\int_{1/2}^{1} t^{\alpha}(1-t)\,dt\right| + m\,|f''(b)|^q\left|\int_{1/2}^{1}(1-t)(1-t^{\alpha})dt\right|\right]^{\frac{1}{q}}.$$

And calculating these integrals, we have

$$I_2 \leq 2^{2^{\frac{3(1-q)}{q}}}\left[\zeta\,|f''(a)|^q + m|\frac{1}{8}-\zeta|\,|f''(b)|^q\right]^{\frac{1}{q}}. \tag{21}$$

Substituting these inequalities for (20) and (21) into inequality (19) and rearranging we complete the proof.

Corollary 6. In Theorem 6 if we choose $\alpha = m = q = 1$, we have

$$\left|\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{32}\left[3\,|f''(a)| + |f''(b)|\right].$$

3 Applications to special means

We now consider the means for arbitrary real numbers α, β ($\alpha \neq \beta$). We take

1 *Arithmetic mean*: $A(\alpha,\beta) = \frac{\alpha+\beta}{2}$.

2 *Logarithmic mean*: $L(\alpha,\beta) = \frac{\alpha-\beta}{\ln|\alpha|-\ln|\beta|}$, $|\alpha| \neq |\beta|$, $\alpha,\beta \neq 0$.

3 *Generalized log-mean*: $L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, n \in Z\backslash\{-1,0\},\ \alpha,\beta \in \mathbb{R}^+$.

4 *Harmonic mean*: $H = H(a,b) = \frac{2\alpha\beta}{\alpha+\beta}$, $\alpha+\beta \neq 0$.

Now, using some results, we give some applications to special means of real numbers.

Proposition 2. Let $a, b \in \mathbb{R}^+$, $a < b$ and $n \in Z\backslash\{-1\}$. Then we have

$$\left|L_n^n(a,b) - A\left[A(a^n,b^n), A^n(a,b)\right]\right| \leq \frac{(b-a)^2}{48} n(n-1)A(a^{n-2},b^{n-2}).$$

Proof. The assertion follows from Corollary 3 for $m = 1$ applied to the (s,m)-convex function $f(x) = x^n$, $x \in \mathbb{R}$.

Proposition 3. Let $a, b \in \mathbb{R}^+$, $a < b$ then we have

$$\left|L^{-1}(a,b) - A\left[H^{-1}(a,b),\ A^{-1}(a,b)\right]\right| \leq \frac{(b-a)^2}{24} H^{-1}(a^3,b^3).$$

Proof. The assertion follows from Corollary 3 for $m = 1$ applied to the (s,m)-convex function $f(x) = \frac{1}{x}$, $x \in \mathbb{R}^+$.

Proposition 4. Let $a, b \in \mathbb{R}^+$, $a < b$ and $n \in Z\backslash\{-1\}$. Then, we have

$$\left|A(a^n,b^n) - A^n(a,b)\right| \leq \frac{(b-a)^2}{8} n(n-1)A(a^{n-2},b^{n-2}).$$

Proof. The assertion follows from Corollary 5 applied to the (s,m)-convex function $f(x) = x^n$, $x \in \mathbb{R}$.

References

1 Mitrinović, D.C., Pečarić, J.E., & Fink, A.M. (1993). Classical and New Inequalities in Analysis (Chapter I, Convex Functions and Jensen Inequalities). Kluwer Academic Publishers. Dordrecht/Boston/ London, 1–19.

2 Toader, G.H. (1985). Some generalisations of the convexity. *Proc. Colloq. Approx. Optim.* Cluj–Napoca, 329–338.

3 Breckner, W. (1978). Stetigkeitsaussagen für eine Klasse Verallgemeinerter Konvexer Funktionen in yopologischen linearen Rumen (German). *Publ. Inst. Math. (Beograd)* (N.S.), 23, 37, 13–20.

4 Miheşan, V.G. (1993). A generalisation of the convexity. Seminar on functional equations, Approximation and Convex., Cluj-Napoca.

5 Dragomir, S., & Pearse, C.E.M. (2000). Selected Topics on Hermite–Hadamard Inequalities and Applications. RGMIA Monographs, Victoria University, ONLINE. *rgmia.org.* Retrieved from http://rgmia.org/papers/monographs/Master.pdf.

6 Özdemir, M.E., Avci, M., & Kavurmacı, H. (2011). Hermite–Hadamard — type inequalities via (α, m)-convexity. *Computer and Mathematics with Applications, 61,* 2614–2620.

7 Kırmacı, U. (2017). On Some Hermite–Hadamard Type Inequlities for Twice differentiable (α, m)-convex functions and Applications, *RGMIA, 20,* Art 51.

8 Bayraktar, B., & Gürbüz, M. (2018). On some integral inequalities for (s, m)-convex functions, *TWMS J. App. Eng. Math.*, Accepted.

9 Park, J. (2011). Generalization of Ostrowski – type inequalities for differentiable real (s, m)-convex mappings. *Far East J. of Math. Sci., 49, 2,* 157–171.

10 Bayraktar B. (2019). Some Integral Inequalities of Hermite–Hadamard type for differentiable (s, m)-convex functions Via Fractional Integrals. *TWMS J. of App. and eng. mathematics*, Accepted.

11 Wang, S.-H., & Wu, S.H. (2018). Some new Hermite–Hadamard type inequalities for operator m-convex and (α, m)-convex functions on the co–ordinates. *Journal of Computational Analysis and Applications, 25, 3,* 474–487.

12 Özdemir, M.E., Kavurmaci–Onalan, H., & Avci–Ardic, M. (2018). New Inequalities of Ostrowski Type for Mappings whose Derivatives are (α, m)-convex via Fractional Integrals. *Thai Journal of Mathematics, 16, 3,* 723–731.

13 Tingsong, D., Yujiao, L., & Zhiqiao, Y. (2017). A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions. *Applied mathematics and computation, 293,* 358–369.

14 Sun, W., & Liu, Q. (2017). New Hermite–Hadamard type inequalities for (α, m)-convex functions and Applications to special means. *Journal Of Mathematical Inequalities, 11, 2,* 383–397.

15 Noor, M.A., & Noor, K.I. (2017). Integral inequalities via generalized (α, m)-convex functions. *Journal of Nonlinear Functional Analysis, No. 32.*

16 Haiping, R., Guofu, W., & Laijun, L. (2017). Sandor Type Fuzzy Inequality Based on the (s, m)-convex function in the second sense. *Symmetry–Basel, 9, 9,* No. 181.

17 Wang, S.H. (2017). New integral inequalities of Hermite–Hadamard type for operator $m-$convex and (α, m)-convex functions. *Journal of Computational Analysis and Applications, 22, 4,* 744–753.

18 Sun, W., & Liu, Q. (2017). New Hermite-Hadamard type inequalities for (α, m)-convex functions and Applications to special means. *Journal of Mathematical Inequalities, 11, 2,* 383–397.

19 Iscan, I. (2016). Hermite–Hadamard type inequalities for harmonically (α, m)-convex functions. *Hacettepe Journal of Mathematics and Statistics, 45, 2,* 381–390.

20 Shuang, Y. Wang, Y., & Qi, F. (2016). Integral inequalities of Simpson's type for (α, m)-convex functions. *Journal of Nonlinear sciences and Applications, 9, 12,* 6364–6370.

21 Shuang, Y., Qi, F., & Wang, Y. (2016). Some inequalities of Hermite–Hadamard type for functions whose second derivatives are (α, m)-convex. *Journal of Nonlinear Sciences and Applications, 9, 1,* 139–148.

22 Baloch, I.A., Işcan, I., & Dragomir, S.S. (2016). Fejer type inequalities for Harmonically (s, m)-convex functions. *International Journal of Analysis and Applications, 12, 2,* 188–197.

23 Baloch, I.A., Imran, A., & Işcan, I. (2015). Some Ostrowski type Inequalities for Harmonically (s, m)-convex functions in second sense. *International Journal of Analysis,* No. 672675.

24 Zhiqiao, Y., Yujiao, L., & Tingsong, D. (2015). A Generalization of Simpson type Inequality Via Differentiable Functions Using (s, m)-convex functions. *Italian Journal of Pure and Applied Mathematics, 35,* 327–338.

25 Baloch, I.A., & Işcan, I. (2015). Some Ostrowski type Inequalities for Harmonically (s, m)-convex functions in second sense. *International Journal of Analysis*, No. 672675, 9. Retrieved from http://dx.doi.org/10.1155/2015/672675.

26 Akdemir, A.O. (2014). Inequalities of Ostrowski's type for m- and (α, m)-logarithmically convex functions via Riemann-Liouville Fractional integrals. *Journal of Computational Analysis and Applications, 16, 2,* 375–383.

27 Park, J. (2014). Hermite–Hadamard type and Simpson type inequalities for the decreasing (α, m)-geometrically convex functions. *Applied Mathematical Sciences, 61–64,* 3181–3195.

28 Shuang, Y., Wang, Y., & Qiy, F. (2014). Some inequalities of Hermite-Hadamard type for functions whose third derivatives are (α, m)-convex. *Journal of Computational Analysis and Applications, 17, 2,* 272–279.

29 Park, J. (2013). Some new Hermite-Hadamard type inequalities for twice differentiable (α, m)-convex functions. *International Journal of Applied Mathematics and Statistics, 45, 15,* 1–11.

30 XI, Bo-Y., Wang, Y., & Qi, F. (2013). Some Integral Inequalities of Hermite-Hadamard type For (s, m)-convex functions. *Transylvanian Journal of Mathematics and Mechanics (TJMM), 5, 1,* 69–84.

31 Tunc, M. (2013). Hermite-hadamard type inequalities via m- and (α, m)-convexity. *Demonstratio Mathematica, 46, 3,* 475–483.

32 Özdemir, M.E., Set, E., & Sarıkaya, M.Z. (2013). Some New Hadamard type Inequalities For Co-Ordinated m-convex and (α, m)-convex functions. *Hacettepe Journal of Mathematics and Statistics, 40, 2,* 219–229.

33 Park, J. (2012). Generalization of Ostrowski — type inequalities for differentiable real (s, m)-convex mappings. *International Journal of Pure and Applied Mathematics, 78, 8,* 1077–1089.

34 Park, J. (2011). Hermite-Hadamard — type inequalities for differentiable (s, m)-convex mappings in the second sense. *Far East Journal of Mathematical Sciences, 52, 1,* 57–73.

35 Sarıkaya, M.Z., & Aktan, N. (2011). On the generalization of some integral inequalities and their applications. *Math. Comput. Modelling, 54,* 2175–2182.

36 Set, E., Sardari, E., Özdemir, M.E., & Rooin, J. (2009). On Generalizations of the Hadamard Inqulity for (α, m)-convex functions, *RGMIA*, 12, 4 Art. 4.

37 Bakula, M.K., Özdemir, M.E., & Pečarić, J.E. (2008). Hadamard type inequalities for m–convex and(α, m)-convex functions. *J. Inequal. Pure and Applied Mathematics, 9, 4,* Art. 96.

38 Bakula, M.K., Pečarić, J.E., & Ribičić, M. (2006). Companion inequalities to Jensen's inequality for m-convex and (α, m)-convex functions. *J. Inequal. Pure and Applied Mathematics, 7, 5,* Art. 194.

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(S, m)-дөңес және (α, m)-дөңес функциялар үшін кейбір жаңа интегралдық теңсіздіктер

Мақалада (s, m)-дөңес және (α, m)-дөңес функциялар үшін бірнеше жаңа интегралдық теңсіздіктер ұсынылған. Бұл нәтижелер жақсы белгілі Эрмит-Адамар типті интегралдық теңсіздікпен, Симпсон типті интегралдық теңсіздікпен және Йенсен типті теңсіздікпен байланысты. Басқаша айтқанда, дөңес функциялардың көрсетілген кластар арқылы осы теңсіздіктер үшін жоғарыдан жаңа бағалар алынды. Мақалада келтірілген нәтижелер дөңес функциялардың анықтамаларын тікелей пайдалануы мен Гельдер типті және дәрежелік орташа типті классикалық интегралдық теңсіздіктерді қолдану арқылы алынды. Жаңа нәтижелермен бірге авторлар әдебиеттегі интегралдық теңсіздіктерге арналған жоғары шекара бағаларын растайтын нәтижелерге қолжеткізді (дербес жағдайда M.Z. Sarıkaya және N. Aktan [35] және U. Kırmacı [7] алынған әдебиеттердегі жақсы белгілі нәтижелер). Мақаланың соңғы бөлімінде арнайы есептеу құралдары үшін алынған бағаларды кейбір қосымшалары келтірілген, яғни әртүрлі шамалар үшін арифметикалық, логарифмдік, жалпыланған логарифмдік орташа және гармоникалық орташа.

Кілт сөздер: дөңес функция, (s, m)-дөңес, (α, m)-дөңес, Эрмит-Адамар теңсіздігі, Йенсен теңсіздігі, Гёльдер теңсіздігі, орташа дәрежелі үшін теңсіздік.

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Некоторые новые интегральные неравенства для (s, m)-выпуклых и (α, m)-выпуклых функций

В статье представлено несколько новых интегральных неравенств для (s, m)-выпуклых и (α, m)-выпуклых функций. Эти результаты связаны с хорошо известным интегральным неравенством типа Эрмита-Адамара, интегральным неравенством типа Симпсона и с неравенством типа Йенсена. Другими словами, получены новые оценки сверху для этих неравенств с использованием указанных классов выпуклых функций. Представленные результаты получены с помощью непосредственно определения выпуклых функций, а также классических интегральных неравенств типа Гельдера и типа степенного среднего. Наряду с новыми результатами авторами получены результаты, подтверждающие существующие в литературе оценки верхних границ для интегральных неравенств (в частности, хорошо известные в литературе результаты M.Z. Sarıkaya и N. Aktan в [35] и U. Kırmacı в [7]). В последнем разделе статьи приведены некоторые приложения полученных оценок для специальных вычислительных средств, а именно: арифметическое, логарифмическое, обобщенное логарифмическое, среднее и среднее гармоническое для различных величин.

Ключевые слова: выпуклая функция, (s, m)-выпуклая, (α, m)-выпуклая, неравенство Эрмита-Адамара, неравенство Йенсена, неравенство Гёльдера, неравенство для среднестепенного.

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Best trigonometric approximation and modulus of smoothness of functions in weighted grand Lebesgue spaces

In this work, first of all, $L_{\omega}^{p),\theta}(\mathbb{T})$ weighted grand Lebesgue spaces and Muckenhoupt weights is defined. The information about properties of these spaces is given. Let T_n be the trigonometric polynomial of best approximation. The approximation of the functions in grand Lebesgue spaces have been investigated by many authors. In this work the relation between fractional derivatives of a T_n trigonımetric polynomial and the best approximation of the function is investigated in weighted grand Lebesgue spaces. In that regard, the neccessary and sufficient condition is expressed in Theorem 1. In addition, in this work in weighted grand Lebesgue spaces a specific operator is defined. Later on, with the help of this operator the fractional modules of smoothness of order r of function f is defined. Also, in this work, using the properties of modulus of smoothness of function, the relationship between the fractional modulus of smoothness of the function and $n-th$ partial and de la Vallée-Poussin sums of its Fourier series in subspace of weighted grand Lebesgue spaces are studied. These results are expressed in Theorem 2.

Keywords: heneralized grand Lebesgue spaces, fractional derivative, fractional moduli of smoothness, $n-th$ partial sums, de la Vallée-Poussin sums, best approximation by trigonometric polynomials.

Introduction and the main results

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$. We denote by $L^p(\mathbb{T})$, $1 \le p \le \infty$, the Lebesgue space of all measurable functions f on $\mathbb{T}$, that is, the space of all such functions for which

$$\|f\|_p = \left(\int_{\mathbb{T}} |f(x)|^p \, dx \right)^{1/p} < \infty.$$

A function ω is called a *weight* on $\mathbb{T}$ if $\omega : \mathbb{T} \to [0,\infty]$ is measurable and $\omega^{-1}(\{0,\infty\})$ has measure zero (with respect to Lebesgue measure).

Let ω be a 2π periodic weight function. We denote by $L_{\omega}^p(\mathbb{T})$, $1 < p < \infty$, the weighted Lebesgue space of all measurable functions on $\mathbb{T}$ for which the norm

$$\|f\|_p = \left(\int_{\mathbb{T}} |f(x)|^p \, \omega dx \right)^{1/p} < \infty.$$

We define a class $L_{\omega}^{p),\theta}(\mathbb{T})$, $\theta > 0$ of 2π periodic measurable functions on $\mathbb{T}$ satisfying the condition

$$\sup_{0<\varepsilon<p-1} \left\{ \frac{\varepsilon^{\theta}}{2\pi} \int_{\mathbb{T}} |f(x)|^{p-\varepsilon} \, \omega(x) dx \right\}^{1/(p-\varepsilon)} < \infty.$$

The class $L_{\omega}^{p),\theta}(\mathbb{T})$, $\theta > 0$, is a Banach space with respect to the norm

$$\|f\|_{L_{\omega}^{p),\theta}(\mathbb{T})} := \sup_{0<\varepsilon<p-1} \left\{ \varepsilon^{\theta} \frac{1}{2\pi} \int_{\mathbb{T}} |f(x)|^{p-\varepsilon} \, \omega(x) dx \right\}^{1/(p-\varepsilon)}. \tag{1}$$

The class $L_{\omega}^{p),\theta}(\mathbb{T})$ with the norm (1) is called as the weighted generalized grand Lebesgue space. Note that non- weighted grand Lebesgue space $L^{p)}(\mathbb{T})$ was introduced by Iwaniec and Sbordone [1]. Information about properties of these spaces can be found in [2–4]. The embeddings

$$L^{p}(\mathbb{T}) \subset L^{p)}(\mathbb{T}) \subset L^{p-\varepsilon},$$

hold. According to [2] $L^{p}(\mathbb{T})$ is not dense in $L^{p)}(\mathbb{T})$. Also, if $\theta_1 < \theta_2$ and $1 < p < \infty$, for weighted generalized grand Lebesgue space, the following relations hold:

$$L_{\omega}^{p}(\mathbb{T}) \subset L_{\omega}^{p),\theta_1}(\mathbb{T}) \subset L_{\omega}^{p),\theta_2}(\mathbb{T}) \subset L_{\omega}^{p-\varepsilon,}(\mathbb{T}).$$

The closure of the space $L^{p}(\mathbb{T})$ by the norm of $L_{\omega}^{p),\theta}(\mathbb{T})$, $\theta > 0$, we denote by $\widetilde{L}_{\omega}^{p,\theta}(\mathbb{T})$.

Let $1 < p < \infty$ and let $A_p(\mathbb{T})$ be the collection of all weights on $\mathbb{T}$ satisfying the condition

$$\sup_{I} \left(\frac{1}{|I|} \int_I \omega(x)^p dx \right)^{1/p} \left(\frac{1}{|I|} \int_I [\omega(x)]^{-1/(p-1)} dx \right)^{p-1} < \infty, \tag{2}$$

where the supremum is taken over all intervals I with length $|I| \leq 2\pi$. The condition (2) is called the *Muckenhoupt* -A_p condition [5] and the weight functions which belong to $A_p(\mathbb{T})$, $(1 < p < \infty)$, are called as the *Muckenhoupt weights.*

Suppose that $f \in L_{\omega}^{p),\theta}$. We define the operator by

$$A_h f(x) := \frac{1}{h} \int\limits_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t) dt, \quad x \in \mathbb{T} \ \ 0 < h \leq 1.$$

Note that $0 < p < \infty$, $\theta > 0$ and $\omega \in A_p$ then the operator A_h is bounded in $L_{\omega}^{p),\theta}$. For $f \in L_{\omega}^{p),\theta}$, we define

$$\sigma_h^r f(x) := (I - A_h)^r f(x) =$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(r+1)}{\Gamma(k+1) \Gamma(r-k+1)} (A_h)^k, \ x, \ h \in \mathbb{T}, \ 0 \leq r,$$

where I is the identity operator and Γ is gamma function.

Let $\omega \in A_p$ and $f \in L_{\omega}^{p),\theta}$. If $0 \leq r$ we can define the *fractional modules of smoothness of order* r of f as

$$\Omega_r(f,\delta)_{p),\theta,\omega} := \sup_{0 \leq h_i, \ t \leq \delta} \left\| \prod_{i=1}^{[r]} (I - A_{h_i}) \sigma_t^{\{r\}} f \right\|_{p),\theta,\omega}, \ \delta \geq 0,$$

where $[r]$ denotes the integer part of the real number r and $\{r\} := r - [r]$.– Note that $\Omega_0(f,\delta)_{p),\theta,\omega} := \|f\|_{p),\theta,\omega}$ and $\prod\limits_{i=1}^{0} (I - A_{h_i}) \sigma_t^r f := \sigma_t^r$ for $0 < r < 1$. The modulus of smoothness $\Omega_r(f,\delta)_{p)\theta,\omega}$, $r \in \mathbb{R}^+$, is a nondecreasing, nonnegative function of δ, and

$$\begin{aligned} \Omega_r(f+g,\delta)_{p),\theta,\omega} &\leq \Omega_\alpha(f,\delta)_{p),\theta,\omega} + \Omega_\alpha(g,\delta)_{p),\theta,\omega}, \\ \lim_{\delta \to 0} \Omega_r(f,\delta)_{p),\theta,\omega} &= 0 \end{aligned}$$

for $f, g \in L_{\omega}^{p),\theta}$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(f,x), \ A_k(f,x) := a_k(f) \cos kx + b_k(f) \sin kx \tag{3}$$

be the Fourier series of the function $f \in L^1(\mathbf{T})$, where $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function f. The nth *partial sums* and *de la Vallée-Poussin sums* of the series (3) are defined, respectively, as:

$$S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k(f,x),$$

$$V_n(f,x) := \frac{1}{n}\sum_{\nu=n}^{2n-1} S_\nu(f,x).$$

We denote by $E_n(f)_{p(.)}$, $(n = 0,1,2,...)$ the best approximation of $f \in L_{2\pi}^{p(.)}$ by trigomonetric polynomials of degree not exceeding n, i.e.

$$E_n(f)_{p),\theta,\omega} := \inf\left\{\|f - T_n\|_{p),\theta,\omega} : T_n \in \Pi_n\right\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

We use the relation $\alpha_n = O(\beta_n)$, $n = 1,2,...$, that is, there exists a constant $C > 0$ such as $\alpha_n \le C\beta_n$, $n = 1,2,....$

The approximation problems of the functions by trigonometric polynomials in grand Lebesgue spaces have been investigated by several authors [6–13].

In the present paper, in weighted generalized grand Lebesgue spaces we investigate the relation between derivatives of a polynomial of best approximation and the best approximation of the function. In addition, relationship between fractional modulus of smoothness of the function and *nth* partial and de la Vallée-Poussin sums of its Fourier series in subspace of weighted grand Lebesgue spaces are studied. Simillar results in different spaces have been investigated in [14–34], and [5]. Note that, in the proof of the main results we use the method as in the proof of [27, 28].

Our main results are the following:

Theorem 1. Let $T_n(f) \in \Pi_n$ be the polynomial of best approximation to f, let r, $\alpha \in \mathbb{R}^+$. In order that

$$\left\|T_n^{(r)}(f)\right\|_{p),\theta,\omega} = O(n^{r-\alpha}) \quad (r > \alpha > 0)$$

it is necessary and sufficient that

$$E_n(f)_{p),\theta,\omega} = O(n^{-\alpha}).$$

Theorem 2. Let $1 < p < \infty$, $\theta > 0$, $r \in \mathbb{R}^+$ and $\omega \in A_p$. If $f \in \widetilde{L}_\omega^{p),\theta}(\mathbb{T})$, then

1.

$$c_4\Omega_r(f,\frac{1}{n})_{p),\theta,\omega} \le n^{-2r}\left\|V_n^{(2r)}(f,\cdot)\right\|_{p),\theta,\omega} + \|f(x) - V_n(f,\cdot)\|_{p),\theta,\omega} \le$$
$$\le c_5\Omega_r(f,\frac{1}{n})_{p),\theta,\omega}, \tag{4}$$

where the constants c_4 and c_5 are dependent on p and r.

2.

$$c_6\Omega_r(f,\frac{1}{n})_{p),\theta,\omega} \le n^{-2r}\left\|S_n^{(2r)}(f,\cdot)\right\|_{p),\theta,\omega} + \|f(x) - S_n(f,\cdot)\|_{p),\theta,\omega} \le$$
$$\le c_7\Omega_r(f,\frac{1}{n})_{p),\theta,\omega}, \tag{5}$$

where the constants c_6 and c_7 are dependent on p and r.

Proofs of the main results

Proof of Theorem 1. Let us assume that

$$E_n(f)_{p),\theta,\omega} = \|f - T_n(f)\|_{p),\theta,\omega} = O(n^{-\alpha}), \quad (\alpha > 0). \tag{6}$$

is satisfied. We can write

$$T_n^{(r)}(x) = T_0^{(r)}(x) + \sum_{\nu=0}^{n-1} \left\{ T_{\nu+1}^{(r)}(x) - T_\nu^{(r)}(x) \right\}. \tag{7}$$

Using Bernstein inequality for the spaces $\widetilde{L}_\omega^{p),\theta}(\mathbb{T})$ in [35] we have

$$\left\| T_n^{(r)}(f) \right\|_{p),\theta,\omega} \leq c_8 n^r \left\| T_n(f) \right\|_{p),\theta,\omega}.$$

From (6), (7) and the last relation we conclude that

$$\left\| T_n^{(r)}(f) \right\|_{p),\theta,\omega} \leq c_9 c_{10} n^r n^{-\alpha} \leq c_{11} n^{r-\alpha}.$$

Now we suppose that

$$\left\| T_n^{(r)}(f) \right\|_{p),\theta,\omega} = O(n^{r-\alpha}). \tag{8}$$

Use of [9] and (8) leads to

$$\left\| T_{2n}(f) - T_n(T_{2n}(f)) \right\|_{p),\theta,\omega} \leq \left\| f - T_{2n}(f) \right\|_{p),\theta,\omega} + \left\| f - T_n(T_{2n}(f)) \right\|_{p),\theta,\omega} \leq$$
$$\leq c_{12} n^{-r} \left\| T_n^{(r)}(f) \right\|_{p),\theta,\omega} \leq c_{13} n^{-\alpha}. \tag{9}$$

On the other hand, since $T_n(T_{2n}(f))$ is a polynomial of order n the following inequality holds:

$$\left\| T_{2n}(f) - T_n(T_{2n}(f)) \right\|_{p),\theta,\omega} = \left\| f - T_n(T_{2n}(f)) - (f - T_{2n}(f)) \right\|_{p),\theta,\omega} \geq$$
$$\geq \left\| f - T_n(T_{2n}(f)) \right\|_{p),\theta,\omega} - \left\| f - T_{2n}(f) \right\|_{p),\theta,\omega} \geq$$
$$\geq E_n(f)_{p),\theta,\omega} - E_{2n}(f)_{p),\theta,\omega} \geq 0. \tag{10}$$

Use of (9) and (10) gives us

$$0 \leq E_n(f)_{p),\theta,\omega} - E_{2n}(f)_{p),\theta,\omega} \leq c_{14} n^{-\alpha}. \tag{11}$$

Condition $E_n(f)_{p),\theta,\omega} \to 0$ is satisfied. Therefore, from the inequality (11) we have

$$\sum_{k=n_0}^{\infty} \left\{ E_{2^k}(f)_{p),\theta,\omega} - E_{2^{k+1}}(f)_{p),\theta,\omega} \right\} \leq c_{15} \sum_{k=n_0}^{\infty} 2^{-k\alpha}.$$

Thus,

$$E_{2^{n_0}}(f)_{p),\theta,\omega} \leq c_{16} 2^{-n_0 \alpha}. \tag{12}$$

By (12) we conclude that $E_n(f)_{p),\theta,\omega} \leq c_{15}(n^{-\alpha})$.

This completes the proof Theorem 1.

Proof of Theorem 2. By [9] the inequality

$$\Omega_r(T_n, \frac{1}{n})_{p),\theta,\omega} \leq c_{17}(p,r) n^{-2r} \left\| T_n^{(2r)} \right\|_{p),\theta,\omega}, \tag{13}$$

holds, where $T_n \in \Pi_n$. On the other hand, using the properties of modulus of smoothness $\Omega_r(f, \frac{1}{n})_{p),\theta,\omega}$ and (13), we find

$$\Omega_r\left(f, \frac{1}{n}\right)_{p),\theta,\omega} \leq \left(\Omega_r\left(f - T_n, \frac{1}{n}\right)_{p),\theta,\omega} + \Omega_r\left(T_n, \frac{1}{n}\right)_{p),\theta,\omega} \right) \leq$$
$$\leq c_{18}(p,r) \left(\left\| f - T_n \right\|_{p),\theta,\omega} + n^{-2r} \left\| T_n^{(2r)} \right\|_{p),\theta,\omega} \right).$$

Now we estimate the modulus of smoothness $\Omega_r(f,\cdot)_{p),\theta,\omega}$ from below. According to reference [10] the following inequalities hold:

$$E_n(f)_{p),\theta,\omega} \leq c_{19}\,(p,r)\,\Omega_r\left(f,\frac{2\pi}{n+1}\right)_{p),\theta,\omega}; \tag{14}$$

$$n^{-2r}\left\|T_n^{(2r)}\right\|_{p),\theta,\omega} \leq c_{20}\,(p,r)\,\Omega_r\left(f,\frac{2\pi}{n+1}\right)_{p),\theta,\omega}. \tag{15}$$

Let $V_n(f,x)$ be de la Vallée-Poussin sums of the series (3) and let $T_n^* \in \Pi_n$ be the polynomial of best approximation to f in $\widetilde{L}_{\omega}^{p),\theta}(\mathbb{T})$, that is $\|f-T_n^*\|_{p),\theta,\omega} = E_n(f)_{p),\theta,\omega}$. Then we get

$$\begin{aligned}\|f-V_n(f,\cdot)\|_{p),\theta,\omega} &\leq \|f-T_n^*\|_{p),\theta,\omega} + \|T_n^*-V_n(f,\cdot)\|_{p),\theta,\omega} \leq \\ &\leq c_{21}(p)E_n(f)_{p),\theta,\omega} + \|V_n(T_n^*-f,\cdot)\|_{p),\theta,\omega} \leq \\ &\leq c_{22}(p)E_n(f)_{p),\theta,\omega}.\end{aligned} \tag{16}$$

Using (14), (15) and (16) we have

$$\begin{aligned}& n^{-2r}\left\|V_n^{(2r)}(f,\cdot)\right\|_{p),\theta,\omega} + \|f-V_n(f,\cdot)\|_{p),\theta,\omega} \leq \\ & \leq c_{23}\,(p,r)\left(\Omega_r(V_n,\frac{1}{n})_{p),\theta,\omega} + E_n(f)_{p),\theta,\omega}\right) \leq \\ & \leq c_{24}\,(p,r)\left(\Omega_r(f,\frac{1}{n})_{p),\theta,\omega} + \Omega_r(f-V_n,\frac{1}{n})_{p),\theta,\omega} + E_n(f)_{p),\theta,\omega}\right) \leq \\ & \leq c_{25}\,(p,r)\,\Omega_r(f,\frac{1}{n})_{p),\theta,\omega}.\end{aligned}$$

which completes the estimation (4) of Theorem 2.

Let T_n be the best approximation polynomial for f, i.e.,

$$E_n(f)_{p),\theta,\omega} = \|f-T_n\|_{p),\theta,\omega}.$$

By [9], Theorem 5; [10], Theorem 2.1 there exists a constant $c_{25}(p)$ such as

$$\|f-S_n(f,\cdot)\|_{p),\theta,\omega} \leq \|f-T_n\|_{p),\theta,\omega} + \|S_n\,(T_n-f)\|_{p),\theta,\omega} \leq c_{26}(p)E_n(f)_{p),\theta,\omega}. \tag{17}$$

Using inequality (17) and the scheme of proof of the estimation (4) we have the estimate (5).

Proof of Theorem 2 is completed.

References

1 Iwaniec, T., & Sbordone, C. (1992). On the integrability of the Jacobian under minimal hypotheses. *Arch. Ration. Mech. Anal. 119, 2,* 129–143.

2 D'onofrio, L., Sbordone, C., & Schiattarella, R. (2013). Grand Sobolev spaces and their application in geometric function theory and PDEs. *J. Fixed Point Theory Appl. 13,* 309–340.

3 Iwaniec, T., & Sbordone, C. (1998). Riesz transforms and elliptic PDEs with VMO coefficients. *J. Anal. Math. 74,* 183–212.

4 Kokilshvili, V. (2011). Boundedness criteria for singular integralas in weighted grand Lebesgue spaces. *J. Math. Sci., 170*, 20–33.

5 Ky, N.X. (1997). Moduli of mean smoothness and approximation with A_p-weights. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 40,* 37–38.

6 Danelia, N., & Kokilasvili, V. (2012). On the approximation of periodic functions within the frame of grand Lebesgue spaces. *Bull. Geoegian Nat. Acad. Sci. 6, 2,* 11–16.

7 Danelia, N.,V., & Kokilashvili, V. (2013). Approximation by trigonometric polynomials in subspace of weighted grand Lebesgue spaces. *Bull. Georg. Natl. Acad. Sci., 7, 1,* 11–15.

8 Danelia, N.,V., & Kokilashvili, V. (2014). Approximation of periodic functions in grand variable exponent Lebesgue spaces. *Proc. A. Razmadze Math. Inst. 164,* 100–103.

9 Danelia, N., Kokilashvili V., & Tsanava, Ts. (2014). Some approximation results in subspace of weighted grand Lebesgue spaces. *Proc. A. Razmadze Math. Inst. 164,* 104–108.

10 Danelia, N., & Kokilashvili, V. (2016). Approximation by trigonometric polynomials in the framework of variable exponent grand Lebesgue spaces. *Georgian Math. J. 23, 1,* 43–53.

11 Israfilov, D.M., & Testici, A. (2016). Approximation in weighted generalized grand Lebesgue spaces. *Colloquim Mathematicum, 143, 1,* 113–126.

12 Israfilov, D.M., & Testici, A. (2017). Approximation in weighted generalized grand Smirnov classes*, Studia Scientiarum Mathematicarum Hungarica, 54 (4),* 471–488.

13 Jafarov, S.Z. (2016). Approximation by trigonometric polynomials in subspace of variable exponent grand Lebesgue spaces. *Global J. Math. 8, 2,* 836–843.

14 Akgün, R. (2010). Sharp Jackson and converse theorems of trigonometric approximation in weighted Lebesgue spaces. *Proc. A. Razmadze Math. Inst. 152,* 1–18.

15 Akgün, R. (2011). Polynomial approximation in weighted Lebesgue spaces. *East. J. Approx. 17, 3,* 253-266.

16 Akgün, A., & Kokilashvili, V. (2011). The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces. *Georgian Math. J. 18, 3,* 399-423.

17 Akgün, R. (2011). Approximation in weighted Orlicz spaces. *Math. Slovaca 61, 4,* 601–618.

18 Akgün, R. (2015). Realization and characterization of modulus of smoothness in weighted Lebesgue spaces. *St. Ptersburg Math. J. 26, 5,* 741–756.

19 Butzer, P.L., & Pawelke, S. (1967). Ableitungen von trigonometrischen Approximations prozessen. *Acta Sci. Math. (Szeged), Vol. 28,* 173–183.

20 Devore, R.A., & Lorentz, G.G. (1993). *Constructive approximation.* Berlin, Springer-Verlag Berlin Heidelberg.

21 Guven, A., & Israfilov, D.M. (2009). Approximation by Means of Fourier trigonometric series in weighted Orlicz spaces. *Adv. Stud. Contemp. Math. (Kyundshang), 19, 2,* 283–295.

22 Guven, A., & Israfilov, D.M. (2010). Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$. *J. Math. Inequal. 4,* 285–290.

23 Israfilov, D.M., & Ali Guven, A. (2006). Approximation by trigonometric polynomials in weighted Orlicz spaces. *Studia Mathematica, 174 (2)*, 147–168.

24 Israfilov, D.M., Kokilaşvili V., & Samko, S.G. (2007). Approximation in weighted Lebesgue spaces and Smirnov spaces with variable exponent. *Proc. A. Razmadze Math. Inst. 143,* 25–35.

25 Jafarov, S.Z. (2015). Linear methods of summing Fourier series and approximation in weighted variable exponent Lebesgue spaces. *Ukr. Math. J, 66, 10,* 1509–1518.

26 Kokilashvili, V.M., & Samko, S.G. (2009). A refined inverse inequality of approximation in weighted variable exponent Lebesgue spaces. *Proc. A. Razmadze Math. Inst. 151,* 134–138.

27 Sunouchı, G. (1968). Derivatives of a polynomial of best approximation. *Jahresbericht d. DMV, 70,* 165–166.

28 Simonov, B.V., & Tikhonov, S.Yu. (2008). Embedding theorems in the constructive theory of approximations. *Sbornik: Mathematics, 199 (9),* 1367–1407.

29 Stechkin, S.B. (1951). On the order of the best approximations of continuous functions. *Izv. Akad. Nauk SSSR Ser. Mat., 15, 3,* 219–242.

30 Sharapudinov, I.I. (2013). Approximation of functions in $L_{2\pi}^{p(x)}$ by trigonometric polynomials. *Izvestiya: Mathematics, 77 (2),* 407–434.

31 Sharapudinov, I.I. (2014). Approximation of functions in variable exponent Lebesgue and Sobolev spaces. *Sbornik: Mathematics, 205 (1–2)*, 291–30.

32 Timan, A.F. (1963). *Theory of approximation of functions of a real variable.* New York: Pergamon Press and MacMillan.

33 Tsanava, T., & Kokilashvili, V. (2012) Some notes on the majorants of Fourier partial sums in new function spaces. *J. Tech. Sci. Technologiers 1,* 29–31.

34 Zygmund, A. (1959). *Trigonometric Series, Parts I and II.* Cambridge: Cambirdge University Press.

35 Kokilashvili, V.M., & Meskhi, A. (2015). On weighted Bernstein type inequality in grand variable exponent Lebesgue spaces. *Math. Ineq. and Appl. 18, 3,* 991–1002.

С.З. Джафаров

Ең жақын тригонометриялық жуықтау және Лебегтің салмақтық гранд-кеңістіктеріндегі функцияның тегістігінің модулі

Мақалада бірінші кезекте Лебегтің салмақтық гранд-кеңістіктері және Макенхаупт $L_{\omega}^{p),\theta}(\mathbb{T})$ салмақтары анықталды. Осы кеңістіктердің қасиеттері жайлы ақпарат берілді. T_n ең жақын жуықтаудың тригонометриялық полиномы болсын. Лебегтің гранд-кеңістіктеріндегі функцияны аппроксимациялау көптеген авторлармен зерттелді. Бұл жұмыста тригонометриялық полиномдағы T_n бөлшек туынды мен Лебегтің салмақтық гранд-кеңістіктеріндегі ең жақын жуықтау арасындағы байланыс қарастырылды. Осыған байланысты қажетті және жеткілікті шарттар 1-теоремада келтірілді. Автор Лебегтің салмақтық гранд-кеңістіктерінде нақты операторды анықтаған. Кейінірек бұл оператордың көмегімен f функциясының r ретті тегістігінің бөлшекті модульдері анықталды. Сонымен қоса бұл жұмыста функцияның тегістігінің модулінің қасиеттерін қолдана отырып, тегістіктің бөлшекті модулі мен Лебегтің салмақтық гранд-кеңістігінің ішкі кеңістігіндегі Фурье қатарының де Валле-Пуссеннің $n-th$ дербес қосындылары арасындағы өзара байланыс қарастырылды. Бұл нәтижелер 2-теоремада келтірілген.

Кілт сөздер: Лебегтің жалпыланған гранд-кеңістіктері, бөлшекті туынды, тегістіктің бөлшекті модульдері, $n-th$ дербес қосындылары, де Валле-Пуссен қосындылары, тригонометриялық полиномдармен ең жақын жуықтау.

С.З. Джафаров

Наилучшее тригонометрическое приближение и модуль гладкости функций в весовых гранд-пространствах Лебега

В статье, в первую очередь, определены весовые гранд-пространства Лебега и веса Макенхаупта $L_{\omega}^{p),\theta}(\mathbb{T})$. Дана информация о свойствах этих пространств. Пусть T_n будет тригонометрическим полиномом наилучшего приближения. Аппроксимация функций в гранд-пространствах Лебега исследовалась многими авторами. В этой работе изучена связь между дробными производными T_n тригонометрического полинома и наилучшим приближением функции в весовых гранд-пространствах Лебега. В связи с этим необходимое и достаточное условие выражено в теореме 1. Кроме того, в этой работе в весовых гранд-пространствах Лебега определен конкретный оператор. Позже с помощью этого оператора будут определены дробные модули гладкости порядка r функции f. Также, используя свойства модуля гладкости функции, автором изучена взаимосвязь между дробным модулем гладкости и $n-th$ частичными суммами де Валле-Пуссена ряда Фурье в подпространстве весового гранд-пространства Лебега. Эти результаты выражены в теореме 2.

Ключевые слова: обобщенные гранд-пространства Лебега, дробная производная, дробные модули гладкости, $n-th$ частичные суммы, суммы де Валле-Пуссена, наилучшее приближение тригонометрическими полиномами.

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Integro-differentiated singularly perturbed equations with fast oscillating coefficients

In the study of various issues related to dynamic stability, with the properties of media with a periodic structure, in the study of other applied problems, one has to deal with differential equations with rapidly oscillating coefficients. Asymptotic integration of differential systems of equations with such coefficients was carried out by the splitting method and the regularization method. In this paper, a system of integro-differential equations is considered. The main objective of the study is to identify the influence of the integral term on the asymptotics of the solution to the original problem. The case of the absence of resonance is considered, i.e. the case when the integer linear combination of frequencies of the rapidly oscillating coefficient does not coincide with the frequency of the spectrum of the limit operator.

Keywords: singularly perturbation, integro-differential equation, rapidly oscillating coefficient, regularization, asymptotic convergence.

Introduction

Consider the following integro-differential system:

$$\varepsilon\frac{dz}{dt} - A(t)z - \varepsilon g(t)\cos\frac{\beta(t)}{\varepsilon}B\left(t\right)z - \int_{t_0}^{t} K\left(t,s\right)z(s,\varepsilon)ds = h(t), \quad z(t_0,\varepsilon) = z^0, \quad t\in[t_0,T], \qquad (1)$$

where $z = \{z_1, z_2\}$, $h\left(t\right) = \{h_1\left(t\right), h_2\left(t\right)\}$, $\beta'\left(t\right) > 0$, $\omega\left(t\right) > 0\,(\forall t\in[t_0,T])$, $g\left(t\right)$ is a scalar function, $A\left(t\right)$ and $B\left(t\right)$ are $(2\times 2)-$ matrices, with $A\left(t\right) = \begin{pmatrix} 0 & 1 \\ -\omega^2\left(t\right) & 0 \end{pmatrix}$, $z^0 = \left\{z_1^0, z_2^0\right\}$, $\varepsilon > 0$ is a small parameter. Such a system in the case $\beta\left(t\right) = 2\gamma\left(t\right), B\left(t\right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ of the absence of an integral term was considered in [1–6]. In the present work, ideas of the regularization method [3–6] are generated on singularly perturbed systems of integro-differential equations with rapidly oscillating coefficients. The study of singularly perturbed integro-differential problems by the regularization method of S.A. Lomov [3, 4] with unstable values of the kernel of an integral operator is reflected in [7–12]. It should also be noted that it is the merit of V.F. Safonov and A.A. Bobodzhanov in the development of the theory of singularly perturbed integro-differential equations [13–16]. In their studies, various problems for integro-differential systems were considered: with diagonal kernel degenerations, with inverse time, with rapidly changing kernels, with rapidly varying kernels, with partial derivatives, etc. [17–21].

In the system the limiting operator $A\left(t\right)$ has a spectrum $\lambda_1\left(t\right) = -i\omega\left(t\right)$, $\lambda_2\left(t\right) = +i\omega\left(t\right)$, $\beta'\left(t\right)$ is a frequency of rapidly oscillating cosine. In the following, functions $\lambda_3\left(t\right) = -i\beta'\left(t\right)$, $\lambda_4\left(t\right) = +i\beta'\left(t\right)$ will be called *the spectrum of a rapidly oscillating coefficient.*

We assume that the following conditions are fulfilled:

1) $\omega\left(t\right), \beta\left(t\right), g\left(t\right) \in C^{\infty}\left(\left[t_0,T\right], \mathbb{C}^1\right), h\left(t\right)\in C^{\infty}\left(\left[t_0,T\right], \mathbb{C}^2\right)$,
$B\left(t\right)\in C^{\infty}\left(\left[t_0,T\right], \mathbb{C}^{2\times 2}\right), K\left(t,s\right)\in C^{\infty}\left(\left[t_0,T\right], \mathbb{C}^{2\times 2}\right)$,

2) for $\forall t\in[t_0,T]$ and $n_3\neq n_4$ inequalities

$$n_3\lambda_3\left(t\right) + n_4\lambda_4\left(t\right) \neq \lambda_j\left(t\right),$$
$$\lambda_k\left(t\right) + n_3\lambda_3\left(t\right) + n_4\lambda_4\left(t\right) \neq \lambda_j\left(t\right),\ k\neq j, k, j = 1,2,$$

for all multi-indices $n = (n_3, n_4)$ with $|n| \equiv n_3 + n_4 \geq 1$ (n_3 and n_4 are non-negative integers) are holds.

We will develop an algorithm for constructing a regularized [3] asymptotic solution of problem (1). Condition 2) is called the *absence of resonance condition.*

1. Regularization of problem (1)

Denote by $\sigma_j = \sigma_j(\varepsilon)$, independent of t magnitudes $\sigma_1 = e^{-\frac{i}{\varepsilon}\beta(t_0)}$, $\sigma_1 = e^{+\frac{i}{\varepsilon}\beta(t_0)}$, and rewrite system (1) as

$$\varepsilon\frac{dz}{dt} - A(t)z - \varepsilon\frac{g(t)}{2}\left(e^{-\frac{i}{\varepsilon}\int_{t_0}^{t}\beta'(\theta)d\theta}\sigma_1 + e^{+\frac{i}{\varepsilon}\int_{t_0}^{t}\beta'(\theta)d\theta}\sigma_2\right)B(t)z -$$

$$-\int_{t_0}^{t}K(t,s)\,z(s,\varepsilon)ds = h(t),\ z(t_0,\varepsilon) = z^0,\ t\in[t_0,T]. \tag{2}$$

We introduce regularizing variables [3, 4]

$$\tau_j = \frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_j(\theta)\,d\theta \equiv \frac{\psi_j(t)}{\varepsilon}, j = \overline{1,4} \tag{3}$$

and instead of problem (2), consider the problem

$$\varepsilon\frac{\partial\tilde z}{\partial t} + \sum_{j=1}^{4}\lambda_j(t)\frac{\partial\tilde z}{\partial\tau_j} - A(t)\tilde z - \varepsilon\frac{g(t)}{2}\left(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2\right)B(t)\tilde z -$$

$$-\int_{t_0}^{t}K(t,s)\,\tilde z(s,\tfrac{\psi(s)}{\varepsilon},\varepsilon)ds = h(t), \qquad \tilde z(t,\tau,\varepsilon)|_{t=t_0,\tau=0} = z^0,\ t\in[t_0,T], \tag{4}$$

for the function $\tilde z = \tilde z(t,\tau,\varepsilon)$, where is indicated (by (3)): $\tau = (\tau_1,\tau_2,\tau_3,\tau_4)$, $\psi = (\psi_1,\psi_2,\psi_3,\psi_4)$. It is clear that if $\tilde z = \tilde z(t,\tau,\varepsilon)$ is a solution to problem (4), then the vector function $z = \tilde z\left(t,\frac{\psi(t)}{\varepsilon},\varepsilon\right)$ is an exact solution to problem (2), therefore, problem (4) is extended with respect to problem (2). However, it cannot be considered fully regularized, since it does not regularize the integral term $J\tilde z = \int_{t_0}^{t}K(t,s)\,\tilde z(s,\frac{\psi(s)}{\varepsilon},\varepsilon)ds$. To regularize the integral operator, we introduce a class M_ε that is asymptotically invariant with respect to the operator $J\tilde z$ [3; 62]. Recall the corresponding concept.

Definition 1. A class M_ε is said to be asymptotically invariant (with $\varepsilon\to+0$) with respect to an operator P_0 if the following conditions are fulfilled:

1) $M_\varepsilon \subset D(P_0)$ with each fixed $\varepsilon > 0$;

2) the image $P_0 g(t,\varepsilon)$ of any element $g(t,\varepsilon)\in M_\varepsilon$ decomposes in a power series

$$P_0 g(t,\varepsilon) = \sum_{n=0}^{\infty}\varepsilon^n g_n(t,\varepsilon)(\varepsilon\to+0, g_n(t,\varepsilon)\in M_\varepsilon, n=0,1,...),$$

convergent asymptotically for $\varepsilon\to+0$ (uniformly with $t\in[t_0,T]$).

From this definition it can be seen that the class M_ε depends on the space U, in which the operator P_0 is defined. In our case $P_0 = J$. For the space U we take the space of vector functions $z(t,\tau)$, represented by sums

$$z(t,\tau,\sigma) = z_0(t,\sigma) + \sum_{i=1}^{4}z_i(t,\sigma)e^{\tau_i} + \sum_{2\le|m|\le N_z}^{*}z^m(t,\sigma)e^{(m,\tau)} + \sum_{j=1}^{2}\sum_{1\le|m|\le N_z}^{*}z^{e_j+m}(t,\sigma)e^{(e_j+m,\tau)},$$

$$\begin{gathered} m = (0,0,m_3,m_4), z_i(t,\sigma),\ z^m(t,\sigma),\ z^{e_j+m}(t,\sigma)\in C^\infty\left([t_0,T],\mathbb{C}^2\right),\\ 1\le|m|\equiv m_3+m_4\le N_z,\ i=\overline{1,4}, j=1,2,\end{gathered} \tag{5}$$

where is denoted: $\lambda(t)\equiv(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$, $(m,\lambda(t))\equiv m_3\lambda_3(t)+m_4\lambda_4(t)$, $(e_j+m,\lambda(t))\equiv\lambda_j(t)+m_3\lambda_3(t)+$ $+m_4\lambda_4(t)$; an asterisk $*$ above the sum sign indicates that the summation for $|m|\ge1$ it occurs only over multi-indices $m=(0,0,m_3,m_4)$ with $m_3\ne m_4$, $e_1=(1,0,0,0)$, $e_2=(0,1,0,0)$, $\sigma=(\sigma_1,\sigma_2)$.

Note that here the degree N_z of the polynomial $z(t,\tau,\sigma)$ relative to the exponentials e^{τ_j} depends on the element z. In addition, the elements of space U depend on bounded in $\varepsilon>0$ terms of constants $\sigma_1=\sigma_1(\varepsilon)$ and $\sigma_2=\sigma_2(\varepsilon)$, and which do not affect the development of the algorithm described below, therefore, in the record of element (5) of this space U, we omit the dependence on $\sigma=(\sigma_1,\sigma_2)$ for brevity. We show that the class $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant with respect to the operator J. The image of the operator on the element (5) of the space U has the form

$$Jz(t,\tau) = \int_{t_0}^{t} K(t,s)\, z_0(s)\, ds + \sum_{i=1}^{4} \int_{t_0}^{t} K(t,s)\, z_i(s)\, e^{\frac{1}{\varepsilon}\int_{t_0}^{s} \lambda_i(\theta) d\theta} ds +$$
$$+ \sum_{2 \le |m| \le N_z}^{*} \int_{t_0}^{t} K(t,s)\, z^m(s)\, e^{\frac{1}{\varepsilon}\int_{t_0}^{s} (m,\lambda(\theta)) d\theta} ds +$$
$$+ \sum_{j=1}^{2} \sum_{1 \le |m| \le N_z}^{*} \int_{t_0}^{t} K(t,s)\, z^{e_j+m}(s)\, e^{\frac{1}{\varepsilon}\int_{t_0}^{s} (e_j+m,\lambda(\theta)) d\theta} ds.$$

Integrating in parts, we will have

$$J_i(t,\varepsilon) = \int_{t_0}^{t} K(t,s)\, z_i(s)\, e^{\frac{1}{\varepsilon}\int_{t_0}^{s} \lambda_i(\theta) d\theta} ds = \varepsilon \int_{t_0}^{t} \frac{K(t,s) z_i(s)}{\lambda_i(s)} d e^{\frac{1}{\varepsilon}\int_{t_0}^{s} \lambda_i(\theta) d\theta} =$$

$$= \varepsilon \frac{K(t,s) z_i(s)}{\lambda_i(s)} e^{\frac{1}{\varepsilon}\int_{t_0}^{s} \lambda_i(\theta) d\theta} |_{s=t_0}^{s=t} - \varepsilon \int_{t_0}^{t} \left(\frac{\partial}{\partial s} \frac{K(t,s) z_i(s)}{\lambda_i(s)} \right) e^{\frac{1}{\varepsilon}\int_{t_0}^{s} \lambda_i(\theta) d\theta} ds =$$

$$= \varepsilon \left[\frac{K(t,t) z_i(t)}{\lambda_i(t)} e^{\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_i(\theta) d\theta} - \frac{K(t,t_0) z_i(t_0)}{\lambda_i(t_0)} \right] - \varepsilon \int_{t_0}^{t} \left(\frac{\partial}{\partial s} \frac{K(t,s) z_i(s)}{\lambda_i(s)} \right) e^{\frac{1}{\varepsilon}\int_{t_0}^{s} \lambda_i(\theta) d\theta} ds.$$

Continuing this process further, we obtain the decomposition

$$J_i(t,\varepsilon) = \sum_{\nu=0}^{\infty} (-1)^{\nu} \varepsilon^{\nu+1} [(I_i^{\nu}(K(t,s)\, z_i(s)))_{s=t}\, e^{\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_i(\theta)) d\theta} - (I_i^{\nu}(K(t,s)\, z_i(s)))_{s=t_0}],$$
$$I_i^0 = \frac{1}{\lambda_i(s)} \cdot, I_i^{\nu} = \frac{1}{\lambda_i(s)} I_i^{\nu-1} \quad (\nu \ge 1,\, i = \overline{1,4}).$$

Applying the integration operation in parts to integrals

$$J_m(t,\varepsilon) = \int_{t_0}^{t} K(t,s)\,(s)\, e^{\frac{1}{\varepsilon}\int_{t_0}^{s} (m,\lambda(\theta)) d\theta} ds,\;\; J_{e_j+m}(t,\varepsilon) = \int_{t_0}^{t} K(t,s)\, z^{e_j+m}(s)\, e^{\frac{1}{\varepsilon}\int_{t_0}^{s} (e_j+m,\lambda(\theta)) d\theta} ds,$$

we note that for all multi-indices $m = (0, 0, m_3, m_4)\,, m_3 \ne m_4$, inequalities

$$(m, \lambda(t)) \equiv m_3 \lambda_3(t) + m_4 \lambda_4(t) \ne 0 \forall t \in [t_0, T]\,, m_3 + m_4 \ge 2$$

are satisfied. In addition, for the same multi-indices we have

$$(e_j + m, \lambda(t)) \ne 0 \forall t \in [t_0, T]\,, j = 1, 2, m_3 \ne m_4, |m| = m_3 + m_4 \ge 1.$$

Indeed, if $(e_1 + m, \lambda(t)) = 0$ for some $t \in [t_0, T]$ and $m_3 \ne m_4, m_3 + m_4 \ge 1$, then $m_3 \lambda_3(t) + m_4 \lambda_4(t) = -\lambda_1(t) = \lambda_2(t)\,, m_3 + m_4 \ge 1$, which contradicts condition 2). And likewise, if $(e_2 + m, \lambda(t)) = 0$ with some $t \in [t_0, T]$ and $m_3 \ne m_4, m_3 + m_4 \ge 1$, then $m_3 \lambda_3(t) + m_4 \lambda_4(t) = -\lambda_1(t) = \lambda_2(t), m_3 + m_4 \ge 1$, which also contradicts condition 2). Therefore, integration by parts in integrals $J_m(t,\varepsilon)\,, J_{e_j+m}(t,\varepsilon)$ is possible. Performing it, we will have:

$$J_m(t,\varepsilon) = \int_{t_0}^{t} K(t,s)\, z^m(s)\, e^{\frac{1}{\varepsilon}\int_{t_0}^{s} (m,\lambda(\theta)) d\theta} ds = \varepsilon \int_{t_0}^{t} \frac{K(t,s) z^m(s)}{(m,\lambda(s))} d e^{\frac{1}{\varepsilon}\int_{t_0}^{s} (m,\lambda(\theta)) d\theta} =$$

$$= \varepsilon \left[\frac{K(t,t) z^m(t)}{(m,\lambda(t))} e^{\frac{1}{\varepsilon}\int_{t_0}^{t} (m,\lambda(\theta)) d\theta} - \frac{K(t,t_0) z^m(t_0)}{(m,\lambda(t_0))} \right] - \varepsilon \int_{t_0}^{t} \frac{\partial}{\partial s} \frac{K(t,s) z^m(s)}{(m,\lambda(s))} e^{\frac{1}{\varepsilon}\int_{t_0}^{s} (m,\lambda(\theta)) d\theta} ds =$$

$$= \sum\nolimits_{\nu=0}^{\infty} (-1)^{\nu} \varepsilon^{\nu+1} [(I_m^{\nu}(K(t,s)\, z^m(s)))_{s=t}\, e^{\frac{1}{\varepsilon}\int_{t_0}^{t} (m,\lambda(\theta)) d\theta} - (I_m^{\nu}(K(t,s)\, z^m(s)))_{s=t_0}];$$

$$I_m^0 = \frac{1}{(m,\lambda(s))} \cdot, I_m^{\nu} = \frac{1}{(m,\lambda(s))} \frac{\partial}{\partial s} I_m^{\nu-1} (\nu \ge 1, |m| \ge 2);$$

$$J_{e_j+m}(t,\varepsilon) = \int_{t_0}^{t} K(t,s)\, z^{e_j+m}(s)\, e^{\frac{1}{\varepsilon}\int_{t_0}^{s} (e_j+m,\lambda(\theta)) d\theta} ds =$$

$$= \varepsilon \int_{t_0}^{t} \frac{K(t,s)\, z^{e_j+m}(s)}{(e_j + m, \lambda(s))} d e^{\frac{1}{\varepsilon}\int_{t_0}^{s} (e_j+m,\lambda(\theta)) d\theta} =$$

$$= -\varepsilon\left[\frac{K(t,t)\,z^{e_j+m}(t)}{(e_j+m,\lambda(t))}e^{\frac{1}{\varepsilon}\int_{t_0}^{s}(e_j+m,\lambda(\theta))d\theta}-\frac{K(t,t_0)\,z^{e_j+m}(t_0)}{(e_j+m,\lambda(t_0))}\right]-$$

$$-\varepsilon\int_{t_0}^{t}\frac{\partial}{\partial s}\frac{K(t,s)\,z^{e_j+m}(s)}{(e_j+m,\lambda(s))}e^{\frac{1}{\varepsilon}\int_{t_0}^{s}(e_j+m,\lambda(\theta))d\theta}ds=$$

$$=\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}[\left(I_{j,m}^{\nu}\left(K(t,s)\,z^{e_j+m}(s)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(e_j+m,\lambda(\theta))d\theta}-$$

$$-\left(I_{j,m}^{\nu}\left(K(t,s)\,z^{e_j+m}(s)\right)\right)_{s=t_0}],$$

$$I_{j,m}^{0}=\frac{1}{(e_j+m,\lambda(s))},\quad I_{j,m}^{\nu}=\frac{1}{(e_j+m,\lambda(s))}\frac{\partial}{\partial s}I_{j,m}^{\nu-1}(\nu\geq 1,|m|\geq 1,j=1,2),$$

Therefore, the image of the operator J on the element (5) of the space U is represented as a series

$$Jz(t,\tau)=\int_{t_0}^{t}K(t,s)\,z_0(s)\,ds+\sum_{i=1}^{4}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}[(I_i^{\nu}(K(t,s)\,z_i(s)))_{s=t}\,e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_i(\theta))d\theta}-$$

$$-(I_i^{\nu}(K(t,s)\,z_i(s)))_{s=t_0}]+\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}[(I_m^{\nu}(K(t,s)\,z^m(s)))_{s=t}\,e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(m,\lambda(\theta))d\theta}-$$

$$-(I_m^{\nu}(K(t,s)\,z^m(s)))_{s=t_0}]+\sum_{j=1}^{2}\sum_{1\leq|m|\leq N_z}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}[\left(I_{j,m}^{\nu}(K(t,s)\,z^{e_j+m}(s))\right)_{s=t}\times$$

$$\times e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(e_j+m,\lambda(\theta))}-\left(I_{j,m}^{\nu}(K(t,s)\,z^{e_j+m}(s))\right)_{s=t_0}]_{\tau=\psi(t)/\varepsilon}.$$

It is easy to show [22; 291–294] that this series converges asymptotically for $\varepsilon\to+0$ (uniformly in $t\in[t_0,T]$). This means that the class M_ε is asymptotically invariant (for $\varepsilon\to+0$) with respect to the operator J.

We introduce operators $R_\nu : U\to U$, acting on each element $z(t,\tau)\in U$ of the form (5) according to the law:

$$R_0z(t,\tau)=\int_{t_0}^{t}K(t,s)\,z_0(s)\,ds, \tag{6_0}$$

$$R_1z(t,\tau)=\sum_{i=1}^{4}[\left(I_i^{0}(K(t,s)\,z_i(s))\right)_{s=t}e^{\tau_i}-\left(I_i^{0}(K(t,s)\,z_i(s))\right)_{s=t_0}]+$$

$$+\sum_{1\leq|m|\leq N_z}^{*}\left(I_m^{0}(K(t,s)\,z^m(s))\right)_{s=t}e^{(m,\tau)}-\left(I_m^{0}(K(t,s)\,z^m(s))\right)_{s=t_0}+$$

$$+\sum_{j=1}^{2}\sum_{1\leq|m|\leq N_z}^{*}[\left(I_{j,m}^{0}(K(t,s)\,z^{e_j+m}(s))\right)_{s=t}e^{(e_j+m,\tau)}-\left(I_{j,m}^{0}(K(t,s)\,z^{e_j+m}(s))\right)_{s=t_0}]; \tag{6_1}$$

$$R_{\nu+1}z(t,\tau)=\sum_{i=1}^{4}(-1)^{\nu}\varepsilon^{\nu+1}[(I_i^{\nu}(K(t,s)\,z_i(s)))_{s=t}\,e^{\tau_i}-(I_i^{\nu}(K(t,s)\,z_i(s)))_{s=t_0}]+$$
$$+\sum_{\nu=0}^{*}(-1)^{\nu}\varepsilon^{\nu+1}[(I_m^{\nu}(K(t,s)\,z^m(s)))_{s=t}\,e^{(m,\tau)}-(I_m^{\nu}(K(t,s)\,z^m(s)))_{s=t_0}]+$$
$$+\sum_{j=1}^{2}\sum_{1\leq|m|\leq N_z}^{*}(-1)^{\nu}[\left(I_{j,m}^{\nu}(K(t,s)\,z^{e_j+m}(s))\right)_{s=t}e^{(e_j+m,\tau)}-\left(I_{j,m}^{\nu}(K(t,s)\,z^{e_j+m}(s))\right)_{s=t_0}],\nu\geq 1. \tag{$6_{\nu+1}$}$$

Now let $\tilde{z}(t,\tau,\varepsilon)$ be an arbitrary continuous function on $(t,\tau)\in[t_0,T]\times\left\{\tau : Re\tau_j\leq 0, j=\overline{1,4}\right\}$ with asymptotic expansion

$$\tilde{z}(t,\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k z_k(t,\tau)\,,\ z_k(t,\tau)\in U, \tag{7}$$

converging as $\varepsilon \to +0$ (uniformly in $(t,\tau) \in [t_0, T] \times \{\tau : Re\tau_j \le 0, j = \overline{1,4}\}$). Then the image $J\tilde{z}(t,\tau,\varepsilon)$ of this function is decomposed into an asymptotic series

$$J\tilde{z}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k Jz_k(t,\tau) = \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} z_s(t,\tau)\,|_{\tau=\psi(t)/\varepsilon}.$$

This equality is the basis for introducing an extension of an operator J on series of the form (7):

$$\tilde{J}\tilde{z}(t,\tau,\varepsilon) \equiv \tilde{J}\left(\sum_{k=0}^{\infty} \varepsilon^k z_k(t,\tau)\right) \triangleq \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} z_s(t,\tau). \tag{8}$$

Although the operator (8) is formally defined, its utility is obvious, since in practice it is usual to construct the N–th approximation of the asymptotic solution of the problem (2), in which impose only N-th partial sums of the series (7), which have not a formal, but a true meaning. Now you can write a problem that is completely regularized with respect to the original problem (2):

$$\varepsilon\frac{\partial\tilde{z}}{\partial t} + \sum_{j=1}^{4} \lambda_j(t)\frac{\partial\tilde{z}}{\partial\tau_j} - A(t)\tilde{z} - \varepsilon\frac{g(t)}{2}\left(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2\right)B\tilde{z} - \tilde{J}\tilde{z} = h(t),$$

$$\tilde{z}(t,\tau,\varepsilon)|_{t=t_0,\tau=0} = z^0,\ \ t \in [t_0, T]. \tag{9}$$

2. Iterative problems and their solvability in space solution of the first iterative problem

Substituting the series (7) into (9) and equating the coefficients with the same degrees, we obtain the following iterative problems:

$$Lz_0(t,\tau) \equiv \sum_{j=1}^{4} \lambda_j(t)\frac{\partial z_0}{\partial\tau_j} - A(t)z_0 - R_0 z_0 = h(t),\ z_0(t_0,0) = z^0; \tag{10_0}$$

$$Lz_1(t,\tau) = -\frac{\partial z_0}{\partial t} + \frac{g(t)}{2}\left(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2\right)B(t)z_0 + R_1 z_0,\ z_1(t_0,0) = 0; \tag{10_1}$$

$$Lz_2(t,\tau) = -\frac{\partial z_1}{\partial t} + \frac{g(t)}{2}\left(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2\right)B(t)z_1 + R_1 z_1 + R_2 z_0,\ z_0(t_0,0) = 0; \tag{10_2}$$

$$\cdots$$

$$Lz_k(t,\tau) = -\frac{\partial z_{k-1}}{\partial t} + \frac{g(t)}{2}\left(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2\right)B(t)z_{k-1} + R_k z_0 + ... + R_1 z_{k-1},\ \ z_k(t_0,0) = 0, k \ge 1. \tag{10_k}$$

Each of the iterative problems (10_k) can be written as

$$Lz(t,\tau) \equiv \sum_{j=1}^{4} \lambda_j(t)\frac{\partial z}{\partial\tau_j} - A(t)z - R_0 z = H(t,\tau),\ z(t_0,0) = z^*, \tag{10}$$

where

$$H(t,\tau) = H_0(t) + \sum_{i=1}^{4} H_i(t)e^{\tau_i} + \sum_{2\le|m|\le N_z}^{*} H^m(t)e^{(m,\tau)} + \sum_{j=1}^{2}\sum_{1\le|m|\le N_H}^{*} H^{e_j+m}(t)e^{(e_j+m,\tau)}$$

is the known vector function of space U, z^* is the known constant vector of the complex space $\mathcal{C}^2$, and the operator R_0 has the form (see (6_0))

$$R_0 z \equiv R_0\left(z_0(t) + \sum_{i=1}^{4} z_i(t)e^{\tau_i} + \sum_{2\le|m|\le N_z}^{*} z^m(t)e^{(m,\tau)} + \right.$$

$$\left. + \sum_{j=1}^{2}\sum_{0\le|m|\le N_z}^{*} z^{e_j+m}(t)e^{(e_j+m,\tau)}\right) \triangleq \int_{t_0}^{t} K(t,s)z_0(s)\,ds.$$

In the future, we will need $\lambda_j(t)$-eigenvectors of the matrix $A(t)$:

$$\varphi_1(t)=\begin{pmatrix}1\\-i\omega(t)\end{pmatrix},\ \varphi_2(t)=\begin{pmatrix}1\\+i\omega(t)\end{pmatrix},$$

and also $\bar\lambda_j(t)$–eigenvectors of the matrix $A^*(t)$:

$$\chi_1(t)=\begin{pmatrix}1\\-\frac{i}{\omega(t)}\end{pmatrix},\ \chi_2(t)=\begin{pmatrix}1\\+\frac{i}{\omega(t)}\end{pmatrix}.$$

These vectors form a biorthogonal system, i.e.

$$(\varphi_k(t),\chi_j(t))=\begin{cases}2,k=j,\\0,k\neq j\end{cases}\quad(k,j=1,2).$$

We introduce scalar (for each $t\in[t_0,T]$) product in space U :

$$\begin{gathered}<z,w>\equiv\\ \equiv<z_0(t)+\sum_{i=1}^{4}z_i(t)e^{\tau_i}+\sum^{*}_{2\le|m|\le N_z}z^m(t)e^{(m,\tau)}+\sum_{j=1}^{2}\sum^{*}_{1\le|m|\le N_z}z^{e_j+m}(t)e^{(e_j+m,\tau)},\\ w_0(t)+\sum_{i=1}^{4}w_i(t)e^{\tau_i}+\sum^{*}_{2\le|m|\le N_w}w^m(t)e^{(m,\tau)}+\sum_{j=1}^{2}\sum^{*}_{1\le|m|\le N_w}w^{e_j+m}(t)e^{(e_j+m,\tau)}>\triangleq\\ \triangleq(z_0(t),w_0(t))+\sum_{i=1}^{4}(z_i(t),w_i(t))+\\ +\sum^{*}_{2\le|m|\le\min(N_z,N_w)}(z^m(t),w^m(t))+\sum_{j=1}^{2}\sum^{*}_{1\le|m|\le\min(N_z,N_w)}\left(z^{e_j+m}(t),w^{e_j+m}(t)\right),\end{gathered}$$

where we denote by $(*,*)$ the usual scalar product in the complex space $\mathbb{C}^2$. Let us prove the following statement.

Theorem 1. Let conditions 1) and 2) be fulfilled and the right-hand side $H(t,\tau)=H_0(t)+\sum_{i=1}^4H_i(t)e^{\tau_i}+$ $+\sum^*_{2\le|m|\le N_z}H^m(t)e^{(m,\tau)}+\sum_{j=1}^2\sum^*_{1\le|m|\le N_H}H^{e_j+m}(t)e^{(e_j+m,\tau)}$ *of system (10) belongs to the space U. Then the system (10) is solvable in U, if and only if*

$$<H(t,\tau),\chi_k(t)e^{\tau_k}>\equiv0,\ k=1,2,\ \forall t\in[t_0,T]. \tag{11}$$

Proof. We will determine the solution of system (10) as an element (5) of the space U :

$$\begin{gathered}z(t,\tau)=z_0(t)+\sum_{i=1}^{4}z_i(t)e^{\tau_i}+\sum^{*}_{2\le|m|\le N_z}z^m(t)e^{(m,\tau)}+\\ +\sum_{j=1}^{2}\sum^{*}_{1\le|m|\le N_H}z^{e_j+m}(t)e^{(e_j+m,\tau)}\equiv\\ \equiv z_0(t)+\sum_{i=1}^{4}z_i(t)e^{\tau_i}+\sum^{*}_{2\le|m|\le N_z}z^m(t)e^{(m,\tau)}+\sum_{k=1}^{2}\sum^{*}_{2\le|m^k|\le N_H}z^{m^k}(t)e^{\left(m^k,\tau\right)},\end{gathered}\tag{12}$$

where for convenience are introduced multi-indices

$$m^1=e_1+m\equiv(1,0,m_3,m_4),m^2=e_2+m\equiv(0,1,m_3!,m_4),\left|m^k\right|=1+m_3+m_4\ge2,$$

m_3 and m_4 are non-negative integer numbers. Substituting (12) into system (10), we will have

$$\begin{gathered}\sum_{i=1}^{4}[\lambda_i(t)I-A(t)]z_i(t)e^{\tau_i}+\sum^{*}_{2\le|m|\le N_z}[(m,\lambda(t))I-A(t)]z^m(t)e^{(m,\tau)}+\\ +\sum_{k=1}^{2}\sum^{*}_{2\le|m^k|\le N_H}\left[\left(m^k,\lambda(t)\right)I-A(t)\right]z^{m^k}(t)e^{\left(m^k,\tau\right)}-A(t)z_0(t)-\int_{t_0}^{t}K(t,s)z_0(s)ds=\\ =H_0(t)+\sum_{i=1}^{4}H_i(t)e^{\tau_i}+\sum^{*}_{2\le|m|\le N_z}H^m(t)e^{(m,\tau)}+\sum_{k=1}^{2}\sum^{*}_{2\le|m^k|\le N_H}H^{m^k}(t)e^{\left(m^k,\tau\right)}.\end{gathered}$$

Equating here the free terms and coefficients separately for identical exponents, we obtain the following systems of equations:

$$-A(t)\,z_0(t) - \int_{t_0}^{t} K(t,s)\,z_0(s)\,ds = H_0(t), \tag{13}$$

$$[\lambda_i(t)\,I - A(t)]\,z_i(t) = H_i(t),\ i = \overline{1,4}; \tag{13_i}$$

$$[(m,\lambda(t))\,I - A(t)]\,z^m(t) = H^m(t),\ m_3 \neq m_4,\ 2 \leq |m| \leq N_H; \tag{13_m}$$

$$\left[\left(m^k,\lambda(t)\right)I - A(t)\right]z^{m^k}(t) = H^{m^k}(t),\ m_3 \neq m_{4,}\ 2 \leq \left|m^k\right| \leq N_H,\ k = 1,2. \tag{14}$$

Since the matrix $A(t)$ is reversible, the system (13) can be written as

$$z_0(t) = \int_{t_0}^{t} \left(-A^{-1}(t)\,K(t,s)\right) z_0(s)\,ds - A^{-1}(t)\,H_0(t)\,. \tag{13_0}$$

Due to the smoothness of the kernel $-A^{-1}(t)\,K(t,s)$ and heterogeneity $-A^{-1}(t)\,H_0(t)$, this Volterra integral system has a unique solution $z_0(t) \in C^\infty\left([t_0,T],\mathbb{C}^2\right)$. The systems ($13_3$) and ($13_4$) also have unique solutions

$$z_i(t) = [\lambda_i(t)\,I - A(t)]^{-1}\,H_i(t) \in C^\infty\left([t_0,T],\mathbb{C}^2\right), i = 3,4,$$

since $\lambda_3(t), \lambda_4(t)$ do not belong to the spectrum of the matrix $A(t)$. Systems (13_1) and (13_2) are solvable in space $C^\infty\left([t_0,T],\mathbb{C}^2\right)$ if and only if there are identities

$$(H_i(t)\,,\chi_i(t)) \equiv 0 \forall t \in [t_0,T]\,, i = 1,2.$$

It is not difficult to see that these identities coincide with identities (11). Further, since $(m,\lambda(t)) \equiv m_3\lambda_3(t) + {} + m_4\lambda_4(t) \neq \lambda_j(t), j = 1,2, |m| = m_3 + m_4 \geq 2, m_3 \neq m_4$ (see condition 2) the absence of resonance), the system (13_m) has a unique solution

$$z^m(t) = [(m,\lambda(t))\,I - A(t)]^{-1}\,H^m(t)\,, 2 \leq |m| \leq N_H \in C^\infty\left([t_0,T]\,,\mathbb{C}^2\right).$$

We now consider systems (14). Let us show that when $\left|m^k\right| \geq 2$ the functions $\left(m^k,\lambda(t)\right)$ are not eigenvalues of the matrix $A(t)$. Indeed, let $\left(m^1,\lambda(t)\right) = \lambda_1(t), \left|m^1\right| \geq 2$. Then

$$\lambda_1(t) + m_3\lambda_3(t) + m_4\lambda_4(t) = \lambda_2(t)\,,\ \ m_3 + m_4 \geq 1,$$

which contradicts condition 2) the absence of resonance. And likewise, equality $\left(m^2,\lambda(t)\right) = \lambda_1(t)\,,\ \left|m^2\right| \geq 2$, $m_3 + m_4 \geq 1$ cannot be fulfilled.

Therefore, when $\left|m^k\right| \geq 2$ the matrix $\left(m^k,\lambda(t)\right) I - A(t)$ is reversible, we get a unique solution of system (14) for $\left|m^k\right| \geq 2$ in the class $C^\infty\left([t_0,T]\,,\mathcal{C}^2\right)$:

$$z^{m^k}(t) = \left[\left(m^k,\lambda(t)\right)I - A(t)\right]^{-1} H^{m^k}(t), 2 \leq \left|m^k\right| \leq N_H, k = 1,2.$$

Thus, condition (11) is necessary and sufficient for the solvability of system (10) in the space U. The theorem is proved.

Remark 1. If identity (11) holds, then under conditions 1) and 2), system (10) has the following solution in the space U:

$$z(t,\tau) = z_0(t) + \sum_{k=1}^{2} \alpha_k(t)\,\varphi_k(t)\,e^{\tau_k} + \frac{(H_1(t)\,,\chi_2(t))}{\lambda_1(t) - \lambda_2(t)}\varphi_2(t)\,e^{\tau_1} +$$

$$+\frac{(H_2(t)\,,\chi_1(t))}{\lambda_2(t) - \lambda_1(t)}\varphi_1(t)\,e^{\tau_2} + \sum_{i=3}^{4} [\lambda_i(t)\,I - A(t)]^{-1}\,H_i(t)\,e^{\tau_i} +$$

$$+ \sum_{2\leq|m|\leq N_H}^{*} [(m,\lambda(t))\,I - A(t)]^{-1}\,H^m(t)\,e^{(m,\tau)} +$$

$$+ \sum_{k=1}^{2} \sum_{1\leq|m|\leq N_H}^{*} [(e_k + m,\lambda(t))\,I - A(t)]^{-1}\,H^{e_k+m}(t)\,e^{(e_k+m,\tau)}, \tag{15}$$

where $\alpha_k(t) \in C^\infty\left([t_0,T]\,,\mathbb{C}^1\right)$ are arbitrary functions, $k = 1,2$, $z_0(t)$ is the solution of an integral system(13_0), $m \equiv (0,0,m_3,m_4)\,, m_3 \neq m_4, |m| = m_3 + m_4 \geq 1.$

3. The unique solvability of the general iterative problem in the space U. Residual term theorem

Let us proceed to the description of the conditions for the unique solvability of system (10) in space Along with problem (10), we consider the system

$$Lw\left(t,\tau\right)=-\frac{\partial z}{\partial t}+\frac{g\left(t\right)}{2}\left(e^{\tau_3}\sigma_1+e^{\tau_4}\sigma_2\right)B\left(t\right)z+Q\left(t,\tau\right), \tag{16}$$

where $z=z(t,\tau)$ is the solution (15) of the system (10), $Q(t,\tau)$ is the well-known function of the space U. The right part of this system:

$$G\left(t,\tau\right)\equiv-\frac{\partial z}{\partial t}+\frac{g\left(t\right)}{2}\left(e^{\tau_3}\sigma_1+e^{\tau_4}\sigma_2\right)B\left(t\right)z+Q\left(t,\tau\right)=-\frac{\partial}{\partial t}[z_0\left(t\right)+\sum_{i=1}^{4}z_i\left(t\right)e^{\tau_i}+$$

$$+\sum_{2\leq|m|\leq N_z}^{*}z^m\left(t\right)e^{(m,\tau)}+\sum_{j=1}^{2}\sum_{1\leq|m|\leq N_z}^{*}z^{e_j+m}\left(t\right)e^{(e_j+m,\tau)}]+\frac{g\left(t\right)}{2}\left(e^{\tau_3}\sigma_1+e^{\tau_4}\sigma_2\right)B\left(t\right)\times$$

$$\times[z_0\left(t\right)+\sum_{i=1}^{4}z_i\left(t\right)e^{\tau_i}+\sum_{2\leq|m|\leq N_z}^{*}z^m\left(t\right)e^{(m,\tau)}+\sum_{j=1}^{2}\sum_{1\leq|m|\leq N_z}^{*}z^{e_j+m}\left(t\right)e^{(e_j+m,\tau)}]+Q\left(t,\tau\right),$$

may not belong to space U, if $z=z\left(t,\tau\right)\in U$. Indeed, taking into account the form (15) of the function $z=z\left(t,\tau\right)\in U$, we will have

$$Z\left(t,\tau\right)\equiv G\left(t,\tau\right)+\frac{\partial z}{\partial t}=\frac{g\left(t\right)}{2}\left(e^{\tau_3}\sigma_1+e^{\tau_4}\sigma_2\right)B\left(t\right)[z_0\left(t\right)+$$

$$+\sum_{i=1}^{4}z_i\left(t\right)e^{\tau_i}+\sum_{2\leq|m|\leq N_z}^{*}z^m\left(t\right)e^{(m,\tau)}+\sum_{j=1}^{2}\sum_{1\leq|m|\leq N_z}^{*}z^{e_j+m}\left(t\right)e^{(e_j+m,\tau)}]=$$

$$=\tfrac{g(t)}{2}B\left(t\right)z_0\left(t\right)\left(e^{\tau_3}\sigma_1+e^{\tau_4}\sigma_2\right)+\sum_{i=3}^{4}\tfrac{g(t)}{2}B\left(t\right)z_i\left(t\right)\left(e^{\tau_i+\tau_3}\sigma_1+e^{\tau_i+\tau_4}\sigma_2\right)+$$

$$+\sum_{k=1}^{2}\tfrac{g(t)}{2}B\left(t\right)z_k\left(t\right)\left(e^{\tau_k+\tau_3}\sigma_1+e^{\tau_k+\tau_4}\sigma_2\right)+$$

$$+\tfrac{g(t)}{2}\left(e^{\tau_3}\sigma_1+e^{\tau_4}\sigma_2\right)B\left(t\right)[\sum_{2\leq|m|\leq N_z}^{*}z^m\left(t\right)e^{(m,\tau)}+\sum_{j=1}^{2}\sum_{1\leq|m|\leq N_z}^{*}z^{e_j+m}\left(t\right)e^{(e_j+m,\tau)}]+$$

$$+Q\left(t,\tau\right).$$

Here are terms with exponents

$$\begin{array}{l}e^{\tau_4+\tau_3}=e^{(m,\tau)}|_{m=(0,0,1,1)},\\ e^{\tau_3+(m,\tau)}\ (\text{if } m_3+1=m_4),\ e^{\tau_4+(m,\tau)}\ (\text{if } m_4+1=m_3)\,,\\ e^{\tau_3+(e_1+m,\tau)}\ (\text{if } m_3+1=m_4),\ e^{\tau_4+(e_2+m,\tau)}\ (\text{if } m_4+1=m_3)\end{array} \tag{$*$}$$

do not belong to space U, since in multi-index $m=(0,0,m_3,m_4)$ of the space U must be $m_3\neq m_4$, $m_3+m_4\geq 1$. Then, according to the well-known theory [3; 234], we embed these terms in the space U according to the following rule (see $(*)$):

$$\begin{array}{l}\widehat{e^{\tau_4+\tau_3}}=e^0=1,\widehat{e^{\tau_3+(m,\tau)}}=\\ =e^0=1\left(m_3+1=m_4,m_3\neq m_4\right),\widehat{e^{\tau_4+(m,\tau)}}=e^0=1\left(m_4+1=m_3,m_3\neq m_4\right),\\ \widehat{e^{\tau_3+(e_1+m,\tau)}}=e^{\tau_1}\left(m_3+1=m_4,m_3\neq m_4\right),\widehat{e^{\tau_4+(e_2+m,\tau)}}=e^{\tau_2}\left(m_4+1=m_3,m_3\neq m_4\right).\end{array} \tag{$**$}$$

In $Z\left(t,\tau\right)$ need of embedding only the terms

$$M\left(t,\tau\right)\equiv\sum_{i=3}^{4}\tfrac{g(t)}{2}B\left(t\right)z_i\left(t\right)\left(e^{\tau_i+\tau_3}\sigma_1+e^{\tau_i+\tau_4}\sigma_2\right)+\sum_{k=1}^{2}\tfrac{g(t)}{2}B\left(t\right)z_k\left(t\right)\left(e^{\tau_k+\tau_3}\sigma_1+e^{\tau_k+\tau_4}\sigma_2\right),$$

$$S\left(t,\tau\right)\equiv\tfrac{g(t)}{2}\left(e^{\tau_3}\sigma_1+e^{\tau_4}\sigma_2\right)B\left(t\right)[\sum_{2\leq|m|\leq N_z}^{*}z^m\left(t\right)e^{(m,\tau)}+\sum_{j=1}^{2}\sum_{1\leq|m|\leq N_z}^{*}z^{e_j+m}\left(t\right)e^{(e_j+m,\tau)}].$$

We describe this embedding in more detail, taking into account formulas (∗∗):

$$M(t,\tau)\equiv\sum_{k=1}^{2}\frac{g(t)}{2}B(t)\,z_k(t)\left(e^{\tau_k+\tau_3}\sigma_1+e^{\tau_k+\tau_4}\sigma_2\right)+$$

$$+\sum_{i=3}^{4}\frac{g(t)}{2}B(t)\,z_i(t)\left(e^{\tau_i+\tau_3}\sigma_1+e^{\tau_i+\tau_4}\sigma_2\right)=$$

$$=\frac{g(t)}{2}B(t)\,[z_1(t)\,e^{\tau_1+\tau_3}\sigma_1+z_1(t)\,e^{\tau_1+\tau_4}\sigma_2+z_2(t)\,e^{\tau_2+\tau_3}\sigma_1+z_2(t)\,e^{\tau_2+\tau_4}\sigma_2+$$

$$+z_3(t)\,e^{2\tau_3}\sigma_1+z_3(t)\,e^{\tau_3+\tau_4}\sigma_2+z_4(t)\,e^{\tau_4+\tau_3}\sigma_1+z_4(t)\,e^{2\tau_4}\sigma_2]\quad\Rightarrow$$

$$\Rightarrow\quad\widehat{M}(t,\tau)=\frac{g(t)}{2}B(t)\,[z_1(t)\,e^{\tau_1+\tau_3}\sigma_1+z_1(t)\,e^{\tau_1+\tau_4}\sigma_2+z_2(t)\,e^{\tau_2+\tau_3}\sigma_1+$$

$$+z_2(t)\,e^{\tau_2+\tau_4}\sigma_2+z_3(t)\,e^{2\tau_3}\sigma_1+z_3(t)\,\sigma_2+z_4(t)\,\sigma_1+z_4(t)\,e^{2\tau_4}\sigma_2$$

(note that in $\widehat{M}(t,\tau)$ there are no members containing e^{τ_1}, e^{τ_2} measurement exponents $|m|=1$);

$$S(t,\tau)\equiv\frac{g(t)}{2}\left(e^{\tau_3}\sigma_1+e^{\tau_4}\sigma_2\right)B(t)\,[\sum_{2\le|m|\le N_z}^{*}z^m(t)\,e^{(m,\tau)}+$$

$$+\sum_{j=1}^{2}\sum_{1\le|m|\le N_z}^{*}z^{e_j+m}(t)\,e^{(e_j+m,\tau)}]=$$

$$=\frac{g(t)}{2}B(t)\,[\sum_{2\le|m|\le N_z}^{*}z^m(t)\left(e^{\tau_3+(m,\tau)}\sigma_1+e^{\tau_4+(m,\tau)}\sigma_2\right)+$$

$$+\sum_{1\le|m|\le N_z}^{*}z^{e_1+m}(t)\left(e^{(e_1+m,\tau)+\tau_3}\sigma_1+e^{(e_1+m,\tau)+\tau_4}\sigma_2\right)+$$

$$+\sum_{1\le|m|\le N_z}^{*}z^{e_2+m}(t)\left(e^{(e_2+m,\tau)+\tau_3}\sigma_1+e^{(e_2+m,\tau)+\tau_4}\sigma_2\right)]\quad\Rightarrow\widehat{S}(t,\tau)=$$

$$=\frac{g(t)}{2}B(t)\sum_{\substack{2\le|m|\le N_z,\\ m_3+1=m_4}}z^m(t)\,\sigma_1+\sum_{\substack{2\le|m|\le N_z,\\ m_4+1=m_3}}z^m(t)\,\sigma_2+\sum_{\substack{2\le|m|\le N_z,\\ m_3+1\ne m_4,\,m_4+1\ne m_3}}^{*}z^m(t)\,e^{(m,\tau)}+$$

$$+\left(\sum_{\substack{1\le|m|\le N_z,\\ m_3+1=m_4}}z^{e_1+m}(t)\,\sigma_1+\sum_{\substack{1\le|m|\le N_z,\\ m_4+1=m_3}}z^{e_1+m}(t)\,\sigma_2\right)e^{\tau_1}+$$

$$+\left(\sum_{\substack{1\le|m|\le N_z,\\ m_3+1=m_4}}z^{e_2+m}(t)\,\sigma_1+\sum_{\substack{1\le|m|\le N_z,\\ m_4+1=m_3}}z^{e_2+m}(t)\,\sigma_2\right)e^{\tau_2}+$$

$$+\sum_{j=1}^{2}\sum_{\substack{1\le|m|\le N_z,\\ m_3+1\ne m_4,\,m_4+1\ne m_3}}^{*}z^{e_j+m}(t)\,e^{(e_j+m,\tau)}].$$

After embedding, the right-hand side of system (20) will look like

$$\widehat{G}(t,\tau) = -\frac{\partial}{\partial t}[z_0(t) + \sum_{i=1}^{4} z_i(t)\, e^{\tau_i} + \sum_{2\le|m|\le N_z}^{*} z^m(t)\, e^{(m,\tau)} +$$

$$+\sum_{j=1}^{2}\sum_{1\le|m|\le N_z}^{*} z^{e_j+m}(t)\, e^{(e_j+m,\tau)}] + \widehat{M}(t,\tau) + \widehat{S}(t,\tau) + Q(t,\tau),$$

moreover, in $\widehat{S}(t,\tau)$ the coefficients at e^{τ_1}, e^{τ_2} do not depend on $z_k(t), k=1,2$. As indicated in [3], the embedding $G(t,\tau) \to \widehat{G}(t,\tau)$ will not affect the accuracy of the construction of asymptotic solutions of problem (2), since $\widehat{Z}(t,\tau)\,|_{\tau=\psi(t)/\varepsilon} \equiv Z(t,\tau)\,|_{\tau=\psi(t)/\varepsilon}$.

Theorem 2. Let conditions 1) and 2) be fulfilled and the right-hand side $H(t,\tau) = H_0(t) + \sum_{i=1}^{4} H_i(t)\, e^{\tau_i} + \sum_{2\le|m|\le N_z}^{*} H^m(t)\, e^{(m,\tau)} + \sum_{j=1}^{2}\sum_{1\le|m|\le N_H}^{*} H^{e_j+m}(t)\, e^{(e_j+m,\tau)} \in U$ *of system (10) satisfy condition (11). Then problem (10) under additional conditions*

$$< \widehat{G}(t,\tau), \chi_k(t)\, e^{\tau_k} > \equiv 0\ \forall t \in [t_0,T],\ k=1,2, \tag{17}$$

where

$$Q(t,\tau) = Q_0(t) + \sum_{k=1}^{4} Q_k(t)\, e^{\tau_k} + \sum_{2\le|m|\le N_z}^{*} Q^m(t)\, e^{(m,\tau)} + \sum_{k=1}^{2}\sum_{1\le|m|\le N_Q}^{*} Q^{e_k+m}(t)\, e^{(e_k+m,\tau)}$$

is the known vector function of space U, *is uniquely solvable in* U.

Proof. Since the right-hand side of system (10) satisfies condition (11), this system has a solution in space U in the form (15), where $\alpha_k(t) \in C^\infty\left([t_0,T], \mathbb{C}^1\right)$ are arbitrary functions so far, $k=1,2$. Submit (15) to the initial condition $z(t_0,0) = z^*$. We get $\sum_{k=1}^{2} \alpha_k(t_0)\, \varphi_k(t_0) = z_*$, where denoted

$$z_* = z^* + A^{-1}(t_0)\, H_0(t_0) - \sum_{i=3}^{4} [\lambda_i(t_0)\, I - A(t_0)]^{-1} H_i(t_0) -$$

$$-\frac{(H^{e_1}(t_0), \chi_2(t_0))}{\lambda_1(t_0)-\lambda_2(t_0)}\varphi_2(t_0) - \frac{(H^{e_2}(t_0), \chi_1(t_0))}{\lambda_2(t_0)-\lambda_1(t_0)}\varphi_1(t_0) -$$

$$-\sum_{2\le|m|\le N_z}^{*} z^m(t_0) - \sum_{k=1}^{2}\sum_{1\le|m^k|\le N_H}^{*}\left[\left(m^k, \lambda(t_0)\right) I - A(t_0)\right]^{-1} H^{m^k}(t_0).$$

Multiplying the equality $\sum_{k=1}^{2} \alpha_k(t_0)\, \varphi_k(t_0) = z_*$ scalarly by $\chi_j(t_0)$ and taking into account the biorthogonality of the systems $\{\varphi_k(t)\}$ and $\{\chi_j(t)\}$, we find the values $\alpha_k(t_0) = \frac{1}{2}(z_*, \chi_k(t_0))$, $k=1,2$. Now we submit the solution (15) to the condition of orthogonality (17). Considering that under these conditions, scalar multiplication performed by vector functions $\chi_k(t)\, e^{\tau_k}$, containing only exponents e^{τ_k}, $k=1,2$, it is necessary to keep in the expression $\widehat{G}(t,\tau)$ only terms with exponents e^{τ_1} and e^{τ_2}. Then condition (17) takes the form

$$< -\frac{\partial}{\partial t}\left(\sum_{k=1}^{2} \alpha_k(t)\, \varphi_k(t)\, e^{\tau_k} + \frac{(H_1(t), \chi_2(t))}{\lambda_1(t)-\lambda_2(t)}\varphi_2(t)\, e^{\tau_1} + \frac{(H_2(t), \chi_1(t))}{\lambda_2(t)-\lambda_1(t)}\varphi_1(t)\, e^{\tau_2}\right) +$$

$$+\left(\sum_{\substack{1\le|m|\le N_z,\\ m_3+1=m_4}} z^{e_1+m}(t)\, \sigma_1 + \sum_{\substack{1\le|m|\le N_z,\\ m_4+1=m_3}} z^{e_1+m}(t)\, \sigma_2\right) e^{\tau_1} +$$

$$+\left(\sum_{\substack{1\le|m|\le N_z,\\ m_3+1=m_4}} z^{e_2+m}(t)\, \sigma_1 + \sum_{\substack{1\le|m|\le N_z,\\ m_4+1=m_3}} z^{e_2+m}(t)\, \sigma_2\right) e^{\tau_2} +$$

$$+Q_1(t)\, e^{\tau_1} + Q_2(t)\, e^{\tau_2}, \chi_k(t)\, e^{\tau_k} > \equiv 0 \quad \forall t \in [t_0,T],\ k=1,2.$$

Performing here scalar multiplication, we obtain linear ordinary differential equations with respect to the functions $\alpha_k(t)$, involved in the solution (15) of system (10). Attaching to them the initial conditions

$\alpha_k(t_0) = \frac{1}{2}(z_*, \chi_k(t_0))$, $k = 1, 2$, computed earlier, we find uniquely the functions $\alpha_k(t) \in C^\infty([t_0, T], \mathbb{C}^1)$, $k = 1, 2$, and, therefore, we construct solution (15) in the space in a unique way. The theorem is proved.

Applying Theorems 1 and 2 to iterative problems (10_k) (in this case, the right-hand sides $H^{(k)}(t, \tau)$ of these problems are embedded in the space U, i.e. $H^{(k)}(t, \tau)$ we replace with $\hat{H}^{(k)}(t, \tau) \in U$), we find uniquely their solutions in space U and construct series (7). Just as in [3], we prove the following statement.

Theorem 3. Suppose that conditions (1) – 2) are satisfied for system (2). Then, when $\varepsilon \in (0, \varepsilon_0]$ $(\varepsilon_0 > 0$ *is sufficiently small), system (2) has a unique solution* $z(t, \varepsilon) \in C^1([0, T], \mathbb{C}^2)$; *in this case, the estimate*

$$||z(t, \varepsilon) - z_{\varepsilon N}(t)||_{C[0,T]} \le c_N \varepsilon^{N+1},$$

holds true, where $z_{\varepsilon N}(t)$ *is the restriction (for* $\tau = \frac{\psi(t)}{\varepsilon}$*) of the N- partial sum of series (9) (with coefficients* $z_k(t, \tau) \in U$, *satisfying the iteration problems* (10_k)), *and the constant* $c_N > 0$ *does not depend on* $\varepsilon \in (0, \varepsilon_0]$.

4. Construction of the solution of the first iteration problem in space U.

Using Theorem 1, we will try to find a solution to the first iteration problem (10_0). Since the right side $h(t)$ of the system (10_0) satisfies condition (11), this system has (according to (15)) a solution in space U in the form

$$z_0(t, \tau) = z_0^{(0)}(t) + \sum_{k=1}^{2} \alpha_k^{(0)}(t)\, \varphi_k(t)\, e^{\tau_k}, \tag{18}$$

where $z_0^{(0)}(t)$ is the solution of the integrated system

$$z_0^{(0)}(t) = \int_{t_0}^{t} \left(-A^{-1}(t)\, K(t, s)\right) z_0^{(0)}(s)\, ds - A^{-1}(t)\, h(t), \tag{19}$$

$\alpha_k^{(0)}(t) \in C^\infty([t_0, T], \mathcal{C}^1)$ are arbitrary functions. Subjecting (18) to the initial condition $z_0(t_0, 0) = z^0$, we will have

$$z_0^{(0)}(t_0) + \sum_{k=1}^{2} \alpha_k^{(0)}(t_0)\, \varphi_k(t_0) = z^0 \Leftrightarrow \sum_{k=1}^{2} \alpha_k^{(0)}(t_0)\, \varphi_k(t_0) = z^0 + A^{-1}(t_0)\, h(t_0).$$

Multiplying this equality scalarly by $\chi_j(t_0)$ and taking into account the biorthogonality of the systems $\{\varphi_k(t)\}$ and $\{\chi_j(t)\}$, we find the values $\alpha_k^{(0)}(t_0) = \frac{1}{2}\left(z^0 + A^{-1}(t_0)\, h(t_0), \chi_k(t_0)\right)$, $k = 1, 2$. To fully compute the functions $\alpha_k^{(0)}(t)$, we proceed to the next iteration problem (10_1). Substituting into it the solution (16) of the system (10_0), we arrive at the following system:

$$L z_1(t, \tau) = -\frac{d}{dt} z_0^{(0)}(t) - \sum_{k=1}^{2} \frac{d}{dt}(\alpha_k^{(0)}(t)\, \varphi_k(t)) e^{\tau_k} +$$
$$+ \frac{g(t)}{2}(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2)\, B(t) \left(z_0^{(0)}(t) + \sum_{k=1}^{2} \alpha_k^{(0)}(t)\, \varphi_k(t)\, e^{\tau_k} \right) +$$
$$+ \sum_{j=1}^{2} \left[\frac{\left(K(t, t)\, \alpha_j^{(0)}(t)\, \varphi_j(t)\right)}{\lambda_j(t)} e^{\tau_j} - \frac{\left(K(t, t_0)\, \alpha_j^{(0)}(t_0)\, \varphi_j(t_0)\right)}{\lambda_j(t_0)} \right] \tag{20}$$

(here we used the expression (6_1) for $R_1 z(t, \tau)$ and took into account that for $z(t, \tau) = z_0(t, \tau)$ only the terms with e^{τ_1} and e^{τ_2} remain in the sum (6_1)). It is not difficult to see that the right side

$$H(t, \tau) = -\tfrac{d}{dt} z_0^{(0)}(t) - \textstyle\sum_{k=1}^{2} \tfrac{d}{dt}(\alpha_k^{(0)}(t)\, \varphi_k(t)) e^{\tau_k} +$$
$$+ \tfrac{g(t)}{2}(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2)\, B(t) \left(z_0^{(0)}(t) + \textstyle\sum_{k=1}^{2} \alpha_k^{(0)}(t)\, \varphi_k(t)\, e^{\tau_k} \right) +$$
$$+ \textstyle\sum_{j=1}^{2} \left[\tfrac{\left(K(t,t)\alpha_j^{(0)}(t)\varphi_j(t)\right)}{\lambda_j(t)} e^{\tau_j} - \tfrac{\left(K(t,t_0)\alpha_j^{(0)}(t_0)\varphi_j(t_0)\right)}{\lambda_j(t_0)} \right]$$

of system (20) belongs to space U. System (20) is solvable in this space U if and only if conditions (11) are satisfied, which in our case take the form

$$\left(-\tfrac{d}{dt}(\alpha_k^{(0)}\left(t\right)\varphi_k\left(t\right)) + \tfrac{\left(K(t,t)\alpha_k^{(0)}(t)\varphi_k(t)\right)}{\lambda_k(t)}, \chi_k\left(t\right)\right) = 0 \;\; \Leftrightarrow$$

$$\Leftrightarrow \;\; 2\tfrac{d\alpha_k^{(0)}(t)}{dt} = \left(\tfrac{(K(t,t)\varphi_k(t))}{\lambda_k(t)} - \dot{\varphi}_k\left(t\right), \chi_k\left(t\right)\right)\alpha_k^{(0)}\left(t\right), k = 1, 2.$$

Attaching to this system the initial conditions $\alpha_k^{(0)}\left(t_0\right) = \frac{1}{2}\left(z^0 + A^{-1}\left(t_0\right)h\left(t_0\right), \chi_k\left(t_0\right)\right)$, we find uniquely functions

$$\alpha_k^{(0)}\left(t\right) = \alpha_k^{(0)}\left(t_0\right)\exp\left\{\frac{1}{2}\int_{t_0}^{t}\left(\frac{\left(K\left(s,s\right)\varphi_k\left(s\right)\right)}{\lambda_k\left(t\right)} - \dot{\varphi}_k\left(s\right), \chi_k\left(s\right)\right)ds\right\}, k = 1, 2,$$

therefore, we uniquely calculate the solution (18) of the problem (10_0) in the space U. Moreover, the main term of the asymptotic of the solution to problem (2) has the form

$$z_{\varepsilon 0}\left(t\right) = z_0^{(0)}\left(t\right) +$$

$$+\sum_{k=1}^{2}\alpha_k^{(0)}\left(t_0\right)\exp\left\{\frac{1}{2}\int_{t_0}^{t}\left(\frac{\left(K\left(s,s\right)\varphi_k\left(s\right)\right)}{\lambda_k\left(s\right)} - \dot{\varphi}_k\left(s\right), \chi_k\left(s\right)\right)ds\right\}\varphi_k\left(t\right)e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_k(\theta)d\theta},$$

where $\alpha_k^{(0)}\left(t_0\right) = \frac{1}{2}\left(z^0 + A^{-1}\left(t_0\right)h\left(t_0\right), \chi_k\left(t_0\right)\right)$, $k = 1, 2$, $z_0^{(0)}\left(t\right)$ is the solution of the integra system (19).

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References

1 Фещенко С.Ф. Асимптотические методы в теории линейных дифференциальных уравнений / С.Ф. Фещенко, Н.И. Шкиль, Л.Д. Николенко. –– Киев: Наук. думка, 1966. — 261 с.

2 Далецкий Ю.Л. Асимптотический метод для некоторых дифференциальных уравнений с осциллирующими коэффициентами / Ю.Л. Далецкий // ДАН СССР. — 1962. — 143 — № 5. — С. 1026–1029.

3 Ломов С.А. Введение в общую теорию сингулярных возмущений / С.А. Ломов. — М.: Наука, 1981. — 400 с.

4 Ломов С.А. Основы математической теории пограничного слоя / С.А. Ломов, И.С. Ломов. — М.: Изд--во МГУ, 2011. — 456 с.

5 Рыжих А.Д. Асимптотическое решение линейного дифференциального уравнения с быстро осциллирующим коэффициентом / А.Д. Рыжих // Тр. МЭИ. — 1977. — 357. — С. 92–94.

6 Рыжих А.Д. Применение метода регуляризации для уравнения с быстро осциллирующими коэффициентами / А.Д. Рыжих // Материалы Всесоюз. конф. по асимп. методам. — Ч. I. — Алма-Ата: Наука, 1979. — С. 64–66.

7 Иманбаев Н.С. Алгоритм метода регуляризации для сингулярно возмущенной задачи с нестабильным значением ядра интегрального оператора / Н.С. Иманбаев, Б.Т. Калимбетов, М.А. Темирбеков // Вестн. Караганд. ун-та. Сер. Математика. — 2013. — № 4. — С. 64–70.

8 Калимбетов Б.Т. Математическое описание внутреннего пограничного слоя для нелинейной интегродифференциальной системы / Б.Т. Калимбетов, Б.И. Ескараева, М.А. Темирбеков // Вестн. Караганд. ун-та. Сер. Математика. — 2014. — № 3. — 75. — С. 77–87.

9 Ескараева Б.И. Дискретный пограничный слой в случае нулевых точек спектра для систем интегродифференциальных уравнений / Б.И. Ескараева, Б.Т. Калимбетов, М.А. Темирбеков // Вестн. Караганд. ун-та. Сер. Математика. — 2014. — № 3. — 75. — С. 88–95.

10 Kalimbetov B.T. Internal boundary layer for integral–differential equations with zero spectrum of the limit operator and rapidly changing kernel / B.T. Kalimbetov, B.I. Yeskarayeva, A.S. Tolep // Applied Mathematical Sciences. — 2015. — 9. — 141–144. — P. 7149–7165.

11 Kalimbetov B.T. Asymptotic solution of singular perturbed problems with an instable spectrum of the limiting operator / B.T. Kalimbetov, M.A. Temirbekov, Zh.O. Habibullaev // Abstract and Applied Analysis. — 2012. — Vol. 2012, No. 120192. — 16 p.

12 Kalimbetov B.T. Regularized asymptotical solutions of integro–differential systems with spectral singularites / B.T. Kalimbetov, N.S. Imanbaev, D.A. Sapakov, L.T. Tashimov // Advances in Difference Equations. — 2013. — Vol. 109. DOI: 10.1186/1687–1847–2013–109.

13 Safonov V.F. A regularization method for systems with instable spectral of the kernel of the integral operator / V.F. Safonov, B.T. Kalimbetov // Differential Equations. — 31.— 4. — 1995. — P. 647–656.

14 Safonov V.F. Regularized asymptotic solutions of the initial problem of systems of integro–partial differential equations / V.F. Safonov, A.A. Bobodzhanov // Mathematical Notes. — 2017. — 102. — 1. — P. 22–30.

15 Safonov V.F. Regularized asymptotics of solutions to integro–differential partial differential equations with rapidly varying kernels / V.F. Safonov, A.A. Bobodzhanov // Ufimsk. Mat. Zh. — 2018. — 10. — 2. — P. 3–12.

16 Bobodzhanov A.A. A generalization of the regularization method to the singularly perturbed integro–differential equations with partial derivatives / A.A. Bobodzhanov, V.F. Safonov // Russian Math. (Iz. VUZ). — 2018. — 62. — 3. — P. 6–17.

17 Bobodzhanov A.A. A problem with inverse time for a singularly perturbed integro–differential equation with diagonal degeneration of the kernel of high order / A.A. Bobodzhanov, V.F. Safonov // Izv. Math. — 2016. — 80. — 2. — P. 285–298.

18 Bobodzhanov A.A. Asymptotic solutions of Fredholm integro–differential equations with rapidly changing kernels and irreversible limit operator / A.A. Bobodzhanov, V.F. Safonov // Russian Math. (Iz. VUZ). — 2015. — 59. — 10. — P. 1–15.

19 Bobodzhanov A.A. The method of normal forms for singularly perturbed systems of Fredholm integro–differential equations with rapidly varying kernels / A.A. Bobodzhanov, V.F. Safonov // Sb. Math. — 2013. — 204. — 7. — P. 979–1002.

20 Safonov V.F. Volterra integral equations with rapidly varying kernels and their asymptotic integration / V.F. Safonov, A.A. Bobodzhanov // Sb. Math. — 2001. — 192. — 8. — P. 1139–1164.

21 Safonov V.F. «Splashes» in Fredholm integro–differential equations with rapidly varying kernels / V.F. Safonov, A.A. Bobodzhanov // Math. Notes. — 2009. — 85. — 2. — P. 153–167.

22 Сафонов В.Ф. Курс высшей математики. Сингулярно возмущенные задачи и метод регуляризации / В.Ф. Сафонов, А.А. Бободжанов. — М.: Издательский дом МЭИ, 2012. — 414 с.

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Жылдам осцилляцияланатын коэффициентті сингуляр ауытқыған интегро-дифференциалдық теңдеулер

Динамикалық орнықтылықпен, периодты құрылымға ие ортамен байланысты және басқа да қолданбалы мәселелерді зерттеулер жылдам осцилляцияланатын коэффициентті дифференциалдық теңдеулермен айналысуды қажет етеді. Мұндай коэффициентті дифференциалдық теңдеулер жүйелерінің асимптотикалық интегралдау бөлшектеу және регуляризация әдістермен жүргізілген. Осы жұмыста интегралды-дифференциалдық теңдеулер жүйесі қарастырылған. Зерттеудің негізгі мақсаты — интегралдық мүшенің алғашқы есептің шешімінің асимптотикасына әсерін зерттеу. Есепте резонанс болмаған жағдай, яғни жылдам осцилляцияланатын коэффициенттің бүтін сызықтық комбинациясының жиіліктері шекті оператордың спектрінің жиілігімен сәйкес келмейтін жағдай, зерттелген.

Кілт сөздер: сингуляр ауытқу, интегро-дифференциалдық теңдеу, жылдам осцилляцияланатын коэффициент, регуляризация, асимптотикалық жинақтылық.

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Интегро-дифференциальные сингулярно возмущенные уравнения с быстро осциллирующими коэффициентами

При исследовании различных вопросов, связанных с динамической устойчивостью, со свойствами сред с периодической структурой, при исследовании других прикладных задач приходится иметь дело с дифференциальными уравнениями с быстро осциллирующими коэффициентами. Асимптотическое интегрирование дифференциальных систем уравнений с такими коэффициентами проводилось методами расщепления и регуляризации. В настоящей работе рассмотрена система интегрально-дифференциальных уравнений. Основная цель исследования состоит в выявлении влияния интегрального члена на асимптотику решения исходной задачи. Изучен случай отсутствия резонанса, т.е. случай, когда целочисленная линейная комбинация частот быстро осциллирующего коэффициента не совпадает с частотой спектра предельного оператора.

Ключевые слова: сингулярное возмущение, интегрально-дифференциальное уравнение, быстро осциллирующий коэффициент, регуляризация, асимптотическая сходимость.

References

1 Feschenko, S.F., Shkil, N.I., & Nikolenko, L.D. (1966). *Asimptoticheskie metody v teorii lineinykh differentsialnykh uravnenii [Asymptotic methods in the theory of linear differential equations].* Kiev: Naukova Dumka [in Russian].

2 Daletsky, Yu.L. (1962). Asimptoticheskii metod dlia nekotorykh differentsialnykh uravnenii s ostsilliruiushchimi koeffitsientami [The asymptotic method for some differential equations with oscillating coefficients]. *Doklady Akademii Nauk SSSR – Reports of the Academy of Science of the USSR, 143, 5,* 1026–1029 [in Russian].

3 Lomov, S.A. (1981). *Vvedenie v obshchuiu teoriiu sinhuliarnykh vozmushchenii [Introduction to the general theory of singular perturbations].* Moscow: Nauka [in Russian].

4 Lomov, S.A., & Lomov, I.S. (2011). *Osnovy matematicheskoi teorii pohranichnoho sloia [Foundations of mathemathical theory of boundary layer].* Moscow: Izdatelstvo MHU [in Russian].

5 Ryzhih, A.D. (1977). Asimptoticheskoe reshenie lineinoho differentsialnoho uravneniia s bystro ostsilliruiushchim koeffitsientom [Asymptotic solution of a linear differential equation with a rapidly oscillating coefficient]. *Trudy MEI – The works of MEI, 357,* 92–94 [in Russian].

6 Ryzhih, A.D. (1979). Primenenie metoda rehuliarizatsii dlia uravneniia s bystro ostsilliruiushchimi koeffitsientami [Application of the regularization method for an equation with rapidly oscillating coefficients]. *Materialy Vsesoiuznoi konferentsii po asimp. metodam – Mater. All-union. conf. by asymp. methods, Part I. Alma-Ata, Science,* 64–66 [in Russian].

7 Imanbaev, N.S., Kalimbetov, B.T., & Temirbekov, M.A. (2013). Alhoritm metoda rehuliarizatsii dlia sinhuliarno vozmushchennoi zadachi s nestabilnym znacheniem yadra intehralnoho operatora [Algorithm of the regularization method for a singularly perturbed problem with unstable value of kernel of the integral operator]. *Vestnik Karahandinskoho universiteta. Seriia Matematika – Bulletin of the Karaganda University. Mathematics series, No. 4,* 64–70 [in Russian].

8 Kalimbetov, B.T., Yeskaraeva, B.I., & Temirbekov, M.A. (2014). Matematicheskoe opisanie vnutrenneho pohranichnoho sloia dlia nelineinoi intehro-differentsialnoi sistemy [Mathematical description of the inner boundary layer for a nonlinear integro-differential system]. *Vestnik Karahandinskoho universiteta. Seriia Matematika – Bulletin of the Karaganda University. Mathematics series, 75, No. 3,* 77–87 [in Russian].

9 Yeskaraeva, B.I., Kalimbetov, B.T., & Temirbekov, M.A. (2014). Diskretnyi pohranichnyi sloi v sluchae nulevykh tochek spektra dlia sistem intehro-differentsialnykh uravnenii [Discrete boundary layer in the case of zero points of the spectrum for systems of integro-differential equations]. *Vestnik Karahandinskoho universiteta. Seriia Matematika – Bulletin of the Karaganda University. Mathematics series, 75, No. 3,* 88–95 [in Russian].

10 Kalimbetov, B.T., Yeskarayeva, B.I., & Tolep A.S. (2015). Internal boundary layer for integral-differential equations with zero spectrum of the limit operator and rapidly changing kernel. *Journal Applied Mathematical Sciences, 9,* 141–144, 7149–7165.

11 Kalimbetov, B.T., Temirbekov, M.A., & Habibullaev, Zh.O. (2012). Asymptotic solution of singular perturbed problems with an instable spectrum of the limiting operator. *Journal Abstract and Applied Analysis, Vol. 2012, No. 120192,* 16.

12 Kalimbetov, B.T., Imanbaev, N.S., Sapakov, D.A., & Tashimov, L.T. (2013). Regularized asymptotical solutions of integro-differential systems with spectral singularites. *Journal Advances in Difference Equations, 109.* DOI: 10.1186/1687–1847–2013–109.

13 Safonov, V.F., & Kalimbetov, B.T. (1995). A regularization method for systems with unstable spectral value of the kernel of the integral operator. *Journal Differential Equations, 31(4),* 647–656.

14 Safonov, V.F., & Bobodzhanov, A.A. (2017). Regularized asymptotic solutions of the initial problem of systems of integro-partial differential equations. *Mathematical Notes, 102, 1,* 22–30.

15 Safonov, V.F., & Bobodzhanov, A.A. (2018). Regularized asymptotics of solutions to integro-differential partial differential equations with rapidly varying kernels. *Ufimsk. Mat. Zh., 10, 2,* 3–12.

16 Bobodzhanov, A.A., & Safonov, V.F. (2018). A generalization of the regularization method to the singularly perturbed integro-differential equations with partial derivatives. *Journal Russian Math. (Izvest. VUZ), 62, 3,* 6–17.

17 Bobodzhanov, A.A., & Safonov, V.F. (2016). A problem with inverse time for a singularly perturbed integro-differential equation with diagonal — of the kernel of high order. *Journal Izvest. Math, 80, 2,* 285–298.

18 Bobodzhanov, A.A., Safonov, V.F. (2015). Asymptotic solutions of Fredholm integro-differential equations with rapidly changing kernels and irreversible limit operator. *Journal Russian Math. (Izvest. VUZ), 59, 10,* 1–15.

19 Bobodzhanov, A.A., & Safonov, V.F. (2013). The method of normal forms for singularly perturbed systems of Fredholm integro-differential equations with rapidly varying kernels. *Sibir. Math J., 204, 7,* 979–1002.

20 Safonov, V.F., & Bobodzhanov, A.A. (2001). Volterra integral equations with rapidly varying kernels and their asymptotic integration. *Sibir. Math. J., 192, 8,* 1139–1164.

21 Safonov, V.F., & Bobodzhanov, A.A. (2009). «Splashes» in Fredholm integro-differential equations with rapidly varying kernels. *Math. Notes, 85, 2,* 153–167.

22 Safonov V.F., Bobodzhanov A.A. (2012). *Kurs vysshei matematiki. Sinhuliarno vozmushchennye zadachi i metod rehuliarizatsii [The course of higher mathematics. Singularly perturbed problems and regularization method].* Moscow: Izdatelskii dom MEI [in Russian].

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On a pseudo-Volterra nonhomogeneous integral equation

In this paper the issues of the solvability of a pseudo-Volterra nonhomogeneous integral equation of the second kind are studied. The solution to the corresponding homogeneous equation and the classes of the uniqueness of the solution are found in [1]. By replacing the right-hand side and the unknown function, the integral equation is reduced to an integral equation, the kernel of which is not «compressible». Using the Laplace transform, the obtained equation is reduced to an ordinary first-order differential equation (linear). Its solution is found. By using the solution of the homogeneous equation the form of a particular solution of the nonhomogeneous differential equation is defined (by the variation method of an arbitrary constant). By using the inverse Laplace transform, a particular solution of the pseudo-Volterra nonhomogeneous integral equation under study is obtained. The case of an nonhomogeneous integral equation with the value of the parameter $k = 1$ is considered and studied. Classes for the right side and the solution of the integral equation are indicated.

Keywords: pseudo-Volterra nonhomogeneous integral equation, class of essentially bounded functions, inverse Laplace transformation, residue.

Introduction

We research the solvability of the following nonhomogeneous pseudo-Volterra integral equation of the second kind

$$\nu(t) - \frac{a}{2\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t\tau}\sqrt{t-\tau}} \exp\left(-\frac{t-\tau}{4a^2}\right) \cdot \nu(\tau) d\tau - \frac{1}{k\,a\sqrt{\pi}} \int_0^t \frac{\sqrt{\tau}}{\sqrt{t(t-\tau)}} \exp\left(-\frac{t-\tau}{4a^2}\right) \cdot \nu(\tau) d\tau = f(t), \quad (1)$$

where $a,\ k$ are positive constants, $f(t)$ is the given function.

1. Reducing the equation (1) to a differential equation in images

Following the results of work [1] after replacements:

$$\frac{1}{\sqrt{t}} \exp\left(\frac{t}{4a^2}\right) \nu(t) = \nu_1(t), \quad \sqrt{t} \exp\left(\frac{t}{4a^2}\right) f(t) = f_1(t) \quad (2)$$

we get the following integral equation

$$t \cdot \nu_1(t) - \frac{a}{2\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} \cdot \nu_1(\tau) d\tau - \frac{1}{k\,a\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} \cdot \tau\, \nu_1(\tau)\, d\tau = f_1(t). \quad (3)$$

Applying the Laplace transform the equation (3) transforms to the following differential equation

$$\left[\frac{1}{k\,a\sqrt{p}} - 1\right] \hat{\nu}_1'(p) - \frac{a}{2\sqrt{p}}\ \hat{\nu}_1(p) = \hat{f}_1(p). \quad (4)$$

The general solution to corresponding homogeneous (4) has the form

$$\hat{\nu}_{1,\ hom}(p) = C \frac{e^{-a\sqrt{p}}}{\left(\sqrt{p} - \frac{1}{ka}\right)^{\frac{1}{k}}},$$

where $\mathcal{C} - const$.

Then the solution of the nonhomogeneous equation (4) is sought in the form

$$\hat{\nu}_1(p) = C(p)\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}. \tag{5}$$

Substituting function (5) into equation (4), we get

$$C(p) = -\int_p^{+\infty} \hat{f}_1(p)\cdot e^{a\sqrt{p}}\ \sqrt{p}\cdot\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}-1}\ dp + C. \tag{6}$$

After substituting (6) into (5), we can write out the partial solution of the nonhomogeneous equation (4) as

$$\hat{\nu}_{1,\,part}(p) = -\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}\int_p^{+\infty}\left(\int_0^{+\infty} e^{-qt}f_1(t)dt\right)\cdot e^{a\sqrt{p}}\ \sqrt{q}\cdot\left(\sqrt{q}-\frac{1}{ka}\right)^{\frac{1}{k}-1}\ dq.$$

Changing the order of integration, we obtain

$$\hat{\nu}_{1,\,part}(p) = -\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}\int_0^{+\infty} f_1(t)\underbrace{\int_p^{+\infty} e^{-qt}e^{a\sqrt{q}}\ \sqrt{q}\cdot\left(\sqrt{q}-\frac{1}{ka}\right)^{\frac{1}{k}-1}\ dq}_{I(p,t)}\ dt.$$

This way, we get

$$\hat{\nu}_{1,\,part}(p) = -\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}\int_0^{+\infty} f_1(t)I(p,\,t)\,dt, \tag{7}$$

where

$$I(p,\,t) = \int_p^{+\infty}\sqrt{q}\cdot\left(\sqrt{q}-\frac{1}{ka}\right)^{\frac{1}{k}-1}\ e^{-qt+a\sqrt{q}}dq. \tag{8}$$

We rewrite integral (8) in the form

$$I(p,\,t) = \underbrace{\int_p^{+\infty}\left(\sqrt{q}-\frac{1}{ka}\right)^{\frac{1}{k}}\ e^{-qt+a\sqrt{q}}dq}_{J_{\frac{1}{k}}(p,\,t)} + \frac{1}{ka}\underbrace{\int_p^{+\infty}\left(\sqrt{q}-\frac{1}{ka}\right)^{\frac{1}{k}-1}\ e^{-qt+a\sqrt{q}}dq}_{J_{\frac{1}{k}-1}(p,\,t)}. \quad\Rightarrow$$

$$I(p,\,t) = J_n(p,\,t) + \frac{1}{ka}J_{n-1}(p,\,t), \tag{9}$$

where

$$J_{n-1}(p,\,t) = \int_p^{+\infty}\left(\sqrt{q}-\frac{1}{ka}\right)^{n-1}\ e^{-qt+a\sqrt{q}}dq = \left\|\begin{array}{ll}\sqrt{q}=x, & q=x^2;\\ dq=2xdx; & \sqrt{p}\le x<+\infty\end{array}\right\| =$$

$$= 2\int_{\sqrt{p}}^{+\infty} x^{-\frac{1}{ka}+\frac{1}{ka}}\left(x-\frac{1}{ka}\right)^{n-1}\ e^{-tx^2+ax}dx = 2\underbrace{\int_{\sqrt{p}}^{+\infty}\left(x-\frac{1}{ka}\right)^{n}e^{-tx^2+ax}dx}_{A_n(p,t)} +$$

$$+\frac{2}{ka}\underbrace{\int_{\sqrt{p}}^{+\infty}\left(x-\frac{1}{ka}\right)^{n-1}\ e^{-tx^2+ax}dx}_{A_{n-1}(p,t)} \quad\Rightarrow$$

$$J_{n-1}(p,\,t) = 2A_n(p,t) + \frac{2}{ka}A_{n-1}(p,t), \tag{10}$$

where

$$A_{n-1}(p,t) = \int_{\sqrt{p}}^{+\infty}\left(x-\frac{1}{ka}\right)^{n-1}\ \exp(-tx^2+ax)dx =$$

$$= \exp\left(\frac{a^2}{4t}\right)\int_{\sqrt{p}}^{+\infty}\left(x-\frac{1}{ka}\right)^{n-1}\exp\left(-t\left(x-\frac{a}{2t}\right)^2\right)dx = \left\| \begin{array}{l} \lambda = x-\frac{a}{2t};\quad d\lambda = dx \\ \sqrt{p}-\frac{a}{2t}<\lambda<+\infty \end{array}\right\| =$$

$$= \exp\left(\frac{a^2}{4t}\right)\underbrace{\int_{\sqrt{p}-\frac{a}{2t}}^{+\infty}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{n-1}e^{-t\lambda^2}\,d\lambda}_{B_{n-1}(p,t)}$$

or

$$A_{n-1}(p,t) = \exp\left(\frac{a^2}{4t}\right)B_{n-1}(p,t), \tag{11}$$

where

$$B_r(p,t) = \int_{\sqrt{p}-\frac{a}{2t}}^{+\infty}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{r}e^{-t\lambda^2}\,d\lambda.$$

We find the last integral when $r = n-1$:

$$B_{n-1}(p,t) = \int_{\sqrt{p}-\frac{a}{2t}}^{+\infty}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{n-1}e^{-t\lambda^2}\,d\lambda = \left\| \begin{array}{l} u = e^{-t\lambda^2};\ dv = \left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{n-1}d\lambda \\ du = -2t\lambda e^{-t\lambda^2}d\lambda;\ v = \frac{1}{n}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{n} \end{array}\right\| =$$

$$= \frac{1}{n}e^{-t\lambda^2}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{n}\Bigg|_{\lambda=\sqrt{p}-\frac{a}{2t}}^{\lambda\to+\infty} + \frac{2t}{n}\int_{\sqrt{p}-\frac{a}{2t}}^{+\infty}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{n}\lambda e^{-t\lambda^2}\,d\lambda =$$

$$= -\frac{1}{n}\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)\left(\sqrt{p}-\frac{1}{ka}\right) + \frac{2t}{n}\left[B_{n+1}(p,\,t)-\left(\frac{a}{2t}-\frac{1}{ka}\right)B_n(p,\,t)\right].\Rightarrow$$

$$B_{n-1}(p,\,t) = -\frac{1}{n}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right) + \frac{2t}{n}\left[B_{n+1}(p,\,t)-\left(\frac{a}{2t}-\frac{1}{ka}\right)B_n(p,\,t)\right]. \tag{12}$$

We substitute the expression (12) into (11):

$$A_{n-1}(p,t) = \exp\left(\frac{a^2}{4t}\right)\left\{-\frac{1}{n}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+\right.$$

$$\left.+\frac{2t}{n}\left[B_{n+1}(p,\,t)-\left(\frac{a}{2t}-\frac{1}{ka}\right)B_n(p,\,t)\right]\right\}. \tag{13}$$

Then

$$A_n(p,\,t) = \exp\left(\frac{a^2}{4t}\right)\left\{-\frac{1}{n+1}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+\right.$$

$$\left.+\frac{2\,t}{n+1}\left[B_{n+2}(p,\,t)+\left(\frac{1}{ka}-\frac{a}{2t}\right)B_{n+1}(p,\,t)\right]\right\}. \tag{14}$$

Substituting the expressions (13) and (14) into (10) we get

$$J_{n-1}(p,\,t) = 2\exp\left(\frac{a^2}{4t}\right)\left\{-\frac{1}{n+1}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+\right.$$

$$+\frac{2\,t}{n+1}\left[B_{n+2}(p,\,t)+\left(\frac{1}{ka}-\frac{a}{2t}\right)B_{n+1}(p,\,t)\right]-\frac{1}{nka}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+$$

$$\left.+\frac{2t}{nka}\left[B_{n+1}(p,\,t)-\left(\frac{a}{2t}-\frac{1}{ka}\right)B_n(p,\,t)\right]\right\}.$$

Or

$$J_{n-1}(p,\,t) = 2\exp\left(\frac{a^2}{4t}\right)\left\{-\frac{nka+n+1}{nka(n+1)}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+\right.$$

$$+\frac{2t}{n+1}B_{n+2}(p,t)+2t\left[\frac{1}{n+1}\left(\frac{1}{ka}-\frac{a}{2t}\right)+\frac{1}{nka}\right]B_{n+1}(p,\,t)+\frac{2t}{n\,k\,a}\left(\frac{1}{ka}-\frac{a}{2t}\right)B_n(p,\,t)\Bigg\}. \tag{15}$$

Then

$$J_n(p,\,t)=2\exp\left(\frac{a^2}{4t}\right)\Bigg\{-\frac{(n+1)ka+n+2}{(n+1)ka(n+2)}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+\frac{2t}{n+2}B_{n+3}(p,t)+$$

$$+2t\left[\frac{1}{n+2}\left(\frac{1}{ka}-\frac{a}{2t}\right)+\frac{1}{(n+1)ka}\right]B_{n+2}(p,\,t)+\frac{2t}{(n+1)k\,a}\left(\frac{1}{ka}-\frac{a}{2t}\right)B_{n+1}(p,\,t)\Bigg\}. \tag{16}$$

Substituting the expressions (15) and (16) into (9) we get:

$$I(p,\,t)=2\exp\left(\frac{a^2}{4t}\right)\Bigg\{-\left(\frac{(n+1)ka+n+2}{(n+1)\,(n+2)ka}+\frac{nka+n+1}{n(n+1)\,(ka)^2}\right)\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+$$

$$+\frac{2t}{n+2}B_{n+3}(p,\,t)+2t\left[\frac{1}{n+2}\left(\frac{1}{ka}-\frac{a}{2t}\right)+\frac{1}{(n+1)ka}+\frac{1}{ka(n+1)}\right]B_{n+2}(p,\,t)+$$

$$+2t\left[\frac{2}{ka(n+1)}\left(\frac{1}{ka}-\frac{a}{2t}\right)+\frac{1}{n(ka)^2}\right]B_{n+1}(p,\,t)+\frac{2t}{n(ka)^2}\left(\frac{1}{ka}-\frac{a}{2t}\right)B_n(p,\,t)\Bigg\}. \tag{17}$$

Substituting the expression (17) into (7) we get $\hat{\nu}_{1,\;part}(p)$:

$$\hat{\nu}_{1,\;part}(p)=-\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}\int_0^{+\infty}f_1(\theta)I(p,\,\theta)\,d\theta, \tag{18}$$

where the function $I(p,\,\theta)$ is defined by formula (17).

We rewrite (18) in the form

$$\hat{\nu}_{1,\;part}(p)=-\int_0^{+\infty}\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}\;I(p,\theta)\;f_1(\theta)\,d\theta. \tag{19}$$

We apply the inverse Laplace transform to (19)

$$\nu_{1,\;part}(t)=\frac{1}{2\pi\,i}\int_{c-i\infty}^{c+i\infty}\hat{\nu}_{1,\;part}(p)\,e^{pt}dp, \tag{20}$$

where the integration is performed along the line $Rep{=}c$, that is parallel to the imaginary axis and is shifted so that all singularities of function $\hat{\nu}_{1,part}$ lie on the left side of it.

Changing in (20) the order of integration θ and t, we get by virtue of the Cauchy residue theorem:

$$\nu_{1,\;part}(t)=-\int_0^{+\infty}f_1(\theta)\sum_{p_r}\underset{p=p_r}{res}\left\{\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}I(p,\;\theta)e^{pt}\right\}d\theta, \tag{21}$$

where p_r is a singular point of a function

$$G(p,\,\theta)=\frac{e^{-a\sqrt{p}+pt}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}I(p,\,\theta), \tag{22}$$

because by virtue of the formula

$$B_r(p,t)=\int_{\sqrt{p}-\frac{a}{2t}}^{+\infty}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^r e^{-t\lambda^2}\,d\lambda.$$

and formula (17) we have

$$\lim_{p\to+\infty}\left\{\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}I(p,\ \theta)e^{pt}\right\}=0,$$

where ∞ is a removable singularity.

Obviously, the singular point of the function (22) (as a function of p) is the point $p=\dfrac{1}{k^2\,a^2}$. We will find the residue of the function $G(p,\,\theta)$ at this point.

We consider the case when $k=1$.

When $k=1$ from (8) we have:

$$I(p,\,\theta)=\int_p^{+\infty}\sqrt{q}\ e^{-q\theta+a\sqrt{q}}dq=2\int_{\sqrt{p}}^{+\infty}x^2e^{x^2\theta+ax}dx=2\exp\left(\frac{a^2}{4\theta}\right)\int_{\sqrt{p}-\frac{a}{2\theta}}^{+\infty}\left(\lambda+\frac{a}{2\theta}\right)^2e^{-\theta\lambda^2}d\lambda=$$

$$=2\exp\left(\frac{a^2}{4\theta}\right)\left[\int_{\sqrt{p}-\frac{a}{2\theta}}^{+\infty}\lambda^2e^{-\theta\lambda^2}d\lambda+\frac{a}{\theta}\int_{\sqrt{p}-\frac{a}{2\theta}}^{+\infty}\lambda\,e^{-\theta\lambda^2}d\lambda+\frac{a^2}{4\theta^2}\int_{\sqrt{p}-\frac{a}{2\theta}}^{+\infty}e^{-\theta\lambda^2}d\lambda\right]=$$

$$=\frac{1}{\theta}\exp\left(\frac{a^2}{4\theta}\right)\left[\left(\sqrt{p}+\frac{a}{2\theta}\right)\exp\left(-\theta\left(\sqrt{p}-\frac{a}{2\theta}\right)^2\right)+\left(1+\frac{a^2}{2\theta}\right)\text{erfc}\left(\sqrt{p}-\frac{a}{2\theta}\right)\right]\Rightarrow$$

$$I(p,\,\theta)=\frac{1}{\theta}\exp\left(\frac{a^2}{4\theta}\right)\left[\left(\sqrt{p}+\frac{a}{2\theta}\right)\exp\left(-\theta\left(\sqrt{p}-\frac{a}{2\theta}\right)^2\right)+\left(1+\frac{a^2}{2\theta}\right)\text{erfc}\left(\sqrt{p}-\frac{a}{2\theta}\right)\right].\tag{23}$$

Then at $p=\dfrac{1}{a^2}$ we obtain from (23):

$$I\left(\frac{1}{a^2};\ \theta\right)=\left(\frac{a}{2\theta^2}+\frac{1}{a\theta}\right)\exp\left(1-\frac{\theta}{a^2}\right)+\left(1+\frac{a^2}{2\theta}\right)\exp\left(\frac{a^2}{4\theta}\right)\text{erfc}\left(\frac{1}{a}-\frac{a}{2\theta}\right).\tag{24}$$

When $k=1$ ($p=\dfrac{1}{a^2}$ is a simple pole) for the function (22) we have in view of (24)

$$\underset{p=\frac{1}{a^2}}{res}\ G(p,\ \theta)=\frac{2}{a}\exp\left(\frac{t}{a^2}-1\right)\cdot I\left(\frac{1}{a^2};\ \theta\right)=$$

$$=\left(\frac{1}{\theta^2}+\frac{1}{a^2\theta}\right)\exp\left(\frac{t-\theta}{a^2}\right)+\left(\frac{2}{a}+\frac{a}{\theta}\right)\exp\left(\frac{t}{a^2}+\frac{a^2}{4\theta}-1\right)\text{erfc}\left(\frac{1}{a}-\frac{a}{2\theta}\right).$$

Then from (21) we get

$$\nu_{1,\,part}(t)=-\int_0^{+\infty}f_1(\theta)\left[\left(\frac{1}{\theta^2}+\frac{2}{a^2\theta}\right)\exp\left(\frac{t-\theta}{a^2}\right)+\right.$$

$$\left.+\left(\frac{2}{a}+\frac{a}{\theta}\right)\exp\left(\frac{t}{a^2}+\frac{a^2}{4\theta}-1\right)\text{erfc}\left(\frac{1}{a}-\frac{a}{2\theta}\right)\right]d\theta.\tag{25}$$

By virtue of replacements (2) from (25) we obtain a particular solution of the initial equation (1)

$$\nu_{part}(t)=-\sqrt{t}\exp\left(-\frac{3t}{4a^2}\right)\int_0^{+\infty}f(\theta)\left[\left(\frac{1}{\theta^{\frac{3}{2}}}+\frac{2}{a^2\sqrt{\theta}}\right)\exp\left(-\frac{3\theta}{4a^2}\right)+\right.$$

$$\left.+\left(\frac{2\sqrt{\theta}}{a}+\frac{a}{\sqrt{\theta}}\right)\exp\left(\frac{a^2}{4\theta}+\frac{\theta}{4a^2}\right)\text{erfc}\left(\frac{1}{a}-\frac{a}{2\theta}\right)\right]d\theta.\tag{26}$$

Thus, the following theorem is proved.

Theorem. The integral equation

$$\nu(t)-\frac{a}{2\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t\tau}\sqrt{t-\tau}}\exp\left(-\frac{t-\tau}{4a^2}\right)\cdot\nu(\tau)d\tau-\frac{1}{a\sqrt{\pi}}\int_0^t\frac{\sqrt{\tau}}{\sqrt{t(t-\tau)}}\exp\left(-\frac{t-\tau}{4a^2}\right)\cdot\nu(\tau)d\tau=f(t)$$

in the weight class of functions

$$\exp\left(-\frac{t}{4a^2}\right)\,\nu(\tau)\,\in L_\infty(0,\ +\infty)$$

at

$$\exp\left(-\frac{t}{4a^2}\right)\,f(t)\,\in L_\infty(0,\ +\infty)$$

has a solution defined by the formula (26).

Remark. Singular homogeneous integral equations were considered in works [2–4]. Their kernels were also «incompressible», but kernels had an another form. In this connection, the weight classes of the solution existence differ from the class of the solution existence for the equation considered in this work. We also note that boundary value problems for a spectrally loaded parabolic equation reduce to this kind of singular integral equations, when the load line moves according to the law $x = t$ [5–10] and problems for essentially loaded equation of heat conduction [11–15].

In works [16, 17] it is shown that the homogeneous Volterra integral equation of the second kind, to which the homogeneous boundary value problem of heat conduction in the degenerating domain is reduced, has a nonzero solution.

In works [18, 19] boundary value problems for heat equation in angular domains with special boundary conditions are studied. The problems are reduced to singular integral equations of Volterra type of the second kind, similar to the equation (1).

A similar kind of integral equation arises in solving the boundary value problems of heat conduction with heat generation, which describe the development of the one-dimensional unsteady heat processes with axial symmetry. More complex equations arise from the model that is based on the system of spherical heat equations in a domain with moving boundary and when studying the Stefan problem [20–23].

To find analytical solutions for classes of transfer problems, special methods or modification of known approaches are needed. Summary of the results accumulated in this area of the analytic theory of the thermal conductivity of solids is given in reviews [24, 25].

References

1 Kosmakova, M.T., Akhmanova, D.M., Iskakov, S.A., Tuleutaeva, Zh.M., & Kasymova, L.Zh. (2019). Solving one pseudo-Volterra integral equation. *Bulletin of the Karaganda University. Mathematics series, 1 (93),* 72–77. DOI: 10.31489/2019M1/72-77.

2 Jenaliyev, M., Amangaliyeva, M., Kosmakova, M., & Ramazanov, M. (2014). About Dirichlet boundary value problem for the heat equation in the infinite angular domain. *Boundary Value Problems, 213,* 1–21. DOI: 10.1186/s13661-014-0213-4.

3 Amangaliyeva, M.M., Dzhenaliev, M.T., Kosmakova, M.T., & Ramazanov M.I. (2015). On one homogeneous problem for the heat equation in an infinite angular domain. *Siberian Mathematical Journal, Vol.56, No. 6*, 982–995. DOI: 10.1134/S0037446615060038.

4 Jenaliyev, M., Amangaliyeva, M., Kosmakova, M., & Ramazanov, M. (2015). On a Volterra equation of the second kind with «incompressible» kernel. *Advances in Difference Equations, 71,* 1–14. DOI: 10.1186/s13662-015-0418-6.

5 Akhmanova, D.M., Dzhenaliev, M.T., & Ramazanov, M.I. (2011). On a particular second kind Volterra integral equation with a spectral parameter. *Siberian Mathematical Journal, 52, 1,* 1–10. DOI: 10.1134 /S0037446606010010.

6 Amangaliyeva, M.M., Akhmanova, D.M., Dzhenaliev, M.T., & Ramazanov, M.I. (2011). Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity. *Differential Equations, 47, 2,* 231–243. DOI: 10.1134/S0012266111020091.

7 Kosmakova, M.T. (2016). On an integral equation of the Dirichlet problem for the heat equation in the degenerating domain. *Bulletin of the Karaganda University. Mathematics series, 1 (81),* 62-67.

8 Dzhenaliev, M.T., & Ramazanov, M.I. (2006). On the boundary value problem for the spectrally loaded heat conduction operator. *Siberian Mathematical Journal, 47, 3,* 433-451. DOI: 10.1007/ s11202-006-0056-z.

9 Dzhenaliev, M.T., & Ramazanov, M.I. (2007a). On a boundary value problem for a spectrally loaded heat operator: I *Differential Equations, 43, 4,* 513-524. DOI: 10.1134/S0012266107040106.

10 Dzhenaliev, M.T., & Ramazanov, M.I. (2007b). On a boundary value problem for a spectrally loaded heat operator: II *Differential Equations, 43, 6,* 806-812. DOI: 10.1134/S0012266107060079.

11 Ramazanov, M.I., Kosmakova, M.T., Romanovsky, V.G., Zhanbusinova, B.H., & Tuleutaeva, Z.M. (2018). Boundary value problems for essentially-loaded parabolic equation. *Bulletin of the Karaganda University. Mathematics series, 4 (92),* 79-86. DOI: 10.31489/2018M4/79-86.

12 Muvasharkhan Jenaliyev, M., & Ramazanov, M. (2016). On a homogeneous parabolic problem in an infinite corner domain. *AIP Conference Proceedings, 1759,* 020085. DOI: 10.1063/1.4959699.

13 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). Uniqueness and non-uniqueness of solutions of the boundary value problems of the heat equation. *AIP Conference Proceedings, 1676,* 020028. DOI: 10.1063/1.4930454.

14 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2014). On the spectrum of Volterra integral equation with the «incompressible» kernel. *AIP Conference Proceedings, 1611,* 127-132. DOI: 10.1063/1.4893816.

15 Kosmakova, M.T., Orumbayeva, N.T., Medeubaev, N.K., & Tuleutaeva, Zh.M. (2018). Problems of Heat Conduction with Different Boundary Conditions in Noncylindrical Domains. *AIP Conference Proceedings, 1997,* UNSP 020071-1. DOI: 10.1063/1.5049065.

16 Akhmanova, D.M., Ramazanov, M.I., & Yergaliyev. M.G. (2018). On an integral equation of the problem of heat conduction with domain boundary moving by law of t = x(2). *Bulletin of the Karaganda University. Mathematics series, 1 (89),* 15-19. DOI: 10.31489/2018M1/15-19.

17 Kosmakova, M.T., Ramazanov, M.I., Tokesheva, A.S., & Khairkulova, A.A. (2016). On the non-uniqueness of solution to the homogeneous boundary value problem for the heat conduction equation in an angular domain. *Bulletin of the Karaganda University. Mathematics series, 4 (84),* 80-87. DOI: 10.31489 /2016M4 /80-87.

18 Jenaliyev, M.T., Iskakov, S.A., & Ramazanov, M.I. (2017). On a parabolic problem in an infinite corner domain. *Bulletin of the Karaganda University. Mathematics series, 1 (85),* 28-35. DOI: 10.31489/2017M1 /28-35.

19 Jenaliyev, M.T., Iskakov, S.A., Ramazanov, M.I., & Tuleutaeva, Z.M. (2018). On the solvability of the first boundary value problem for the loaded equation of heat conduction. *Bulletin of the Karaganda University. Mathematics series, 1 (89),* 33-41.

20 Sarsengeldin, M., Kharin, S., Rayev, Zh., & Khairullin, Y. (2018). Mathematical model of heat transfer in opening electrical contacts with tunnel effect. *Filomat, 32(3),* 1003-1008. DOI: 10.2298/FIL1803003S.

21 Kharin, S.N., Sarsengeldin, M.M., & Nouri, N. (2016). Analytical solution of two-phase spherical Stefan problem by heat polynomials and integral error functions. *AIP Conference Proceedings, 1759,* 020031. DOI: 10.1063/1.4959645.

22 Kharin, S.N. (2015). Mathematical model of electrical contact bouncing. *AIP Conference Proceedings, 1676,* 020019. DOI: 10.1063/14930445.

23 Kharin, S., Nouri, N., & Bizjak, M. (2009). Effect of Vapour Force at the Blow-Open Process in Double-Break Contacts. *Ieee transactions on components and packaging technologies, 32, 1,* 180-190. DOI: 10.1109 /TCAPT.2009.2013986.

24 Kartashov, E.M. (2000). New integral relationships for analytical solutions to parabolic equations in noncylindrical domains. *Doklady mathematics, 62, 2,* 288-292.

25 Kartashov, E.M. (1996). Method of Green functions in solving boundary value problems of the parabolical equation in non-cylindrical regions. *Doklady akademii nauk, 351, 1,* 32-36.

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Біртекті емес псевдо-Вольтерра интегралдық теңдеуі туралы

Мақалада екінші текті біртекті емес псевдо-Вольтерра теңдеуінің шешімі болады ма деген сұрақтар қарастырылды. Теңдеуге сәйкес біртекті теңдеудің шешімі және шешімінің жалғыздығының класы [1] жұмыста табылған. Берілген интегралдық теңдеудің оң жағы мен ізделінді функцияны ауыстыру арқылы, ядросы «қысылған» болмайтын интегралдық теңдеу түріне келді. Алынған теңдеу Лаплас түрлендіруінің көмегімен бірінші ретті қарапайым (сызықты) дифференцалдық теңдеуге алып келеді. Оның шешімі табылды. Біртекті теңдеудің шешімінің көмегімен біртекті емес дифференциалдық теңдеудің дербес шешімінің түрі анықталды (тұрақтыны вариациялау әдісімен). Лапластың кері түрлендіруін қолданып, зерттеліп отырған біртекті емес псевдо-Вольтерра интегралдық теңдеуінің дербес шешімі алынды. Біртекті емес интегралдық теңдеуінің параметрінің мәні $k = 1$ болған жағдайы зерттелді. Интегралдық теңдеудің оң жағы мен шешімі үшін кластары көрсетілді.

Кілт сөздер: біртекті емес псевдо-Вольтерра интегралдық теңдеуі, маңызды шектелген функциялар класы, Лапластың кері түрлендіруі, шегерім.

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Об одном псевдо-Вольтерровом неоднородном интегральном уравнении

В статье исследованы вопросы разрешимости псевдо-Вольтеррового неоднородного интегрального уравнения второго рода. Решение соответствующего однородного уравнения и классы единственности решения найдены в работе [1]. С помощью замен правой части и искомой функции интегральное уравнение сведено к интегральному уравнению, ядро которого не является «сжимаемым». С помощью преобразования Лапласа полученное уравнение сведено к обыкновенному дифференциальному уравнению первого порядка (линейному). Найдено его решение. С помощью решения однородного уравнения определен вид частного решения неоднородного дифференциального уравнения (методом вариации произвольной постоянной). Применением обратного преобразования Лапласа получено частное решение исследуемого псевдо-Вольтеррового неоднородного интегрального уравнения. Рассмотрен и исследован случай неоднородного интегрального уравнения при значении параметра $k = 1$. Указаны классы для правой части и решения интегрального уравнения.

Ключевые слова: псевдо-Вольтеррово неоднородное интегральное уравнение, класс существенно ограниченных функций, обратное преобразование Лапласа, вычет.

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Conditions of coercive solvability of third-order differential equation with unbounded intermediate coefficients

In this paper we study the following equation $-y''' + r(x)\,y'' + q(x)\,y' + s(x)\,y = f(x)$, where the intermediate coefficients r and q do not depend on s. We give the conditions of the coercive solvability for $f \in L_2(-\infty, +\infty)$ of this equation. For the solution y, we obtained the following maximal regularity estimate: $\|y'''\|_2 + \|ry''\|_2 + \|qy'\|_2 + \|sy\|_2 \le C\,\|f\|_2$, where $\|\cdot\|_2$ is the norm of $L_2(-\infty, +\infty)$. This estimate is important for study of the qwasilinear third-order differential equation in $(-\infty, +\infty)$. We investigate some binomial degenerate differential equations and we prove that they are coercive solvable. Here we apply the method of the separability theory for differential operators in a Hilbert space, wich was developed by M. Otelbaev. Using these auxillary statements and some well-known Hardy type weighted integral inequalities, we obtain the desired result. In contrast to the preliminary results, we do not assume that the coefficient s is strict positive, the results are also valid in the case that $s = 0$.

Keywords: differential equation, unbounded coefficients, maximal regularity, separability.

1 Introduction and Main Theorems

We consider the following linear differential equation:

$$L_0 y \equiv -y''' + r(x)\,y'' + q(x)\,y' + s(x)\,y = f(x), \tag{1.1}$$

where $x \in R = (-\infty, +\infty)$ and $f \in L_2 := L_2(R)$. By $C_{loc}^{(j)}(R)$ $(j = 0,\ 1,\ 2,\ \ldots)$ we denote the set of the j-times continuously differentiable functions in the every compact, $C_{loc}^{(0)}(R) = C_{loc}(R)$ is the set of the continuous functions. We assume that $r \in C_{loc}^{(2)}(R)$, $q \in C_{loc}^{(1)}(R)$, $s \in C_{loc}(R)$ in (1.1) are real functions, in general, they are unbounded. We denote by L the closure in L_2 of the operator L_0 defined on the set of three times continuously differentiable functions with compact support $C_0^{(3)}(R)$.

Definition 1.1. The function $y \in L_2$ is called a solution of the equation (1.1), if $y \in D(L)$ and $Ly = f$.

In future, by C, C_1, C_2 and etc. we will denote the positive constants, which, generally speaking, are different in the different places.

Definition 1.2. The solution y of the equation (1.1) is called a maximally L_2-regular, if the following estimate holds:

$$\|y'''\|_2 + \|ry''\|_2 + \|qy'\|_2 + \|sy\|_2 \le C\,\|f\|_2, \tag{1.2}$$

where $\|\cdot\|_2$ is a norm in L_2. The inequality (1.2) is called the maximal L_2-regularity estimate. If (1.2) holds, then the operator L is said to be separable in L_2.

The purpose of this work is to find the sufficient conditions for correct solvability of the equation (1.1) and the fullfilment of the estimate (1.2) for a solution of the equation (1.1). The important examples of the equation (1.1) are the Korteweg-de Vries equation (linearized) and its modifications that describe the wave propagation and are used in the problems of a gas dynamics (see [1] and the references therein), as well as the composite type equations that are used in the hydrodynamics and hydromechanics [2]. Furthermore, the equation (1.1) appear in the case that we apply the Fourier method to the partial differential equations of mathematical physics. The other applications of the third-order differential equations can be seen in [3–6].

The smoothness problems for solutions of the equation (1.1) are of a great interest. The case of the bounded domains and smooth scalar coefficients are well understood and sufficiently well described in the known literature. In the case that the domain is unbounded, although the solution of the odd-order equation (1.1) is local smooth, but it may not belong to the Sobolev spaces. This fact causes some difficulties for study of (1.1).

The estimate (1.2) is very important for the study of singular nonlinear differential equations [7]. The maximal regularity problem for the second-order partial differential equations were investigated by P.C. Kunstmann, W. Arendt, M. Duelli, G. Metafune, D. Pallara, J. Prüss, R. Schnaubelt, A. Rhandi [7–10] in the case that their intermediate coefficients are unbounded, although they were controlled by the potential. This problem for a degenerate second-order differential operator was studied in [11].

The maximal regularity (separability) problem for a singular third-order equation has been investigated, mainly, for the following two-terms equation (see [12-19] and the references therein)

$$Ky = -y''' + s(x)y = F(x)\ (x \in (-\infty, +\infty)). \tag{1.3}$$

In [12–19] were obtained conditions for the continuous invertibility and separability of the operator K in $L_p(R)$ $(1 < p < +\infty)$. However, we can not use their results to investigate of the equation (1.1) with unbounded intermediate coefficients. In general, in the case that the intermediate coefficients have more faster growth, the equations (1.1) and (1.3) are different. For example, the solution of (1.1) belongs to L_2 in the case only that the functions r and q satisfy some additional conditions. The question of maximum regularity for other elliptic and parabolic equations defined in infinite domains has been investigated in many papers [20–35].

In the present paper, we consider the following two cases for the intermediate coefficients r and q of (1.1):
a) the growth of the function r does not depend on q and s;
b) the growth of the function q does not depend on r and s.

For continuous functions p and $v \neq 0$, we denote

$$\alpha_{p,v,j}(x) = \left(\int_0^x |p(t)|^2 dt\right)^{\frac{1}{2}} \cdot \left(\int_x^{+\infty} t^{2j} v^{-2}(t)\, dt\right)^{\frac{1}{2}}, x > 0,$$

$$\beta_{p,v,j}(\tau) = \left(\int_\tau^0 |p(t)|^2 dt\right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^\tau t^{2j} v^{-2}(t)\, dt\right)^{\frac{1}{2}}, \tau < 0,$$

$$\gamma_{p,v,j} = \max\left(\sup_{x>0} \alpha_{p,v,j}(x), \sup_{\tau<0} \beta_{p,v,j}(\tau)\right) (j = 0, 1),$$

Theorem 1.1. Assume that the functions r, q and s satisfy the following conditions:

$$r \in C_{loc}^{(2)}(R),\ |r| \geq 1,\ q \in C_{loc}^{(1)}(R),\ s \in C_{loc}(R), \tag{1.4}$$

$$\gamma_{1,\sqrt{|r|},1} < \infty,\ C_0^{-1} \leq \frac{r(x)}{r(\eta)} \leq C_0,\ \forall x, \eta \in R : |x - \eta| \leq 1,\ C_0 > 1, \tag{1.5}$$

$$\gamma_{q,r,0} < \infty,\ \gamma_{s,r,1} < \infty. \tag{1.6}$$

Then for any right-hand side $f \in L_2$ there exists a unique solution y of the equation (1.1). Moreover, for y the estimate (1.2) holds.

In the theorem the growth of coefficients q and s are controlled by r.

Remark 1.1. The condition $|r| \geq 1$ in (1.4) can be replaced by the inequality $|r| \geq \delta > 0$. To show this statement it is enough to put $x = \sqrt{\delta}t$ in (1.1), where $t \in R$.

The following equation:

$$-y''' + \left(7x^2 + 3\right)^4 y'' + \left(2x^3 - 3x^2 + 1\right) y' + x^3 y = f_1, f_1 \in L_2, \tag{1.7}$$

satisfies (see Example 2.1 below) the conditions of Theorem 1.1, consequently, the equation (1.7) is uniquely solvable, and for its solution y the following estimate holds:

$$\|y'''\|_2 + \left\|\left(7x^2 + 3\right)^4 y''\right\|_2 + \left\|\left(2x^3 - 3x^2 + 1\right) y'\right\|_2 + \left\|x^3 y\right\|_2 \leq C \|f_1\|_2 . \tag{1.8}$$

In the following theorem the growth of r and s are controlled by coefficient q.

Theorem 1.2. Assume that the functions $r \in C_{loc}^{(2)}(R)$, $q \in C_{loc}^{(1)}(R)$ and $s \in C_{loc}(R)$ satisfy the following conditions:

$$q \geq 1,\ \gamma_{1,\sqrt{q},0} < \infty,\ C_1^{-1} \leq \frac{q(x)}{q(\eta)} \leq C_1,\ \forall x, \eta \in R : |x - \eta| \leq 1,\ (C_1 > 1); \tag{1.9}$$

$$\gamma_{s,q,0} < \infty; \tag{1.10}$$

$$2\left(r^2 + 2r'\right) \leq q. \tag{1.11}$$

Then for any $f \in L_2$ there exists a unique solution y of the equation (1.1). Moreover, for y the estimate (1.2) holds.

Remark 1.2. In Theorem 1.2 the condition $q \geq 1$ can be replaced by $q \geq \delta > 0$. To show this statement it is enough to put $x = \delta t$ in the equation (1.1).

The conditions of Theorem 1.2 satisfy the coefficients of the following equation:

$$-y''' + x^3 \cos^2 x^2 y'' + \left[18(1 + x^6)\right] y' + 3x^4 y = f_2 (x) \tag{1.12}$$

(see Example 3.1 below).

2 The case that the coefficient r is growing independently

In this section we investigate the equation (1.1) in the case that the growth of the function r does not depend on q and s. First, we consider the following linear two term differential equation:

$$l_0 y = -y''' + r(x)y'' = h(x), \tag{2.1}$$

where $x \in R$, $h \in L_2$, and $r \in C^{(2)}_{loc}(R)$. We denote by l the closure in L_2 of the operator l_0 defined on the set of three times continuously differentiable functions with compact support $C^{(3)}_0(R)$.

Definition 2.1. The function $y \in L_2$ is called a solution of the equation (2.1), if $y \in D(l)$ and $ly = h$.

The following statement is proved in [36].

Lemma 2.1. Let the function r be a twice continuously differentiable function and it satisfies the following conditions:

$$r \geq \delta > 0, \ \gamma_{1,\sqrt{r},1} < \infty,$$

$$C^{-1} \leq \frac{r(x)}{r(\eta)} \leq C \ \forall x, \eta \in R: \ |x - \eta| \leq 1, \ C > 1.$$

Then for any right — hand side $h \in L_2$ there exists a unique solution y of the equation (2.1). Moreover, for y the following estimate holds (i.e. y is maximally L_2- regular):

$$\|y'''\|_2 + \|ry''\|_2 \leq C_l \|h\|_2 .$$

Proof of Theorem 1.1. We put $x = at$ ($0 > 0$, t is new variable) in the equation (1.1). Then (1.1) become the following form:

$$\tilde{L}_a \tilde{y} = -\tilde{y}''' + a\tilde{r}(t)\tilde{y}'' + a^2 \tilde{q}(t)\tilde{y}' + a^3 \tilde{s}(t)\tilde{y} = \tilde{f}(t), \tag{2.2}$$

where $y(at) = \tilde{y}(t)$, $r(at) = \tilde{r}(t)$, $q(at) = \tilde{q}(t)$, $s(at) = \tilde{s}(t)$, $a^3 f(at) = \tilde{f}(t)$ and $a^3 L_a y = \tilde{L}_a \tilde{y}$. First, we consider the following equation:

$$l_{0a} \tilde{y} = -\tilde{y}''' + a\tilde{r}(t)\tilde{y}'' = \tilde{h}(t). \tag{2.3}$$

We denote by l_a a closure in L_2 of the operator l_{0a} defined in $C^{(3)}_0(R)$. We have $a^{-1}\tilde{r}(t) \geq \delta > 0$. By Remark 1.1 and Lemma 2.1, for any function $\tilde{h} \in L_2$ there exists a unique solution $\tilde{y}$ of the equation (2.3) and for $\tilde{y}$ the following estimate holds:

$$\|\tilde{y}'''\|_2 + \|a\tilde{r}\tilde{y}''\|_2 \leq C_{l_a} \|l_a \tilde{y}\|_2, \forall \tilde{y} \in D(l_a). \tag{2.4}$$

Using (2.4), by Theorem 2.1 in [37] and Lemma 2.1 [11], we have

$$\left\|a^2 \tilde{q}\tilde{y}'\right\|_2 \leq 2a\gamma_{\tilde{q},\tilde{r},0} C_{l_a} \|l_a \tilde{y}\|_2 \tag{2.5}$$

and

$$\left\|a^3 \tilde{s}\tilde{y}\right\|_2 \leq 2a^2 \gamma_{\tilde{s},\tilde{r},1} C_{l_a} \|l_a \tilde{y}\|_2 . \tag{2.6}$$

If we choose

$$a = \left[2\left(\gamma_{\tilde{q},\tilde{r},0} + a\gamma_{\tilde{s},\tilde{r},1}\right) C_{l_a} + 1\right]^{-1},$$

then, by (2.5) and (2.6), the following estimate holds:

$$\left\|a^2 \tilde{q}\tilde{y}'\right\|_2 + \left\|a^3 \tilde{s}\tilde{y}\right\|_2 \leq \theta \|l_a \tilde{y}\|_2 , \tag{2.7}$$

where

$$0 < \theta = \frac{2\left(\gamma_{\tilde{q},\,\tilde{r},\,0} + \gamma_{\tilde{s},\,\tilde{r},\,1}\right) C_{l_a}}{2\left(\gamma_{\tilde{q},\,\tilde{r},\,0} + \gamma_{\tilde{s},\,\tilde{r},\,1}\right) C_{l_a} + 1} < 1.$$

Then by Lemma 2.1 and the well-known perturbation theorem [38; 196], there exists a unique solution of the equation (2.2).

Now, we show the maximal L_2-regularity estimate for a solution of the equation (2.2). By (2.7),

$$\|l_a\tilde{y}\|_2 \le \left\|l_a\tilde{y} + a^2\tilde{q}\tilde{y}' + a^3\tilde{s}\tilde{y}\right\|_2 + \left\|a^2\tilde{q}\tilde{y}' + a^3\tilde{s}\tilde{y}\right\|_2 \le \left\|\tilde{L}_a\tilde{y}\right\|_2 + \theta\,\|l_a\tilde{y}\|_2$$

$(0 < \alpha < 1)$. Consequently

$$\|l_a\tilde{y}\|_2 \le \frac{1}{1-\theta}\left\|\tilde{L}_a\tilde{y}\right\|_2. \tag{2.8}$$

By the estimates (2.4), (2.7) and (2.8),

$$\|\tilde{y}'''\|_2 + \|a\tilde{r}\tilde{y}''\|_2 + \left\|a^2\tilde{q}\tilde{y}'\right\|_2 + \left\|a^3\tilde{s}\tilde{y}\right\|_2 \le \frac{C_{l_a}+\theta}{1-\theta}\left\|\tilde{L}_a\tilde{y}\right\|_2. \tag{2.9}$$

(2.9) is the desired estimate for a solution $\tilde{y}$ of the equation (2.2). By replacing $t = \frac{x}{a}$, we get that there exists a unique solution y of the equation (1.1), moreover, for it the estimate (1.7) holds.

Example 2.1. We consider the following equation

$$-y''' + \left(7x^2+3\right)^4 y'' + \left(2x^3 - 3x^2 + 1\right) y' + x^3 y = f_1(x),$$

where $x \in R$, $f_1(x) \in L_2$. Here, $r = \left(7x^2+3\right)^4$, $q = 2x^3 - 3x^2 + 1$ and $s = x^3$. The intermediate coefficients r and q satisfy conditions (1.4), (1.5), and (1.6) of Theorem 1.1. In fact, since the function $\left(7x^2+3\right)^4$ is even, for any $x > 0$

$$\alpha_{1,\,\sqrt{r},\,1}(x) = \beta_{1,\,\sqrt{r},\,1}(-x) \le \sqrt{x}\left(\int_x^{+\infty}\frac{dt}{(7t^2+3)^4}\right)^{\frac{1}{2}} \le \frac{\sqrt{x}}{(7x^2+3)^3}\left(\int_x^{+\infty}\frac{dt}{7t^2+3}\right)^{\frac{1}{2}} < \infty.$$

Analogously, we obtain

$$\alpha_{q,\,r,\,0}(x) = \beta_{q,\,r,\,0}(-x) \le \left(\int_0^x 3\left(4t^6 + 9t^4 + 1\right)dt\right)^{\frac{1}{2}}\left(\int_x^{+\infty}\frac{dt}{(7t^2+3)^8}\right)^{\frac{1}{2}} \le$$

$$\le \frac{\left(\frac{12}{7}x^7 + \frac{9}{5}x^5 + x\right)^{\frac{1}{2}}}{\left(7x^2+3\right)^7}\left(\int_x^{+\infty}\frac{dt}{7t^2+3}\right)^{\frac{1}{2}} < \infty$$

and

$$\alpha_{s,\,r,\,1}(x) = \beta_{s,\,r,\,1}(-x) \le \left(\int_0^x \left|t^3\right|^2 dt\right)^{\frac{1}{2}}\left(\int_x^{+\infty} t^2\left(7t^2+3\right)^{-8}dt\right)^{\frac{1}{2}} < \infty.$$

For any $x,\,\eta \in R$ such that $|x-\eta| \le 1$

$$\frac{\left(7x^2+3\right)^4}{\left(7\eta^2+3\right)^4} \le \frac{\left[6\left(7\eta^2+3\right)\right]^4}{\left(7\eta^2+3\right)^4} = 1296.$$

So, by Theorem 1.1, for any $f_1 \in L_2$ there exists a unique solution y of the equation (1.7) and for it the estimate (1.8) holds.

3 The case that the coefficient q is growing independently

In this section we consider the equation (1.1) in the case that the function q is fast growing function. First, we consider the following differential equation:

$$\tilde{l}_0 y = -y''' + q\,(x)\,y' = u\,(x),\ x \in R,\ u \in L_2. \tag{3.1}$$

We denote by $\tilde{l}$ a closure in L_2 of the operator $\tilde{l}_0 y = -y''' + q\,(x)\,y'$ defined in $C_0^{(3)}(R)$. The element $y \in D(\tilde{l})$ such as $\tilde{l}y = u$, is called a solution of the equation (3.1).

Lemma 3.1. If $q\,(x)$ is continuously differentiable function such as

$$q \geq 1, \gamma_{1,\sqrt{q},0} < \infty, \tag{3.2}$$

then for any $u \in L_2$ there exists a unique solution y of the equation (3.1). Moreover, for y the following estimate holds:

$$\|\sqrt{q}y'\|_2 + \|y\|_2 \leq C \left\|\tilde{l}y\right\|_2. \tag{3.3}$$

Proof. Let $y \in C_0^{(3)}(R)$. Integrating by parts, we have

$$\left(\tilde{l}_0 y, y'\right) = \|y''\|_2^2 + \|\sqrt{q}y'\|_2^2. \tag{3.4}$$

Taking into account the condition (3.2), by the Holder inequality, we obtain

$$\left|\left(\tilde{l}_0 y, y'\right)\right| \leq \left\|\frac{1}{\sqrt{q}}\tilde{l}_0 y\right\|_2 \|\sqrt{q}y'\|_2.$$

Then by (3.2) and (3.4),

$$\|y\|_2 \leq 2\gamma_{1,\sqrt{q},0}\,\|\sqrt{q}y'\|_2 \leq 2\gamma_{1,\sqrt{q},0}\left\|\frac{1}{\sqrt{q}}\tilde{l}_0 y\right\|_2$$

and

$$\|y\|_2 + \|\sqrt{q}y'\|_2 \leq \left[2\gamma_{1,\sqrt{q},0} + 1\right] \left\|\tilde{l}_0 y\right\|_2,\ y \in C_0^{(3)}(R). \tag{3.5}$$

Further, we show that the estimate (3.5) holds for any $y \in D(l)$. Let $\{y_n\}_{n=1}^{\infty} \subset C_0^{(3)}(R)$ such sequence that

$$\|y_n - y\|_2 \to 0, \left\|\tilde{l}_0 y_n - ly\right\|_2 \to 0\ (n \to \infty). \tag{3.6}$$

By (3.5), for any $y_n,\, y_m \in C_0^{(3)}(R)$

$$\|y_n\|_2 + \|\sqrt{q}y_n'\|_2 \leq \left[2\gamma_{1,\sqrt{q},0} + 1\right] \left\|\tilde{l}_0 y_n\right\|_2 \tag{3.7}$$

and

$$\|y_n - y_m\|_2 + \|\sqrt{q}\,(y_n' - y_m')\|_2 \leq \left[2\gamma_{1,\sqrt{q},0} + 1\right] \left\|\tilde{l}_0 y_n - \tilde{l}_0 y_m\right\|_2. \tag{3.8}$$

We denote by $W^1_{2,\sqrt{q}}(R)$ the completion of $C_0^{(3)}\,(R)$ with respect to the norm $\|y\|_W = \left\|\sqrt{q}\,y'\right\|_2 + \|y\|_2$. According to (3.8), $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $W^1_{2,\sqrt{q}}(R)$. $W^1_{2,\sqrt{q}}(R)$ is a Banach space, therefore there exists an element z such as $\|y_n - z\|_W \to 0\ (n \to \infty)$. Then by (3.6), $z \in D(l)$, furthermore, z is a solution of (3.1). Passing to the limit at $n \to \infty$ in (3.7), we obtain the inequality (3.3) for z with $C = 2\gamma_{1,\sqrt{q},0} + 1$.

By (3.3) and Definition 2.1, there exists the inverse $\tilde{l}^{-1}$ to the operator $\tilde{l}$. So, a solution of the equation (3.1) is unique.

We show, that for any $u \in L_2$ there exists a solution of the equation (3.1). By Definition 2.1, it is sufficient to prove that $R(\tilde{l}) = L_2$. Assume the contrary, let $R(\tilde{l}) \neq L_2$. Then there exists the non-zero element $z\,(x) \in R(\tilde{l})^{\perp}$: $\left(\tilde{l}y, z\right) = 0$ for any $y \in C_0^{(3)}\,(R)$. On the other hand

$$\left(\tilde{l}y,\, z\right) = \int_R y\left(\bar{z}''' - \left[q\,(x)\,\bar{z}\right]'\right) dx,\ \forall y \in C_0^{(3)}\,(R).$$

$C_0^{(3)}(R)$ is dense in L_2, so we have

$$z'' - qz = C_1. \tag{3.9}$$

From (3.9), taking into account that $q(x) \in C_{loc}^{(1)}(R)$, we have that $z(x) \in C_{loc}^{(3)}(R)$. We consider two cases with respect to C_1.

1. $C_1 \neq 0$. Then, we can assume, that $C_1 = 1$:

$$z'' - q(x)z = 1, x \in R. \tag{3.10}$$

The general solution z of this equation belongs to $C_{loc}^{(3)}(R)$ and is represented in the following form:

$$z(x) = C_2 z_1(x) + C_3 z_2(x) + \int_{-\infty}^{+\infty} G(x,t)\, dt,$$

where $z_1(x)$ and $z_2(x)$ are two linearly independent solutions of the homogeneous equation $z'' - q(x)z = 0$ and

$$G(x,t) = \begin{cases} z_1(x) z_2(t), & x \le t, \\ z_2(x) z_1(t), & x > t \end{cases}$$

is the Green function of the Sturm-Liouville operator. It is known that $z_1(x) > 0$ and $z_2(x) > 0$. By well-known comparison theorem and maximum principle, for any $x \in R$ the following estimates hold:

$$\begin{cases} z_1(x) \ge K^{-1} e^x, \ 0 < z_2 \le K e^x, \ x > 0, \\ z_2(x) \ge K^{-1} e^x, \ 0 < z_1 \le K e^x, \ x < 0, \\ z_1'(x) > 0, \ z_2'(x) < 0, \end{cases}$$

hence $0 < G(x,t) \le C_4 e^{-|x-t|}$. By condition $z \in L_2$, we obtain $C_2 = 0$ and $C_3 = 0$. So,

$$z(x) = \int_{-\infty}^{+\infty} G(x,t)\, dt > 0.$$

By (3.10), $z'' = 1 + q(x)z \ge 1$. Let $a \in R$ such as $z(a) = k > 0$ and $z'(a) = m > 0$. By (3.2) and (3.10),

$$z(x) - k = m(x-a) + \frac{(x-a)^2}{2} + \int_a^x \left(\int_a^t qz(s)\, ds \right) dt \ge \frac{(x-a)^2}{2} \quad \forall x > a.$$

So, $z \notin L_2$.

2. Let $C_1 = 0$. Then the solution z of the equation (3.9) is represented as follows:

$$z(x) = C_4 z_1(x) + C_5 z_2(x), \ x \in R.$$

As mentioned above, $z_1(x) \to +\infty$, $z_2(x) \to 0$ $(x \to +\infty)$, and $z_2(x) \to +\infty$, $z_1(x) \to 0$ $(x \to -\infty)$. We have $C_4 = 0$ and $C_5 = 0$. So $z(x) = 0, \ x \in R$.

We have obtained contradictions, which show that $R(\tilde{l}) = L_2$.

Lemma 3.2. Assume that the function q satisfies conditions of Lemma 3.1 and

$$C_0^{-1} \le \frac{q(x)}{q(\eta)} \le C_0 \ \ \forall x, \eta \in R : |x - \eta| \le 1 \ (C_0 > 1). \tag{3.11}$$

Then for the solution y of the equation (3.1) the following estimate holds:

$$\|y'''\|_2 + \|qy'\|_2 \le C \left\| \tilde{l} y \right\|_2. \tag{3.12}$$

Proof. Let y be a solution of the equation (3.1). By (3.3), $y' \in L_2$. Assume, that $y' = z$, then we obtain the following Sturm-Liouville equation:

$$\Im z = -z'' + q(x) z = \tilde{u}(x).$$

By conditions of Lemma, the solution z of the last equation satisfies the following estimate [39; 199]:

$$\|z''\|_2 + \|qz\|_2 \le C \|\tilde{u}\|_2.$$

Then for the solution y of (3.1), we obtain the estimate (3.12).

Next, we will consider the following equation

$$y''' + q(x)y' + s(x)y = u_0(x). \tag{3.13}$$

Lemma 3.3. Let $q(x)$ be a continuously differentiable function, and $s(x)$ be a continuous function. Assume that the conditions (3.2) and (3.11) and the following condition

$$\gamma_{s,q,0} < \infty \tag{3.14}$$

hold. Then for any $u_0 \in L_2$ there exists the unique solution y of the equation (3.14). Moreover, y satisfies the following estimate:

$$\|y'''\|_2 + \|qy'\|_2 + \|sy\|_2 \le C\,\|u\|_2\,. \tag{3.15}$$

Proof. In (3.13) we put $x = at$, where $a > 0$ and $t \in R$. Then

$$a^3\tilde{l}y = -y'''_{ttt}(at) + a^2 q(at)\, y'_t(at) + a^3 s(at)\, y(at) = a^3 u(at).$$

If we introduce the notations

$$y(at) = \tilde{y}(t)\,,\ q(at) = \tilde{q}(t)\,,\ s(at) = \tilde{s}(t)\,,\ a^3 u(at) = \tilde{u}(t)\, a^3\tilde{l}y = \tilde{\tilde{l}}\tilde{y},$$

then (3.13) become the following form:

$$\tilde{\tilde{l}}\tilde{y} = -\tilde{y}''' + a^2\tilde{q}\tilde{y}' + a^3\tilde{s}\tilde{y} = \tilde{u}. \tag{3.16}$$

We denote by $\tilde{\tilde{l}}_a$ a closure in L_2 of the differential expression $\tilde{\tilde{l}}_{0a}\tilde{y} = -\tilde{y}''' + a^2\tilde{q}\tilde{y}'$ defined on $C_0^{(3)}(R)$. Since $a^2\tilde{q}(t) \ge \delta > 0$, by Lemma 3.1 and Remark 1.1, the operator $\tilde{\tilde{l}}_a$ is continuously invertible and the following estimate holds:

$$\|\tilde{y}'''\|_2 + \left\|a^2\tilde{q}\tilde{y}'\right\|_2 \le C_{\tilde{\tilde{l}}_a}\left\|\tilde{\tilde{l}}_a\tilde{y}\right\|_2,\ \ \forall \tilde{y} \in D\left(\tilde{\tilde{l}}_a\right). \tag{3.17}$$

Taking into account the condition (3.14) and Lemma 3.1, we have

$$\left\|a^3\tilde{s}\tilde{y}\right\|_2 \le a^{-1}\gamma_{\tilde{s},\tilde{q},0} C_{\tilde{\tilde{l}}_a}\left\|\tilde{\tilde{l}}_a\tilde{y}\right\|_2.$$

By (3.16), $\tilde{\tilde{l}} = \tilde{\tilde{l}}_a + a^3\tilde{s}E$. Choosing the number a such as $a = 2C_{\tilde{\tilde{l}}_a}\left(1 + \gamma_{\tilde{s},\tilde{q},0}\right)$, we obtain

$$\left\|a^3\tilde{s}\tilde{y}\right\|_2 \le \alpha\left\|\tilde{\tilde{l}}_a\tilde{y}\right\|_2,\ 0 < \alpha \le \frac{1}{2}. \tag{3.18}$$

Then, by the well-known perturbation theorem (for example, see Theorem 1.16 [38; 196]), there exists the inverse operator $\left(\tilde{\tilde{l}}_a + a^3\tilde{s}E\right)^{-1}$ and the equality $R\left(\tilde{\tilde{l}}_a + a^3\tilde{s}E\right) = L_2$ is true. By estimates (3.17) and (3.18),

$$\|\tilde{y}'''\|_2 + \left\|a^2\tilde{q}\tilde{y}'\right\|_2 + \left\|a^3\tilde{s}\tilde{y}\right\|_2 \le \left(C_{\tilde{\tilde{l}}_a} + \frac{1}{2}\right)\left\|\tilde{\tilde{l}}_a\tilde{y}\right\|_2. \tag{3.19}$$

On the other hand, by (3.18),

$$\left\|\tilde{\tilde{l}}_a\tilde{y}\right\|_2 \le \left\|\left(\tilde{\tilde{l}}_a + a^3\tilde{s}E\right)\tilde{y}\right\|_2 + \frac{1}{2}\left\|\tilde{\tilde{l}}_a\tilde{y}\right\|_2,$$

i.e.

$$\left\|\tilde{\tilde{l}}_a\tilde{y}\right\|_2 \le 2\left\|\left(\tilde{\tilde{l}}_a + a^3\tilde{s}E\right)\tilde{y}\right\|_2. \tag{3.20}$$

The estimates (3.19) and (3.20) imply

$$\|\tilde{y}'''\|_2 + \left\|a^2\tilde{q}\tilde{y}'\right\|_2 + \left\|a^3\tilde{s}\tilde{y}\right\|_2 \le C\,\|\tilde{u}\|_2\,,\ C = 2\left(C_{\tilde{\tilde{l}}_a} + \frac{1}{2}\right).$$

By replacing $t = a^{-1}x$, we obtain the estimate (3.15).

Proof of Theorem 1.2. If the conditions (1.9) and (1.10) hold, then by Lemma 3.3, the operator

$$\tilde{l}y = -y''' + q(x)\,y' + s(x)\,y$$

is continuously invertible, and for any $y \in D(\tilde{l})$ the following estimate holds:

$$\|y'''\|_2 + \|qy'\|_2 + \|sy\|_2 \le C_{\tilde{l}} \left\|\tilde{l}y\right\|_2. \tag{3.21}$$

Taking into account the condition (1.11), for any $y \in C_0^{(3)}(R)$ we obtain

$$\|ry''\|_2^2 \le \frac{1}{4}\|y'''\|_2^2 + \left\|r^2y'\right\|_2^2 + \|2r'y'\|_2^2 + \frac{1}{4}\|ry''\|_2^2.$$

Then, by (3.15),

$$\|ry''\|_2^2 \le \frac{1}{3}\left(\|y'''\|_2^2 + \left\|2\left(r^2+2r'\right)y'\right\|_2^2 + \|sy\|_2^2\right) \le \frac{1}{3}\left\|\tilde{\tilde{l}}y\right\|_2^2.$$

so,

$$\|ry''\|_2 \le \frac{1}{\sqrt{3}}\left\|\tilde{\tilde{l}}y\right\|_2. \tag{3.22}$$

It is clear that this inequality holds for any $y \in D(\tilde{\tilde{l}})$. Then by Theorem 1.16 [38; 196] the operator $Ly = \tilde{\tilde{l}}y + ry''$ is closed and invertible, and its inverse L^{-1} is defined in all of L_2. By (3.22), for any $y \in C_0^{(3)}(R)$

$$\left\|\tilde{l}y\right\|_2 \le \frac{\sqrt{3}}{\sqrt{3}-1}\|Ly\|_2.$$

Then, by (3.21) and (3.22), for any $y \in C_0^{(3)}(R)$ holds the estimate (1.2), where $C_L = \frac{1+\sqrt{3}\,C_L}{\sqrt{3}-1}$. Taking into account that the operator L is closed, we obtain that the last estimate holds for a solution of the equation (1.1).

Example 3.1. We consider the following equation

$$-y''' + x^3\cos^2 x^2 y'' + \left[18(1+x^6)\right]y' + 3x^4y = f_2(x),\ x \in R,\ f_2(x) \in L_2.$$

The coefficients $r = x^3\cos^2 x^2$, $q = 18(1+x^6)$ and $s = 3x^4$ of this equation satisfy all of the conditions of Theorem 1.2. In fact,

$$\alpha_{1,\sqrt{q},0}(x) = \beta_{1,\sqrt{q},0}(-x) \le \sqrt{x}\left(\int_x^{+\infty}\frac{dt}{18(t^6+1)}\right)^{\frac{1}{2}} < \infty,\ x > 0.$$

For any $x, \eta \in R$ such that $|x-\eta| \le 1$ we obtain

$$\frac{18(1+x^6)}{18(1+\eta^6)} \le \frac{1+(1+\eta)^6}{1+\eta^6} < \infty.$$

Further,

$$\alpha_{s,q,0}(x) = \beta_{s,q,0}(-x) \le x^{\frac{9}{2}}\left(\int_x^{+\infty}\frac{dt}{18^2(t^6+1)^2}\right)^{\frac{1}{2}} < \infty,\ x > 0.$$

$$2(r^2+2r') = 2(x^6 + 8\,|x|^4 + 1) \le 18(x^6+1) = q.$$

So, by Theorem 1.2, for any $f_2 \in L_2$ there exists the unique solution y of the equation (1.12), and for y the following estimate holds:

$$\|-y'''\|_2 + \|x^3\cos^2 x^2y''\|_2 + \|\left[18(1+x^6)\right]y'\|_2 + \|3x^4y\|_2 \le C\|f_2\|_2.$$

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References

1 Galaktionov V.A. Ordered invariant sets for KdV-type nonlinear evolution equations/ V.A. Galaktionov // Computational Mathematics and Mathematical Physics. — 1999. — Vol. 39, № 9. — P. 1564–1570.

2 Kozhanov A.I. Composite type equations and inverse problems / A.I. Kozhanov // VSP, Utrecht, the Netherlands. — 1999.

3 Maxworthy T.A Solitary wave theory of the great red spot and other observed features in the Jovian atmosphere / T. Maxworthy, L.G. Redekopp // Icarus. — 1976. — Vol. 29, Issue 2. — P. 261–271. DOI: 10.1016/0019-1035(76)90056-7.

4 Perel'man T.L. A modified Korteweg-de Vries equation in electrohydrodynamics / T.L. Perel'man, A.X. Fridman, M.M. El'yashevich // Journal of Experimental and Theoretical Physics. — 1974. — Vol. 66. — P. 1316–1323.

5 Liu C.H. Propagation of pulse trains through a random / C.H. Liu, A.W. Weznik, K.C. Yeh // Ieee Transactions on Antennas and Propagation. — 1974. — Vol. 22, Issue 4. — P. 184–187. DOI: 10.1109/TAP. 1974.1140830.

6 Taimanov I.A. Finite-gap solutions of the modified novikov-veselov equations: Their spectral properties and applications / I.A. Taimanov // Siberian Mathematical Journal. — 1999. — Vol. 40, № 6. — P. 1146–1156. DOI: 10.1007/BF02677539.

7 Kunstmann P.C. Maximal L_p-regularity for Parabolic Equations, Fourier Multiplier Theorems and H^∞-functional Calculus / P.C. Kunstmann, L. Weis // Functional Analytic Methods for Evolution Equations. Lecture Notes in Mathematics, 1855. — Springer-Verlag, Berlin, Heidelberg, 2004. — P. 65–311. DOI: 10.1007/978-3-540-44653-8-2.

8 Arendt W. Maximal L_p-regularity for parabolic and elliptic equations on the line / W. Arendt, M. Duelli // Journal of Evolution Equations. — 2006. — Vol. 6, Issue 4. — P. 773–790. DOI: 10.1007/s00028-006-0292-5.

9 Metafune G. L_p-theory for elliptic operators on R^d with singular coefficients / G. Metafune, D. Pallara, J. Prüss, R. Schnaubelt // Z. Anal. Anwendungen. Journal for Analysis and its Applications. — 2005. — Vol. 24, № 3. — P. 497–521.

10 Metafune G. L_p-regularity for elliptic operators with unbounded coefficients / G. Metafune, J. Prüss, R. Schnaubelt, A. Rhandi // Advances in Differential Equations. — 2005. — Vol. 10, No. 10. — P. 1131–1164.

11 Ospanov K. Separation and the existence for second order nonlinear differential equation / K. Ospanov, R. Akhmetkaliyeva // Electronic Journal of Qualitative Theory of Differential Equations. — 2012, No. 66. — P. 1–12.

12 Аманова Т.Т. О разделимости одного дифференциального оператора / Аманова Т.Т. // Известия АН КазССР. Сер. физ.-мат. — 1981. — № 3. — С. 48–51.

13 Аманова Т.Т. Гладкость решения одного нелинейного дифференциального уравнения / Т.Т. Аманова, М.В. Муратбеков // Известия АН КазССР. Сер. физ.-мат. — 1983, № 5. — С. 4–7.

14 Биргебаев А. О разделимости нелинейного дифференциального оператора третьего порядка / А. Биргебаев, М. Отелбаев // Известия АН КазССР. Сер. физ.-мат. — 1984, № 3. — С. 11– 13.

15 Тогочуев А.Ж. О суммируемости решений дифференциальных уравнений нечетного порядка с весом / А.Ж. Тогочуев // Изв. АН КазССР. Сер. физ.-мат. — 1985, № 5. — С. 55–58.

16 Сапенов М. О суммируемости с весом решений двучленных дифференциальных уравнений / М. Сапенов, L.A. Shuster // Изв. АН КазССР. Сер. физ.-мат. — 1987, № 1. — С. 38–42.

17 Айткожа Ж.Ж. О разрешимости одного класса нелинейных сингулярных уравнений третьего порядка / Ж.Ж. Айткожа, М.В. Муратбеков, К.Н. Оспанов // Вестн. ЕНУ им. Л.Н. Гумилева. — 2005, № 6 (46). — С. 10–15.

18 Muratbekov M.B. Coercive solvability of odd-order differential equations and its applications / M.B. Muratbekov, M.M. Muratbekov, K.N. Ospanov // Doklady Mathematics. — 2010. — Vol. 82, No. 3. — P. 909–911. DOI: 10.1134/S1064562410060189.

19 Akhmetkaliyeva R.D. Some new results concerning a class of third order differential equations / R.D. Akhmetkaliyeva, L.E. Persson, K. Ospanov, P. Woll // Applicable Analysis: An International Journal. — 2015. — Vol. 94, Issue 2. — P. 419–434. DOI: 10.1080/00036811.2014.898375.

20 Albanese A. L_p-uniqueness for elliptic operators with unbounded coefficients in R^N / A. Albanese, L. Lorenzi, E.M. Mangino // Journal of Functional Analysis. — 2009. — Vol. 256, No. 4. — P. 1238–1257. DOI: 10.1016/j.jfa.2008.07.022.

21 Albanese A.A. Cores for Feller semigroups with an invariant measure / A.A. Albanese, E.M. Mangino // Journal of Differential Equations. — 2008. — Vol. 244, No. 11. — P. 2980–2982. DOI: 10.1016/j.jde.2005.09.014.

22 Da Prato G. On a class of self-adjoint elliptic operators in L_2 spaces with respect to invariant measure / G. Da Prato, A. Lunardi // Journal of Differential Equations. — 2007. — Vol. 234, No. 1. — P. 54–79. DOI: 10.1016/j.jde.2006.11.017.

23 Da Prato G. Maximal L_p regularity for elliptic equations with unbounded coefficients / G. Da Prato, V. Vespri // Journal of Differential Equations. — 2002. — Vol. 49, No. 6. — P. 747–755. DOI: 10.1016/S0362-546X(01)00133-X.

24 Hieber M. Global properties of generalized Ornstein–Uhlenbeck operators on $L_p(R^N, R^N)$ with more than linearly growing coefficients / M. Hieber, L. Lorenzi, J. Pruss, A. Rhandi, R. Schnaubelt // Journal of Mathematical Analysis and Applications. — 2009. — Vol. 350, No. 1. — P. 100–121. DOI: 10.1016/j.jmaa.2008.09.011

25 Lunardi A. On the Ornstein–Uhlenbeck operator in L_2 spaces with respect to invariant measures / A. Lunardi // Transactions of the American Mathematical Society. — 1997. — Vol. 349, No. 1. — P. 155–169.

26 Ospanov K.N. Qualitative and approximate characteristics of solutions of Beltrami-type systems / K.N. Ospanov // Complex variables and elliptic equations. — 2015. — Vol. 60, Issue 7. — P. 1005–1014. DOI: 10.1080/17476933.2015.1025066.

27 Ospanov K.N. On the nonlinear generalized Cauchy-Riemann system on the whole plane / K.N. Ospanov // Siberian Mathematical Journal. — 1997. — Vol. 38, Issue 2. — P. 314–319. DOI: 10.1007/BF02674630.

28 Ospanov K. L1-maximal regularity for quasilinear second order differential equation with damped term / K. Ospanov // Electronic Journal of Qualitative Theory of Differential Equations. — 2015. — Vol. 39. — P. 1–9.

29 Wu L.M. A new topological approach to the L^∞-uniqueness of operators and the L^1-uniqueness of Fokker–Planck equations/ L.M. Wu, Y. Zhang // Journal of Functional Analysis. — 2006. — Vol. 241, No. 2. — P. 557–610. DOI: 10.1016/j.jfa.2006.04.020.

30 Metafune G. L_p-spectrum of Ornstein–Uhlenbeck operators / G. Metafune // Annali della Scuola normale superiore di Pisa, Classe di scienze. — 2001. — Vol. 30, No. 1. — P. 97–124.

31 Metafune G. Long time behavior of heat kernels of operators with unbounded drift terms / G. Metafune, E.M. Ouhabaz, D. Pallara // Journal of Mathematical Analysis and Applications. — 2011. — Vol. 377, No. 1. — P. 170–179. DOI: 10.1016/j.jmaa.2010.10.023.

32 Metafune G. Spectrum of Ornstein–Uhlenbeck operators in L_p spaces with respect to invariant measures / G. Metafune, D. Pallara, E. Priola // Journal of Functional Analysis. — 2002. — Vol. 196, No. 1. — P. 40–60. DOI: 10.1006/jfan.2002.3978.

33 Metafune G. The domain of the Ornstein–Uhlenbeck operator on an L_p-space with invariant measure / G. Metafune, J. Pruss, A. Rhandi, R. Schnaubelt // Annali della Scuola Normale Superiore di Pisa - Classe di Scienze. — 2002. — Vol. 1, No. 2. — P. 471–485.

34 Ospanov K.N. On Coercive Solvability of a Third-Order Differential Equation with Oscillating Intermediate Coefficient / K.N. Ospanov, Zh.B. Yeskabylova // AIP Conference Proceedings. — 2018. — Vol.1997. — P. 020055-1-020055-4. DOI: 10.1063/1.5049049.

35 Ospanov K.N. On smoothness property of third-order differential operator / K.N. Ospanov, Zh.B. Yeskabylova // AIP Conference Proceedings. — 2017. — Vol. 1880. — P. 040009-1-040009-6. DOI: 10.1063/1.5000625.

36 Оспанов К.Н. Үшінші ретті бір нұқсанды дифференциалдық оператордың анықталу облысын сипаттау / К.Н. Оспанов, Ж.Б. Ескабылова // Л.Н. Гумилев атындағы ЕҰУ хабаршысы, I-б. — 2017. — Т. 121, № 6. — 30–36-б.

37 Apyshev O.D. On the spectrum of a class of differential operators and some imbedding theorems / O.D. Apyshev, M. Otelbaev // Mathematics of the USSR-Izvestiya. — 1980. — Vol. 15, No. 1. — P. 739–764. DOI: 10.1070/IM1980v015n01ABEH001190.

38 Kato T. Perturbation theory for linear operators / T. Kato. — Springer, 1995.

39 Мынбаев К.Т. Весовые функциональные пространства и спектр дифференциальных операторов / К.Т. Мынбаев, М. Отелбаев. — М.: Наука, 1990.

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Аралық коэффициенттері шенелмеген үшінші ретті дифференциалдық теңдеудің коэрцитивті шешілу шарттары

Мақалада келесі теңдеу қарастырылған: $-y''' + r(x)\,y'' + q(x)\,y' + s(x)\,y = f(x)$, мұндағы r және q — аралық коэффициенттер, s-қа бағынбайды. Осы теңдеудің $f \in L_2(-\infty, +\infty)$ үшін коэрцитивті шешілу шарттары келтірілген. Және y шешім үшіні келесі максималды регулярлық баға алынған: $\|y'''\|_2 + \|ry''\|_2 + \|qy'\|_2 + \|sy\|_2 \leq C\,\|f\|_2$,- мұндағы $\|\cdot\|_2$ — $L_2(-\infty, +\infty)$-дегі норма. Бұл бағалау $(-\infty, +\infty)$ аралығындағы үшінші ретті квазисызықты дифференциалдық теңдеуді зерттеуде маңызды рөл атқарады. Кейбір екімүшелі нұқсанды дифференциалдық теңдеулер қарастырылып, олардың коэрцитивті шешілуі дәлелденді. Бұл жерде М. Өтелбаев жасаған Гильберт кеңістігіндегі дифференциалдық оператордың бөліктену теориясы әдісі қолданылды. Осы көмекші тұжырымдарды және кейбір белгілі Харди типті салмақты интегралдық теңсіздіктер арқылы қажетті нәтижеге қолжеткізілді. Осыған дейін алынған нәтижелермен салыстырғанда авторлар s потенциалы қатаң оң деп ұйғарып, нәтижелері $s = 0$ жағдайы үшін де орынды деп қорытынды жасады.

Кілт сөздер: дифференциалдық теңдеу, шенелмеген коэффициенттер, максималды регулярлы, бөліктену.

К.Н. Оспанов, Ж.Б. Ескабылова

Условия коэрцитивной разрешимости дифференциального уравнения третьего порядка с неограниченными промежуточными коэффициентами

В статье рассмотрено уравнение: $-y''' + r(x)\,y'' + q(x)\,y' + s(x)\,y = f(x)$, в котором промежуточные коэффициенты r и q не зависят от s. Приведены условия коэрцитивной разрешимости этого уравнения для $f \in L_2(-\infty, +\infty)$. Для решения y получена следующая оценка максимальной регулярности: $\|y'''\|_2 + \|ry''\|_2 + \|qy'\|_2 + \|sy\|_2 \leq C\,\|f\|_2$, где $\|\cdot\|_2$ — норма в $L_2(-\infty, +\infty)$. Эта оценка важна для изучения квазилинейного дифференциального уравнения третьего порядка в $(-\infty, +\infty)$. Исследованы некоторые двучленные вырожденные дифференциальные уравнения и доказаны, что они являются коэрцитивно разрешимыми. Здесь применен метод теории разделимости дифференциальных операторов в гильбертовом пространстве, разработанный М. Отелбаевым. С помощью этих вспомогательных утверждений и некоторых известных весовых интегральных неравенств типа Харди получен желаемый результат. В отличие от предварительных результатов, авторы предполагают, что потенциал s является строго положительным, результаты также справедливы в случае, когда $s = 0$.

Ключевые слова: дифференциальное уравнение, неограниченные коэффициенты, максимальная регулярность, разделимость.

References

1 Galaktionov, V.A. (1999). Ordered invariant sets for KdV-type nonlinear evolution equations. *Computational Mathematics and Mathematical Physics, Vol. 39, No. 9,* 1564–1570.

2 Kozhanov, A.I. (1999). Composite type equations and inverse problems. *VSP, Utrecht, the Netherlands.*

3 Maxworthy, T., & Redekopp, L.G. (1976). A solitary wave theory of the great red spot and other observed features in the Jovian atmosphere. *Icarus, Vol. 29, Issue 2,* 261–271. DOI: 10.1016/0019-1035(76)90056-7.

4 Perel'man, T.L., Fridman, A.X., & El'yashevich, M.M. (1974). A modified Korteweg-de Vries equation in electrohydrodynamics. *Journal of Experimental and Theoretical Physics, Vol. 66,* 1316–1323.

5 Liu, C.H., Weznik, A.W., & Yeh, K.C. (1974). Propagation of pulse trains through a random. *Ieee Transactions on Antennas and Propagation, Vol. 22, Issue 4,* 184–187. DOI: 10.1109/TAP.1974.1140830.

6 Taimanov, I.A. (1999). Finite-gap solutions of the modified Novikov-Veselov equations: Their spectral properties and applications. *Siberian Mathematical Journal, Vol. 40, 6,* 1146–1156. DOI: 10.1007/BF02677539.

7 Kunstmann, P.C., & Weis, L. (2004). Maximal L_p-regularity for Parabolic Equations, Fourier Multiplier Theorems and H^∞-functional Calculus. *Functional Analytic Methods for Evolution Equations. Lecture Notes in Mathematics,* 1855. Springer-Verlag, Berlin, Heidelberg, 5–311. DOI: 10.1007 /978-3-540-44653-8-2.

8 Arendt, W., & Duelli, M. (2006). Maximal L_p-regularity for parabolic and elliptic equations on the line. *Journal of Evolution Equations, Vol. 6, Issue 4,* 773–790. DOI: 10.1007/s00028-006-0292-5.

9 Metafune, G., Pallara, D., Prüss, J., & Schnaubelt, R. (2005). L_p-theory for elliptic operators on R^d with singular coefficients. *Z. Anal. Anwendungen. Journal for Analysis and its Applications, Vol. 24, No. 3,* 497–521.

10 Metafune, G., Prüss, J., Schnaubelt, R., & Rhandi, A. (2005). L_p-regularity for elliptic operators with unbounded coefficients. *Advances in Differential Equations, Vol. 10, No. 10,* 1131–1164.

11 Ospanov, K., & Akhmetkaliyeva, R. (2012). Separation and the existence for second order nonlinear differential equation. *Electronic Journal of Qualitative Theory of Differential Equations, No. 66,* 1–12.

12 Amanova, T.T. (1981). O razdelimosti odnoho differentsialnoho operatora [On the separability of a single differential operator]. *Izvestiia AN KazSSR. Seriia fiziko-matematicheskaiia – News of AS of Kazakh SSR. Physics and Mathematics Series, 3,* 48–51 [in Russian].

13 Amanova, T.T., & Muratbekov, M.B. (1983). Hladkost resheniia odnoho nelineinoho differentsialnoho uravneniia [The smoothness of solution of a single nonlinear differential equation]. *Izvestiia AN KazSSR. Seriia fiziko-matematicheskaiia – News of AS of Kazakh SSR. Physics and Mathematics Series, 5,* 4–7 [in Russian].

14 Birgebaev, A., & Otelbaev, M. (1984). O razdelimosti nelineinoho differentsialnoho operatora treteho poriadka [On the separability of a third-order nonlinear differential operator]. *Izvestiia AN KazSSR. Seriia fiziko-matematicheskaiia – News of AS of Kazakh SSR. Physics and Mathematics Series, 3,* 11–13 [in Russian].

15 Togochuev, A.Zh. (1985). O summiruemosti reshenii differentsialnykh uravnenii nechetnoho poriadka s vesom [On the summability of solutions of differential equations of odd order with weight]. *Izvestiia AN KazSSR. Seriia fiziko-matematicheskaiia – News of AS of Kazakh SSR. Physics and Mathematics Series, 5,* 55–58 [in Russian].

16 Sapenov, M., & Shuster, L.A. (1987). O summiruemosti s vesom reshenii dvuchlennykh differentsialnykh uravnenii [On summability with the weight of solutions of two-term differential equations]. *Izvestiia AN KazSSR. Seriia fiziko-matematicheskaiia – News of AS of Kazakh SSR. Physics and Mathematics Series, 1,* 38–42 [in Russian].

17 Aitkozha, Zh.Zh., Muratbekov, M.B., & Ospanov, K.N. (2005). O razreshimosti odnoho klassa nelineinykh sinhuliarnykh uravnenii treteho poriadka [On the solvability of a single class of third order nonlinear singular equations]. *Vestnik ENU imeni L.N. Gumilyova – Bulletin of L.N. Gumilyov ENU, 6(46),* 10–15 [in Russian].

18 Muratbekov, M.B., Muratbekov, M.M., & Ospanov, K.N. (2010). Coercive Solvability of odd-order differential equations and its applications. *Doklady mathematics, Vol. 82, No. 3,* 909–911. DOI: 10.1134/S1064562410060189.

19 Akhmetkaliyeva, R.D., Persson, L.E., Ospanov, K., & Woll P. (2015). Some new results concerning a class of third order differential equations. *Applicable Analysis: An International Journal, Vol. 94, Issue 2,* 419–434. DOI: 10.1080/00036811.2014.898375.

20 Albanese, A., Lorenzi, L., & Mangino, E.M. (2009). L_p-uniqueness for elliptic operators with unbounded coefficients in R^N. *Journal of Functional Analysis, Vol. 256, No. 4,* 1238–1257. DOI: 10.1016/j.jfa.-2008.07.022.

21 Albanese, A.A., & Mangino, E.M. (2008). Cores for Feller semigroups with an invariant measure. *Journal of Differential Equations, Vol. 244, No. 11,* 2980–2982. DOI: 10.1016/j.jde.2005.09.014.

22 Da Prato, G., & Lunardi, A. (2007). On a class of self-adjoint elliptic operators in L_2 spaces with respect to invariant measure. *Journal of Differential Equations, Vol. 234, 1,* 54–79. DOI: 10.1016/j.jde.2006.11.017.

23 Da Prato, G., & Vespri, V. (2002). Maximal L_p-regularity for elliptic equations with unbounded coefficients. *Journal of Differential Equations, Vol. 49, No. 6,* 747–755. DOI: 10.1016/S0362-546X(01) 00133-X.

24 Hieber, M., Lorenzi, L., Pruss, J., Rhandi, A., & Schnaubelt, R. (2009). Global properties of generalized Ornstein–Uhlenbeck operators on $L_p(R^N, R^N)$ with more than linearly growing coefficients. *Journal of Mathematical Analysis and Applications, Vol. 350, No. 1,* 100–121. DOI: 10.1016/j.jmaa.2008.09.011.

25 Lunardi, A. (1997). On the Ornstein–Uhlenbeck operator in L_2 spaces with respect to invariant measures. *Transactions of the American Mathematical Society, Vol. 349, No. 1,* 155–169.

26 Ospanov, K.N. (2015). Qualitative and approximate characteristics of solutions of Beltrami-type systems. *Complex variables and elliptic equations, Vol. 60, Issue 7,* 1005–1014. DOI: 10.1080/ 17476933.2015.1025066.

27 Ospanov, K.N. (1997). On the nonlinear generalized Cauchy-Riemann system on the whole plane. *Siberian Mathematical Journal, Vol. 38, Issue 2,* 314–319. DOI: 10.1007/BF02674630.

28 Ospanov, K. (2015). L1-maximal regularity for quasilinear second order differential equation with damped term. *Electronic Journal of Qualitative Theory of Differential Equations, Vol. 39, 1–9.*

29 Wu, L.M., & Zhang, Y. (2006). A new topological approach to the L^∞-uniqueness of operators and the L^1-uniqueness of Fokker–Planck equations. *Journal of Functional Analysis, Vol. 241, No. 2,* 557–610. DOI: 10.1016/j.jfa.2006.04.020.

30 Metafune, G. (2001). L_p-spectrum of Ornstein–Uhlenbeck operators. *Annali della Scuola normale superiore di Pisa, Classe di scienze, Vol. 30, No. 1,* 97–124.

31 Metafune, G., Ouhabaz, E.M., & Pallara, D. (2011). Long time behavior of heat kernels of operators with unbounded drift terms. *Journal of Mathematical Analysis and Applications, Vol. 377, No. 1,* 170–179. DOI: 10.1016/j.jmaa.2010.10.023.

32 Metafune, G., Metafune, G., Pallara, D., & Priola, E. (2002). Spectrum of Ornstein–Uhlenbeck operators in L_p spaces with respect to invariant measures. *Journal of Functional Analysis, Vol. 196, No. 1,* 40–60. DOI: 10.1006/jfan.2002.3978.

33 Metafune, G., Pruss, J., Rhandi, A., & Schnaubelt, R. (2002). The domain of the Ornstein–Uhlenbeck operator on an L_p-space with invariant measure. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Vol. 1, No. 2,* 471–485.

34 Ospanov, K.N., & Yeskabylova, Zh.B. (2018). On Coercive Solvability of a Third-Order Differential Equation with Oscillating Intermediate Coefficient. *AIP Conference Proceedings, Vol. 1997, 020055-1-020055-4.* DOI: 10.1063/1.5049049.

35 Ospanov, K.N., & Yeskabylova, Zh.B. (2017). On smoothness property of third-order differential operator. *AIP Conference Proceedings, Vol. 1880,* 040009-1-040009-6. DOI: 10.1063/1.5000625.

36 Ospanov, K.N., & Yeskabylova, Zh.B. (2017). Ushinshi retti bir nuksandy differencialdyk operatordyn anyktalu oblysyn sipattau [Description of the domain of definition of a third order degenerate differential operator]. *Vestnik ENU imeni L.N. Gumileva – Bulletin of L.N. Gumilyov ENU, Vol. 121, No. 6,* 30–36 [in Kazakh].

37 Apyshev, O.D., & Otelbaev, M. (1980). On the spectrum of a class of differential operators and some imbedding theorems. *Mathematics of the USSR-Izvestiia, Vol. 15, 1,* 739–764. DOI: 10.1070/IM1980v015n01ABEH001190.

38 Kato, T. (1995). *Perturbation theory for linear operators.* Springer.

39 Mynbaev, K.T., & Otelbaev, M. (1990). *Vesovye funktsionalnye prostranstva i spektr differentsialnykh operatorov [Weighted functional spaces and the spectrum of the differential operators]*. Moscow: Nauka [in Russian].

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On multi-periodic solutions of quasilinear autonomous systems with an operator of differentiation on the Lyapunov's vector field

A quasilinear autonomous system with an operator of differentiation with respect to the characteristic directions of time and space variables associated with a Lyapunov's vector field is considered. The question of the existence of multi-periodic solutions on time variables is investigated, when the matrix of a linear system along characteristics has the property of exponential stability. And the non-linear part of the system is sufficiently smooth. In the note, on the basis of Lyapunov's method, the necessary properties of the characteristics of the system with the specified differentiation operator were substantiated; theorems on the existence and uniqueness of multi-periodic solutions of linear homogeneous and nonhomogeneous systems were proved; sufficient conditions for the existence of a unique multi-periodic solution of a quasilinear system were established. In the study of a nonlinear system, the method of contraction mapping was used.

Key words: multi-periodic solutions, autonomous system, operator of differentiation, Lyapunov's vector field.

Introduction

It is known that many phenomena connected by a continuous medium are described by systems of partial differential equations. In many cases, these systems are quasilinear, and these phenomena (sound, light, electromagnetic, gas and hydromechanical) are oscillatory-wave in nature. Consequently, the study of solutions of such systems with oscillatory properties over both time and space variables belong to an important part of the theory of equations in ordinary and partial derivatives. The foundations of this theory were laid in the classical works of A.M. Lyapunov, H.Poincaré and the fundamental research of Andronov-Witt-Khaykin, Krylov-Bogoliubov-Mitropolsky-Samoilenko, Kolmogorov-Arnold-Moser, etc.

A peculiar approach to the problems of the theory of oscillations was proposed in the works of V. Kharasakhal and D.U. Umbetzhanov [1–8], based on a deep connection between an almost periodic function of one variable and a periodic function of many variables, called a multi-periodic function, where the problems are quasi-periodic solutions of ordinary differential equations, are studied on the basis of multi-periodic solutions of systems of the partial differential equations of the first order. In this connection, we note that many quite serious results, known from oscillatory solutions of ordinary differential equations, they are extended to the case of multi-periodic solutions of partial differential equations [9–20], which were further developed in the articles [21–23].

We note, that some information on multi-periodic solutions of systems of the partial differential equations is contained in the literature review of the fundamental work [24], where the number of papers by one of the authors is presented.

We also note, that many theoretical questions of physics and technology are based on oscillatory processes. In particular, we pay attention to the works [25, 26], where an interesting research was conducted of problems from hydromechanics and control theory related to oscillatory processes described by the differential and integro-differential equations. These equations are attractive because it is possible for them to consider the problem of multi-periodic solutions and use the methods outlined in this article.

Of particular interest is the work [27], where the equations with a differentiation operator along the directions of a vector field on a torus are considered and conditions for the existence of their periodic solutions are established. Note that the differential operator under consideration is similar to the differentiation operator, which given in this note.

The methods of Poincaré-Lyapunov and Hamilton-Jacobi for integrating and researching of the periodic solutions are the basis of the methodology for studying the problem of this work. It is obvious, that the sources of multi-periodic solutions of the differential equations are their periodic solutions with different rationally incommensurable frequencies. In this regard, our attention is drawn to the problems studied in the articles [28, 29] and some commonality of their study methods with the methods of this work.

One of the common ways to investigate the periodic solutions is to use the methods of boundary value problems for the differential equations. In the works [30–35] for investigating the oscillatory solutions of some equations of various types of mathematical physics was used, a technique calling the method of parameterization. We note that the equations under consideration are representable as systems of equations of first-order derivatives.

In this article, we consider the quasilinear system of equations with a differentiation operator along the directions of the vector fields, where the characteristic directions of the differentiation operator along the time and space variables are independent, with the space variables being differentiated along the directions defined by the Lyapunov's system.

In the case of a non-autonomous system, the frequencies of the desired multi-periodic oscillations are mainly determined by the system itself. Consequently, the frequencies and their number are known in advance.

In this autonomous case, the main difficulty of the considering problem is related to the uncertainty of the frequency of periodic oscillations, which are components of the desired multi-periodic oscillations. This difficulty was surmountable that the characteristic vector field satisfies the conditions of the Lyapunov's system. Although, systems of the partial differential equations that do not contain time variables are often found in the scientific literature, but the problem of this note on the formulation is new and is being investigated for the first time.

We consider the autonomous system

$$Dx = P(\zeta)x + f(\zeta, x), \tag{1}$$

with differentiation operator

$$D = \frac{\partial}{\partial \tau} + \left\langle e, \frac{\partial}{\partial \overline{\tau}} \right\rangle + \left\langle J\zeta + \psi(\zeta), \frac{\partial}{\partial \zeta} \right\rangle, \tag{2}$$

where $x = (x_1, \ldots, x_n) \in R^n$ are unknown vector-functions with respect to the time $\tau \in R$, $\overline{\tau} = (\tau_1, \ldots, \tau_m) \in R^m$ and space $\zeta = (\zeta_0, \ldots, \zeta_k)$, $\zeta_j = (\xi_j, \eta_j) \in R^2$, $j = \overline{0,k}$, variables; $\left\langle e, \frac{\partial}{\partial \overline{\tau}} \right\rangle$ is the scalar product of m-vectors $e = (1, \ldots, 1)$ and $\frac{\partial}{\partial \overline{\tau}} = \left(\frac{\partial}{\partial \tau_1}, \ldots, \frac{\partial}{\partial \tau_m} \right)$; J is a $(2k+2)$-dimensional constant matrix; $\psi(\zeta)$ is a $(2k+2)$-vector-function given in a δ-neighborhood R_δ^{2k+2} of a point $\zeta = 0$ in Euclidean space R^{2k+2}; $\frac{\partial}{\partial \zeta} = \left(\frac{\partial}{\partial \zeta_0}, \ldots, \frac{\partial}{\partial \zeta_k} \right)$, $\frac{\partial}{\partial \zeta_j} = \left(\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \eta_j} \right)$, $j = \overline{0,k}$, is a vector operator.

The matrix $P(\zeta) = [p_{ij}(\zeta)]_1^n$ is holomorphic in the R_ε^{2n+2} neighborhood of the point $\zeta = 0$:

$$P(\zeta) = \sum_{j=0}^{+\infty} \frac{1}{j!} \left\langle \zeta, \frac{\partial}{\partial \zeta} \right\rangle^j P(0), \zeta \in R_\varepsilon^{2k+2}, \tag{3}$$

where $\varepsilon > 0$ is some constant and $\delta = \delta(\varepsilon) > 0$ is sufficiently small.

The vector-function $f(\zeta, x)$ has the following properties of continuity and smoothness

$$f(\zeta, x) \in C_\zeta^{(e)} \left(R_\varepsilon^{2k+2} \times R_\Delta^n \right) \tag{4}$$

with bounded matrix of Jacobi

$$\left| \frac{\partial f(\zeta, x)}{\partial x} \right| \leq c, \ (\zeta, x) \in \overline{R}_\varepsilon^{2k+2} \times \overline{R}_\Delta^n, \tag{5}$$

where $c > 0$ is a constant, $\overline{R}_\varepsilon^{2k+2} \times \overline{R}_\Delta^n$ is the closure of the region $R_\varepsilon^{2k+2} \times R_\Delta^n$.

Thus, set the problem to clarify the conditions the $(\theta, \overline{\theta})$-periodicity of solutions of the system (1) when conditions (3), (4), and (5) are performed.

The differentiation operator along the directions of the diagonal of time and space variables on the Lyapunov's vector field

Differentiation by the operator D is conducted along directions of vector fields of time variables

$$\frac{d\overline{\tau}}{d\tau} = e \tag{6}$$

and space variables

$$\frac{d\zeta}{d\tau} = J\zeta + \psi(\zeta), \tag{7}$$

associated with the time variable $\tau \in R$.

The characteristic of the vector equation (6), outgoing from the point $\overline{\tau}_0 = (\tau_1^0, \ldots, \tau_m^0)$ when $\tau = \tau_0$ is determined by the relation $\overline{\tau} = \overline{\tau}_0 + e(\tau - \tau_0)$. For our purpose, it's useful to take as the initial point $\overline{\tau}_0 = e\tau_0$. Therefore, we have

$$\overline{\tau} = e\tau. \tag{8}$$

It should also be noted here that the dimension m of the time vector $\overline{\tau}$ is related to the dimension of the common frequency basis of the family periodic solutions of the autonomous system (7), which cannot be specified in advance. In our case, we note that $m = k$.

The vector field (7) can be characterized by the following properties:

a) The matrix J can be represented in the form

$$J = diag[\nu_0 I_2, \ldots, \nu_k I_2],\ \langle q, \nu\rangle \neq 0,\ q \in Z^{k+1},\ q \neq 0, \tag{9}$$

where $I_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a two-dimensional symplectic unit, $\nu_j, j = \overline{0,k}$, are incommensurable frequencies, $q = (q_0, \ldots, q_k) \in Z^{k+1}$ is an integer vector, $\nu = (\nu_0, \ldots, \nu_k)$ is vector, Z is the set of integers.

b) The vector function $\psi(\zeta)$ is formed by a given scalar holomorphic function $\Psi(\zeta)$ in some δ-neighborhood R_δ^{2k+2} of the point $\zeta = 0$ in Euclidean space R^{2k+2} by applying an operator $I\frac{\partial}{\partial\zeta}$ with $(2k+2)$-matrix $I = diag[I_2, \ldots, I_2]$, whose decomposition of the function of which $\Psi(\zeta)$ begins with a homogeneous form of at least the third degree:

$$\begin{gathered} \psi(\zeta) = I\tfrac{\partial}{\partial\zeta}\Psi(\zeta), \zeta \in R_\delta^{2k+2}, \\ \Psi(\zeta) = \sum_{j=3}^{+\infty} \tfrac{1}{j!}\left\langle \zeta_j, \tfrac{\partial}{\partial\zeta}\right\rangle^j \Psi(0). \end{gathered} \tag{10}$$

It is obvious, that the vector field (7) under the conditions (9) and (10) belongs to the class of Lyapunov's systems.

By conditions (9) and (10) can be represented system (7) to the scalar form

$$\begin{cases} \dfrac{d\xi_j}{d\tau} = -\nu_j\eta_j - \dfrac{\partial\Psi(\zeta)}{\partial\eta_j}, \\ \dfrac{d\eta_j}{d\tau} = \nu_j\xi_j + \dfrac{\partial\Psi(\zeta)}{\partial\xi_j},\ j = \overline{0,k}. \end{cases} \tag{11}$$

with the first integral

$$H(\zeta) = \sum_{j=0}^{k} \frac{\nu_j}{2}(\xi_j^2 + \eta_j^2) + \Psi(\zeta). \tag{12}$$

By the first integral (12) the system (11) can be written in the form of a canonical system

$$\begin{cases} \dfrac{d\xi_j}{d\tau} = -\dfrac{\partial H(\zeta)}{\partial\eta_j}, \\ \dfrac{d\eta_j}{d\tau} = \dfrac{\partial H(\zeta)}{\partial\xi_j},\ j = \overline{0,k}. \end{cases} \tag{13}$$

According to the Lyapunov's method [1, 2], the variables $(\xi_l,\eta_l), l\neq j, 0\leq l\leq k$ as functions (ξ_j,η_j) with a fixed number j can be determined from the system of the partial differential equations

$$\begin{cases} \dfrac{\partial H}{\partial \xi_j}\dfrac{\partial x_l}{\partial \eta_j}-\dfrac{\partial H}{\partial \eta_j}\dfrac{\partial x_l}{\partial \xi_j}=-\dfrac{\partial H}{\partial \eta_j}, \\ \dfrac{\partial H}{\partial \xi_j}\dfrac{\partial y_l}{\partial \eta_j}-\dfrac{\partial H}{\partial \eta_j}\dfrac{\partial y_l}{\partial \xi_j}=\dfrac{\partial H}{\partial \xi_j}, \; l\neq j, \; 0\leq l\leq k, \end{cases} \tag{14}$$

where $H=H(\zeta)$ is the Hamiltonian (12) of the systems (13).

System (14) with the initial condition $(\xi_l,\eta_l)=(0,0)$ for $(\xi_j,\eta_j)=0$, under conditions (9) and (10) allows an unique holomorphic solution $(\xi_l^*(\xi_j,\eta_j),\eta_l^*(\xi_j,\eta_j))=\zeta_l^*(\xi_j,\eta_j)$ in the sufficiently small neighborhood R_δ^2 of the point $\zeta_l=0$ in the plane R^2 for fixed values $l\neq j$.

We obtain the function

$$H_j(\xi_j,\eta_j)=H\left(\zeta_0^*(\xi_j,\eta_j),\ldots,\zeta_{j-1}^*(\xi_j,\eta_j),\zeta_j,\zeta_{j+1}^*(\xi_j,\eta_j),\ldots,\zeta_k^*(\xi_j,\eta_j)\right). \tag{15}$$

by substituting found solutions $\zeta_0^*(\xi_j,\eta_j),\ldots,\zeta_{j-1}^*(\xi_j,\eta_j),\zeta_{j+1}^*(\xi_j,\eta_j),\ldots,\zeta_k^*(\xi_j,\eta_j)$ of the systems (14) to the Hamiltonian $H(\zeta)$. Also, we set up a function

$$g_j(\xi_j,\eta_j)=1+\sum_{l\neq j}\left(\frac{\partial \xi_l^*(\xi_j,\eta_j)}{\partial \xi_j}\frac{\partial \eta_l^*(\xi_j,\eta_j)}{\partial \eta_j}-\frac{\partial \xi_l^*(\xi_j,\eta_j)}{\partial \eta_j}\frac{\partial \eta_l^*(\xi_j,\eta_j)}{\partial \xi_j}\right). \tag{16}$$

On the basis of functions (15) and (16), we consider the system of ordinary differential equations

$$\begin{cases} g_j(\xi_j,\eta_j)\dfrac{d\xi_j}{d\tau_j}=-\dfrac{\partial H_j(\xi_j,\eta_j)}{\partial \eta_j}, \\ g_j(\xi_j,\eta_j)\dfrac{d\eta_j}{d\tau_j}=\dfrac{\partial H_j(\xi_j,\eta_j)}{\partial \xi_j}, \end{cases} \tag{17}$$

which is a Lyapunov's system corresponding to the frequency ν_j. Therefore, system (17) defines a two-parameter family of periodic solutions

$$\begin{aligned} \xi_j &= h_j^{'}(\tau_j-\tau_0,\xi_j^0,\eta_j^0), \\ \eta_j &= h_j^{''}(\tau_j-\tau_0,\xi_j^0,\eta_j^0) \end{aligned} \tag{18}$$

with arbitrary initial values $(\xi_j,\eta_j)\big|_{\tau_j=\tau_0}=(\xi_j^0,\eta_j^0)$ from a sufficiently small neighborhood R_δ^2, and a period

$$\theta_j=\frac{2\pi}{\nu_j}\left(1+c_j^{(1)}H_j\left(\xi_j^0,\eta_j^0\right)+c_j^{(2)}\left[H_j\left(\xi_j^0,\eta_j^0\right)\right]^2+\ldots\right), \tag{19}$$

which the coefficients $c_j^{(1)},c_j^{(2)},\ldots$ didn't depend on the initial data, and they are $\theta_j=\dfrac{2\pi}{\nu_j}$ when $(\xi_j^0,\eta_j^0)=(0,0)$.

Thus, by changing j from 0 to k and using the Lyapunov's method, we obtain all $1+k$ periodic solutions (18) of the system (17). These solutions are components of a multi-periodic solution ζ of the system (7) in the form

$$\zeta=\Big(h_0^{'}\Big(\tau-\tau_0,\xi_0^0,\eta_0^0\Big),h_0^{''}\Big(\tau-\tau_0,\xi_0^0,\eta_0^0\Big),\ldots,h_k^{'}\Big(\tau_k-\tau_0,\xi_k^0,\eta_k^0\Big),h_k^{''}\Big(\tau_k-\tau_0,\xi_k^0,\eta_k^0\Big)\Big)\equiv$$

$$\equiv h\Big(\tau-\tau_0,\tau_1-\tau_0,\ldots,\tau_k-\tau_0,\xi_0^0,\eta_0^0,\ldots,\xi_k^0,\eta_k^0\Big) \tag{20}$$

with initial condition

$$\zeta\big|_{\tau=\tau_1=\ldots=\tau_k=\tau_0}=\left(\xi_0^0,\eta_0^0,\ldots,\xi_k^0,\eta_k^0\right)=\zeta_0 \tag{21}$$

and vector-period $\theta=(\theta_0,\theta_1,\ldots,\theta_k)$ with components (19) on a vector variable $\overline{\tau}=(\tau,\tau_1,\ldots,\tau_k)$.

We are passing to the vector notation from (20)–(21), and then the solution ζ of the system (7) is represented in the form

$$\zeta = h(\tau - \tau_0, \overline{\tau} - e\tau_0, \zeta_0), \tag{22}$$

where $h(0, \overline{0}, \zeta_0) = \zeta_0$. Equation (22) together with equation (8), represent the characteristics of the operator (2). Thus, the following assertion is substantiated.

1^0. Under conditions (9) and (10), operator (2) is the operator of differentiation with respect to τ of the functions $x(\tau, \overline{\tau}, \zeta)$ of the along direction of the main diagonal (8) of the time variables and along multi-periodically closed curves (22) with respect to space variables.

Therefore, the function Dx along the characteristic, given by relation (22), determines the rate of change of the function $x = x(\zeta)$ with respect to τ:

$$Dx\big|_{\zeta=h} = \frac{dx(h)}{d\tau}.$$

The statement 1^0 allows us to go from the differential equations with operator D to the integral equations, defined along the characteristics.

By the uniqueness property the solutions of the system (7), from the equation of characteristics (22) we have the expression

$$\zeta_0 = h(\tau_0 - \tau, e\tau_0 - \overline{\tau}, \zeta), \tag{23}$$

which is the first integral of the operator $D : Dh = 0$.

In addition, based on the same property, we obtain the group property of the characteristic in the form

$$h\left(\tau - s, \overline{\tau} - es, h(s - \tau_0, es - e\tau_0, \zeta_0)\right) = h\left(\tau - \tau_0, \overline{\tau} - e\tau_0, \zeta_0\right) \tag{24}$$

with $s \in R$.

Then we have the following statement.

2^0. Under the conditions of paragraph 1^0 the function $x\left(s - \tau_0, \overline{\tau} - e\tau_0, h(s - \tau_0, es - e\tau_0, \zeta_0)\right)$ taking into account property (24), is proceed to the function $x\left(s - \tau, es - \overline{\tau}, h(s - \tau, es - \overline{\tau}, \zeta)\right)$ with parameter $s \in R$, and variables $(\tau, \overline{\tau}, \zeta)$, where the given function defined along the characteristic

$$\tau = s - \tau_0,$$

$$\overline{\tau} = es - e\tau_0,$$

$$\zeta = h(s - \tau_0, es - e\tau_0, \zeta_0)$$

with a parameter s based on the first integrals of the systems (6) and (7) in the form

$$\tau_0 = \tau,$$

$$e\tau_0 = \overline{\tau},$$

$$\zeta_0 = h(\tau_0 - \tau, e\tau_0 - \overline{\tau}, \zeta),$$

obtained by relations (8) and (23).

Paragraph 2^0 allows leaving expressions defined along the characteristics of the operator D to the space of variables $(\tau, \overline{\tau}, \zeta)$.

Further, by the periodicity the characteristics (18) of the operator D in the period (19), the property of multi-periodicity of the vector-function (22) can be represented as

$$h(\tau + \theta, \overline{\tau} + q\overline{\theta}, \zeta) = h(\tau, \overline{\tau}, \zeta), \tag{25}$$

where the vector-period $(\theta, \overline{\theta}) = (\theta_0, \theta_1, \ldots, \theta_k)$ with the components $\theta = \theta_0, \overline{\theta} = (\theta_1, \ldots, \theta_k)$ is defined by the relation (19), and the periods $\theta_j, j = \overline{0, k}$ depend on the initial data $\zeta = \zeta_0$ of the characteristics of the operator (2), $q \in Z^m$.

We note, that when the question of multi-periodicity, we consider periods $(\theta, \overline{\theta})$ in the space of variables $(\tau, \overline{\tau}, \zeta) \in R \times R^k \times R_\delta^{k+1}$, by replacing the initial data ζ_0 to the corresponding value (23), and we get functions with respect to variables $(\tau, \overline{\tau}, \zeta)$, and parameter $\tau_0 \in R$:

$$\begin{aligned} \theta &= \theta(\tau_0, \tau, \overline{\tau}, \zeta), \\ \overline{\theta} &= \overline{\theta}(\tau_0, \tau, \overline{\tau}, \zeta). \end{aligned} \tag{26}$$

Since expressions (23) represent the first integral, that is, we have identity

$$Dh(\tau_0-\tau, e\tau_0-\overline{\tau},\zeta)=0, \tag{27}$$

then periods (26) are also first integrals, and therefore, we have identity relations

$$\begin{aligned} &D\theta(\tau_0,\tau,\overline{\tau},\zeta)=0,\\ &D\overline{\theta}(\tau_0,\tau,\overline{\tau},\zeta)=0. \end{aligned} \tag{28}$$

Thus, as a consequence of the identities (27) and (28), we can formulate the following statement.

3^0. Under the conditions of the preceding paragraphs, any smooth function $f(h)$ has the properties

$$Df\big(h\,(s-\tau, es-\overline{\tau},\zeta)\big)=0, \tag{29}$$

$$f\big(h\,\big(s-\tau+\theta, es-\overline{\tau}+q\overline{\theta},\zeta\big)\big)=f\big(h\,(s-\tau, es-\overline{\tau},\zeta)\big), \tag{30}$$

where $s\in R$ is the parameter, and $q\in Z^m$, $h(s-\tau, es-\overline{\tau},\zeta)$ is the integral function (23).

In deducing relation (29) it was taken into account that if h is the first integral, and then the smooth function $f(h)$ is also an integral. Identity (30) follows from identity (25).

Homogeneous linear system

We consider a linear system

$$Dx=P(\zeta)x, \tag{31}$$

with respect to the unknown vector-function $x=(x_1,\dots,x_n)$, where the operator D is defined by the formula (2) with properties (9) and (10); $P(\zeta)=[p_{ij}(\zeta)]_1^n$ is holomorphic matrix in the R_ε^{2n+2} neighborhood of the point $\zeta=0$, and satisfies condition (3).

The system (31) along the characteristics (22) represents a system of the ordinary differential equations with the multi-periodic matrix $P(h(\tau-s,\overline{\tau}-es,\zeta_0))$ with respect to $(\tau,\overline{\tau})$ with period $(\theta,\overline{\theta})$.

Then it is possible to determine the matrix X of the linear system (31)on the basis of the integral equation

$$X(\tau_0,\tau,\overline{\tau},\zeta)=E+\int\limits_{\tau_0}^{\tau}P(h(s-\tau, es-\overline{\tau},\zeta))X(\tau_0,s,es,\zeta)\,ds, \tag{32}$$

where E is the unit n-matrix, $\tau_0\in R$, $\tau\in R$, $\overline{\tau}\in R^k$, $\zeta\in R_\varepsilon^{2k+2}$ for sufficiently small $\varepsilon>0$, $X(\tau_0,\tau_0,e\tau_0,\zeta)=E$.

Obviously, the matriciant $X(\tau_0,\tau,\overline{\tau},\zeta)$ is holomorphic with respect to ζ by virtue of (3) and $(\theta,\theta,\overline{\theta})$-periodic by $(\tau_0,\tau,\overline{\tau})$

$$X(\tau_0+\theta,\tau+\theta,\overline{\tau}+e\theta,\zeta)=X(\tau_0,\tau,\overline{\tau},\zeta). \tag{33}$$

Further, suppose that the matrix $P(\zeta)$ provides the property of the exponential stability of system (31) in the form

$$|X(\tau_0,\tau,\overline{\tau},\zeta)|\le ae^{-\alpha(\tau-\tau_0)},\ \tau\ge\tau_0 \tag{34}$$

with constants $a\ge 1,\alpha>0$, where $\tau_0\in R$. If we take into account the solution $x=x(\tau_0,\tau,\overline{\tau},\zeta)$ of the system (31) with an initial condition that turns into the initial smooth function $u(\zeta)$, when $\tau=\tau_0$ in the form

$$x\big|_{\tau=\tau_0}=u(\zeta)\in C_\zeta^{(1)}\left(R_\varepsilon^{2k+2}\right)$$

expressed by

$$x(\tau_0,\tau,\overline{\tau},\zeta)=X(\tau_0,\tau,\overline{\tau},\zeta)u(h(\tau_0-\tau, e\tau_0-\overline{\tau},\zeta)), \tag{35}$$

then from the condition (34) implies the absence of the multi-periodic solution of the system (31), that is different from zero.

Lemma 1. Let conditions (3), (9), (10), and (34) be satisfied. Then the homogeneous linear system (31) has no multi-periodic solution, except for the trivial one.

Proof. Indeed, from the representation of solutions (35) and condition (34) follows that for fixed values ζ any solution with nonzero initial data $u\neq 0$ is unbounded.

Therefore, such a solution cannot be multi-periodicity. It only follows that $u=0$ is the only multi-periodic solution of the system (31).

Nonhomogeneous linear system

Now we consider the system in the form

$$Dx = P(\zeta)x + f(\zeta) \tag{36}$$

with free term $f(\zeta)$ of holomorphic in R_ε^{2k+2}:

$$f(\zeta) = \sum_{j=0}^{+\infty} \frac{1}{j!}\left\langle \zeta, \frac{\partial}{\partial\zeta}\right\rangle^j f(0), \zeta \in R_\varepsilon^{2k+2}, \tag{37}$$

where $\varepsilon > 0$ is some constant.

Theorem 1. Let conditions (3), (9), (10), (34), and (37) be satisfied. Then the system (36) has the unique $(\theta, \overline{\theta})$-periodic solution holomorphic with respect to $\zeta \in R_\delta^{2k+2}$ for sufficiently small $\delta = \delta(\varepsilon) > 0$

$$x^*(\tau, \overline{\tau}, \zeta) = \int_{-\infty}^{\tau} X(s, \tau, \overline{\tau}, \zeta) f(h(s-\tau, es-\overline{\tau}, \zeta))\, ds, \tag{38}$$

where $\delta = \delta(\varepsilon)$ is chosen such that, for $\zeta \in R_\delta^{2k+2}$ the inequality $|h(\tau, \overline{\tau}, \zeta)| < \varepsilon$ is satisfied.

Proof. Indeed, by a direct verification, we see, that function (38), which is determined by condition (34), in the form of an improper integral is the solution of the system (36). In this case, it is necessary to take into account that the matriciant X satisfies the matrix equation (32), and the integral h has the properties (22) and (23). The periodicity of the solution (37) with respect to τ with a period θ is checked on the basis of the property (33), and the $\overline{\theta}$-periodicity with respect to $\overline{\tau}$ follows from the property (25). The property of holomorphy follows from the holomorphy of the integral h of the matrix P, and the function f given in conditions (3), (10), and (37).

In conclusion, we note that if conditions (3) and (37) about the holomorphy of the matrix $P(\zeta)$ and vector-function $f(\zeta)$ are replaced by the conditions of their continuous differentiability, then generalizing Theorem 1, we can get the result about the existence of the multi-periodic solution (38), without the property of holomorphy.

Therefore, we have the following theorem.

Theorem 2. Let the matrix function $P(\zeta)$ be continuously differentiable in R_ε^{2k+2}:

$$P(\zeta) \in C_\zeta^{(e)}\left(R_\varepsilon^{2k+2}\right), \tag{39}$$

and the free term $f(\zeta)$ of the system also has the same property:

$$f(\zeta) \in C_\zeta^{(e)}\left(R_\varepsilon^{2k+2}\right), \tag{40}$$

where $C_\zeta^{(e)}\left(R_\varepsilon^{2k+2}\right)$ is the class of smooth functions of order $e = (1, \dots, 1)$ in R_ε^{2k+2}. If satisfied conditions (9), (10) and (34), then relation (38) represents the unique $(\theta, \overline{\theta})$-periodic solution of the system (36) for sufficiently small $\delta = \delta(\varepsilon) > 0$, when $\zeta \in R_\delta^{2k+2}$.

The proof of the Theorem 2 is similar to the proof of the Theorem 1, with the difference, that the smoothness of the solutions is everywhere provided by the conditions (39), (40).

Now, additionally we consider the case when the free term f, except $\zeta \in R_\delta^{2k+2}$, depends on $(\tau, \overline{\tau}) \in R \times R^k$.

Then we have a non-autonomous system of equations

$$Dx = P(\zeta)x + f(\tau, \overline{\tau}, \zeta), \tag{41}$$

where the vector-function $f(\tau, \overline{\tau}, \zeta)$ has the property

$$f(\tau+\theta, \overline{\tau}+q\overline{\theta}, \zeta) = f(\tau, \overline{\tau}, \zeta) \in C_{\tau,\overline{\tau},\zeta}^{(1,\overline{e},e)}\left(R \times R^k \times R_\varepsilon^{2k+2}\right), \tag{42}$$

$q \in Z^k$, $\overline{e} = (1, \dots, 1)$ is k-vector, $e = (1, \dots, 1)$ is $(2k+2)$-vector, $\varepsilon = const > 0$.

Theorem 3. Let conditions (9), (10), (34), (39), and (42) be satisfied. Then system (41) allows the unique $(\theta, \overline{\theta})$-periodic solution in the form

$$x^*(\tau, \overline{\tau}, \zeta) = \int_{-\infty}^{\tau} X(s, \tau, \overline{\tau}, \zeta) f(s, es, h(s-\tau, es-\overline{\tau}, \zeta))\, ds, \tag{43}$$

where $(\tau, \overline{\tau}, \zeta) \in R \times R^k \times R_\delta^{2k+2}$, $\delta = \delta(\varepsilon)$ is a sufficiently small positive number.

The proof is conducted similarly to the proof of the Theorems 1 and 2, and therefore we will not prove that theorem.

Here, since the free term f is periodically with respect to $(\tau, \overline{\tau})$ with the same periods $(\theta, \overline{\theta})$ as the integral $h(\tau, \overline{\tau}, \zeta)$, then the oscillations described by the system (41), and doesn't undergo any other changes. We will need this case when studying a nonlinear autonomous system.

Nonlinear system

Let us consider the question of the existence of the multi-periodic solution of the system (1) satisfying conditions (3), (4), and (5). From conditions (4) and (5) follows that

$$|f(\zeta, x) - f(\zeta, y)| \leq c|x - y|, \tag{44}$$

$$|f(\zeta, x)| \leq b + c|x|, \tag{45}$$

where $(\zeta, x) \in \overline{R}_{\varepsilon}^{2k+2} \times \overline{R}_{\Delta}^{n}$. Let the constants α, a, b, c and Δ be related by

$$a(b + c\Delta) < \alpha\Delta. \tag{46}$$

The value $\delta = \delta(\varepsilon) > 0$ is chosen such that, $R_{\delta}^{2k+2} \subset R_{\varepsilon}^{2k+2}$.

We consider the space $S_{\delta,\Delta}^{\theta,\overline{\theta}}$ of vector-functions $x(\tau, \overline{\tau}, \zeta)$, which continuous for each $(\tau, \overline{\tau}, \zeta) \in R \times R^k \times R_{\delta}^{2k+2}$, $(\theta, \overline{\theta})$-periodic for $(\tau, \overline{\tau})$ and bounded by number $\Delta > 0$ on the norm

$$\|x\| = \sup_{R \times R^k \times \overline{R}_{\delta}^{2k+2}} \max_{j} |x_j(\tau, \overline{\tau}, \zeta)| \leq \Delta.$$

We define the operator in this space

$$(Fx)(\tau, \overline{\tau}, \zeta) = \int\limits_{-\infty}^{\tau} X(s, \tau, \overline{\tau}, \zeta) f\left(h(s - \tau, es - \overline{\tau}, \zeta), x(s, es, h(s - \tau, es - \overline{\tau}, \zeta))\right) ds. \tag{47}$$

Lemma 2. Let conditions (4), (5), (9), (10), (34), (39) and (46) be satisfied. Then the operator F in the space $S_{\delta,\Delta}^{\theta,\overline{\theta}}$ has the unique fixed point for sufficiently small $\delta > 0$.

By virtue the conditions (34) and (45) the improper integral (47) converges uniformly. Consequently, by virtue property (4) the function $(Fx)(\tau, \overline{\tau}, \zeta)$ is continuous for all arguments.

After shifting τ to period θ, by virtue periodicity $x(\tau, \overline{\tau}, \zeta)$ with respect to τ, and the property (33) of the matriciant X, by replacing τ with $\tau + \theta$, we are convinced that function (47) is also θ-periodic with respect to τ. The periodicity of Fx with respect to $\overline{\tau}$ with the period $\overline{\theta}$ directly follows from the $\overline{\theta}$-periodicity of the matriciant X, given in (33), and the integral h by $\overline{\tau}$ according to property (25).

Using the estimates (34), (45), and (46) from expression (47), we obtain

$$|(Fx)(\tau, \overline{\tau}, \zeta)| \leq \frac{a}{\alpha}(b + c\Delta) < \Delta.$$

Consequently, the function Fx bounded by the number $\Delta > 0$. Thus, we were convinced that the operator F reflects into itself of the space $S_{\delta,\Delta}^{\theta,\overline{\theta}}$.

Further, on the basis of (44), from the representation (47) of the operator F we have the inequality

$$|(Fx)(\tau, \overline{\tau}, \zeta) - (Fy)(\tau, \overline{\tau}, \zeta)| \leq \frac{ac}{\alpha}\|x - y\|.$$

Consequently, by virtue of condition (46) $d = \dfrac{ac}{\alpha} < 1$, the estimate

$$|Fx - Fy| \leq d\|x - y\|$$

shows that the operator F is a contraction operator.

Obviously, the space $S^{\theta,\overline{\theta}}_{\delta,\Delta}$ is complete, and then the operator F has the unique fixed point in this space

$$x^*(\tau,\overline{\tau},\zeta) = (Fx^*)(\tau,\overline{\tau},\zeta) \in S^{\theta,\overline{\theta}}_{\delta,\Delta}. \tag{48}$$

Theorem 4. Under the conditions of the Lemma 2, the system (1) has the unique $(\theta,\overline{\theta})$-periodic solution for sufficiently small $\delta > 0$.

For the proof, we show, that the $(\theta,\overline{\theta})$-periodic by $(\tau,\overline{\tau})$ solution $x_*(\tau,\overline{\tau},\zeta)$ of the system (1) satisfies to the integral equation

$$x(\tau,\overline{\tau},\zeta) = (Fx)(\tau,\overline{\tau},\zeta). \tag{49}$$

Indeed, using this solution $x_*(\tau,\overline{\tau},\zeta)$, we consider a linear system

$$Dx = P(\zeta)x + f(\zeta, x_*(\tau,\overline{\tau},\zeta)) \tag{50}$$

by the form (41).

Therefore, in accordance with the Theorem 3, system (50), according to formula (43), has the unique $(\theta,\overline{\theta})$-periodic solution x of the operator expression

$$x = (Fx_*)(\tau,\overline{\tau},\zeta), \tag{51}$$

for sufficiently small $\delta > 0$.

Since the solution $x_*(\tau,\overline{\tau},\zeta)$ satisfies the system (1), it is also solution of the equation (50).

Consequently, by virtue of the uniqueness of $(\theta,\overline{\theta})$-periodic solutions from (51), we have

$$x_*(\tau,\overline{\tau},\zeta) = (Fx_*)(\tau,\overline{\tau},\zeta). \tag{52}$$

i.e. showed that $x_*(\tau,\overline{\tau},\zeta)$ is solution of the integral equation (49).

But as shown in the Lemma 2, it has the unique multi-periodic solution. Consequently, from the identities (48) and (52) we have

$$x_*(\tau,\overline{\tau},\zeta) = x^*(\tau,\overline{\tau},\zeta).$$

Thus, the solution $x^*(\tau,\overline{\tau},\zeta)$ has the smoothness property for all arguments and is determined by the integral equation (49). The theorem is completely proved.

References

1 Харасахал В.Х. Почти-периодические решения обыкновенных дифференциальных уравнений / В.Х. Харасахал. — Алма-Ата: Наука, 1970. — 200 с.

2 Kharasakhal V.K. A generalization of a theorem of Bohr and Neugebauer / V.K. Kharasakhal // Journal of Applied Mathematics and Mechanics. — 1959. — Vol. 23, 3. — P. 844-845. DOI:10.1016/0021-8928(59) 90178-9.

3 Kharasakhal V.K. Almost-periodic solutions of nonlinear systems of differential equations / V.K. Kharasakhal // Journal of Applied Mathematics and Mechanics. — 1960. — Vol. 24, 3. — P. 838–842. DOI:10.1016/0021-8928(60)90194-5.

4 Kharasakhal V.K. On quasi-periodic solutions of systems of ordinary differential equations / V.K. Kharasakhal // Journal of Applied Mathematics and Mechanics. — 1963. — Vol. 27, 4. — P. 1019–1034. DOI:10.1016/0021-8928(63)90183-7.

5 Kharasakhal V.K. Application of the point transformation method to quasi-periodic oscillations of nonlinear systems / V.K. Kharasakhal // Journal of Applied Mathematics and Mechanics. — 1965. — Vol. 29, 3. — P. 703–707. DOI:10. 1016/0021-8928(65)90077-8.

6 Умбетжанов Д.У. Почти многопериодические решения дифференциальных уравнений в частных производных / Д.У. Умбетжанов. — Алма-Ата: Наука, 1979. — 211 с.

7 Umbetzhanov D.U. On almost-multiperiodic solutions of a certain integrodifferential equation of transport type / D.U. Umbetzhanov // Ukrainian Mathematical Journal. — 1989. — Vol. 41, 1. — P. 72–77. DOI:10.1007/BF01060650.

8 Umbetzhanov D.U. On a space of potentials of fractional smoothness with densities from the Stepanov class / D.U. Umbetzhanov // Ukrainian Mathematical Journal. — 1995. — Vol. 47, 10. — P. 1599–1605. DOI:10.1007/BF01060159.

9 Мухамбетова А.А. Устойчивость решений систем дифференциальных уравнений с многомерным временем / А.А. Мухамбетова, Ж.А. Сартабанов. — Актобе: Принт А, 2007. — 168 с.

10 Кульжумиева А.А. Периодические решения систем дифференциальных уравнений с многомерным временем / А.А. Кульжумиева, Ж.А. Сартабанов. — Уральск: РИЦ ЗКГУ, 2013. — 168 с.

11 Sartabanov Zh. Pseudoperiodic solutions of a system of integrodifferential equations / Zh. Sartabanov// Ukrainian Mathematical Journal. — 1989. — Vol. 41, 1. — P. 116–120.

12 Berzhanov A.B. Solution of a countable system of quasilinear partial differential equations multiperiodic in a part of variables / A.B. Berzhanov, E.K. Kurmangaliev // Ukrainian Mathematical Journal, 2009. — Vol. 61, 2. — P. 336–345. DOI:10.1007/s11253-009-0202-4.

13 Kenzhebaev K.K. Multiperiodic Solution of a Boundary-Value Problem for one Class of Parabolic Equations with Multidimensional Time / K.K. Kenzhebaev, G.A. Abdikalikova, A.B. Berzhanov // Ukrainian Mathematical Journal. — 2014. — Vol. 66, 5. — P. 719–731. DOI:10.1007/s11253-014-0967-y.

14 Abdikalikova G. On multiperiodicity and almost periodicity of solutions of boundary value problem for system of parabolic type equation / G. Abdikalikova, A. Berzhanov // AIP Conference Proceedings. — 2014. — Vol. 1611. — P. 58–62. DOI:10.1063/1.4893804.

15 Muhambetova B.Zh. Multiperiodic solutions of systems of equations with one quasi-linear differential operator in partial derivatives of the first order / B.Zh. Muhambetova, Zh.A. Sartabanov, A.A. Kulzhumieva // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 78, 2. — P. 112–117.

16 Kulzhumiyeva A.A. On reducibility of linear D-e-system with constant coefficients on the diagonal to D-e-system with Jordan matrix in the case of equivalence of its higher order one equation / A.A. Kulzhumiyeva, Zh.A. Sartabanov // Bulletin of the Karaganda University. Mathematics series. — 2016. — Vol. 84, 4. — P. 88–93.

17 Kulzhumiyeva A.A. On multiperiodic integrals of a linear system with the differentiation operator in the direction of the main diagonal in the space of independent variables / A.A. Kulzhumiyeva, Z.A. Sartabanov // Eurasian Mathematical Journal. — 2017. — Vol. 8, 1. — P. 67–75.

18 Kulzhumiyeva A.A. Reduction of linear homogeneous D-e-systems to the Jordan canonical form / A.A. Kulzhumiyeva, Zh.A. Sartabanov // News of the National Academy of Sciences of the Republic of Kazakhstan. Series Physico-Mathematical. — 2017. — Vol. 5, 315. — P. 5–12.

19 Sartabanov Z.A. The multi-period solution of a linear system of equations with the operator of differentiation along the main diagonal of the space of independent variables and delayed arguments / Z.A. Sartabanov // AIP Conference Proceedings. — 2017. — Vol. 1880, 040020. DOI:10.1063/1.5000636.

20 Kulzhumiyeva A.A. General bounded multiperiodic solutions of linear equations with differential operator in the direction of the mail diagonal / A.A. Kulzhumiyeva, Zh.A. Sartabanov // Bulletin of the Karaganda University. Mathematics series. — 2018. — Vol. 92, 4. — P. 44–53. DOI:10.31489/ 2018M4/44-53.

21 Sartabanov Z.A. Multiperiodic solutions of autonomous systems with operator of differentiation on the Lyapunov's vector field / Z.A. Sartabanov, B.Z. Omarova // AIP Conference Proceedings. 2018. — Vol. 1997, 020041. DOI:10.1063/1.5049035.

22 Sartabanov Zh.A. Multiperiodic solutions of a linear autonomous system with an operator of differentiation with respect to directions of the Lyapunov vector field / Zh.A. Sartabanov, B.Zh. Omarova // The proceedings of the XVIII International Scientific Conference on Differential Equations «Yerugin's Reading-2018». — Minsk: Institute of Math. NAS of Belarus, 2018. — Part 2. — P. 45, 46.

23 Omarova B. Multiperiodic solutions of the autonomous systems with the operator of differentiation along diagonal of the time and space variables on the Lyapunov's vector field / B. Omarova, Zh. Sartabanov // MADEA International Conference. Abstracts. — Bishkek: KTMU, 2018. — P. 94, 95.

24 Vejvoda O. Partial differential equations: time-periodic solutions / O. Vejvoda, L. Herrmann, V. Lovicar, M. Sova, I. Straskaba, M. Stedry. — Springer Netherlands, 1982. — 358 p.

25 Štědrý M. Time periodic solutions of a one-dimensional two-phase Stefan problem / M. Štědrý, O. Vejvoda // Annali di Matematica Pura ed Applicatal. — 1981. — Vol. 127, 1. — P. 67–78. DOI:10.1007/BF01811719.

26 Kerimbekov A. On the solvability of a nonlinear tracking problem under boundary control for the elastic

oscillations described by fredholm integro-differential equations / A. Kerimbekov, E. Abdyldaeva // IFIP Advances in Information and Communication Technology. — 2016. — Vol. 494. — P. 312–321. DOI:10.1007/978-3-319-55795-3_29.

27 Bergamasco A.P. Existence and Regularity of Periodic Solutions to Certain First-Order Partial differential equations / A.P. Bergamasco, P.L. Dattori da Silva, R.B. Gonzalez // Journal of Fourier Analysis and Applications. — 2017. — Vol. 23, 1. — P. 65–90. DOI:10.1007/s00041-016-9463-0.

28 Joyal P. Invariance of Poincaré-Lyapunov polynomials under the group of rotations / P. Joyal // Electronic Journal of Differential Equation. — 1998. — Vol. 1998. — P. 1–8.

29 Fonda A. Multiple periodic solutions of hamiltonian systems confined in a box / A. Fonda, A. Sfecci // Discrete and Continuous Dynamical Systems — Series A. — 2017. — Vol. 37, 3. — P. 1425–1436. DOI:10.3934/dcds.2017059.

30 Asanova A.T. Almost periodic solution of a semilinear parabolic equation / A.T. Asanova // Differential Equations. — 1998. — Vol. 34, 12. — P. 1705–1707.

31 Asanova A.T. On a bounded almost periodic solution of a semilinear parabolic equation / A.T. Asanova // Ukrainian Mathematical Journal. — 2000. — Vol. 52, 6. — P. 950–952. DOI:10.1007/ BF02591788.

32 Asanova A.T. Periodic solutions of systems of hyperbolic equations bounded on a plane / A.T. Asanova, D.S. Dzhumabaev // Ukrainian Mathematical Journal. — 2004. — Vol. 56, 4. — P. 682–694. DOI:10.1007 /s11253-005-0103-0.

33 Asanova A.T. Periodic solutions in the plane of systems of second-order hyperbolic equations / A.T. Asanova // Mathematical Notes. — 2017. — Vol. 101, 1–2. — P. 39–47. DOI:10.1134/ S0001434617010047.

34 Orumbayeva N.T. On an algorithm of finding periodical boundary value problem for system of the quasi-linear of hyperbolic equations / N.T. Orumbayeva // Siberian Electronic Mathematical Reports. — 2013. — Vol. 10. — P. 464–474.

35 Orumbayeva N.T. On Solvability of Non-Linear Semi-Periodic Boundary-Value Problem for System of Hyperbolic Equations / N.T. Orumbayeva // Russian mathematics. — 2016. — Vol. 60, 9. — P. 23–37. DOI:10.3103/S1066369X16090036.

Ж.А. Сартабанов, Б.Ж. Омарова

Ляпунов векторлық өрісі бойынша дифференциалдау операторлы квазисызықты автономдық жүйенің көппериодты шешімі туралы

Ляпунов векторлық өрісімен байланысты уақыт және кеңістік айнымалылы сипаттама бағыты бойынша дифференциалдау операторлы квазисызықты автономдық жүйе қарастырылды. Сызықты жүйенің матрицанты сипаттама бойында экспоненциалды орнықтылық қасиетке ие болғанда уақыт айнымалысы бойынша көппериодты шешімнің бар болуы туралы сұрақ зерттелді. Ал жүйенің сызықты емес бөлігі жеткілікті жатық болады. Мақалада Ляпунов әдісі негізінде көрсетілген дифференциалдау операторлы жүйенің сипаттамасының қажетті қасиеттері негізделді; біртекті және біртексіз сызықты жүйенің көппериодты шешімінің бар болуы және жалғыздығы туралы теорема дәлелденді; квазисызықты жүйенің жалғыз көппериодты шешімінің бар болуының жеткілікті шарты анықталды. Сызықты емес жүйені зерттеу барысында сығушы бейнелеу әдісі қолданылды.

Кілт сөздер: көппериодты шешім, автономдық жүйе, дифференциалдау операторы, Ляпунов векторлық өрісі.

Ж.А. Сартабанов, Б.Ж. Омарова

О многопериодических решениях квазилинейных автономных систем с оператором дифференцирования по векторному полю Ляпунова

Рассмотрена квазилинейная автономная система с оператором дифференцирования по характеристическим направлениям временных и пространственных переменных, связанных с векторным полем Ляпунова. Исследован вопрос о существовании многопериодических по временным переменным решений, когда матрицант линейной системы вдоль характеристик обладает свойством экспоненциальной устойчивости. А нелинейная часть системы является достаточно гладкой. В статье на основе метода Ляпунова обоснованы необходимые свойства характеристик системы с указанным оператором дифференцирования; доказаны теоремы о существовании и единственности многопериодических решений линейных однородных и неоднородных систем; установлены достаточные условия существования единственного многопериодического решения квазилинейной системы. При исследовании нелинейной системы использован метод сжатых отображений.

Ключевые слова: многопериодическое решение, автономная система, оператор дифференцирования, векторное поле Ляпунова.

References

1 Kharasakhal V.Kh. (1970). *Pochti-periodicheskie resheniia obyknovennykh differentsialnykh uravnenii [Almost-periodic solutions of ordinary differential equations].* Alma-Ata: Nauka [in Russian].

2 Kharasakhal, V.Kh. (1959). A generalization of a theorem of Bohr and Neugebauer. *Journal of Applied Mathematics and Mechanics, Vol. 23, 3,* 844-845. DOI:10.1016/0021-8928(59)90178-9.

3 Kharasakhal, V.Kh. (1960). Almost-periodic solutions of nonlinear systems of differential equations. *Journal of Applied Mathematics and Mechanics, Vol. 24, 3,* 838-842. DOI:10.1016/0021-8928(60)90194-5.

4 Kharasakhal, V.Kh. (1963). On quasi-periodic solutions of systems of ordinary differential equations. *Journal of Applied Mathematics and Mechanics, Vol. 27, 4,* 1019-1034. DOI:10.1016/0021-8928(63)90183-7.

5 Kharasakhal, V.Kh. (1965). Application of the point transformation method to quasi-periodic oscillations of nonlinear systems. *Journal of Applied Mathematics and Mechanics, Vol. 29, 3,* 703-707. DOI:10.1016 /0021-8928(65)90077-8.

6 Umbetzhanov, D.U. (1979). *Pochti mnohoperiodicheskie resheniia differentsialnykh uravnenii v chastnykh proizvodnykh [Almost multiperiodic solutions of partial differential equations].* Alma-Ata: Nauka [in Russian].

7 Umbetzhanov, D.U. (1989). On almost-multiperiodic solutions of a certain integrodifferential equation of transport type. *Ukrainian Mathematical Journal, Vol. 41, 1,* 72-77. DOI:10.1007/BF01060650.

8 Umbetzhanov, D.U. (1995). On a space of potentials of fractional smoothness with densities from the Stepanov class. *Ukrainian Mathematical Journal, Vol. 47, 10,* 1599–1605. DOI:10.1007/BF01060159.

9 Mukhambetova A.A., & Sartabanov Zh.A. (2007). *Ustoichivost reshenii sistem differentsialnykh uravnenii s mnohomernym vremenem [Stability of solutions of the systems of differential equations with multidimensional time].* Aktobe: Print A [in Russian].

10 Kulzhumieva A.A., & Sartabanov Zh.A. (2013). *Periodicheskie resheniia sistem differentsialnykh uravnenii s mnohomernym vremenem [Periodic solutions of the systems of differential equations with multidimensional time].* Uralsk: RITs ZKGU [in Russian].

11 Sartabanov, Zh. (1989). Pseudoperiodic solutions of a system of integrodifferential equations. *Ukrainian Mathematical Journal, Vol. 41, 1,* 116-120.

12 Berzhanov, A.B., & Kurmangaliev, E.K. (2009). Solution of a countable system of quasilinear partial differential equations multiperiodic in a part of variables. *Ukrainian Mathematical Journal, Vol. 61, 2,* 336-345. DOI:10.1007/s11253-009-0202-4.

13 Kenzhebaev, K.K., Abdikalikova, G.A., & Berzhanov, A.B. (2014). Multiperiodic Solution of a Boundary-Value Problem for one Class of Parabolic Equations with Multidimensional Time. *Ukrainian Mathematical Journal, Vol. 66, 5,* 719-731. DOI:10.1007/s11253-014-0967-y.

14 Abdikalikova, G., & Berzhanov, A. (2014). On multiperiodicity and almost periodicity of solutions of boundary value problem for system of parabolic type equation. *AIP Conference Proceedings, Vol. 1611,* 58-62. DOI:10.1063/1.4893804.

15 Muhambetova, B.Zh., Sartabanov, Zh.A., & Kulzhumieva, A.A. (2015). Multiperiodic solutions of systems of equations with one quasi-linear differential operator in partial derivatives of the first order. *Bulletin of the Karaganda university. Mathematics Series, Vol. 78, 2,* 112-117.

16 Kulzhumiyeva, A.A., & Sartabanov, Zh.A. (2016). On reducibility of linear D-e-system with constant coefficients on the diagonal to D-e-system with Jordan matrix in the case of equivalence of its higher order one equation. *Bulletin of the Karaganda university. Mathematics Series, Vol. 84, 4,* 88-93.

17 Kulzhumiyeva, A.A., & Sartabanov, Z.A. (2017). On multiperiodic integrals of a linear system with the differentiation operator in the direction of the main diagonal in the space of independent variables. *Eurasian Mathematical Journal, Vol. 8, 1,* 67-75.

18 Kulzhumiyeva, A.A., & Sartabanov, Zh.A. (2017). Reduction of linear homogeneous D-e-systems to the Jordan canonical form. *News of the national academy of sciences of the Republic of Kazakhstan. Physico-mathematical series, Vol. 5, 315,* 5-12.

19 Sartabanov, Z.A. (2017). The multi-period solution of a linear system of equations with the operator of differentiation along the main diagonal of the space of independent variables and delayed arguments. *AIP Conference Proceedings, Vol. 1880, 040020.* DOI:10.1063/1.5000636.

20 Kulzhumiyeva, A.A., & Sartabanov, Zh.A. (2018). General bounded multiperiodic solutions of linear equations with differential operator in the direction of the mail diagonal. *Bulletin of the Karaganda university. Mathematics Series, Vol. 92, 4,* 44-53. DOI:10.31489/2018M4/44-53.

21 Sartabanov, Z.A., & Omarova, B.Z. (2018). Multiperiodic solutions of autonomous systems with operator of differentiation on the Lyapunov's vector field. *AIP Conference Proceedings, Vol. 1997, 020041.* DOI: 10.1063/1.5049035.

22 Sartabanov, Zh.A. & Omarova, B.Zh. (2018). Multiperiodic solutions of a linear autonomous system with an operator of differentiation with respect to directions of the Lyapunov vector field. *The proceedings of the XVIII International Scientific Conference on Differential Equations «Yerugin's Reading-2018».* Minsk: Institute of Math. NAS of Belarus, 45, 46.

23 Omarova, B. & Sartabanov, Zh. (2018). Multiperiodic solutions of the autonomous systems with the operator of differentiation along diagonal of the time and space variables on the Lyapunov's vector field. *MADEA International Conference. Abstracts.* (pp. 94-95). Bishkek: KTMU.

24 Vejvoda, O., Herrmann, L., Lovicar, V., Sova, M., Straskaba, I., & Stedry M. (1982). *Partial differential equations: time-periodic solutions.* Springer, Netherlands.

25 Štědrý, M., & Vejvoda, O. (1981). Time periodic solutions of a one-dimensional two-phase Stefan problem. *Annali di Matematica Pura ed Applicata, Vol. 127, 1,* 67-78. DOI:10.1007/BF01811719.

26 Kerimbekov, A., & Abdyldaeva, E. (2016). On the solvability of a nonlinear tracking problem under boundary control for the elastic oscillations described by fredholm integro-differential equations. *IFIP Advances in Information and Communication Technology, Vol. 494,* 312-321. Doi:10.1007/978-3-319-55795-3_29.

27 Bergamasco, A.P., Dattori da Silva, P.L., & Gonzalez, R.B. (2017). Existence and Regularity of Periodic Solutions to Certain First-Order Partial Differential Equations. *Journal of Fourier Analysis and Applications, Vol. 23, 1,* 65–90. DOI:10.1007/s00041-016-9463-0.

28 Joyal, P. (1998). Invariance of Poincaré-Lyapunov polynomials under the group of rotations *Electronic Journal of Differential Equations, Vol. 1998,* 1–8.

29 Fonda, A., & Sfecci A. (2017). Multiple periodic solutions of hamiltonian systems confined in a box. *Discrete and Continuous Dynamical Systems. Series A, Vol. 37,* 3, 1425-1436. DOI:10.3934/. dcds.2017059

30 Asanova, A.T. (1998). Almost periodic solution of a semilinear parabolic equation. *Differential Equations, Vol.34, 12,* 1705-1707.

31 Asanova, A.T. (2000). On a bounded almost periodic solution of a semilinear parabolic equation. *Ukrainian Mathematical Journal, Vol. 52, 6,* 950-952. DOI:10.1007/BF02591788.

32 Asanova, A.T., & Dzhumabaev, D.S. (2004). Periodic solutions of systems of hyperbolic equations bounded on a plane. *Ukrainian Mathematical Journal, Vol. 56, 4,* 682-694. DOI:10.1007/s11253-005-0103-0.

33 Asanova, A.T. (2017). Periodic solutions in the plane of systems of second-order hyperbolic equations. *Mathematical Notes, Vol. 101, 1-2,* 39-47. DOI:10.1134/S0001434617010047.

34 Orumbayeva, N.T. (2013). On an algorithm of finding periodical boundary value problem for system of the quasi-linear of hyperbolic equations. *Siberian Electronic Mathematical Reports, Vol. 10,* 464-474.

35 Orumbayeva, N.T. (2016). On Solvability of Non-Linear Semi-Periodic Boundary-Value Problem for System of Hyperbolic Equations. *Russian Mathematics, Vol. 60, 9,* 23-37. DOI:10.3103/S1066369X16090036.

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The atomic definable subsets of semantic model

In this paper some properties of small models, generally speaking, not necessarily complete theories and their relationship with each other were considered. Under small models we will understand some modifications of the concepts of countable atomic and prime models. These models were defined in the study of countable models of complete theories. Studies were conducted by analogy with the classic result of R. Vaught on countable-prime models of complete theories, but by other technical means. This work is oriented on the syntactic properties of special subsets of the semantic model of some Jonsson theory. A new concept was also introduced, as a model-theoretic «rheostat», in order to obtain results related to the refinement of concept of atomicity within the framework of Jonsson theories. Thus, the main purpose of this article is to formally define this «rheostat» and to obtain on the basis of this concept the results having a relation to the refinement of the concept of atomicity in the frame of Jonsson theories.

Keywords: Jonsson theory, semantic model, existentially prime model, atomic model, convexity.

The study of properties of the so-called small models for various subclasses of inductive theories is connecting with given work. Each time the difference in these subclasses always depends on the special conditions imposed on some inductive theory. These conditions are of the following type: the joint embedding property, the amalgam property, convexity, an existential primeness, a certain type of completeness, perfectness in the sense of Jonsson theory, if the considered inductive theory is such. All of these conditions are connected with considered theory, but are not purely syntactic, since the above mentioned definitions of conditions relate to the class of models of considered theory, for example, convexity and an existential primeness. On the other hand, the main accent of this work is oriented on the syntactic properties of special subsets of the semantic model of some Jonsson theory.

We consider small models which are some modifications of the concepts of countable atomic and prime models defined in the study of countable models of complete theories. This subject was originally defined after R. Vaught's classical work [1] where it was proved that a model is atomic if and only if it is countable and prime. In [2] D. Baldwin and D. Kueker considered a more general situation in the sense of theory and in the sense of small models.

Namely, the theory was assumed only complete for some type of sentences, i.e. generally speaking, not the complete and the concepts of atomicity and primeness of model, in contrast to the concept of atomicity and primeness of the model from [1], were also more general because the atoms of a theory were considered within the formulas lattice and not the Boolean algebra of the Lindenbaum-Tarsky. Also, instead of elementary embeddings, isomorphic embeddings were considered. Further, we give these definitions and consider in detail their differences.

As the results of [2] showed, unfortunately, it was not possible to obtain, analogue of the above mentioned, the result describing a small models from [1]. Thus, the study of the behavior of the small models in the class of incomplete theories is an actual problem and in general it has not been solved yet. Moreover, due to the examples in [2], there is a confidence that this problem will not be solved in the formulated framework of [2]. In this regard, one of the authors of this article in [3–6] formulated the problem of characterization of countable atomic and prime models in the study of Jonsson theories, its class, first of all, is a natural subclass of inductive theories, secondly, there are a lot of classical natural examples from algebra which satisfy the conditions of Jonsson theories. Thus, it seems to us it is possible to narrow down the scope of the above mentioned problem, both in the syntactic and semantic sense in fairly wide subclass of inductive theories to advance a search for a positive solution of this problem. It is necessary to find acceptable conditions that connect corresponding concepts of atomicity and algebraic primeness of models in the framework of studying the data restrictions on the theory.

In this introduction we would like to announce the main idea of our approach to the study of the description of small models within the above-mentioned. Since, by virtue of the above concepts of the atomic model, in senses of [1] and [2], there are essentially different, we propose to consider in a certain sense a «continuous» transition from the notion of model atomicity in the sense of [1] to the notion of model atomicity in the sense of [2]. Thus, we tried gradually to isolate the metamorphosis of the transition of these concepts into each other in stages. That is, to construct conditionally in a certain sense a model-theoretic «rheostat» by moving, we get the concepts of atomicity for complete and Jonsson theories, gradually. Thus, the main purpose of this article is formally to define this «rheostat» and to obtain, on the basis of this concept, results concerning to the refinement of the concept of atomicity within of Jonsson theories.

We give the following definitions of concepts and related results from that part of model theory that are necessary for the study of Jonsson theories.

Definition 1. A theory T is a Jonsson if:

1) theory T has infinite models;
2) theory T is inductive;
3) theory T has the joint embedding property (JEP);
4) theory T has the property of amalgam (AP).

Examples of Jonsson theories are:

1) the group Theory,
2) the theory of Abelian groups,
3) the theory of fields of fixed characteristic,
4) the theory of Boolean algebras,
5) the theory of polygons over a fixed monoid,
6) the theory of modules over a fixed ring,
7) the theory of linear order.

The following definition of the universality and homogeneity of model allocates semantic invariant of any Jonsson theory, namely its semantic model. Moreover, it turned out that the saturation or non-saturation of this model significantly changes the structural properties of both the Jonsson theory itself and its class of models.

Definition 2. Let $\kappa \geq \omega$. Model M of theory T is said to be κ-universal for T, if each model T with the power strictly less κ isomorphically imbedded in M; κ-homogeneous for T, if for any two models A and A_1 of theory T, which are submodels of M with the power strictly less then κ and for isomorphism $f : A \to A_1$ for each extension B of model A, which is a submodel of M and is model of T with the power strictly less then κ there exists the extension B_1 of model A_1, which is a submodel of M and an isomorphism $g : B \to B_1$ which extends f.

Definition 3. Model C of Jonsson theory T is said to be semantic model, if it is ω^+- homogeneous-universal.

As can be seen from the definition of the Jonsson theory, this theory is not complete. But nevertheless, with the help of its semantic invariant (semantic model) we can always determine the center of Jonsson theory, which is a complete theory.

Definition 4. The center of Jonsson theory T is said to be an elementary theory of its semantic model. And denoted through T^*, i.e. $T^* = Th(C)$.

The following two facts speak about the «good» exclusivity of the semantic model.

Fact 1. Each Jonsson theory T has k^+- homogeneous-universal model of power 2^k. Conversely, if a theory T is inductive and has infinite model and ω^+- homogeneous-universal model then the theory T is a Jonsson theory.

Fact 2. Let T be a Jonsson theory. Two k-homogeneous-universal models M and M_1 of T are elementary equivalent.

It is well known from the course of model theory that a saturated model is always a homogeneous-universal model, the reverse is also true. But this definition of homogeneous-universal model [7; 299] is considered as a rule in the framework in the study of complete theory. In the framework of the study of Jonsson theory, we are dealing with a particular case of the definition of a homogeneous-universal model belonging to B. Jonsson [8]. The concept of a saturated model is the same in both cases. By virtue of a more general situation of homogeneous-universality in the case of Jonsson theory, we do not have a saturation criterion in terms of homogeneous-universal as in [7; 299]. Therefore, those Jonsson theories, the semantic model of which is saturated, allocate in a special subclass of class of all Jonsson theories, and such theories are called perfect. We give a definition of perfectness of Jonsson theory.

Definition 5. Jonsson theory T is said to be a perfect theory, if each semantic model of theory T is saturated model of T^*.

The first author of this article obtained a result describing the perfect Jonsson theory.

Theorem 1 [9]. Let T be a Jonsson theory. Then the following conditions are equivalent:

1) Theory T is perfect;

2) Theory T^* is a model companion of theory T.

From the above list of Jonsson theories, the following examples 2)-4), 6),7) are examples of a perfect Jonsson theory. But, for example, group theory is not such, due to the fact that it does not have a model companion.

Let E_T be a class of all existentially closed models of Jonsson theory T.

This class of models in general case for an arbitrary theory can be empty. But the following result [10; 367] is well known, who says that any inductive theory has a nonempty class of existentially closed models. Since the Jonsson theory is a subclass of the class of inductive theories, we can say that E_T is a non-empty class.

In the case of a perfect Jonsson theory, the class of models of center of this theory coincides with E_T. This follows from the following theorem.

Theorem 2 [9]. If T is a perfect Jonsson theory then $E_T = ModT^*$.

Let L be a countable language of first order. Let T be Jonsson theory in the language L and its semantic model is C.

Let us turn to the definition of central concept of this article. Namely, the concept of a $(\nabla_1, \nabla_2) - cl$ atomic set.

Let T be some Jonsson theory in a fixed language and C — its semantic model.

Definition 6. Model A of a theory T is said to be existentially closed if for any model B and any existential formula $\varphi(\overline{x})$ with constants of A we have $A \models \exists \overline{x} \varphi(\overline{x})$ provided that A is a submodel of B and $B \models \exists \overline{x} \varphi(\overline{x})$.

Definition 7. A is an algebraically prime model of theory T, if A is a model of T and A may be isomorphically embedded in each model of the theory T.

Definition 8. The inductive theory T is said to be the existentially prime if: 1) it has an algebraically prime model, the class of its AP (algebraically prime models) denote by AP_T; 2) class E_T non trivial intersects with class AP_T, i.e. $AP_T \bigcap E_T \neq 0$.

Definition 9. The theory T is said to be convex if for any its model A and for any family $\{B_i \mid i \in I\}$ of substructures of A, which are models of the theory T, the intersection $\bigcap_{i \in I} B_i$ is a model of T, provided it is non-empty. If in addition such an intersection is never empty, then T is said to be strongly convex.

Definition 10. A model is said to be atomic if every tuple of its elements satisfies some complete formula.

Definition 11. A formula $\varphi(\overline{x})$ is a Δ-formula, if exist existential formulas (from Σ) $\psi_1(\overline{x})$ and $\psi_2(\overline{x})$ such as

$$T \models (\varphi \leftrightarrow \psi_1) \quad \text{и} \quad T \models (\neg\varphi \leftrightarrow \psi_2).$$

Definition 12.

(i) $(A, a_0, ..., a_{n-1}) \Rightarrow_\Gamma (B, b_0, ..., b_{n-1})$ means that for every formula $\varphi(x_1, ..., x_{n-1})$ of Γ, if $A \models \varphi(\overline{a})$, then $B \models \varphi(\overline{b})$.

(ii) $(A, \overline{a}) \equiv_\Gamma (B, \overline{b})$ means that $(A, \overline{a}) \Rightarrow_\Gamma (B, \overline{b})$ and $(B, \overline{b}) \Rightarrow_\Gamma (A, \overline{a})$.

As classes Γ we consider $\triangle$ or Σ.

The following definition of an atomic model refers to [1].

Consider a complete theory T in L. A formula $\varphi(x_1...x_n)$ is said to be complete (in T) if and only if for every formula $\psi(x_1...x_n)$ exactly one of

$$T \models \varphi \rightarrow \psi, \quad T \models \varphi \rightarrow \neg\psi$$

holds. A formula $\theta(x_1...x_n)$ is said to be completable (in T) if and only if there is a complete formula $\varphi(x_1...x_n)$ with $T \models \varphi \rightarrow \theta$. If $\theta(x_1...x_n)$ is not completable it is said to be incompleatable.

A theory T is said to be atomic if and only if every formula of L which is consistent with T is completable in T. A model A is said to be an atomic model if and only if every n- tuple $a_1...a_n \in A$ satisfies a complete formula in $Th(A)$.

Definition 13. A model is said to be atomic if every tuple of its elements satisfies some complete formula. In connection with the new concept of atomicity from [2], the following concept will be analogous to the definition of a complete formula.

Definition 14. A formula $\varphi(x_1, ..., x_n)$ is complete for Γ-formulas if φ is consistent with T and for every formula $\psi(x_1, ..., x_n)$ in Γ, having no more free variables than φ, or

$$T \models \forall \overline{x}(\varphi \to \psi).$$

Equivalently, a consistent $\varphi(\overline{x})$ is complete for Γ-formulas provided whenever as $\psi(\overline{x})$ is a Γ- formula and $(\varphi \wedge \psi)$ is consistent with T, then $T \models (\varphi \to \psi)$.

And the concept of the atomic model from [1] is transformed into the following concept from [2].

Definition 15. B is a $(\Gamma_1, \Gamma_2,)$ — atomic model of T, if B is a model of T and for every n every n-tuple of elements of A satisfies some formula from B in Γ_1, which is complete for Γ_2-formulas.

The following notion of a weakly atomic model from [2] is a generalization of above definition.

Definition 16. B is a weak (Γ_1, Γ_2) — atomic model of T, if B is a model of T and for every n every n-tuple $\overline{a}$a of elements of A satisfies in B some formula $\varphi(\overline{x})$ of Γ_1 such as $T \models (\varphi \to \psi)$ as soon as $\psi(\overline{x})$ of Γ_2 and $B \models \psi(\overline{a})$.

In this paper we will not give examples of the (Γ_1, Γ_2) — atomic model and the weak (Γ_1, Γ_2) atomic model, leaving the reader to do this on their own, referring to a sufficient, the number of examples of these concepts given in [2].

Before discussing the obtained results, concerning to $(\nabla_1, \nabla_2) - cl$ atomic models, we note that we fix some Jonsson theory T and its semantic model C in the countable language L and $\nabla_1, \nabla_2 \subseteq L : (\nabla_1, \nabla_2)$ actually those sets consist of $\exists, \forall, \forall\exists$ — formulas which are consistent with T, that is, any finite subset of formulas from ∇_1, ∇_2 are consistent with T. Let $A \subseteq C$.

Let cl is some closure operator defining a pregeometry over C (for example $cl = acl$ or $cl = dcl$). It is clear that such operator is a special case of the closure operator and its example is a closure operator defined on any linear space as a linear shell.

We also assume that the pregeometry given by the cl operator is modular [8].

Definition 17. A set A is said to be $(\nabla_1, \nabla_2) - cl$ atomic in the theory T, if

1) $\forall a \in A, \exists \varphi \in \nabla_1$ such as for any formula $\psi \in \nabla_2$ follows that φ is complete formula for ψ and $C \models \varphi(a)$;

2) $cl(A) = M, M \in E_T$, and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$ atomic model of theory T.

Definition 18. A set A is said to be weakly $(\nabla_1, \nabla_2) - cl$ is atomic in T, if

1) $\forall a \in A, \exists \varphi \in \nabla_1$ such as in $C \models \varphi(a)$ for any formula $\psi \in \nabla_2$ follows that $T \models (\varphi \to \psi)$ whenever $\psi(x)$ of ∇_2 and $C \models \psi(a)$;

2) $cl(A) = M, M \in E_T$, and obtained model M is said to be weakly $(\nabla_1, \nabla_2) - cl$ atomic model of theory T.

It is easy to understand that definitions 17 and 18 are naturally generalized the notion of atomicity and weak atomicity to be ∇_1-atom and weak ∇_1-atom for any tuple of finite length from set A.

Let $i \in \{1, 2\}$, $M_i = cl(A_i)$, where $A_i = (\nabla_1, \nabla_2)$ is a $cl-$ atomic set . $a_0, ..., a_{n-1} \in A_1$, $b_0, ..., b_{n-1} \in A_2$.

Definition 19.

(i) $(M_1, a_0, ..., a_{n-1}) \Rightarrow_\nabla (M_2, b_0, ..., b_{n-1})$ means that for every formula $\varphi(x_1, ..., x_{n-1})$ of ∇, if $M_1 \models \varphi(\overline{a})$, then $M_2 \models \varphi(\overline{b})$.

(ii) $(M_1, \overline{a}) \equiv_\nabla (M_2, \overline{b})$ means that $(M_1, \overline{a}) \Rightarrow_\nabla (M_2, \overline{b})$ and $(M_1, \overline{b}) \Rightarrow_\nabla (M_1, \overline{a})$.

Definition 20. A set A is said to be $(\nabla_1, \nabla_2) - cl$-algebraically prime in the theory T, if

1) If A is $(\nabla_1, \nabla_2) - cl$-atomic set in T;

2) $cl(A) = M, M \in AP_T$, and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$ algebraically prime model of theory T.

From the definition of an algebraically prime set in the theory T follows that the Jonsson theory T which has an algebraically prime set is automatically existentially prime. It is easy to understand that an example of such a theory is the theory of linear spaces.

Definition 21. The set A is said to be $(\nabla_1, \nabla_2) - cl$-core in the theory T, if

1) A is (∇_1, ∇_2) a cl - atomic set in the theory T;

2) $cl(A) = M$, and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$ core model of theory T.

Definition 22. (a) A - $(\nabla_1, \nabla_2) - cl$-atomic set in theory T is said to be A - $(\nabla_1, \nabla_2) - cl$-$\Sigma$-nice-set in theory T, $\forall A' : A'$ - $(\nabla_1, \nabla_2) - cl$-atomic set in theory T, if

1) $cl(A) = M \in E_T \cap AP_T$, and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$-$\Sigma$-nice model of theory T.

2) for all $a_0, ..., a_{n-1} \in A$, $b_0, ..., b_{n-1} \in A'$, if $(M, a_0, ..., a_{n-1}) \Rightarrow_\exists (M', b_0, ..., b_{n-1})$, then $\forall a_n \in A$, $\exists b_n \in A'$ such as $(M, a_0, ..., a_n) \Rightarrow_\exists (M', b_0, ..., b_n)$, where $M' = cl(A')$.

(b) A - $(\nabla_1, \nabla_2) - cl - \Sigma^*$- nice-set in theory T if the condition in (a) holds with '$\Rightarrow_\exists$' replaced both places it occurs by '$\equiv_\exists$'

and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$-$\Sigma^*$- nice model of theory T.

(c) A - $(\nabla_1, \nabla_2) - cl - \Delta$- nice-set in theory T if the condition in (a) holds with '$\Rightarrow_\Delta$' replaced both places it occurs by '$\equiv_\Delta$', where $\Delta \subseteq L, \Delta = \forall \cap \exists$.

and obtained model M is said to be $(\nabla_1, \nabla_2) - cl - \Delta$-nice model of theory T.

Principle of «rheostat».

Let two countable models A_1, A_2 of some Jonsson theory T be given. Moreover, A_1 is an atomic model in the sense of [1], and X is $(\nabla_1, \nabla_2) - cl$ -algebraically prime set of theory T and $cl(X) = A_2$. Since $\nabla_1 = \nabla_2 = L$, then $A_1 \cong A_2$.

By the definition of (∇_1, ∇_2) - algebraic primeness of the set X, the model A_2 is both existentially closed and algebraically prime. Thus, the model A_2 is isomorphically embedded in the model A_1. If by condition the model A_1 is countably atomic, then according to the Vaught's theorem, A_1 is prime, i.e. it is elementarily embedded in the model A_2. Thus, the models A_1, A_2 differ from each other only by the interior of the set X. This follows from the fact that any element of $a \in A_2 \backslash X$ implements some principal type, since $a \in cl(X)$. That is, all countable atomic models in the sense of [1] are isomorphic to each other, then by increasing X we find more elements that do not realize the principal type and, accordingly, $cl(X)$ is not an atomic model in the sense of [1]. Thus, the principle of rheostat is that, by increasing the power of the set X, we move away from the notion of atomicity in the sense of [1] and on the contrary, decreasing the power of the set X we move away from the notion of atomicity in the sense of [2].

Let $APC \in$ {atomic, algebraically prime, core}. Thus, by specifying the set X as $(\nabla_1, \nabla_2) - cl - APC$, (where APC is a semantic property), we can also specify atomicity in the sense [2] in relation to atomicity in the sense of [1]. And according to the usage using of the principle of «rheostat» after the APC property is defined, we obtain the corresponding concepts of atomic models, the role of which is played by A_2 from the principle of «rheostat».

Two authors of this article have got the following result which connected with this topic and this result will publish in near future.

Theorem 1. Let T be complete for $\exists$-sentences, a strongly convex Jonsson perfect theory and let A is $(\nabla_1, \nabla_2) - cl$-atomic set in T.

Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \wedge (vi), (i) \Rightarrow (i)^* \Rightarrow (v) \Rightarrow (vi), (ii) \Rightarrow (ii)^* \Rightarrow (vi)$, $(i)^* \Rightarrow (ii)^*$ and $(iv)^* \Rightarrow (iv)$, where:

(i) A is $(\Delta, \Sigma) - cl$-atomic set in theory T,

$(i)^*$ A is weakly $(\Delta, \Pi) - cl$-atomic set in theory T,

(ii) A is $(\Sigma, \Sigma) - cl$-atomic set in theory T,

$(ii)^*$ A is weakly $(\Sigma, \Pi) - cl$-atomic set in theory T,

(iii) A is weakly $(\Sigma, \Sigma) - cl$-atomic set in theory T,

(iv) $cl(A) \in AP_T$,

$(iv)^*$ A is core set in theory T,

(v) A is weakly $(\Delta, \Delta) - cl$-atomic set in theory T,

(vi) A is weakly $(\Sigma, \Delta) - cl$-atomic set in theory T.

In according to the above mentioned notions, we have the following theorems. Those results are very close to investigation around atomicity and algebraically primeness in the frame of [2]. Nevertheless even if algebraically primeness is the same, but the combinations of APC-atomicity differ from atomicity from [2].

Theorem 2. Let T be complete for $\exists$-sentences, a strongly convex Jonsson perfect theory and let M is $(\nabla_1, \nabla_2) - cl$-atomic model in T.

(a) Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (vi), (i) \Rightarrow (i)^* \Rightarrow (v) \Rightarrow (vi), (ii) \Rightarrow (ii)^* \Rightarrow (vi)$, and $(i)^* \Rightarrow (ii)^*$ and $(iv)^* \Rightarrow (iv)$, where:

(i) M is $(\Delta, \Sigma) - cl$-atomic model in theory T,

$(i)^*$ M is a weakly $(\Delta, \Pi) - cl$-atomic model in theory T

(ii) M is a $(\Sigma, \Sigma) - cl$-atomic model in theory T,

$(ii)^*$ M is a weakly $(\Sigma, \Pi) - cl$-atomic model in theory T,

(iii) M is a weakly $(\Sigma, \Sigma) - cl$-atomic model in theory T,

(iv) $M \in AP_T$,

$(iv)^*$ M is core model in theory T,

(v) M is $(\Delta, \Delta) - cl$-atomic model in theory T,

(vi) M is $(\Sigma, \Delta) - cl$-atomic model in theory T,

(b) If T is complete for $\forall\exists$ sentences, then $(i) \Leftrightarrow (i)^*$ and $(ii) \Leftrightarrow (ii)^*$.

Proof. We have to note that for us sufficiently work with the $(\Sigma, \Sigma) - cl$-atomic set, i.e. the elements from outside of this set in any type of atomic model of this theory will realize the principal type and this fact allowed say for us that such element belongs to any type of atomic model. It is sufficient to get that existence of any kind of atomic model in our meaning following from fact that a given theory has countable atomic models of any type of it. Therefore, all parts of this Theorem are immediate from Theorem 1 and the existence of such sets admit for us corresponding type of atomicity. We have only one exception: that is the implication $(iv) \Rightarrow (vi)$. So, assume that T has an algebraically prime model B. Let $\bar{b}_n \in B$ and let $\{\psi_i(\overline{x}) : i \in \omega\}$ list all the Δ-formulas satisfied by $\bar{b}$ in B. Since B is a.p. we know there is no model of T satisfying $\forall\overline{x} \bigvee_i \neg\psi_i$. Since $\neg\psi_i$ is existential we can apply above pointed Theorem 1 to get an existential formula $\psi(\overline{x})$ consistent with T such as $T \models (\psi \to \psi_i)$ for all $i \in \omega$. This ψ is hence complete for Δ-formulas, and also implies any open formula satisfied by $\overline{a}$ in A. Every open formula consistent with T is satisfied by some $a_0, ..., a_n \in A$, and therefore is implied by some existential formula complete for Δ-formulas. In the connections with the fact that we obtained $(\Sigma, \Sigma) - cl$-algebraically prime model in theory T we can apply Theorem 2.2 from [2] (iii) in order to show that T has a $(\Sigma, \Delta) - cl$-atomic model.

Lemma 1 [11]. Let T be complete for existential sentences perfect Jonsson theory.

1) If A is weakly $(\nabla, \Delta) - cl$-atomic set in the theory T, then A is $(\nabla, \Delta) - cl$-atomic set,

2) If A is weak $(\nabla, \Delta) - cl$-atomic set in the theory T, then A is $(\nabla, \Delta) - cl$-atomic set.

Before we will prove the theorem 2 let us note the following Remark.

Remark 1 [11]. By the perfectness of T, we can apply Lemma 1 and then, by Lemma 1, we can replace ∇_i on Δ, where $i \in \{1, 2\}$. Due to the strongly convexity of the theory, the theory T has a unique core model. This follows from the fact that if the theory satisfies the property of joint embedding and is additionally strongly convex, then its core model in the theory T is unique up to isomorphism [7]. Based on this fact, we can conclude that under the conditions of this theorem we have a unique core model, since its existence follows from strongly convexity, and its uniqueness follows from the combination with Jonssonness.

Theorem 3. Let T be complete for $\exists$-sentences a strongly convex Jonsson perfect theory and let M is $(\nabla_1, \nabla_2) - cl$-atomic model in T.

(a) Then $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(ii) \Rightarrow (ii)^*$ where:

(i) M is $(\Sigma, \Sigma) - cl$-atomic model in theory T,

(ii) M is $(\nabla_1, \nabla_2) - cl$-$\Sigma^*$-nice-model in theory T,

$(ii)^*$ M is e.c. and $(\nabla_1, \nabla_2) - cl$-$\Sigma$-nice-model in theory T,

(iii) M is weak $(\Sigma, \Pi) - cl$-atomic model in theory T,

(b) If T is complete for $\forall\exists$ sentences, then $(i), (ii), (ii)^*$ and (iii) are all equivalent.

Proof. Easy to note that proof of this theorem follows from Theorem 1, Theorem 2, Remark 1 and Theorem 3.2 from [2].

From above results of Theorem 2 and Theorem 3 we can conclude that the mechanism of «rheostat» for atomicity behave regard itself very predictable: if more of elements from $(\nabla_1, \nabla_2) - cl$-atomic set inside in an atomic model by meaning of [1] we obtain more far property of atomicity from [1] and more close to atomicity from [2].

All concepts, that are not defined here, can be extracted from [9, 11].

References

1 Vaught R. Denumerable models of complete theories in Infinitistic Methode / R. Vaught // Pergamon. — London, 1961. — P. 303–321.

2 Baldwin J.T. Algebraically prime models / J.T. Baldwin, D.W. Kueker // Ann. Math. Logic. — 1981. — 20. — P. 289–330.

3 Yeshkeyev A.R. Strongly minimal Jonsson sets and their properties / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 80, No. 4. — P. 47–51.

4 Poizat B. Positive Jonsson Theories / B. Poizat, A.R. Yeshkeyev // Logica Universalis. — 2018. — Vol. 12, No. 1-2. — P. 101–127.

5 Yeshkeyev A.R. Properties of lattices of the existential formulas of Jonsson fragments / A.R. Yeshkeyev, M.T. Kasymetova // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79, No. 3. — P. 25-32.

6 Yeshkeyev A.R. On lattice of existential formulas for fragment of Jonsson set / A.R. Yeshkeyev, O.I. Ulbrikht // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79, No. 3. — P. 33–39.

7 Кейслер Х.Дж. Теория моделей / Х.Дж. Кейслер , Ч.Ч. Чэн. — М.: Мир, 1977. — 614 с.

8 Jonsson B. Homogeneous universal relational systems / B. Jonsson. — Math. Scand, 1960. — P. 137–142.

9 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во КарГУ, 2016. — 370 с.

10 Hodges W.A. Model Theory / W.A. Hodges. — Cambridge University Press, 1993. — P. 772.

11 Yeshkeyev A.R. ∇-cl-atomic and prime sets / Yeshkeyev A.R., Issayeva A.K. // Bulletin of the Karaganda University. Mathematics series. — 2019. — Vol. 93, No. 1. — P. 88–94.

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Семантикалық модельдің атомдық анықталған ішкі жиындары

Мақалада кішігірім модельдердің кейбір қасиеттері, жалпы айтқанда, олар толық болу міндетті емес және олардың өзара қарым-қатынасы қарастырылған. Кішігірім модельдер деп толық теориялардың саналымды модельдерін зерттеу кезінде анықталған саналымды атомдық және жай модельдер ұғымдарының кейбір модификацияларын айтамыз. Зерттеулер Р. Вооттың саналымды жай модельдердің толық теориялары туралы классикалық нәтижесі бойынша, бірақ басқа техникалық құралдар қолдануымен жүргізілді. Бұл мақала кейбір йонсондық теорияның семантикалық моделінің арнайы ішкі жиындарының синтаксистік қасиеттеріне бағытталған. Йонсон теориясының шеңберінде атомдық тұжырымдаманы нақтылауға байланысты нәтижелер алу үшін модельдік-теориялық «реостат» ретінде жаңа тұжырым енгізілді. Осылайша, бұл мақаланың негізгі мақсаты — осы «реостатты» формалды түрде анықтау және осы ұғымның негізінде йонсондық теориялар аясында атомдық ұғымды нақтылауға түсініктемесі бар нәтижелерді алу.

Кілт сөздер: йонсондық теория, семантикалық модель, экзистенциалды жай модель, атомдық модель, дөңестілік.

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Атомные определимые подмножества семантической модели

В статье были рассмотрены некоторые свойства малых моделей, вообще говоря, необязательно полных теорий и их связь между собой. Под малыми моделями будем понимать некоторые модификации понятий счетных атомных и простых моделей, определенных при изучении счетных моделей полных теорий. Исследования проводились по аналогии с классическим результатом Р. Воота о счетно-простых моделях полных теорий, но другими техническими средствами. Данная работа ориентирована на синтаксические свойства специальных подмножеств семантической модели некоторой йонсоновской теории. Также было введено новое понятие — как теоретико-модельный «реостат», с целью получить результаты, имеющие отношение к уточнению понятия атомности в рамках йонсоновских теорий. Таким образом, основная цель данной статьи — формально определить данный «реостат» и получить на основе данного понятия результаты, имеющие отношение к уточнению понятия атомности в рамках йонсоновских теорий.

Ключевые слова: йонсоновская теория, семантическая модель, экзистенциально простая модель, атомная модель, выпуклость.

References

1 Vaught, R. (1961). *Denumerable models of complete theories in Infinitistic Methode.* Pergamon. London.

2 Baldwin, J.T., & Kueker, D.W. (1981). Algebraically prime models. *Ann. Math. Logic., 20,* 289–330.

3 Yeshkeyev, A.R. (2015). Strongly minimal Jonsson sets and their properties. *Bulletin of the Karaganda University. Mathematics series, Vol. 80, 4,* 47–51.

4 Poizat, B., & Yeshkeyev, A.R.(2018). Positive Jonsson Theories. *Logica Universalis, Vol. 12, 1-2,* 101–127.

5 Yeshkeyev, A.R., & Kasymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3,* 25–32.

6 Yeshkeyev, A.R., & Ulbrikht, O.I. (2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3,* 33–39.

7 Keisler, H.J., & Chang, C.C. (1977). *Teoriia modelei [Model Theory].* Moscow: Mir [in Russian].

8 Jonsson, B. (1960). *Homogeneous universal relational systems.* Math. Scand.

9 Yeshkeyev, A.R., & Kasymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models].* Karaganda: Izdatelstvo KarHU [in Russian].

10 Hodges, W.A. (1993). *Model Theory.* Cambridge University Press.

11 Yeshkeyev, A.R., & Issayeva, A.K. (2019). ∇-cl-atomic and prime sets. *Bulletin of the Karaganda University. Mathematics series, Vol. 93, 1,* 88–94.

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The J-minimal sets in the hereditary theories

Our attention in given article will be paid to the study of model-theoretic properties of hereditary Jonsson theories, while we consider Jonsson theories that retain jonsonness under any admissible enrichment. In given paper new concepts of «essential type», «essential geometric base» are introduced, the orbital types and strongly minimal sets within the framework of special subsets of the semantic model, on which a closure operator is given, defining the special geometry of Jonsson are considered. The results for the J-strongly minimal types of the semantic model in the case, when these sets are separated from the orbits of the central types of Jonsson hereditary theories are also obtained.

Keywords: Jonsson theory, semantic model, hereditary theory, J-minimal set, J-strongly minimal set, permissible enrichment, central type, orbital type, essential type.

It is well known that any first-order theory can be transformed in easy way to Jonsson theory. Thus, if we consider some model of an arbitrary signature, then the choice of all Jonsson theories of this model in the special class is a natural admissibility, and the study of the model-theoretic properties of such class is undoubtedly actual problem. This is due to the fact that in the classical model theory there are historically established two approaches to the study of theories and their classes of models. In the first case, a class of complete theories is usually considered and, correspondingly, their models are studied. In the second case, we consider a class of theories, generally speaking incomplete theories, but with some additional properties, while naturally there are limitations and under study to the classes of models of such theories. Today, more research in model theory is associated with the study of model-theoretic properties of a class of theories of the first kind, i.e. complete theories, as well and the study of their classes of models.

An essential example of the theory from the second case is the class of inductive theories of a fixed signature, and this class has a subclass of Jonsson theories that define natural algebraic conditions, this subclass was so named after a well-known expert in the field of universal algebra and model theory of B. Jonsson. Among the theories of Jonsson, perfect theories of Jonsson are best studied [1]. Since Jonsson theories as a rule, are incomplete, and isomorphic embedding and various kinds of homomorphisms are considered as morphisms [2], the technical apparatus of the study of such theories is less developed than the apparatus of studying complete theories, in connection with the transfer of ideas, concepts and, correspondingly, results from the field of complete theories [3, 4] in the field of Jonsson theories, of course, is an interesting challenge. Model-theoretic properties of this class are good enough studied both from the standpoint of model theory and from the side of universal algebra, and many classical examples of classes of algebra satisfy the requirements of this subclass, namely, Jonsson conditions. Our attention in this article will be paid specifically to the study of model-theoretic properties of theories for this class and the corresponding classes of models. As it turned out, in the case of perfect Jonsson theories, it is enough to study the classes of existentially closed models of these theories. In arbitrary case, Jonsson theory always has some semantic invariant – the semantic model of this theory. Correspondingly, there is a syntactic invariant – the elementary theory of given semantic model. The natural interest is the study of special formula subsets of this semantic model. In the case when the pregeometry given on the set of all subsets of the considered semantic model is modular, and the enrichment of the language saves the properties of jonssonnes and some other important model-theoretic properties (for example, the definability of the type for the considered type of stability), we will deal with admissible and hereditary types of enrichments of Jonsson theory.

In this article, given problem is considered with respect to special kinds of enrichments of the signature and wherein the received central types.

Since our main goal in this article is to consider special properties of central types, we will work with some enrichments of signatures in which some fixed Jonsson theory is given. As it turned out, not all enrichments preserve the property of the jonssonnes of the theory. In this regard, we will consider only those Jonsson theories that retain their jonssonnes with any enrichment of the signature. We shall call such Jonsson theories hereditary. In some cases, the requirement of modularity for the Jonsson theory is sufficient to keep it conserved.

We give the basic necessary definitions of concepts and associated with them the obtained previously results.

Let T be a some Jonsson theory, C be its semantic model.

Definition 1 [5; 289]. Let C be as above and let cl: $P(C) \to P(C)$ be an operator on the power set of C. We say that (C, cl) is a pregeometry if the following conditions are satisfied.

i) If $A \subseteq C$, then $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$.

ii) If $A \subseteq B \subseteq C$, then $cl(A) \subseteq cl(B)$.

iii) (exchange) if $A \subseteq C$, $a, b \in C$ and $a \in cl(A \cup \{b\})$, then $a \in cl(A)$, $b \in cl(A \cup \{a\})$.

iv) (finite character) if $A \subseteq C$ and $a \in cl(A)$, then there is a finite $A_0 \subseteq A$ such that $a \in cl(A_0)$.

We say that $A \subseteq C$ is closed if $cl(A) = A$.

If D is strongly minimal, we can associate a pregeometry by defining $cl(A) = acl(A) \cap D$ for $A \subseteq D$.

We can generalize basic ideas about independence and dimension from strongly minimal sets to arbitrary pregeometries for any subset of fix semantic model of some Jonsson theory.

Let as call (X, cl) – Jonsson pregeometry (further J-pregeometry) if $X \subseteq C$ and C and T as above.

Definition 2. If (X, cl) is a Jonsson pregeometry, we say that A is Jonsson independent if $a \notin cl(A \setminus \{a\})$ for all $a \in A$ and that B is a J-basis for Y if $B \subseteq Y$ is J-independent and $Y \subseteq acl(B)$.

Definition 3. We say that a J-pregeometry (X, cl) is J-geometry if $cl(\emptyset) = \emptyset$ and $cl(\{x\}) = \{x\}$ for any $x \in X$.

If (X, cl) is a J-pregeometry, then we can naturally define a J-geometry. Let $X_0 = X \setminus cl(\emptyset)$. Consider the relation $\sim$ on X_0 given by $a \sim b$ iff $cl(\{a\}) = cl(\{b\})$. By exchange, $\sim$ is an equivalence relation. Let $\widehat{X}$ be $X_0/\sim$. Define $\widehat{cl}$ on $\widehat{X}$ by $\widehat{cl}(A/\sim) = \{b/\sim : b \in cl(A)\}$.

Definition 4. Let (X, cl) be J-pregeometry. We say that (X, cl) is trivial if $cl(A) = Y_{a \in A} cl\{a\}$ for any $A \subseteq X$. We say that (X, cl) is modular if for any finite-dimensional closed $Jdim(A \cup B) = Jdim(A) + Jdim(B) - -Jdim(A \cap B)$.

We say that (X, cl) is locally modular if (X, cl_a) is modular for some $a \in X$.

Definition 5. We say that (X, cl) is modular if for any finite-dimensional closed $A, B \subseteq X$

$$dim(A \cup B) = dimA + dimB - dim(A \cap B).$$

Definition 6. If $X = C$ and (X, cl) is a modular, then Jonsson theory T is called modular.

We work actually with the following types of sets.

Definition 7. Let $X \subseteq C$. We will say that a set X is $\nabla - cl$-Jonsson subset of C, if X satisfies the following conditions:

1) X is ∇ – definable set (this means that there is a formula from ∇, the solution of which in the C is the set X, where $\nabla \subseteq L$, that is, ∇ is a view of formula, for example $\exists, \forall, \forall\exists$ and so on);

2) $cl(X) = M$, $M \in E_T$, where cl is some closure operator defining a pregeometry over C (for example $cl = acl$ or $cl = dcl$).

Definition 8. An enrichment $\overline{T}$ of the Jonsson theory T is said to be permissible if any ∇-type (it means that ∇ subset of language L_σ and any formula from this type belongs to ∇) in this enrichment is definable in the framework of $\overline{T}_\Gamma$-stability.

Definition 9. Jonsson theory is said to be hereditary, if in any of its permissible enrichment, any expansion of it in this enrichment will be Jonsson theory.

Let $S^{(1)}_{\nabla}(X)$ be the set of all complete 1-types over the set X, formulas which belong to ∇. Let $X \subseteq M$, $M \in E_T$.

Definition 10. Type $p \in S^{(1)}_{\nabla}(X)$ is called essential if for any set Y, $Y \subseteq N$, $N \in E_T$, such that $X \subseteq Y$ in T exists only unique type $q \in S^{(1)}_{\nabla}(Y)$ and the type q is a J-nonforking extension of type p.

Let $p, q \in S^{(1)}_{\nabla}(X)$, $\mathfrak{A} \in E_T$ and $X \subseteq A$. The relation $p \leq_A q$ is means that for any model $\mathfrak{B} \in E_T$, such that $\mathfrak{B} \supseteq \mathfrak{A}$, from the realizability of q in $B \setminus A$ implies the realizability of p in $B \setminus A$. The relation $p \equiv q$ means that for any model $\mathfrak{A} \in E_T$, $X \subseteq A$, has $p \leq_A q$ and $q \leq_A p$. We denote the set $\{q | q \in S^{(1)}_{\nabla}(X), p \equiv q\}$ by $[p]$, and the set $\{[p] | p \in S^{(1)}_{\nabla}(X)\}$ denote by $S^{(1)}_{\nabla}[X]$. We write $[p] \leq_A [q]$, if $p \leq_A q$. The types p, q are called independent if for any $\mathfrak{A} \in E_T$, $X \subseteq A$, don't have a place neither $p \leq_A q$, nor $q \leq_A p$. If p and q are independent, then we say that $[p]$ and $[q]$ are independent.

The following definition gives the concept of a basis among the above types.

Definition 11. The set $B = \{[p_i] \in S^{(1)}_{\nabla}[X]/i \in I\}$ is called base for $S^{(1)}_{\nabla}[X]$ if:

(1) $[p_i]$ and $[q_j]$ independent for $i \neq j$;

(2) for any $[q] \in S^{(1)}_{\nabla}[X]$ and $\mathfrak{A} \in E_T$, $X \subseteq A$, exists $i \in I$ such that $[p_i] \leq_A [q]$

Definition 12. The base of the theory T is the base for $S^{(1)}_{\nabla}[\varnothing]$ (if it exists). The base B of theory T is called essential if for any $[p] \in B$ exists an essential type $q \in [p]$.

Definition 13. We will call the essential base of the types of Jonsson theory T geometric if the following conditions are satisfied:

1) $\forall p \in S^{(1)}_{\nabla}(X)$, where $X \subseteq C$, C as above and (C, cl) – J-geometry;

2) the concept of independence in the sense of geometry generated by a strongly minimal central type will coincide with the concept of independence (C, cl) – J-geometry (coincidence of the concept of a base in terms of strong minimality, pregeometry and central types that form an essential base, wherein the orbits of the central types that are their solutions in the semantic model).

It is necessary to take into account the condition of completeness of the considered theories order to after enrichment they can be associated with their central types (they are complete). In [6] the class of existentially closed models and the properties of a forking on a subset of these models are considered. The types considered in this paper were complete existential types. In this connection, we will notice, that we need $\exists$-completeness for the theories under consideration, and the fact, that independence in the sense of forking in [6] for existentially closed models is consistent with independence in the sense of geometry mentioned above.

In [7], the notion of Jonsson spectrum for abelian groups was considered. We want to define the concept of Jonsson spectrum for an arbitrary case.

Let $\mathfrak{A}$ be an arbitrary model of some fixed signature σ.

Then the $JSp(\mathfrak{A}) = \{T|\ \mathfrak{A} \in ModT,\ T -$ Jonsson theory of the signature $\sigma\}$ is the Jonsson spectrum of the model $\mathfrak{A}$. We will say that two models $\mathfrak{A}, \mathfrak{B}$ are cosemantic among themselves if they have the same semantic model. Symbolically $\mathfrak{A} \bowtie \mathfrak{B}$. It is easy to understand that this relation $\bowtie$ is the equivalence relation between models, which generalizes the concept of elementary equivalence [2]. Therefore we consider the factor set $JSp(\mathfrak{A})/_{\bowtie}$ for the model $\mathfrak{A}$.

Further we will work only with permissible enrichments and we will consider such enrichments for the Jonsson spectrums, which consist only of hereditary Jonsson theories.

Consider some enrichment of the signature σ and consider the central type of this enrichment for all Jonsson theories $T \in JSp(\mathfrak{A})$.

Let C be the semantic model of the theory T, $A \subseteq C$. Let $\sigma_\Gamma = \sigma \bigcup \Gamma$, where $\Gamma = \{P\} \bigcup \{c\}$. Let $\overline{T} = Th_{\forall\exists}\left(C, c_a\right)_{a \in C} \bigcup Th_{\forall\exists}(E_T) \bigcup \{P\left(c\right)\} \bigcup \{"P \subseteq"\}$, where $\{"P \subseteq"\}$ is an infinite set of sentences expressing the this fact that the interpretation of the symbol P is an existentially closed submodel in the language of the signature σ_Γ. I.e. the interpretation of the symbol P is the solution of the following equation $P(C) = M \in E_T$ in the language σ_Γ. By virtue of the hereditary of the theory T the theory $\overline{T}$ will be a Jonsson theory. Consider all the completions of the theory $\overline{T}$ in the signature language σ_Γ. Since $\overline{T}$ is a Jonsson theory, it has its center, and we denote it by $\overline{T}^*$ and this center is one of the above completions of the theory $\overline{T}$. At restriction of the signature σ_Γ to $\sigma \cup P$, due to the laws of first-order logic, since the constant c already does not belong to this signature, we can replace this constant on a symbol of variable, for example x. And then the theory $\overline{T}^*$ becomes a complete 1-type for the variable x. This type we will call the central type of the theory $\overline{T}$ in this enrichment. This enrichment is denoted by $\odot$.

Next, we will be back to $JSp(\mathfrak{A})/\bowtie$ for the model $\mathfrak{A}$ of an arbitrary signature and consider the admissible enrichment of this signature with the help of a predicate and a constant and consider the central type for each theory Δ from $JSp(\mathfrak{A})$.

For the ∇-complete Jonsson theory, we will define the concept of J–strongly minimality [8].

Definition 14. Let $\mathfrak{M}$ be an existentially closed model of T and $\varphi(\overline{x})$ be a non-algebraic ∇-formula.

1. The set $\varphi(\mathfrak{M})$ is called J-minimal in $\mathfrak{M}$ if for all ∇-formulas $\psi(\overline{x})$ the intersection $\varphi(\mathfrak{M}) \wedge \psi(\mathfrak{M})$ is either finite or cofinite in $\varphi(\mathfrak{M})$.

2. The formula $\varphi(\overline{x})$ is J-strongly minimal if $\varphi(\overline{x})$ defines a J-minimal set in all existentially closed extensions of $\mathfrak{M}$. In this case, we also call the definable set $\varphi(\mathfrak{M})$ is J-strongly minimal.

3. A non-algebraic type in $S^{(1)}_{\nabla}(T)$ containing a J-strongly minimal formula is called J-strongly minimal.

4. A Jonsson theory T is J-strongly minimal if any its existentially closed model is J-strongly minimal.

Clearly, J-strongly minimality is preserved under definable bijections; i.e. if A and B are definable subsets of $\mathfrak{M}^k$, $\mathfrak{M}^m$ defined by $\varphi, \psi \in \Delta$, correspondingly, such as there is a definable bijection between A and B, then if φ is J-strongly minimal so is ψ.

We note another useful fact in the case of a perfect Jonsson theory, if f is an automorphism of the structure C, leaving all elements of the set $A, f \in Aut_A(C)$, then f obviously translates into itself each A-definable subset and therefore translates into itself all full types over A, due to the saturation of the semantic model C. The reverse is also true: if $\bar{A}, \bar{d} \in C^n$, then $tp\left(\bar{c}/A\right) = tp\left(\bar{d}/A\right)$, if there exists such $f \in Aut_A\left(C\right)$, that $f\left(\bar{c}\right) = \bar{d}$.

In a saturated model the complete n-types over A correspond exactly to the orbits of the n elements under automorphisms fixing A elementwise. And correspondingly, when the theory is complete for existential sentences in the language L, then this is also true for existential types.

Further, it is convenient to work inside the semantic model C of Jonsson theory C, containing all the others.

Further, any set of parameters A is considered a subset in C. The model M is a subset of C that is the carrier of an existentially closed substructure. This means that any $L(M)$ – existential formula $\varphi(x)$, satisfable in C, also holds on some element from M. The parameters of formulas further always belong to C, and we write $\models \varphi$ if $C \models \varphi$.

The following fact is well known.

Lemma. A definable set D is definable over the set A, if it is invariant with respect to all automorphisms of the model C, leaving in place each element from A. (Let's call them automorphisms over A)

It follows that the definable closure $dcl(A)$ of the set A, i.e. the set of all elements definable over A, coincides with the set of elements invariant with respect to all automorphisms over A.

The element b, contained in a finite A-definable set, is called algebraic over A. It follows that an element b is algebraic over A, if it has only a finite number of conjugates over A.

The set $acl(A)$, consisting of all elements algebraic over A, is called the algebraic closure of the set A.

Next, we will consider in the language of concepts of a pure pair [9] the concept of the above-mentioned central type and, correspondingly, model-theoretic concepts (for example, stability and nonforking type extensions) with this connection.

The approach to types through automorphisms of a saturated model has been known for a long time, but this was determined, firstly, for complete types and for complete theories.

In our case, after the above enrichment, we are dealing with the central types (they are complete) of Jonsson theories and Jonsson theories, which are complete for the $\nabla-$ sentences.

Instead of a monster model, we turn to the semantic model of some Jonsson theory and then consider its automorphism group.

We give definition of some important model-theoretic concepts in the language pure pair (A, G), where A is some subsets of the semantic model and G is automorphism group of semantic model.

Let (A, G) be an arbitrary pure pair $X \subseteq A$.

1. $G_x \leftrightharpoons \{g \in G : \forall x \in X(g(x) = x)\}$. It is obvious that $G_x \subseteq G$.

2. If $Y \subseteq A$ then $G_x(Y) \leftrightharpoons \{g(Y) : g \in G_x\}$. If $Y = \{a\}$ then we will use the record $G_x(a)$. $G_x(Y)$ is called G_x orbit Y.

3. If $0 < n < \omega$ then $O^n(X) \leftrightharpoons \{G_x(\overline{a} : \overline{a} \in A^n)\}$.

4. $acl(X) \leftrightharpoons \{a \in A : |G_x(a)| < \omega\}$.

5. The sequence $E = \langle \overline{e}_i : i < A\rangle$ finite sequences (tuples) the same length is called indistinguishable over X if:

a) $\overline{e}_i \neq \overline{e}_j$ for all $i < j < a$;

b) for any sequence $\langle i_k : k < m < \omega\rangle$ indices such as $i_k < i_s \Leftrightarrow k < s$ for all $k, s \leq m$ exist $g \in G_x$ such as $g(\langle e_k : k \leq m\rangle)$, $\langle e_{i_k} : k \leq m\rangle$.

6. If $(I; <)$ is linearly ordered set of indices, then the sequence $E = \langle \overline{e}_i : i \in I\rangle$ is called indistinguishable over X if for all $I_0 \subseteq I$ such as $ord\langle I_0 = \omega\rangle$, $E = \langle e_i : i \in I\rangle$ is indistinguishable over X sequence.

7. The set $E = \langle \overline{e}_i : i \in I\rangle$ sequences of the same length are said to be indistinguishable over X if:

a) $\overline{e}_i \neq \overline{e_j}$ herewith $i \neq j$;

b) for any $F, D \subseteq E$ such as $|F| = |D| < \omega$ and any bijection $\psi : F \to D$ exist $g \in G_x$ such as $\psi \in g$

8. If $X \subseteq Y$, $p \in O^n(Y)$ then p is called:

a) splitting over X if there exist such $\overline{a}, \overline{b} \in Y$ that $G_x(\overline{a}) = G_x(\overline{b})$, but for any $\overline{c} \in pG_{x\cup\overline{c}}(\overline{a}) \cap G_{x\cup\overline{c}}(\overline{b}) = = \overline{c} \in p(\phi)$;

b) strictly splitting over X if there exist such an indistinguishable over X infinite sequence $E = \langle \overline{a}_i : i < \omega\rangle$ in A that $\overline{a}_0, \overline{a}_1 \in Y$ and for any $\overline{c} \in p$ occurs $G_{x\cup\overline{c}}(\overline{a}) \cap G_{x\cup\overline{c}}(\overline{b}) = \overline{c} \in p(\phi)$;

c) branching over $X(p \curlywedge X)$ if there is such $Z \supseteq Y$ that $|Z\backslash Y| < \omega$ and for any $q \in O^n(Z)$ from the fact that $q \leq p$ follows that q is strictly splitting over X.

9. Subset $X \subseteq A$ is called λ – saturated if $\forall Y \subseteq X, \forall p \in O^1(Y) < \lambda \Rightarrow X \cap p \neq (\phi)$

10. Pure pair (A, G) is called λ – stable if $\forall X \subseteq A(|X| \leq \lambda \Rightarrow |O^1(X)| \leq \lambda)$.

11. Let $O(A) \leftrightharpoons \bigcup\{ \bigcup\limits_{n<\omega} O^n(X) : |X| < |A|\}$.

By induction, we define the rank function $L : O(A) \to Ord \cup \{\infty\}$:

a) $L(p) \geq 0$ for all $p \in O(A)$;

b) if α is the limit ordinal then $L(p) \geq a$ if $L(p) \geq \beta$ for all $\beta < \alpha$;

c) if $\alpha = \beta + 1$, $p \in O^n(X)$ then $L(p) \geq \alpha$ if $L(p) \geq \beta$ and there exists such $Y \subseteq A$, $q \in O^n(Y)$ that $X \subseteq Y$, $q \subseteq p$, $L(q) \geq \beta$ and $q \curlywedge X$;

d) $L(p) = \alpha \Leftrightarrow L(p) \geq \alpha \vee L(p) \ngeq \alpha + 1$;

e) $L(p) = \infty \Leftrightarrow L(p) \geq \alpha$ for all ordinals α.

12. If $\overline{a}, \overline{b} \in A^n$ then $\overrightarrow{v}(\overline{a}, X) = \overrightarrow{v}(\overline{b}, X)$ means that there exists such $Y, p \in O^n(Y)$ that $X \subseteq Y$, $Y \omega X \subseteq Y$ is saturated $p \curlywedge X$, $\overline{a}, \overline{b} \in p$.

13. $V^n(X) \leftrightharpoons \{\overrightarrow{v}(\overline{a}, X;\ \overline{a} \in A^n\}$, $V(X) = \bigcup_{n<\omega} V^n(X)$ If $p \in O^n(X)$ then $V_p \leftrightharpoons \{\overrightarrow{v}\overline{a}, X) : \overline{a} \in p\}$.

14. If $X \subseteq Y$, $\overrightarrow{\omega} \in V^n(X), \overrightarrow{u} \in V^n(Y)$ then
$\overrightarrow{\omega} < \overrightarrow{u} \leftrightharpoons \forall \overline{a}, \overline{b} \in A^n(\overrightarrow{\omega} = \overrightarrow{v}(a, X) \vee \overrightarrow{u} = \overrightarrow{v}(\overline{b}, Y) \Rightarrow \overrightarrow{v}(\overline{a}, X) = \overrightarrow{v}(\overline{b}, X) \wedge G_\gamma(\overline{b}) \curlywedge X)$.

15. The sequence $\langle \overline{a}_i : i < a \rangle$ is called the Morley sequence over X, generated $\overline{u}$ from $V^n(X)$ if $\overrightarrow{u} < \overrightarrow{v}(\overline{a}_i, X \cup \bigcup_{j<i} \overline{a}_j)$ for all $i < a$.

16. Let's call $\overrightarrow{u}, \overrightarrow{\omega} \in V(X)$ almost orthogonal (we denote by $\overrightarrow{u} \perp^\alpha \overrightarrow{\omega}$ if $\forall \overline{a}, \overline{b} \in A(\overrightarrow{u} = \overrightarrow{v}(\overline{a}, X) \wedge \overrightarrow{\omega} = = \overrightarrow{v}(\overline{b}, X)) \Rightarrow G_{X \cup \overline{b}}(\overline{a}) \curlywedge X$.

17. Let's call $p, q \in O(X)$ almost orthogonal (we denote by $p \perp^\alpha q$ if $\overrightarrow{u} \perp^\alpha \overrightarrow{\omega}$ for all $\overrightarrow{u} \in V_p$, $\overrightarrow{\omega} \in V_q$).

18. Let's call $\overrightarrow{u} \in V(X), \overrightarrow{\omega} \in V(Y)$ orthogonal (we denote by $\overrightarrow{u} \perp^\alpha \overrightarrow{\omega}$) if $\forall Z \forall \overrightarrow{u}, \overline{\omega}_1 \in V(Z)(X \cup Y \subseteq \subseteq Z \wedge \overrightarrow{u} < \overrightarrow{u}_1 \wedge \overline{\omega} < \overline{\omega}_1 \Rightarrow \overrightarrow{u}_1 \perp^\alpha \overline{\omega}_1)$.

19. Let's call $p \in O(X)$, $q \in O(Y)$ orthogonal (we denote by $p \perp q$) if $q \subseteq V_p \forall \overrightarrow{\omega} \in V_q(\overrightarrow{u} \perp \overrightarrow{\omega})$.

20. Let's call $p \in O(X)$ regular if $\forall Y \forall q \in O(Y)(X \in Y \wedge q \subseteq p \wedge q \curlywedge X \Rightarrow p \perp q)$.

All concepts introduced in this way related to nonforking extensions of types of Jonsson theory naturally give Jonsson analogs of theorems for complete theories.

Let the above permissible enrichment of Jonsson spectrum $JSp(A)/\bowtie$ of an arbitrary model $\mathfrak{A}$ of some fixed signature σ be given. Moreover, $PJSp(A)/\bowtie \subseteq JSp(A)/\bowtie$ is a perfect Jonsson spectrum, i.e. those Jonsson theories of the $\mathfrak{A}$ model are perfect. Also among the theories from $JSp(A)/\bowtie$ we choose those theories that are $\nabla-$ complete, $\nabla-$ is the type of formulas of the type $\forall, \exists, \forall\exists$. Thus, we will work with both perfect and not perfect Jonsson theories, but only theories from the class $\nabla-$complete and hereditary.

When working with existentially closed models by fixed Jonsson theory, the following well-known fact is important.

Theorem 1 [10; 185]. Let L be a first-order language and T be a theory in L. Suppose that T has JEP, and let A, B be e.c. models of T. Then every $\forall_2$ sentence of L which is true in A is true in B too.

Consider the class $[T] \in JSp(A)/_{\bowtie}$ and $Th_{\forall\exists}(E_\Delta) = T_0$ (an analog of the Kaiser hull), where $\Delta \in [T]$. It is clear that this theory is Jonsson and also belongs to $[T]$. Let $\overline{T}_0$ be the enrichment of T_0 in the enrichment of $\odot$. Then its center is $\overline{T}_0^*$ and P_Δ^c is its central type. Correspondingly, all Jonsson theories $\Delta \in [T]$ have their corresponding types, which we will denote by p_Δ^c. Let $Th(C)$ (where C is a semantic model of class $[T]$) denote by $\mathbb{T}$ and correspondingly $\overline{\mathbb{T}}$ is the enrichment of $\mathbb{T}$ and $\overline{\mathbb{T}}^*$ is the center of $\overline{\mathbb{T}}$, $P_{\mathbb{T}}^C$ is its center type. Thus, we can consider the class of central types for each Jonsson theory $\Delta \in [T]$, and we denote this class by $\mathbb{P}_{[T]}^C$, and moreover $P_{\mathbb{T}}^C \in \mathbb{P}_{[T]}^C$. Note that for perfect Jonsson theories $\Delta \in [T]$ their central types are equal to each other.

Consider an essential geometric base consisting of central types for the case when the orbits of these central types distinguish J — strongly minimal subsets of the semantic model C. And let the orbital central types form a base in the sense of $C - cl$ geometry, where $cl = acl$, $cl = dcl$ on J – strongly minimal subsets of the semantic model C.

Within the above sets of types, we get the following result:

Let T be a Δ – complete hereditary Jonsson theory in the above enrichment $\odot$, C — its semantic model.

Theorem 2. For any $\Delta \in [T]$, $\nabla = \exists$, if $\psi(x) \in \nabla$ is a formula of L, then the following is equivalent.

(a) $\psi(A)$ is J-strongly minimal, $A \in E_\Delta$;

(b) for every existentially closed model $B \in E_\Delta$, $\psi(B)$ is J-minimal in B;

(c) $\psi(C)$ is J-minimal in C.

Proof. (b) implies (a) by the definitions, and (a) implies (c) since one can note that A as existentially closed submodel of C. To complete the proof it suffices to show that if C is $\omega-\nabla$-saturated (it means that saturateness regarding ∇-types) and $\psi(C)$ is J-minimal in C, then $\psi(C)$ is J-strongly minimal in C. (When T is perfect, we know that C is $\omega-\nabla$-saturated) if T not perfect then $\overline{T}$ not perfect, but $\psi(C)$ J-minimal in C which means that it has no more than a countable number ∇-type. Then model has $\omega-\nabla-$ saturated model. Suppose to the contrary that B is an elementary extension of C and there is a formula $\phi(x,\overline{b})$ with parameters $\overline{b}$ in B, such as $\psi(B)\cap\phi(B,\overline{b})$ and $\psi(B)\backslash\phi(B,\overline{b})$ are both infinite. (Here we are temporarily dropping the assumption that every model is an elementary substructure of C.) Let $\overline{a}$ be the parameters of ψ. By virtue of Theorem 1, $(C,\overline{a})\underset{\forall\exists}{\equiv}(B,\overline{a})$. Then since C is $\omega-\nabla$-saturated, there is a tuple $\overline{c}$ in C such as $(C,\overline{a},\overline{c})\underset{\forall\exists}{\equiv}(B,\overline{a},\overline{b})$. This implies that both $\psi(C)\cap\phi(C,\overline{c})$ and $\psi(C)\backslash\phi(C,\overline{c})$ are infinite, contradicting the assumption that $\psi(C)$ is J-minimal in C.

Theorem 3. For every $\Delta\in[T]$, $P^C_{\mathbb{T}}\in\mathbb{P}^C_{[T]}$ if $P^C_{\mathbb{T}}$ is J-strongly minimal non-algebraic type, thenp^c_Δ will be J-minimal non-algebraic type.

Proof. The proof follows from Theorem 2.

All concepts that are not defined in this article can be extracted from [1].

References

1 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во КарГУ, 2016. – 346 с.

2 Poizat B. Positive Jonsson Theories / B. Poizat, A.R. Yeshkeyev // Logica Universalis. — 2018. — Vol. 12, No. 1-2. — P. 101-127.

3 Yeshkeyev A.R. Properties of lattices of the existential formulas of Jonsson fragments / A.R. Yeshkeyev, M.T. Kasymetova // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79, No. 3. — P. 25-32.

4 Yeshkeyev A.R. On lattice of existential formulas for fragment of Jonsson set / A.R. Yeshkeyev, O.I. Ulbrikht // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79, No. 3. — P. 33-39.

5 Marker D. Model Theory: In introduction / D. Marker. — Springer-Verlag. New York. Inc., 2002. — P. 342.

6 Pillay A. Forking in the category of existentially closed structures / A. Pillay. Connection between Model Theory and Algebraic and Analytic Geometry (A. Macintyre, ed), Quaderni di Matematica 6, 2000.

7 Ешкеев А.Р. JSp-косемантичность и JSB свойство абелевых групп / А.Р. Ешкеев, О.И. Ульбрихт // Siberian Electronic Mathematical Reports [Электронный ресурс]. — Режим доступа: http://semr.math.nsc.ru. Вып. 13(2016). — С. 861-874.

8 Yeshkeyev A.R. Strongly minimal Jonsson sets and their properties / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 80, No. 4. —P. 47-51.

9 Ешкеев А.Р. Йонсоновские множества и их некоторые теоретико-модельные свойства. — Караганда: Изд-во КарГУ, 2014.

10 Hodges W. A shorter model theory / W. Hodges. — Cambridge: Cambridge University Press, 1997. — P. 310.

А.Р. Ешкеев, М.Т. Омарова, Г.Е. Жумабекова

Мұралы теориялардағы J-минималды жиындар

Мақалада басты назар мұралы йонсондық теориялардың, сонымен қатар кез келген байытуда йонсондықты сақтайтын йонсондық теориялардың модельді-теориялық қасиеттерін зерттеуге бөлінген. Авторлар «елеулі тип», «елеулі геометриялық база» сияқты жаңа ұғымдары енгізіп, арнайы йонсондық геометрияны анықтайтын түйықталу операторы берілген семантикалық модельдің арнайы ішкі жиындарды аясында орбиталдық типтер мен қатты минималдық жиындарды қарастырды. Және осы жиындар йонсондық мұралы теорияның орталық типінің орбитасынан алынған жағдайда семантикалық модельдегі J-қатты минималдық типтерге қатысты нәтижелер алынды.

Кілт сөздер: йонсондық теория, семантикалық модель, мұралы теория, J-минималды жиын, J-қатты минималды жиын, рұқсат етілген байыту, централды тип, орбиталдық тип, елеулі тип.

А.Р. Ешкеев, М.Т. Омарова, Г.Е. Жумабекова

J-минимальные множества в наследственных теориях

В статье уделено внимание изучению теоретико-модельных свойств наследственных йонсоновских теорий, при этом рассмотрены йонсоновские теории, которые сохраняют йонсоновость при любом допустимом обогащении. Авторами введены новые понятия «существенный тип», «существенная геометрическая база», рассмотрены орбитальные типы и сильно минимальные множества в рамках специальных подмножеств семантической модели, на которых задан оператор замыкания, определяющий специальную йонсоновскую геометрию. Также получены результаты для J-сильно минимальных типов семантической модели в случае, когда эти множества выделены из орбит центральных типов йонсоновских наследственных теорий.

Ключевые слова: йонсоновская теория, семантическая модель, наследственная теория, J-минимальное множество, J-сильно минимальное множество, допустимое обогащение, центральный тип, орбитальный тип, существенный тип.

References

1 Yeshkeyev, A.R., & Kassymetova, M.T. (2016). *Yonsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models]*. Karaganda: Izdatelstvo KarHU [in Russian].

2 Poizat, B., & Yeshkeyev, A.R. (2018). Positive Jonsson Theories. *Logica Universalis, Vol. 12, 1–2,* 101–127.

3 Yeshkeyev, A.R. & Kassymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3*, 25-32.

4 Yeshkeyev, A.R. & Ulbrikht, O. I. (2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3,* 33-39.

5 Marker, D. (2002). *Model Theory: In introduction.* Springer-Verlag. New York. Inc.

6 Pillay, A. (2000). *Forking in the category of existentially closed structures.* Quaderni di Matematica 6.

7 Yeshkeyev, A.R. & Ulbrikht, O.I. (2016). JSp-kosemantichnost i JSB svoistvo abelevykh hrupp [JSp-cosemantic and JSB property of abelian groups]. *Siberian Electronic Mathematical Reports.* Retrieved from http://semr.math.nsc.ru [in Russian].

8 Yeshkeyev, A.R. (2015). Strongly minimal Jonsson sets and their properties. *Bulletin of the Karaganda University. Mathematics series, Vol. 80, 4,* 47–51.

9 Yeshkeyev, A.R. (2014) *Yonsonovskie mnozhestva i ikh nekotorye teoretiko-modelnye svoistva [Jonsson sets and its model-theoretic properties]*. Karaganda: Izdatelstvo KarHU [in Russian].

10 Hodges, W. (1997). *A shorter model theory.* Cambridge: University of Cambridge.

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Justification of the dependencies for calculating gripping forces of multifaceted unresharpenable plates in the holder of a cutoff tool at their lateral installation

In the article mathematical dependencies to determine the gripping force of the cutting plate in the socket of assembled cutoff tools with the lateral installation of multifaceted unresharpenable plates (MUP) are proposed for the first time, which makes it impossible to move the plate in any direction while the cutting forces acting on it. Moreover, the expressions are obtained to determine the minimum height of the intersection of the cutoff tool socket head, which is important at the stage of creating a methodology for designing this type of tool.

Keywords: assembled cutoff tool, multifaceted unresharpenable plate, mechanical mounting.

Introduction

In modern machining of metals by cutting the most progressive are assembled tools with mechanical mounting of multifaceted unresharpenable plates. One of the main advantages of this type of tools is that when one cutting edge is worn, a cutting plate rotates around its own axis to enable the operation of another, which significantly reduces the time of tool adjustment, since in this case there is no need to remove it from the tool holder, which is very relevant for modern machine tools with numerical control and automated lines. Application in industry of assembled metal cutting tools with mechanical mounting of plates allows increasing productivity of processing, since the cutting speed of these tools is 1,5 – 2 times higher than that of brazed ones [1–5]. Therefore, the widespread use of assembled metal cutting tools is important for modern machine building.

However, for modern cutoff tools with mechanical mounting, predominately, single- or double-blade cutting plates of a specific non-technological complex shape are used. Cutoff tools equipped with multifaceted plates of three-, four- or five-faced shape [6–8] are used mainly for cutting rods with a diameter of up to 12 mm due to lateral installation of cutting plates on the tool case (Fig. 1), which significantly restricts the overhang of a cutting part and, as a result, the scope of their application.

In order to eliminate the disadvantages of listed above, the authors [9–14] for the first time proposed a new design of assembled cutoff tool with lateral installation of multifaceted unresharpenable plates (Fig. 2), consisting of a holder 1, hook 2, screw 3, and multifaceted unresharpenable plate 4. In this design of the cutting tool, locating and fixing of MUP is carried out only on the thrust surfaces, which makes it possible to perform cutting of rods with a diameter of up to 30 mm.

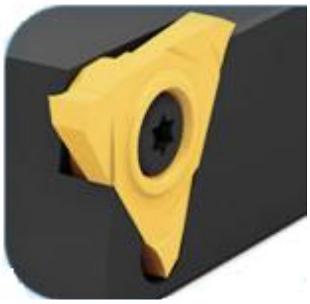


Figure 1. Groove cutters with lateral mounting of multifaceted plates

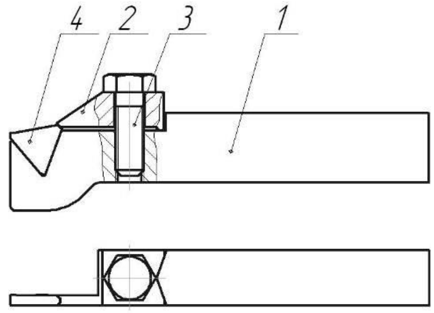


Figure 2. Design of assembled cutoff tool with lateral installation of multifaceted plates

Relevance. The main factors influencing the reliability of assembled metal cutting tools are the installation scheme, mounting and fixing of multifaceted unresharpenable plates. The provision of the necessary gripping force is a crucial task, which depends on the performance of the cutting tool under the action of the resulting cutting force.

Given kind of plate installation is proposed for the first time, in order to ensure the efficiency of this tool, it is necessary to obtain mathematical dependencies that will enable to determine the required strength of attaching the plate in the socket, which makes it impossible to move the plate in any direction while cutting forces acting on it, and to determine minimum height of the intersection of the cutter head socket.

Research results

The clamping of a plate should exclude the movement of MUP under the action of the forces that occur during cutting. Primarily, their action is determined by the physical and chemical properties of the material being processed (hardness, chemical composition), on the basis of which the cutting modes are specified, and by the heterogeneity of the distribution in it of the constituent elements, as well as the structural and geometric parameters of the cutting part and the conditions of the plate installation.

The cutting plate during the cutting process is exposed to active forces: the resulting cutting force P_p and the force of gripping $F_{gr.}$ the cutting plate, which are balanced by the reactions R_1 and R_2 of supporting surfaces of the plate socket (Fig. 3, Fig. 4). Since the forces acting on the plate form, a balanced plane system, the algebraic sums of the projections of these forces on the coordinate axis are equal to zero. The plate is under the action of a system of arbitrarily located forces, for the equilibrium of which the fulfillment of the three following conditions is required [15–18]:

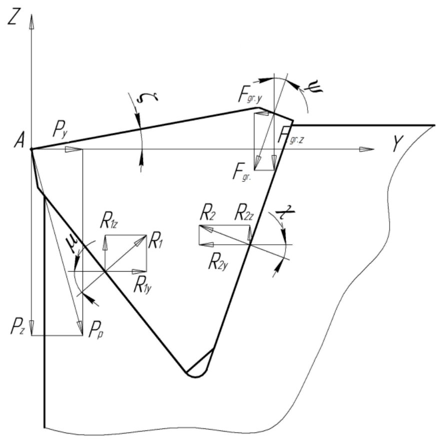


Figure 3. Scheme of active forces acting on the cutting plate

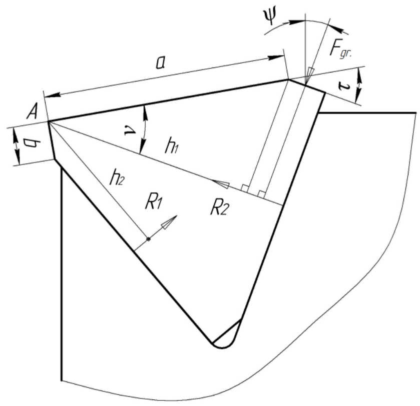


Figure 4. Scheme to determine the moments

$$\sum F_{iz} = 0; -P_z - F_{\text{gr.}} \cdot \cos\psi + R_2 \cdot \sin\chi + R_1 \cdot \sin\mu = 0; \quad (1)$$

$$\sum F_{iy} = 0; P_y - F_{\text{gr.}} \cdot \sin\psi - R_2 \cdot \cos\chi + R_1 \cdot \cos\mu = 0; \quad (2)$$

$$\sum (F) = 0; -F_{\text{gr.}} \cdot \mathrm{h}_1 + \mathrm{R}_1 \cdot \mathrm{h}_2 = 0. \quad (3)$$

From the equation (1) we derive the gripping force $F_{\text{gr.}}$:

$$F_{\text{gr.}} = \frac{\mathrm{R}_1 \cdot \sin\mu + R_2 \cdot \sin\chi - Pz}{\cos\psi}. \quad (4)$$

From the equation (2) we determine the reaction R_2:

$$R_2 = \frac{R_1 \cdot \cos\mu + Py - F_{\text{gr.}} \cdot \sin\psi}{\cos\chi}. \quad (5)$$

From the equation (3) we determine the reaction R_1:

$$R_1 = \frac{\mathrm{F}_{\text{gr.}} \cdot h_1}{h_2}. \quad (6)$$

We apply the equation (6) to the equation (5)

$$R_2 = \frac{\frac{F_{\text{gr.}} \cdot \mathrm{h}_1}{\mathrm{h}_2} \cos\mu + P_y - F_{\text{gr.}} \cdot \sin\psi}{\cos\chi} = \frac{h_1 \cdot \mathrm{F}_{\text{gr.}} \cdot \cos\mu + \mathrm{h}_2 \cdot P_y - h_2 \cdot F_{\text{gr.}} \cdot \sin\psi}{h_2 \cdot \cos\chi}. \quad (7)$$

The obtained equations (6) and (7) we apply to the equation (4) to determine the gripping force of a cutting plate:

$$F_{\text{gr.}} = \frac{\frac{\sin\chi(\mathrm{h}_1 \cdot \mathrm{F}_{\text{gr.}} \cdot \cos\mu + h_2 \cdot \mathrm{P}_{\mathrm{y}} - \mathrm{h}_2 \cdot \mathrm{F}_{\text{gr.}} \cdot \sin\psi)}{\mathrm{h}_2 \cdot \cos\chi} + \frac{F_{\text{gr.}} \cdot \mathrm{h}_1}{h_2} \cdot \sin\mu - P_z}{\cos\psi};$$

$$F_{\text{gr.}} = \frac{\frac{tg\chi(\mathrm{h}_1 \cdot \mathrm{F}_{\text{gr.}} \cdot \cos\mu + h_2 \cdot \mathrm{P}_{\mathrm{y}} - \mathrm{h}_2 \cdot \mathrm{F}_{\text{gr.}} \cdot \sin\psi)}{\mathrm{h}_2} + \frac{F_{\text{gr.}} \cdot \mathrm{h}_1}{h_2} \cdot \sin\mu - P_z}{\cos\psi};$$

$$F_{\text{gr.}} \cdot cos\psi = tg\chi \frac{\mathrm{h}_1 \cdot \mathrm{F}_{\text{gr.}}}{\mathrm{h}_2} \cdot \cos\mu + \mathrm{P}_{\mathrm{y}} \cdot tg\chi - tg\chi \frac{\mathrm{h}_2 \cdot \mathrm{F}_{\text{gr.}}}{\mathrm{h}_2} \sin\psi + \frac{F_{\text{gr.}} \cdot \mathrm{h}_1}{h_2} \cdot \sin\mu - P_z;$$

$$F_{\text{gr.}} \cdot cos\psi = tg\chi \frac{\mathrm{h}_1 \cdot \mathrm{F}_{\text{gr.}}}{\mathrm{h}_2} \cdot \cos\mu - tg\chi \frac{\mathrm{h}_2 \cdot \mathrm{F}_{\text{gr.}}}{\mathrm{h}_2} \sin\psi - \frac{F_{\text{gr.}} \cdot \mathrm{h}_1}{h_2} \cdot \sin\mu = \mathrm{P}_{\mathrm{y}} \cdot tg\chi - P_z;$$

$$F_{\text{gr.}} = \frac{\mathrm{h}_2 \cdot cos\psi - tg\chi \cdot \mathrm{h}_1 \cdot \cos\mu - tg\chi \cdot \sin\psi - \mathrm{h}_1 \cdot \sin\mu}{\mathrm{h}_2} = \mathrm{P}_{\mathrm{y}} \cdot tg\chi - P_z;$$

$$F_{\text{gr.}} = \frac{\mathrm{h}_2 \cdot (\mathrm{P}_{\mathrm{y}} \cdot tg\chi - P_z)}{\mathrm{h}_2 \cdot cos\psi - tg\chi \cdot \mathrm{h}_1 \cdot \cos\mu - tg\chi \cdot \sin\psi - \mathrm{h}_1 \cdot \sin\mu}. \quad (8)$$

We determine the arms of forces according to Figure 4:

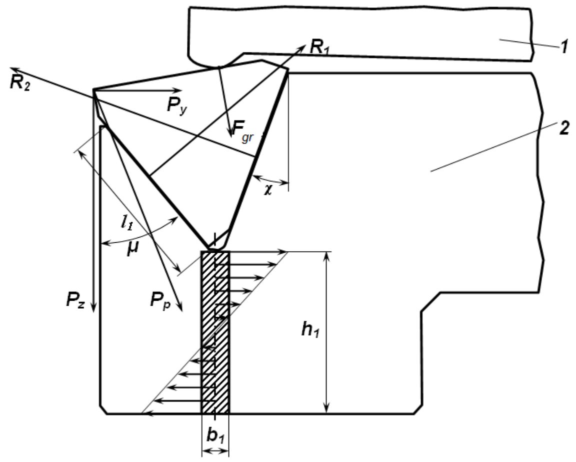
$$h_1 = a \cdot \cos\nu + \frac{\mathrm{b}}{2};$$

$$h_2 = b \cdot \cos\tau + \frac{a}{2}.$$

We apply the determined values of the arms of forces to the equation and obtain the mathematical dependency that allows determining the necessary gripping force of a plate at the given type of its installation (8):

$$F_{\text{gr.}} = \frac{(b \cdot \cos\tau + \frac{a}{2}) \cdot (\mathrm{P}_{\mathrm{y}} \cdot tg\chi - P_z)}{(b \cdot \cos\tau + \frac{a}{2}) \cdot cos\psi - tg\chi \cdot \cos\mu(a \cdot \cos\nu + \frac{b}{2}) - tg\chi \cdot \sin\psi - \sin\mu \cdot (a \cdot \cos\nu + \frac{b}{2})}.$$

However, the disadvantage of this plate mounting scheme (Fig. 4) is the action of the plate gripping force F_{gr} upon the front wall of the tool socket [19–22]. In order to eliminate this drawback, the following scheme of three-faceted plate (Fig. 5) was proposed. According to this scheme, the front socket wall is made at an angle $\mu = 40°$, and the back one – at an angle $\chi = 20°$, which, under the chosen scheme of hook installation removes the action of the gripping force F_{gr} on the front socket wall. Thus, the components of the cutting forces P_z, P_y and the resulting cutting force P_p are fully compensated by the reactions of the supports of the front R_1, R_{1z}, R_{1y} and back R_2, R_{2z}, R_{2y} socket walls [23–25].



1 – hook; 2 – cutter case

Figure 5. Scheme for calculating stresses and the minimum intersection of the cutter head

The equations of interaction of these forces are as follows:

$$P_z = R_1 \cdot \sin\mu + R_2 \cdot \sin\chi; \tag{9}$$

$$P_y = -R_1 \cdot \cos\mu + R_2 \cdot \cos\chi. \tag{10}$$

From the equation (10) we determine the support reaction R_2 of the back socket wall:

$$R_2 = \frac{P_y + R_1\cos\mu}{\cos\chi}.$$

Applying it to the equation (9), after performing transformations we obtain:

$$\begin{aligned}
&P_z = R_1 \cdot \sin\mu + \tfrac{P_y + R_1\cos\mu}{\cos\chi}\sin\chi = \\
&P_z\cos\chi = R_1\sin\mu\cos\chi + P_y\sin\chi + R_1\cos\mu\sin\chi = \\
&P_z\cos\chi - P_y\sin\chi = R_1\sin\mu\cos\chi + R_1\cos\mu\sin\chi = \\
&P_z\cos\chi - P_y\sin\chi = R_1(\sin\mu\cos\chi + \cos\mu\sin\chi) = \\
&P_z\cos\chi - P_y\sin\chi = R_1\sin(\mu+\chi).
\end{aligned} \tag{11}$$

From the equation (11) we determine the support reaction R_1 of the front socket wall:

$$R_1 = \frac{P_z\cos\chi - P_y\sin\chi}{\sin(\mu+\chi)}.$$

The most dangerous is the bending stress σ_{bs} in the minimum intersection of the cutter head on the bottom of the socket, from the action of the reaction force R_2 on its front wall, which we consider to be applied in the middle of this wall, having a length $l_1 = 14$ mm:

$$\sigma_{bs} = \frac{R_1\sin\mu \cdot \frac{1}{2}l_1\sin\mu}{W_x}, \tag{12}$$

where W_x – the moment of resistance to the bend of the minimum intersection on the head of the cutter, which has a height h_1 and a thickness $b_1 = 2.5$ mm:

$$W_x = \frac{b_1h_1^2}{6}. \tag{13}$$

After applying the equation (13) to the equation (12), we obtain the following:

$$\sigma_{bs} = \frac{6R_1\sin\mu \cdot \frac{1}{2}l_1\sin\mu}{b_1h_1^2} = \frac{3 \cdot R_1 \cdot l_1\sin^2\mu}{b_1h_1^2}. \tag{14}$$

From the equation (14) we obtain the minimum height of the intersection h_1:

$$h_1 = \sqrt{\frac{3 \cdot R_1 \cdot l_1 \sin^2 \mu}{b_1 \left[\sigma_{bs}\right]}},$$

where $[\sigma_{bs}]$ – maximum allowable bending stresses, MPa.

Conclusions

As a result of the performed researches, mathematical dependencies to determine the gripping force of the cutting plate in the socket of assembled cutoff tools with the lateral installation of multifaceted unresharpenable plates, are proposed for the first time which makes it impossible to move the plate in any direction while the cutting forces acting on it. Moreover, the expressions are obtained to determine the minimum height of the intersection of the cutoff tool socket head, which is important at the stage of creating a methodology for designing this type of tool.

References

1 Arslan, H., Er, A.O., Orhan, S. & Aslan, E. (2016). Tool Condition Monitoring in Turning Using Statistical Parameters of Vibration Signal. *International Journal of Acoustics and Vibration, 21(4),* 371–378. Retrieved from https://doi.org/10.20855/ijav.2016.21.4432.

2 Barry, J. & Byrne, G. (2001). Cutting tool wear in the machining of hardened steels Part II: cubic boron nitride cutting tool wear. *Wear, 247(2),* 152–160. Retrieved from https://doi.org/10.1016/s0043-1648(00)00528-7.

3 Boim, N.G. & Sokolov, I.N. (1984). The use of super-hard material and ceramic cutting tools in machine-tool construction. *Soviet Engineering Research, 4(7),* 55–56.

4 Chen, W.Y. (2000). Cutting forces and surface finish when machining medium hardness steel using CBN tools. *International Journal of Machine Tools & Manufacture, 40(3),* 455–466. Retrieved from https://doi.org/10.1016/s0890-6955(99)00011-5.

5 Chou, Y.K. & Evans, C.J. (1997). Tool wear mechanism in continuous cutting of hardened tool steels. *Wear, 212(1),* 59–65. Retrieved from https://doi.org/10.1016/s0043-1648(97)00139-7.

6 Chou, Y.K. & Song, H. (2004). Tool nose radius effects on finish hard turning. *Journal of Materials Processing Technology, 148(2),* 259–268. Retrieved from https://doi.org/10.1016/j.jmatprotec.2003.10.029.

7 Dasgupta, S. & Mukherjee, S. (2016). Effects of machining parameters on tool life and its optimization in turning mild steel with brazed carbide cutting tool. *International Conference on Advances in Materials and Manufacturing Applications (IConAMMA).* Retrieved from https://doi.org/10.1088/1757-899X/149/l/012005.

8 Gao, D., Liao, Z.R., Lv, Z.K. & Lu, Y. (2015). Multi-scale statistical signal processing of cutting force in cutting tool condition monitoring. *International Journal of Advanced Manufacturing Technology, 80(9–12),* 1843–1853. Retrieved from https://doi.org/10.1007/s00170-015-7116-0.

9 Grzesik, W., Wanat, T. & Sme. (2005). Hard turning of quenched alloy steel parts using conventional and wiper ceramic inserts. *Transactions of the North American Manufacturing Research Institution of SME, Vol 33,* 9–16.

10 Huddle, D. (2002). A new direction for turning — Plunge turning can be a cost-effective grinding alternative. *Manufacturing Engineering, 128(4),* 76–83.

11 Jiang, X.C., Li, B., Mao, X.Y., Hao, C.H. & Liu, H.Q. (2018). Tool condition monitoring based on dynamic sensitivity of a tool-workpiece system. *International Journal of Advanced Manufacturing Technology, 98(5-8),* 1441–1460. Retrieved from https://doi.org/10.1007/s00170-018-2252-y.

12 Klimenko, S.A., Mukovoz, Y.A., Lyashko, V.A., Vashchenko, A.N. & Ogorodnik, V.V. (1992). On the wear mechanism of cubic boron-nitride base cutting tools. *Wear, 157(1),* 1–7. Retrieved from https://doi.org/10.1016/0043-1648(92)90183-9.

13 Klimenko, S.A., Mukovoz, Y.A. & Polonsky, L.G. (1996). Cutting tools of superhard materials. *Advanced Ceramic Tools for Machining Application - Ii, 114,* 1–66. Retrieved from https://doi.org/ 10.4028/www.scientific.net/KEM.114.1.

14 Kramer, B.M. & Suh, N.P. (1980). Tool wear by solution — a quantitative understanding. *Journal of Engineering for Industry-Transactions of the Asme, 102(4),* 303–309. Retrieved from https://doi.org/10. 1115 /1.3183869.

15 Kumar, A.S., Durai, A.R. & Sornakumar, T. (2003). Machinability of hardened steel using alumina based ceramic cutting tools. *International Journal of Refractory Metals & Hard Materials, 21(3–4),* 109–117. Retrieved from https://doi.org/10.1016/s0263-4368(03)00004-0.

16 Letot, C., Serra, R., Dossevi, M. & Dehombreux, P. (2016). Cutting tools reliability and residual life prediction from degradation indicators in turning process. *International Journal of Advanced Manufacturing Technology, 86(1–4),* 495–506. Retrieved from https://doi.org/10.1007/s00170-015-8158-z.

17 LoCasto, S., LoValvo, E., Lucchini, E., Maschio, S., Piacentini, M. & Ruisi, V.F. (1996). Machining of steel with advanced ceramic cutting-tools. *Advanced Ceramic Tools for Machining Application - Ii, 114,* 105–133. Retrieved from https://doi.org/10.4028/www.scientific.net/KEM.114.105.

18 Matsumoto, Y., Barash, M.M. & Liu, C.R. (1987). Cutting mechanism during machining of hardened steel. *Materials Science and Technology, 3(4),* 299–305. Retrieved from https://doi.org/10.1179/mst.1987.3.4.299.

19 Nastasenko, V.O., Babiy, M.V. & Protsenko, V.O. (2018). Mathematical model of cutting process of cutting tools with a side-mounted multifaceted, requiring no sharpening plates. *Bulletin of the Karaganda University. Mathematics series, 92(4),* 132–138. Retrieved from https://doi.org/10.31489/ 2018m4/132-138.

20 Ozel, T., Hsu, T.K. & Zeren, E. (2005). Effects of cutting edge geometry, workpiece hardness, feed rate and cutting speed on surface roughness and forces in finish turning of hardened AISIH13 steel. *International Journal of Advanced Manufacturing Technology, 25(3–4),* 262–269. Retrieved from https://doi.org/10.1007 /s00170-003-1878-5.

21 Ozel, T., Karpat, Y., Figueira, L. & Davim, J.P. (2007). Modelling of surface finish and tool flank wear in turning of AISID2 steel with ceramic wiper inserts. *Journal of Materials Processing Technology, 189(1–3),* 192–198. Retrieved from https://doi.org/10.1016/j.jmatprotec.2007.01.021.

22 Schaal, N., Kuster, F. & Wegener, K. (2015). Springback in metal cutting with high cutting speeds. *15th Cirp Conference on Modelling of Machining Operations (15th Cmmo), 31,* 24–28. Retrieved from https://doi.org/10.1016/j.procir.2015.03.065.

23 Shintani, K., Ueki, M. & Fujimura, Y. (1989). Optimum cutting tool geometry when interrupted cutting carburized steel by cbn tool. *International Journal of Machine Tools & Manufacture, 29(3),* 415–423. Retrieved from https://doi.org/10.1016/0890-6955(89)90010-2.

24 Thangarasu, S.K., Shankar, S., Thomas, A.T. & Sridhar, G. (2017). Prediction of Cutting Force in Turning Process-an Experimental Approach. *International Conference on Advances in Materials and Manufacturing Applications (IConAMMA).* Retrieved from https://doi.org/10.1088/1757-899x/310 /1/012119.

25 Zak, K. (2017). Cutting mechanics and surface finish for turning with differently shaped cbn tools. *Archive of Mechanical Engineering, 64(3),* 347–357. Retrieved from https://doi.org/10.1515/meceng-2017-0021.

М.В. Бабий, В.О. Настасенко, В.О. Проценко, Д.А. Зинченко

Кескідегі кескіш ұстағышындағы бүйірлік орнатуда көпқырлы қайта жоңылмайтын пластиналарды бекіту күшін есептеуге арналған тәуелділіктерді негіздеу

Мақалада алғаш рет кесу күшінің әсерінен болатын пластинаның кез келген бағыттағы қозғалысын болдырмайтын, бүйірлік орнатуда көпқырлы қайта жонылмайтын пластиналарды кескідегі кескіштер жиынтығының орнына кескіш пластинаны бекітуге қажетті күшті анықтау үшін математикалық тәуелділік ұсынылды. Сонымен бірге мұндай құрылғының жобалау әдістемесін құру кезінде маңызды болатын, кескіш басы орнының қимасының минималды түрде биіктігін анықтауға қажетті өрнек алынды.

Кілт сөздер: кескіннің кескіш жиынтығы, көпқырлы қайта кескіш пластина, механикалық қондырғы.

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Обоснование зависимостей для расчета сил закрепления многогранных неперетачиваемых пластин в державке отрезного резца при их боковой установке

В статье впервые предложены математические зависимости для определения необходимой силы закрепления режущей пластины в гнезде сборных отрезных резцов с боковой установкой многогранных неперетачиваемых пластин, которая сделает невозможным движение пластины в любом направлении при воздействии на нее сил резания. Также получены выражения для определения минимальной высоты сечения гнезда головки резца, что является важным на этапе создания методики проектирования данного вида инструмента.

Ключевые слова: сборный отрезной резец, многогранная неперетачиваемая режущая пластина, механическое крепление.

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Two-dimensional thermo-viscoelastic waves in layered media

Dynamic problems of deformation of solids have been the subject of numerous studies in the CIS and abroad. The rejection of a number of simplifying assumptions made in the cited and other published works leads to the need for further refinement and improvement of mechanical and mathematical models describing the kinematics and stress state of both the drummer and the barrier. Further, the axisymmetric collision of a cylindrical indenter with an obstacle in the form of a package of isotropic plates containing free cavities and rigid inclusions is numerically investigated within the framework of the coupled theory of thermo-viscoelasticity. Various formulations of the problems of the theory of elasticity and thermo-viscoelasticity are possible. However, the used formulation in velocities and stresses is one of the most universal, since it allows solving the main boundary value problems (including mixed ones) by a uniform way. The paper gives a grid-characteristic scheme and its convergence. In accordance with the theory of A.A. Samarskii, the stability in the energy norm of the grid problem is proved.

Keywords: two-dimensional thermo-viscoelastic waves, stability of a difference scheme, convergence of a solution of a difference problem, indenter, deformation, tensor, stresses.

Introduction

Let a deformable continuous (or hollow) cylinder of finite length h_0 as $t < 0$ simultaneously performs translational (with velocity V_0) and rotational (with angular velocity ω_0) motion. At the initial moment of time $t = 0$, the rotating indenter with its flat base normally collides with the surface of a multilayer plate (barrier) weakened by cylindrical cavities and inclusions.

To describe the dynamic behavior of an isotropic medium, we use the relations

$$\dot{\sigma} + \frac{\sigma}{\theta_1} = 3K(\dot{\varepsilon} + 3\alpha\,\dot{\mathrm{T}}); \dot{S}_{ij} + \frac{S}{\theta_2} = 2\mu\,\dot{e}_{ij},$$

where σ — is the sum of normal voltages; ε — volumetric deformation; S_{ij}, e_{ij} — components of deviators of symmetric stress and strain tensors; T — temperature increment; $\mathrm{K} = \lambda + \frac{2}{3}\mu$ — elastic modulus of bulk expansion — compression (λ, μ — Lame parameters); α — linear thermal expansion coefficient; θ_1, θ_2 — relaxation times for ball and deviatoric stresses; a dot above the letters means time differentiation.

Under the conditions of axial symmetry, the written system, which is supplemented by three equations of motion and the equation of heat conduction, is equivalent to the following dependencies, containing, as unknown displacement velocities, stresses and temperatures:

$$\begin{gathered}
\rho\,\dot{u} = \frac{\partial \sigma_r}{\partial r} + \frac{\partial r_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\varphi}{r};\\
\rho\,\dot{\vartheta} = \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial r_z}{\partial z} + \frac{\sigma_{rz}}{r};\\
\rho\,\dot{w} = \frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{\partial r_{z\varphi}}{\partial z} + \frac{2\sigma_{r\varphi}}{r};\\
\dot{\sigma}_r + \eta\sigma_r - \beta\sigma = \lambda\,\dot{\varepsilon} + 2\mu\,\dot{\varepsilon}_r - p\,\dot{\mathrm{T}}; s\\
\dot{\sigma}_\varphi + \eta\sigma_\varphi - \beta\sigma = \lambda\,\dot{\varepsilon} + 2\mu\,\dot{\varepsilon}_\varphi - p\,\dot{\mathrm{T}};\\
\dot{\sigma}_z + \eta\sigma_z - \beta\sigma = \lambda\,\dot{\varepsilon} + 2\mu\,\dot{\varepsilon}_z - p\,\dot{\mathrm{T}};
\end{gathered} \qquad (1)$$

$$\overset{\bullet}{\sigma}_{r\varphi} + \eta\sigma_{r\varphi} = 2\mu\,\overset{\bullet}{\varepsilon}_{r\varphi}; \overset{\bullet}{\sigma}_{z\varphi} + \eta\sigma_{z\varphi} = 2\mu\,\overset{\bullet}{\varepsilon}_{z\varphi}; \overset{\bullet}{\sigma}_{rz} + \eta\sigma_{rz} = 2\mu\,\overset{\bullet}{\varepsilon}_{rz}; \quad (2)$$

$$c\,\overset{\bullet}{\mathrm{T}} = \mathrm{K}\left(\frac{\partial^2 \mathrm{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{T}}{\partial r} + \frac{\partial^2 \mathrm{T}}{\partial z^2}\right) - p\mathrm{T}\,\overset{\bullet}{\varepsilon} + W^*, \quad (3)$$

where

$$\underset{r}{\overset{\bullet}{\varepsilon}} = \frac{\partial u}{\partial r}; \underset{\varphi}{\overset{\bullet}{\varepsilon}} = \frac{u}{r}; \underset{z}{\overset{\bullet}{\varepsilon}} = \frac{\partial \vartheta}{\partial z}; \overset{\bullet}{\varepsilon} = \underset{r}{\overset{\bullet}{\varepsilon}} + \underset{\varphi}{\overset{\bullet}{\varepsilon}} + \underset{z}{\overset{\bullet}{\varepsilon}}; \underset{r\varphi}{\overset{\bullet}{\varepsilon}} = \frac{1}{2}\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right); \underset{z\varphi}{\overset{\bullet}{\varepsilon}} = \frac{1}{2}\frac{\partial w}{\partial z};$$

$$\underset{rz}{\overset{\bullet}{\varepsilon}} = \frac{1}{2}\left(\frac{\partial \vartheta}{\partial r} + \frac{\partial u}{\partial z}\right); \sigma = \sigma_r + \sigma_\varphi + \sigma_z; \eta = \frac{1}{\theta_2}; \beta = \frac{1}{3}\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right); p = 3\mathrm{K}\alpha;$$

$$W^* = \frac{1}{2\mu\theta_2}s_{ij}s_{ij} + \frac{\sigma^2}{9\mathrm{K}\theta_1}; s_{ij}s_{ij} = s_r^2 + s_\varphi^2 + s_z^2 + 2\left(\sigma_{r\varphi}^2 + \sigma_{z\varphi}^2 + \sigma_{rz}^2\right); s_r = \sigma_r - \frac{\sigma}{3}; s_\varphi = \sigma_\varphi - \frac{\sigma}{3};$$

$$s_z = \sigma_z - \frac{\sigma}{3}.$$

Here u, ϑ, w are the components of the velocity vector in the direction of the coordinate axes: r, z, φ, σ_{ij} are the stress components; ρ is density; K is coefficient of thermal conductivity; c is specific volumetric heat capacity with constant strain tensor; p is parameter of connectivity of deformation fields and temperature; W^* is the energy dissipation function [1], which takes a zero value for a perfectly elastic medium ($\eta = \beta = 0$). The connectivity equation is nonlinear due to the presence of a member $p\mathrm{T}\,\overset{\bullet}{\varepsilon}$ in it. It should also be pointed out that the third, seventh and eighth equations form an independent system with respect to the quantities w, $\sigma_{r\varphi}$, $\sigma_{z\varphi}$.

Various formulations of the problems of the theory of elasticity and thermo-viscoelasticity are possible [2, 3]. However, the formulation used in velocities and voltages is one of the most universal, since it allows one to solve the main boundary problems (including mixed ones) in a uniform way. Among the first publications, where a similar form was proposed for writing the defining equations for a linearly elastic medium under plane strain conditions, the work [4] should be noted.

With a small thermal perturbation, the thermo-viscouselastic properties of the material can be considered being independent of temperature [5, 6]. In [5], for example, it was shown that when $\mathrm{T} \leq 390\,\mathrm{K}$ the value of the viscosity coefficient $\eta^* = 2\mu\theta_2$ for pure aluminum remains unchanged.

The boundary conditions for the considered contact problem are formulated as follows. The outer boundaries of the deformable mechanical system, as well as the walls of the internal cavities, are free from external forces: $\sigma_z = \sigma_{rz} = \sigma_{z\varphi} = 0$ and $\sigma_r = \sigma_{rz} = \sigma_{r\varphi} = 0$ (for boundaries that are parallel to the axis r and z, respectively). If the region $r_1 < r < r_2, z_1 < z < z_2$ is a rigid inclusion with density ρ_*, then at the points of its boundaries

$$u = 0;$$

$$\overset{\bullet}{\vartheta} = \frac{2\pi}{\mathrm{M}_*}\left\{\int_{r_1}^{r_2}\left[\sigma_2(r,z_2) - \sigma_z(r,z_1)\right] r dr + \int_{z_1}^{z_2}\left[r_2\sigma_{rz}(r_2,z) - r_1\sigma_{rz}(r_1,z)\right] dz\right\};$$

$$\overset{\bullet}{w} = \frac{2\pi r}{\mathrm{M}_*\left(r_1^2 + r_2^2\right)}\left\{\int_{r_1}^{r_2}\left[\sigma_{z\varphi}(r,z_2) - \sigma_{z\varphi}(r,z_1)\right] r^2 dr + \int_{z_1}^{z_2}\left[r_2^2\sigma_{r\varphi}(r_2,z) - r_1^2\sigma_{r\varphi}(r_1,z)\right] dz\right\}; \quad (4)$$

$$\mathrm{M}_* = \pi\left(r_2^2 - r_1^2\right)(z_2 - z_1)\,\rho_*.$$

It is easy to notice that with a sufficiently large inclusion density, located in the initially quiescent medium, $\vartheta \approx w \approx 0$.

On a circular platform, collisions $r_0 \leq r \leq R,\ z = 0$ can be performed as conditions for rigid coupling of the end face of the striker with the surface of the obstacle

$$[u] = [\vartheta] = [w] = [\sigma_z] = [\sigma_{rz}] = [\sigma_{z\varphi}] = 0; \quad (5)$$

$$\left\{\ [u] = \lim u\,(z-0) - \lim u\,(z+0)\right\},$$

and the boundary conditions that simulate the absence of friction forces between interacting bodies (smooth impact). In this case, the first and fifth equations in (5) are replaced by

$$\sigma_{rz} = 0. \quad (6)$$

If, in the course of a smooth stroke, the voltage σ_z or $\sigma_{z\varphi}$ vanishes at any point of the contact zone, then the type of boundary conditions will change and thereafter, for this point, respectively, it is assumed $\sigma_z = 0$ or $\sigma_{z\varphi} = 0$.

On the flat boundaries of the compound of dissimilar materials that make up the layered package, the conditions of rigid adhesion are carried out (5).

It is assumed that the heat exchange of the mechanical system with the environment, the temperature of which is considered constant ($\mathrm{T}_0 = const$), is carried out according to Newton's law with the nonlinear dependence of the heat transfer coefficient on temperature:

$$-\mathrm{K}_n \frac{\partial \mathrm{T}_n}{\partial n} = \alpha_s \left(\mathrm{T}_n - \mathrm{T}_0\right), \tag{7}$$

where

$$\alpha_s = 0,23 \cdot 10^{-6} \xi \left(\frac{\mathrm{T}_0 + \mathrm{T}_n}{2}\right)^3 + 1,4 \left(\frac{\mathrm{T}_n + \mathrm{T}_0}{h_0}\right)^{\frac{1}{4}} \cdot \left(\frac{\mathrm{B}B}{m^2 \mathrm{K}}\right).$$

Here T_0 is the surface temperature (n is the direction of the normal to the surface at the boundary points); ξ is the degree of blackness of the body. The temperature dependence of the total heat transfer coefficient, which reflects the processes of heat transfer by convection and radiation, is borrowed from [6]. Accounting for nonlinearity in the boundary conditions is due to the fact that, unlike other thermophysical parameters, the coefficient is most sensitive to temperature changes and its value can vary within very wide limits [7].

At the collision site and at the interfaces between the layers there is an ideal thermal contact location:

$$[\mathrm{T}] = \left[\mathrm{K} \frac{\partial \mathrm{T}}{\partial z}\right] = 0. \tag{8}$$

At the moment of time $t = 0$ the colliding bodies are free from stresses, and for the impactor the initial velocities of the translational and rotational motions are given:

$$\vartheta\left(r, z\right) = V_0, \ \ w\left(r, z\right) = w_0 r \ \left(r_0 \leq r \leq R, \ \ -h_0 \leq z \leq 0\right).$$

The initial temperatures of the striker and the obstacle are respectively equal to T^* and T_*

Note that in the contact interaction of solids, where the fast wave process is usually considered up to $10^{-5} - 10^{-3} C$, the temperature field initiated only by dynamic mechanical effects, can be calculated in the adiabatic approximation. Boundary and initial conditions for temperature are necessary only in the cooling problem, when the heated cylinder comes in contact with the surface of the plate and the calculation of their thermal state is carried out over rather long time intervals.

For the numerical solution of the mixed boundary-value problem (1)–(8), we construct an explicit difference scheme based on the grid-characteristic approach and the principle of the electrothermal analogy. The expediency of using an explicit scheme is due to the fact that implicit schemes have a lower resolution when calculating transient processes in deformable media. Implicit counting, as a rule, does not impose restrictions on the size of the time step, since in the overwhelming majority they are absolutely stable. However, the region of dependence of difference equations for them is greatly expanded, as a result of which the profiles of wave fronts are substantially smoothed out and the whole picture of unsteady wave motion turns out to be blurred. Moreover, the algorithms of implicit schemes are much more complicated and their implementation requires much more computational resources.

The domain of applicability of implicit schemes seems to be limited to the class of steady motions, when time plays a purely auxiliary role in the calculations. In addition, in some problems, some countable regions can be calculated using implicit schemes, while others can be calculated using explicit ones.

The construction of a difference scheme begins with the construction of a difference grid, according to which the calculation will be carried out. For this, the area of change of continuous arguments r is divided into rectangular cells with sides h_{ri} and h_{zj} $(i = 1, 2, ..., I;\ j = 1, 2, ..., J)$, each cell is assigned a number $\left(i - \frac{1}{2},\ j - \frac{1}{2}\right)$. The calculation is carried out by successive steps in time. The values of the desired functions on the time layer $(n+1)\,\tau$ are determined at fixed grid nodes corresponding to the geometric center of the cells, but by known solution on the previous layer $n\tau$.

Using the central differences for the approximation of the first derivatives but spatial variables, we replace the hyperbolic equations of the system (1) with their finite-difference analogues:

$$
\begin{gathered}
u^{n+1}_{i-\frac{1}{2},j-\frac{1}{2}} = u^{n}_{i-\frac{1}{2},j-\frac{1}{2}} + \\
+\frac{\tau}{\rho_{i-\frac{1}{2},j-\frac{1}{2}}}\left[\frac{\left(\sum^{n+\frac{1}{2}}_{ri,j-\frac{1}{2}}-\sum^{n+\frac{1}{2}}_{ri-1,j-\frac{1}{2}}\right)}{h_{ri}}+\frac{\left(\sum^{n+\frac{1}{2}}_{rzi-\frac{1}{2},j}-\sum^{n+\frac{1}{2}}_{rzi-\frac{1}{2},j-1}\right)}{h_{zj}}+\frac{\left(\sigma^{n}_{ri-\frac{1}{2},j-\frac{1}{2}}-\sigma^{n}_{\varphi i-\frac{1}{2},j-\frac{1}{2}}\right)}{r_{i-\frac{1}{2}}}\right]; \\
\vartheta^{n+1}_{i-\frac{1}{2},j-\frac{1}{2}} = \vartheta^{n}_{i-\frac{1}{2},j-\frac{1}{2}} + \\
+\frac{\tau}{\rho_{i-\frac{1}{2},j-\frac{1}{2}}}\left[\frac{\left(\sum^{n+\frac{1}{2}}_{rzi,j-\frac{1}{2}}-\sum^{n+\frac{1}{2}}_{rzi-1,j-\frac{1}{2}}\right)}{h_{ri}}+\frac{\left(\sum^{n+\frac{1}{2}}_{zi-\frac{1}{2},j}-\sum^{n+\frac{1}{2}}_{zi-\frac{1}{2},j-1}\right)}{h_{zj}}+\frac{\sigma^{n}_{rzi-\frac{1}{2},j-\frac{1}{2}}}{r_{i-\frac{1}{2}}}\right]; \\
\omega^{n+1}_{i-\frac{1}{2},j-\frac{1}{2}} = \omega^{n}_{i-\frac{1}{2},j-\frac{1}{2}} + \\
+\frac{\tau}{\rho_{i-\frac{1}{2},j-\frac{1}{2}}}\left[\frac{\left(\sum^{n+\frac{1}{2}}_{r\varphi i,j-\frac{1}{2}}-\sum^{n+\frac{1}{2}}_{r\varphi i-1,j-\frac{1}{2}}\right)}{h_{ri}}+\frac{\left(\sum^{n+\frac{1}{2}}_{z\varphi i-\frac{1}{2},j}-\sum^{n+\frac{1}{2}}_{z\varphi i-\frac{1}{2},j-1}\right)}{h_{zj}}+\frac{2\sigma^{n}_{r\varphi i-\frac{1}{2},j-\frac{1}{2}}}{r_{i-\frac{1}{2}}}\right]; \\
\sigma^{n+1}_{ri-\frac{1}{2},j-\frac{1}{2}} = \left(1-\tau\eta_{i-\frac{1}{2},j-\frac{1}{2}}\right)\cdot\sigma^{n}_{ri-\frac{1}{2},j-\frac{1}{2}}+\tau\beta_{i-\frac{1}{2},j-\frac{1}{2}}\sigma^{n}_{i-\frac{1}{2},j-\frac{1}{2}}+ \\
+\tau\left[\frac{\left(\lambda_{i-\frac{1}{2},j-\frac{1}{2}}+2\mu_{i-\frac{1}{2},j-\frac{1}{2}}\right)\cdot\left(U^{n+\frac{1}{2}}_{i,j-\frac{1}{2}}-U^{n+\frac{1}{2}}_{i-1,j-\frac{1}{2}}\right)}{h_{ri}}+\lambda_{i-\frac{1}{2},j-\frac{1}{2}}\left(\frac{\left(V^{n+\frac{1}{2}}_{i-\frac{1}{2},j}-V^{n+\frac{1}{2}}_{i-\frac{1}{2},j-1}\right)}{h_{zj}}+\frac{u^{n}_{i-\frac{1}{2},j-\frac{1}{2}}}{r_{i-\frac{1}{2}}}\right)\right]- \\
-p_{i-\frac{1}{2},j-\frac{1}{2}}\cdot\left(\mathrm{T}^{n+1}_{i-\frac{1}{2},j-\frac{1}{2}}-\mathrm{T}^{n}_{i-\frac{1}{2},j-\frac{1}{2}}\right); \\
\sigma^{n+1}_{\varphi i-\frac{1}{2},j-\frac{1}{2}} = \left(1-\tau\eta_{i-\frac{1}{2},j-\frac{1}{2}}\right)\cdot\sigma^{n}_{\varphi i-\frac{1}{2},j-\frac{1}{2}}+\tau\beta_{i-\frac{1}{2},j-\frac{1}{2}}\sigma^{n}_{i-\frac{1}{2},j-\frac{1}{2}}+ \\
+\tau\left[\frac{\left(\lambda_{i-\frac{1}{2},j-\frac{1}{2}}+2\mu_{i-\frac{1}{2},j-\frac{1}{2}}\right)\cdot\left(u^{n}_{i-\frac{1}{2},j-\frac{1}{2}}\right)}{r_{i-\frac{1}{2}}}+\lambda_{i-\frac{1}{2},j-\frac{1}{2}}\left(\frac{\left(U^{n+\frac{1}{2}}_{i,j-\frac{1}{2}}-U^{n+\frac{1}{2}}_{i-1,j-\frac{1}{2}}\right)}{h_{ri}}+\frac{\left(V^{n+\frac{1}{2}}_{i-\frac{1}{2},j}-V^{n+\frac{1}{2}}_{i-\frac{1}{2},j-1}\right)}{h_{zj}}\right)\right]- \qquad (9)\\
-p_{i-\frac{1}{2},j-\frac{1}{2}}\cdot\left(\mathrm{T}^{n+1}_{i-\frac{1}{2},j-\frac{1}{2}}-\mathrm{T}^{n}_{i-\frac{1}{2},j-\frac{1}{2}}\right); \\
\sigma^{n+1}_{zi-\frac{1}{2},j-\frac{1}{2}} = \left(1-\tau\eta_{i-\frac{1}{2},j-\frac{1}{2}}\right)\cdot\sigma^{n}_{zi-\frac{1}{2},j-\frac{1}{2}}+\tau\beta_{i-\frac{1}{2},j-\frac{1}{2}}\sigma^{n}_{i-\frac{1}{2},j-\frac{1}{2}}+ \\
+\tau\left[\frac{\left(\lambda_{i-\frac{1}{2},j-\frac{1}{2}}+2\mu_{i-\frac{1}{2},j-\frac{1}{2}}\right)\cdot\left(V^{n+\frac{1}{2}}_{i-\frac{1}{2},j}-V^{n+\frac{1}{2}}_{i-\frac{1}{2},j-1}\right)}{h_{zj}}+\lambda_{i-\frac{1}{2},j-\frac{1}{2}}\left(\frac{\left(U^{n+\frac{1}{2}}_{i,j-\frac{1}{2}}-U^{n+\frac{1}{2}}_{i-1,j-\frac{1}{2}}\right)}{h_{ri}}+\frac{\left(u^{n}_{i-\frac{1}{2},j-\frac{1}{2}}\right)}{r_{i-\frac{1}{2}}}\right)\right]- \\
-p_{i-\frac{1}{2},j-\frac{1}{2}}\cdot\left(\mathrm{T}^{n+1}_{i-\frac{1}{2},j-\frac{1}{2}}-\mathrm{T}^{n}_{i-\frac{1}{2},j-\frac{1}{2}}\right); \\
\sigma^{n+1}_{r\varphi i-\frac{1}{2},j-\frac{1}{2}} = \left(1-\tau\eta_{i-\frac{1}{2},j-\frac{1}{2}}\right)\cdot\sigma^{n}_{r\varphi i-\frac{1}{2},j-\frac{1}{2}}+\tau\mu_{i-\frac{1}{2},j-\frac{1}{2}}\left(\left(\frac{\left(W^{n+\frac{1}{2}}_{i,j-\frac{1}{2}}-W^{n+\frac{1}{2}}_{i-1,j-\frac{1}{2}}\right)}{h_{ri}}+\frac{\left(w^{n}_{i-\frac{1}{2},j-\frac{1}{2}}\right)}{r_{i-\frac{1}{2}}}\right)\right); \\
\sigma^{n+1}_{z\varphi i-\frac{1}{2},j-\frac{1}{2}} = \left(1-\tau\eta_{i-\frac{1}{2},j-\frac{1}{2}}\right)\cdot\sigma^{n}_{z\varphi i-\frac{1}{2},j-\frac{1}{2}}+\tau\mu_{i-\frac{1}{2},j-\frac{1}{2}}\frac{\left(W^{n+\frac{1}{2}}_{i,j-\frac{1}{2}}-W^{n+\frac{1}{2}}_{i-1,j-\frac{1}{2}}\right)}{h_{zj}}; \\
\sigma^{n+1}_{rzi-\frac{1}{2},j-\frac{1}{2}} = \left(1-\tau\eta_{i-\frac{1}{2},j-\frac{1}{2}}\right)\cdot\sigma^{n}_{rzi-\frac{1}{2},j-\frac{1}{2}}+\tau\mu_{i-\frac{1}{2},j-\frac{1}{2}}\left(\frac{\left(U^{n+\frac{1}{2}}_{i-\frac{1}{2},j}-U^{n+\frac{1}{2}}_{i-\frac{1}{2},j-1}\right)}{h_{zj}}+\frac{\left(V^{n+\frac{1}{2}}_{i,j-\frac{1}{2}}-V^{n+\frac{1}{2}}_{i-1,j-\frac{1}{2}}\right)}{h_{ri}}\right);
\end{gathered}
$$

In the two-layer difference scheme (9), the auxiliary «large» quantities U, V, W, Z, defined at the points of the boundaries of the rectangular cells, are calculated using coordinate wise splitting of the spatial two-dimensional equations (1) and using one-dimensional relations on the characteristics [8].

After obvious transformations, the difference scheme for the heat equation takes the form

$$\frac{C_{i-\frac{1}{2},j-\frac{1}{2}}\left(\mathrm{T}^{n+1}_{i-\frac{1}{2},j-\frac{1}{2}}-\mathrm{T}^{n}_{i-\frac{1}{2},j-\frac{1}{2}}\right)}{\tau}=$$

$$=-\mathrm{T}^{n}_{i-\frac{1}{2},j-\frac{1}{2}}\cdot p_{i-\frac{1}{2},j-\frac{1}{2}}\left[\frac{\left(U^{n+\frac{1}{2}}_{i,j-\frac{1}{2}}-U^{n+\frac{1}{2}}_{i-1,j-\frac{1}{2}}\right)}{h_{ri}}+\frac{\left(V^{n+\frac{1}{2}}_{i-\frac{1}{2},j}-V^{n+\frac{1}{2}}_{i-\frac{1}{2},j-1}\right)}{h_{zj}}+\frac{\left(u^{n}_{i-\frac{1}{2},j-\frac{1}{2}}\right)}{r_{i-\frac{1}{2}}}\right]+\frac{1}{d_{i-\frac{1}{2},j-\frac{1}{2}}}\times$$

$$\times\left(\alpha^{-}_{ri-\frac{1}{2},j-\frac{1}{2}}\mathrm{T}^{n}_{i-\frac{3}{2},j-\frac{1}{2}}+\alpha^{+}_{ri-\frac{1}{2},j-\frac{1}{2}}\mathrm{T}^{n}_{i+\frac{1}{2},j-\frac{1}{2}}+\right.$$

$$\left.+\alpha^{-}_{zi-\frac{1}{2},j-\frac{1}{2}}\mathrm{T}^{n}_{i-\frac{1}{2},j-\frac{3}{2}}+\alpha^{+}_{zi-\frac{1}{2},j-\frac{1}{2}}\mathrm{T}^{n}_{i-\frac{1}{2},j+\frac{1}{2}}-\gamma_{i-\frac{1}{2},j-\frac{1}{2}}\mathrm{T}^{n}_{i-\frac{1}{2},j-\frac{1}{2}}\right)+W^{*n}_{i-\frac{1}{2},j-\frac{1}{2}};$$

$$\left(\gamma_{i-\frac{1}{2},j-\frac{1}{2}}=\alpha^{-}_{ri-\frac{1}{2},j-\frac{1}{2}}+\alpha^{+}_{ri-\frac{1}{2},j-\frac{1}{2}}+\alpha^{-}_{zi-\frac{1}{2},j-\frac{1}{2}}+\alpha^{+}_{zi-\frac{1}{2},j-\frac{1}{2}}\right). \tag{10}$$

Here $\gamma_{i-\frac{1}{2},j-\frac{1}{2}}$ is the total thermal conductivity of the cell.

In proving the stability of the difference scheme from the initial data, we first consider the uncoupled viscoelastic problem without taking into account temperature additions $\frac{p\partial \mathrm{T}}{\partial t}$ in the relationship between normal stresses and strains.

A two-layer difference scheme (9) corresponds to a transition operator H, that translates a solution vector $\vec{F}$ on a temporary layer $t_n = n\tau$ into a vector $\vec{F}_1$ on a layer $t_{n+1} = t_n + \tau$

$$\vec{F}_1 = H\vec{F}. \tag{11}$$

The scheme is stable on the initial data, if the condition is met

$$\|\mathrm{H}\| \le 1. \tag{12}$$

The operator norm is determined by the energy norm of the vector $\vec{F}$:

$$\left\|\vec{F}\right\|^2=\mathrm{K}\,(t)+\mathrm{P}\,(t)\,,\ \|\mathrm{H}\|=\sup\left\|\mathrm{H}\vec{F}\right\|,\ \left\|\vec{F}\right\|=1, \tag{13}$$

where

$$\mathrm{K}\,(t)=\frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{J}\rho_{i-\frac{1}{2},j-\frac{1}{2}}\left(u^2_{i-\frac{1}{2},j-\frac{1}{2}}+v^2_{i-\frac{1}{2},j-\frac{1}{2}}+w^2_{i-\frac{1}{2},j-\frac{1}{2}}\right)\cdot h_{ri}\cdot h_{zj};$$

$$\mathrm{P}\,(t)=\tfrac{1}{2}\left(\vec{\sigma},\vec{\varepsilon}\right)=\tfrac{1}{2}\left(\sigma_r\varepsilon_r+\sigma_\varphi\varepsilon_\varphi+\sigma_z\varepsilon_z+\sigma_{r\varphi}\varepsilon_{r\varphi}+\sigma_{z\varphi}\varepsilon_{z\varphi}+\sigma_{rz}\varepsilon_{rz}\right)=$$

$$=\tfrac{1}{2}\textstyle\sum_{i=1}^{I}\sum_{j=1}^{J}\left\{\alpha\sigma_r^2+2q\sigma_r\sigma_z+\alpha\sigma_z^2+\frac{1}{\mu}\left[\sigma^2_{r\varphi}+\sigma^2_{z\varphi}+\sigma^2_{rz}\right]+\frac{1}{\mathrm{E}}\left[\sigma_\varphi-\nu\left(\sigma_r+\sigma_z\right)\right]^2\right\}_{i-\frac{1}{2},j-\frac{1}{2}}\times$$
$$\times h_{ri}\cdot h_{zj};$$

$$\alpha=\frac{a^2}{4\rho b^2\left(a^2-b^2\right)};\ \ q=\frac{2b^2-a^2}{4\rho b^2\left(a^2-b^2\right)};\ \mathrm{E}=\frac{\alpha-2q}{\left(q-\alpha\right)^2};\ \nu=\frac{q}{q-\alpha}.$$

Here $\mathrm{K}\,(t)$ and $\mathrm{P}\,(t)$ up to a constant factor corresponds to a discrete analogue of kinetic and potential energy.

Equality (13) can be written in matrix form:

$$\left\|\vec{F}\right\|^2_{\mathrm{A}}=\frac{1}{2}\left(\mathrm{A}\vec{F},\vec{F}\right),$$

where the scalar product is defined as follows:

$$\left(\vec{F},\vec{W}\right)=\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\sum_{k=1}^{g}F_{k,i-\frac{1}{2},j-\frac{1}{2}}W_{k,i-\frac{1}{2},j-\frac{1}{2}}\right)\cdot h_{ri}\cdot h_{zj},$$

and a positive definite matrix A has the form

$$\mathrm{A}=\begin{Vmatrix}
\rho & & & & & & & & 0\\
 & \rho & & & & & & & \\
 & & \rho & & & & & & \\
 & & & \frac{1}{\mathrm{E}} & -\frac{\nu}{\mathrm{E}} & -\frac{\nu}{\mathrm{E}} & & & \\
 & & & -\frac{\nu}{\mathrm{E}} & \frac{1}{\mathrm{E}} & -\frac{\nu}{\mathrm{E}} & & & \\
 & & & -\frac{\nu}{\mathrm{E}} & -\frac{\nu}{\mathrm{E}} & \frac{1}{\mathrm{E}} & & & \\
 & & & & & & \frac{1}{\mu} & & \\
 & & & & & & & \frac{1}{\mu} & \\
 & 0 & & & & & & & \frac{1}{\mu}
\end{Vmatrix}.$$

From (11), (12) we have

$$\left\|\vec{F}\left(t_{n+1}\right)\right\|_{\mathrm{A}}=\left\|\mathrm{H}\vec{F}\left(t_{n}\right)\right\|_{\mathrm{A}}\leq\|\mathrm{H}\|_{\mathrm{A}}\left\|\vec{F}\left(t_{n}\right)\right\|_{\mathrm{A}}\leq\left\|\vec{F}\left(t_{n}\right)\right\|_{\mathrm{A}}\leq\left\|\vec{F}\left(t_{n-1}\right)\right\|_{\mathrm{A}}\leq\ldots\leq\left\|\vec{F}\left(t_{1}\right)\right\|_{\mathrm{A}}\leq\left\|\vec{F}\left(0\right)\right\|_{\mathrm{A}},$$

which means the stability of the scheme according to the initial data.

The operator H can be represented as

$$\mathrm{H}=I+\tau\left(\mathrm{H}_r+\mathrm{H}_z+\mathrm{H}_{\mathrm{B}}\right)=\left(1-\frac{\tau}{\tau_r}-\frac{\tau}{\tau_z}-\frac{\tau}{\tau_{\mathrm{B}}}\right)\cdot I+\frac{\tau}{\tau_r}\left(I+\tau_r\mathrm{H}_r\right)+\frac{\tau}{\tau_z}\left(I+\tau_z\mathrm{H}_z\right)+\frac{\tau}{\tau_{\mathrm{B}}}\left(I+\tau_{\mathrm{B}}\mathrm{H}_{\mathrm{B}}\right).$$

Here I is the identity operator; H_r, H_z are operators containing differential derivatives only with respect to spatial variables r and z, respectively; τ_r, τ_z are time steps of «one-dimensional» schemes, corresponding to the operator $I+\tau_r\mathrm{H}_r$ and $I+\tau_z\mathrm{H}_z$ ($\tau_r>0$, $\tau_z>0$); the matrix of coefficients, taking into account viscous properties of the medium,

$$\mathrm{H}_{\mathrm{B}}=\begin{Vmatrix}
0 & & & & & & & & 0\\
 & 0 & & & & & & & \\
 & & 0 & & & & & & \\
 & & & -\eta+\beta & \beta & \beta & & & \\
 & & & \beta & -\eta+\beta & \beta & & & \\
 & & & \beta & \beta & -\eta+\beta & & & \\
 & & & & & & \eta & & \\
 & & & & & & & \eta & \\
 & 0 & & & & & & & \eta
\end{Vmatrix}.$$

As by defining the norm

$$\|\mathrm{H}\|_{\mathrm{A}}\leq\left|1-\frac{\tau}{\tau_r}-\frac{\tau}{\tau_z}-\frac{\tau}{\tau_{\mathrm{B}}}\right|\|I\|_{\mathrm{A}}+\frac{\tau}{\tau_r}\left\|I+\tau_r\mathrm{H}_r\right\|_{\mathrm{A}}+\frac{\tau}{\tau_r}\left\|I+\tau_z\mathrm{H}_z\right\|_{\mathrm{A}}+\frac{\tau}{\tau_{\mathrm{B}}}\left\|I+\tau_{\mathrm{B}}\mathrm{H}_{\mathrm{B}}\right\|_{\mathrm{A}},$$

then for the stability of the scheme specified by the operator H, it is enough to choose a step τ based on the condition

$$1-\frac{\tau}{\tau_r}-\frac{\tau}{\tau_z}-\frac{\tau}{\tau_{\mathrm{B}}}\geq 0, \tag{14}$$

if at the same time the steps τ_r, τ_z, τ_{B} ensure the implementation of inequalities

$$\left\|I+\tau_r\mathrm{H}_r\right\|_{\mathrm{A}}\leq 1,\ \left\|I+\tau_z\mathrm{H}_z\right\|_{\mathrm{A}}\leq 1,\ \left\|I+\tau_{\mathrm{B}}\mathrm{H}_{\mathrm{B}}\right\|_{\mathrm{A}}\leq 1. \tag{15}$$

Let's consider the last inequality in (15), in which the matrix $I+\tau_{\mathrm{B}}\mathrm{H}_{\mathrm{B}}$ defines the difference scheme

$$\frac{\vec{F}_1-\vec{F}}{\tau_{\mathrm{B}}}=\mathrm{H}_{\mathrm{B}}\vec{F}\ \ 8;8\ \ \vec{F}_1=\left(I+\tau_{\mathrm{B}}\mathrm{H}_{\mathrm{B}}\right)\cdot\vec{F}.$$

It is known from the properties of the norm of the matrix that

$$\left\| I + \tau_{\text{B}} \text{H}_{\text{B}} \right\|_{\text{A}} \leq \lambda_m,$$

where λ_m- the maximum modulo eigenvalue of the matrix $I + \tau_{\text{B}} \text{H}_{\text{B}}$. Therefore, to satisfy the inequality in question, it suffices to require meeting conditions

$$|1 + \tau_{\text{B}} \eta| \leq 1; \;\; |1 + \tau_{\text{B}} (\beta - \eta)| \leq 1,$$

of which we have

$$\tau_{\text{B}} \leq \min \left\{ \frac{2}{\eta}, \frac{2}{\beta - \eta} \right\}.$$

Since the parameters η, β depend on the indices i, j and $\eta \geq \beta \geq 0$, then, denoting $\eta_m = \max\limits_{i,j} \eta_{i-\frac{1}{2}, j-\frac{1}{2}}$, we get $\tau_{\text{B}} \leq \frac{2}{\eta_m}$.

Then, based on the constraint (14) for the stability of a two-dimensional scheme with respect to the initial data, it suffices that the size of the time step satisfies the inequality

$$\tau \left(\frac{1}{\tau_r} + \frac{1}{\tau_z} + \frac{\eta_m}{2} \right) \leq 1, \tag{16}$$

where

$$\tau_r = \min_{i,j} \frac{h_{ri}}{a_{i-\frac{1}{2}, j-\frac{1}{2}}}, \;\; \tau_z = \min_{i,j} \frac{h_{zj}}{a_{i-\frac{1}{2}, j-\frac{1}{2}}}.$$

Thus, the stability of the proposed difference scheme (9), further in accordance with the theory of A.A. Samarskii [9], the convergence of the solution of the difference scheme (9) to the solution of the differential problem (1), (2) is obtained.

References

1 Победря Б.Е. О связанных задачах в механике сплошной среды / Б.Е. Победря // Упругость и неупругость: сб. науч. тр. — Вып. 2. — М.: Изд-во ЛГУ, 1971. — С. 224–253.

2 Горский Н.М. О решении динамических задач теории упругости в напряжениях и скоростях смещений / Н.М. Горский / Численные методы механики сплошной среды. — Т. 3, № 3. — Новосибирск: ВЦ СО АН СССР, 1972. — С. 32–41.

3 Горский Н.М., Коновалов А.Н. О разностных методах решения динамических задач теории упругости: труды III Всесоюз. конф. по численным методам решения задач теории упругости и пластичности. — Т. I. — Новосибирск: ВЦ СО АН СССР, 1974. — С. 68-84.

4 Clifton K.I. A difference method for plane problems in dynamic elasticity / K.I. Clifton. — Juart. Appl. Math. — 1967. — Vol. 25. — N 1. — P. 97–116.

5 Боли Б. Теория температурных напряжений / Б. Боли, Дж. Уэйнер. — М.: Мир, 1964. — 517 с.

6 Дульнев Г.Н. Теплообмен в радиоэлектронных аппаратах / Г.Н. Дульнев, Э.М. Семяшкин. — Л.: Энергия, 1968. — 360 с.

7 Коваленко А.Д. Термоупругость / А.Д. Коваленко. — Киев: Выща шк., 1975. — 216 с.

8 Годунов С.К. Численное решение многомерных задач газовой динамики / С.К. Годунов, А.В. Забродин, М.Я. Иванов и др. — М.: Наука, 1976. — 400 с.

9 Самарский А.А. Теория разностных схем: учеб. пос. / А.А. Самарский. — М.: Наука, 1977. — 658 с.

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Қабатты ортадағы екіөлшемді термотұтқырсерпімді толқындар

Қатты дененің деформациясының динамикалық мәселелері ТМД және шетелдерде көптеген зерттеулердің тақырыбы болды. Аталған және басқа жарияланған жұмыстарда қабылданған жорамалдарды жеңілдету, соқтығу мен кедергі сияқты, кинематика мен кернеу жағдайын сипаттайтын механикалық және математикалық модельдерді одан әрі жетілдіру мен жақсарту қажеттілігіне әкеледі. Сонымен қатар термотұтқырсерпімді теориясының шеңберінде еркін қуыстар мен қатаң кірмелерді қамтитын изотроптық пластиналар түрінде кедергі бар цилиндрлік индетанттың осьтік симметриялық соқтығы сандық түрде зерттелді. Серпімділік теориясының және қысымның икемділігі туралы теориялық есептердің әртүрлі тұжырымдамалары болуы мүмкін. Дегенмен, жылдамдық пен кернеулерде пайдаланылатын формулалар әмбебап болып табылады, себебі ол негізгі шекаралық шарттарды (аралас шарттарды қоса) біркелкі түрде шешуге мүмкіндік туғызады. Бұл жұмыста тор-сипаттамалық схема және оның конвергенциясы берілген. А.А. Самарскийдің теориясына сәйкес, тор есебі арқылы энергетикалық нормадағы орнықтылығы дәлелденген.

Кілт сөздер: екіөлшемді термотұтқырсерпімді толқындар, айырымдық схеманың орнықтылығы, айырымдық есептің шешімінің жинақтылығы, индентор, деформация, тензор, кернеу.

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Двумерные термовязкоупругие волны в слоистых средах

Динамические задачи о деформировании твердых тел явились предметом многочисленных исследований в СНГ и за рубежом. Отказ от ряда упрощающих предположений, принятых в цитируемых и других опубликованных работах, приводит к необходимости дальнейшего уточнения и совершенствования механико-математических моделей, описывающих кинематику и напряженное состояние как ударника, так и преграды. Далее в рамках связанной теории термовязкоупругости численно исследовано осесимметричное соударение цилиндрического индентора с препятствием в виде пакета изотропных пластин, содержащего свободные полости и жесткие включения. Возможны различные формулировки задач теории упругости и термовязкоупругости. Однако используемая постановка в скоростях и напряжениях является одной из наиболее универсальных, так как позволяет решать основные граничные задачи (в том числе и смешанные) единообразным способом. В работе даны сеточно-характеристическая схема и ее сходимость. В соответствии с теорией А.А. Самарского, доказана устойчивость в энергетической норме сеточной задачи.

Ключевые слова: двумерные термовязкоупругие волны, устойчивость разностной схемы, сходимость решения разностной задачи, индентор, деформация, тензор, напряжения.

References

1 Pobedria, B.E. (1971). O sviazannykh zadachakh v mekhanike sploshnoi sredy [On related problems in continuum mechanics]. *Upruhost i neupruhost – Elasticity and inelasticity, Issue 2,* 224–253. Moscow: Izdatelstvo LHU[in Russian].

2 Gorskii, N.M. (1972). O reshenii dinamicheskikh zadach teorii upruhosti v napriazheniiakh i skorostiakh smeshchenii [On the solution of dynamic problems of the theory of elasticity in stresses and velocities of displacements]. *Chislennye metody mekhaniki sploshnoi sredy – Numerical method in continuum mechanics, Vol. 3, 3,* 32–41. Novosibirsk: VTs SO AN SSSR [in Russian].

3 Gorskii, N.M., & Konovalov, A.N. (1974). O raznostnykh metodakh resheniia dinamicheskikh zadach teorii upruhosti [Differential method of solving dynamic problems of elasticity theory]. *Trudy III Vsesoiuznoi konferentsii po chislennym metodam resheniia zadach teorii upruhosti i plastichnosti – Proceeding of III Allunion conference on numerical metods in elasticity and plastic theory, 1,* 68–84. Novosibirsk: VTs SO AN SSSR [in Russian].

4 Clifton, K.I. (1967) A difference method for plane problems in dynamic elasticity. *Juart. Appl. Math., Vol. 25, 1,* 97–116.

5 Boli, B., & Ueiner, Dzh. (1960). *Teoriia temperaturnykh napriazhenii [Theory of Thermal Stresses].* Moscow: Mir [in Russian].

6 Dulnev, G.N., & Semiashkin, E.M. (1968). *Teploobmen v radioelektronnykh apparatakh [Heat transfer in electronic devices].* Leningrad: Enerhia [in Russian].

7 Kovalenko, A.D. (1975). *Termoupruhost [Thermoelasticity].* Kiev: Vishzha shkola [in Russian].

8 Goduno,v S.K., Zabrodin, A.V., Ivanov, M.Ia. & et al. (1976). *Chislennoe reshenie mnohomernykh zadach hazovoi dinamiki [Computational solution of multidimensional gas dynamics problems].* Moscow: Nauka [in Russian].

9 Samarskii, A.A. (1977). *Teoriia raznostnykh skhem [The Theory of difference scheme].* Moscow: Nauka [in Russian].

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On the calculation of rectangular plates by the trigonometric series

The article is devoted to the question of applying the method of single trigonometric series to solving the plate bending problems. In this article the structure of this method is described, its main components are highlighted, the classical approach of calculating rectangular plates hinged supported on two parallel sides and with arbitrary boundary conditions on each of the other two sides is characterized. The mathematical apparatus of the method of single trigonometric series is presented in the volume necessary for calculating the plates. The detailed example of calculating a rectangular plate by the stated method is given. The article is focused mainly on students and undergraduates engaged in research work in the field of mechanics and applied mathematics.

Keywords: bending of a rectangular plate, plate deflection function, boundary conditions of plate, trigonometric series method, the solution of Levi.

Plate bending problems play an important role in construction, engineering, aviation, shipbuilding, etc. Construction and technics are the branches of activity of the industrial complex which always were, are and will remain in demand by the country's economy; therefore, issues related to the theoretical studies of such problems remain relevant and have important practical value [1].

Many analytical and numerical calculation methods are used to study the problems of plate bending [2, 3]. An exact solution in analytical form for such problems is possible only in some particular cases of the geometrical type of the plate, the load and the conditions for its fixation on the supports, therefore, for engineering practice, approximate, but sufficiently accurate methods for solving the considered boundary value problem are of special importance.

When considering the plate bending problems, the methods of double and single trigonometric series are the most interesting because of connection with their possible numerical implementation in the Maple software package [4].

The solution in double trigonometric series (Navier's solution) is typically used for rectangular plates, freely or hinged supported around the entire contour. The solution in single trigonometric series (Levi's solution) allows to perform the calculation of a plate hinged supported on two parallel sides and with arbitrary boundary conditions on each of the other two sides.

We consider the case of a plate $0 \le x \le a$, $0 \le y \le b$, in which only two opposite edges have a hinge support (for example, $x = 0$ and $x = a$) and the other two edges have arbitrary boundary conditions.

We present the desired function of plate deflections $W(x, y)$ in the form of a single trigonometric series

$$W(x,y) = \sum_{n=1}^{\infty} Y_n \sin \omega_n x, \tag{1}$$

where $\omega_n = \frac{n\pi}{a}$, $Y_n = Y_n(y)$ is an unknown function, which is chosen so that expression (1) satisfies the resolving equation of S. Germain

$$D\,\Delta\Delta\,W = q\,(x,y), \tag{2}$$

and the conditions of fixing on the edges $y = 0$ and $y = b$. Here D is the cylindrical rigidity of the plate, q is the intensity of the external distributed load, $\Delta\Delta\,W$ is a biharmonic operator.

The deflection and the bending moment along the hinged supported edges must be equal to zero, so the boundary conditions have the following form when $x = 0$ and $x = a$ [5]

$$W = 0, \qquad \frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} = 0; \tag{3}$$

where ν is Poisson's coefficient. It is obvious that expression (1) satisfies the boundary conditions (3), which are given on the sides $x = 0$, $x = a$ of the plate.

We present the load function $q(x, y)$ in a form of a trigonometric series

$$q(x,y) = \sum_{n=1}^{\infty} q_n(y) \sin \omega_n x, \tag{4}$$

where

$$q_n(y) = \frac{2}{a} \int_o^a q(x,y) \cdot \sin \omega_n x dx. \tag{5}$$

Substituting formulas (1) and (4) into the basic differential equation (2), we obtain

$$\sum_{n=1}^{\infty} (\omega_n^4 Y_n - 2\,\omega_n^2 Y_n'' + Y_n^{IV}) \sin \omega_n x = \frac{1}{D} \sum_{n=1}^{\infty} q_n \sin \omega_n x. \tag{6}$$

Obviously, the relation (6) will be satisfied if

$$Y_n^{IV} - 2\omega_n^2 Y_n'' + \omega_n^4 Y_n = \frac{q_n}{D}. \tag{7}$$

The ordinary differential equation (7) allows us to determine an unknown function Y_n for any number n of expansion. Its general solution can be written as

$$Y_n(y) = A_n \cdot ch\omega_n y + B_n \cdot sh\omega_n y + C_n \cdot y \cdot ch\omega_n y + D_n \cdot y \cdot sh\omega_n y + \varphi_n(y), \tag{8}$$

where $A_n,\ B_n,\ C_n,\ D_n$ are arbitrary integration constants, and φ_n is a partial integral depending on the type q_n and, therefore, on a given external load q.

To determine the four integration constants $A_n,\ B_n,\ C_n,\ D_n$, the boundary conditions defined at the edges of the plate $y = 0$, $y = b$ are used and this boundary conditions, of course, can be different. In the general case, this leads to the solving a system of algebraic equations with respect to unknowns $A_n,\ B_n,\ C_n,\ D_n$.

The order of this system will increase if the load is given in the direction of the y-axis by a discontinuous law. For example, if the load breaks the plate in the direction of the y-axis into k sections. For each section we will have four unknowns $A_n,\ B_n,\ C_n,\ D_n$, and their total number will be equal to $4k$. Thus, to determine the integration constants, it is necessary to create a system of $4k$ algebraic equations here, four of which will reflect the boundary conditions at the edges of the plate, and $4k$-4 other equations will be the conjugation conditions of the k sections. To overcome the noted inconvenience, the solution of equation (7) should not be represented in the form of (8), but this solution should be presented in the form of the method of initial parameters. In this case, for any law of load distribution, to find the integration constants (initial parameters), it will be necessary to solve a system of only two algebraic equations [6].

After finding the coefficients $A_n,\ B_n,\ C_n,\ D_n$ and determining the function $Y_n(y)$ by the formula (8), the plate deflections can be found by the formula (1) in the form of a series, so bending moments, torque, as well as, transverse forces will be written as

$$\begin{gathered}
M_x(x,y) = -D \sum_{n=1}^{\infty} (\nu Y_n'' - \omega_n^2 Y_n) \sin \omega_n x, \\
M_y(x,y) = -D \sum_{n=1}^{\infty} (Y_n'' - \nu \omega_n^2 Y_n\ ) \sin \omega_n x, \\
M_{xy}(x,y) = -D(1-\nu) \sum_{n=1}^{\infty} \omega_n Y_n'\ \cos \omega_n x, \\
Q_x(x,y) = -D \sum_{n=1}^{\infty} \omega_n (Y_n'' - \omega_n^2 Y_n) \cos \omega_n x, \\
Q_y(x,y) = -D \sum_{n=1}^{\infty} (Y_n''' - \omega_n^2 Y_n') \sin \omega_n x\ .
\end{gathered} \tag{9}$$

Consider the case of a uniformly distributed load of the constant intensity $q = const$. Using the formula (5), we obtain

$$q_n = \begin{cases} 0\,; & n = 2m, \quad m = 1,\ 2, ... \\ \frac{4q}{n\pi}; & n = 2m-1, \quad m = 1,\ 2, ... \end{cases} \tag{10}$$

Then, taking into account (10), the partial integral of equation (7) can be written as

$$\varphi_n = \begin{cases} 0\,; & n = 2m, \quad m = 1,\ 2, ... \\ \frac{4q}{n\pi\omega_n^4 D}; & n = 2m-1, \quad m = 1,\ 2, ... \end{cases} \tag{11}$$

It can be seen from (11) that for even n, the homogeneous differential equation (7) has only trivial solution, so in the case of a uniformly distributed load of constant intensity the deflection function $W(x,y)$ takes the form

$$W(x,y) = \sum_{m=1}^{\infty} [A_{2m-1} ch\omega_{2m-1} y + B_{2m-1} sh\omega_{2m-1} y + y\,(C_{2m-1} ch\omega_{2m-1} y + D_{2m-1} sh\omega_{2m-1} y) +$$

$$+ \frac{4q}{\pi D(2m-1)\omega_{2m-1}^4}\ ] \cdot \sin\omega_{2m-1} x, \tag{12}$$

where the coefficients $A_n,\ B_n,\ C_n,\ D_n$ depend on the given boundary conditions of the plate edges $y = 0$ and $y = b$.

As an example of calculating the coefficients of the plate $A_n,\ B_n,\ C_n,\ D_n$, we consider the case when one of the sides of the plate parallel to the x-axis is supported by an elastic contour, and the other side is rigidly pinched. The elastic contour may be, for example, a beam, bending under the action of pressures applied to it.

Denote by EJ the rigidity of the beam, then on the elastically supported edge of the plate $y = 0$ the boundary conditions take the form [5]

$$\left(\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} \right)\Bigg|_{y=0} = 0,$$

$$D \left[\frac{\partial^3 W}{\partial y^3} + (2-\nu) \frac{\partial^3 W}{\partial x^2 \partial y} \right]\Bigg|_{y=0} = \left(EJ \frac{\partial^4 W}{\partial x^4} \right)\Bigg|_{y=0}. \tag{13}$$

On the rigidly pinched edge of the plate $y = b$, the boundary conditions are written as

$$W|_{y=b} = 0, \qquad \frac{\partial W}{\partial y}\Bigg|_{y=b} = 0 \tag{14}$$

Note that from the relation

$$\sum_{n=1}^{\infty} F_n(y) \sin\omega_n x = G(x,y), \tag{15}$$

when multiplying (15) by $\sin\omega_k x$, integrating with respect to x from 0 to a and replacing k by n, we receive

$$F_n(y) = \frac{2}{a} \int_0^a G(x,y) \sin\omega_n x dx. \tag{16}$$

From the boundary conditions (13), (14), taking into account (15), (16) we obtain that the required function $Y_n(y)$ must satisfy the following relations

$$\begin{cases} Y_n''(0) - \nu \cdot \omega_n^2 \cdot Y_n(0) = 0, \\ Y_n'''(0) - (2-\nu) \cdot \omega_n^2 \cdot Y_n'(0) - \frac{EJ}{D} \cdot \omega_n^4 \cdot Y_n(0) = 0, \\ Y_n(b) = 0, \\ Y_n'(b) = 0. \end{cases} \tag{17}$$

From (17) and (8) we obtain a system of algebraic equations to determine the coefficients $A_n,\ B_n,\ C_n,\ D_n$

$$\begin{cases} (1-\nu) \cdot \omega_n \cdot A_n + 2D_n = f_1, \\ \frac{EJ}{D} \cdot \omega_n^2 \cdot A_n + (1-\nu) \cdot \omega_n \cdot B_n - (1+\nu) \cdot C_n = f_2, \\ A_n \cdot ch\omega_n b + B_n \cdot sh\omega_n b + C_n \cdot b \cdot ch\omega_n b + D_n \cdot b \cdot sh\omega_n b = f_3, \\ A_n \cdot \omega_n \cdot sh\omega_n b + B_n \cdot \omega_n \cdot ch\omega_n b + C_n(ch\omega_n b + b \cdot \omega_n \cdot sh\omega_n b) + D_n(sh\omega_n b + b \cdot \omega_n \cdot ch\omega_n b) = f_4, \end{cases} \tag{18}$$

where

$$\begin{cases} f_1 = -\frac{1}{\omega_n}\varphi_n''(0) + \nu \cdot \omega_n \cdot \varphi_n(0), \\ f_2 = \frac{1}{\omega_n^2}\varphi_n'''(0) - (2-\nu)\cdot \varphi_n'(0) - \frac{EJ}{D}\omega_n^2 \cdot \varphi_n(0), \\ f_3 = -\varphi_n(b), \\ f_4 = -\varphi_n'(b). \end{cases} \quad (19)$$

After transformations, the system (18), (19) takes the form

$$\begin{cases} A_n + \frac{2}{(1-\nu)\omega_n}D_n = g_1, \\ B_n - \frac{1+\nu}{(1-\nu)\omega_n}C_n - \frac{2EJ}{D(1-\nu)^2}D_n = g_2, \\ \left[\frac{1+\nu}{(1-\nu)\omega_n}sh\omega_n b + bch\omega_n b\right]C_n + \left[\frac{\tau}{D(1-\nu)^2}sh\omega_n b - \frac{2}{(1-\nu)\omega_n}ch\omega_n b\right]D_n = g_3, \\ \left[\frac{2}{1-\nu}ch\omega_n b + b\omega_n sh\omega_n b\right]C_n + \left[\frac{\tau\omega_n}{D(1-\nu)^2}ch\omega_n b - \frac{1+\nu}{1-\nu}sh\omega_n b\right]D_n = g_4, \end{cases} \quad (20)$$

where

$$g_1 = \frac{1}{(1-\nu)\omega_n}f_1;$$

$$g_2 = -\frac{EJ}{D(1-\nu)^2}f_1 + \frac{1}{(1-\nu)\omega_n}f_2;$$

$$g_3 = \left[\frac{EJ}{D(1-\nu)^2}sh\omega_n b - \frac{1}{(1-\nu)\omega_n}ch\omega_n b\right]\cdot f_1 - \frac{sh\omega_n b}{(1-\nu)\omega_n}f_2 + f_3; \quad (21)$$

$$g_4 = \left[\frac{EJ\omega_n}{D(1-\nu)^2}ch\omega_n b - \frac{1}{1-\nu}sh\omega_n b\right]\cdot f_1 - \frac{ch\omega_n b}{1-\nu}f_2 + f_4;$$

$$\tau = 2EJ + bD(1-\nu)^2.$$

From the last two equations of the system (20), (21), we find the values of the coefficients C_n and D_n :

$$C_n = \frac{\left[\tau\omega_n ch\omega_n b - D(1-\nu^2)sh\omega_n b\right]g_3 + \left[\frac{2D}{\omega_n}(1-\nu)ch\omega_n b - \tau\cdot sh\omega_n b\right]g_4}{\left(b\tau\omega_n + \frac{4D}{\omega_n}\right)ch^2\omega_n b + \left[Db(1-\nu)^2 - \tau\right]sh\omega_n b\cdot ch\omega_n b - \left[\frac{D}{\omega_n}(1+\nu)^2 + b\tau\omega_n\right]sh^2\omega_n b};$$

$$D_n = \frac{D(1-\nu)\left\{-\left[2ch\omega_n b + b\omega_n(1-\nu)sh\omega_n b\right]g_3 + \left[\frac{1+\nu}{\omega_n}sh\omega_n b + b(1-\nu)ch\omega_n b\right]g_4\right\}}{\left(b\tau\omega_n + \frac{4D}{\omega_n}\right)ch^2\omega_n b + \left[Db(1-\nu)^2 - \tau\right]sh\omega_n b\cdot ch\omega_n b - \left[\frac{D}{\omega_n}(1+\nu)^2 + b\tau\omega_n\right]sh^2\omega_n b}. \quad (22)$$

Using the first equations of the system (20)

$$A_n = g_1 - \frac{2}{(1-\nu)\omega_n}D_n,$$

$$B_n = g_2 + \frac{1+\nu}{(1-\nu)\omega_n}C_n + \frac{2EJ}{D(1-\nu)^2}D_n,$$

and taking into account (22), we obtain the values of the coefficients A_n and B_n in the following forms

$$A_n = g_1 - \frac{2}{(1-\nu)\omega_n}\cdot\frac{D(1-\nu)\left\{-\left[2ch\omega_n b + b\omega_n(1-\nu)sh\omega_n b\right]g_3 + \left[\frac{1+\nu}{\omega_n}sh\omega_n b + b(1-\nu)ch\omega_n b\right]g_4\right\}}{\left(b\tau\omega_n + \frac{4D}{\omega_n}\right)ch^2\omega_n b + \left[Db(1-\nu)^2 - \tau\right]sh\omega_n b\cdot ch\omega_n b - \left[\frac{D}{\omega_n}(1+\nu)^2 + b\tau\omega_n\right]sh^2\omega_n b};$$

$$B_n = g_2 + \frac{1+\nu}{(1-\nu)\omega_n}\cdot\frac{\left[\tau\omega_n ch\omega_n b - D(1-\nu^2)sh\omega_n b\right]g_3 + \left[\frac{2D}{\omega_n}(1-\nu)ch\omega_n b - \tau\cdot sh\omega_n b\right]g_4}{\left(b\tau\omega_n + \frac{4D}{\omega_n}\right)ch^2\omega_n b + \left[Db(1-\nu)^2 - \tau\right]sh\omega_n b\cdot ch\omega_n b - \left[\frac{D}{\omega_n}(1+\nu)^2 + b\tau\omega_n\right]sh^2\omega_n b} +$$

$$+\frac{2EJ}{1+\nu}\cdot\frac{-\left[2ch\omega_n b + b\omega_n(1-\nu)sh\omega_n b\right]g_3 + \left[\frac{1+\nu}{\omega_n}sh\omega_n b + b(1-\nu)ch\omega_n b\right]g_4}{\left(b\tau\omega_n + \frac{4D}{\omega_n}\right)ch^2\omega_n b + \left[Db(1-\nu)^2 - \tau\right]sh\omega_n b\cdot ch\omega_n b - \left[\frac{D}{\omega_n}(1+\nu)^2 + b\tau\omega_n\right]sh^2\omega_n b}. \quad (23)$$

Due to the bulkiness of formulas (22), (23) for the determination of the coefficients $A_n,\ B_n,\ C_n,\ D_n$ in the general case, and, consequently, due to the inconvenience and complexity of further use of these formulas, it is recommended that all calculations of the constants $A_n,\ B_n,\ C_n,\ D_n$ are carried out for specific numerical values of the system coefficients (20), (21) in each particular case with given numerical parameters.

Substitution of the found coefficients $A_n,\ B_n,\ C_n,\ D_n$ in (8), (9), (10) gives the function of plate deflections $W(x,y)$, bending moments and torques, as well as transverse forces in the form of trigonometric series in the case where one of the sides of the plate parallel to the x-axis is supported by an elastic contour, and another side is rigidly pinched.

In the case of a uniformly distributed load of constant intensity q, the deflection function has the form (12) with coefficients (22), (23).

In principle, the method of single trigonometric series is more accurate than the previously considered Navier's method [6], since in this method the required function $W(x,y)$ is approximated by trigonometric functions only in one direction, and in another direction the function $W(x,y)$ is sought precisely from the differential equation (7). This can be seen from a comparison of the results obtained by the two methods for the previously considered problem of bending a square plate, hinged around the entire contour, in Table [6].

It should be noted that with one term of expansion (1) in the single trigonometric series method, not only the values of the deflection $W(x,y)$ and bending moment M_y are significantly clarified, but also the value of bending moment of another direction M_x are greatly improved. Note that in both of the considered methods, the convergence of the series will be higher and the accuracy will be greater, than better a given load $q(x,y)$ can be represented by expansion in trigonometric functions [6].

References

1 Завьялов В.Н. Основы строительной механики пластин / В.Н. Завьялов, Е.А. Мартынов, В.М. Романовский. — Омск: СибАДИ, 2012. — 116 с.

2 Аханова А.С. О расчете пластин разложением в ряд функции прогибов / А.С. Аханова, Г.А. Есенбаева, Н.К. Турсынгалиев // Вестн. Караганд. ун-та. Сер. Математика. — 2016. — №2 (82). — С. 15–22.

3 Есенбаева Г.А. О расчете прямоугольных пластин вариационным методом / Г.А. Есенбаева, А.А. Смаилова // Вестн. Караганд. ун-та. Сер. Математика. — 2016. — №2 (82). — С. 56–62.

4 Косауров А.П. Анализ и особенности методов при расчете пластин и оболочек на изгиб / А.П. Косауров, П.В. Тимофеев. - М.: Фонд «Основание», 2013. — 17 с.

5 Тимошенко С.П. Теория упругости / С.П. Тимошенко, Дж. Гудьер. — М.: Наука, 1979. — 560 с.

6 Животов А.Г. О расчете прямоугольных пластин методом тригонометрических рядов / А.Г. Животов, Г.А. Есенбаева // Вестн. Караганд. ун-та. Сер. Математика. — 2015. — №3 (79). — С. 44–49.

Г.А. Есенбаева, Ф.М. Аханов, Т.Х. Макажанова

Тікбұрышты пластиналарды тригонометриялық қатарлармен есептеу туралы

Мақала пластиналардың иілуі туралы есептерді шешуге дара тригонометриялық қатарлар әдісін қолдану мәселесіне арналған. Осы әдіс құрылымы келтірілді, оның негізгі компоненттері көрсетілді, тікбұрышты пластиналарды есептеуге классикалық әдісті сипаттайды, екі параллель жақтары топсалы бекітілген және басқа екі жақтары кез келген шекаралық жағдаймен анықталды. Дара тригонометриялық қатарлар әдісінің математикалық аппараты пластиналарды есептеу үшін қажетті көлемде ұсынылған. Берілген әдіспен тікбұрышты пластинаны есептеудің егжей-тегжейлі мысалы келтірілген. Бұл мақала, негізінен, механика және қолданбалы математика саласындағы ғылыми-зерттеу жұмыстарымен айналысатын студенттер мен магистранттарға бағытталған.

Кілт сөздер: тікбұрышты пластинаның иілуі, пластинаның иілу функциясы, пластинаның шекаралық шарттары, тригонометриялық қатарлар әдісі, Леви шешімі.

Г.А. Есенбаева, Ф.М. Аханов, Т.Х. Макажанова

О расчете прямоугольных пластин тригонометрическими рядами

Статья посвящена вопросу применения метода одинарных тригонометрических рядов к решению задач об изгибе пластин. Авторами представлена структура данного метода, выделены его основные компоненты, охарактеризован классический подход расчёта прямоугольных пластин, шарнирно опертых по двум параллельным сторонам и с произвольными граничными условиями на каждой из двух других сторон. Математический аппарат метода одинарных тригонометрических рядов представлен в необходимом для расчёта пластин объеме. Приведен подробный пример расчета прямоугольной пластины изложенным методом. Данная статья ориентирована, главным образом, на студентов и магистрантов, занимающихся научно-исследовательской работой в области механики и прикладной математики.

Ключевые слова: изгиб прямоугольной пластины, функция прогиба пластины, граничные условия пластины, метод тригонометрических рядов, решение Леви.

References

1 Zavyalov, V.N., Martynov, E.A., & Romanovsky, V.M. (2012). *Osnovy stroitelnoi mekhaniki plastin [Basics of structural mechanics of plates].* Omsk: SibADI [in Russian].

2 Akhanova, A.S., Yessenbayeva, G.A., & Tursyngaliyev, N.K. (2016). O raschete plastin razlozheniem v riad funktsii prohibov [On the calculation of plates by the series representation of the deflection function]. *Vestnik Karahandinskoho universiteta. Seriia Matematika – Bulletin of the Karaganda University. Mathematics series, 2(82),* 15-22 [in Russian].

3 Yessenbayeva, G.A., & Smailova, A.A. (2016). O raschete priamouholnykh plastin variatsionnym metodom [On the calculation of rectangular plates by the variation method]. *Vestnik Karagandinskoho universiteta. Seriia Matematika – Bulletin of the Karaganda University. Mathematics series, 2(82),* 56-62 [in Russian].

4 Kosaurov, A.P., & Timofeev, P.V. (2013). *Analiz i osobennosti metodov pri raschete plastin i obolochek na izhib [Analysis and features of methods for calculation of plates and shells for bending].* Moscow: Fond «Osnovanie» [in Russian].

5 Timoshenko, S.P., & Goodier, J. (1979). *Teoriia upruhosti [Theory of elasticity].* Moscow: Nauka [in Russian].

6 Zhivotov, A.G., & Yessenbayeva, G.A. (2015). O raschete priamouholnykh plastin metodom trihonometricheskikh riadov [On the calculation of rectangular plates by the method of trigonometric series]. *Vestnik Karahandinskoho universiteta. Seriia Matematika – Bulletin of the Karaganda University. Mathematics series, 3(79),* 44–49 [in Russian].

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Digitalization of healthcare system of Kazakhstan

The article is devoted to the development of digital technologies in the field of healthcare of Kazakhstan. The process of automation of a medical institution hospital is considered. The functionality of the developed information system and the prospects for the implementation of this system in this area are described. The developed information system allows it to accumulate the medical and statistical information in the databases of the hospital and use in practice, automates the management of the hospital workflow. The capabilities of the system consist in registering patients in a hospital, maintaining an electronic medical history and obtaining complete statistical reports. The patient's electronic medical history contains his personal data, diagnoses, recommended procedures and medications. The introduction of information and communication technologies in the health sector will improve the quality of care and significantly speed up the work of staff.

Keywords: health digitalization, hospital automation, information system functionality.

The health of every person, as a component of the health of the entire population, becomes a factor that determines not only the usefulness of its existence, but also the potential of its capabilities [1].

President N. Nazarbayev in his message to citizens of Kazakhstan emphasized that one of the directions of state policy at the new stage of development of our country should be the improvement of the quality of medical services and the development of a high-tech healthcare system. The quality of medical services is a complex concept and depends on many reasons, among which the material and technical equipment of medical organizations, the level of professionalism and motivation of clinical specialists to improve it, the introduction of modern technologies to manage and organize medical care, the introduction of effective methods of paying for medical help. Improving the management of the quality of medical services occupies an important place in the context of the strategic development of health care in Kazakhstan until 2020 [2].

Today, digital technologies are beginning to change the most conservative sphere of human activity – health care. As part of the «Digital Kazakhstan» program, domestic clinics face the challenge of introducing digital health care. Already this year, the full transition to the electronic format of medical services is announced [3].

The transition to such a model will help digitization, omni-channel, the use of big data and the use of artificial intelligence to process them. Working with data will provide an opportunity to improve the quality of medical care, reduce treatment time, and, at the same time, increase the medical activity of citizens and lead to an increase in the number of patients. But all these changes will not be possible as long as patient data is not collected in digital form.

Health care is generally one of the most difficult branches. It is very conservative. The difficulty is also associated with a large amount of accumulated data, and unstructured. At the same time, all this is connected with human health – the topic is very sensitive. In the coming years, we are waiting for a radical transformation of this industry. Health care will begin to interact more and more with information technology and management issues. It has ceased to be merely treating people. Health care will come to digitalization, it is inevitable. For centuries, people thought about the need to go to the doctor in a negative way, when something bad happened. This approach will change. There appears a service component – rating of doctors. The doctor ceases to be the person from whom only knowledge is required. Convenience, good service, digitalization, big data – all this is necessary for medicine today [4–10].

Health becomes a service. We cannot escape from this: people who are accustomed to receive this service in other areas of life today want the same from health care too – so that they can be comfortable, understandable, with feedback. An affiliate model will determine the future development of medicine.

Kazakhstan is just entering into the trends that will determine the development of health care in the future. Modern medicine seeks to create ecosystem, where patients anticipate until they get sick, but visit doctors at healthy state, just to maintain and strengthen health. In addition, the experience of other industries already dictates increased requirements for service in medicine – people want low-cost, personalized medical services. So far, the health care sector has remained aloof from current market changes. But this is the same service sector, and patients, customers of these services, want convenience, proactivity, electronic services, mobility [11, 12].

E-medicine will be a huge breakthrough for the industry. All the data on patients that will be collected and systematized will help in the future to receive better service, correct diagnosis and the appointment of effective treatment not only in Kazakhstan, but also in any location in the world.

The government provides proper financial support and a certain «moral pressure», which is necessary now. All people will come to digitalization. And the presence of good competition in terms of medical systems will be only an advantage. Every Head of a medical organization will be able to choose a system that suits him. The issue of improving the quality of medical care is relevant and most common among the problems of the organization and healthcare institutions management. In large multidisciplinary hospitals, the need to create a system of rapid and effective interaction between the services and departments involved in the examination and treatment of patients is particularly vivid. So far, in the field of medical care, information systems are designed as unique for each organization and, as a rule, are focused on statistical data processing and partial automation of administrative and business activities [13–18].

The process of automating a hospital of a medical institution was reviewed in this article. The object of automation is the Karaganda Regional Clinical Hospital. Analysis of the current day hospital data processing system of this hospital showed that data processing is mainly done manually. Installed computers are mainly used as typewriters for printing statements. Documents are stored in paper form, that complicates their processing and storage [19].

Having analyzed the existing hospital system, began to develop own information system «Hospital» using the high-level programming language Delphi 7.0. The developed software product is designed to automate the activities of day hospital, allows accumulating the medical and statistical information in the databases of the hospital and using in practice, automates the management of documents. The «Stationary» automated information system has an easy-to-use interface, i.e. this system can be used not only by specially trained users, but also by medical personnel [20, 21].

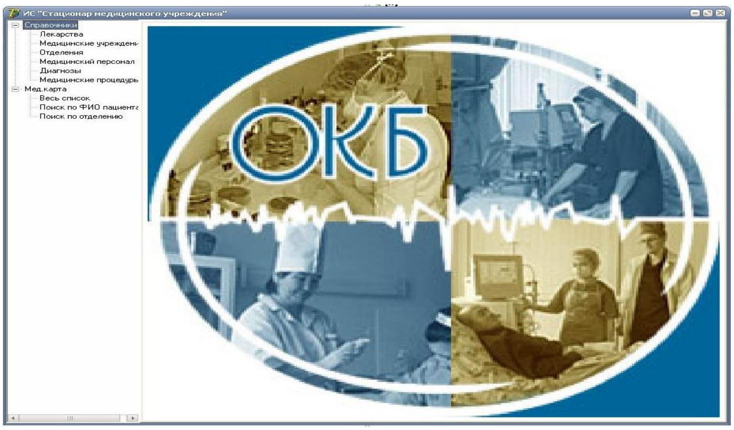

Figure 1. The main boot form

The application launch is carried out by double clicking the left mouse button on the MedStacionar.exe file shortcut. After that, the main boot form will appear on the screen (Fig. 1).

The program menu is presented in the form of a hierarchical tree, where all commands are divided into two groups:

– Reference books;
– Medical card.

All menu commands are selected by double-clicking.

The «Directory» menu provides work with the main system directories:

– Medicines;
– Medical institutions;
– Departments;
– Medical staff;
– Diagnoses;
– Medical procedures.

The «Medication» directory contains a list of medications (Fig. 2). The window has a navigator for working with records: navigating through the records, adding, editing and deleting records.

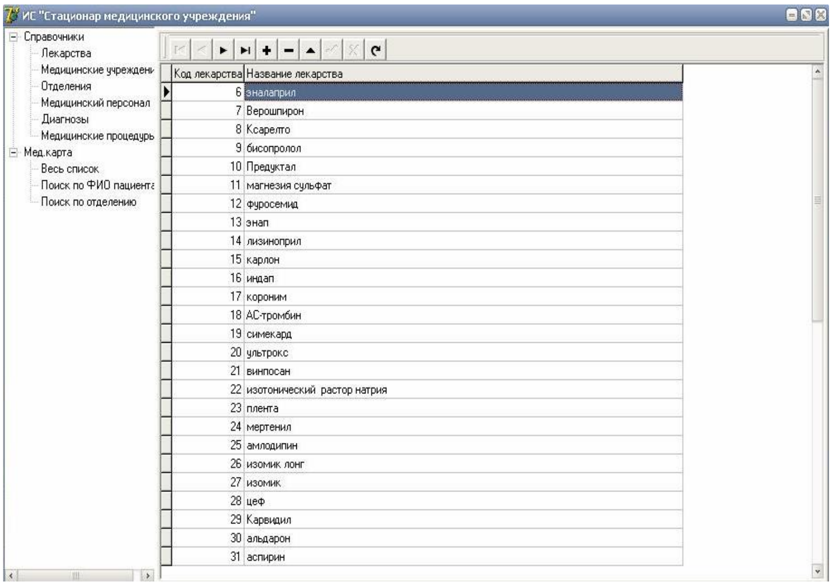

Figure 2. The «Medication» directory window

The directory «Medical institutions» contains a list of medical institutions with which the hospital works. Directory «Branch» contains a list of departments of the hospital. The reference book «Medical personnel» contains a list of medical personnel of the hospital with indication of the position. The «Diagnoses reference book» contains a list of diagnoses and their symptoms. The reference book «Medical procedures» contains a list of medical procedures that are conducted by the hospital.

The menu command «Medical card» contains three commands for working with medical cards:

– The whole list;
– Search by name of the patient;
– Search by branch.

When you select the entire list command, the patient's medical record window will be opened (Fig. 3).

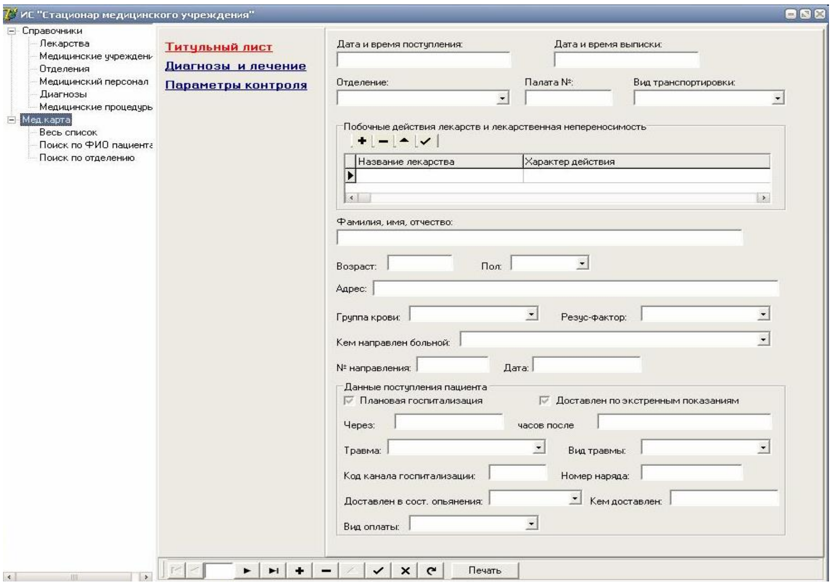

Figure 3. The window of the medical record of the patient

The patient's medical record consists of three sections:

– Title page;

– Diagnoses and treatment;

– Control parameters.

The «Title page» contains general information about the patient, the order of his arrival, the presence of allergic reactions to drugs. The «Diagnoses and Treatment» section contains a list of the diagnoses that have been made, the procedures prescribed and the medications for treating the established diagnosis assigned to the patient. Section «Control parameters» contains daily information on the patient's condition.

At the bottom of the «Medical Record» window there is a navigator for working with records, a card number and a button for printing a card. When you select the print command, medical record data will be imported into Word (Fig. 4).

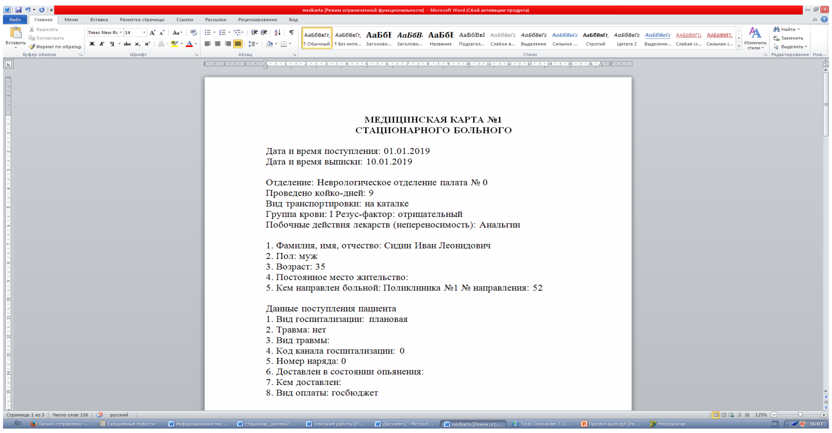

Figure 4. Medical record extract

When selecting the «Search by name of the patient» command, a window opens that asks to enter the name of the desired patient. Then the found medical record is opened. The search by department is the same.

The developed software product is designed to automate the activities of a day hospital at the Regional Clinical Hospital. In this institution, the information system «Hospital» was tested and is at the implementation stage.

The developed software product is designed to automate the activities of a hospital medical institution, allows it to accumulate in the hospital databases and use in practical work medical and statistical information, automates the management of hospital records.

System functionality:

1. Accounting of hospital patients.

2. Accounting of hospital patients receiving medical care in accordance with various types of payment: for compulsory health insurance, with their own expense, on voluntary medical insurance, under contracts with organizations, with the expense of budget funds.

3. Keeping an electronic medical record; print medical history in the required format.

4. The accumulation in the database of full information about the case of hospitalization, including:

- information about diagnoses;
- information on the assigned and executed medical services and surgical interventions;
- results of tests and diagnostic examinations of patients;
- information about the patient's daily condition;
- information about patient transfers within the hospital;
- information on the appointment and issuance of medicines;
- data on sick leave.

5. The accumulation of information about outpatient services provided to hospital patients.

6. Obtaining complete and consistent statistical reporting on approved forms based on common information resources.

The program does not require any special equipment in addition to a computer and printer, which will simplify the implementation process.

All the necessary work on the implementation of methods of access to information stored in the database, its modification, maintaining the database in a coherent form is hidden inside and the user does not need to know about it in order to successfully solve the whole range of emerging tasks related to the use of information stored in the database. Moreover, the program interface simplifies the work with the database as much as possible (up to a choice from the proposed number of options). Even accessing the database with complex queries is carried out in such a way that the structure of the returned data is visible even before its execution. The system independently tests the records in the database and brings the database to a complete state, eliminating possible errors. All routine operations of this kind are taken by the machine, which no doubt saves the efforts and time of the end user [22–26].

The development of this information system is relevant:

- first, in the age of information breakthrough, it is impossible to imagine any serious organization without a computer and data processing programs;
- second, it is minimization of manual labor and information processing on paper;
- third, reduction of time for searching and processing the necessary information;
- fourth, saving money.

References

1 Sackett D.L. Evidence-based medicine: what it is and what it isn't / D.L. Sackett, W.M. Rosenberg, W.S. Richardson, R.B. Hynes, J.A. Muirgray // BMJ. – 1996. – 312. – P. 71–72.

2 Государственная программа развития здравоохранения Республики Казахстан «Денсаулық» на 2016–2019 годы [Электронный ресурс]. — Режим доступа: http://adilet.zan.kz/rus/docs/U1600000176.

3 Государственная программа «Цифровой Казахстан» [Электронный ресурс]. — Режим доступа: http://adilet.zan.kz/rus/docs/P1700000827.

4 Rodrigues, J. Information systems: the key to evidence-based health practice / J. Rodrigues // Bull. World Health Organ. — 2000. – 78. – No. 11. — P. 1344–1351.

5 Sackett D.L. Evidence-based medicine: how to practice and teach EBM / D.L. Sackett, S.E. Straus, W.S. Richardson, P. Glasziou, R.B. Haynes // BMJ. — 2000.

6 Jeffrey A. History and development of evidence-based medicine / A. Jeffrey, C. Timothy // World J. Surg. – 2005. – 29. – P. 547–553.

7 Rube T. Electronic health information: The key to evidence-based medicine and improved patient care / T. Rube // Gov. Insights. — 2008.

8 Hunt D.L. Effects of computer-based clinical decision support systems on physician performance and patient outcomes: a systematic review / D.L. Hunt, R.B. Haynes, S.E. Hanna // JAMA. – 1998. – 280. – P. 1339–1346.

9 Bukken S. An informatics infrastructure is essential for evidence-based practice / S. Bukken // J. Am. Med. Inform. Assoc. — 2001. — 8. — P. 199–201.

10 Wells L. Role of information technology in evidence-based medicine: advantages and limitations / L. Wells // Internet J. Healthc. Admin. — 2007. — 4. — No. 2.

11 Сайт о казахстанском рынке информационных технологий [Электронный ресурс]. — Режим доступа: https://profit.kz.

12 Бабич Т.Н. Прогнозирование и планирование в условиях рынка: учеб. пос. / Т.Н. Бабич, И.А. Козьева, Ю.В. Вертакова, Э.Н. Кузьбожев. — М.: НИЦ ИНФРА-М, 2013. — 336 с.

13 Aminpour F. Medical informatics: concepts and applications / F. Aminpour // Iran. J. Med. Educ. – 2004. – 4, No. 2. – P. 95–103.

14 Mueller M.L. Workflow analysis and evidence-based medicine: Towards integration of knowledge-based functions in hospital information systems / M.L. Mueller, T. Ganslandt, T. Frankewitsch, C.F. Krieglstein, N. Senninger, H.U. Prokosch // Proc AMIA Symp. – 1999. – P. 330–334.

15 Hemiear M. Improving patient safety with technology / M. Hemiear // Int. J. Med. Inform. — 2004. – 73, No. 7. – P. 543–546.

16 Humphreys L. Electronic health record meets digital library: a new environment for achieving an old goal / L. Humphreys // J. Am. Med. Inform. Assoc. – 2000. – 7, No. 5. – P. 444–452.

17 Humphreys B.L. The UMLS project: making the conceptual connection between users and the information they need / B.L. Humphreys, D. Lindberg // Bull. Med. Libr. Assoc. – 1993. – 81, No. 2. – P. 170–177.

18 Ahmadi M. Evaluation of hospital information system in the medical records department / M. Ahmadi, M. Barabadi, M. Kamkar Haghighy // Health Inf. Manag. – 2010. – 7, No. 1. – P. 16–23.

19 Официальный сайт областной клинической больницы [Электронный ресурс]. — Режим доступа: https://okbkar.kz.

20 Шаяхметова Б.К. Технология создания программ для сложных систем / Б.К.Шаяхметова. — Астана: МПА Туран-Профи, 2010. — 170 с.

21 Вирт Н. Алгоритмы + структуры программ = программы / Н.Вирт. — М.: Мир, 1985. — 406 с.

22 Clamp S. Electronic health record: is the evidence base any use? / S. Clamp, J. Keen // Med. Inform. Internet Med. – 2007. – 32, No. 1. – P. 5–10.

23 Wright A. Clinical decision support capabilities of commercially-available clinical information systems / A. Wright, D.F. Sittig, J.S. Ash, S. Sharma, J.E. Pang, B. Middleton // J. Am. Med. Inform. Assoc. – 2009. – 16, No. 5. – P. 637–644.

24 Young J.M. Evidence-based medicine in general practice: beliefs and barriers among Australian GPs / J.M. Young, J.E. Ward // J. Eval. Clin. Pract. – 2001. – 7, No. 2. – P. 201–210.

25 Hersh R. How well do physicians use electronic information retrieval systems? A framework for investigation and systematic review / R. Hersh, H. Hickam // JAMA . – 1998. – 280, No. 15. P. 1347–1352.

26 Mansson J. Collection and retrieval of structured clinical data from electronic patient records in general practice / J. Mansson, G. Nilsson, C. Bjorkelund, L.E. Strender // Scand. J. Prim. Care. – 2004. – 22. – P. 6–10.

М.У. Баешова, А.М. Омаров

Қазақстанның денсаулық сақтау жүйесін цифрландыру

Мақалада медициналық мекеме стационарының процесін автоматтандыру үрдісі қарастырылған. Стационар қызметін автоматтандыратын ақпараттық жүйе әзірленді. Құрастырылған программалық өнімнің функционалдық мүмкіндіктері мен аталған өнімді медициналық қызмет көрсету саласына ендіру болашағы сипатталған. Құрастырылған ақпараттық жүйе медициналық және статистикалық ақпаратты стационар базасында жинақтауға және практикалық жұмысында пайдалануға мүмкіндік береді, стационардың құжат алмасуын басқаруды автоматтандырады. Жүйенің мүмкіндіктері стационарда науқастарды тіркеуден, электрондық аурулар тарихын жүргізуден және толық статистикалық есептерді алудан тұрады. Науқастың электрондық аурулар тарихында оның жеке деректері, қойылған диагноздары, ұсынылған процедуралары мен дәрі-дәрмектер тізімі бар. Денсаулық сақтау саласына ақпараттық-коммуникациялық технологияларды енгізу медициналық көмектің сапасын арттыруға және қызметкерлердің жұмысын едәуір жеделдетуге мүмкіндік берді.

Кілт сөздер: денсаулық сақтауды цифрландыру, стационарды автоматтандыру, ақпараттық жүйенің функционалды мүмкіндіктері.

М.У. Баешова, А.М. Омаров

Цифровизация системы здравоохранения Казахстана

В статье рассмотрен процесс автоматизации стационара медицинского учреждения. Описаны функциональные возможности разработанного программного продукта и перспективы внедрения данного продукта в область медицинского обслуживания. Разработанная информационная система позволяет накапливать в базах данных стационара и использовать в практической работе медицинскую и статистическую информацию, автоматизирует ведение документооборота стационара. Возможности системы заключаются в учете пациентов стационара, ведении электронной истории болезни и получении полной статистической отчетности. Электронная история болезни пациента содержит его личные данные, поставленные диагнозы, рекомендованные процедуры и лекарства. Внедрение информационно-коммуникационных технологий в сферу здравоохранения позволит улучшить качество обслуживания и заметно ускорить работу персонала.

Ключевые слова: цифровизация здравоохранения, автоматизация стационара, функциональные возможности информационной системы.

References

1 Sackett, D.L., Rosenberg, W.M., Richardson, W.S., Hynes, R.B., & Muirgray, J.A. (1996). Evidence-based medicine: what it is and what it isn't. *BMJ, 312,* 71-72.

2 Hosudarstvennaia prohramma razvitiia zdravookhraneniia Respubliki Kazakhstan «Densaulyk» na 2016–2019 hody [The state program for the development of health care in the Republic of Kazakhstan «Densaulyk» for 2016–2019]. (n.d.). *adilet.zan.kz.* Retrieved from http://adilet.zan.kz/rus/docs/V1600000176 [in Russian].

3 Hosudarstvennaia prohramma «Tsifrovoi Kazakhstan» [The state program «Digital Kazakhstan»]. (n.d.). *adilet.zan.kz.* Retrieved from http://adilet.zan.kz/rus/docs/P1700000827 [in Russian].

4 Rodrigues, J. (2000). Information systems: the key to evidence-based health practice. *Bull. World Health Organ, 78, 11,* 1344-1351.

5 Sackett, D.L., Straus, S.E., Richardson, W.S., Glasziou, P., & Haynes, R.B. (2000). Evidence-Based Medicine: How to Practice and Teach EBM. *BMJ, 452,* 152-154.

6 Jeffrey, A., & Timothy, C. (2005). History and development of evidence-based medicine. *World J. Surg, 29,* 547-553.

7 Rube, T. (2008). Electronic health information: The key to evidence-based medicine and improved patient care. *Gov. Insights.*

8 Hunt, D.L., Haynes, R.B., & Hanna, S.E. (1998). Effects of computer-based clinical decision support systems on physician performance and patient outcomes: a systematic review. *JAMA, 280,* 1339-1346.

9 Bukken, S. (2001). An informatics infrastructure is essential for evidence-based practice. *J. Am. Med. Inform. Assoc, 8,* 199-201.

10 Wells, L. (2007). Role of information technology in evidence based medicine: advantages and limitations. *Internet J. Healthc. Admin, 4,* 2.

11 Sait o kazakhstanskom rynke informatsionnykh tekhnolohii [Site about the Kazakhstan market of information technologies]. *profit.kz.* Retrieved from http://profit.kz [in Russian].

12 Babich, T.N., Kozyeva, I.A., Vertakova, Yu.V. & Kuzbozhev, E.N. (2013). *Prohnozirovanie i planirovanie v usloviiakh rynka [Forecasting and planning in market conditions].* Moscow: NITs INFRA-M [in Russian].

13 Aminpour, F. (2004). Medical informatics: concepts and applications. *Iran. J. Med. Educ., 4, 2,* 95-103.

14 Mueller, M.L., Ganslandt, T., Frankewitsch, T., Krieglstein, C.F., Senninger, N., & Prokosch, H.U. (1999). Workflow analysis and evidence-based medicine: Towards integration of knowledge-based functions in hospital information systems. *Proc AMIA Symp,* 330-334.

15 Hemiear, M. (2004). Improving patient safety with technology. *Int. J. Med. Inform, 73, 7,* 543-546.

16 Humphreys, L. (2000). Electronic health record meets digital library: a new environment for achieving an old goal. *J. Am. Med. Inform. Asso, 7, 5,* 444-452.

17 Humphreys, B.L., & Lindberg, D. (1993). The UMLS project: making the conceptual connection between users and the information they need. *Bull. Med. Libr. Assoc, 81, 2,* 170-177.

18 Ahmadi, M., Barabadi, M., & Kamkar Haghighy, M. (2010). Evaluation of hospital information system in the medical records department. *Health Inf. Manag, 7,* 16-23.

19 Ofitsialnyi sait oblastnoi klinicheskoi bolnitsy [Official site of the regional clinical hospital]. *okbkar.kz.* Retrieved from https://okbkar.kz [in Russian].

20 Shayakhmetova, B.K. (2010). *Tekhnolohiia sozdaniia prohramm dlia slozhnykh sistem [Technology of creating programs for complex systems].* Astana: Turan-Pro [in Russian].

21 Virt, N. (1985). *Alhoritmy + struktury prohramm = prohrammy [Algorithms + program structure = program].* Moscow: Mir [in Russian].

22 Clamp, S., & Keen, J. (2007). Electronic health record: is the evidence base any use? *Med. Inform. Internet Med, 32, 1,* 5-10.

23 Wright, A., Sittig, D.F., Ash, J.S., Sharma, S., Pang, J.E., & Middleton, B. (2009). Clinical decision support capabilities of commercially-available clinical information systems. *J. Am. Med. Inform. Assoc. 16, 5,* 637-644.

24 Young, J.M., & Ward, J.E. (2001). Evidence-based medicine in general practice: beliefs and barriers among Australian GPs. *J. Eval. Clin. Pract, 7, 2,* 201-210.

25 Hersh, R., & Hickam, H. (1998). How well do physicians use electronic information retrieval systems? A framework for investigation and systematic review. *JAMA, 280, 15,* 1347-1352.

26 Mansson, J., Nilsson, G., Bjorkelund, C., & Strender, L.E. (2004). Collection and retrieval of structured clinical data from electronic patient records in general practice. *Scand. J. Prim. Care, 22,* 6-10.

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Mathematical modeling of the source and environment response for the equation of geoelectrics

In this paper an algorithm is proposed for determining the source of excitation of electromagnetic waves emitted by the Ground-penetrating radar (GPR) device as a function of time. A mathematical model for solving this problem was constructed and tested on model data. We have built an algorithm for constructing a source function based on real georadar data. For this purpose, the results of experimental studies conducted in field conditions using the Loza-V GPR. Experiments were carried out in the medium: air-sand. The received signal of the response of the medium was processed from interference and noise. For this purpose, we use frequency filtering, signal averaging, amplitude correction for processing radarograms. In the future, the obtained table form of the disturbance signal will be used by us to study inhomogeneous media, including the study of localized objects. The series of calculations for the considered problems are given.

Keywords: inverse problems, source modeling, Maxwell equation, frequency filtering, radarogram processing, numerical results.

1 Method for solving inverse coefficient problems

The result of the GPR survey is a set of single traces (signals) recorded by the receiving antenna at each position of the GPR. To solve engineering problems, it is necessary to have the amplitude of the signal depending on the depth of its reflection, while the original radarogram is the dependence of the signal amplitude on the reflection time.

As you know, all radars can only record the time of reflection from the boundaries of objects or inhomogeneous media. On the other hand, it is a function of time or otherwise referred to as additional information (the response of the environment). This function is additional information for solving inverse problems, in our case determining the geological section. We present below the research methodology, which is one of the well-known methods for solving such problems.

The optimization method for solving inverse problems of electrical exploration is one of the most effective and widely used methods in practice.

The main ideas of the method were proposed by A.N. Tikhonov, M.M. Lavrentiev, V.K. Ivanov, G.I. Marchuk, A.S. Alekseev and many of their students and followers.

A. Bamberger, G. Ghavent, P. Lailly [1] used and investigated the conjugate gradient method to solve the one-dimensional inverse seismic problem. The essence of the method is to minimize the quadratic function of the discrepancy of observed and calculated fields. The application of the method to field data is given in the articles of A. Bamberger, G. Chavent, Ch. Hemon, and P. Lailly [2]. The method of least squares was also used to reverse the data obtained during the registration of seismic fields, see D.A. Cook, W.A. Schneider [3], and W.W. Johonson, H.H. Nogami [4].

In the article of G. Chavent, M. Dupuy, P. Lemonnier [5] the optimization method was applied to the problem of determining the distribution of magnetic permeability.

In the article by F. Santosa, W. Symes, G. Raggio [6] considered the problem of determining the acoustic impedance of a layered medium from reflected seismograms in which low-frequency components are small or are absent.

A wide range of inverse problems of geoelectric and numerical methods for solving these problems (including the optimization method) are presented in the monograph of V.G. Romanov, S.I. Kabanikhin [7].

For the numerical solution of inverse problems of geoelectric in layered and vertically inhomogeneous media, the method of conjugate gradients is applied in the article of K.T. Iskakov, S.I. Kabanikhin [8].

The series of articles [9–12] is devoted to experimental and numerical studies of solutions of the direct and inverse problem of non-linear diffusion associated with the process of drying of porous materials. The experiments were conducted by a team of authors in the laboratory of magnetic and spin phenomena of the International tomographic center SB RAS and the Institute of Catalysis SB RAS. The results of experimental and numerical studies are published in [9, 10] and problem statements and the development of algorithms in [11, 12].

The complex includes an algorithm for calculating the concentration (direct problem) in the processes of drying (adsorption) of various porous materials (aluminium oxide, silica gel) containing liquid (water, acetone, etc.), adsorption of water vapor by selective water sorbents (silica gel or aluminium oxide, containing calcium chloride).

The results of numerical calculations of the solutions of the direct and inverse problems are in satisfactory agreement with the measurement data obtained experimentally.

The questions of the conditional stability of the inverse problem for the hyperbolic equation with Lipschitz constant depending on the depth, on the priori given norm of the unknown coefficient and the norm of the data of the inverse problem in space H_1 are reflected in [13].

The Fourier collocation method for reconstruction of the source function F(x) depending on the spatial variable is developed and tested on model examples. The inverse problem of source identification depending on the spatial variable F(x) in the one-dimensional wave equation is considered. The connection between this problem and the problem of georadar data interpretation (GPR) is shown [14, 15]. Articles [16, 17] are devoted to visualization of the obtained data for subsequent interpretation and obtaining reliable information about underground geological environments, natural and artificial inhomo-geneities, anomalies, objects. Visualization of the processed data of GPR and the description of the program of processing of signals of GPR for interpretation of subsurface geological environments are given.

In [18, 19] an effective algorithm (layer-by-layer conversion method) for solving the inverse problem for the geoelectric equation in the frequency domain for the simultaneous determination of the dielectric permittivity and medium conductivity, as well as determining the boundaries of discontinuity is presented.

The differentiability of the discrepancy functional is shown with allowance for media interface boundaries. With this in mind, the well-known algorithm for layer-by-layer conversion is generalized. Based on the critical frequency, a range is set for which two functions can be defined simultaneously.

In [20] the questions of conditional stability, the solution of the inverse coefficient problem for the equation of geoelectrics are studied. To study stability, the initial inverse problem is reduced to a system of the second kind of Volterra integral equations. The class of correctness of solutions of the inverse problem and the class of input data is introduced. The estimation of the conditional stability of the solution of the inverse problem from the input data in the normal space H_1 is obtained.

In [21], the residual functional for the numerical solution of the inverse problem for the equations of the theory of elasticity was investigated. The medium model is horizontally layered. The differentiability of the function of the discrepancy by the coordinate of the point of discontinuity of the medium for the equations of the theory of elasticity is proved. An explicit analytical expression for this derivative is obtained. This allows the gradient method to determine the coordinates of the gaps and the thickness of the layers.

Methods of estimation of various parameters of geophysical models were considered by D.W. Marquardt [22], Y.M. Chen, J.H. Seinfeld [23], G. Stoyan [24] and many others. Therefore, we note the review article of B. Ursin, K.A. Berteussen [25] and the bibliography available in it.

2 Modeling the source of disturbances

At a georadar research the device registers the signals received by the receiving antenna as a set of single routes in the form of the image — radarogramma [26, 27]. To solve engineering problems, it is necessary to have the dependence of the signal amplitude on the depth of its reflection, and the original radarogram is the dependence of the signal amplitude on the travel time. On an other hand, it is necessary to clear the signal from various kinds of noise that hides the useful signal. For this purpose, we use frequency filtering, signal averaging, amplitude correction [17] to process radarograms.

To solve the inverse problem of GPR, it is necessary to know the time dependence of the signal entering the environment [15]. However, due to the complex interaction of the antennas with the medium, this function depends on the medium to which the signal goes. Therefore, existing georadars do not provide this kind of information before the soundings. In this paper it is proposed to determine the source function on the basis of test experiments conducted on a homogeneous medium.

We formulate a simplified mathematical model of GPR in the wave approximation [28, 29]. As it is common in geophysics, assume that the medium fills the half-space $z > 0$ and the half-space $z < 0$ corresponds to the air. Let the electrical permittivity ε of the medium depend on the coordinate z only, magnetic permittivity $\mu = \mu_0 = const > 0$ in the whole space and the conductivity is negligible. Let the current source with intensity

$$j^{ex}(t) = \Phi(t)\,\delta(z)\,, \Phi(t) = 0, \text{if} \quad t \leq 0,$$

$$\Phi(t) \in C^2\,[0.\infty]\,, \Phi''(+0) \neq 0$$

be placed at the boundary $z = 0$ and directed along the axis y. Then it follows from Maxwell's equations [6], that the electromagnetic field depends on (z, t) only. The field has an electric component $E_2(z, t)$ along the axis y, and a magnetic component $H_1(z, t)$ along the axis x that satisfy the Cauchy problem:

$$\frac{\partial H_1}{\partial z} = \varepsilon(z)\,\frac{\partial E_2}{\partial t} + \delta(z)\,\Phi(t)\,, \frac{\partial E_2}{\partial z} = \mu(z)\,\frac{\partial H_1}{\partial t}, (E_2, H_1)_{t<0} = 0. \tag{1}$$

We assume below $\mu(z) = \mu_0 > 0$. By taking first derivatives with respect to t from first equation and with respect to z from second one in (1) and eliminating $\partial^2 H_1/\partial t \partial z$ we get:

$$\partial E^2 \partial z^2 = \mu_0 \varepsilon(z)\,\tfrac{\partial E^2}{\partial t^2} + \mu_0 \delta(z)\;\Phi'(t)\,, E_{2|t<0} = 0.$$

Denote by

$$c(z) = 1/\sqrt{\mu_0 \varepsilon(z)}.$$

The purpose of GPR studies is to determine the electrical properties of the medium. Within the framework of our model, we need to recover the specific permittivity function $\varepsilon(z)$. As shown in [15], knowledge of the source function $\Phi(t)$ allows to approximate the function $c(z)$ based on measurements of the field strength on the surface of the medium. Further, according to the $c(z)$ distribution is easy to compute the function $\varepsilon(z)$.

Consider an environment in which the distribution of specific electrical conductivity is subject to the following representation

$$c^{-2}(z) = \begin{cases} c_0^{-2}, & if \quad z < 0 \\ c_1^{-2}, & if \quad z \geq 0 \end{cases}$$

$$c_0, c_1 = \text{const}.$$

Here c_0 is the speed of propagation of the signal in the air, and c_1- the speed of the radio signal in a homogeneous environment.

$$E_{zz} = \frac{1}{c^{-2}(z)} E_{zz} + \mu_0 \Phi(t) \delta(z), \quad (z \in \Re, t > -\infty), \tag{2}$$

$$E\,|_{t<0} \equiv 0. \tag{3}$$

In [15] it is shown that in this case the distribution of the electric field in the medium $E_2(z, t) = U(z, t)$ is a generalized solution of the Cauchy (2)–(3).
Then the solution of the problem is given by the formula:

$$E_{zz}(z, t) = -\frac{\mu_0 c_0 c_1}{c_0 + c_1} \begin{cases} \Phi\left(t + \frac{z}{c_0}\right), & z < 0 \\ \Phi\left(t - \frac{z}{c_0}\right), & z > 0 \end{cases}$$

It can be checked directly that the function $E(z, t)$ is continuous anywhere and twice continuously differentiable in the half spaces $\Re^2_- = \{(z, t) \mid z < 0, \ \ t \in \Re\}, \Re^2_+ = \{(z, t) \mid \ z > 0, \ \ t \in \Re\}$ and its first derivatives at $z = 0$ are expressed as

$$E_z(-0, t) = -\frac{\mu_0 c_1}{c_0 + c_1} \Phi'(t)\,, E_z(+0, t) = -\frac{\mu_0 c_1}{c_0 + c_1} \Phi'(t),$$

i.e.

$$E_z(+0, t) - E_z(-0, t) = \mu_0 \Phi'(t)\,. \tag{4}$$

The last formula confirms that the second derivative E_{zz} is represented as the singular function $\mu_0\Phi'(t)\delta(z)$ and a regular one.

From formula (4) it follows that the field strength in the GPR problem of a homogeneous half-space is determined by the formula

$$g(t) = E(0,t) = -\frac{\mu_0 c_0 c_1}{c_0 + c_1}\Phi(t), (t > 0). \qquad (5)$$

In GPR studies, readings are taken from the receiving antenna. From formula (5), it follows that the excitation signal generates a field with intensity (5) at the boundary of the medium. This field acts on the receiving antenna, causing alternating current in it. This means that the readings recorded by the device must be proportional to the function $\Phi(t)$.

Thus, before conducting a survey on the object under study, it is necessary to have the data obtained for a homogeneous half-space. This will determine the type of source function and use it for further interpretation. Next, the problem arises with approximation of a smooth function measured on a homogeneous half-space of data. As shown in [15], a second derivative of the function $\Phi(t)$ is needed for further interpretation. Therefore, it is desirable to approximate the table-set measured data with a smooth function of a simple form, for example, the following:

$$\Phi(t) = A\sin(\omega t + \beta)\exp(-\gamma t) - A\sin\beta. \qquad (6)$$

When data is approximated, it is necessary to select the parameters ω, β, γ in function (6).

3 The results of experimental studies

As part of the research project under contract No. 132 dated 12.03.2018, experimental studies were conducted in accordance with clause 8 of the calendar plan. To simulate a source of radiation emitted by Loza-V georadar, a site of a sand pit was selected, with a geoelectrical section was previously known. According to GPR data, namely additional information, the inverse problem of source modeling will be subsequently solved. For this purpose, experiments were conducted using different antennas. The expedition was done by Professor of ENU named after L.N. Gumilyov K.T. Iskakov and by senior teacher of KazNPU named after AbayB.B. Sholpanbayev on January 3, 2018. The experiments were conducted on the sandy quarry LLP «Bek», located 30 km away from the city of Almaty in the direction towards the town of Kapchagai. Radarograms were processed by the 2nd year doctoral candidate D.K. Tokseit The spectral analysis of the radarograms was performed by the senior teacher S.A. Boranbayev Objectives of the study: geophysical examination of the structure of the underlying layers of a homogeneous medium-river sand, modelling of the impulse source from the Loza-V device; determination of the spectral characteristics of signals emitted by antennas: 0.5 m, 1 m, 1.5 m, 3 m; interpretation of a series of radarograms obtained in result of sounding required to solve the inverse problem of source recovery. We present the experimental data: 1.5 m antenna, 7 tracks, 49 measurements, a step of 0.20-0.2 m are used. In Figure 1, 7 tracks are marked and are denoted as: A0-A6.

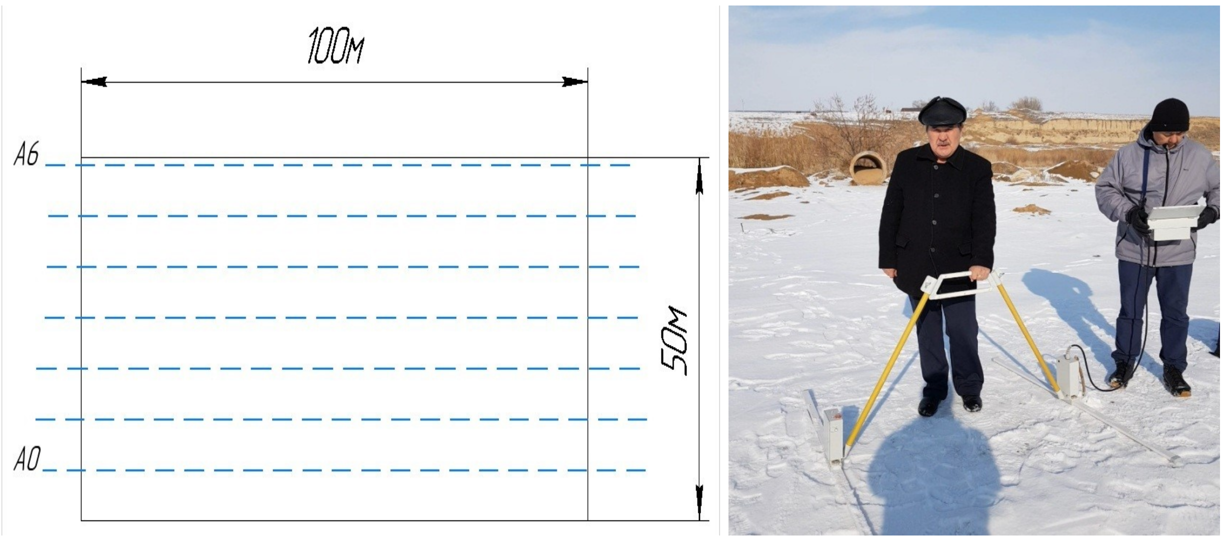

Figure 1. Measurement scheme for experiment № 1 (antenna 1.5 m)

Figure 2 shows the radarogram performed by the program «Krot». The results of the studies with antenna sweep: 1.5 m antenna, 7 tracks, 49 measurements, step 0.20-0.25 m.

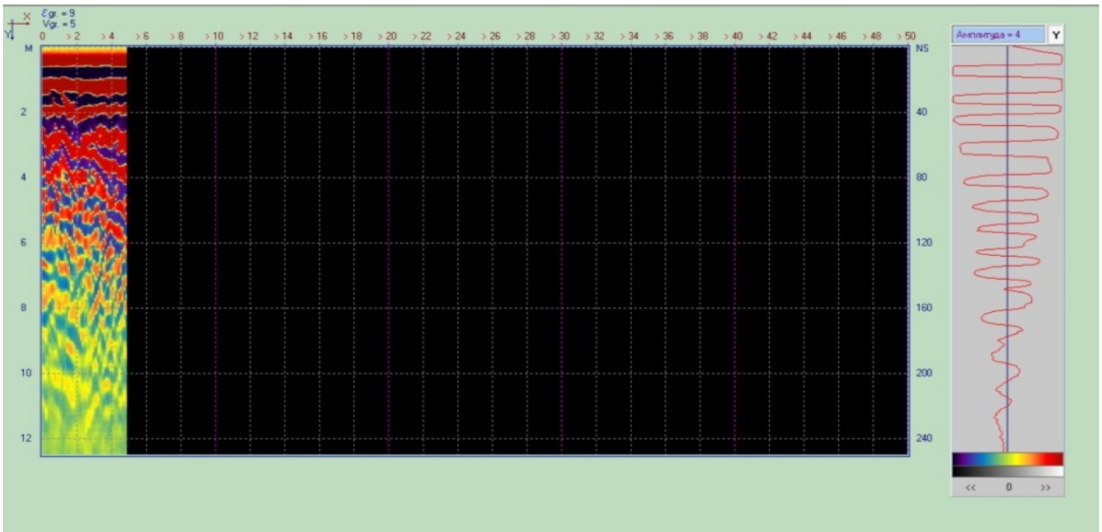

Figure 2. Radarogram of the experiment

4 Radarogram processing — medium response

The graphs of the traces and spectra of the radarograms will be carried out according to the algorithms given in [2]. Figure 3 shows the graphs of the paths and spectrum of radarograms obtained with a 1.5 m antenna at a frequency of 100 MHz.

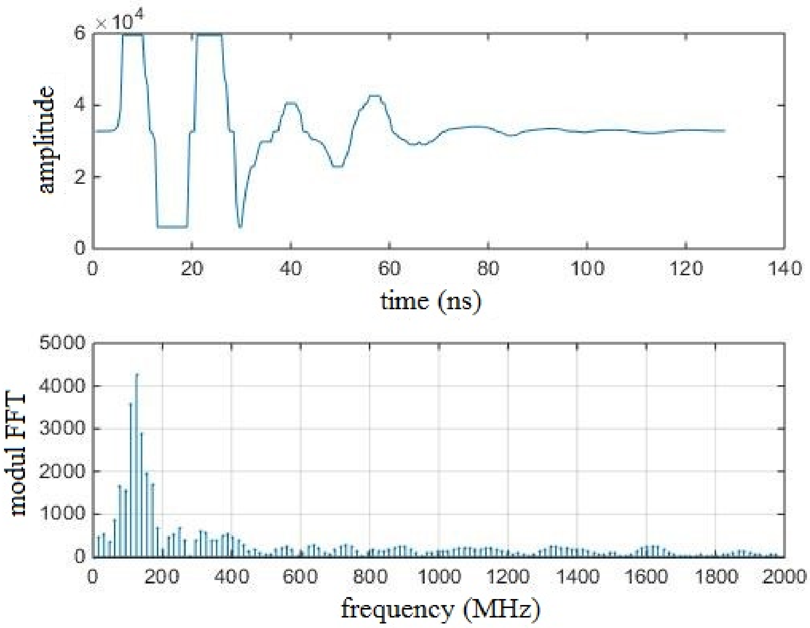

Figure 3. Graphs of the route and spectra of radarograms (1.5 m antenna, 100 MHz)

Radarogram is an ultra-wideband radio signal. As can be seen from the graphs, the main spectral components of the radarogram, which influence the amplitude of the signal, are located around the center frequency of the GPR antenna. The basic information about the subsurface environment lies in the amplitude of the signal corresponding to the time of receiving the signal. Experimental studies were conducted on the sandy LLP «Bek», located 30 km from Almaty city in the direction towards the city of Kapchagai. Experimental studies were conducted in a sand homogeneous medium at a site of 100 m — 50 m, using the Loza-V georadar. 147 radarograms on 7 tracks were filmed, in increments of 0.2-0.25 meters using antennas with a sweep of 0.5 m; 1 m; 1.5 m. The length of the profiles in each case is 100. With antenna — 3 m received three profiles. Finally, as a confirmation of the calculations, measurements were taken on a hill in which a cut of the medium is visible. Three profiles were received. As a result of the interpretation of the radarograms, a series of responses of the media were obtained, with the use of various antennas, which will be used to solve the inverse problem of source recovery (Table value). A spectral analysis of the received radarograms using the software package was carried out.

5 Numerical results

By the formula (5), we determine

$$\Phi(t) = g(t)/\kappa, \quad \text{where} \quad \kappa = -\frac{\mu_0 c_0 c_1}{c_0 + c_1}.$$

Here c_0 is the speed of propagation of the signal in the air $c_0 = 0,3\frac{m}{ns}$, and c_1 — the speed of the radio signal $c_1 = 0,122\frac{m}{ns} - 0,15\frac{m}{ns}$ in a homogeneous environment in the sand m/ns, *magnetic permittivity in the whole sand* $\mu_0 = 1$. Then,

$$\kappa = -\frac{0,3 \cdot 0,15}{0,45} = -0,1\frac{m}{ns}.$$

We present the data of the Loza-V GPR, in the form of Table 1. Measurements were carried out with a 1.5 m antenna, frequency 100 MHz. The amplitude values of the signal are ALn1 = 0 ... 32768 ... 65535, the amplitudes of the time step are 0.5 ns.).

Table 1

Signal amplitude values

32772	32772	32772	32772	32772	32772	32772	32772	32772	32772	32772	32772	32772	32772
32772	32772	32777	32777	32777	32777	32777	32777	32777	32777	32777	32777	32777	32777
32777	32777	32777	32777	32783	32783	32783	32783	32783	32783	32783	32783	32783	32783
32783	32783	32783	32783	32783	32783	32795	32795	32795	32795	32795	32795	32795	32795
32795	32795	32795	32795	32795	32795	32795	32795	32806	32806	32806	32806	32806	32806
32806	32806	32806	32806	32806	32806	32806	32806	32806	32806	32829	32829	32829	32829
32829	32829	32829	32829	32829	32829	32829	32829	32829	32829	32829	32829	32856	32856
32856	32856	32856	32856	32856	32856	32856	32856	32856	32856	32856	32856	32856	32856
32892	32892	32892	32892	32892	32892	32892	32892	32892	32892	32892	32892	32892	32892
32892	32892	33100	33100	33100	33100	33100	33100	33100	33100	33100	33100	33100	33100
33100	33100	33100	33100	34248	34248	34248	34248	34248	34248	34248	34248	34248	34248
34248	34248	34248	34248	34248	34248	38848	38848	38848	38848	38848	38848	38848	38848
38848	38848	38848	38848	38848	38848	38848	38848	59504	59504	59504	59504	59504	59504
59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504
59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504
59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504
59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504
59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504	59504
59504	59504	59504	59504										

We carry out data normalization in the range from 0 to 1, with this purpose we will divide all the data in Table 1 by the maximum amplitude value. The data obtained are shown in Table 2.

Table 2

Normalized amplitude values of signals

0.50 4185	0.50 4185	0.50 4185	0.50 4185	0.50 4185	0.50 4185	0.50 4185	0.50 4185	0.50 4185	0.50 4185	0.50 4185	0.50 4185	0.50 4185	0.50 4185	0.50 4185
0.50 4185	0.50 4262	0.50 4262	0.50 4262	0.50 4262	0.50 4262	0.50 4262	0.50 4262	0.50 4262	0.50 4262	0.50 4262	0.50 4262	0.50 4262	0.50 4262	0.50 4262
0.50 4262	0.50 4262	0.50 4354	0.50 4354	0.50 4354	0.50 4354	0.50 4354	0.50 4354	0.50 4354	0.50 4354	0.50 4354	0.50 4354	0.50 4354	0.50 4354	0.50 4354
0.50 4354	0.50 4354	0.50 4354	0.50 4538	0.50 4538	0.50 4538	0.50 4538	0.50 4538	0.50 4538	0.50 4538	0.50 4538	0.50 4538	0.50 4538	0.50 4538	0.50 4538
0.50 4538	0.50 4538	0.50 4538	0.50 4538	0.50 4708	0.50 4708	0.50 4708	0.50 4708	0.50 4708	0.50 4708	0.50 4708	0.50 4708	0.50 4708	0.50 4708	0.50 4708
0.50 4708	0.50 4708	0.50 4708	0.50 4708	0.50 4708	0.50 5062	0.50 5062	0.50 5062	0.50 5062	0.50 5062	0.50 5062	0.50 5062	0.50 5062	0.50 5062	0.50 5062
0.50 5062	0.50 5062	0.50 5062	0.50 5062	0.50 5062	0.50 5062	0.50 5477	0.50 5477	0.50 5477	0.50 5477	0.50 5477	0.50 5477	0.50 5477	0.50 5477	0.50 5477
0.50 5477	0.50 5477	0.50 5477	0.50 5477	0.50 5477	0.50 5477	0.50 5477	0.50 6031	0.50 6031	0.50 6031	0.50 6031	0.50 6031	0.50 6031	0.50 6031	0.50 6031
0.50 6031	0.50 6031	0.50 6031	0.50 6031	0.50 6031	0.50 6031	0.50 6031	0.50 6031	0.50 9231	0.50 9231	0.50 9231	0.50 9231	0.50 9231	0.50 9231	0.50 9231
0.50 9231	0.50 9231	0.50 9231	0.50 9231	0.50 9231	0.50 9231	0.50 9231	0.50 9231	0.50 9231	0.52 6892	0.52 6892	0.52 6892	0.52 6892	0.52 6892	0.52 6892
0.52 6892	0.52 6892	0.52 6892	0.52 6892	0.52 6892	0.52 6892	0.52 6892	0.52 6892	0.52 6892	0.52 6892	0.59 7662	0.59 7662	0.59 7662	0.59 7662	0.59 7662
0.59 7662	0.59 7662	0.59 7662	0.59 7662	0.59 7662	0.59 7662	0.59 7662	0.59 7662	0.59 7662	0.59 7662	0.59 7662	0.91 5446	0.91 5446	0.91 5446	0.91 5446
0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446
0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446
0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446
0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446
0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446	0.91 5446
0.91 5446														

Then the environment response schedule:

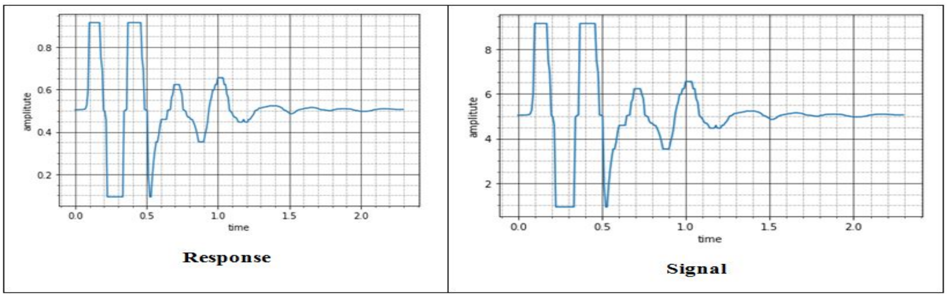

Figure 4. The response of the environment (according to Table 2.).
Disturbance source graph. (Calculated by the formula (5))

Below in Figure 5. the signal graph with coefficients minus k = -0.1 is shown.

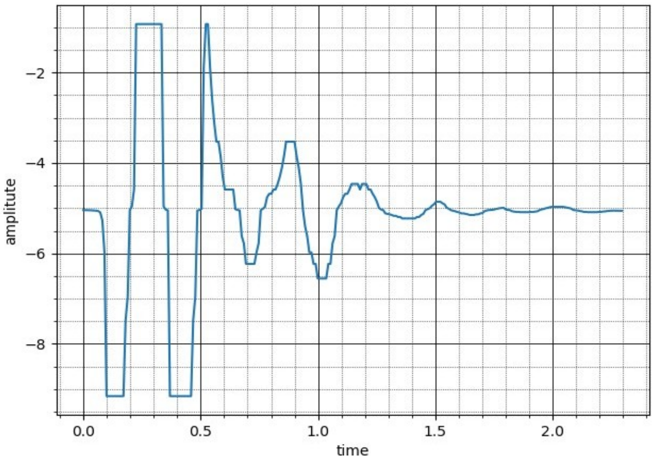

Figure 5. Signal graph with a minus sign

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References

1 Bamberger A. About the stability of the inverse problem in 1-D wave equations – applications to the interpretation of seismic profiles / A. Bamberger, G. Chavent, P. Lailly // Applied Mathematic Optimization. — 1979. — 5, 1. — P. 1–47.

2 Bamberger A. Inversion of normal incidence seismograms / A. Bamberger, G. Chavent, Ch. Hemon, P. Lailly // Geophysics. – 1982. – 47, 5 – P. 757–770.

3 Cook D.A. Generalized linear inversion of reflection seismic data / D.A. Cook, W.A. Schneider // Geophysics. — 1983. — 48. — P. 665–676.

4 Johonson W.W. One-dimensional inversion of seismic data / W.W. Johonson, H.H. Nogami // Abstracts of Papers Presented at the 52 Annual International SEG Meeting in Dallas. — Texas, 1982.

5 Chavent G. History matching by use of optimal control theory / G. Chavent, M. Dupuy, P. Lemonnier // Full Meeting of Society of Petroleum Engineers. — Las Vegas. — 1975. — 2. — P. 75–86.

6 Santosa F. Inversion of Band Limited Reflection Seismograms Using Stacking Velocities as Constraints. / F. Santosa, W.W. Symes, G. Raggio // Cornell ONR/SRO Report. — № 29. — Arlington, Virginia. — 1985.

7 Romanov V.G. Inverse Problems for Maxwell's Equations / V.G. Romanov, S.I. Kabanikhin. // Utrecht, The Netherlands: VSP. — 1994. — 249 p.

8 Iskakov K.T. The solution of one-dimensional inverse problem of geoelectrics by the method of conjugate gradients / K.T. Iskakov, S.I. Kabanikhin // Journal Theoretical and Applied Mechanics. – 1992. — 2, 3. — P. 197–221.

9 Koptyug I.V. A quantitative NMR imaging study of mass transport in porous solids during drying / I.V. Koptyug, S.I. Kabanikhin, K.T. Iskakov, V.B. Fenelonov, L.Yu. Khitrina, R.Z. Sagdeev, V.N. Parmon // Journal Chemical Engineering Science. — 2000. — 55 — P. 1559–1571.

10 Koptyug I.V. An H NMR microimaging study of water vapor sorption by individual porous pellets / I.V. Koptyug, L.Yu. Khitrina, Yu.I. Aristov, M.M. Tokarev, K.T. Iskakov, V.N. Parmon, R.Z. Sagdeev // Journal of Physical Chemistry. B. — 2000. — 104 — P. 1695–1700.

11 Kabanikhin S.I. Inverse problem for a quasi-linear equation of diffusion / S.I. Kabanikhin, I.V. Koptyug, K.T. Iskakov, R.Z. Sagdeev // Journal of Inverse Ill-Posed Problems. — 1998. — 6, 4 — P. 335–353.

12 Kabanikhin S.I. Inverse problem for diffusion transport of water upon single pellet moisture sorption / S.I. Kabanikhin, I.V. Koptyug, K.T. Iskakov, R.Z. Sagdeev // International Journal of Nonlinear Sciences and Numerical Simulation. – 2000. — 1, 1 — P. 31–42.

13 Kabanikhin S.I. H_1-conditional stability with explicit Lipschitz constant for a one-dimensional inverse acoustic problem / S.I. Kabanikhin, K.T. Iskakov, M. Yamamoto //Journal of Inverse and Ill-Posed Problems. – 2001. — 9, 3. — P. 249–267.

14 Mukanova B. Fourier Collocation Algorithm for identification of a spacewise dependent source in wawe equation from Neumann-type measured data [Electronic resouce] / B. Mukanova, A. Hasanov // Applied Numerical Mathematics. — 2017. — 111. — P. 49-63. — Access mode: http://www.sciencedirect.com/science/article/pii/S0168927416301738

15 Mukanova B.G. Inverse source problem for wave equation and GPR data interpretation problem / B.G. Mukanova, V.G. Romanov // Eurasian Journal of Mathematical and Computer Applications. — 2016. — 4, 3 — P. 15–28.

16 Iskakov K.T. Experimental data of research using ground-penetrating radar «Zond-12c» and interpretation Georadarograms [Electronic resouce]. / K.T. Iskakov, S. Boranbayev, Z. Alimbayeva, B. Isin // Acta physica polonica a. — 2016. — 130. — P. 322-324. — Access mode: http://przyrbwn. icm. edu. pl/APP/PDF /130/a130z1p087.pdf

17 Iskakov K.T. Wavelet processing and filtering of the radargram trace / K.T. Iskakov, S.A. Boranbaev, N. Uzakkyzy // Eurasian Journal of Mathematical and Computer Applications. — 2017. — 5, No. 4. — P. 43–54.

18 Oralbekova Zh.O. Existence of the residual functional derivative with respect to a coordinate of gap point of medium [Electronic resouce]. / Zh.O. Oralbekova, K.T. Iskakov, A.L. Karchevsky // Applied and Computational Mathematics. — 2013. — 12, 2. — P. 222-233. — Access mode: http://www.acmij.az/view.php? lang=az&menu= journal&id=323.

19 Karchevsky A.L. Solution of the inverse problem of subsurface electric exploration for horizontally stratified medium. / A.L. Karchevsky, Zh.O. Oralbekova, K.T. Iskakov // International Journal of Applied Mathematics. — 2013. — No. 432121. — P.1-9.

20 Iskakov K.T., Oralbekova Zh.O. Resolving Power of Algorithm for Solving the Coefficient Inverse Problem for the Geoelectric Equation Mathematical Problems in Engineering. — 2014. — No. 545689. — P. 1–9. DOI: 10.1155/2014/545689

21 Karchevsky A.L. Reconstruction of pressure velocities and boundaries of thin layers in thinly-stratified layers / A.L. Karchevsky // Journal of Inverse and Ill-Posed Problems. — 2010. — Vol. 18. — P. 371–388.

22 Marquardt D.W. An algorithm for least-squares estimation of nonlinear parameters / D.W. Marquardt // Journals SIAM. — 1963. — 11. — P. 431–441.

23 Chen Y.M., Seinfeld J.H. Estimation of spatially varying parameters in partial differential equations / Y.M. Chen., J. H. Seinfeld // International Journal of Control. — 1972. — 15, 3 — P. 487–495.

24 Stoyan G. Numerical experiments on the identification of heat conduction coefficients / G. Stoyan. Theory of Nonlinear Operators Proc.Fifth Intern. Summer School, Berlin, 1977). – Berlin: Akademie-Verlag, 1978. – P. 259–268.

25 Ursin B. Comparison of some inverse methods for wave propagation in layered media / B. Ursin, K.A. Berteussen // Proceedings of the IEEE. – 1986. – 74, 3 – P. 7–19.

26 Финкельштейн М.И. Основы радиолокации: учебник для вузов / М.И. Финкельштейн. — 2–е изд., перераб. и доп. — М.: Радио и связь, 1983.— 536 с.

27 Владов М.Л. Введение в георадиолокацию: учеб. пос. / М.Л. Владов, А.В. Старовойтов. — М.: Изд-во МГУ, 2004. — 153 с.

28 Александров П.Н. Теоретические основы георадарного метода / П.Н. Александров. — М.: Физматлит., 2017. — 112 с.

29 Светов Б.С. Основы геоэлектрики / Б.С. Светов. — М.: Наука, 2008. — 656 с.

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Геоэлектрика теңдеуі үшін негіздің және орта реакциясын математикалық модельдеу

Мақалада GPR құралы шығаратын электрмагниттік толқындардың қозғалыс негізі уақыт функциясы ретінде анықтау үшін алгоритм ұсынылған. Бұл мәселені шешуге арналған математикалық модель модельдік деректер бойынша құрастырылып, сыналды. Нақты георадар деректеріне негізделген бастапқы функция құру үшін алгоритм таңдалды. Осы мақсатта Лоза-В георадарын қолданып, бұл салада жүргізілген эксперименттік зерттеулердің нәтижелері пайдаланылды. «Ауа — құм» ортаға эксперимент жүргізілді. Ортадан алынған дабыл кедергі және шудан тазартылған. Бұл үшін біз радиолокацияларды өңдеу үшін жиілік сүзгісін, сигналдың орташалануын, амплитудалық түзетуді қолданылды. Болашақта біркелкі емес медианы зерттеуге, соның ішінде оқшауланған нысандарды анықтауға алынған бұзылулардың кестелік толқындық формасы пайдаланылды. Қарастырылған мәселелер бойынша есептер сериясы келтірілген.

Кілт сөздер: кері есептер, көздерді модельдеу, Максвелл теңдеуі, жиілікті сүзу, радиограмманы өңдеу, сандық нәтижелер.

К.Т. Искаков, Б.Г. Муканова, А.С. Бердышев, А.С. Кембай, Д.К. Токсеит

Математическое моделирование источника и отклика среды для уравнения геоэлектрики

В статье предложен алгоритм определения источника возбуждения электромагнитных волн, излучаемых георадиолокационным прибором, как функции времени. Математическая модель для решения этой задачи построена и апробирована на модельных данных. Авторами построен алгоритм по построению функции источника на основе реальных данных георадара. В этих целях использованы результаты экспериментальных исследований, проведенных в полевых условиях с применением георадара Лоза-В. Проведены эксперименты в среде «воздух — песок». Полученный сигнал отклика среды обработан от помех и шумов. Для обработки радарограмм использованы частотная фильтрация, усреднение сигналов, коррекция амплитуд. В дальнейшем полученная табличная форма сигнала возмущения будет использована для исследования неоднородных сред, в том числе и для изучения локализованных объектов. Приведены серии расчетов для рассматриваемых задач.

Ключевые слова: обратные задачи, моделирование источников, уравнение Максвелла, частотная фильтрация, обработка радарограммы, численные результаты.

References

1 Bamberger, A., Chavent, G. & Lailly, P. (1979). About the stability of the inverse problem in 1-D wave equations — applications to the interpretation of seismic profiles. *Applied Mathematic Optimization*, 5, 1, 1—47.

2 Bamberger, A., Chavent, G., Hemon, Ch. & Lailly, P. (1982). Inversion of normal incidence seismograms. *Geophysics, 47, 5,* 757–770.

3 Cook, D.A. & Schneider, W.A. (1983). Generalized linear inversion of reflection seismic data. *Geophysics, 48,* 665–676.

4 Johonson, W.W. & Nogami, H.H. (1982). One-dimensional inversion of seismic data. *Abstracts of Papers Presented at the 52 Annual International SEG Meeting in Dallas*, Texas.

5 Chavent, G., Dupuy, M. & Lemonnier, P.(1975). History matching by use of optimal control theory. *Full Meeting of Society of Petroleum Engineers.* Las-Vegas, 2, 75–86.

6 Santosa, F., Symes, W.W. & Raggio, G.(1985). Inversion of Band Limited Reflection Seismograms Using Stacking Velocities as Constraints. *Cornell ONR/SRO Report,* Arlington, Virginia.

7 Romanov,V.G. & Kabanikhin, S.I. (1994). Inverse Problems for Maxwell's Equations. Utrecht, The Netherlands: VSP, 249 p.

8 Iskakov, K.T. & Kabanikhin, S.I. (1992). The solution of one-dimensional inverse problem of geoelectrics by the method of conjugate gradients. *Journal Theoretical and Applied Mechanics, 2, 3,* 197–221.

9 Koptyug, I.V., Kabanikhin, S.I., Iskakov, K.T., Fenelonov, V.B., Khitrina, L.Yu., Sagdeev, R.Z. & Parmon, V.N. (2000). A quantitative NMR imaging study of mass transport in porous solids during drying. *Journal Chemical Engineering Science, 55,* 1559—1571.

10 Koptyug, I.V., Khitrina, L.Yu., Aristov, Yu.I., Tokarev, M.M., Iskakov, K.T., Parmon, V.N., & Sagdeev, R.Z. (2000). An H NMR microimaging study of water vapor sorption by individual porous pellets. *Journal of Physical Chemistry B., Vol 104,* 1695–1700.

11 Kabanikhin, S.I., Koptyug, I.V., Iskakov, K.T. & Sagdeev, R.Z. (1998). Inverse problem for a quasi-linear equation of diffusion. *Journal of Inverse Ill-Posed Problems, 6, 4,* 335—353.

12 Kabanikhin, S.I., Koptyug, I.V., Iskakov, K.T., & Sagdeev, R.Z. (2000). Inverse problem for diffusion transport of water upon single pellet moisture sorption. *International Journal of Nonlinear Sciences and Numerical Simulation, Vol 1, 1,* 31–42.

13 Kabanikhin, S.I., Iskakov, K.T. & Yamamoto, M. (2001). H_1-conditional stability with explicit Lipschitz constant for a one-dimensional inverse acoustic problem. *Journal of Inverse and Ill-Posed Problems, 9, №3,* 249–267.

14 Mukanova, B.G., & Hasanov, A. (2017). Fourier Collocation Algorithm for identification of a spacewise dependent source in wawe equation from Neumann-type measured data. *Applied Numerical Mathematics, 111,* 49–63. Retrieved from http://www.sciencedirect.com/science/article/ pii/S0168927416301738.

15 Mukanova, B.G., & Romanov, V.G. (2016). Inverse source problem for wave equation and GPR data interpretation problem. *Eurasian Journal of Mathematical and Computer Applications, 4, 3,* 15–28.

16 Iskakov, K.T., Boranbayev, S.A., Alimbayeva, Z., & Isin, B. (2016). Experimental data of research using ground-penetrating radar «Zond-12c» and interpretation Georadarograms. *Acta physica polonica a., 130,* 322–324. Retrieved from http://przyrbwn.icm.edu.pl/APP/PDF/130/a130z1 p087.pdf.

17 Iskakov, K.T., Boranbaev, S.A. & Uzakkyzy, N. (2017). Wavelet processing and filtering of the radargram trace. *Eurasian Journal of Mathematical and Computer Applications, 5, 4,* 43–54.

18 Oralbekova, Zh.O., Iskakov, K.T., & Karchevsky, A.L. (2013). Existence of the residual functional derivative with respect to a coordinate of gap point of medium. *Applied and Computational Mathematics, 12, 2,* 222–233. Retrieved from http://www.acmij.az/view.php? lang=az&menu= journal&id=323.

19 Karchevsky, A.L., Oralbekova, Zh.O., & Iskakov, K.T. (2013). Solution of the inverse problem of subsurface electric exploration for horizontally stratified medium. *International Journal of Applied Mathematics*, No. 432121, 1-9.

20 Iskakov, K.T., & Oralbekova, Zh.O. (2014). Resolving Power of Algorithm for Solving the Coefficient. *Inverse Problem for the Geoelectric Equation Mathematical Problems in Engineering*, No. 545689, 1-9. DOI: 10.1155/2014/545689.

21 Karchevsky, A.L. (2010). Reconstruction of pressure velocities and boundaries of thin layers in thinly-stratified layers. *Journal of Inverse and Ill-Posed Problems, 18,* 371–388.

22 Marquardt, D.W. (1963). An algorithm for least-squares estimation of nonlinear parameters. *Journals SIAM, 11,* 431–441.

23 Chen, Y.M., & Seinfeld J.H. (1972). Estimation of spatially varying parameters in partial differential equations. *International Journal of Control, 15, 3,* 487–495.

24 Stoyan, G. (1978). Numerical experiments on the identification of heat conduction coefficients. *Theory of Nonlinear Operators.* Proc. Fifth Intern. Summer School, Berlin: Akademie-Verlag, 1977, 259–268.

25 B. Ursin, & Berteussen, K.A. (1986). Comparison of some inverse methods for wave propagation in layered media. *Proceedings of the IEEE, Vol 74, 3,* 7–19.

26 Finkelshtein, M.I. (1983). *Osnovy radiolokatsii: uchebnik dlia vuzov [Fundamentals of radar: the textbook for high schools].* (2nd ed.). Moscow: Radio i sviaz [in Russian].

27 Vladov, M.L (2004). *Vvedenie v heoradiolokatsiiu [Introduction to ground penetrating radar].* Moscow: Izdatelstvo MHU [in Russian].

28 Aleksandrov, P.N. (2017). *Teoreticheskie osnovy heoradarnoho metoda [The theoretical basis of the GPR method]*. Moscow: Fizmatlit [in Russian].

29 Svetov, B.S. (2008). *Osnovy heoelektriki [Basics of geoelectrics]*. Moscow: Nauka [in Russian].

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Models of interaction of cryptography and chaotic dynamics

Cryptography deals with the problem of information protection by its transformation, providing the protection of information (by means of computational techniques), i.e. a set of agreed encryptian tools. Under the cryptosystem in the narrow sense we will understand dynamical systems with a nonlinear function and spatial states, it is represented by a differential equation. Some conditions of the dynamic system, the Lyapunov exponent, as a measure of sensitivity are considered. Identification of the interconnection between objects of study in the theory of chaos and cryptography is revealed; the conclusions about the possibility of using the trajectory of dynamical systems with the chaos for the representation and the transmission of information.

Keywords: cryptography, information protection, encryption, nonlinear function, dynamical system, cryptosystem, parameters, trajectory, transformation, chaotic system, measure of sensitivity.

Introduction

Cryptography deals with the protection of information data through its transformation. Cryptography solves the problems of confidentiality, integrity, authentication, and a number of others that are associated with them. Cryptography actually examines methods for encrypting information, generating digital signatures, and key management certificates.

A cryptographic system, in a broader sense, is an infrastructure that guarantees the protection of information data by means of computer technology, a set of coordinated methods of encryption, transfer and key management, authentication and other elements. A cryptosystem is a hardware-software complex that interacts with a person. It should be noted that scientists working on the protection of information in the conditions of deterministic chaos, the formation of models and descriptions of software applications: E.N. Lorenz [1], M.S. Baptista [2], A. Abel, W. Schwarz [3], K.M. Cuomo, A.V. Oppenheim, S.H. Strogatz [4], K.M. Cuomo, A.V. Oppenheim, S.H. Strogatz [5], L. Kocarev, U. Parlitz [6], Jr.E. Rosa, S. Hayes, C. Grebogi [7], I.P. Marino, Jr.E. Rosa, C. Grebogi [8], I.P. Marino, L. Lopez, M.A.F. Sanjuan [9], L. Kocarev, K.S. Halle, K. Eckert, L. Chua, U. [10], A. Dmitriev, A. Panas, S. Starkov [11], L.A.B. Torres, L.A. Aguirre [12], A.Yu. Loskutov, A.I. Shishmarev [13], A.Yu. Loskutov, V.M. Tereshko, K.A. Vasiliev [14], L. Mariot, A. Leporati, L. Manzoni, G. Mauri, A.E. Porreca, C. Zandron [15], L. Mariot, A. Leporati, A. Dennunzio, E. Formenti [16], A. Leporati [17], L. Mariot, S. Picek, A. Leporati, D. Jakobovic [18].

Purpose of the study. Analyze the relationship for the transmission and presentation of information in terms of cryptography between chaos and objects of complex dynamic systems.

Material and research methods

In the mathematical representation, the cryptosystem $S = (X, Y, K, f)$ is a kind of information conversion $f : X \times K \to Y$, set on the sets of initial states X, keys K and final states Y. The state $x \in X$ encodes some useful information. The sets $X = Y =\subset \{0, 1\}*, K \subset \{0, 1\}*$ had been studied in computer cryptography, and the transformation f had been studied by means of an algorithm (program) implemented on a Turing machine.

Transformation f is studied as iterations of a cryptographic algorithm (Fig. 1). In this case, the cryptosystem implements a sequence of the set of states $x_0, x_1, ..., x_i, ...$, where $x_i = f(x_{(i-1)}, k) = f^i(x_0, k), x_0 \in X, k \in K$, and the sequence of the set of states is called the system trajectory in Figure 1. The entire trajectory to the same is found by the parameter k and the initial state of the system x_0.

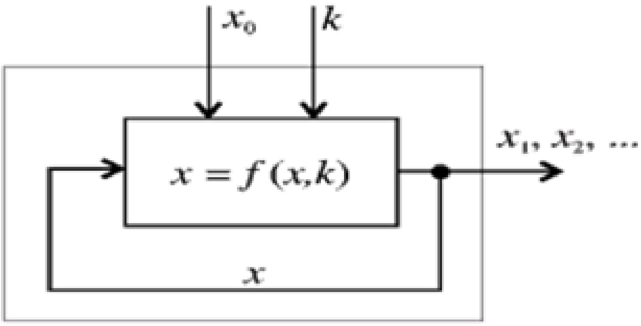


Figure 1. Model of a cryptosystem

The successive transformation of the system states as a result of the use of a certain elementary function of the same type$f$ can be observed in in-line and block ciphers, one-way functions, and pseudorandom number generators. These systems are components of a cryptosystem in a broad sense [19].

The cryptosystem, in a broader sense, is a dynamic system $\langle f, X, k\rangle$ with a non-linear function f, the state space X and the parameter space K.

The dynamic system of continuous time and continuous state $S = \langle X, K, f\rangle$, depends on the parameters and is given in the form of a differential equation:

$$\frac{dx}{dt} = f(x,k), x \in X \subseteq R^d, k \subseteq K \subseteq R^{dk}, \tag{1}$$

where $f : X \times K \to Y$ is a smooth vector function, K is the space of control parameters and X is the state space. The system (1) for a separate initial requirement x_0 satisfies the requirement of the presence and uniqueness of the solution $x \quad (t, x_0) = x_0$, where $x \quad (0, x_0) = x_0$. The curve $\varphi_t(t, x_0)$ corresponding to this solution is a trajectory.

The dynamic system of discrete time (continuous state) can be specified as an iterative function:

$$x_{n+1} = f(x_n, k), x_n \in X \subseteq R^d, k \subseteq R^{dk}, \quad n = 0, 1, 2, ..., \tag{2}$$

where x_i is discrete states of the system. The trajectory $\varphi(i, x_0)$ is a sequence of the set $x_0, x_1, ...$ Expression (2), you could notice that it seems with a cryptographic iteration function used in block ciphers, in cryptographic and dynamic systems, pseudo-random generators, studied in iterative transformation of information data that depend on the parameter [20]. Then, the parameter k is reduced in the notation of the system (X, f), and the iteration function $f(x)$. The result of the n-times use of $f(x)$ is written in the form:

$$x_n = f((...(x_0)...)) = f^n(x_0), x_0, x_n \in X.$$

Researchers identify some properties under which chaotic behavior occurs in the system. Namely, the required criterion is made by two classical features — topological transitivity and sensitivity to the initial requirements.

The definition of «chaotic system» has the following interpretation: a dynamic system $\langle X, f\rangle$ is considered chaotic when the following criteria are met:

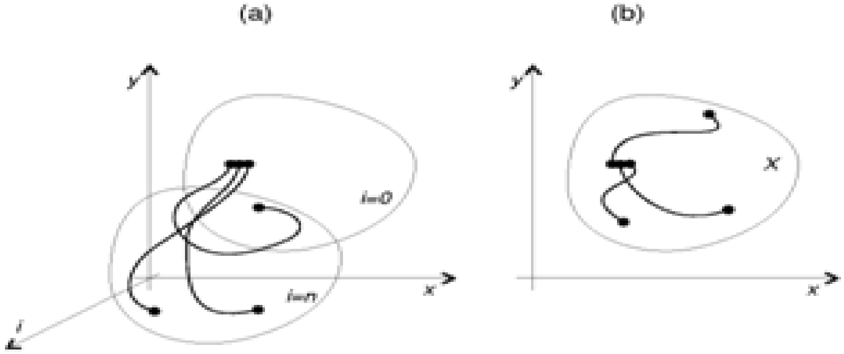
1) The function f is sensitive to the initial criteria, if there is $\delta > 0, n \geq 0$, that for different $x \in X$and its neighborhood H_x there is $y \in H_x$ for which

$$f^n(x) - f^n(y) \vee \sigma;$$

2) The function $f : X \to X$ on some metric set is topologically transitive $X \subset R_d$, if for different open sets $U, V \subset X$ there are $n \geq 0$, such as

$$f^n(U) \bigcap V \neq 0.$$

A dynamic system, in other words, is called chaotic, if all its trajectories are the limit, but instantly diverge at each point of the phase space (Fig. 2).



a) Temporary space; (b) Phase space

Figure 2. Two-dimensional chaotic system

The results of the study and their discussion. The above examples of cryptosystems are similar to chaotic systems: topological transitivity is necessary, firstly, to maintain the state of the cryptosystem within the limits allowed by the information carrier, to «cover» the entire set of ciphertext states as well. The susceptibility to the initial conditions corresponds to the susceptibility of a cryptosystem to a pseudo-random generator or plaintext. From here, as in cryptography, and in the theory of chaos, they come into contact with systems in which even a small change in the initial conditions leads to significant changes along the entire trajectory.

The concept of susceptibility to initial conditions is introduced into the understanding of a chaotic system. This indicator, as a Lyapunov exponent $\lambda(x_0)$, determined for each point $x \in X$, becomes a measure of susceptibility, in other words, determines the speed of the exponential divergence of trajectories, which are located in the vicinity of point x_0

$$f^n(x_0+\varepsilon) - f^n(x_0) \vee \varepsilon * \varepsilon^{n\lambda(x_0)}$$

in a one-dimensional system where ε is a small deviation from the initial state of the point x_0, and n is a certain number of iterations (or discrete time). For the general case, the value of λ depends on the initial conditions of the point x_0, hence the definition of the averaged value is necessary. For systems that preserve measure, λ is constant for all trajectories. Lyapunov's indicator, in practice, can be calculated as the limit

$$\lambda(x_0) = \lim_{x\to\infty} \lim_{t\to\infty} \frac{1}{n} \vee \frac{f^n(x_0+\varepsilon) - f^n(x_0)}{\varepsilon} \vee;$$

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k}^{n} \log \vee f'(x_k) \vee \lim_{n\to\infty} \frac{1}{n} \prod_{k-1}^{n} (x_k) \vee .$$

The derivative $f'(x_k)$, for each k, sets how soon the function f will change relative to the growth of the value of the argument from x_k to x_{k+1}. The limit will be equal to the average value of the logarithm of the derivative after n performed iterations and will show the value of the rate of divergence of the trajectories during the discrete time period. A positive indicator ($\lambda > 0$) is an indicator of the chaotic behavior of the system [21].

For a d-dimensional system, the set $\lambda = \{\lambda_1, ..., \lambda_d\}$ is formed and a more complex behavior is created that is not qualitatively different from the one-dimensional case.

To take into account the accuracy (resolution) of observation, the Kolmogorov-Sinai-h_{KS} entropy, given below, becomes more necessary information.

The value of the Lyapunov indicator, from the point of view of cryptography, becomes a measure of the cryptographic efficiency of the system. More precisely, the larger the value of λ, the smaller the number of iterations needed to obtain a given degree of mixing or spraying information. Existing traditional cryptosystems (pseudo-random generators, encryption schemes) should be studied as dynamic systems that transform information (Table).

Table

Relationship between objects of study in cryptography and chaos theory

Cryptography	Chaos theory
Pseudo-chaotic system	Chaotic system
-finite number of states	-infinite number of states
-finite number of iterations	-infinite number of iterations
-nonlinear transform	-nonlinear transform
Plaintext	Initial state
Key	Initial conditions and parameters
Entanglement	Asymptotic independence of the initial and final states
Ciphertext	Final state
Spraying	Sensitivity to initial conditions and parameters, mixing

From the side of objects and research accents between the theory of chaos and cryptography there are fundamental differences:

1) cryptography analyzes the obtained result of a finite number of iterative transformations ($n < \infty$), as chaos theory (discrete and continuous) studies the asymptotic behavior of the system ($n \to \bullet\infty$);

2) in cryptography it is advised to use all sorts of combinations for independent variables (the system is as unpredictable as possible) and work with spaces with integer dimensions (Fig. 3). Regarding classical chaotic systems, they are displayed in the form of some object or set of phase space, which is endowed with a fractional dimension (in essence, is a fractal);

3) in computer cryptography, the study of a system is carried out at a certain finite number of states, and the multiple state space of a chaotic system is formed with an infinite set of continuous or discrete values. It follows that absolutely all the models of chaos implemented on a computer are very approximate [22].

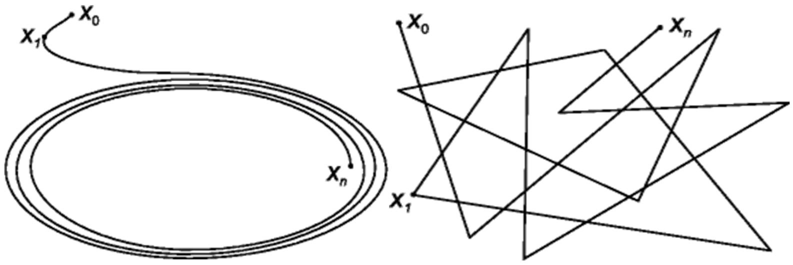


Figure 3. Phase portraits of the chaotic and cryptographic systems

Optimal security (perfect security) of an object will take place only in the situation when it is completely unpredictable for a cryptanalyst (external observer). All of this implies that the likely outcomes (all states) are very equiprobable and are not dependent on past states. In other words, the sequence of states is established by a uniform law of probability distribution and it does not have patterns (correlations). The term «absolute unpredictability» is equivalent to the concept of «true chance». Random sequence is called «white noise». The source of this white noise can be the chaotic system itself, with a rather large number of degrees of freedom (for example, a closed system with a so-called ideal gas).

Certain practical security in the current world is formed by cryptography systems, which, to some extent, will be less than ideal (due to operational and economic feasibility). The definitions of unpredictability and randomness are respectively replaced by polynomial (computational) and pseudo-randomness, unpredictability. A pseudo-random object should not at all differ from a truly random object obtained by means of computational facilities available to an external observer. By analogy, the behavior of a computationally unpredictable object cannot be predicted by the computational means used by the observer. From here, you can prove that a pseudo-random object will be computationally unpredictable.

Therefore, a truly random object will be pseudo-random and algorithmically random. The definitions of algorithmic randomness are also different from each other: pseudo-randomness: a compact generator creates a pseudo-random string, but an external observer cannot predict the sequence and create this generator. The Universe, nature and matter appear as natural chaos, possessing colossal dimensionality, lack of coverage of the «system of iterative functions» and an infinite number of states. The entropy of these systems, thanks to self-organization, is much less than that of the «completely random» system of a corresponding scale. Multidimensional and chaotic systems cannot be used in encryption, since they are not reproduced. Key generation (without the possibility of repetition), on the other hand, through «natural» chaos (for example, the thermal noise of a computer in a system unit) is widely used today [23].

The deterministic chaos that we use in encryption is endowed with a very small dimension and an infinite number of states. Obviously, such systems are likely to be more predictable than the variant of natural chaos, and they can be modeled by humans in computing systems. To create a calculated estimate of the randomness of such systems, we will make a consistent consideration of the Kolmogorov-Sinai entropy (tightly interconnected with the Lyapunov exponent and algorithmic complexity) and find, moreover, that deterministic chaos leads algorithmically random sequences. In the mixing system, even more so, the numerical sample $x_n, x_{n+k}, x_{n+2k}, x_{n+3k}...$ will be asymptotically ($k \to \infty$) random, that is, with an increase in the value of k, the members of the sample become all less dependent.

Conclusions

Thus, in our analysis, we found a close relationship between the objects of study in chaos theory and cryptography; the conclusion is made with evidence about the apparent probability of applying the trajectory of dynamic systems with chaos for the transmission and presentation of information:

1) the well-known and studied signs of chaotic systems (ergodicity, exponential divergence of trajectories, mixing) can also be fully applied in cryptography for the development of new encryption schemes;

2) the choice of the value of the control parameter in cryptographic applications makes it possible to set the unpredictability of the system, in other words, if the chaotic mapping parameter is used as the key, then the entire space of the probable keys for the assumption of keys is required to correspond to the chaotic one.

References

1 Lorenz E.N. Deterministic Nonperiodic Flow / E.N. Lorenz // J. Athmos. Sci. — 2000. — 20. — P. 130.

2 Baptista M.S. Cryptography with chaos / M.S. Baptista // Phys. Lett. A. — 1998. — 240. — P. 50-54.

3 Abel A. Chaos communication principles, schemes, and system analysis / A. Abel, W. Schwarz // Proc. IEEE. — 2002. — 90. — No. 5. — P. 691–710.

4 Cuomo K.M. Synchronization of Lorenz-based chaotic circuits with application to communications / K.M. Cuomo, A.V. Oppenheim, S.H. Strogatz // IEEE Trans. Circ. Syst. II. — 1993. — 40. — No. 10. — P. 626–633.

5 Cuomo K.M. Circuit implementation of synchronized chaos with applications to communications / K.M. Cuomo, A.V. Oppenheim // Phys. Rev. Lett. — 1993. — 71, No. 1. — P. 65–68.

6 Kocarev L. General approach for chaotic synchronization with applications to telecommunication / L. Kocarev, U. Parlitz // Phys. Rev. Lett. — 1995. — 74, No. 25. — P 5028–5031.

7 Rosa Jr. E. Noise ltering in communication with chaos / Jr.E. Rosa, S. Hayes, C. Grebogi // Phys. Rev. Lett. — 1997. — 78, No. 7. — P. 1247-1250.

8 Marino I.P. Exploiting the natural redundancy of chaotic signals in communication systems / I.P. Marino, Jr.E. Rosa, C. Grebogi // Phys. Rev. Lett. — 2000. — 85, No. 12. — P. 2629–2632.

9 Marino I.P. Channel coding in communications using chaos / I.P. Marino, L. Lopez, M.A.F. Sanjuan // Physics Letters A. — 2002. — 295, No. 4. — P. 185–191.

10 Kocarev L. Experimental demonstration of secure communications via chaotic synchronization / L. Kocarev, K.S. Halle, K. Eckert, L. Chua, U. Parlitz // Int. J. Bifurcation and Chaos. — 1992. — 2, 3. — P. 709.

11 Dmitriev A. Experiments on speech and music signals transmission using chaos / A. Dmitriev, A. Panas, S. Starkov // Int. Journal of Bifurcation and Chaos. — 1995. — 5, 4. — P. 1249–1254.

12 Torres L.A.B. Extended chaos control method applied to Chua circuit / L.A.B. Torres, L.A. Aguirre // Electronics Letters. — 1999. — 35. — No. 10. — P. 768–770.

13 Loskutov A.Yu. Control of dynamical systems behavior by parametric perturbations an analytic approach / A.Yu. Loskutov, A.I. Shishmarev // Chaos. — 2004. — 4. — No. 2. — P. 351–355.

14 Loskutov A.Yu. Stabilization of chaotic dynamics of one-dimensional maps by cyclic parametric transformation / A.Yu. Loskutov, V.M. Tereshko, K.A. Vasiliev // Int. J. Bi. and Chaos. — 2006. — 6, No. 4. — P. 725–735.

15 Leporati A.C. Characterizing the complexity of tissue P systems with fission rules / A.C. Leporati, L. Manzoni, G. Mauri, A.E. Porreca, C. Zandron // Journal of Computer and System Sciences. — 2017. — 90. — P. 115-128.

16 Mariot L.A. Computing the periods of preimages in surjective cellular automata / L.A. Mariot, A. Leporati, A. Dennunzio, E. Formenti, // Natural Computing. — 2017. — 16, No. 3. — P. 367–381.

17 Mariot, L.A cryptographic and coding-theoretic perspective on the global rules of cellular automata / L.A. Mariot, A. Leporati // Natural Computing. – 2018. – 17. – №3. – P. 487-498.

18 Mariot L.A. Cellular automata based S-boxes / L.A. Mariot // Cryptography and Communications. — 2019. — 11, No. 1. — P. 41–62.

19 Колесов В.В. Шифрование цифровой информации при использовании генераторов с хаотической динамикой / В.В. Колесов, Н.Н. Залогин, Г.М. Воронцов // РЭ. — 2008. — 53, № 4, — С. 459–467.

20 Птицын Н. Приложение детерминированного хаоса в криптографии / Н. Птицын. — М.: Изд-во Москов. техн. ун-та им. Н.У. Баумана, 2009. – С. 70.

21 Марков А.С. Конвергенция средств защиты информации / А.С. Марков, А.А. Фадин // Защита информации: Инсайд. — 2013. — № 4. — С.53–55.

22 Бейсенби М.А., Тайлак Б.Е., Томилова Н.И. Формализация взаимосвязи детерминированного хаоса и криптографии // Фундаментальные и прикладные исследования, разработка и применение высоких технологий в промышленности: материалы XIII Междунар. науч.-практ. конф. — СПб.: СПб НИУ ИТМО, 2012. — С. 168–171.

23 Когай Г.Д. Криптографические системы по управлению детерминированным хаосом / Г.Д. Когай, Т.Л. Тен // Гамбург, 2014. — С. 132-134.

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Криптография және хаос динамиканың өзарабайланыс модельдері

Криптография ақпаратты қорғауды қамтамасыз ететін (есептеуіш техника құралдары арқылы) оны түрлендіру жолымен ақпаратты қорғау мәселесімен айналысады, яғни келісілген шифрлау құралдарының жиынтығы. Тар мағынада криптожүйе — сызықты емес функциясы және кеңістік жағдайлары бар динамикалық жүйе, әдетте ол дифференциалдық, теңдеумен ұсынылған. Динамикалық жүйенің кейбір шарттары, сезімталдық өлшемі ретінде Ляпунов көрсеткіші қарастырылды. Хаос және криптография теориясындағы зерттеу нысандары арасындағы өзара байланыс анықталды; ақпаратты ұсыну және беру үшін хаос пен динамикалық жүйелердің траекториясын пайдалану мүмкіндіктері туралы қорытынды жасалды.

Кілт сөздер: криптография, ақпаратты қорғау, шифрлеу, бейсызық функция, динамикалық жүйе, криптожүйе, параметрлер, траектория, түрлендіру, хаотикалық жүйе, сезімталдық мөлшері.

Б.К. Шаяхметова, Т.Л. Тен, Г.Д. Когай, Ш.Е. Омарова

Модели взаимосвязи криптографии и хаотической динамики

Криптография занимается проблемой защиты информации путем ее преобразования (средствами вычислительной техники), т.е. является совокупностью согласованных средств шифрования. Под криптосистемой в узком смысле будем понимать динамическую систему с нелинейной функцией и пространством состояний, обычно она представлена дифференциальным уравнением. Рассмотрены некоторые условия динамической системы, показатель Ляпунова как мера чувствительности. Выявлена взаимосвязь между объектами изучения в теории хаоса и криптографии; сделаны выводы относительно возможности использования траектории динамических систем с хаосом для предоставления и передачи информации.

Ключевые слова: криптография, защита информации, шифрование, нелинейная функция, динамическая система, криптосистема, параметры, траектория, преобразование, хаотическая система, мера чувствительности.

References

1 Lorenz, E.N. (2000). Deterministic Nonperiodic Flow. *J. Athmos. Sci, 20,* 130.

2 Baptista, M.S. (1998). Cryptography with chaos. *Phys. Lett. A., 240,* 50-54.

3 Abel, A., & Schwarz, W. (2002). Chaos communication principles, schemes, and system analysis. *Proc. Ieee, 90, 5,* 691-710.

4 Cuomo, K.M., Oppenheim, A.V., & Strogatz, S.H. (1993). Synchronization of Lorenz-based chaotic circuits with application to communications. *Ieee Trans. Circ. Syst. II, 40, 10,* 626-633.

5 Cuomo, K.M., & Oppenheim, A.V. (1993). Circuit implementation of synchronized chaos with applications to communications. *Phys. Rev. Lett., 71, 1,* 65-68.

6 Kocarev, L., & Parlitz, U. (1995). General approach for chaotic synchronization with applications to telecommunication. *Phys. Rev. Lett., 74, 25,* 5028-5031.

7 Rosa, Jr.E., Hayes, S., & Grebogi, C. (1997). Noise ltering in communication with chaos. *Phys. Rev. Lett., 78, 7,* 1247-1250.

8 Marino, I.P., Rosa, Jr.E., & Grebogi, C. (2000). Exploiting the natural redundancy of chaotic signals in communication systems. *Phys. Rev. Lett., 85, 12,* 2629-2632.

9 Marino, I.P., Lopez, L., & Sanjuan, M.A.F. (2002). Channel coding in communications using chaos. *Physics Letters A, 295, 4,* 185-191.

10 Kocarev, L., Halle, K.S., Eckert, K., Chua, L., & Parlitz, U. (1992). Experimental demonstration of secure communications via chaotic synchronization. *Int. J. Bifurcation and Chaos, 2, 3,* 709-713.

11 Dmitriev, A., Panas, A., & Starkov, S. (1995). Experiments on speech and music signals transmission using chaos. *Int. Journal of Bifurcation and Chaos, 5, 4,* 1249-1254.

12 Torres, L.A.B., & Aguirre, L.A. (1999). Extended chaos control method applied to Chua circuit. *Electronics Letters, 35,10,* 768-770.

13 Loskutov, A.Yu., & Shishmarev, A.I. (2004). Control of dynamical systems behavior by parametric perturbations an analytic approach. *Chaos, 4, 2,* 351-355.

14 Loskutov, A.Yu., Tereshko, V.M., & Vasiliev, K.A. (2006). Stabilization of chaotic dynamics of one-dimensional maps by cyclic parametric transformation. *Int. Journal of Bifurcation and Chaos, 6, 4,* 725-735.

15 Leporati, A., Manzoni, L., Mauri, G., Porreca, & A.E., Zandron, C. (2017). Characterizing the complexity of tissue P systems with fission rules. *Journal of Computer and System Sciences, 90,* 115-128.

16 Mariot, L., Leporati, A., Dennunzio, A., & Formenti, E. (2017). Computing the periods of preimages in surjective cellular automata. *Natural Computing, 16, 3,* 367-381.

17 Mariot, L., & Leporati, A. (2018). A cryptographic and coding-theoretic perspective on the global rules of cellular automata. *Natural Computing, 17, 3,* 487-498.

18 Mariot, L., Picek, S., Leporati, A., & Jakobovic, D. (2019). Cellular automata based S-boxes. *Cryptography and Communications, 11, 1,* 41-62.

19 Kolesov, V.V., Zalogin, N.N., & Vorontsov, G.M. (2008). Shifrovanie tsifrovoi informatsii pri ispolzovanii heneratorov s khaoticheskoi dinamikoi [Encryption of digital information using generators with chaotic dynamics]. *RE, 53, 4,* 459-467 [in Russian].

20 Ptitsyn, N. (2009). *Prilozhenie determinirovannoho khaosa v kriptohrafii [Application of deterministic chaos in cryptography].* Maoscow: Izd-vo Mosk. tekh. univ. im. N.U. Baumana [in Russian].

21 Markov, A.S., & Fadin, A.A. (2013). Konverhentsiia sredstv zashchity informatsii [Convergence of information protection means]. *Zashchita informatsii. Insaid, 4,* 53-55 [in Russian].

22 Beisenbi, M.A., Taylak, B.E., & Tomilova, N.I. (2012). Formalizatsiia vzaimosviazi determinirovannoho khaosa i kriptohrafii [Formalization of the relationship of deterministic chaos and cryptography]. *Materials of the XIII International Scientific and Practical Conference «Fundamental and applied research, development and application of high technologies in industry».* Saint Petersburg: St. Petersburg NRU ITMO[in Russian].

23 Kogay, G.D., & Ten, T.L. (2014). Kriptohraficheskie sistemy po upravleniiu determinirovannym khaosom [Cryptographic systems for managing deterministic chaos]. *Hamburg*, 218, 132-134 [in Russian].

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The usage of the method of mathematical statistics in the process of optimization of the content of antifriction composite materials

The optimum content of microtalc (dispersion d = 7...10 μm) and silver carbonate (dispersion d = 0,5 μm) fillers in polymeric materials was determined by the method of mathematical statistics to form an adhesive and functional layers of protective coatings. The mathematical models of physicomechanical and thermophysical characteristics of composites were obtained by the method of statistical processing of the results of the investigation materials.

Keywords: composite, mathematical statistics, modulus of elasticity, destruction, optimization.

Introduction

Statement of the problem

The formation of constructional materials, including polymer ones, with the necessary complex of improved properties is an important problem for today [1–12]. This problem is solved by selecting a range of fillers content in materials, which is achieved using the method of mathematical statistics. Experimental studies related to the optimization of the composition of protective coatings are, as a rule, multifactorial (optimization of composites properties and fillers content). Methods of mathematical statistics allow to adequately assess the content of several fillers of different dispersion, taking into account technological factors, a complex of physicomechanical, thermophysical properties and reliability indicators [13–19].

Analysis of recent researches and publications

The input of fillers of various nature and dispersion into the binder is one of the methods for improving the properties of composite materials (CM) based on an epoxy matrix. Previously, we investigated the influence of fillers of different nature and dispersion on the physic-mechanical and thermophysical properties of CM [20–23]. The optimum content of microdispersed (7...10 μm) and nanodispersed (100...500 μm) fillers particles of various nature was established to form coatings of different functional purpose with increased exploitation characteristics.

The results of the experimental studies were statistically processed using the Statgraphics application package to predict the properties and optimize the content of each filler in the PCM.

The purpose of the work is to determine the most optimal mass part of the filler, using multicriteria selection methods for each type of filler.

Experimental results and their discussion

During the experiment, the influence of two factors (the content of microtalc (MT) and silver carbonate (SC)) on physicomechanical (modulus of elasticity under bending, destructive bending stresses) and thermophysical (heat resistance (by Martens), temperature of the start of the destruction process) properties of PCM were studied.

Output data for statistical processing of the research results of PCM 2, consisting of the epoxy diane resin ED-20 grade (q = 100 wt %), hardened by polyethylene polyamine (q = 10 wt %) and filled with particles of MT and SC, are given in Table 1.

Table 1

Output data for statistical processing of the research results of PCM properties

The level of variation	Variable factors	
	Microtalc content, q_1, wt %	Silver carbonate content, q_2, wt %
Upper	80	0,7
Lower	60	0,3

Output data and results of the implementation of the mathematical model in accordance with the study of PCM properties are given in Table 2.

Table 2

Output data and results of the implementation of the mathematical model in the experimental research of PCM properties

Number of experiment	Factors		Response			
	q_1 (A)	q_2 (B)	E	σ_b	T	T_0
	wt %	wt %	GPa	MPa	K	K
1	70	0,78	7,2	32,8	376	624
2	84	0,5	6,8	22,6	370	618
3	55	0,5	6,6	32,5	369	617
4	70	0,5	6,3	27,8	371	621
5	80	0,7	6,9	24,9	374	620
6	80	0,3	6,9	26,3	375	620
7	70	0,22	6,4	29,2	372	620
8	60	0,3	5,9	27,4	361	617
9	60	0,7	5,8	29,8	362	618
10	70	0,3	6,4	28,1	372	620

Note. A and B denote the content of the factors q_1 and q_2 (Fig. 1).

To determine the significance of the factors, we used Pareto maps (Fig. 1, a-d) and graphs of normal probabilistic distribution (Fig. 1, e-h).

It is shown on Pareto maps (Fig. 1, a-d) that those factors and their combinations (columns corresponding to them on Pareto maps) that cross the vertical line with 95 % confidence probability have statistically significant effects.

Also, the analysis of the obtained graphs of diagnostic of prediction values errors (Fig. 1, e-h) shows that the factors and their combinations, which are substantially deviating from the straight of normal distribution, are significant in the mathematical model, in contrast to other factors located directly at the distribution straight line. These results confirm the conclusions of the significance of the factors of mathematical model, which were made using the Pareto map (Fig. 1, a-d).

Excluding insignificant factors and their combinations, we received surfaces of responses for physicomechanical (E, σ_b) and thermophysical (T, T_0) properties of PCM (Fig. 2, a-d), as well as contour graphs (Fig. 2, e-h).

Mathematical models of physic-mechanical (E, σ_b, W) and thermophysical (T, T_0) properties of PCM are given in Table 3.

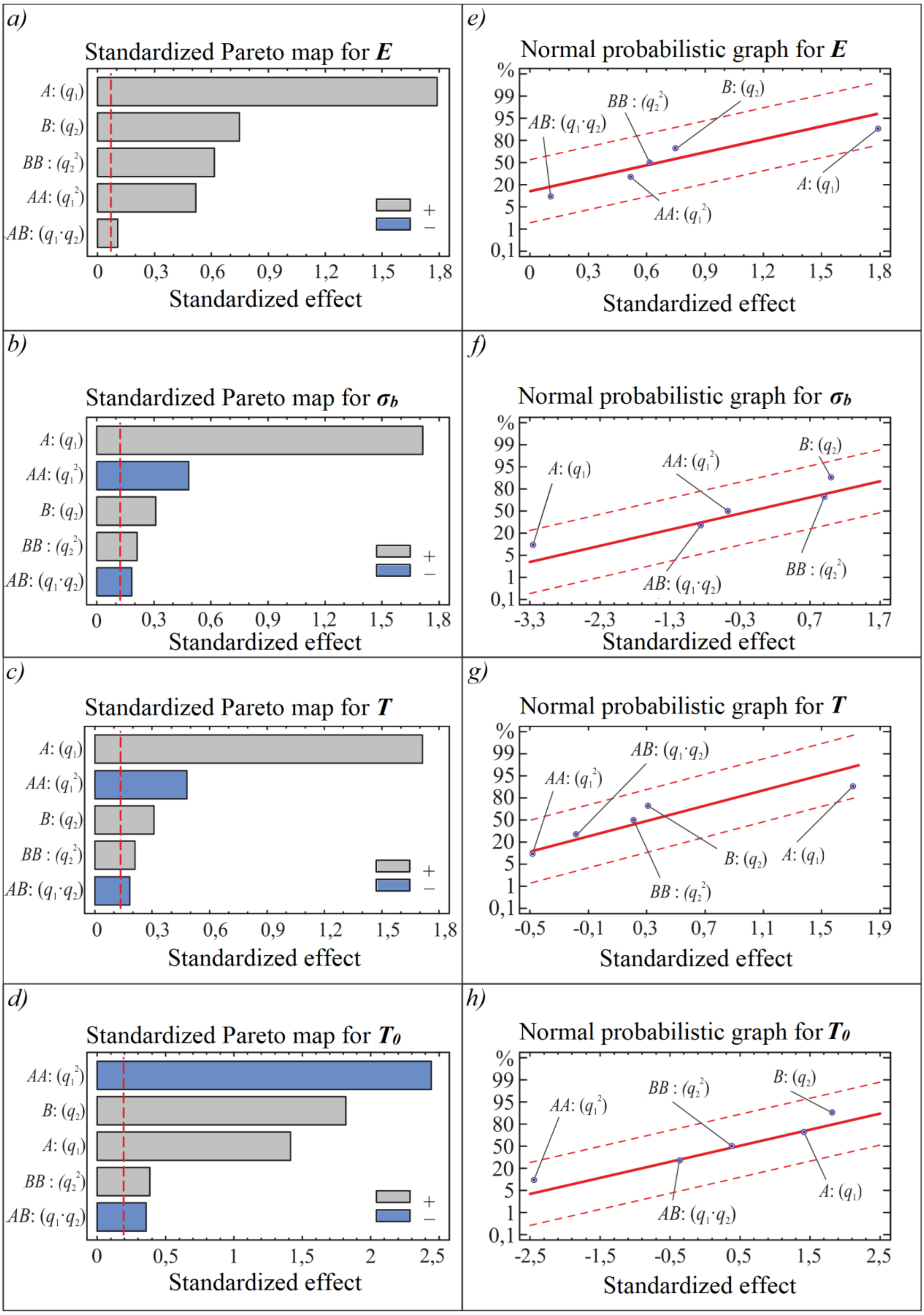

Figure 1. Pareto maps (a-d) and graphs of diagnostic of deviation of prediction values errors of the output parameter from the normal distribution (e-h) for the responses E, σ_b, Ti T_0

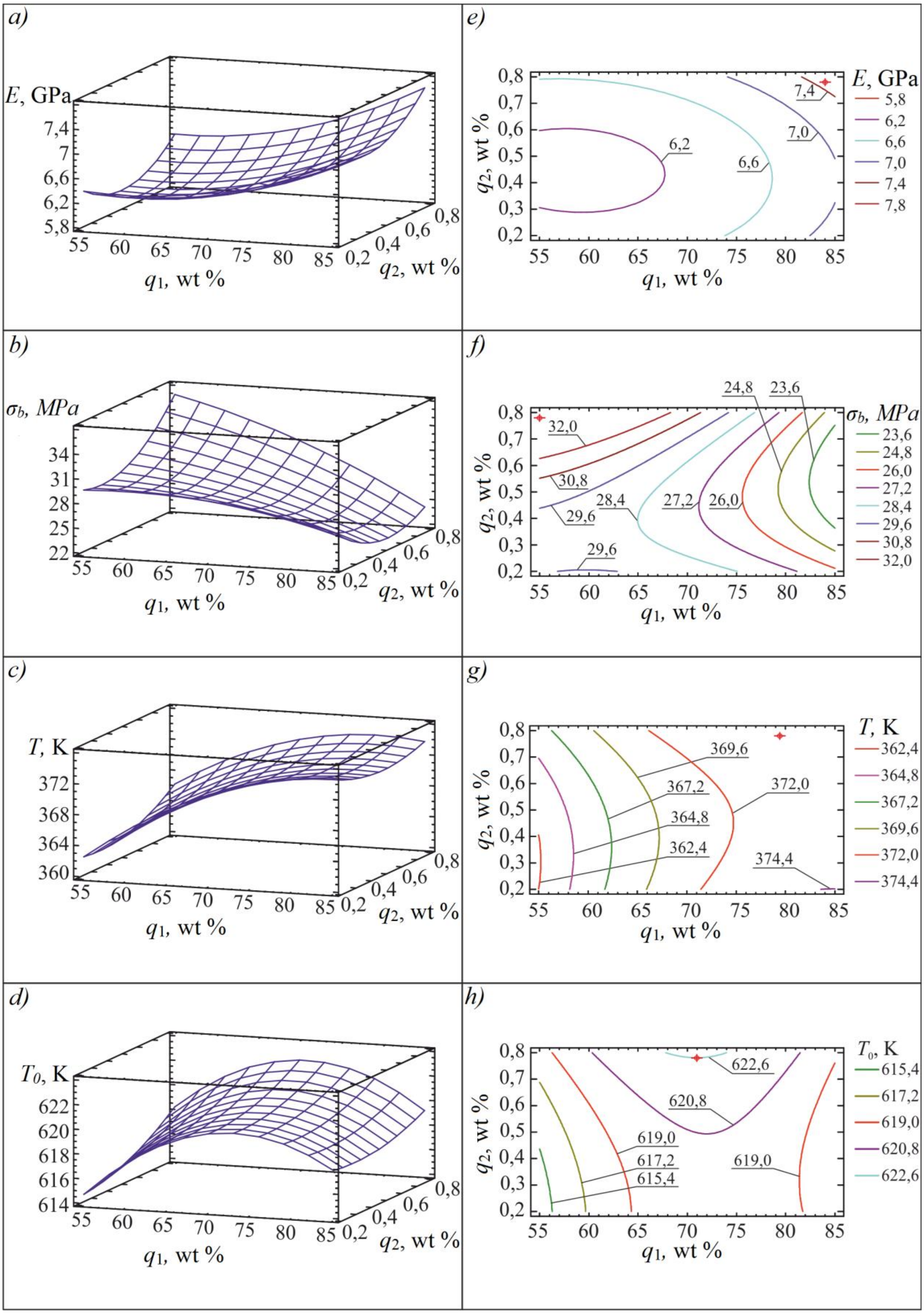

Figure 2. Response surfaces (a-d) and contour graphs (e-h)
for the response E, σ_b, T and T_0, shown in Table 4

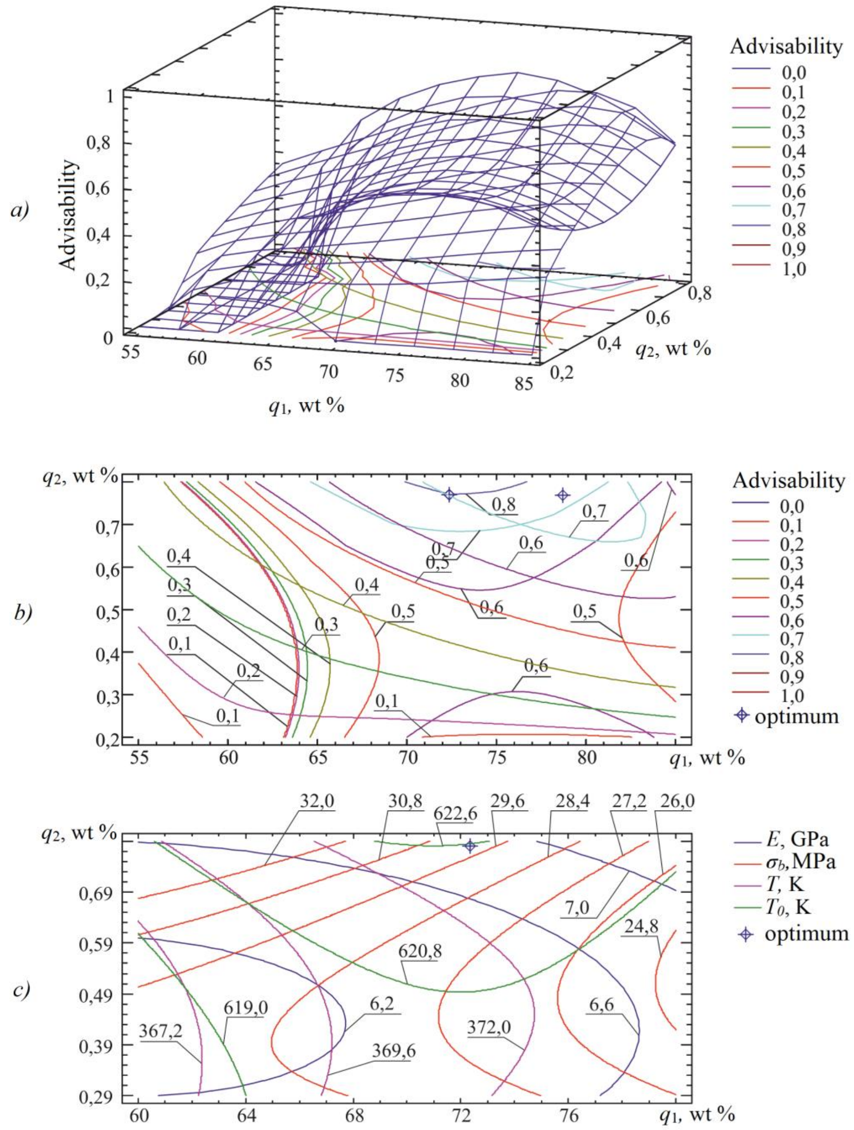

Figure 3. Surface of advisability (a), contour graph of advisability (b) and contour graphs (c) for the responses E, σ_b, T and T_0

The optimum values of indicators of physicomechanical and thermophysical properties of PCM at the corresponding content of the fillers (microtalc – q_1 and silver carbonate – q_2) according to the data of statistical processing are given in Table 4.

Table 3

Mathematical models of physicomechanical and thermophysical properties of PCM

Regression model	Determination coefficient R^2, %	Adjusted coefficient R^2_{adj}, %
$E = 11,59 - 0,153 \cdot q_1 - 4,52 \cdot q_2 + 0,00126 \cdot q_1^2 + 0,0125 \cdot q_1 \cdot q_2 + 4,25 \cdot q_2^2$	99,8	89,6
$\sigma_b = 7,61 + 0,740 \cdot q_1 + 7,78 \cdot q_2 - 0,0054 \cdot q_1^2 - 0,475 \cdot q_1 \cdot q_2 + 29,22 \cdot q_2^2$	99,8	93,1
$T = 275,7 + 2,346 \cdot q_1 + 3,589 \cdot q_2 - 0,0135 \cdot q_1^2 - 0,25 \cdot q_1 \cdot q_2 + 16,75 \cdot q_2^2$	99,9	90,4
$T_0 = 525,1 + 2,596 \cdot q_1 + 5,219 \cdot q_2 - 0,0176 \cdot q_1^2 - 0,25 \cdot q_1 \cdot q_2 + 7,83 \cdot q_2^2$	99,9	96,1

After obtaining polynomial regression equations (Table 3), connecting dependent and independent variables, the mathematical model was optimized with the simultaneous consideration of all response – indicators of physicomechanical and thermophysical properties of PCM for the purpose to determine the optimum content of fillers. The function of advisability (preferred use) was evaluated throughout the range of this model. The results of optimization are given on Figure 3 and in Table 5.

Table 4

The optimum values of indicators of physicomechanical and thermophysical properties of PCM

Optimum values		Content of fillers	
		q_1, wt %	q_2, wt %
E_opt, GPa	7,49	84	0,78
$\sigma_{b\ opt}$,MPa	35,5	55	0,78
T_{opt}, K	374,1	79,4	0,78
$T_{0\ opt}$, K	622,6	70,9	0,78

Note. q_1 – the content of microtalc in PCM; q_2 – the content of silver carbonate in PCM.

When optimizing the combination of experimental factors was determined for all given responses by maximizing each of them.

The maximum value of the generalized advisability of D_{opt} = 0,798 (corresponding to the permissible and good quality level on the scale of advisability) was obtained as a result of the optimization performed for PCM, at which the content of fillers in PCM is:

q_1 = 72,3 wt % – the content of microtalc;

q_2 = 0,78 wt % – the content of silver carbonate.

The values of responses for the specified content of the fillers in the PCM are:

E = 7,19 GPa – modulus of elasticity under bending;

σ_b = 34,1 MPa – destructive bending stresses;

T = 373,7 K – heat resistance (by Martens);

T_0 = 623 K – temperature of the start of the destruction process.

Comparing the values obtained as a result of optimization, with the values given in Table 4, it can be argued that the relative error will be: for modulus of elasticity under bending – 4,0 %; for destructive bending stresses – 3,9 %; for heat resistance (by Martens) – 0,2 %; for temperature of the start of the destruction process – 0,1 %. This allows us to confirm about the adequacy of the received data and their consistency with the results of optimization by the criterion of advisability.

Table 5

Optimization results for PCM

№	Simultaneous combination of responses to determine advisability				Partial advisability for the appropriate optimization parameter				Generalized advisability
	Y_j				d_i				$D = \sqrt[n]{\prod_{i=1}^{n} d_i}$
	E	σ_b	T	T_0	$d_1(E)$	$d_2(\sigma_b)$	$d_3(T)$	$d_4(T_0)$	–
	GPa	MPa	K	K	–	–	–	–	–
1	0,692	0,692	0,692	0,692	0,798	0,692	0,692	0,692	0,798
2	0,521	0,066	0,441	0,130	0,385	0,521	0,066	0,441	0,385
3	0,420	0,677	0,392	0,066	0,472	0,420	0,677	0,392	0,472
4	0,264	0,375	0,488	0,420	0,551	0,264	0,375	0,488	0,551
5	0,569	0,177	0,619	0,316	0,548	0,569	0,177	0,619	0,548
6	0,569	0,268	0,657	0,316	0,589	0,569	0,268	0,657	0,589
7	0,316	0,475	0,534	0,316	0,570	0,316	0,475	0,534	0,570
8	0,095	0,346	0,066	0,066	0,258	0,095	0,346	0,066	0,258
9	0,066	0,516	0,093	0,130	0,303	0,066	0,516	0,093	0,303
10	0,316	0,397	0,534	0,316	0,554	0,316	0,397	0,534	0,554

Conclusions

The mathematical models of physicomechanical and thermophysical characteristics of composites were obtained by the method of statistical processing of the results of the study materials. As a result of optimization the property metrics for the material with particles of microtalc ($q = 70\ldots80$ wt %) and silver carbonate ($q = 0{,}7\ldots1{,}0$ wt %) are: modulus of elasticity under bending – $E = 7{,}2$ GPa; destructive bending stresses – $\sigma_b = 34{,}1$ MPa; heat resistance – $T = 373{,}7$ K; temperature of the start of the destruction process – $T_0 = 623$ K.

References

1 Kal'ba E.M. Adhesion strength of polymer-containing protective coatings / E.M. Kal'ba, A.V. Buketov, P.P. Savchuk, S.M. Holotenko // Materials Science. — 1999. — 35(1). — P. 125–128.

2 Savchuk P.P. Structure and functional properties of epoxy composites reinforced with superfine particles / P.P. Savchuk, A.G. Kostornov // Powder Metallurgy and Metal Ceramics. — 2009. — 48(9-10). — P. 555–559.

3 Savchuk P.P. Rheological properties of epoxy composites with different reinforcement contents / P.P. Savchuk, A.G. Kostornov // Powder Metallurgy and Metal Ceramics. — 2009. — 48(3-4). — P. 157–161.

4 Savchuk P.P. Friction wear of modified epoxy composites / P.P. Savchuk, A.G. Kostornov, V.P. Kashitskii, O.L. Sadova // Powder Metallurgy and Metal Ceramics. — 2014. — 53(3-4). — P. 205–209.

5 Savchuk P.P. Friction wear of modified epoxy composites / P.P. Savchuk, V.P. Kashitskii, O.L. Sadova, O.M. Lyushuk // Powder Metallurgy and Metal Ceramics. — 2017. — 56(7-8). — P. 448–455.

6 Yanovsky Yu.G. Special features of the structure of carbon nanotubes in polymer composite media / Yu.G. Yanovsky, G.V. Kozlov, Z.B. Zhirikova, V.Z. Aloev, Yu.N. Karnet // Nanomechanics Science and Technology: An International Journal. — 2012. — 3(2). — P. 99–124.

7 Aygubova A. Effective length of a nanofiller and the degree of reinforcement of polymer/carbon nanotubes (nanofibers) nanocomposites / A. Aygubova, Y. Karnet, G. Kozlov, G. Magomedov // Nanomechanics Science and Technology: An International Journal. — 2016. — 7(4). — P. 349–354.

8 Burya A.I. Temperature and density determination for heat flux in the area of the shaft-bushing contact / A.I. Burya, B. Peleshenko, I. Rula // Journal of Friction and Wear. — 2010. — 31(6). — P. 443–448.

9 Dolbin I.V. The structure and thermal stability of polymer materials: A fractal model / I.V. Dolbin, A.I. Burya, G.V. Kozlov // High Temperature — 2007. — 45(3). — P. 313–316.

10 Burya A.I. Friction and wear of organoplastics based on aromatic polyamide of phenylone type / A.I. Burya, O.A. Naberezhnaya // Journal of Friction and Wear. — 2016. — 37(3). — P. 259–262.

11 Kozlov G.V. Interactions between polymers and carbon nanotubes and their impact on the properties of nanocomposites / G.V. Kozlov, I.V. Dolbin, Yu.N. Karnet, A.N. Vlasov // Composites: Mechanics, Computations, Applications: An International Journal. — 2018. — 9(3). — P. 239–246.

12 Kozlov G.V. Mechanism of growth of interfacial regions in polymer / carbon nanotube nanocomposites / G.V. Kozlov, A.N. Vlasov, I.V. Dolbin, Yu.N. Karnet // Composites: Mechanics, Computations, Applications: An International Journal. — 2019. — 10(3). — P. 213–220.

13 Kozlov G.V. The effect of nanofiller structure on the reinforcement degree of polymer/carbon nanotubes nanocomposites / G.V. Kozlov, I.V. Dolbin, Yu.N. Karnet // Nanoscience and Technology: An International Journal. — 2017. — 8(2). — P. 123–131.

14 Valiev H.H. Application of quantum-chemical modeling results in experimental investigations of silicone composites / H.H. Valiev, V.V. Vorobyev, Yu.N. Karnet, Yu.V. Kornev, O.B. Yumaschev // Materials Physics and Mechanics. — 2017. — 32. — P. 293–297.

15 Kozlov G.V. The effective length of a nanofiller and reinforcement degree of polymer/carbon nanotubes (nanofilaments) nanocomposites / G.V. Kozlov, A.Ch. Aygubova, G.M. Magomedov, Yu.N. Karnet // Nanoscience and Technology: An International Journal. — 2015. — 6(3). — P. 203–208.

16 Yudin V.E. Effect of the nanoparticles with various structures and morphologies on the thermo-mechanical properties of epoxy-based carbon fiber-reinforced plastics / V.E. Yudin, E.S. Molchanov, K. Kydralieva, E.M. Ivankova, V.Yu. Elokhovsky, G.V. Vaganov // Nanoscience and Technology: An International Journal. — 2013. — 4(1). — P. 47–58.

17 Valiev H.H. Atomic force microscopy and physical - mechanical properties of new elastomer composites / H.H. Valiev, Yu.N. Karnet, N.L. Kochurov, M.S. Parshina, N.A. Semenov, O.B. Yumaschev, Yu.G. Yanovsky // Materials Physics and Mechanics. — 2016. — 26. — P. 45–48.

18 Mikitaev A.K. Fractal models of the reinforcement of polymer particulate-filled nanostructured composites by nanoparticle aggregates / A.K. Mikitaev, Yu.G. Yanovsky, G.V. Kozlov, Yu.N. Karnet // Nanoscience and Technology: An International Journal. — 2013. — 4(1). — P. 47–58.

19 Kornev Yu.V. Physical model of the behavior of polymer nanostructured composites in nanoindentation / Yu.V. Kornev, O.V. Boiko, G.V. Kozlov, Yu.G. Yanovsky, Yu.N. Karnet // Nanoscience and Technology: An International Journal. — 2010. — 1(4). — P. 291–300.

20 Sapronov O.O. Features of structural processes in epoxy composites filled with silver carbonate on increase in temperature / O.O. Sapronov, A.V. Buketov, D.O. Zinchenko, V.M. Yatsyuk // Composites: Mechanics, Computations, Applications. An International Journal. — 2017. — № 8(1). — P. 47–66.

21 Букетов А.В. К вопросу оптимизации ингредиентов композитных материалов на основе эпоксидиановой смолы / А.В. Букетов, А.В. Шарко, Д.А. Зинченко, Д.М. Степанчиков // Вестн. Караганд. ун-та. Сер. Математика. — 2017. — 2(86). — С. 37–44.

22 Buketov A.V. Investigation of the hydroabrasive wear of epoxy composites with two-component filler / A.V. Buketov, O.O. Sapronov, M.V. Brailo, D.O. Zinchenko, V.D. Nihalatii // Materials Science. — 2017. — 53(1). — P. 62–66.

23 Букетов А.В. Исследование физико-механических свойств композитных материалов на основе эпоксидиановой смолы, отвержденной полиэтиленполиамином, с добавлением пластификатора-антипирена трихлорэтилфосфата / А.В. Букетов, А.В. Акимов, Д.А. Зинченко // Вестн. Хмельницкого нац. ун-та. Технические науки. — 2015. — № 5. — С. 126–133.

24 Букетов А.В.Влияние микродисперсного карбоната серебра на свойства эпоксикомпозитов / А.В. Букетов, А.А. Сапронов, Д.А. Зинченко, В.Н. Яцюк // Механика композиционных материалов и конструкций. — 2015. — С. 534–548.

25 Букетов А.В. Дослідження впливу мікродисперсних антифрикційних наповнювачів на адгезійну міцність епоксикомпозитів / А.В. Букетов, Д.О. Зінченко, В.Д. Нігалатій // Наукові нотатки: Міжвузівський збірник — Луцькю, 2016. — № 53. — С. 17–23.

26 Buketov A.V. Nanomodified dispersed particles- and synthetic fibers-filled epoxy composite materials for the metal-polymer tribosystems of transport vehicles / A.V. Buketov, D.O. Zinchenko, S.O. Smetankin // Nanomechanics Science and Technology: An International Journal. — 2017. — 8(1). — P. 41–54.

27 Букетов А.В. Вплив синтетичних дискретних волокон на властивості епоксидних композитів для захисних покриттів / А.В. Букетов, Д.О. Зінченко, О.В. Шарко // Журнал нано- та електронної фізики. Суми. – 2017. — Т. 9. — № 2. — С. 02014-1–02014-5.

28 Zinchenko D.O. Nanomodified epoxycomposite materials for metal-polymeric tribosystems of transport vehicles / D.O. Zinchenko, S.O. Smetankin // Nanoscience and Technology: An International Journal. — 2018. — 9(1). — P. 9–17.

29 Buketov A.V. Enhancing performance characteristics of equipment of sea and river transport by using epoxy composites / A.V. Buketov, P.O. Maruschak, O.O. Sapronov, D.O. Zinchenko, V.M. Yatsyuk, S. Panin // TRANSPORT. — Vilnius Gediminas Technical University (VGTU), 2016. — Vol. 31(3). — P. 333–342.

30 Букетов А.В. Исследование влияния нанодисперсного оксида серебра на структуру и теплофизические свойства полимерных композитных материалов / А.В. Букетов, А.А. Сапронов, Д.А. Зинченко, В.Н. Яцюк // Пластические массы. — 2017. — № 5, 6. — С. 28–37.

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Математикалық статистика әдісімен композиттік материалдар құрамын оңтайландыру

Мақалада математикалық статистика әдістерімен қорғаныс жабындарының адгезиялық және функционалдық қабаттарын қалыптастыру үшін полимерлік материалдар толтырғыштары микротальктің (дисперсиялығы d = 7...10 μm) және күміс карбонатының (дисперсиялығы d = 0,5 μm) тиімді құрамы анықталған. Материалдарды зерттеу нәтижелерін статистикалық өңдеу әдістері арқылы композиттердің физика-механикалық және жылуфизикалық сипаттамаларының математикалық моделі алынды.

Кілт сөздер: композит, математикалық статистика, серпімділік модулі, деструкция, оңтайландыру.

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Оптимизация состава композитных материалов методом математической статистики

В статье методом математической статистики определено оптимальное содержание в полимерных материалах наполнителей микроталька (дисперсностью d = 7. . . 10 мкм) и карбоната серебра (дисперсностью d = 0,5 мкм) для формирования адгезионного и функционального слоев защитных покрытий. Методом статистической обработки результатов исследования материалов получены математические модели физико-механических и теплофизических характеристик композитов.

Ключевые слова: композит, математическая статистика, модуль упругости, деструкция, оптимизация.

References

1 Kal'ba E.M., Buketov A.V., Savchuk P.P., & Holotenko S.M. (1999). Adhesion strength of polymer-containing protective coatings. *Materials Science, 35, 1,* 125–128.

2 Savchuk P.P., & Kostornov A.G. (2009). Structure and functional properties of epoxy composites reinforced with superfine particles. *Powder Metallurgy and Metal Ceramics, 48,* 9-10, 555–559.

3 Savchuk P.P., & Kostornov A.G. (2009). Rheological properties of epoxy composites with different reinforcement contents. *Powder Metallurgy and Metal Ceramics, 48,* 3-4, 157–161.

4 Savchuk P.P., Kostornov A.G., Kashitskii V.P., & Sadova O.L. (2014). Friction wear of modified epoxy composites, *Powder Metallurgy and Metal Ceramics, 53,* 3-4, 205–209.

5 Savchuk P.P., Kashitskii V.P., Sadova O.L., & Lyushuk O.M. (2017). Friction wear of modified epoxy composites. *Powder Metallurgy and Metal Ceramics, 56,* 7-8, 448–455.

6 Yanovsky Yu.G., Kozlov G.V., Zhirikova Z.B., Aloev V.Z., & Karnet Yu.N. (2012). Special features of the structure of carbon nanotubes in polymer composite media. *Nanomechanics Science and Technology: An International Journal, 3, 2,* 99–124.

7 Aygubova A., Karnet Y., Kozlov G., & Magomedov G. (2016). Effective length of a nanofiller and the degree of reinforcement of polymer/carbon nanotubes (nanofibers) nanocomposites. *Nanomechanics Science and Technology: An International Journal, 7, 4,* 349–354.

8 Burya A., Peleshenko B., & Rula I. (2010). Temperature and density determination for heat flux in the area of the shaft-bushing contact. *Journal of Friction and Wear, 31, 6,* 443–448.

9 Dolbin I.V., Burya A.I., & Kozlov G.V. (2017). The structure and thermal stability of polymer materials: A fractal model. *High Temperature, 45, 3,* 313–316.

10 Burya A.I., & Naberezhnaya O.A. (2016). Friction and wear of organoplastics based on aromatic polyamide of phenylone type. *Journal of Friction and Wear, 37, 3,* 259–262.

11 Kozlov G.V., Dolbin I.V., Karnet Yu.N., & Vlasov A.N. (2018). Interactions between polymers and carbon nanotubes and their impact on the properties of nanocomposites. *Composites: Mechanics, Computations, Applications: An International Journal, 9, 3,* 239–246.

12 Kozlov G.V., Vlasov A.N., Dolbin I.V., & Karnet Yu.N. (2019). Mechanism of growth of interfacial regions in polymer / carbon nanotube nanocomposites. *Composites: Mechanics, Computations, Applications: An International Journal, 10, 3,* 213–220.

13 Kozlov G.V., Dolbin I.V., & Karnet Yu.N. (2017). The effect of nanofiller structure on the reinforcement degree of polymer/carbon nanotubes nanocomposites. *Nanoscience and Technology: An International Journal, 8, 2,* 123–131.

14 Valiev H.H., Vorobyev V.V., Karnet Yu.N., Kornev Yu.V., & Yumaschev O.B. (2017). Application of quantum-chemical modeling results in experimental investigations of silicone composites. *Materials Physics and Mechanics, 32,* 293–297.

15 Kozlov G.V., Aygubova A.Ch., Magomedov G.M., & Karnet Yu.N. (2015). The effective length of a nanofiller and reinforcement degree of polymer/carbon nanotubes (nanofilaments) nanocomposites. *Nanoscience and Technology: An International Journal, 6, 3,* 203–208.

16 Yudin V.E., Molchanov E.S., Kydralieva K., Ivankova E.M., Elokhovsky V.Yu., & Vaganov G.V. (2013). Effect of the nanoparticles with various structures and morphologies on the thermo-mechanical properties of epoxy-based carbon fiber-reinforced plastics. *Nanoscience and Technology: An International Journal, 4, 1,* 47–58.

17 Valiev H.H., Karnet Yu.N., Kochurov N.L., Parshina M.S., Semenov N.A., Yumaschev O.B., & Yanovsky Yu.G. (2016). Atomic force microscopy and physical-mechanical properties of new elastomer composites. *Materials Physics and Mechanics, 26,* 45–48.

18 Mikitaev A.K., Yanovsky Yu.G., Kozlov G.V., & Karnet Yu.N. (2013). Fractal models of the reinforcement of polymer particulate-filled nanostructured composites by nanoparticle aggregates. *Nanoscience and Technology: An International Journal, 4, 1,* 47–58.

19 Kornev Yu.V., Boiko O.V., Kozlov G.V., Yanovsky Yu.G., & Karnet Yu.N. (2010). Physical model of the behavior of polymer nanostructured composites in nanoindentation. *Nanoscience and Technology: An International Journal, 1, 4,* 291–300.

20 Sapronov O.O., Buketov A.V., Zinchenko D.A., & Yatsyuk V.M. (2017). Features of structural processes in epoxy composites filled with silver carbonate on increase in temperature. *Composites: Mechanics, Computations, Applications. An International Journal, 8,1,* 47–66.

21 Buketov, A.V., Sharko, A.V., Zinchenko, D.A., & Stepanchikov, D.M. (2017). K voprosu optimizatsii inhredientov kompozitnykh materialov na osnove epoksidianovoi smoly [On the issue of optimization of ingredients of composite materials based on epoxy resin]. *Vestnik Karahandinskoho universiteta. Seriia Matematika — Bulletin of the Karaganda University. Mathematics series, 2, 86,* 37–44 [in Russian].

22 Buketov A.V., Sapronov O.O., Brailo M.V., Zinchenko D.O., & Nihalatii V.D. (2017). Investigation of the hydroabrasive wear of epoxy composites with two-component filler. *Materials Science, 53, 1,* 62–66.

23 Buketov, A.V., Akimov, A.V., & Zinchenko, D.A. (2015). Issledovanie fiziko-mekhanicheskikh svoistv kompozitnykh materialov na osnove epoksidianovoi smoly, otverzhdennoi polietilenpoliaminom, s dobavleniem plastifikatora-antipirena trikhloretilfosfata [Investigation of physical and mechanical properties of composite materials based on epoxy resin, cured with polyethylene polyamine, with the addition of plasticizer-flame retardant trichloroethyl phosphate] *Vestnik Khmelnitskoho natsionalnoho universiteta. Tekhnicheskie nauki – Bulletin of Khmelnitsky National University. Technical science. – Khmelnitsky: KhNU, No. 5*, 126–133 [in Russian].

24 Buketov, A.V., Sapronov, A.A., Zinchenko, D.A., & Iatsiuk, V.N. (2015). Vliianie mikrodispersnoho karbonata serebra na svoistva epoksikompozitov. [Effect of microdispersed silver carbonate on the properties of epoxy composites]. *Mekhanika kompozitsionnykh materialov i konstruktsii. — Mechanics of composite materials and structures, 4*, 534–548 [in Russian].

25 Buketov, A.V., Zinchenko, D.O., & Nigalatii, V.D. (2016). Doslidzhennia vplivu mikrodispersnikh antifriktsiinikh napovniuvachiv na adgeziinu mitsnist epoksikompozitiv [Investigation of the effect of microdisperse antifriction fillers on the adhesive strength of epoxy composites]. *Mizhvuzivskii zbirnik «Naukovi notatki». Lutsk — Interuniversity collection «Research notes». Lutsk, 53*, 17–23 [in Ukrainian].

26 Buketov, A.V., Zinchenko, D.O., & Smetankin, S.O. (2017). Nanomodified dispersed particles- and synthetic fibers-filled epoxy composite materials for the metal- polymer tribosystems of transport vehicles. *Nanomechanics Science and Technology: An International Journal, 8, 1,* 41–54.

27 Buketov, A.V., Zinchenko, D.O., & Sharko, O.V. (2017). Vpliv sintetichnikh diskretnikh volokon na vlastivosti epoksidnikh kompozitiv dlia zakhisnikh pokrittiv. [Influence of synthetic discrete fibers on the properties of epoxy composites for protective coatings]. *Zhurnal nano- ta elektronnoï fiziki. Journal of nano-and electronic physics. Sumy-2017, Vol. 9, No. 2,* 02014-1 – 02014-5 [in Ukrainian].

28 Zinchenko, D.O., & Smetankin, S.O. (2018). Nanomodified epoxycomposite materials for metal-polymeric tribosystems of transport vehicles. *Nanoscience and Technology: An International Journal, 9, 1,* 9–17.

29 Buketov, A.V., Maruschak, P.O., Sapronov, O.O., Zinchenko, D.O., Yatsyuk, V.M., & Panin, S. (2016). Enhancing performance characteristics of equipment of sea and river transport by using epoxy composites. *TRANSPORT. Vilnius Gediminas Technical University (VGTU), 31, 3,* 333–342.

30 Buketov, A.V., Sapronov, A.A., Zinchenko, D.A., Iatsiuk, V.N. (2017). Issledovanie vliianiia nanodispersnoho oksida serebra na strukturu i teplofizicheskie svoistva polimernykh kompozitnykh materialov [Investigation of the effect of nanodispersed silver oxide on the structure and thermophysical properties of polymer composite materials]. *Plasticheskie massy. — Plastic masses*, 5–6, 28–37 [in Russian].

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