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The natural solvability of the Navier-Stokes equations

It is known that the three-dimensional Navier-Stokes equations (ENS) the existence theorem of smooth solutions in the presence of smooth data for the whole with respect to time has not proved and the uniqueness theorem is violated in the class of generalized solutions. In a number of works by the author of this article, the results of search studies on the justification of the maximum principle for three-dimensional ENS are given. Over time, these studies have improved and later the justice of the simplest principle for maximum was shown for three-dimensional ENS. A further continuation of the search led to the determination of the relationship between pressure and the square of the velocity vector modulus from the properties of the ENS solutions. On the basis of this the answers to many problematic issues related to the solvability of the ENS were found. And in particular, in the selected spaces, the uniqueness of the weak and the existence of strong solutions of the problem for the three-dimensional Navier-Stokes equations for the whole of time are proved.

Keywords: Navier-Stokes equations, pressure in the Navier-Stokes equations, the uniqueness of weak generalized solutions, the existence of strong solutions.

Some introductory information

Unsolved problems in the theory of Navier-Stokes equations homogeneous fluid are given in [1, 2], etc. The initial-boundary value problem for Navier-Stokes equations [1] with respect to the velocity vector $\mathbf{U} = (U_1, U_2, U_3)$ and the pressure P in the domain $Q = (0, T] \times \Omega$:

$$\frac{\partial \mathbf{U}}{\partial t} - \mu\Delta\mathbf{U} + (\mathbf{U}, \nabla)\mathbf{U} + \nabla P = \mathbf{f}(t, \mathbf{x}),\ \operatorname{div}\mathbf{U} = 0; \tag{1a}$$

$$\mathbf{U}(0, \mathbf{x}) = \mathbf{\Phi}(\mathbf{x}),\ \mathbf{U}(t, \mathbf{x})\big|_{\mathbf{x}\in\partial\Omega} = 0, \tag{1b}$$

where $\mathbf{x} \in \Omega \subset R_3$; Ω — is a convex domain and $\partial\Omega$ is the boundary of Ω, $t \in [0, T]$, $T < \infty$; $\mathring{\mathbf{J}}(\Omega)$ — space solenoidal vectors; $\mathbf{L}_\infty(Q)-$ is the subspace of $\mathbf{C}(\bar{Q})$. $W^k_{p,0}(\Omega)$ is the Sobolev space functions equal to zero on $\partial\Omega$; The input data $\mathbf{f}$ and $\mathbf{\Phi}$ of the problem (1) meet the requirements:

i) $\mathbf{f}(t, \mathbf{x}) \in \mathbf{L}_\infty(0, T; \mathbf{L}_p(\Omega)) \cap \mathring{\mathbf{J}}(Q)$;

ii) $\mathbf{\Phi}(\mathbf{x}) \in \mathbf{L}_p(\Omega) \cap \mathbf{W}^1_{2,0}(\Omega) \cap \mathring{\mathbf{J}}(\Omega)$, $\forall p$.

Further, we use the Holder inequalities

$$\Big|\int\limits_\Omega UV\,\mathbf{dx}\Big| \le \Big(\int\limits_\Omega |U|^p\,\mathbf{dx}\Big)^{\frac{1}{p}} \Big(\int\limits_\Omega |V|^q\,\mathbf{dx}\Big)^{\frac{1}{q}} \tag{2}$$

and Jung for pair products

$$UV \le \frac{1}{\epsilon p}|U|^p + \frac{\epsilon}{q}|V|^q, \quad \epsilon > 0, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{3}$$

in addition, the integration by parts formula

$$\int\limits_{\Omega} V \Delta U \, \mathbf{dx} = -\int\limits_{\Omega} \nabla V \, \nabla U \, \mathbf{dx} + \int\limits_{\partial\Omega} V \frac{\partial U}{\partial \mathbf{n}} \, \mathbf{dx}. \tag{4}$$

1 Explicit relation between pressure and square of the velocity vector module

From the properties of the solutions of the problem (1) a quadratic form connecting the pressure $P(t, \mathbf{x})$ from the components of the vector of the speed $\mathbf{U}(t, \mathbf{x})$:

$$\big(\mathbf{v}, \ \mathbf{B}\mathbf{v}'\big) = 0, \tag{5}$$

where $\mathbf{B} = \big\|U_\alpha U_\beta + \delta_\alpha^\beta P\big\|_{\alpha,\,\beta=1,2,3}$ — is the symmetric matrix; $\mathbf{v} = (v_1, v_2, v_3)$ — an arbitrary vector; the components $\{v_\alpha\}$ consist of arbitrary numbers such that $\sum\limits_{\alpha=1}^{3} v_\alpha^2 \ne 0$; $\mathbf{v}' = \big(v_1, \, v_2, \, v_3\big)'$ — vector column; δ_α^β — the Kronecker symbol.

Using the orthogonal matrix $\mathbf{T}$ from eigenvectors matrix $\mathbf{B}$ the quadratic form (5) is reduced to the sum of squares

$$\big(z_1, z_2, z_3\big)\mathbf{\Lambda}\big(z_1, z_2, z_3\big)' \equiv \big(\mathbf{z}, \mathbf{\Lambda z}\big) \equiv \sum_{\alpha=1}^{3} \lambda_\alpha z_\alpha^2 = 0, \tag{6}$$

where $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3\}$, The columns $\mathbf{T}$ consist of the eigenvectors of the corresponding to the eigenvalues of the matrix $\mathbf{B}$, $\mathbf{z} = \mathbf{T}'\mathbf{v}$. Notice, that $\mathbf{z} = \big(z_1, z_2, z_3\big)-$ is also an arbitrary vector.

Elements of the matrix $\mathbf{\Lambda}$ are determined from of the characteristic equation of the matrix $\mathbf{B}$, that is $\big|\mathbf{B} - \mathbf{\Lambda}\big| = 0$. Whence we obtain cubic equation $\lambda^3 + a\lambda^2 + b\lambda + c = 0$, where $a = -(\big|\mathbf{U}\big|^2 + 3P)$; $b = 2\big|\mathbf{U}\big|^2 P + 3P^2$; $c = -(\big|\mathbf{U}\big|^2 + P)P^2$. The solution of this equation we find using the Cardano formula. For this, substitution $\lambda = y - a/3$ we arrive at the «incomplete» type $y^3 + ry + d = 0$, $r = -1/3|\mathbf{U}|^4$, $d = -2/(27)|\mathbf{U}|^6$. Whence follows, that the discriminant of this equation is equal to zero, that is $D = (r/3)^3 + (d/2)^2 = 0$. Which means all the roots of the «incomplete» the equations are real, and two of them are equal to each other. In fact, $A = \sqrt[3]{-d/2} = 1/3\big|\mathbf{U}\big|^2$; $B = \sqrt[3]{-d/2} = 1/3|\mathbf{U}|^2$; $y_1 = A + B = 2/3|\mathbf{U}|^2$; $y_{2/3} = -(A+B)/2 = -1/3|\mathbf{U}|^2$. From here we find the roots λ_α, $\alpha = \overline{1,3}$ of the original cubic equations: $\lambda_1 = \big|\mathbf{U}\big|^2 + P$; $\lambda_2 = P$; $\lambda_3 = P$.

Now we rewrite the quadratic form reduced to the sum of squares (6)

$$\big(\mathbf{z}, \mathbf{\Lambda z}\big) \equiv \sum_{\alpha=1}^{3} \lambda_\alpha z_\alpha^2 = 0, \ \forall \mathbf{z}.$$

This relation is zero if and only then when $\lambda_\alpha = 0$, $\alpha = \overline{1,3}$. Where does it follow that $\lambda_1 = \big|\mathbf{U}\big|^2 + P = 0$, $\lambda_2 \equiv P_2 = 0$, $\lambda_3 \equiv P_3 = 0$. From here

$$P_1(t, \mathbf{x}) = -\big|\mathbf{U}\big|^2 \equiv -2E; \quad P_2(t, \mathbf{x}) = 0; \quad P_3(t, \mathbf{x}) = 0. \tag{7}$$

2 Estimations of the solution of the problem (1)

Theorem 1. If the input data of the problem (1) satisfy the requirements **i**), **ii**)*, then for the solutions of the problem (1) the following estimate holds:*

$$\|\mathbf{U}\|_{\mathbf{L}_\infty(Q)} \le \|\mathbf{\Phi}\|_{\mathbf{L}_\infty(\Omega)} + T\|\mathbf{f}\|_{\mathbf{L}_\infty(Q)} \equiv A, \ \forall T < \infty. \tag{8}$$

Proof. We multiply the scalar equation (1a) by the vector function $pE^{p-1}\mathbf{U}$, the product is integrated over the domain Ω and use the identity $E^p = \frac{1}{2^p}|\mathbf{U}|^{2p}$, then

$$\frac{1}{2^p}\frac{d}{dt}\int\limits_{\Omega} |\mathbf{U}|^{2p} \, \mathbf{dx} - p\mu \int\limits_{\Omega} \Delta \mathbf{U}\mathbf{U} E^{p-1} \mathbf{dx} + p \int\limits_{\Omega} (\mathbf{U}, \nabla)\mathbf{U} E^{p-1} \mathbf{U} \mathbf{dx} +$$

$$+p\int\limits_{\Omega} E^{p-1}\nabla P\mathbf{U}\mathbf{dx}\ = p\int\limits_{\Omega} E^{p-1}\mathbf{U}\mathbf{f}\mathbf{dx},\quad t\in(0,T]. \tag{9}$$

Each term (9) is transformed by integration by parts (4). In estimating the fourth term on the left-hand side of we take into account (7). We estimate the right-hand side by Holder's inequality (2) and, as a result, we get:

$$p\int\limits_{\Omega}(\frac{\partial\mathbf{U}}{\partial t},\mathbf{U})E^{p-1}d\mathbf{x}=\frac{1}{2^p}\frac{d}{dt}\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x}; \tag{10}$$

$$-p\mu\int\limits_{\Omega}(\Delta\mathbf{U},\mathbf{U})E^{p-1}d\mathbf{x}=p\mu\int\limits_{\Omega}E^{p-1}\sum_{\alpha=1}^{3}(\nabla U_\alpha)^2d\mathbf{x}+p(p-1)\mu\int\limits_{\Omega}E^{p-2}(\nabla E)^2d\mathbf{x}\geq 0'; \tag{11}$$

$$p\int\limits_{\Omega}(\mathbf{U},\nabla)\mathbf{U}E^{p-1}\mathbf{U}d\mathbf{x}=\int\limits_{\Omega}\mathbf{U}\nabla E^pd\mathbf{x}=-\int\limits_{\Omega}div\mathbf{U}E^pd\mathbf{x}+\int\limits_{\partial\Omega}(\mathbf{U},\mathbf{n})E^pd\mathbf{x}=0'; \tag{12}$$

$$p\int\limits_{\Omega}E^{p-1}\mathbf{U}\nabla Pd\mathbf{x}=-2p\int\limits_{\Omega}E^{p-1}\mathbf{U}\nabla Ed\mathbf{x}=-2\int\limits_{\Omega}\mathbf{U}\nabla E^pd\mathbf{x}=0; \tag{13}$$

$$p\int\limits_{\Omega}E^{p-1}\mathbf{U}\mathbf{f}d\mathbf{x}\leq\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{f}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}. \tag{14}$$

From the identity (9), taking into account the relation (10)–(14), we have the estimate

$$\frac{1}{2^p}\frac{d}{dt}\int\limits_{\Omega}|\mathbf{U}|^{2p}\,\mathbf{dx}+p\mu\int\limits_{\Omega}E^{p-1}\sum_{\alpha=1}^{3}(\nabla U_\alpha)^2d\mathbf{x}+p(p-1)\mu\int\limits_{\Omega}E^{p-2}(\nabla E)^2d\mathbf{x}\leq$$

$$\leq\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{f}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}},\quad t\in(0,T]. \tag{15}$$

Because of the nonnegativity of the second and third terms on the left-hand side of (15), from which we proceed to the strengthened inequality.

$$\frac{1}{2^p}\frac{d}{dt}\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\leq\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{f}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}. \tag{16}$$

Both parts (16), dividing by a positive integral $\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}$, we write

$$\frac{d}{dt}\left(\int\limits_{\Omega}\left|\mathbf{U}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}\leq\left(\int\limits_{\Omega}\left|\mathbf{f}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}.$$

We integrate over t in the range from 0 to t and taking parity of exponents, leaving p behind it, we obtain

$$\left(\int\limits_{\Omega}\left|\mathbf{U}(t,\mathbf{x})\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}}\leq\left(\int\limits_{\Omega}\left|\mathbf{\Phi}(\mathbf{x})\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}}+\int\limits_{0}^{t}\left(\int\limits_{\Omega}\left|\mathbf{f}(\tau,\mathbf{x})\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}}d\tau,\forall p=2m,m\in N.$$

Hence we have

$$\left\|\mathbf{U}\right\|_{L_\infty(0,T;L_p(\Omega))}\leq\left\|\mathbf{\Phi}(\mathbf{x})\right\|_{L_p(\Omega)}+T\left|\mathbf{f}\right\|_{L_\infty(0,T;L_p(\Omega))},\ \forall p=2m. \tag{17}$$

Whence for $p=\infty$ we arrive at the proof of the theorem 1.

Corollary 1. For the solutions of the problem (1) the following estimates hold:

$$\|\mathbf{U}\|_{\mathbf{L}_\infty(0,T;L_2(\Omega))} \leq \|\mathbf{\Phi}\|_{\mathbf{L}_2(\Omega)} + T\|\mathbf{f}\|_{\mathbf{L}_\infty(0,T;\mathbf{L}_2(\Omega))} \equiv A_1,\ \forall T < \infty; \tag{18}$$

$$\int\limits_0^t \sum_{\alpha=1}^3 \big\|\nabla U_\alpha(\tau)\big\|^2_{L_2(\Omega)} d\tau \leq \frac{1}{\mu}\Big(\big\|\mathbf{\Phi}\big\|^2_{L_2(\Omega)} + T/2(1+T)\big\|\mathbf{f}\big\|^2_{L_\infty(0,T;L_2(\Omega))}\Big) = A_2; \tag{19}$$

$$\big\|\mathbf{U}\big\|^3_{L_\infty(0,T;L_4(\Omega))} \leq 3\Big(\big\|\mathbf{\Phi}\big\|^3_{L_4(\Omega)} + T^3\big\|\mathbf{f}\big\|^3_{L_\infty(0,T;L_4(\Omega))}\Big) = A_3; \tag{20}$$

$$\int\limits_0^t \big\|\nabla E(\tau)\big\|^2_{L_2(\Omega)} d\tau \leq \frac{1}{2\mu}\Big(\big\|\mathbf{\Phi}\big\|_{L_{4(\Omega)}} + TA_3\big\|\mathbf{f}\big\|_{L_\infty(0,T;L_4(\Omega))}\Big) = A_4, \quad t \in (0,T]. \tag{21}$$

Proof. Estimates (18), (20) follow from (17) respectively for $p = 1$ and $p = 4$. To prove (19) from (15) for $p = 1$, we have

$$\frac{1}{2}\frac{d}{dt}\int\limits_\Omega |\mathbf{U}|^2\,\mathbf{dx} + \mu\int\limits_\Omega \sum_{\alpha=1}^3 (\nabla U_\alpha)^2 d\mathbf{x} \leq \Big(\int\limits_\Omega \big|\mathbf{U}\big|^2 d\mathbf{x}\Big)^{\frac{1}{2}}\Big(\int\limits_\Omega \big|\mathbf{f}\big|^2 d\mathbf{x}\Big)^{\frac{1}{2}}.$$

We integrate over t in the range from 0 to t,

$$\frac{1}{2}\int\limits_\Omega |\mathbf{U}(t)|^2\,\mathbf{dx} + \mu\int\limits_0^t \sum_{\alpha=1}^3 \int\limits_\Omega (\nabla U_\alpha)^2 d\mathbf{x}d\tau \leq \frac{1}{2}\int\limits_\Omega |\Phi|^2\,\mathbf{dx} + \int\limits_0^t (\int\limits_\Omega \big|\mathbf{U}\big|^2 d\mathbf{x})^{\frac{1}{2}}(\int\limits_\Omega \big|\mathbf{f}\big|^2 d\mathbf{x})^{\frac{1}{2}} d\tau, \quad t \in (0,T].$$

Hence, since the first integral on the left-hand side is nonnegative, we get

$$\mu\int\limits_0^t \sum_{\alpha=1}^3 \int\limits_\Omega \big\|\nabla U_\alpha(\tau)\big\|^2_{L_2(\Omega)} d\tau \leq \frac{1}{2}\big\|\mathbf{\Phi}\big\|^2_{L_2(\Omega)} + \big\|\mathbf{U}\big\|_{L_\infty(0,T;L_2(\Omega))}\int\limits_0^t \big\|\mathbf{f}(\tau)\big\|_{L_2(\Omega)} d\tau, \quad t \in (0,T].$$

From this, using the inequalities $2ab \leq (a^2 + b^2)$, (18), we arrive at (19).

To prove (20), the estimate (15) is written for $p = 2$

$$\frac{1}{4}\frac{d}{dt}\int\limits_\Omega |\mathbf{U}|^4\,\mathbf{dx} + \mu\int\limits_\Omega E\sum_{\alpha=1}^3 (\nabla U_\alpha)^2 d\mathbf{x} + 2\mu\int\limits_\Omega (\nabla E)^2 d\mathbf{x} \leq \Big(\int\limits_\Omega \big|\mathbf{U}\big|^4 d\mathbf{x}\Big)^{\frac{3}{4}}\Big(\int\limits_\Omega \big|\mathbf{f}\big|^4 d\mathbf{x}\Big)^{\frac{1}{4}}, \quad t \in (0,T].$$

We integrate over t in the range from 0 to t, and then, as in the previous case, we find

$$2\mu\int\limits_0^t \int\limits_\Omega (\nabla E)^2 d\mathbf{x}d\tau \leq \big\|\mathbf{\Phi}\big\|_{L_{4(\Omega)}} + T\,A_3\big\|\mathbf{f}\big\|_{L_\infty(0,T;L_4(\Omega))}, \quad t \in (0,T].$$

Hence we come to (21).

3 Weak generalized solutions

We multiply the equation (1a) by an arbitrary vector-valued function

$$\mathbf{Z}(t,\mathbf{x}) \in \mathbf{L}_\infty(Q) \cap W_2^1(Q) \cap \mathring{\mathbf{J}}(Q),$$

equal to zero for $\big(t = T\big) \wedge \big(\mathbf{x} \in \partial\Omega\big)$. The product is integrable over the domain $Q = [0,T] \times \Omega$ and with by integrating by parts, taking into account the conditions (1) from We transfer the first two terms from $\mathbf{U}$ to $\mathbf{Z}$. As a result, we get

$$\int\limits_Q \Big(-\mathbf{U}\,\frac{\partial \mathbf{Z}}{\partial t} + \mu\sum_{k=1}^3 \nabla U_k \nabla Z_k + \big((\mathbf{U},\nabla)\mathbf{U} + \nabla P\big)\mathbf{Z}\Big)\,\mathbf{dx}\,dt =$$

$$= \int_\Omega \mathbf{\Phi}\mathbf{Z}(0,\mathbf{x})\mathbf{dx} + \int_Q \mathbf{fZdx}\,dt. \tag{22}$$

Definition 1. We call the vector-function $\mathbf{U}$ *of spaces weakly generalized solution of the initial-boundary value problem for the Navier-Stokes equations (1a) from the spaces*

$$\mathbf{U} \in \mathbf{L}_\infty(Q) \cap \mathbf{L}_\infty\big(0,T;\mathbf{W}^1_{2,0}(\Omega)\big) \cap \mathring{\mathbf{J}}(Q);\ \forall t \in [0,T]\big) \tag{23}$$

and satisfying the identity (22) for any

$$\mathbf{Z}(t,\mathbf{x}) \in \mathbf{L}_\infty(Q) \cap W^1_2(Q) \cap \mathring{\mathbf{J}}(Q) \wedge \left(\mathbf{Z}\,\Big|_{(t=T)\wedge(\mathbf{x}\in\partial\Omega)} = 0\right).$$

The validity of the definition 1 follows from the fact that all the integrals occurring in (22) are finite for any $\mathbf{Z}$, from the class indicated.

From Theorem 1 and Corollary 1, the uniqueness of weak generalized solutions of the problem (1).

Theorem 2. If the input data $\mathbf{f}$ and $\mathbf{\Phi}$ satisfy the requirements **i**) and **ii**), then the problem (1) has the unique weak generalized solution $\mathbf{U}$ satisfying the identity (22) for any $\mathbf{Z}$ from the definition 1.

Proof. Let the functions $\mathbf{U}$ and $\mathbf{U}^*$ be two solutions of the problem (1). We set $\mathbf{V} = \mathbf{U} - \mathbf{U}^*$; $\nabla R = 2\nabla(E^* - E)$, then we have:

$$\frac{\partial \mathbf{V}}{\partial t} - \mu\Delta\mathbf{V} + (\mathbf{V},\nabla)\mathbf{U} + (\mathbf{U}^*,\nabla)\mathbf{V} + \nabla R = 0; \tag{24a}$$

$$\mathbf{V}(0,\mathbf{x}) = 0, \quad \mathbf{V}(t,\mathbf{x})\big|_{\partial\Omega} = 0, \quad \mathbf{x} \in \partial\Omega. \tag{24b}$$

From the equations (24a) we pass to the identity

$$\int_{Q_t} \left(\frac{\partial \mathbf{V}}{\partial t}\mathbf{V} - \mu\Delta\mathbf{V}\mathbf{V} + (\mathbf{V},\nabla)\mathbf{U}\mathbf{V} + (\mathbf{U}^*,\nabla)\mathbf{V}\mathbf{V} + \nabla R\mathbf{V}\right)\mathbf{dx}d\tau = 0, \quad \forall t \in (0,T]. \tag{25}$$

We transform all terms by integration by parts. As $\mathbf{U},\mathbf{U}^* \in \mathring{\mathbf{J}}(Q)$, thereby $\mathbf{V} \in \mathring{\mathbf{J}}(Q)$, then

$$\int_{Q_t} (\mathbf{U}^*,\nabla)\mathbf{V}\mathbf{V}\mathbf{dx} = 0, \quad \int_{Q_t} \nabla R\mathbf{V}\mathbf{dx} = 0.$$

From (25) we find

$$\frac{1}{2}\big\|\mathbf{V}(t)\big\|^2_{\mathbf{L}_2(\Omega)} + \mu\sum_{k=1}^3 \int_0^t \|\nabla V_k(\tau)\|^2_{\mathbf{L}_2(\Omega)}d\tau = -\int_{Q_t}\sum_{k,\beta=1}^3 V_\beta\frac{\partial V_k}{\partial x_\beta}U_k\mathbf{dx}d\tau. \tag{26}$$

The integral on the right-hand side is estimated successively by the Holder inequality for $p = \infty$ and $q = 1$, as well as Young's (3) for $p = 2$ as a result we obtain the chain of inequalities

$$\Big|\int_{Q_t}\sum_{k,\beta=1}^3 V_\beta\frac{\partial V_k}{\partial x_\beta}U_k\mathbf{dx}d\tau\Big| \le \max_k \|U_k\|_{L_\infty(Q)}\sum_{k,\beta=1}^3\int_{Q_t}\Big|\frac{\partial V_k}{\partial x_\beta}\Big|\big|V_\beta\big|\mathbf{dx}d\tau \le$$

$$\le A\epsilon/2\sum_{k,\beta=1}^3\int_0^t\Big\|\frac{\partial V_k}{\partial x_\beta}\Big\|^2_{L_2(\Omega)}d\tau + A_5\int_0^t\sum_{\beta=1}^3\big\|V_\beta\big\|^2_{L_2(\Omega)}d\tau \le$$

$$\le A\epsilon/2\sum_{k=1}^3\int_0^t\big\|\nabla V_k(\tau)\big\|^2_{\mathbf{L}_2(\Omega)}d\tau + A_5\int_0^t\big\|\mathbf{V}(\tau)\big\|^2_{\mathbf{L}_2(\Omega)}d\tau, \quad A_5 = 3A/(2\epsilon).$$

Taking into account the estimates (8), (19) and, using the latter for $\epsilon = 2\mu/A$ from (26), we find

$$\|\mathbf{V}(t)\|^2_{\mathbf{L}_2(\Omega)} \le A_5 \int_0^t \|\mathbf{V}(\tau)\|^2_{\mathbf{L}_2(\Omega)} d\tau,\ A_5 = 3A^2/(4\mu), \quad \forall t \in (0,T].$$

Whence we have $\frac{d}{dt}\Big(\exp(-A_5 t)\|\mathbf{V}(t)\|^2_{\mathbf{L}_2(\Omega)}\Big) \le 0$. From this inequality we conclude that $\mathbf{V} \equiv 0$, $\forall t \in (0,T]$, that is, that the solutions $\mathbf{U}$ and $\mathbf{U}^*$ match. The theorem 2 is proved.

4 Strong solutions

Definition 2. If in a domain Q a weak generalized solution of the initial-boundary value problem for Navier-Stokes equations has all possible generalized derivatives of the same order as the equations themselves, then this solution is called strong.

Theorem 3. If the input data of the problem (1) satisfies the requirements **i**), **ii**) and $\partial\Omega \in C^2$, then the problem (1) has a unique strong generalized solution $\mathbf{U}$ from spaces

$$\mathbf{U} \in \mathbf{W}^{2,1}_{2,0}(Q) \cap \mathring{\mathbf{J}}_\infty(Q), \forall t \in [0,T]\big),$$

satisfying the equations (1a) almost everywhere in Q, and for them the following estimates hold:

$$\|\mathbf{U}_t\|^2_{\mathbf{L}_2(Q)} \le \mu \sum_{k=1}^{3} \|\nabla\Phi_k\|^2_{\mathbf{L}_2(\Omega)} + 3T\|\mathbf{f}\|^2_{\mathbf{L}_\infty(0,T;\mathbf{L}_2(\Omega))} + 3(AA_2 + 4A_4) \equiv A_6; \tag{27}$$

$$\|\Delta\mathbf{U}\|^2_{\mathbf{L}_2(Q)} \le A_6/\mu^2 \equiv A_7; \tag{28}$$

$$\|\nabla U_k\|^2_{\mathbf{L}_\infty(0,T;\mathbf{L}_2(\Omega))} \le A_6/\mu \equiv A_8,\ k = \overline{1,3}; \tag{29}$$

$$\|\mathbf{U}\|_{\mathbf{L}_2(0,T;\mathbf{W}^2_2(\Omega))} \le A_9\|\Delta\mathbf{U}\|_{\mathbf{L}_2(Q)}; \quad A_9 - const. \tag{30}$$

Proof. In order to establish the inequalities (27) from the equation (1a), we pass to the identity

$$\int_{Q_t} \big(\mathbf{U}_t - \mu\Delta\mathbf{U}\big)^2 \mathbf{dx}\, d\tau = \int_{Q_t} \big(\mathbf{f} - (\mathbf{U},\nabla)\mathbf{U} + 2\nabla E\big)^2 \mathbf{dx} d\tau. \tag{31}$$

We will square the integrands. After that the pair product on the left side is transformed by integration by parts. On the right side, Young's inequality for $\epsilon = 1$ and $p = 2$. Then from (31) we pass to the inequality

From the last inequality, taking into account estimates

$$3\int_0^t \int_\Omega |(\mathbf{U},\nabla)\mathbf{U}|^2 \mathbf{dx} d\tau \le 3\|\mathbf{U}\|_{L_\infty(Q)} \sum_{k=1}^{3} \int_0^t \|\nabla U_k(\tau)\|^2_{L_2(\Omega)} d\tau = 3AA_2$$

and the estimates (8), (19) and (21), we obtain estimates (27)–(29) for strong generalized solutions of the problem (1). And note that (29) is slightly better than estimates (19).

Since the boundary of the domain $\partial\Omega \in C^2$ is found to be an estimate (30), using the inequalities from [1; 26], which is valid for any functions $U(x) \in W^2_2(\Omega) \cap W^2_{2,0}(\Omega)$: The theorem 3 is proved.

Remark. As a result, we were convinced that the properties (7) together with the estimate (8) allows one to find answers to many problematic questions connected with the solvability of the problem (1). In addition, (8) confirms the validity of the maximum principle for (1a) shown in [3–6] and on the basis of which the obtained results from the same papers.

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Ә.Ш. Ақыш (Ақышев)

Навье-Стокс теңдеулерінің табиғи жағдайда шешілетіндігі

Үшөлшемді Навье-Стокс теңдеулерінің (НСТ) деректері сыптығыр болғанымен, ұзақ уақыт бойы сыптығыр шешімдерінің табылатындығы дәлелденбегені және жалпылама шешімдер класында жалқылық теоремасының орындалмайтыны туралы мәліметтер белгілі. Үшөлшемді НСТ-не максимум қағидасын негіздеуге мақала авторының біраз жұмыстарында зерттеу ізденістерінің нәтижелері келтірілген. Бұл зерттеулер жылдар бойы жетілдіре дамытылып, нәтижесінде НСТ-ға максимум қағидасының өте жеңіл түрі орындалатындығы көрсетілген. Ізденісті жалғастыру барысында НСТ шешімдерінің қасиеттерінен қысым мен жылдамдық векторы модулі квадратының арақатынас байланысы табылған. Бұл нәтиже негізінде НСТ-ның шешілетіндігі жөніндегі көптеген өзекті мәселелерге жауап алынды. Зерттеушінің таңдаған кеңістігінде үшөлшемді НСТ-ға қойылған есептің әлсіз шешімінің жалқылығы мен әлді шешімінің ұзақ бойы табылатындығы дәлелденген.

Кілт сөздер: Навье-Стокс теңдеулері, Навье-Стокс теңдеулеріндегі қысым, әлсіз жалпылама шешімнің жалқылығы, әлді шешімнің табылатындығы.

А.Ш. Акыш (Акишев)

Естественная разрешимость уравнений Навье-Стокса

Известно, что для трехмерных уравнений Навье-Стокса (УНС) не доказаны существование в целом по времени гладких решений при наличии гладких данных, а в классе обобщенных решений о нарушении теорема единственности. Автором статьи ранее приведены результаты поисковых исследований по обоснованию принципа максимума для трехмерных УНС. Со временем результаты этих исследований улучшались, и впоследствии была доказана справедливость простейшего принципа максимума для трехмерных УНС. Дальнейшее исследование позволило установить из свойств решений УНС соотношение между давлением и квадратом модуля вектора скорости, на основе чего найдены ответы на многие проблемные вопросы, связанные с разрешимостью УНС. В частности, в выбранных пространствах доказаны единственность слабых и существование сильных решений задачи для трехмерных уравнений Навье-Стокса в целом по времени.

Ключевые слова: уравнения Навье-Стокса, давление в уравнениях Навье-Стокса, единственность слабых обобщенных решений, существование сильных решений.

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Maximal regularity and compactness conditions for a high order system of difference equations

In this paper we study an infinite linear system of difference equations of high even order with the right-hand side from the Hilbert space of numerical sequences. Sequences formed from the coefficients of the equations of the system for the same orders of difference can be unlimited, and their growth may not be subject to the growth of the potential. The previously developed methods, which essentially use the dominant potential growth in the difference systems of Sturm-Liouville type equations, do not pass here, since In the case under consideration, the potential may turn out to be zero, or not having a definite sign by a sequence. We give conditions for the correct solvability of the system, as well as optimal estimates of the norms of the solution and its differences up to the highest order. Conditions for the compactness of the resolvent of the corresponding system of a degenerate operator are obtained. We prove some difference weight inequalities of Hardy type having independent scientific interest. They are used in the proof of the main results of the paper. It is shown that, in comparison with degenerate differential equations, in the case of a difference system, it is possible to remove the condition for oscillations of the coefficients of the system.

Keywords: difference system, intermediate coefficient, correctness of solution, maximum regularity, compactness of resolution.

1 Introduction and main results

The present paper is devoted to the study of the correct solvability and differential properties of the solution of the following high order infinite system of difference equations:

$$L_0 y = \Delta^{(2n)} y + r\Delta^{(2n-1)} + s\overline{\Delta^{(2n-1)} y} + \sum_{j=1}^{2n-1} \left(Q^{(j)} \Delta^{(2n-j-1)} y + P^{(j)} \overline{\Delta^{(2n-j-1)}} \right) = f, \tag{1.1}$$

where

$$y = \{y_k\}_{k=-\infty}^{+\infty}, \quad \Delta_+ y_k = y_{(k+1)} - y_k, \quad \Delta^{(2)} y_k = \Delta_- \Delta_+ y_k = y_{k+1} - 2y_k + y_{k-1}\,(k \in Z)\,,$$

$$\Delta^{(2s)} y = \Delta^{(2)} \Delta^{(2s-2)} y, \quad \Delta^{(2s-1)} y = \Delta_+ \underbrace{\Delta^{(2)} \Delta^{(2)} \dots \Delta^{(2)}}_{(s-1)}\,(s \in N)\,,$$

and

$$r = \{\mathrm{diag}, r_{jj}\}_{j=-\infty}^{+\infty}, \quad s = \{\mathrm{diag}, s_{jj}\}_{j=-\infty}^{+\infty},$$

$$Q^{(\theta)} = \{\mathrm{diag}, q_{jj}^{(\theta)}\}_{j=-\infty}^{+\infty}, \quad P^{(\theta)} = \{\mathrm{diag}, p_{jj}^{(\theta)}\}_{j=-\infty}^{+\infty}, \quad \theta = \overline{1, 2n-1}$$

are given diagonal matrices, $f \in l_2$.

Many dynamic problems in practice, according to the nature of the formulation, are given either to differential equations, or to infinite difference systems, or to differential-difference equations. Functional analysis accelerated the development of the theory of infinite systems of difference equations. Recently, much attention is paid to the study of differential equations and second-order systems with unlimited intermediate coefficients, in view of their important applications. In addition, questions of modeling the propagation of vibrations in viscoelastic and compressible media [1, 2], as well as some problems in the theory of stochastic processes and stochastic differential equations [3–6] lead to them. Known representatives of such equations are, for example, the Fokker-Planck equation and the Oinstein-Uhlenbeck equation used to describe the Brownian motion. Along with this, a number of issues of the solution of second-order systems depend on infinite higher-order difference equations.

We present the main results of this paper. Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $u = \{\text{diag}, u_n\}_{n=-\infty}^{+\infty}$, $v = \{\text{diag}, v_n\}_{n=-\infty}^{+\infty}$, sequence of real numbers. We introduce the following notations.

$$T_{m,u,v} = \sup_{n=0,1,2,\dots} \left(\sum_{j=0}^{n} |u_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=n}^{+\infty} \left(j^{(m-1)p'} \right)^{p'} |v_j|^{-p'} \right)^{\frac{1}{p'}};$$

$$T''_{m,u,v} = \sup_{\tau<0} \left(\sum_{j=\tau}^{0} |u_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=-\infty}^{\tau} |j|^{(m-1)p'} |v_j|^{-p'} \right)^{\frac{1}{p'}};$$

$$\gamma_{m,u,v} = \sqrt[p]{\max\left[(T_{m,u,v})^p, (T''_{m,u,v})^p \right]}, \ (m = 2, 3, \dots).$$

We denote by $\widetilde{l}$ the set of all finite sequences of real numbers.

Definition 1.1. $y = \{y_j\}_{j=-\infty}^{+\infty} \in l_2$ is called a solution of the system (1.1), if there exists a sequence $\{z^{(k)}\}_{k=1}^{+\infty} \subset \widetilde{l}$ such that $\|z^{(k)} - y\|_2 \to 0$, $\|L_0 z^{(k)} - f\|_2 \to 0$ $(k \to +\infty)$.

Theorem 1.1. Let the sequences $\widetilde{r} = \{r_{jj}\}_{j=-\infty}^{+\infty}$, $\widetilde{s} = \{s_{jj}\}_{j=-\infty}^{+\infty}$, $\widetilde{Q}^{(\theta)} = \{q_{jj}^{\theta}\}_{j=-\infty}^{+\infty}$, $\widetilde{P}^{(\theta)} = \{p_{jj}^{\theta}\}_{j=-\infty}^{+\infty}$; $(\theta = \overline{1, 2n-1})$ satisfy the following conditions:

$$\max\left(\gamma_{2n-1,e,\widetilde{r}}, \gamma_{\theta,\widetilde{Q}^{(\theta)},\widetilde{r}}, \gamma_{\theta,\widetilde{P}^{(\theta)},\widetilde{r}} \right) < \infty \quad (\theta = \overline{1, 2n-1}); \tag{1.2}$$

$$|s_{jj}| \le \alpha_1 r_{jj} \quad (j \in Z), \quad 0 < \alpha_1 < \frac{1}{5\sqrt{2}}, \tag{1.3}$$

where $e = \{e_n\}_{n=-\infty}^{+\infty}$, $e_n = 1 \ \forall n \in Z$. Then there exists a unique solution of the system (1.1). Furthermore, the solution satisfies the following estimate:

$$\left\| \Delta^{(2n)} y \right\|_2 + \left\| r \Delta^{(2n-1)} y \right\|_2 + \left\| s \overline{\Delta^{(2n-1)} y} \right\|_2 + $$
$$+ \sum_{j=1}^{2n-1} \left(\left\| Q^{(j)} \Delta^{(2n-j-1)} y \right\|_2 + \left\| P^{(j)} \Delta^{(2n-j-1)} y \right\|_2 \right) \le C_1 \|L_0 y\|_2 . \tag{1.4}$$

We denote by L the closure in l_2 of the following difference expression

$$L_0 y = \Delta^{2n} y + r \Delta^{(2n-1)} y + s \overline{\Delta^{(2n-1)} y} + \sum_{j=1}^{2n-1} \left(Q^{(j)} \Delta^{(2n-j-1)} y + P^{(j)} \overline{\Delta^{(2n-j-1)} y} \right),$$

originally defined on a set $\widetilde{l}$ of all finite sequences. If the conditions of Theorem 1.1 hold, then there exists an inverse operator L^{-1} to L and it is continuous. The following assertion is important in questions of the approximate solution of the system (1.1).

Theorem 1.2. Let all of the conditions of Theorem 1.1 be satisfied and

$$\lim_{n\to\infty} \left(n \cdot \sum_{j=n}^{+\infty} r_{jj}^{-2} \right) = 0; \tag{1.5}$$

$$\lim_{k\to\infty} \left(k \cdot \sum_{j=-\infty}^{k} r_{jj}^{-2} \right) = 0. \tag{1.6}$$

Then the operator L_{-1} is compact in space l_2.

First part of this paper is devoted to study of the new difference weighted Hardy inequalities. In the second part, we apply this results together with the operator methods and theorems on small perturbations, to the proof of Theorems 1.1 and 1.2. A review of previous results obtained in these directions is contained in [7]. When $n = 1$ and $h = 1$, Theorem 1.1 coincides with the results of [7]. In the case of higher-order elliptic differential equations, the question of maximal regularity (coercivity) was considered in [8, 9]. One of the achievements of Theorem 1.1 obtained for the difference analogue of these equations, is that it removes the restriction on the oscillations of the coefficients. Qualitative research on systems of infinite difference equations can be found in [7, 10] and references therein, and the weight inequalities associated with these systems and questions of compactness in Sobolev difference spaces can be found in [11, 12].

2 Some weighted difference inequalities

Let $\widetilde{l}_+ = \{\{w_n\}_{n=-\infty}^{+\infty} \in \widetilde{l} : w_k = 0, \forall k < 0\}$

Lemma 2.1. Let $y = \{y_n\}_{n=-\infty}^{+\infty} \in \widetilde{l}_+$, and the numbers $P_{s,k}$ $(s \in N, k = 0,1,2,\dots)$ are defined as follows:

$$P_{1,k} = 1, P_{2,k} = k, P_{3,k} = \frac{k(k+1)}{2}, P_{m,k} = \sum_{j=0}^{k} P_{m-1,j} \quad (m = 4,5,\dots). \tag{2.1}$$

Then holds the following equality:

$$y_n = \sum_{k-n}^{+\infty} P_{m,k-n}(-\Delta)^{(m)} y_k \quad (n = 0,1,2,\dots), \tag{2.2}$$

where m — is a fixed natural number.

Proof. Let $\{a_k\} \in \widetilde{l}_+$. If

$$y_n = \sum_{k=n}^{+\infty} a_k,$$

then $a_n = -\Delta y_n$. Therefore,

$$y_n = \sum_{k=n}^{+\infty} (-\Delta)\, y_k \quad (n = 0,1,2,\dots). \tag{2.3}$$

If we put $-\Delta y_k = z_k$, then by (2.3)

$$z_k = \sum_{s=k}^{+\infty} (-\Delta)\, z_s.$$

From here

$$y_n = \sum_{k=n}^{+\infty} \sum_{s=k}^{+\infty} (-\Delta)^{(2)}\, y_s = \sum_{s=n}^{+\infty} \sum_{k=n}^{s} (-\Delta)^{(2)}\, y_s =$$

$$= \sum_{s=n}^{+\infty} (s-n)\, (-\Delta)^{(2)}\, y_s, \quad n \geq 1.$$

Continuing this process, we obtain equalities (2.1) and (2.2). The lemma is proved.

We take the sequence $v = \{v_j\}_{j=0}^{+\infty}$, $v_j \neq 0$ $(j = 0,1,\dots)$. We denote by $\widetilde{H}_{p,v}^{(k)}$ $(1 < p < \infty, k \in N)$ the space with a norm

$$\|a\|_{\widetilde{H}_{p,v}^{(k)}} = \left(\sum_{s=0}^{+\infty} \left| v_s \Delta^{(k)} a_s \right|^p \right)^{\frac{1}{p}} \left(a = \{a_s\}_{s=0}^{+\infty}\right).$$

Lemma 2.2. If $y = \{y_n\}_{n=-\infty}^{+\infty} \in \widetilde{l}_+$ and $(m \in N)$, then

$$\sup_{\|y\|_{\widetilde{H}_{p,v}^{(m)}} = 1} |y_n| = \left(\sum_{s=n}^{+\infty} |P_{m,s-n}|^{p'} \right)^{\frac{1}{p'}} (n = 0,1,2,\dots,). \tag{2.4}$$

Proof. From (2.2) by Holder inequality:

$$|y_n| \leq \left(\sum_{k=n}^{+\infty} P_{m,k-n}^{p'} |v_k|^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{k=n}^{+\infty} |v_k|^p \left| (-\Delta)^{(m)}\, y_k \right|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=n}^{+\infty} P_{m,k-n}^{p'} |v_k|^{-p'} \right)^{\frac{1}{p'}} \|y\|_{\widetilde{H}_{p,v}^{(m)}},$$

so

$$\sup_{\|y\|_{\widetilde{H}_{p,v}^{(m)}} = 1} |y_n| \leq \left(\sum_{k=n}^{+\infty} P_{m,k-n}^{p'} |v_k|^{-p'} \right)^{\frac{1}{p'}}. \tag{2.5}$$

We choose the sequence $\widetilde{y} = \{\widetilde{y_{n,j}}\}_{j=0}^{+\infty}$ by the equalities:

$$(-\Delta)^{(m)}\, \widetilde{y}_{n,j} = \begin{cases} P_{m,j-n}^{p'-1} \left|v_j\right|^{-p'}, \text{if } j \in [n, N],\ N \geq n+1; \\ 0, \text{if } j \notin [n, N]. \end{cases} \tag{2.6}$$

Then by (2.2) $\widetilde{y}_{n,j} = \sum_{s=j}^{N} P_{m,s-j} \left(P_{m,s-j}\right)^{p'-1} \left|v_s\right|^{-p'}$, and when $j = n$

$$\widetilde{y}_{n,n} = \sum_{s=n}^{N} P_{m,s-n}^{p'} \left|v_s\right|^{-p'}. \tag{2.7}$$

Further

$$\|y\|_{\widetilde{H}_{p,v}^{(m)}}^{p} = \sum_{s=n}^{N} \left[|v_s| \left(P_{m,s-j}^{p'-1} \left|v_s\right|^{-p'} \right) \right]^{p} = \sum_{s=n}^{N} P_{m,s-n}^{(p'-1)p} \left|v_s\right|^{(1-p')p} = \sum_{s=n}^{N} P_{m,s-n}^{p'} \left|v_s\right|^{-p'}.$$

By this equality and (2.7),

$$\sup_{\|y\|_{\widetilde{H}_{p,v}^{(m)}}=1} |y_n| \geq \frac{|\widetilde{y}_{n,n}|}{\|\widetilde{y}_{n,n}\|_{\widetilde{H}_{p,v}^{(m)}}} = \frac{\sum_{s=n}^{N} P_{m,s-n}^{p'} \left|v_s\right|^{-p'}}{\left[\sum_{s=n}^{N} P_{m,s-n}^{p'} \left|v_s\right|^{-p'}\right]^{\frac{1}{p}}} = \left(\sum_{s=n}^{N} P_{m,s-n}^{p'} \left|v_s\right|^{-p'} \right)^{\frac{1}{p'}}. \tag{2.8}$$

From (2.8) and (2.5), since N is any number not less than $n+1$, we obtain (2.4). The lemma is proved. Consider the sum $S_m(n) = 1^m + 2^m + \cdots + (n-1)^m + n^m$ $(m, n = 1, 2, \ldots)$.

Lemma 2.3. If $m, n = 1, 2, \ldots, n \geq 2m+1$, then the following inequalities hold:

$$\frac{1}{10(m+1}\,(n+1)^{m+1} \leq S_m(n) \leq \frac{1}{m+1}\,(n+1)^{m+1}. \tag{2.9}$$

Proof. Equalities $S_1(n) = \frac{n(n+1}{2}$, $S_2(n) = \frac{n(n+1)(n+1/2)}{3}$ and

$$C_{m+1}^{1} S_m(n) = (n+1)^{m+1} - C_{m+1}^{2} S_{m-1}(n) - C_{m+1}^{3} S_{m-2}(n) - \cdots - C_{m+1}^{m-2} S_3(n) -$$

$$-C_{m+1}^{m-1} S_2(n) - C_{m+1}^{m} S_1(n) - n - 1; \tag{2.10}$$

$C_k^r = \frac{k!}{r!(k-r)!}$ $(k, r \in N, k \geq r)$ are well known. By (2.10):

$$S_m(n) \leq \frac{1}{m+1}(n+1)^{m+1}. \tag{2.11}$$

The lower estimate for $S_m(n)$ follows easily from (2.10) and (2.11). The lemma is proved.

Lemma 2.4. If $s \geq n \geq 2m+1$, $m \geq 2$ $(m, n, s \in N)$, then

$$\frac{1}{10^{m-2}(m-1)!}(s-n+m-2)^{m-1} \leq P_{m,s-n} \leq \frac{1}{(m-1)!}(s-n+m-2)^{m-1}. \tag{2.12}$$

Proof. If $m = 2$, then $P_{2,s-n} = s - n$ and is satisfied (2.12). Suppose that (2.12) holds for $m = k$:

$$\frac{1}{10^{k-2}(k-1)!} \sum_{j=n}^{s} (j-n+k-2)^{k-1} \leq P_{k,s-n} \leq \frac{1}{(k-1)!} \sum_{j=n}^{s} (j-n+k-2)^{k-1}.$$

Then by Lemma 2.1,

$$\frac{1}{10^{k-2}(k-1)!} S_{k-1}(s-n+k-2) = \frac{1}{10^{k-2}(k-1)!}(s-n+k-2)^{k-1} \leq$$

$$\leq P_{k+1,s-n} \leq \frac{1}{(k-1)!}(s-n+k-2)^{k-1} = \frac{1}{(k-1)!} S_{k-1}(s-n+k-2).$$

By inequality (2.9), we have

$$\frac{1}{10^{k-1}k!}(s-n+k-1)^k \le P_{k+1,s-n} \le \frac{1}{k!}(s-n+k-1)^k.$$

Thus, inequalities (2.12) also hold, when $m=k+1$. The principle of mathematical induction proves the lemma.

From (2.12), since m is a fixed number, we obtain the following assertion.

Corollary 2.1. If $m \ge 2$, then there exists a positive number j_0 , that for all $j \ge j_0$ the following inequalities hold

$$A_+ j^{m-1} \le P_{m,j} \le B_+ j^{m-1}, \tag{2.13}$$

where A_+ and B_+ are positive constants.

Theorem 2.1. Let $1<p<\infty$, $1/p+1/p'=1$, $m \ge 2$ and

$$T_{m,u,v} = \sup_{n=0,1,2,\dots} \left(\sum_{j=0}^{n} |u_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=n}^{+\infty} \left(j^{(m-1)p'}\right)^{p'} |v_j|^{-p'}\right)^{\frac{1}{p'}} < \infty.$$

Then

$$\left(\sum_{n=0}^{+\infty} |u_n y_n|^p\right)^{\frac{1}{p}} \le C_{m,u,v} \left(\sum_{n=0} +\infty \left|v_n \left(-\Delta\right)^{(m)} y_n\right|^p\right)^{\frac{1}{p}}, \forall y = \{y_n\}_{n=0}^{\infty} \in \widetilde{l}_+. \tag{2.14}$$

In addition, if $C_{m,u,v}$ is the smallest constant satisfying (2.14), then

$$A_+ T_{0,m,u,v} \le C_{m,u,v} \le B_+ p^{\frac{1}{p}} \left(p'\right)^{\frac{1}{p'}} T_{m,u,v}, \tag{2.15}$$

where

$$T_{0,m,u,v} = \sup_{n=0,1,2,\dots} \left(\sum_{j=0}^{n} |u_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=n}^{+\infty} \left((j-n)^{(m-1)p'}\right)^{p'} |v_j|^{-p'}\right)^{\frac{1}{p'}}$$

and A_+ и B_+ are constants in (2.13).

Proof. It suffices to verify that inequalities (2.15) hold. If we use (2.2), (2.13) and the well-known weighted difference Hardy type theorem [11]

$$\left(\sum_{n=0}^{+\infty} |u_n y_n|^p\right)^{\frac{1}{p}} = \left(\sum_{n=0}^{+\infty} \left|u_n \sum_{k=n}^{+\infty} P_{m,k} v_k^{-1} \left(v_k(-\Delta)^{(m)} y_k\right)\right|^p\right)^{\frac{1}{p}} \le$$

$$\le p^{\frac{1}{p}} (p')^{\frac{1}{p'}} \sup_{n=0,1,2,\dots} \left[\left(\sum_{j=0}^{n} |u_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=n}^{+\infty} \left|P_{m,j} v_j^{-1}\right|^{p'}\right)^{\frac{1}{p'}}\right] \left(\sum_{j=0}^{+\infty} \left|v_j (-\Delta)^{(m)} y_j\right|^p\right)^{\frac{1}{p}} \le$$

$$\le B_+ p^{\frac{1}{p}} {p'}^{\frac{1}{p'}} T_{m,u,v} \left(\sum_{j=0}^{+\infty} |v_j (-\Delta)^m y_j|^p\right)^{\frac{1}{p}}.$$

This implies the right-hand inequality in (2.15). Now we prove the left-hand inequality in (2.15). For the numbers $\widetilde{y}_{n,n}$ $(n=0,1,\dots)$ chosen above, by virtue of (2.7), we have

$$\sum_{n=0}^{+\infty} |u_n \widetilde{y}_{n,n}|^p = \sum_{n=0}^{+\infty} |u_n|^p \left|\sum_{s=n}^{N} P_{m,s-n}^{p'} |v_s|^{-p'}\right|^p \ge \sum_{j=0}^{n} |u_j|^p \left(\sum_{s=n+1}^{N} P_{m,s-n}^{p'} |v_s|^{-p'}\right)^p.$$

And by (2.2) and (2.6),

$$\|\widetilde{y}\|^p_{\widetilde{H}^{(m)}_{p,v}} = \sum_{s=n+1}^{N} \left[|v_s| \left(-\Delta\right)^{(m)} \widetilde{y}_s\right]^p = \sum_{s=n+1}^{N} |v_s|^p \left(P_{m,s-n}\right)^{(p'-1)p} |v_s|^{-p'p} = \sum_{s=n+1}^{N} \left(P_{m,s-n}\right)^{p'} |v_s|^{-p'}.$$

Consequently

$$\sum_{n=0}^{+\infty}|u_n\widetilde{y}_{n,n}|^p\geq\sum_{j=0}^{n}|u_j|^p\left(\sum_{s=n+1}^{N}(P_{m,s-n})^{-p'}\right)^{p-1}\cdot\|\widetilde{y}\|^p_{\widetilde{H}^{(m)}_{p,v}}.$$

According to our choice, N is any positive integer. Therefore, by (2.14), we obtain the estimate $A_+T_{0,m,u,v}\leq$ $\leq C_{m,u,v}$. The theorem is proved.

Let $\widetilde{l}_+=\{\{w_n\}_{n=-\infty}^{+\infty}\in\widetilde{l}:w_k=0\quad\forall k\geq 0\}$. The following assertion is proved similarly to Lemma 2.1 and Lemma 2.4.

Lemma 2.5. If $m\geq 2$, $n\leq -m-1$, $j\leq n$ $(m,n,j\in Z)$, then for each $y=\{y_n\}_{n=-\infty}^{0}\in\widetilde{l}_-$ the following equality holds:

$$y_n=(-1)^m\sum_{j=-\infty}^{n}P_{l,-,n-j}\,(-\Delta)^{(m)}\,y_j(n\in Z),$$

where $P_{m,-,n-j}$ $(m=1,2,\dots)$ are defined by

$$P_{1,-,n-j}=1,P_{k,-,n-j}=\sum_{s=j}^{n}P_{k-1,-,s-j}(k=2,3,\dots)$$

and they satisfy the following estimates:

$$\frac{1}{10^{m-1}\cdot(m-1)!}\,(n-j+m-2)^{m-1}\leq P_{m,-,n-j}\leq\frac{1}{(m-1)!}\,(n-j+m-2)^{m-1}.$$

Corollary 2.2. If $m\geq 2$, then there exists a number $j_0<0$, such that for all $j\leq j_0$ hold the following inequalities:

$$A_-\,|j|^{m-1}\leq P_{m,j}\leq B_-\,|j|^{m-1}\,, \tag{2.16}$$

where and are positive constants.

We denote by $\widehat{H}^{(k)}_{p,v}$ the space with the norm

$$\|a\|_{\widehat{H}^{(k)}_{p,v}}=\left(\sum_{s=-\infty}^{0}\left|v_s\Delta^{(k)}a_s\right|^p\right)^{\frac{1}{p}}\,(a=\{a_s\}_{s=-\infty}^{0}).$$

Using Lemma 2.5 we prove the following assertion. *Lemma 2.6.* Let $y=\{y_n\}_{n=-\infty}^{0}\in\widetilde{l}_-$. Then

$$\sup_{\|y\|_{\widehat{H}^{(m)}_{p,v}}=1}|y_n|=\left(\sum_{s=-\infty}^{n}|P_{m,-,n-s}|^{p'}\,v_s^{-p'}\right)^{\frac{1}{p'}}\,(n=0,-1,-2,\dots).$$

Theorem 2.2 Let $1<p<\infty$, $1/p+1/p'=1$, $m\geq 2$ and

$$T''_{m,u,v}=\sup_{\tau<0}\left(\sum_{j=\tau}^{0}|u_j|^p\right)^{\frac{1}{p}}\left(\sum_{j=-\infty}^{\tau}|j|^{(m-1)p'}\,|v_j|^{-p'}\right)^{\frac{1}{p'}}<\infty.$$

Then for $y=\{y_n\}_{n=-\infty}^{0}\in\widetilde{l}_-$ the following inequality holds:

$$\left(\sum_{n=-\infty}^{0}|u_ny_n|^p\right)^{\frac{1}{p}}\leq C_1\left(\sum_{n=-\infty}0\left|v_n\,(-\Delta)^{(m)}\,y_n\right|^p\right)^{\frac{1}{p}}. \tag{2.17}$$

In addition, if C_1 is the smallest constant satisfying (2.17), then

$$A_-T''_{0,m,u,v}\leq C_1\leq B_-p^{\frac{1}{p}}\,(p')^{\frac{1}{p'}}\,T''_{m,u,v}.$$

Here A_- and B_- are constants in (2.16), and where

$$T''_{0,m,u,v} = \sup_{n=0,-1,-2,\dots} \left(\sum_{j=\tau}^{0} |u_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=-\infty}^{\tau} |(\tau - j|^{(m-1)p'} |v_j|^{-p'} \right)^{\frac{1}{p'}} .$$

This theorem is proved on the basis of Theorem 2.1, by performing some simple substitutions. We introduce the notations

$$\gamma_{m,u,v,B} = \sqrt[p]{\max \left[(B_+ T_{m,u,v})^p , \left(B_- T''_{m,u,v} \right)^p \right]}$$

and

$$\gamma'_{m,u,v,B} = \sqrt[p]{\min \left[(A_+ T_{0,m,u,v})^p , \left(A_- T''_{0,m,u,v} \right)^p \right]},$$

where A_+ , B_+ ,A_-, B_- are constants in (2.13) and (2.16).

Theorem 2.3 Let the sequences u, v satisfy the condition $\gamma_{m,u,v,B} < \infty$. Then for $y_j \in \widetilde{l}$ the folloving inequalities hold:

$$\sum_{j=-\infty}^{+\infty} |u_j y_j|^p \, leq C^p_{2,m,u,v} \sum_{j=-\infty}^{+\infty} \left| v_j \Delta^{(m)} y_j \right|^p . \tag{2.18}$$

In addition, if $C_{2,m,u,v}$ is a smallest constant for which (2.18) is true, then

$$\gamma'_{m,u,v,A} \le C_{2,m,u,v} \le p^{1/p} \left(p' \right)^{1/p'} \gamma_{m,u,v,B}. \tag{2.19}$$

Proof. By Theorem 2.1 and Theorem 2.2, we obtain estimates (2.14) and (2.17)

$$\sum_{n=0}^{+\infty} |u_n y_n|^p \, leq C^p_{m,u,v} \sum_{n=0}^{+\infty} \left| v_n \left(-\Delta \right)^{(m)} y_n \right|^p$$

and

$$\sum_{j=-\infty}^{-1} |u_n y_n|^p \, leq C^p_{1,m,u,v} \sum_{n=-\infty}^{-1} \left| v_n \Delta^{(m)} y_n \right|^p .$$

Summing them, we have (2.18). Now we estimate $C_{2,m,u,v}$. According to (2.14) and (2.17),

$$\sum_{j=-\infty}^{+\infty} |u_n y_n|^p \le \left[B_+ p^{\frac{1}{p}} \left(p' \right)^{\frac{1}{p'}} T_{m,u,v} \right]^p \sum_{n=0}^{\infty} \left| v_n \left(-\Delta \right)^{(m)} y_n \right|^p + \left[B_- p^{\frac{1}{p}} \left(p' \right)^{\frac{1}{p'}} T''_{m,u,v} \right]^p \sum_{n=-\infty}^{0} \left| v_n \Delta^{(m)} y_n \right|^p \le$$

$$\le p^{\frac{1}{p}} \left(p' \right)^{p-1} \gamma^p_{m,u,v,B} \sum_{n=-\infty}^{+\infty} \left| v_n \Delta^{(m)} y_n \right|^p .$$

In this way, $C_{2,m,u,v} \le p^{\frac{1}{p}} \left(p' \right)^{p-1} \gamma_{m,u,v,B}$.

By Theorem 2.1 and Theorem 2.2, respectively, $C_{m,u,v} \ge T_{0,m,u,v}$ and $C_{1,m,u,v} \ge T''_{0,m,u,v}$. In other words, there are sequences $z = \{z_n\}_{n=-\infty}^{1} \in \widetilde{l}_-$ and $\theta = \{\theta_k\}_{k=0}^{+\infty} \in \widetilde{l}_+$ such that

$$\sum_{n=-\infty}^{-1} |u_n z_n|^p \ge \left(A_- T''_{0,m,u,v} \right)^p \sum_{n=-\infty}^{-1} \left| v_n \Delta^{(m)} z_n \right|^p ;$$

$$\sum_{n=0}^{+\infty} |u_n \theta_n|^p \ge \left(A_+ T_{0,m,u,v} \right)^p \sum_{n=0}^{+\infty} \left| v_n \Delta^{(m)} \theta_n \right|^p .$$

Then, denoting $z_k = \theta_k$ $(k = 0, 1, \dots)$, for $\widetilde{z} = \{z_n\}_{n=-\infty}^{+\infty} \in \widetilde{l}$, we obtain

$$\sum_{n=-\infty}^{+\infty} |u_n z_n|^p \ge \left(A_+ T_{0,m,u,v} \right)^p \sum_{n=0}^{+\infty} \left| v_n \Delta^{(m)} z_n \right|^p + \left(A_- T''_{0,m,u,v} \right)^p \sum_{n=-\infty}^{-1} \left| v_n \Delta^{(m)} z_n \right|^p \ge$$

$$\ge \left(\gamma'_{m,u,v,A} \right)^p \sum_{n=-\infty}^{+\infty} \left| v_n \Delta^{(m)} z_n \right|^p .$$

This implies the left-hand inequality in (2.19). The theorem is proved.

3 Coercive estimates for a degenerate system of difference equations

Consider the following system

$$l_0 y_j = \Delta^{2n} y_j + r_{jj}\Delta^{2n-1} y_j = F_j, j \in Z. \tag{3.1}$$

Let $y = \{y_j\}_{j=-\infty}^{+\infty} \in \widetilde{l}$. From (3.1) we obtain

$$\sum_{j\in Z} \Delta^{2n} y_j \cdot \Delta^{2n-1} y_j + \sum_{j\in Z} r_{jj} \left[\Delta^{2n-1} y_j\right]^2 = \sum_{j\in Z} l_0 y_j \cdot r_j \Delta^{2n-1} y_j. \tag{3.2}$$

Putting $\Delta^{2n-1} y_j = z_j$, we rewrite this equality in the following form:

$$\sum_{j\in Z} \Delta_- z_j \cdot z_j + \sum_{j\in Z} r_j z_j^2 = \sum_{j\in Z} l_0 y_j \cdot z_j,$$

It is easy to see that the expression $A = \sum_{j\in Z} \Delta_- z_j \cdot z_j$ satisfies the equality $A = \sum_{j\in Z} \left[\Delta_- z_j\right]^2 - A$, so

$$A = \frac{1}{2}\sum_{j\in Z} \left[\Delta_- z_j\right]^2 = \frac{1}{2}\sum_{j\in Z} \left[\Delta^{2n} y_j\right]^2.$$

Then (3.2) implies the following estimate:

$$\frac{1}{2}\sum_{j\in Z} \left[\Delta^{2n} y_j\right]^2 + \sum_{j\in Z} r_{jj} \left[\Delta^{2n-1} y_j\right]^2 \le \left(\sum_{j\in Z} \left[\frac{l_0 y_j}{\sqrt{r_{jj}}}\right]^2\right)^{\frac{1}{2}} \left(\sum_{j\in Z} \left[\sqrt{r_{jj}}\Delta^{(2n-1)} y_j\right]^2\right)^{\frac{1}{2}}, \tag{3.3}$$

in particular,

$$\left(\sum_{j\in Z} \left[\sqrt{r_{jj}}\Delta^{(2n-1)} y_j\right]^2\right)^{\frac{1}{2}} \le \left(\sum_{j\in Z} \left[l_0 y_j \frac{1}{\sqrt{r_{jj}}}\right]^2\right)^{\frac{1}{2}}.$$

Using this inequality and condition $r_{jj} \ge 1$, from (3.3) we have

$$\frac{1}{2}\sum_{j\in Z} \left[\Delta^{2n} y_j\right]^2 + \sum_{j\in Z} r_{jj} \left[\Delta^{2n-1} y_j\right]^2 \le \sum_{j\in Z} \left[l_0 y_j\right]^2. \tag{3.4}$$

Then, in view of (3.1),

$$\sum_{j\in Z} r_{jj}^2 \left[\Delta^{2n-1} y_j\right]^2 \le 3\sum_{j\in Z} \left(l_0 y_j\right)^2.$$

From this and (3.4)

$$\left\|\Delta^{(2n)} y\right\|_2^2 + \left\|r\Delta^{(2n-1)} y\right\|_2^2 \le 5\left\|l_0 y\right\|_2^2,$$

then

$$\left\|\Delta^{(2n)} y\right\|_2 + \left\|r\Delta^{(2n-1)} y\right\|_2 \le 5\sqrt{2}\left\|l_0 y\right\|_2, \quad y \in \widetilde{l}. \tag{3.5}$$

We put $\widetilde{r} = \{r_{jj}\}_{j=-\infty}^{+\infty}$. If $\gamma_{2n-1,e,\widetilde{r}} < \infty$, then by Theorem 2.3,

$$\|y\|_2 \le 2\gamma_{2n-1,e,\widetilde{r}} \left\|r\Delta^{(2n-1)} y\right\|_2, y \in \widetilde{l}.$$

Taking this into account, from (3.5) we obtain

$$\left\|\Delta^{(2n)} y\right\|_2 + \left\|r\Delta^{(2n-1)} y\right\|_2 + \|y\|_2 \le \left(2\gamma_{2n-1,e,\widetilde{r}} + \sqrt{10}\right)\left\|l_0 y\right\|_2, y \in \widetilde{l}. \tag{3.6}$$

The following assertion is proved by a standard method.

Lemma 3.1. Let $r_{jj} \geq 1$ $(j \in Z)$ and $\gamma_{2n-1,e,\widetilde{r}} < \infty$. Then the operator l_0 $(D(l_0) = \widetilde{l})$ corresponding to the system (3.1) is closable in the norm of l_2 . We denote by l the closure of l_0 in l_2 .

Lemma 3.2. Let $r_{jj} \geq 1$ $(j \in Z)$ and $\gamma_{2n-1,e,\widetilde{r}} < \infty$. Then the inequality (3.6) holds for each $y \in D(l)$.

Proof If $y \in D(l)$, then there exists a sequence $\{y^{(k)}\}_{k=1}^{\infty}$ such that

$$\left\|y^{(k)} - y\right\|_2 \to 0, \left\|l_0 y^{(k)} - ly\right\|_2 \to 0$$

as $k \to \infty$. According to (3.6)

$$\left\|\Delta^{(2n)} y^{(k)}\right\|_2 + \left\|r\Delta^{(2n-1)} y^{(k)}\right\|_2 + \left\|y^{(k)}\right\|_2 \leq \left(2\gamma_{2n-1,e,\widetilde{r}} + \sqrt{10}\right) \left\|ly^{(k)} - ly^{(m)}\right\|_2 . \tag{3.7}$$

We denote by $\dot{w}_2^{(2n)}$ the completion of $\widetilde{l}$ in the norm $\|v\|_w = \left\|\Delta^{(2n)} v\right\|_2 + \left\|r\Delta^{(2n-1)} v\right\|_2 + \|v\|_2$. $\dot{w}_2^{(2n)}$ is a difference Sobolev space with weight. By (3.7), for $\forall k, m \in N$ we obtain

$$\left\|y^{(k)} - y^{(m)}\right\|_w \leq \left(2\gamma_{2n-1,e,\widetilde{r}} + \sqrt{10}\right) \left\|ly^{(k)} - ly^{(m)}\right\|_2 .$$

Consequently, the sequence $\{y^{(k)}\}_{k=1}^{\infty} \in \widetilde{l}$ is fundamental in a Banach space $w_2^{(2n)}$, so, there exists an element $v \in w_2^{(2n)}$ such that $\left\|y^{(k)} - v\right\|_w \to 0$ $(k \to \infty)$. According to our choice $\left\|ly^{(k)} - ly\right\|_2 \to 0$ $(k \to \infty)$. Then $v \in D(l)$ and $ly = lv$. By (3.7),

$$\|v\|_w \leq \left(2\gamma_{2n-1,e,\widetilde{r}} + \sqrt{10}\right) \|lv\|_2 = \left(2\gamma_{2n-1,e,\widetilde{r}} + \sqrt{10}\right) \|ly\|_2 . \tag{3.8}$$

Therefore $D(l) \subseteq w_2^{(2n)}$. In this way, $y \in w_2^{(2n)}$ and $v = y$. Then by (3.8), we obtain the inequality (3.6). The lemma is proved.

Definition 3.1. The element $y = \{y_j\}_{j=-\infty}^{+\infty}$ is called a solution of the system (3.1), if there is a sequence $\{z^{(k)}\}_{k=1}^{+\infty}$ such that the following relations are satisfied:

$$\left\|z^{(k)-y}\right\|_2 \to 0, \left\|l_0 z^{(k)-F}\right\|_2 \to 0, (k \to +\infty).$$

It's clear that $y = \{y_j h\}_{j=-\infty}^{+\infty} \in l_2$ is a solution of (3.1) if and only if $y \in D(l)$ and $ly = f$.

Theorem 3.1. Let $n \geq 2$, and the sequence $\widetilde{r} = \{r_{jj}\}_{j=-\infty}^{+\infty}$ satisfies the conditions $r_{jj} \geq 1$ $(j \in Z)$ and $\gamma_{2n-1,e,\widetilde{r}} < \infty$. Then the system (3.1) has a unique solution $y \in l_2$. In addition, for the solution y holds (3.6).

Proof. By lemma 3.2 and Definition 3.1, the inequality (3.6) holds for a solution of (3.1). The solution of the system (3.1) is unique. In fact, if $z_1, z_2 \in l_2$ are two solutions of (3.1), then for $w = z_1 - z_2$ we have $lw = 0$. By (3.6), $\|w\|_2 = 0$, so $z_1 = z_2$. Now we will prove that the solution of the system (3.1) does exist. It suffices to show that $R(l) = l_2$. Assume the contrary, let $R(l) \neq l_2$. Then there exists an element $v \in l_2 \quad R(l)$, $v \neq 0$, such that $(l_0 y, v) = 0 \forall y \in D(l_0)$. Since the set $D(l_0) = \widetilde{l}$ is dense in l_2 , we have

$$\Delta^{(2n)} v_j - \Delta^{(2n-1)} \left(r_{jj} v_j\right) = 0, \forall j \in Z. \tag{3.9}$$

By choice, $v \in l_2$, therefore, $\lim_{|j|\to\infty} |v_j|^2 = 0$. Consequently $\forall \varepsilon > 0$, $\exists j_0 : \forall j \geq j_0$ $\left|\Delta^{(2n)} v_j\right| < \varepsilon$. Then by (3.9)

$$\Delta^{(2n-1)} \left(r_{jj} v_j\right) = 0.$$

According to the conditions imposed on the sequence $r = \{r_{jj}\}_{(j=-\infty}^{+\infty}$, the solution of this equation belonging to l_2 , is only $v = 0$. We obtain the contradiction. The theorem is proved.

4 Proofs of the main theorems

Proof of Theorem 1.1. By condition (1.2), hold (1.3). Then by Theorem 3.1, the minimal closed operator $\dot{l}$ in l_2 defined by equality $\dot{l}y = \Delta^{(2n)} y + r\Delta^{(2n-1)} y$, is continuously invertible and (3.5) holds for each $y \in D(\dot{l})$. In view of (3.5) and (1.3)

$$\left\|s\overline{\Delta^{(2n-1)} y}\right\|_2 < \alpha_1 \left\|r\Delta^{(2n-1)} y\right\|_2 \leq 5\sqrt{2}\alpha_1 \left\|\dot{l}y\right\|_2 . \tag{4.1}$$

Then, by (1.3), (4.1) and the well-known theorem on small perturbations, $\widehat{l}y = y + s\dot{\overline{\Delta^{(2n-1)}y}}$ is an closed and continuously invertible operator. And for $y \in D(\widehat{l})$ holds the inequality

$$\left\|\Delta^{(2n)}y\right\|_2 + \left\|r\Delta^{(2n-1)}y\right\|_2 + \left\|s\overline{\Delta^{(2n-1)}y}\right\|_2 \leq 5\sqrt{2}(1+\alpha_1)\left\|\dot{l}y\right\|_2. \tag{4.2}$$

On the other hand,

$$\left\|\dot{l}y\right\|_2 \leq \left\|\widehat{l}y\right\|_2 + \left\|s\overline{\Delta^{(2n-1)}y}\right\|_2 < \left\|\widehat{l}y\right\|_2 + 5\sqrt{2}\alpha_1\left\|\dot{l}y\right\|_2,$$

so

$$\left\|\dot{l}y\right\|_2 \leq \frac{1}{1-5\sqrt{2}\alpha_1}\left\|\widehat{l}y\right\|_2 \, \forall y \in D(\widehat{l}).$$

Then from (4.2)

$$\left\|\Delta^{(2n)}y\right\|_2 + \left\|r\Delta^{(2n-1)}y\right\|_2 + \left\|s\overline{\Delta^{(2n-1)}y}\right\|_2 \leq \frac{5\sqrt{2}(1+\alpha_1)}{1-5\sqrt{2}\alpha_1)}\left\|\widehat{1}y\right\|_2 \forall y \in D(\widehat{l}). \tag{4.3}$$

Let, now, k be a positive constant, and $\tau = \frac{1}{k}$. If the values of $\widetilde{y}_{j\tau}$ $(j \in Z)$ are chosen such that $\widetilde{y}_{j\tau} = y_{j\tau}$ $(j\tau \in Z)$, then $\Delta_+ y_j = y_{j+1} - y_j = k\left(\widetilde{y}_{(j+1)\tau} - \widetilde{y}_{j\tau}\right) = k\Delta_+\widetilde{y}_{j\tau}$, $\Delta^{(2)}y_j = \Delta_+\left(\Delta_- y_j\right) = k^2\Delta^{(2)}\widetilde{y}_{j\tau}$ and $\Delta_-\left(\Delta^{(2)}y_j\right) = k^3\Delta^{(3)}\widetilde{y}_{j\tau}$. Also $\Delta^{(m)}y_j = k^m\Delta^{(m)}\widetilde{y}_{j\tau}$, $m \in N$. Therefore, if we introduce quantities $\widehat{r}_{jj\tau}$, $\widehat{s}_{jj\tau}$, $\widehat{q}_{jj\tau}$, $\widehat{p}_{jj\tau}$, $\widehat{f}_{jj\tau}$ $(j \in Z)$ according to $\widehat{r}_{jj\tau} = r_{jj\tau}$, $\widehat{s}_{jj\tau} = s_{jj\tau}$, $\widehat{q}_{jj\tau} = q_{jj\tau}$, $\widehat{p}_{jj\tau} = p_{jj\tau}$, $\widehat{f}_{jj\tau} = f_{jj\tau}$ $(j\tau = \frac{j}{k} \in Z)$, then equation (1.1) reduces to the form:

$$\widehat{L}_0 y = \Delta^{(2n)}\widetilde{y} + \frac{1}{k}\widehat{r}\Delta^{(2n-1)}\widetilde{y} + \frac{1}{k}\overline{\widehat{s}\Delta^{(2n-1)}\widetilde{y}} +$$
$$+\sum_{j=1}^{2n-1}\frac{1}{k^{j+1}}\left(\widehat{Q}^{(j)}\Delta^{2n-j-1}\widetilde{y} + \widehat{P}^{(j)}\overline{\Delta^{2n-j-1}\widetilde{y}}\right) = k^{-2n}\widehat{f}, \widehat{f} \in l_{2,\tau}, \tag{4.4}$$

where $\widetilde{y} = \{\widetilde{y}_{j\tau}\}_{j=-\infty}^{+\infty}$; $\widehat{r} = \{\mathrm{diag}, \widehat{r}_{jj\tau}\}_{j=-\infty}^{+\infty}$; $\widehat{s} = \{\mathrm{diag}, \widehat{s}_{jj\tau}\}_{j=-\infty}^{+\infty}$; $\widehat{Q}^{(\theta)} = \{\mathrm{diag}, \widehat{q}_{jj\tau}^{(\theta)}\}_{j=-\infty}^{+\infty}$; $\widehat{P}^{(\theta)} = \{\mathrm{diag}, \widehat{p}_{jj\tau}^{(\theta)}\}_{j=-\infty}^{+\infty}$; $(\theta = \overline{1, 2n-1})$, $\widetilde{f} = \{\widetilde{f}_{j\tau}\}_{j=-\infty}^{+\infty}$. We rewrite (4.4) in the following form:

$$\widehat{l}\widetilde{y} + \sum_{j=1}^{2n-1}\frac{1}{k^{j+1}}\left(\widehat{Q}^{(j)}\Delta^{2n-j-1}\widetilde{y} + \widehat{P}^{(j)}\overline{\Delta^{2n-j-1}\widetilde{y}}\right) = \frac{1}{2n}\widehat{f}, \widehat{f} \in l_{2,\tau}.$$

By condition (1.2) and Theorem 2.3,

$$\sum_{j=1}^{2n-1}\left\|\frac{1}{k^{j+1}}\widehat{Q}^{(j)}\Delta^{2n-j-1}\widetilde{y}\right\|_{2,\tau} \leq$$
$$\leq (2n-1)max_{j=\overline{1,2n-1}}\left[\frac{1}{k^{j+1}}\gamma_{(2n-1),\widehat{Q}^{(j)},[min(A_-,A_+)]^{-1}\widehat{r}}\right]\left\|\widehat{r}\Delta^{(2n-1)}\widetilde{y}\right\|_{2,\tau} \tag{4.5}$$

and

$$\sum_{j=1}^{2n-1}\left\|\frac{1}{k^{j+1}}\widehat{P}^{(j)}\Delta^{2n-j-1}\widetilde{y}\right\|_{2,\tau} \leq$$
$$\leq (2n-1)max_{j=\overline{1,2n-1}}\left[\frac{1}{k^{j+1}}\gamma_{(2n-1),\widehat{P}^{(j)},[min(A_-,A_+)]^{-1}\widehat{r}}\right]\left\|\widehat{r}\Delta^{(2n-1)}\widetilde{y}\right\|_{2,\tau}. \tag{4.6}$$

We denote

$$G(\Phi, \widetilde{r}, n, \tau) = (2n-1), max_{j=\overline{1,2n-1}}\left[\frac{1}{k^{j+1}}\gamma_{(2n-1),\Phi,[min(A_-,A_+)]^{-1}\widehat{r}}\right],$$

where Φ is the matrix equal to either $\widehat{Q}^{(\theta)}$, or $\widehat{P}^{(\theta)}$ $(\theta = \overline{1, 2n-1})$. If we choose k such that

$$k^{2n-1} > \frac{5\sqrt{2}(1+\alpha_1)}{\alpha(1-5\sqrt{2}\alpha_1)}\left\{4(2n-1)max\left[max_{\theta=\overline{1,2n-1}}G(\widehat{P}^{(\theta)}, \widehat{r}, n, \theta)\right]\right\}\frac{5\sqrt{2}(1+\alpha_1)}{(1-5\sqrt{2}\alpha_1)k^{2n-1}}\left\|\widehat{1}\widetilde{y}\right\|_2$$

for some $\alpha \in (0,1)$, then according to (4.5) and (4.6),

$$\left\|\sum_{j=1}^{2n-1}\frac{1}{k^{j+1}}\left(\widehat{Q}^{(j)}\Delta^{2n-j-1}\widetilde{y}+\widehat{P}^{(j)}\overline{\Delta^{2n-j-1}\widetilde{y}}\right)\right\|_{2,\tau}\le$$

$$\le\left\{4(2n-1)max\left[max_{\theta=\overline{1,2n-1}}G(\widehat{Q}^{(\theta)},\widehat{r},n,\theta),max_{\theta=\overline{1,2n-1}}G(\widehat{P}^{(\theta)},\widehat{r},n,\theta)\right]\right\}\frac{5\sqrt{2}(1+\alpha_1)}{(1-5\sqrt{2}\alpha_1)k^{2n-1}}\left\|\widehat{l}\widetilde{y}\right\|_{2,\tau}<$$

$$<\alpha\left\|\widehat{l}\widetilde{y}\right\|_{2,\tau}. \tag{4.7}$$

Since the operator $\widehat{l}$ is closed, from the inequality (4.7), first, by the theorem on small perturbations it follows that the operator $\widehat{L}_0$ is closed. We denote its closure by $\widehat{L}$. Second, it follows that the operator $\widehat{L}$ is continuously invertible. Then, since $\widehat{L}\widetilde{y}=Ly$, the operator L is also closed and continuously invertible. Thus, by Definition 3.1, the solution of the equation (1.1) exists and is unique.

Let $\widetilde{y}\in D(\widehat{L}_0$. Then from inequalities (4.3) and (4.7) we obtain:

$$\left\|\Delta^{(2n)}\widetilde{y}\right\|_{2,\tau}+\left\|k^{-1}\widehat{r}\Delta^{(2n-1)}\widetilde{y}\right\|_{2,\tau}+\left\|k^{-1}\widehat{s}\overline{\Delta^{(2n-1)}\widetilde{y}}\right\|_{2,\tau}+\sum_{j=1}^{2n-1}\left\|\frac{1}{k^{j+1}}\widehat{Q}^{(j)}\Delta^{2n-j-1}\widetilde{y}\right\|_{2,\tau}+$$

$$+\sum_{j=1}^{2n-1}\left\|\frac{1}{k^{j+1}}\widehat{P}^{(j)}\Delta^{2n-j-1}\widetilde{y}\right\|_{2,\tau}\le\left(\frac{5\sqrt{2}(1+\alpha_1)}{1-5\sqrt{2}\alpha_1}+\alpha\right)\left\|\widehat{l}\widetilde{y}\right\|_{2,\tau}. \tag{4.8}$$

Further

$$\left\|\widehat{l}\widetilde{y}\right\|_{2,\tau}\le\left\|\widehat{l}\widetilde{y}+\sum_{j=1}^{2n-2}\frac{1}{k^{j+1}}\left(\widehat{Q}^{(j)}\Delta^{2n-j-1}\widetilde{y}+\widehat{P}^{(j)}\overline{\Delta^{2n-j-1}\widetilde{y}}\right)\right\|_{2,\tau}+$$

$$+\left\|\sum_{j=1}^{2n-2}\frac{1}{k^{j+1}}\left(\widehat{Q}^{(j)}\Delta^{2n-j-1}\widetilde{y}+\widehat{P}^{(j)}\overline{\Delta^{2n-j-1}\widetilde{y}}\right)\right\|_{2,\tau}\le\left\|\widehat{L}\widetilde{y}\right\|_{2,\tau}+\alpha\left\|\widehat{l}\widetilde{y}\right\|_{2,\tau};$$

$$\left\|\widehat{l}\widetilde{y}\right\|_{2,\tau}\le\frac{1}{1-\alpha}\left\|\widehat{L}\widetilde{y}\right\|_{2,\tau}. \tag{4.9}$$

By virtue of (4.8) and (4.9)

$$\left\|\Delta^{(2n)}\widetilde{y}\right\|_{2,\tau}+\left\|\frac{1}{k}\widehat{r}\Delta^{(2n-1)}\widetilde{y}\right\|_{2,\tau}+\left\|\frac{1}{k}\widehat{s}\overline{\Delta^{(2n-1)}\widetilde{y}}\right\|_{2,\tau}+\sum_{j=1}^{2n-1}\frac{1}{k^{j+1}}\left(\left\|\widehat{Q}^{(j)}\Delta^{2n-j-1}\widetilde{y}\right\|_{2,\tau}+\left\|\widehat{P}^{(j)}\overline{\Delta^{2n-j-1}\widetilde{y}}\right\|_{2,\tau}\right)\le$$

$$\le\frac{(1-5\sqrt{2}\alpha_1)\alpha+5\sqrt{2}(1+\alpha_1)}{(1-\alpha)(1-5\sqrt{2}\alpha_1)}\left\|\widehat{L}\widetilde{y}\right\|_{2,\tau}. \tag{4.10}$$

From (4.10), making the substitutions, we obtain the inequality (1.4) for each $\widetilde{y}\in D(L)$, and, in addition, for the solution of the system (1.1). The theorem is proved. *Proof of Theorem 1.2.* By virtue of the inequality (1.4), the operator L^{-1} displays the entire space l_2 on the difference Sobolev space $H_{2,\widetilde{r}}^{(2n)}$ with the norm $\left\|\Delta^{(2n)}y\right\|_2+\left\|r\Delta^{(2n-1)}y\right\|_2$. According to the results of [12, 13], when conditions (1.5) and (1.6) are satisfied, an embedding operator of a weighted space $H_{2,\widetilde{r}}^{(2n)}$ in l_2 is compact. Therefore, $L^{-1}:l_2\to l_2$ is a compact operator. The theorem is proved. *Remark 4.1* If $h>0$, then the assertions of Theorem 1.1 and Theorem 1.2 are also satisfied for an infinite difference system

$$L_0y=h^{-2n}\Delta^{(2n)}y+h^{-2n+1}r\Delta^{(2n-1)}y+h^{-2n+1}s\overline{\Delta^{(2n-1)}y}+$$

$$+\sum_{j=1}^{2n-1}\left(Q^{(j)}h^{-(2n-j-1)}\Delta^{(2n-j-1)}y+P^{(j)}h^{-(2n-j-1)}\overline{\Delta^{(2n-j-1)y}}\right)=f,$$

where $y=\{y_{jh}\}_{j=-\infty}^{+\infty}$; $\Delta_+y_{kh}=y_{(k+1)h}-y_{kh}$; $\Delta^{(2)}y_{kh}=\Delta_-\Delta_+y_{kh}=y_{(k+1)h}-2y_{kh}+y_{(k-1)h}$ and $r=(r_{jh,jh})_{j\in Z}\,(r_{jh,jh}\ge1)$; $s=(s_{jh,jh})_{j\in Z}$; $Q^{\theta}=\left(q_{jh,jh}^{(\theta)}\right)_{j\in Z}$; $P^{\theta}=\left(p_{jh,jh}^{(\theta)}\right)_{j\in Z}$ $\left(\theta=\overline{1,2n-1}\right)$ are diagonal matrices, and $f \in l_2(h)$. Here $l_2(h)$ is the space of the real numerical sequences $f = \{f_{jh}\}_{j=-\infty}^{+\infty}$ with the norm $\|f\|_{2,h} = \left(\sum_{j=-\infty}^{+\infty} f_{jh}^2 h\right)^{1/2}$.

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Д.Р. Бейсенова, Қ.Н. Оспанов

Жоғарғы ретті айырымдық теңдеулер жүйесі үшін максималды регулярлық және шағын шарттары

Мақалада жұп ретті шексіз айырымдық теңдеулер жүйесі қарастырылған, жүйенің оң жағы гильберт кеңістігінің сандық тізбектерінен тұрады. Жұмыста қарастырылған жүйе теңдеулерінің коэффициенттерінен құралған тізбектер бірдей ретті айырымдарда шексіз болуы мүмкін, сонымен бірге олардың өсуі потенциалдың өсуіне бағынбауы мүмкін. Біздің қарастырып отырған жағдайда потенциал нольдік тізбек, немесе таңбасы белгілі болмауы мүмкін, сондықтан Штурм-Лиувилль типті айырымдық теңдеулер жүйесінің потенциалы өсуін қолданатын бұрын-соңды жасалған әдістер бұл жерде қолданыла алмайды. Мақала авторлары жүйенің корректілі шешілуінің шарттарын келтіріп, шешімнің және оның айырмаларының нормаларының тиімді бағаларын берді. Нұқсанды оператор жүйесіне сәйкес резольвентаның компактылығы шарттары алынды. Өзіндік ғылыми қызығушылық тудыратын кейбір салмақты айырымдық Харди типтес теңсіздіктер дәлелденген. Олар жұмыстың негізгі нәтижелерін дәлелдеу барысында пайдаланылған. Нұқсанды дифференциалдық теңдеулермен салыстырғанда айырымдық жүйе жағдайында жүйе коэффициенттері тербелісіне қойылған шартты алып тастауға болатыны көрсетілген.

Кілт сөздер: айырымдық жүйе, аралық коэффициент, шешімнің корректілігі, максималды регулярлық, резольвентаның компактылығы.

Д.Р. Бейсенова, К.Н. Оспанов

Условия максимальной регулярности и компактности для системы разностных уравнений высокого порядка

В статье исследована бесконечная линейная система разностных уравнений высокого четного порядка с правой частью из гильбертова пространства числовых последовательностей. Последовательности, образованные из коэффициентов уравнений системы при одинаковых порядках разностей, могут быть неограниченными, а также их рост может не подчиняться росту потенциала. Ранее разработанные методы, существенно использующие доминирующий рост потенциала в разностных системах уравнений типа Штурма-Лиувилля, здесь не подходят, так как в рассматриваемом нами случае потенциал может оказаться нулевым, или не имеющим определенного знака последовательности. Авторами статьи приведены условия корректной разрешимости системы, а также оптимальные оценки норм решения и его разностей вплоть до самого старшего порядка. Получены условия компактности резольвенты, соответствующей системе вырожденного оператора. Доказаны некоторые разностные весовые неравенства типа Харди, имеющие самостоятельный научный интерес. Они использованы в доказательстве основных результатов работы. Показано, что, по сравнению с вырожденными дифференциальными уравнениями, в случае разностной системы удается снять условие на колебания коэффициентов системы.

Ключевые слова: разностная система, промежуточный коэффициент, корректность решения, максимальная регулярность, компактность разрешения.

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The tri-harmonic Neumann problem

In this article investigated the tri-harmonic Neumann function for the unit dics. For harmonics functions the Neumann's boundary problem is well studied and solved under certain conditions through Neumann's function, sometimes it is also called Green's function of the second order. Any case of finding of Green function of the corresponding boundary value problem is very important for this or that area D as it contains extensive information, allowing to write out a large number of analytical solutions in the form of integrated ratios. At the same time the specified procedure makes the main difficulty at the solution Dirichlet and Neumann problems and in an explicit form Green function is known only for a small number of simple areas. The harmonics Green function with itself consistently leads to the subsequent polyharmonic Green function which can be used to solve the subsequent Dirichlet problem for higher order of the Poisson equation. Methods of integrated transformation have received tri-harmonic Neumann function in explicit form for the unit disc of the complex plane with biharmonic Neumann function. With Neumann's function an integrated idea is given by development for the tri-harmonic operator. Above-mentioned polyharmonic Green function for the unit disc gives rise to the solution some specific polyharmonic objective of Dirichlet problem. In the same way harmonic Neumann function with itself consistently leads to the subsequent polyharmonic Neumann function. Received in the present article result allows to expect interesting prospects in further development of the analytical theory of boundary valua problems in complex analysis for the equations of elliptic type.

Keywords: Neumann function, Green function, harmonic function, potential, field, the Dirichlet problem.

The Neumann function for the Laplacian of the unit disc is given as

$$N_1(z,\zeta) = -\log|(\zeta - z)(1 - z\bar{\zeta})|^2. \tag{1}$$

This function are related to the fundamental solution of the Laplacian. The Neumann function on the boundary satisfies

$$\partial_{\nu_z} N_1(z,\zeta) = (z\partial_z + \bar{z}\partial_{\bar{z}})N_1(z,\zeta) = -2.$$

Neumann boundary conditions are given via outer normal derivatives ∂_ν. For the unit disc this is

$$\partial_\nu = z\partial_z + \bar{z}\partial_{\bar{z}}.$$

Typical for Neumann problems is that they are in general not well-posed. They are neither always solvable nor uniquely solvable. As well solvability conditions have to be determined as normalization conditions to be posed.

The bi-harmonic Neumann function has the form [1], [2].

$$-N_2(z,\zeta) = |\zeta - z|^2\left[\log|(\zeta - z)(1 - z\bar{\zeta})|^2 - 4\right] + 4\sum_{k=2}^{+\infty}\frac{(z\bar{\zeta})^k + (\bar{z}\zeta)^k}{k^2} +$$

$$+2\left[z\bar{\zeta} + \bar{z}\zeta\right]\log|1 - z\bar{\zeta}|^2 - (1 + |z|^2)(1 + |\zeta|^2)\left[\frac{\log(1 - z\bar{\zeta})}{z\bar{\zeta}} + \frac{\log(1 - \bar{z}\zeta)}{\bar{z}\zeta}\right] \tag{2}$$

and satisfies the Neumann problem

$$\partial_z\partial_{\bar{z}} N_2(z,\zeta) = N_1(z,\zeta) \text{ in D for fixed } \zeta \in \overline{\text{D}};$$

$$\partial_{\nu_z} N_2(z,\zeta) = 2(1 - |\zeta|^2) \text{ on } \partial\text{D for fixed } \zeta \in \overline{\text{D}}$$

and the normalization condition

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} N_2(z,\zeta)\frac{dz}{z}=0.$$

Moreover, N_2 is symmetric in z and ζ, $N_2(z,\zeta)=N_2(\zeta,z)$.

Theorem 1. The Neumann problem

$$(\partial_z\partial_{\bar z})^2 w=f \text{ in } \mathbb{D},\ \partial_\nu w=\gamma_0,\ \partial_\nu\partial_z\partial_{\bar z}w=\gamma_1 \text{ on } \partial\mathbb{D};$$

$$\frac{1}{2\pi i}\int_{|\zeta|=1} w(\zeta)\frac{d\zeta}{\zeta}=c_0,\ \frac{1}{2\pi i}\int_{|\zeta|=1} w_{\zeta\bar\zeta}(\zeta)\frac{d\zeta}{\zeta}=c_1$$

for $f\in L_p(\mathbb{D},\mathbb{C})$, $2<p$, $\gamma_0,\gamma_1\in C(\partial\mathbb{D};\mathbb{C}), c_0,c_1\in\mathbb{C}$ is uniquely solvable if and only if

$$\frac{1}{2\pi i}\int_{|\zeta|=1}\gamma_0(\zeta)\frac{d\zeta}{\zeta}=2c_1-\frac{2}{\pi}\int_{|\zeta|<1}(1-|\zeta|^2)f(\zeta)d\xi d\eta$$

and

$$\frac{1}{2\pi i}\int_{|\zeta|=1}\gamma_1(\zeta)\frac{d\zeta}{\zeta}=\frac{2}{\pi}\int_{|\zeta|<1}f(\zeta)d\xi d\eta.$$

The solution is given as

$$w(z)=c_0-(1-|z|^2)c_1+\frac{1}{4\pi i}\int_{|\zeta|=1}\{N_1(z,\zeta)\gamma_0(\zeta)+N_2(z,\zeta)\gamma_1(\zeta)\}\frac{d\zeta}{\zeta}-$$
$$-\frac{1}{\pi}\int_{|\zeta|<1}N_2(z,\zeta)f(\zeta)d\xi d\eta. \tag{3}$$

Definition. The Neumann-3 function for the unit disc $\mathbb{D}$ is

$$N_3(z,\zeta)=-\frac{1}{4}|\zeta-z|^4\log|(\zeta-z)(1-z\bar\zeta)|^2+n_3(z,\zeta), \tag{4}$$

where $n_3(z,\zeta)$ is tri-harmonic in both variables with proper boundary behavior. The properties of the third Neumann function are

$$\partial_z\partial_{\bar z}N_3(z,\zeta)=N_2(z,\zeta) \text{ in } \mathbb{D}\backslash\{\zeta\} \text{ for } \zeta\in\mathbb{D};$$
$$\partial_\nu N_3(z,\zeta)=-\frac{1}{2}(1-|\zeta|^2)^2-\frac{1}{2}\partial_\nu N_2(z,\zeta) \text{ on } \partial\mathbb{D} \text{ for } \zeta\in\mathbb{D},$$

where

$$\partial_{\nu_z}N_2(z,\zeta)=2(1-|\zeta|^2) \text{ on } \partial\mathbb{D} \text{ for } \zeta\in\mathbb{D},$$

so that

$$\partial_{\nu_z}N_3(z,\zeta)=-\left[\frac{1}{2}(1-|\zeta|^2)^2+(1-|\zeta|^2)\right];$$
$$\frac{1}{2\pi i}\int_{|z|=1}N_3(z,\zeta)\frac{dz}{z}=0 \text{ for } \zeta\in\mathbb{D};$$
$$N_3(z,\zeta)=N_3(\zeta,z) \text{ for } z,\zeta\in\mathbb{D}.$$

It is important that the normal derivative of $N_3(z,\zeta)$ with respect to z does depend on ζ but not on z. In order to find $N_3(z,\zeta)$ in a proper way some particular Neumann problems are investigated. We must calculate function $n_3(z,\zeta)$. From a formula (3) we will express function:

$$n_3(z,\zeta)=N_3(z,\zeta)+\frac{1}{4}|\zeta-z|^4\log|(\zeta-z)(1-z\bar\zeta)|^2. \tag{5}$$

We will prove some properties to which function (5) satisfies:

$$\partial_z n_3(z,\zeta) = \partial_z N_3(z,\zeta) - \frac{1}{2}(\zeta - z)(\overline{\zeta - z})^2 \log|(\zeta - z)(1 - z\bar{\zeta})|^2 -$$
$$-\frac{1}{4}|\zeta - z|^4 \left(\frac{1}{\zeta - z} + \frac{\bar{\zeta}}{1 - z\bar{\zeta}}\right);$$

$$\partial_z\partial_{\bar{z}} n_3(z,\zeta) = N_2(z,\zeta) + |\zeta - z|^2 \log|(\zeta - z)(1 - z\bar{\zeta})|^2 + 2|\zeta - z|^2 -$$
$$-\frac{1}{2}|\zeta - z|^2(1 - |\zeta|^2)\left(\frac{1}{1 - z\bar{\zeta}} + \frac{1}{1 - \bar{z}\zeta}\right);$$

$$(\partial_z\partial_{\bar{z}})^2 n_3(z,\zeta) = 6 - (1 - |\zeta|^2)\left(\frac{1}{1 - z\bar{\zeta}} + \frac{1}{1 - \bar{z}\zeta}\right) - \frac{1}{2}(1 - |\zeta|^2)^2\left(\frac{1}{(1 - \bar{z}\zeta)^2} + \frac{1}{(1 - \bar{z}\zeta)^2}\right),$$

for $|z| = 1$ then

$$\partial_\nu n_3(z,\zeta) = -\frac{1}{2}(1 - |\zeta|^2)^2 - (1 - |\zeta|^2) + \frac{1}{2}|\zeta - z|^2(2 - z\bar{\zeta} - \bar{z}\zeta)\log|(\zeta - z)(1 - z\bar{\zeta})|^2 + \frac{1}{2}|\zeta - z|^4;$$

$$\partial_\nu\partial_z\partial_{\bar{z}} n_3(z,\zeta) = 4(2 - z\bar{\zeta} - \bar{z}\zeta) + 2(2 - z\bar{\zeta} - \bar{z}\zeta)\log|1 - z\bar{\zeta}|^2 - (1 - |\zeta|^2) +$$
$$+\frac{1}{2}(1 - |\zeta|^2)(2 - z\bar{\zeta} - \bar{z}\zeta) - (1 - |\zeta|^2)\left(\frac{1 - \bar{z}\zeta}{1 - z\bar{\zeta}} + \frac{1 - z\bar{\zeta}}{1 - \bar{z}\zeta}\right)$$

follows.

Next the first solvability conditions of Theorem is verified.

$$\frac{1}{2\pi i}\int_{|z|=1} \partial_\nu n_3(z,\zeta)\frac{dz}{z} = 2c_1 - \frac{2}{\pi}\int_{|z|<1}(1 - |z|^2)\partial_z\partial_{\bar{z}} n_3(z,\zeta)dxdy.$$

At the beginning consider the left-hand side

$$\frac{1}{2\pi i}\int_{|z|=1} \partial_\nu n_3(z,\zeta)\frac{dz}{z} = \frac{1}{2\pi i}\int_{|z|=1}\left\{-\frac{1}{2}(1 - |\zeta|^2)^2 - (1 - |\zeta|^2) +\right.$$
$$\left.+\frac{1}{2}|\zeta - z|^2(2 - z\bar{\zeta} - \bar{z}\zeta)\log|(\zeta - z)(1 - z\bar{\zeta})|^2 + \frac{1}{2}|\zeta - z|^4\right\}\frac{dz}{z} =$$
$$= -\frac{1}{2}(1 - |\zeta|^2)^2 - (1 - |\zeta|^2) + 3|\zeta|^2 + \frac{1}{2}|\zeta|^4 + 3|\zeta|^2 +$$
$$+\frac{1}{2}|\zeta|^4 + 1 + 4|\zeta|^2 + |\zeta|^4 = 10|\zeta|^2 + |\zeta|^4 - 1.$$

Also to solve the right-side of the condition, namely:

$$2c_1 - \frac{2}{\pi}\int_{|z|<1}(1 - |z|^2)\partial_z\partial_{\bar{z}} n_3(z,\zeta)dxdy,$$

where

$$c_1 = \frac{1}{2\pi i}\int_{|z|=1}\partial_z\partial_{\bar{z}} n_3(z,\zeta)\frac{dz}{z} =$$
$$= \frac{1}{2\pi i}\int_{|\zeta|=1}\left\{N_2(z,\zeta) + |\zeta - z|^2\log|(\zeta - z)(1 - z\bar{\zeta})|^2 +\right.$$
$$\left.+2|\zeta - z|^2 - \frac{1}{2}|\zeta - z|^2(1 - |\zeta|^2)\left[\frac{1}{1 - z\bar{\zeta}} + \frac{1}{1 - \bar{z}\zeta}\right]\right\}\frac{dz}{z} =$$

$$= 4|\zeta|^2 + 2 + 2|\zeta|^2 - \frac{1}{2}(1-|\zeta|^2) - \frac{1}{2}(1-|\zeta|^2) = 7|\zeta|^2 + 1$$

and solving separately:

$$\frac{2}{\pi}\int_{|z|<1}(1-|z|^2)(\partial_z\partial_{\bar z})^2 n_3(z,\zeta)dxdy =$$

$$= \frac{2}{\pi}\int_{|z|<1}(1-|z|^2)\left\{6-(1-|\zeta|^2)\left(\frac{1}{1-z\bar\zeta}+\frac{1}{1-\bar z\zeta}\right)-\right.$$

$$\left.-\frac{1}{2}(1-|\zeta|^2)^2\left(\frac{1}{(1-\bar z\zeta)^2}+\frac{1}{(1-z\bar\zeta)^2}\right)\right\}dxdy =$$

$$= 6 - 2(1-|\zeta|^2) - (1-|\zeta|^2)^2 = 3 + 4|\zeta|^2 - |\zeta|^4$$

it follows that

$$2c_1 - \frac{2}{\pi}\int_{|\zeta|<1}(1-|z|^2)(\partial_z\partial_{\bar z})^2 n_3(z,\zeta)d\xi d\eta = 10|\zeta|^2 + |\zeta|^4 - 1.$$

We have proved the validity of the first condition:

$$10|\zeta|^2 + |\zeta|^4 - 1 = 10|\zeta|^2 + |\zeta|^4 - 1.$$

In the next step we verify the second solvability condition of Theorem 1:

$$\frac{1}{2\pi i}\int_{|z|=1}\partial_\nu\partial_z\partial_{\bar z}n_3(z,\zeta)\frac{dz}{z} = \frac{2}{\pi}\int_{|z|<1}(\partial_z\partial_{\bar z})^2 n_3(z,\zeta)dxdy.$$

The left-hand side is

$$\frac{1}{2\pi i}\int_{|z|=1}\partial_\nu\partial_z\partial_{\bar z}n_3(z,\zeta)\frac{dz}{z} =$$

$$= \frac{1}{2\pi i}\int_{|z|=1}\left\{4(2-z\bar\zeta-\bar z\zeta) + 2(2-z\bar\zeta-\bar z\zeta)\log|1-z\bar\zeta|^2 - (1-|\zeta|^2)+\right.$$

$$\left.+\frac{1}{2}(1-|\zeta|^2)(2-z\bar\zeta-\bar z\zeta) - (1-|\zeta|^2)\left(\frac{1-\bar z\zeta}{1-z\bar\zeta}+\frac{1-z\bar\zeta}{1-\bar z\zeta}\right)\right\}\frac{dz}{z} =$$

$$= 8 + 4|\zeta|^2 - (1-|\zeta|^2)^2 - (1-|\zeta|^2)(1-|\zeta|^2) = 6 + 8|\zeta|^2 - 2|\zeta|^4.$$

Then evaluating the right-hand side of the condition shows

$$\frac{2}{\pi}\int_{|\zeta|<1}(\partial_z\partial_{\bar z})^2 n_3(z,\zeta)dxdy =$$

$$= \frac{2}{\pi}\int_{|\zeta|<1}\left\{6-(1-|\zeta|^2)\left(\frac{1}{1-z\bar\zeta}+\frac{1}{1-\bar z\zeta}\right)-\right.$$

$$\left.-\frac{1}{2}(1-|\zeta|^2)^2\left(\frac{1}{(1-z\bar\zeta)^2}+\frac{1}{(1-\bar z\zeta)^2}\right)\right\}dxdy =$$

$$= 12 - 4(1-|\zeta|^2) - 2(1-|\zeta|^2)^2 = 8 + 4|\zeta|^2 - 2 + 4|\zeta|^2 - 2|\zeta|^4 = 6 + 8|\zeta|^2 - 2|\zeta|^4.$$

Hence the second conditions is satisfied, i.e.

$$6 + 8|\zeta|^2 - 2|\zeta|^4 = 6 + 8|\zeta|^2 - 2|\zeta|^4.$$

In order to find the solution of Theorem 1, we also must calculate

$$c_0 = \frac{1}{2\pi i}\int_{|\zeta|=1} n_3(z,\zeta)\frac{d\zeta}{\zeta} =$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1}\left\{N_3(z,\zeta) + \frac{1}{4}|\zeta - z|^4 \log|(\zeta - z)(1 - z\bar{\zeta})|^2\right\}\frac{d\zeta}{\zeta}.$$

Evaluating this integral shows $c_0 = 2|\zeta|^2 - \frac{3}{2}|\zeta|^4$.

Thus we have verified all the necessary and sufficient conditions of solvability of Theorem 1. According to Theorem 1 the function $n_3(z,\zeta)$ is given as [3]:

$$n_3(z,\zeta) = c_0 + (1-|z|^2)c_1 - \frac{1}{4\pi i}\int_{|\tilde{\zeta}|=1}\Big\{N_1(z,\tilde{\zeta})\partial_\nu n_3(\tilde{\zeta},\zeta) +$$

$$+N_2(z,\tilde{\zeta})\partial_\nu\partial_{\tilde{\zeta}}\partial_{\bar{\tilde{\zeta}}} n_3(\tilde{\zeta},\zeta)\Big\}\frac{d\tilde{\zeta}}{\tilde{\zeta}} - \frac{1}{\pi}\int_{|\zeta|<1} N_2(z,\tilde{\zeta})f(\tilde{\zeta})d\tilde{\xi}d\tilde{\eta} =$$

$$= 2|\zeta|^2 - \frac{3}{2}|\zeta|^4 + (1-|z|^2)(7|\zeta|^2+1) + \frac{1}{4\pi i}\int_{|\tilde{\zeta}|=1}\Big\{\Big(\log|(\tilde{\zeta}-z)(1-z\bar{\tilde{\zeta}})^2|\Big)\times$$

$$\times\Big(-\frac{1}{2}(1-|\zeta|^2)^2 - (1-|\zeta|^2) + \frac{1}{2}|\zeta-\tilde{\zeta}|^2(2-\zeta\bar{\tilde{\zeta}} - \bar{\zeta}\tilde{\zeta})\log|(\zeta-\tilde{\zeta})(1-\tilde{\zeta}\bar{\zeta})|^2 +$$

$$+\frac{1}{2}|\zeta-\tilde{\zeta}|^4\Big)\Big\}\frac{d\tilde{\zeta}}{\tilde{\zeta}} + \frac{1}{4\pi i}\int_{|\zeta|=1}\Big\{\Big(|\tilde{\zeta}-z|^2[\log|(\tilde{\zeta}-z)(1-z\bar{\tilde{\zeta}})|^2 - 4] +$$

$$+4\sum_{k=2}^{+\infty}\frac{(z\bar{\tilde{\zeta}})^k + (\bar{z}\tilde{\zeta})^k}{k^2} + 2(z\bar{\tilde{\zeta}} + \bar{z}\tilde{\zeta})\log|1-z\bar{\tilde{\zeta}}|^2 - (1+|z|^2)(1+|\zeta|^2)\Big[\frac{\log(1-z\bar{\tilde{\zeta}})}{z\bar{\tilde{\zeta}}} +$$

$$+\frac{\log(1-\bar{z}\tilde{\zeta})}{\bar{z}\tilde{\zeta}}\Big]\Big)\Big(4(2-\tilde{\zeta}\bar{\zeta} - \bar{\tilde{\zeta}}\zeta) + 2(2-\tilde{\zeta}\bar{\zeta} - \bar{\tilde{\zeta}})\log|1-\tilde{\zeta}\bar{\zeta}|^2 - (1-|\zeta|^2) +$$

$$+\frac{1}{2}(1-|\zeta|^2)(2-\tilde{\zeta}\bar{\zeta} - \bar{\tilde{\zeta}}\zeta) - (1-|\zeta|^2)\Big(\frac{1-\bar{\tilde{\zeta}}\zeta}{1-\tilde{\zeta}\bar{\zeta}} + \frac{1-\tilde{\zeta}\bar{\zeta}}{1-\bar{\tilde{\zeta}}\zeta}\Big)\Big)\Big\}\frac{d\tilde{\zeta}}{\tilde{\zeta}} -$$

$$-\frac{1}{\pi}\int_{|\zeta|<1}\Big\{\Big(|\tilde{\zeta}-z|^2[\log|(\tilde{\zeta}-z)(1-z\bar{\tilde{\zeta}})|^2 - 4] + 4\sum_{k=2}^{+\infty}\frac{(z\bar{\tilde{\zeta}})^k + (\bar{z}\tilde{\zeta})^k}{k^2} +$$

$$+2(z\bar{\tilde{\zeta}} + \bar{z}\tilde{\zeta})\log|1-z\bar{\tilde{\zeta}}|^2 - (1+|z|^2)(1+|\zeta|^2)\Big[\frac{\log(1-z\bar{\tilde{\zeta}})}{z\bar{\tilde{\zeta}}} +$$

$$+\frac{\log(1-\bar{z}\tilde{\zeta})}{\bar{z}\tilde{\zeta}}\Big]\Big)(6 - (1-|\zeta|^2)\Big(\frac{1}{1-\tilde{\zeta}\bar{\zeta}} + \frac{1}{1-\bar{\tilde{\zeta}}\zeta}\Big) -$$

$$-\frac{1}{2}(1-|\zeta|^2)^2\Big(\frac{1}{(1-\bar{\tilde{\zeta}}\zeta)^2} + \frac{1}{(1-\tilde{\zeta}\bar{\zeta})^2}\Big)\Big)\Big\}d\tilde{\xi}d\tilde{\eta}. \tag{6}$$

Then (6) we will insert in (4), from here we will receive [4]:

$$N_3(z,\zeta) = -\frac{1}{4}|\zeta-z|^4\log|(\zeta-z)(1-z\bar{\zeta})|^2 + n_3(z,\zeta) = \frac{3}{2}\left(|\zeta|^4 + |z|^4\right) + 5\left(|\zeta|^2+1\right)\left(|z|^2+1\right) +$$

$$+2\left(|\zeta|^2 + |z|^2 + 6\right) + \frac{1}{4}\left(|\zeta|^2 - 1\right)\left(|z|^2-1\right)\left(\bar{z}\zeta + z\bar{\zeta}\right) + \frac{1}{4}|\zeta-z|^4\log\left|\frac{1-z\bar{\zeta}}{\zeta-z}\right|^2 -$$

$$-\left[2\left(|z|^2+2\right)\left(|\zeta|^2+2\right) + \frac{1}{2}\left(|\zeta|^4+|z|^4\right)\right]\log|1-z\bar{\zeta}|^2 +$$

$$+\left[\frac{1}{2}\left(|\zeta|^2+|z|^2\right)\left(|z|^2+1\right)\left(|\zeta|^2+1\right)+4\left(|\zeta|^2+|z|^2+2\right)\right]\times$$

$$\times\left[\frac{\log\left(1-\bar{z}\zeta\right)}{\bar{z}\zeta}+\frac{\log\left(1-z\bar{\zeta}\right)}{z\bar{\zeta}}\right]-$$

$$-\frac{1}{4}\left(|z|^4+1\right)\left(|\zeta|^4+1\right)\left[\frac{\log\left(1-\bar{z}\zeta\right)}{\left(\bar{z}\zeta\right)^2}+\frac{\log\left(1-z\bar{\zeta}\right)}{\left(z\bar{\zeta}\right)^2}+\frac{1}{\bar{z}\zeta}+\frac{1}{z\bar{\zeta}}\right]+$$

$$+\sum_{l=0}^{\infty}\left\{\left[\frac{8}{(l+1)^3}-\frac{\left(|\zeta|^2+1\right)\left(|z|^2+1\right)}{(l+2)^2}-\frac{4\,|z|^2+4\,|\zeta|^2+6}{(l+1)^6}\right]\left[\left(\bar{z}\zeta\right)^{l+1}+\left(z\bar{\zeta}\right)^{l+1}\right]\right\},\qquad z,\zeta\in D. \tag{7}$$

Theorem 2. The tri-harmonic Neumann problem

$$(\partial_z\partial_{\bar{z}})^3 w=f \text{ in } \mathbb{D},\ f\in L_p(\mathbb{D};\mathbb{C}),\ 2<p<+\infty;$$

$$\partial_\nu w=\gamma_0,\ \partial_\nu\partial_z\partial_{\bar{z}}w=\gamma_1\ \partial_\nu(\partial_z\partial_{\bar{z}})^2 w=\gamma_2 \text{ on } \partial\mathbb{D},\ \gamma_0,\gamma_1,\gamma_2\in C(\partial\mathbb{D};\mathbb{C}),$$

satisfying

$$\frac{1}{2\pi i}\int_{|\zeta|=1} w(\zeta)\frac{d\zeta}{\zeta}=c_0,\ \frac{1}{2\pi i}\int_{|\zeta|=1}\partial_\zeta\partial_{\bar{\zeta}}w(\zeta)\frac{d\zeta}{\zeta}=c_1,\ \frac{1}{2\pi i}\int_{|\zeta|=1}(\partial_\zeta\partial_{\bar{\zeta}})^2 w(\zeta)\frac{d\zeta}{\zeta}=c_2$$

is uniquely solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\zeta)\frac{d\zeta}{\zeta}=2c_1-c_2-\frac{1}{16\pi i}\int_{\partial\mathbb{D}}\gamma_2(\zeta)\frac{d\zeta}{\zeta}+\frac{1}{\pi}\int_{\mathbb{D}}\left((1-|\zeta|^2)^2-\frac{1}{2}\right)f(\zeta)d\xi d\eta;$$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{d\zeta}{\zeta}=c_1-2c_2-\frac{2}{\pi}\int_{\mathbb{D}}(1-|\zeta|^2)f(\zeta)d\xi d\eta$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_2(\zeta)\frac{d\zeta}{\zeta}=\frac{2}{\pi}\int_{\mathbb{D}}f(\zeta)d\xi d\eta.$$

The solution is given as

$$w(z)=c_0-c_1(1-|z|^2)-c_2\left(\frac{1}{4}(1-|z|^2)^2+\frac{1}{2}(1-|z|^2)\right)+\frac{1}{4\pi i}\int_{\partial\mathbb{D}}\{N_1(z,\zeta)\gamma_0(\zeta)+$$

$$+N_2(z,\zeta)\gamma_1(\zeta)+N_3(z,\zeta)\gamma_2(\zeta)\}\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int_{\mathbb{D}}f(\zeta)N_3(z,\zeta)d\xi d\eta.$$

Proof. Rewriting the Neumann-3 problem as the system

$$(\partial_z\partial_{\bar{z}})^2 w=\omega \text{ in } \mathbb{D},\ \partial_\nu w=\gamma_0,\ \partial_\nu\partial_z\partial_{\bar{z}}w=\gamma_1 \text{ on } \partial\mathbb{D};$$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} w(\zeta)\frac{d\zeta}{\zeta}=c_0,\ \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\partial_\zeta\partial_{\bar{\zeta}}w(\zeta)\frac{d\zeta}{\zeta}=c_1$$

and

$$\partial_z\partial_{\bar{z}}\omega=f \text{ in } \mathbb{D},\ \partial_\nu\omega=\gamma_2 \text{ on } \partial\mathbb{D},\ \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\omega(\zeta)\frac{d\zeta}{\zeta}=c_2$$

leads to the solvability conditions

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\zeta)\frac{d\zeta}{\zeta}=2c_1-\frac{2}{\pi}\int_{\mathbb{D}}(1-|\zeta|^2)\omega(\zeta)d\xi d\eta \tag{8}$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{d\zeta}{\zeta}=\frac{2}{\pi}\int_{\mathbb{D}}\omega(\zeta)d\xi d\eta. \tag{9}$$

The solution then is

$$w(z)=c_0-(1-|z|^2)c_1+\frac{1}{4\pi i}\int_{\partial\mathbb{D}}\{\gamma_0(\zeta)N_1(z,\zeta)+\gamma_1(\zeta)N_2(z,\zeta)\}\frac{d\zeta}{\zeta}-$$
$$-\frac{1}{\pi}\int_{\mathbb{D}}\omega(\zeta)N_2(z,\zeta)d\xi d\eta; \tag{10}$$

$$\omega(\zeta)=c_2+\frac{1}{4\pi i}\int_{\partial\mathbb{D}}\gamma_2(\tilde{\zeta})N_1(\zeta,\tilde{\zeta})\frac{d\tilde{\zeta}}{\tilde{\zeta}}-\frac{1}{\pi}\int_{\mathbb{D}}N_1(\zeta,\tilde{\zeta})f(\tilde{\zeta})d\tilde{\xi}d\tilde{\eta}.$$

Inserting ω into the (8)-(9) conditions gives

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\zeta)\frac{d\zeta}{\zeta}=2c_1-\frac{1}{\pi}\int_{\mathbb{D}}(1-|\zeta|^2)\left\{c_2+\frac{1}{4\pi i}\int_{\partial\mathbb{D}}\gamma_2(\tilde{\zeta})N_1(\zeta,\tilde{\zeta})\frac{d\tilde{\zeta}}{\tilde{\zeta}}-\right.$$
$$\left.-\frac{1}{\pi}\int_{\mathbb{D}}N_1(\zeta,\tilde{\zeta})f(\tilde{\zeta})d\tilde{\xi}d\tilde{\eta}\right\}d\xi d\eta$$

with

$$\frac{1}{\pi}\int_{\mathbb{D}}(1-|\zeta|^2)N_1(\zeta,\tilde{\zeta})d\xi d\eta=\frac{1}{2}\left(1-\frac{1}{2}|\zeta|^2\right)^2-\frac{1}{4}.$$

Then inserting ω into (10) shows

$$w(z)=c_0-(1-|z|^2)c_1-c_2\left(\frac{1}{\pi}\int_{\mathbb{D}}N_2(z,\zeta)d\xi d\eta\right)+\frac{1}{4\pi i}\int_{\partial\mathbb{D}}\{\gamma_0(\zeta)N_1(z,\zeta)+$$
$$+\gamma_1(\zeta)N_2(z,\zeta)\}\frac{d\zeta}{\zeta}-\frac{1}{4\pi i}\int_{\partial\mathbb{D}}\gamma_2(\tilde{\zeta})\frac{1}{\pi}\int_{\mathbb{D}}N_1(\zeta,\tilde{\zeta})N_2(z,\tilde{\zeta})d\xi d\eta\frac{d\zeta}{\zeta}+$$
$$+\frac{1}{\pi}\int_{\mathbb{D}}f(\tilde{\zeta})\frac{1}{\pi}\int_{\mathbb{D}}N_1(\zeta,\tilde{\zeta})N_2(z,\zeta)d\xi d\eta d\tilde{\xi}d\tilde{\eta}.$$

So,we get

$$w(z)=c_0-c_1(1-|z|^2)-c_2\left(\frac{1}{4}(1-|z|^2)^2+\frac{1}{2}(1-|z|^2)\right)+\frac{1}{4\pi i}\int_{\partial\mathbb{D}}\{N_1(z,\zeta)\gamma_0+$$
$$+N_2(z,\zeta)\gamma_1(\zeta)+N_3(z,\zeta)\gamma_2(\zeta)\}\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int_{D}f(\zeta)N_3(z,\zeta)d\xi d\eta,$$

where $N_1(z,\zeta)$, $N_2(z,\zeta)$, $N_3(z,\zeta)$ are given (1), (2) and (7) respectively.

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Тригармоникалық Нейман есебі

Лаплас теңдеуі тиесілі элиптикалық типтес теңдеулері физикалық қолданылулары маңызды рөл атқарады. Оларға сығылмайтын сұйықтықтың ықтимал қозғалысы, электрстатикалық өрістің потенциалы, стационар жылу, диффузиялық үдерістер, өрістің потенциал өрісі, аэромеханиканың мәселелері жатады. Екінші ретті сызықтық эллиптикалық теңдеулері үшін, оның ішінде Лаплас теңдеуі үшін Дирихле және Нейман есептері негізгі шектік есептер болып табылады. Бұл шектік есептердің нақты шешімдерін табу үшін әртүрлі аналитикалық әдістер бар, мысалға, интегралдық теңдеулер әдістері, интегралдық түрлендіру әдістері, бейнелеу әдісі және т.с.с. Берілген D облысы үшін тиісті шектік есептің Грин функциясын табуда кез келген жағдайы өте маңызды, себебі ол көптеген аналитикалық шешімдерді интегралдық түрлендіру түрінде жазуға мүмкіндік беретін ауқымды ақпарат сақтайды. Айтылған әдістер Дирихле және Нейман есептерін шешуде негізгі қиындықтар туғызады және Грин функциясының айқын түрдегі шешімі қарапайым облыстар үшін белгілі. Бірлік шеңбері үшін Грин функциясы алғаш рет Алманзи еңбектерінде шешімін тапқан, бұл нәтиже Дирихле есебін шешуде алғашқы қадамдарының бірі болып саналады. Сонымен қоса гибрид бигармоникалық Грин функциясы туралы Г. Бегер еңбектерінде кездеседі және Грин функциясының гибрид полигармоникалық түрлерінің әртүрлі болуы еңбектерінде кездеседі. Мақала авторы тригармоникалық Нейман есебін бірлік шеңберде зерттеген. Нейман есебі гармоникалық функциялар үшін жақсы зерттелген және Нейман функциясы арқылы нақты шарттарда шешілген, кейбір жағдайларда бұл функцияны екінші ретті Грин функциясы деп атайды. Тригармоникалық Нейман функциясы бигармоникалық Нейман функциясымен кешен жазықтықтың бірлік шеңберінде айқын түрде табылған. Нейман функциясымен интегралдық көрінісі тригармоникалық оператордың дамуына жол береді. Мақалада алынған нәтиже эллиптикалық теңдеулер үшін шекаралық есептердің аналитикалық теориясын әрі қарай дамытудың қызықты болашағын болжауға мүмкіндік береді.

Кілт сөздер: Нейман функциясы, Грин функциясы, гармоникалық функция, потенциал, өріс, Дирихле есебі.

С.К. Бургумбаева

Тригармоническая задача Неймана

В статье исследована тригармоническая функция Неймана на единичной окружности. Для гармонических функций задача Неймана хорошо изучена и решена при определенных условиях через функцию Неймана, иногда ее также называют функцией Грина второго порядка. Всякий случай нахождения функции Грина соответствующей краевой задачи для той или иной области D весьма важен, так как содержит обширную информацию, позволяя выписать большое число аналитических решений в виде интегральных соотношений. В то же время указанная процедура составляет основную трудность при решении задач Дирихле и Неймана, и в явном виде функция Грина известна только для небольшого числа простых областей. Гармоническая функция Грина с собой последовательно приводит к последующей полигармонической функции Грина, которая может использоваться для решения последующей проблемы Дирихле для более высокого порядка уравнения Пуассона. Методами интегрального преобразования получена тригармоническая функция Неймана в явном виде на единичной окружности комплексной плоскости с бигармонической функцией Неймана. С функцией Неймана интегральное представление дает развитие для тригармонического оператора. Упомянутая выше полигармоническая функция Грина на единичной окружности дает начало решению некоторой конкретной полигармонической задачи Дирихле. Точно так же гармоническая функция Неймана с собой последовательно приводит к последующей полигармонической функции Неймана. Полученный автором результат позволяет предвидеть интересные перспективы в дальнейшем развитии аналитической теории краевых задач в комплексном анализе для уравнений эллиптического типа.

Ключевые слова: функция Неймана, функция Грина, гармоническая функция, потенциал, поле, задача Дирихле.

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Stable perturbations of boundary problems for differential equations

In this paper, we expand the class of nondegenerate two-point boundary value problems for the Sturm-Liouville equation, which have a complete system of eigenfunctions and associated functions in special function spaces. Such spaces depend on the length of support of the potential of the Sturm-Liouville equation. The formulated results clarify well-known results of V.A. Marchenko. Two-point boundary value problems for the Sturm-Liouville equation are divided into degenerate and nondegenerate boundary conditions in the sense of V.A. Marchenko. The main result of V.A. Marchenko asserts that systems of eigenfunctions and associated functions of nondegenerate boundary value problems for the Sturm-Liouville equation form a complete system of functions in the space of square-summable functions. In this paper, the result of V.A. Marchenko is clarified in the following direction. There are operators with a complete system of eigenfunctions and associated functions in the space of square-summable functions among the degenerate boundary value problems in the sense of V.A. Marchenko. The presence of the completeness property depends on the length of support of the measure which is antisymmetry to the potential of the Sturm-Liouville equation.

Keywords: finite nonempty set, eigenvalue, three-point boundary value problem, Volterra operator, nondegenerate boundary condition.

1 Introduction

Let $n = 2\mu$ and let us consider a self-adjoint differential expression on the interval $[0,1]$ with real coefficients:

$$l\left(y\right) \equiv \left(p_0 y^{(\mu)}\right)^{(\mu)} + \left(p_1 y^{(\mu-1)}\right)^{(\mu-1)} + \cdots + \left(p_{\mu-1} y^{(1)}\right)^{(1)} + p_\mu y,$$

where $p_{0,} \cdots ,\ p_\mu$ are sufficiently smooth functions.

Let us given a system of linear functionals $\{U_1, \cdots ,\ U_n\}$. Let B_p denote the operator generated by the differential expression $l(\cdot)$ and boundary conditions

$$U_j\left(y\right) = 0,\ j = 1, \cdots , n.$$

In this paper, we study the question: how do coefficients $p_{0,} \cdots, p_\mu$ of the operator B_p influence the structure of the spectrum of the operator and the completeness of the system of root functions in $L_2(0,\ 1)$?

As usual, we introduce the fundamental system of solutions $\{y_1(x,\lambda), \cdots ,\ y_n(x,\ \lambda)\}$ of a homogeneous differential equation $l\left(y\right) = \lambda \cdots\ y(x)$ with initial conditions $y_k^{(j-1)}\left(0\right) =\ \delta_{jk}$. Denote by $\Delta_P(\lambda)$ the characteristic determinant of the operator B_p, which is given by the following formula

$$\Delta_P\left(\lambda\right) = det\left\{\|U_j(y_k)\|\right\}.$$

It is well known that $\Delta_P\left(\lambda\right)$ is an entire function of exponential type by a parameter λ. Zeros of the entire function $\Delta_P\left(\lambda\right)$ uniquely characterize the spectrum of the operator B_p. If λ_s is a eigenvalue of the operator B_p with multiplicity m_s, then λ_s is a zero of $\Delta_P\left(\lambda\right)$ with multiplicity m_s. Inverse statement is also true.

Basically, zeros of an entire function $\Delta_P(\lambda)$ can be represented as:

– empty set;
– finite nonempty set;
– a countable set without end points;
– the set coinciding with the complex plane.

The first case is realized when the operator B_p corresponds to the Cauchy problem. Note that, there are examples of operators with an empty spectrum even it is given with boundary conditions

$$U_j\left(y\right) = 0,\ j = 1, \cdots , n,$$

which is not the Cauchy conditions. Operators with an empty spectrum are often called Volterra operators. Examples of the Volterra operators representing three-point boundary value problems for second-order differential operators can be found in the work of S.A. Dzhumabaev, D.B. Nurakhmetov [1].

However, there are still no examples of differential operators with nonempty finite spectrum. It is proved that if the spectrum of the operator B_p represents a nonempty finite set, then its power is not greater than μ. In the case of $n = 2$. T.Sh. Kalmenov and A. Shaldanbayev [2] proved that there is no boundary value problem with a nonempty finite spectrum.

When the spectrum of the operator B_p is a countable set without finite limit points, then the operator B_p is said to have a discrete spectrum. Operators with discrete spectrum occur frequently.

More rarely, there are operators with a spectrum that coincides with the whole complex plane. Similar examples of two-point boundary value problems for higher order differential operators were given in [3].

A detail investigation of the second order differential operators with two-point boundary conditions were studied in the monograph by V.A. Marchenko [4]. The main result of [4] that interests us is that there nondegenerate two-point boundary conditions for a second order differential equation were selected and also it was proved the completeness of the corresponding system of root functions in $L_2(0,1)$.

In this work, it is clarified the result of V.A. Marchenko. We select operators from the class of degenerate boundary conditions for a second order differential equation which have complete system of root functions in a special functional space.

2 Two-point boundary value problems for the Sturm-Liouville equation

Consider an eigenvalue problem

$$-y^{(2)} + q\left(x\right)y = \ \lambda\ y\ ,\ 0 < x < 1$$

with boundary conditions

$$U_j\left(y\right) = \ a_{i1}y\left(0\right) + \ a_{i2}y^{(1)}\left(0\right) + a_{i3}y\left(1\right) + \ a_{i4}y^{(1)}\left(1\right) = 0\ ,\ i = 1, 2,$$

where $q(x)$ is a summable function on $(0,1), a_{ij}$ are complex numbers. The boundary conditions are identified using the matrix

$$A = \left\| \begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{array} \right\|,\quad rank(A) = 2.$$

We will denote by A_{ij} the minor of the matrix A composed of columns with numbers i and j. We also introduce the function

$$\hat{q}\left(x\right) = q\left(x\right),\ 0 \le x \le \frac{1}{2},\ \ \hat{q}\left(x\right) = q\left(1-x\right),\ \frac{1}{2} < x \le 1.$$

The difference $q\left(x\right) - \hat{q}\left(x\right)$ is denoted by $Q(x)$. It is clear that

$$Q(x) \equiv 0,\ 0 \le x \le \frac{1}{2}.$$

We introduce solutions $\hat{c}\left(x,\lambda\right), \hat{s}\left(x,\lambda\right)$ of a homogeneous equation

$$-y^{(2)} + \hat{q}\left(x\right)y = \lambda y,\ 0 < x < 1$$

with Cauchy conditions

$$\hat{c}\left(\frac{1}{2},\lambda\right) = 1,\ \hat{s}\left(\frac{1}{2},\lambda\right) = 0,\ \hat{c'}\left(\frac{1}{2},\lambda\right) = 0,\ \hat{s'}\left(\frac{1}{2},\lambda\right) = 1.$$

Similarly, we introduce the solutions $c\left(x,\lambda\right),\ s\left(x,\lambda\right)$ of the homogeneous equation

$$-y^{(2)} + q\left(x\right)y = \ \lambda\ y\ ,\ 0 < x < 1.$$

It is easy to verify that for all $0 < x < 1$ the following relations are true

$$\hat{c}\,(x,\lambda) = \hat{c}\,(1-x,\lambda)\,,\, \hat{s}\,(x,\lambda) = -\hat{s}\,(1-x,\lambda)\,.$$

It can also be understood that for all $0 < x < 1$ integral equations are valid

$$c\,(x,\lambda) = \hat{c}\,(x,\lambda) + \int_{0,5}^{x} \begin{vmatrix} \hat{c}\,(t,\lambda) & \hat{s}\,(t,\lambda) \\ \hat{c}\,(x,\lambda) & \hat{s}\,(x,\lambda) \end{vmatrix} Q(t)c\,(t,\lambda)\,dt;$$

$$s\,(x,\lambda) = \hat{s}\,(x,\lambda) + \int_{0,5}^{x} \begin{vmatrix} \hat{c}\,(t,\lambda) & \hat{s}\,(t,\lambda) \\ \hat{c}\,(x,\lambda) & \hat{s}\,(x,\lambda) \end{vmatrix} Q\,(t)\,s\,(t,\lambda)\,dt.$$

The characteristic determinant is given by the formula

$$\Delta_q\,(\lambda) = det\left\{\|U_j\,(c\,(\cdot,\lambda)) \quad U_j\,(s\,(\cdot,\lambda))\,, \quad j = 1,\ 2\|\right\}.$$

Let

$$r\,(x,\lambda) = \int_{0,5}^{x} \begin{vmatrix} \hat{c}\,(t,\lambda) & \hat{s}\,(t,\lambda) \\ \hat{c}\,(x,\lambda) & \hat{s}\,(x,\lambda) \end{vmatrix} Q(t)c\,(t,\lambda)\,dt;$$

$$p\,(x,\lambda) = \int_{0,5}^{x} \begin{vmatrix} \hat{c}\,(t,\lambda) & \hat{s}\,(t,\lambda) \\ \hat{c}\,(x,\lambda) & \hat{s}\,(x,\lambda) \end{vmatrix} Q\,(t)\,s\,(t,\lambda)\,dt.$$

Similar to the work of V.A. Marchenko [4], let us formulate a statement in the case of a symmetric potential $\hat{q}\,(x)$.

Lemma 1. We have

$$\Delta_{\hat{q}}\,(\lambda) = A_{12} + A_{34} + A_{14} + A_{32} + 2A_{31}\hat{c}\,(0,\lambda)\,\hat{s}\,(0,\lambda) +$$

$$+2\,(A_{14} + A_{32})\,\hat{s}\,(0,\lambda)\,\hat{c}'\,(0,\lambda) + 2A_{24}\hat{c}'\,(0,\lambda)\,\hat{s}'\,(0,\lambda)\,.$$

We now write the representation of the characteristic determinant in the case of an arbitrary potential.

Lemma 2. The following formula holds

$$\Delta_q(\lambda) = \Delta_{\hat{q}}(\lambda) + \Delta_{\hat{q}}(\lambda)\int_{0,5}^{1} dt \int_{0.5}^{1} dz Q(t)Q(z)c(z,\lambda)s(t,\lambda) \begin{vmatrix} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ \hat{c}(z,\lambda) & \hat{s}(z,\lambda) \end{vmatrix} +$$

$$+\int_{0.5}^{1} Q(t)s(t,\lambda)\{A_{13}\hat{c}(t,\ \lambda)\hat{s}(0,\lambda)\hat{s}(0,\lambda) + (A_{23}{+}A_{41})\hat{c}(t,\lambda)\hat{s}(0,\lambda)\hat{s'}(0,\lambda){+}A_{42}\hat{c}(t,\ \lambda)\hat{s'}(0,\lambda)\hat{s'}(0,\lambda)+$$

$$+(A_{23}{+}A_{43})\hat{s}(t,\lambda) + A_{13}\hat{s}(t,\ \lambda)\hat{c}(0,\lambda)\hat{s}(0,\lambda)+$$

$$+(A_{23}{+}A_{41})\hat{s}(t,\ \lambda)\hat{s}(0,\lambda)\hat{c'}(0,\lambda){+}A_{42}\hat{s}(t,\lambda)\hat{s'}(0,\lambda)\hat{c'}(0,\lambda)\}dt+$$

$$+\int_{0.5}^{1} Q(t)c(t,\lambda)\{A_{31}\hat{c}(t,\lambda)\hat{s}(0,\lambda)\hat{c}(0,\ \lambda) + (A_{32}{+}A_{14})\hat{c}(t,\lambda)\hat{s}(0,\lambda)\hat{c'}(0,\lambda){+}A_{24}\hat{c}(t,\lambda)\hat{c'}(0,\lambda)\hat{s'}(0,\lambda)+$$

$$+(A_{14}{+}A_{34})\hat{c}(t,\lambda) + A_{31}\hat{s}(t,\lambda)\hat{c}(0,\lambda)\hat{c}(0,\lambda)+$$

$$+(A_{32}{+}A_{14})\hat{s}(t,\lambda)\hat{c}(0,\lambda)\hat{c'}(0,\lambda){+}A_{24}\hat{s}(t,\lambda)\hat{c'}(0,\lambda)\hat{c'}(0,\lambda)\}dt.$$

It follows from the result of V.A. Marchenko that the matrix A defines nondegenerate boundary conditions if at least one of the following conditions hold:

$A_{24} \neq 0.$

$A_{24} = 0,\ A_{14} + A_{32} \neq 0.$

$A_{24} = 0,\ A_{14} + A_{32} = 0,\ A_{31} \neq 0$

and the system of eigenfunctions and associated functions of the Sturm-Liouville problem represents a complete system in the space $L_2(0,1)$.

Let matrix A define degenerate boundary conditions, i.e.

$$A_{24} = 0,\ A_{14} + A_{32} = 0,\ A_{31} = 0.$$

Then, Lemma 2 implies

$$\Delta_q(\lambda) = (A_{12} + A_{34})$$
$$\left(1 + \int_{0,5}^{1} dt \int_{0.5}^{1} dz Q(t) Q(z) c(z,\lambda) s(t,\lambda) \begin{vmatrix} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ \hat{c}(z,\lambda) & \hat{s}(z,\lambda) \end{vmatrix}\right) +$$
$$+ \int_{0.5}^{1} Q(t) s(t,\lambda) \{(A_{14} - A_{34}) \hat{s}(t, \lambda)\} dt +$$
$$+ \int_{0.5}^{1} Q(t) c(t,\lambda) \{(A_{14} + A_{34}) \hat{c}(t, \lambda)\} dt.$$

Since $rankA = 2$, it follows that $A_{12} \neq 0$.

Hence, for $A_{12} \neq 0$, it follows from the last relation that

$$\frac{\Delta_q(\lambda)}{A_{12}} = \left(1 - \theta^2\right) \times$$
$$\times \left(1 + \int_{0,5}^{1} dt \int_{0.5}^{1} dz Q(t) Q(z) c(z,\lambda) s(t,\lambda) \begin{vmatrix} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ \hat{c}(z,\lambda) & \hat{s}(z,\lambda) \end{vmatrix}\right) -$$
$$-\theta(1+\theta) \int_{0.5}^{1} Q(t) s(t,\lambda) \hat{s}(t, \lambda) dt -$$
$$-\theta(1-\theta) \int_{0.5}^{1} Q(t) c(t,\lambda) \hat{c}(t, \lambda) dt$$

for some θ.

In the case of degenerate boundary conditions, it is necessary to investigate completeness of the system of root functions in $L_2(0,1)$. As in [4], we introduce the solutions of a homogeneous equation

$$-y^{(2)} + q(x) y = \lambda y, \ 0 < x < 1$$

by formulas

$$\omega_1(x,\lambda) = \begin{vmatrix} c(x,\lambda) & s(x,\lambda) \\ U_2(c(\cdot,\lambda)) & U_2(s(\cdot,\lambda)) \end{vmatrix};$$
$$\omega_2(x,\lambda) = \begin{vmatrix} U_1(c(\cdot,\lambda)) & U_1(s(\cdot,\lambda)) \\ c(x,\lambda) & s(x,\lambda) \end{vmatrix}.$$

The following representations are also useful

$$\omega_1(x,\lambda) = \begin{vmatrix} c(x,\lambda) & s(x,\lambda) \\ U_2(\hat{c}(\cdot,\lambda)) & U_2(\hat{s}(\cdot,\lambda)) \end{vmatrix} + \begin{vmatrix} c(x,\lambda) & s(x,\lambda) \\ U_2(r(\cdot,\lambda)) & U_2(p(\cdot,\lambda)) \end{vmatrix} =$$
$$= \left\{ a_{21} \begin{vmatrix} c(x,\lambda) & s(x,\lambda) \\ (1+\theta)\hat{c}(0,\lambda) & (1-\theta)\hat{s}(0,\lambda) \end{vmatrix} + a_{22} \begin{vmatrix} c(x,\lambda) & s(x,\lambda) \\ (1+\theta)\hat{c}'(0, \lambda) & (1-\theta)\hat{s}'(0, \lambda) \end{vmatrix} \right\} +$$
$$+\theta \int_{0,5}^{1} \begin{vmatrix} c(t,\lambda) & s(t,\lambda) \\ c(x,\lambda) & s(x,\lambda) \end{vmatrix} Q(t) \left\{ a_{21} \begin{vmatrix} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ -\hat{c}(0,\lambda) & \hat{s}(0,\lambda) \end{vmatrix} - a_{22} \begin{vmatrix} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ \hat{c}'(0, \lambda) & -\hat{s}'(0, \lambda) \end{vmatrix} \right\} dt;$$
$$\omega_2(x, \lambda) = \left\{ a_{11} \begin{vmatrix} (1+\theta)\hat{c}(0,\lambda) & (1-\theta)\hat{s}(0,\lambda) \\ c(x,\lambda) & s(x,\lambda) \end{vmatrix} + a_{12} \begin{vmatrix} (1+\theta)\hat{c}'(0, \lambda) & (1-\theta)\hat{s}'(0, \lambda) \\ c(x,\lambda) & s(x,\lambda) \end{vmatrix} \right\} -$$
$$-\theta \int_{0,5}^{1} \begin{vmatrix} c(t,\lambda) & s(t,\lambda) \\ c(x,\lambda) & s(x,\lambda) \end{vmatrix} Q(t) \left\{ a_{11} \begin{vmatrix} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ -\hat{c}(0,\lambda) & \hat{s}(0,\lambda) \end{vmatrix} - a_{12} \begin{vmatrix} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ \hat{c}'(0, \lambda) & -\hat{s}'(0, \lambda) \end{vmatrix} \right\} dt.$$

Main result: Let

$$r = max_{x \in suppQ} > \frac{1}{2} \text{ and } A_{24} = 0, \ A_{14} + A_{32} = 0, \ A_{31} = 0.$$

The system of eigenfunctions and associated functions of the two-point boundary value problem for the Sturm-Liouville equation is complete in the space $L_2(\frac{1}{2} - r, \frac{1}{2} + r)$. The proof of the main result is carried out by the method of V.A. Marchenko [4] with the involvement of some modifications.

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Дифференциалдық теңдеулер үшін шекаралық есептердің орнықты ауытқулары

Мақалада Штурм-Лиувилл теңдеуіне ақаулы емес екінүктелі шекаралық есептердің кластары кеңейтілді. Кеңейтілген класқа тиісті есептердің меншекті және қосалқы функциялар жүйелері арнайы функционалдық кеңістіктерде толық болады. Авторлармен құрастырылған арнайы кеңістіктер Штурм-Лиувилл теңдеуінің потенциалының тасушысының ұзындығына тәуелді. Мақаладағы нәтижелер В.А. Марченконың белгілі жетістіктерін кеңейтеді. Штурм-Лиувилл теңдеуіне сәйкес екінүктелі шекаралық шарттар, В.А. Марченко бойынша, ақаулы және ақаулы емес шеттік шарттарға бөлінеді. В.А. Марченконың негізгі тұжырымы бойынша, Штурм-Лиувилл теңдеуіне сәйкес ақаулы емес шекаралық есептердің меншікті және қосалқы функциялар жүйесі арнайы функционалдық кеңістікте толық жүйе құрайды. В.А. Марченко бойынша, ақаулы шекаралық есептердің арасында меншікті және қосалқы функциялар жүйелері арнайы кеңістіктерде толық болуы мүмкін екендігі көрсетілген.

Кілт сөздер: ақырлы бос емес жиынтық, меншікті мән, үшнүктелі шекаралық есептер, Вольтерра операторлары, ерекше емес шекаралық шарттар.

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Устойчивые возмущения граничных задач для дифференциальных уравнений

В статье расширен класс невырожденных двухточечных граничных задач для уравнения Штурма-Лиувилля, имеющих полную систему собственных и присоединенных функций в специальных функциональных пространствах. Указанные специальные пространства зависят от длины носителя потенциала уравнения Штурма-Лиувилля. Сформулированные результаты уточняют известные результаты В.А. Марченко. Двухточечные краевые задачи для уравнения Штурма-Лиувилля делятся на вырожденные и невырожденные, по В.А. Марченко, граничные условия. Основной результат В.А. Марченко утверждает, что системы собственных и присоединенных функций невырожденных граничных задач для уравнения Штурма-Лиувилля в пространстве квадратично суммируемых функций образуют полную систему функций. Авторами статьи уточнен результат В.А. Марченко в следующем направлении. Среди вырожденных граничных задач, по В.А. Марченко, имеются задачи с полной системой собственных и присоединенных функций в пространстве квадратично суммируемых функций. Наличие свойства полноты зависит от длины носителя меры антисимметрии носителя потенциала уравнения Штурма-Лиувилля.

Ключевые слова: конечное непустое множество, собственное значение, трехточечные краевые задачи, вольтерровы операторы, невырожденные граничные условия.

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General bounded multiperiodic solutions of linear equation with differential operator in the direction of the main diagonal

In this article we determine the structure of the general solution of a n-th order linear equation with differential operator in the direction of the main diagonal in a space of independent variables, and with coefficients being constant on the characteristic of this operator under some condition on its eigenvalues. It is assumed that the coefficients and a given vector-function have the properties of periodicity and smoothness, where periods are rationally incommensurable positive constants. First, we study the homogeneous equation that reduces to a homogeneous linear system. Moreover, on this base, in terms of eigenvalues we establish conditions of existence of solutions being periodic with respect to all independent variables (so-called *multiperiodic* solutions). We give an integral representation of the multiperiodic solution of nonhomogeneous equation. The conditions for existence and uniqueness of the bounded and multiperiodic solutions of the n-th order linear nonhomogeneous equation are established. It is shown that the bounded solution of the nonhomogeneous equation is periodic in all variable solutions with a variable bounded period. This is one of the specific features of the equations with differential operator in the direction of the main diagonal.

Keywords: linear equation, differential operator, multiperiodic solution, integral representation.

Introduction

Let $x(\tau,t)$ be a function of variables $\tau\in(-\infty,+\infty)=\mathbb{R}$ and $t=(t_1,\dots,t_m)\in\mathbb{R}\times\dots\times\mathbb{R}=\mathbb{R}^m$, $D_e=\frac{\partial}{\partial\tau}+\left\langle e,\frac{\partial}{\partial t}\right\rangle$ the operator determined by the scalar product $\langle\cdot,\cdot\rangle$ of m-dimensional vectors $e=(1,\dots,1)$ and $\frac{\partial}{\partial t}=\left(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_m}\right)$.

The operator D_e is called the differential operator in the direction of the main diagonal or of vector field $\frac{dt}{d\tau}=e$ with characteristic $t=e(\tau-s)+\sigma$, where $(s,\sigma)\in\mathbb{R}\times\mathbb{R}^m$, $\sigma=(\sigma_1,\dots,\sigma_m)$. Obviously, $\sigma=t-e(\tau-s)$ is a base integral of the vector field, consequently, $D_eh(\sigma)=0$ for any differentiable function $h(\sigma)$. In particular, we have $D_e\sigma=0$. A function of the type $h(\sigma)$ is called a function constant on characteristic.

The study of oscillation solutions of systems of first order partial differential equations are importance in mathematics as in theoretical and applied aspect. For example, (θ,ω)-periodic systems in (τ,t) of the form

$$D_ex=f(\tau,t,x)$$

with differential operator D_e are closely connected with the theory of multifrequency oscillations [1–3], where $\omega=(\omega_1,\dots,\omega_m)$, $\omega_0=\theta$, $\omega_1,\dots,\omega_m$ are positive incommensurable constants. (θ,ω,ω)-periodicity in (τ,t,σ) of solutions of systems has been studied in [4–6].

In [7] a method of studying (θ,ω)-periodic solutions of such systems has been offered. A further study of these problems has brought forth the systems with characteristic $\sigma=t-e(\tau-s)$, of the form

$$D_ex=g(\tau,t,\sigma,x),$$

see [8]. After substitution $\tau-s\mapsto\tau$ we have $\sigma=t-e\tau$, with ω-periodicity in t of the systems being still valid. Consequently, $g(\tau,t,\sigma,x)$ is ω-periodic as well in σ. Let us remark that, generally speaking, with any fixed value of t, this system is quasi-periodic in τ.

A system of such form can be obtained from n-order linear equations

$$D_e^nx+a_1(\sigma)D_e^{n-1}x+...+a_n(\sigma)x=b(\tau,t,\sigma),\tag{1}$$

with operators $D_e^jx=D_e(D_e^{j-1}x)$, $j=\overline{1,n}$, where $a_j(\sigma)$, $j=\overline{1,m}$, and given vector-functions $b(\tau,t,\sigma)$ have the properties:

$$a_j(\sigma+q\omega)=a_j(\sigma)\in C_\sigma^{(1)}(\mathbb{R}^m),\quad j=\overline{1,m},\quad q\in\mathbb{Z}^m;\tag{2}$$

$$b(\tau+\theta, t+q\omega, \sigma+q\omega) = b(\tau, t, \sigma) \in C^{(0,1,1)}_{\tau,t,\sigma}(\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^m), \quad q\in\mathbb{Z}^m \tag{3}$$

with multiple vector-period $q\omega = (q_1\omega_1, \ldots, q_m\omega_m)$, $q = (q_1, \ldots, q_m)$, $q_j \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, $j = \overline{1,m}$, $\sigma = t - e\tau$.

Our basic objects of study here are the structure of the general solution to the equation (1) and, on this base, of its (θ, ω, ω)-periodic solutions.

Linear homogeneous equation

If $b = 0$, (1) becomes a homogeneous equation

$$D_e^n x + a_1(\sigma) D_e^{n-1} x + \ldots + a_n(\sigma) x = 0 \tag{4}$$

with corresponding characteristic polynomial equation in λ

$$H_n(\sigma, \lambda) = \lambda^n + a_1(\sigma)\lambda^{n-1} + \ldots + a_n(\sigma) = 0. \tag{5}$$

Suppose that all roots $\lambda = \lambda(\sigma)$ of (5) are in $\mathbb{R}^m$ and have the following properties.

1^0. Roots are either zero everywhere in $\mathbb{R}^m$, or roots are different from zero in $\mathbb{R}^m$: $\lambda(\sigma) = 0$, $\sigma\in\mathbb{R}^m$ or $\lambda(\sigma) \neq 0$, $\sigma\in\mathbb{R}^m$.

2^0. Roots are separated: $\inf\left|\lambda'(\sigma) - \lambda''(\sigma)\right| \geq \delta = const > 0$ for any pair of roots $\lambda'(\sigma)$ and $\lambda''(\sigma)$.

3^0. Roots are periodic with period ω: $\lambda(\sigma + q\omega) = \lambda(\sigma)$, $\sigma\in\mathbb{R}^m$.

4^0. Roots are continuously differentiable: $\lambda(\sigma) \in C^{(1)}_\sigma(\mathbb{R}^m)$.

It is not difficult to notice that the property 2^0 implies there exist exactly n roots of the equation (5) counted with multiplicity.

Assuming (2) and $1^0 - 4^0$, we aim to describe the structure of the set of solutions of the equation (4).

For this purpose, having put $x = y_1$, $D_e y_1 = y_2$, $D_e y_2 = y_3$, ..., $D_e y_{n-1} = y_n$, we obtain the equation (4) in the form of a linear system

$$D_e y = A(\sigma) y, \tag{6}$$

where $y = (y_1, \ldots, y_n)$ a vector, $A(\sigma)$ an $(n\times n)$-matrix of the form

$$A(\sigma) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n(\sigma) & -a_{n-1}(\sigma) & -a_{n-2}(\sigma) & \ldots & -a_2(\sigma) & -a_1(\sigma) \end{pmatrix}.$$

Obviously, the characteristic polynomials of the equation (4) and system (6) coincide $\det[\lambda E - A(\sigma)] = H_n(\sigma, \lambda)$. Consequently, the roots $\lambda = \lambda(\sigma)$ of equation (5) are eigenvalues of the matrix $A(\sigma)$.

We shall describe the structure of the set solutions of the system (6) and, consequently, of equations (4) in dependence on multiplicity of eigenvalues.

The case of simple eigenvalues. $\lambda_i(\sigma) \neq \lambda_j(\sigma)$, $i \neq j$, $i, j = \overline{1,n}$ and $\lambda_i(\sigma)$ are real-valued functions. It is easy to check that the Vandermonde matrix $T(\sigma)$, formed with eigenvalues $\lambda_j = \lambda_j(\sigma)$, $j = \overline{1,n}$, $\sigma\in\mathbb{R}^m$, and the diagonal matrix $J(\sigma) = \operatorname{diag}[\lambda_1(\sigma), \ldots, \lambda_n(\sigma)]$ satisfy the identity

$$A(\sigma)T(\sigma) = T(\sigma)J(\sigma).$$

Moreover, $\det T(\sigma) = \prod\limits_{1\leq j<i\leq n} [\lambda_i(\sigma) - \lambda_j(\sigma)] \neq 0$, $\sigma\in\mathbb{R}^m$, thanks to 1^0 and 2^0. Consequently, the matrix $T(\sigma)$ is reversible for $\sigma\in\mathbb{R}^m$ and with substitution $y = T(\sigma)z$ the system (6) boils down to the system

$$D_e z = J(\sigma) z, \tag{7}$$

which in scalar form can be written in the form $D_e z_\alpha = \lambda_\alpha(\sigma) z_\alpha$, $\alpha = \overline{1,n}$, where $z = (z_1, \ldots, z_n)$. From this system we get

$$z_\alpha = C_\alpha(\sigma)\exp\left[\tau\lambda_\alpha(\sigma)\right] \tag{8}$$

with arbitrary differentiable function $C_\alpha(\sigma)$. Taking into account (8), the general solution $z = z(\tau, \sigma)$ of the system (7) is of the form

$$z(\tau, \sigma) = Z(\tau, \sigma) C(\sigma), \tag{9}$$

where $Z(\tau,\sigma) = \operatorname{diag}\left[e^{\tau\lambda_1(\sigma)},\ldots,e^{\tau\lambda_n(\sigma)}\right]$ is a matrix, $C(\sigma) = (C_1(\sigma),...,C_n(\sigma))$ an arbitrary differentiable vector-function.

Then, substituting (9) into $y = T(\sigma)z$ we obtain the general solution of the system (6) in the form

$$y(\tau,\sigma) = T(\sigma)Z(\tau,\sigma)C(\sigma). \tag{10}$$

Further, from the relation (10) we determine the structure of the solution of the equation (4):

$$x(\tau,\sigma) = y_1(\tau,\sigma) = \sum_{j=1}^{n} e^{\lambda_j(\sigma)\tau}C_j(\sigma), \tag{11}$$

where it is taken into account that t_{1j}, the entries in the first row of the matrix $T(\sigma)$, are equal 1, $C_j(\sigma)$ are components of arbitrary differentiable vector-function $C(\sigma)$.

We remark that, in view of 3^0 and 4^0, the matrix $T(\sigma)$ is ω-periodic and continuously differentiable.

Theorem 1. Assume the conditions (2) and $1^0 - 4^0$ hold. Then in the case of simple eigenvalues, the solution x of equation (4) satisfying the initial condition

$$x|_{\tau=0} = u_1(\sigma), \quad D_e x|_{\tau=0} = u_2(\sigma), \quad \ldots, \quad D_e^{n-1}x|_{\tau=0} = u_n(\sigma) \tag{12}$$

can be presented in the form (11), where $u_j(\sigma)$, $j = \overline{1,n}$, are given differentiable ω-periodic functions.

Proof. Indeed, to determine the solution of the problem (4), (12) we act step by step with the operator D_e on the solution (11) and use the condition (12). Then we shall obtain a linear system of algebraic equations of the form $\sum\limits_{j=1}^{n} \lambda_j^{\alpha}(\sigma)C_j(\sigma) = u_2(\sigma)$, $(\alpha = \overline{1,n-1})$ with coefficients matrix being the Vandermonde matrix $T(\sigma)$. Consequently, $C(\sigma) = T^{-1}(\sigma)u(\sigma)$, where $u(\sigma) = (u_1(\sigma),\ldots,u_n(\sigma))$.

Therefore, the initial problem (4), (12) is uniquely solvable and its solution can be presented in the form (11). Consequently, the relation (11) is itself the general solution of equation (4).

The case of multiple eigenvalues. In view of $1^0 - 2^0$ $\lambda_i(\sigma)$, $i = \overline{1,r}$ have multiplicity k_i independent of $\sigma \in \mathbb{R}^m$ (we shall assume the eigenvalues are real-valued), where $k_1 + \cdots + k_n = n$.

A) We start with the particular case when $\lambda(\sigma)$ is a unique, with multiplicity n, root of the equation (4). We introduce Jordan block $J(\sigma)$, corresponding to the eigenvalue $\lambda(\sigma)$: $J(\sigma) = \lambda(\sigma)E + I$, where E is the identity matrix of order n, I is the matrix with units on the superdiagonal and the rest of entries being zero.

Let $T(\sigma)$ be the matrix with entries $t_{ij}(\sigma)$, $i,j = \overline{1,n}$, of the form

$$t_{ij}(\sigma) = \begin{cases} \sum\limits_{k=1}^{j} C_{i-1}^{k-1}\lambda^{i-k}(\sigma), & j \le i; \\ \sum\limits_{k=1}^{i} C_{i-1}^{k-1}\lambda^{i-k}(\sigma) = t_{ii}, & j > i, \end{cases}$$

where $C_i^j = \frac{i(i-1)\ldots(i-j+1)}{j!}$, $j \le i$ is a binomial coefficient.

It is not difficult to check that $A(\sigma)T(\sigma) = T(\sigma)J(\sigma)$, moreover $\det T(\sigma) = 1$. Consequently, a transformation of system (6) of the form (7) produces the system $D_e z = (\lambda(\sigma)E + I)z$. Then general solution $z(\tau,\sigma)$ of this system can be presented as

$$z(\tau,\sigma) = e^{\tau\lambda(\sigma)}Z(\tau)C(\sigma), \tag{13}$$

where $Z(\tau)$ is an $(n \times n)$-matrix of the form

$$Z(\tau) = \begin{pmatrix} 1 & \frac{\tau}{1!} & \frac{\tau^2}{2!} & \ldots & \frac{\tau^{n-1}}{(n-1)!} \\ 0 & 1 & \frac{\tau}{1!} & \ldots & \frac{\tau^{n-2}}{(n-2)!} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix},$$

$C(\sigma) = (C_1(\sigma),\ldots,C_n(\sigma))$ an arbitrary differentiable vector-function.

Therefore, by (7) and (13), we have a general solution $x(\tau,\sigma)$ of equation (4) in the form

$$x(\tau,\sigma) = \sum_{j=1}^{n} C_j(\sigma)\frac{\tau^{j-1}}{(j-1)!}e^{\tau\lambda(\sigma)} \tag{14}$$

with arbitrary differentiable functions $C_j(\sigma)$, $j=\overline{1,n}$. Obviously, with initial problem (4), (12) we have $C_j(\sigma)=u_j(\sigma)$, $j=\overline{1,n}$.

Consequently, one can formulate the result on the structure of the general solution (14) of the equation (4) in the case A).

Theorem 2. Assume (2) and 1^0-4^0. Then in the case of one eigenvalue $\lambda(\sigma)$ of multiplicity n, the solution $x(\tau,\sigma)$ of (4), (12) has the form (14).

Proof. Indeed, acting with operator D_e on relation (14), then using condition (12) we get an equation $(E+\lambda I_1)C(\sigma)=u(\sigma)$, where E is the identity matrix, I_1 is the matrix with units on the subdiagonal.

Obviously, $\det(E+\lambda I_1)=1$. Consequently, $C(\sigma)$ is uniquely defined by $u(\sigma)$. Theorem 2 is proved.

Before we pass to the general case, under the conditions as in Theorem 2 we first by study nonhomogeneous equation corresponding to equation (4) with quasilinear polynomial in τ of the form

$$D_e^n x+a_1(\sigma)D_e^{n-1}x+...+a_n(\sigma)x=\sum_{j=0}^{k}C_j^*(\sigma)\tau^j e^{\tau\mu(\sigma)}, \tag{15}$$

where coefficients $C_j^*(\sigma)$, $j=\overline{0,k}$, and index $\mu(\sigma)$ are differentiable ω-periodic functions with $\mu(\sigma)\neq\lambda(\sigma)$.

Since $\lambda(\sigma)$ is an n-multiple root of the characteristic equation (5), using the symbolic operator $D_e-\lambda(\sigma)$ the equation (15) can be given in the following form:

$$[D_e-\lambda(\sigma)]^n x=\sum_{j=0}^{k}C_j^*(\sigma)\tau^j e^{\tau\mu(\sigma)}. \tag{16}$$

In order to solve the equation (16), first we make the substitution $x=e^{\tau\mu(\sigma)}y$, and bring it to the form

$$\sum_{j=0}^{n}C_n^j[\mu(\sigma)-\lambda(\sigma)]^j D_e^{n-j}y=\sum_{j=0}^{k}C_j^*(\sigma)\tau^j, \tag{17}$$

where C_n^j is, as before, the binomial coefficient.

We set a particular solution $y^*(\tau,\sigma)$ of (17) with undetermined coefficients $v_j(\sigma)$, $j=\overline{0,k}$, as

$$y^*(\tau,\sigma)=\sum_{j=0}^{k}v_j(\sigma)\tau^j. \tag{18}$$

Then these coefficients are defined by recurrence relations and, as $\mu(\sigma)\neq\lambda(\sigma)$, they have unique presentation through coefficients $C_j^*(\sigma)$, $j=\overline{0,k}$:

$$v_j(\sigma)=\frac{1}{j!}v_j^*(\sigma)\equiv[\mu(\sigma)-\lambda(\sigma)]^{-j}\pi_j(\sigma,C_0^*(\sigma),\ldots,C_j^*(\sigma)), \tag{19}$$

where $\pi_j(\sigma,C_0^*(\sigma),\ldots,C_j^*(\sigma))$ are linear with respect to $C_0^*(\sigma),\ldots,C_j^*(\sigma)$ and ω-periodic in σ, differentiable functions. Having substituted (19) into (18) we get a particular solution $y^*(\tau,\sigma)$ of equation (17):

$$y^*(\tau,\sigma)=\sum_{j=0}^{k}v_j^*(\sigma)\frac{\tau^j}{j!}, \tag{20}$$

while in view of $x=e^{\tau\mu(\sigma)}y$ we have a particular solution $x^*(\tau,\sigma)$ of equation (16), consequently, equations (15) in the form of

$$x^*(\tau,\sigma)=y^*(\tau,\sigma)e^{\tau\mu(\sigma)} \tag{21}$$

with multipliers (20).

Since equation (15) is linear, its general solution $x(\tau,\sigma)$ is the sum of general solution (14) of the homogeneous equation (4) and a particular solution (21) of nonhomogeneous equation (15):

$$x(\tau,\sigma)=\sum_{j=1}^{n}C_j(\sigma)\frac{\tau^{j-1}}{(j-1)!}e^{\tau\lambda(\sigma)}+\sum_{j=0}^{k}v_j^*(\sigma)\frac{\tau^j}{j!}e^{\tau\mu(\sigma)} \tag{22}$$

with arbitrary differentiable coefficients $C_j(\sigma)$, $j=\overline{1,n}$.

As it was shown in the proof Theorem 2, analogously one can prove uniquely the initial problem for equation (15) with condition (12). Consequently, the relation (22) describes the structure of the general solution of the equation (15).

Corollary. Under the same conditions as in Theorem 2, the general solution $x(\tau,\sigma)$ of (15) is of the form (22).

B) We pass with our considerations to the general case, when roots $\lambda_1(\sigma),\ldots,\lambda_r(\sigma)$ have multiplicities $k_1,\ldots,k_r$ respectively, $k_1+\cdots+k_n=n$, and satisfy conditions 1^0-4^0.

In this case we will determine the structure of the general solution $x(\tau,\sigma)$ of equation (4) provided condition (2) holds. Using the symbolic operator

$$L(D_e)=D_e^n+a_1(\sigma)D_e^{n-1}+...+a_n(\sigma)=[D_e-\lambda_1(\sigma)]^{k_1}\ldots[D_e-\lambda_r(\sigma)]^{k_r},$$

we shall present (4) in the form

$$[D_e-\lambda_1(\sigma)]^{k_1}\ldots[D_e-\lambda_{r-1}(\sigma)]^{k_{r-1}}[D_e-\lambda_r(\sigma)]^{k_r}x=0. \tag{23}$$

We shall prove that the general solution $x(\tau,\sigma)$ of (4), and so of (23), has the form

$$x(\tau,\sigma)=\sum_{j=1}^{k_1}C_j(\sigma)\frac{\tau^{j-1}e^{\tau\lambda_1(\sigma)}}{(j-1)!}+...+\sum_{j=1}^{k_r}C_{n-k_r+j}(\sigma)\frac{\tau^{j-1}e^{\tau\lambda_r(\sigma)}}{(j-1)!}. \tag{24}$$

For the proof we will use the induction method. For $r=1$ the formula (24) holds due to Theorem 2. We shall assume that it true for $r-1$ and prove it for r. For this purpose, in equation (24) we put $[D_e-\lambda_r(\sigma)]^{k_r}x=z$ and get an equation

$$[D_e-\lambda_1(\sigma)]^{k_1}\ldots[D_e-\lambda_{r-1}(\sigma)]^{k_{r-1}}z=0. \tag{25}$$

Since the formula (24) holds for $r-1$ eigenvalues, the equation (25) has a general solution $z(\tau,\sigma)$ of the form

$$z(\tau,\sigma)=\sum_{j=1}^{k_1}C_j(\sigma)\frac{\tau^{j-1}e^{\tau\lambda_1(\sigma)}}{(j-1)!}+...+\sum_{j=1}^{k_{r-1}}C_{n-k_r-kr-1+j}(\sigma)\frac{\tau^{j-1}e^{\tau\lambda_{r-1}(\sigma)}}{(j-1)!}. \tag{26}$$

Further, having put the expression (26) into $[D_e-\lambda_r(\sigma)]^{k_r}x=z$ we get a nonhomogeneous equation

$$[D_e-\lambda_r(\sigma)]^r x=\sum_{j=1}^{k_1}C_j(\sigma)\frac{\tau^{j-1}}{(j-1)!}e^{\tau\lambda_1(\sigma)}+...+\sum_{j=1}^{k_{r-1}}C_{n-k_r-kr-1+j}(\sigma)\frac{\tau^{j-1}}{(j-1)!}e^{\tau\lambda_{r-1}(\sigma)}. \tag{27}$$

Now, in order to solve the equation (27) it is necessary to apply the corollary of Theorem 2 to solution of each of the equations

$$[D_e-\lambda_r(\sigma)]^r x=\sum_{j=1}^{k_1}C_j(\sigma)\frac{\tau^{j-1}}{(j-1)!}e^{\tau\lambda_1(\sigma)};$$

....

$$[D_e-\lambda_r(\sigma)]^r x=\sum_{j=1}^{k_{r-1}}C_{n-k_r-kr-1+j}(\sigma)\frac{\tau^{j-1}}{(j-1)!}e^{\tau\lambda_{r-1}(\sigma)}$$

and use by the superposition principle. Obviously, as coefficients of quasipolynomial solutions of these equations depend linearly on the arbitrary coefficients of their righthand parts, they are also arbitrary. The formula (24) is proved. Consequently, the solution of (4), (12) is of the form (24). This result will be formulated as the theorem.

Theorem 3. Assume the conditions (2) and 1^0-4^0 hold. Then the general solution $x(\tau,\sigma)$ of equation (4) has the form (24), with arbitrary differentiable coefficients $C_j(\sigma)$, $j=\overline{1,n}$.

The case of complex eigenvalues. Let the roots of equation (4) $\lambda_j^{\pm}(\sigma)=\alpha_j(\sigma)\pm i\beta_j(\sigma)$, $j=\overline{1,p}$ have multiplicity k_j, $j=\overline{1,p}$, $2k_1+\cdots+2k_p=n_1\le n$ and its have sets of real parts $\{\alpha_j(\sigma)\}$ and imaginary parts $\{\beta_j(\sigma)\}$ both of which possess the properties 1^0-4^0.

Since the complex solution $x(\tau,\sigma)=v(\tau,\sigma)+i\omega(\tau,\sigma)$ has real and imaginary parts, $\operatorname{Re}x(\tau,\sigma)=v(\tau,\sigma)$ and $\operatorname{Im}x(\tau,\sigma)=\omega(\tau,\sigma)$, being solutions of (4), to any pair of complex coupled roots $\lambda_j^{\pm}(\sigma)=\alpha_j(\sigma)\pm i\beta_j(\sigma)$ there corresponds the solution

$$x^j(\tau,\sigma)=\left[P_j(\tau,\sigma)\cos(\beta_j(\sigma)\tau)+Q_j(\tau,\sigma)\sin(\beta_j(\sigma)\tau)\right]e^{\tau\alpha_j(\sigma)}, \tag{28}$$

where $P_j(\tau,\sigma)=\sum\limits_{k=1}^{k_j} p_k^{(j)}(\sigma)\frac{\tau^{k-1}}{(k-1)!}$ and $Q_j(\tau,\sigma)=\sum\limits_{k=1}^{k_j} q_k^{(j)}(\sigma)\frac{\tau^{k-1}}{(k-1)!}$ with arbitrary coefficients $p_k^{(j)}(\sigma)$ and $q_k^{(j)}(\sigma)$, $k=\overline{1,k_j}$, $j=\overline{1,p}$.

Consequently, the general solution $x(\tau,\sigma)$ of equation (4) in the case of complex roots has the form

$$x(\tau,\sigma)=\sum_{j=1}^{p} x^{(j)}(\tau,\sigma)+\sum_{j=n_1+1}^{n} x^{(j)}(\tau,\sigma), \tag{29}$$

where $x^{(j)}(\tau,\sigma)$, $j>n_1$ are solutions corresponding to real roots and for $j=\overline{1,p}$ $x^{(j)}(\tau,\sigma)$ are defined by the relation (28).

Therefore, the following theorem is proved.

Theorem 4. Suppose that, under condition (2), the equation (4) has complex eigenvalues $\lambda_j(\sigma)=\alpha_j(\sigma)\pm i\beta_j(\sigma)$, $j=\overline{1,p}$ of multiplicity k_j and real eigenvalues satisfying the properties 1^0-4^0. Then the general solution $x(\tau,\sigma)$ of equation (4) is defined by relations (28) and (29).

Notice that in the case of Theorem 3 and Theorem 4, endowed with initial conditions (12), it is possible to show the unique solubility of the initial problem (4), (12).

Let $x^{(j)}(\tau,\sigma)$, $j=\overline{1,n}$ be solutions of the equation (4) satisfying the initial conditions

$$D_e^k x^{(j)}(\tau,\sigma)\big|_{\tau=\tau_0}=\begin{cases}0, & k\neq j-1;\\ 1, & k=j-1,\end{cases} \tag{30}$$

where $k=\overline{0,n-1}$.

Such a system of solutions we shall call a normalized fundamental system of solutions of (4).

Theorem 5. Under conditions as in Theorem 4 the unique solution $x(\tau,\sigma)$ of problem (4), (12) is defined by

$$x(\tau,\sigma)=\sum_{j=1}^{n} u_j(\sigma)x^{(j)}(\tau,\sigma), \tag{31}$$

where $x^{(j)}(\tau,\sigma)$, $j=\overline{1,n}$ is a normalized fundamental system of the solutions.

Indeed, it is not difficult to check that (31) fulfills (4) and in the view (30) the initial condition (12). Linear combinations of solutions from normalized fundamental system satisfying condition (12) are uniquely described the relation (31).

Linear nonhomogeneous equation

Assume the conditions (2), (3) and 1^0-4^0 hold. On the base of Theorem 5 we introduce the solution $X(\tau,t,\sigma,s,\sigma+es)$ of equation (4) satisfying the initial condition

$$D_e^k X(s,\sigma+es,\sigma,s,\sigma+es)=0,\quad (k=\overline{0,n-2}),\quad D_e^{n-1}X(s,\sigma+es,\sigma,s,\sigma+es)=1 \tag{32}$$

and function $x^0(\tau,t,\sigma)=\int\limits_0^\tau X(\tau,t,\sigma,s,\sigma+es)b(s,\sigma+es,\sigma)ds$. It is easy to check that $x^0(\tau,t,\sigma)$ in view of (32) satisfy the equation (1) with zero initial condition $D_e^k x^0(s,\sigma+es,\sigma)=0,\quad k=\overline{0,n-1}$.

Therefore, unique solution $x(\tau,t,\sigma)$ of equation (1) with initial condition (12) is defined by

$$x(\tau,t,\sigma)=\sum_{j=1}^{n} u_j(\sigma)x^{(j)}(\tau,\sigma)+x^0(\tau,t,\sigma). \tag{33}$$

This result will be formulated as Theorem.

Theorem 6. Assume the conditions (2) and (3) hold, sets of real eigenvalues and complex eigenvalues possess the properties 1^0-4^0. Then initial problem for equation (1) with condition (12) has unique solution (33).

The bounded and periodic solutions

When is known structures of the general solution to the equations (1) and (4) then can present conditions of existence of bounded and periodic solutions in terms of eigenvalues.

Let's start to consider homogeneous equation (4) with condition (2). It is limited solutions of problem of the form (4), (12).

Theorem 7. The solution $x(\tau,\sigma)$ of problem (4), (12) under condition (2) is ω-periodic in $\sigma \in R^m$.

Proof of the Theorem 7 follows from ω-periodic of initial functions $u_j(\sigma)$, $j=\overline{1,n}$ and eigenvalues $\lambda_j(\sigma)$, $j=\overline{1,n}$.

Theorem 8. Under the conditions (2), 1^0-4^0 and when real parts of eigenvalues are different from zero $Re\,\lambda_j(\sigma)\neq 0$, $j=\overline{1,n}$ then equation (4) hasn't bounded therefore periodic solutions except zero.

It is not difficult to show that under the conditions of Theorem 8 will be found constant $\gamma>0$, $\Gamma>0$ and any solution $x^j(\tau,\sigma)$ of equation (4) entering into the fundamental system is satisfied by estimation

$$|x^j(\tau,\sigma)|\leq \Gamma e^{-\gamma|\tau|}. \qquad (34)$$

Then as in Theorem 5 in view of (34) follows unbounded of all solutions $x(\tau,\sigma)$ of problem (4), (12) except zero.

Further allow that equation (4) has only imaginary eigenvalues $\lambda_{1,2}(\sigma)=\pm i\beta(\sigma)\neq 0$ which by the conditions on Theorem 4 satisfied bounded solution of the form $x^*(\tau,\sigma)=C_1(\sigma)\cos(\beta_j(\sigma)\tau)+C_2(\sigma)\sin(\beta_j(\sigma)\tau)$ with arbitrary ω-periodic in σ coefficients $C_1(\sigma)$ and $C_2(\sigma)$. Obviously that this solution (θ,ω)-periodic in (τ,σ), where $\theta=\frac{2\pi}{\beta(\sigma)}\equiv\theta(\sigma)$ is ω-periodic differentiable function.

Therefore in this case the bounded solution $x^*(\tau,\sigma)$ of equation being periodic in τ with variable bounded period $\theta(\sigma)=\omega_0(\sigma)$. We note that it is one of specific particularities of equation with operator D_e.

In this case (ω_0,ω)-periodic solutions consist double-parameter family where parameters are ω-periodic in σ functions $C_1=C_1(\sigma)$ and $C_2=C_2(\sigma)$. If equation (4) has zero eigenvalue $\lambda=0$ then it is in view of Theorem 3 satisfied one-parameter family ω-periodic in σ of solutions $x=C(\sigma)$.

Theorem 9. If under the conditions as in Theorem 4 the equation (4) has zero $\lambda_1=0$ and only imaginary eigenvalues $\lambda_j(\sigma)=\pm i\beta_j(\sigma)$, $j=\overline{2,p}$ then it is allowed bounded solutions

$$x(\tau,\sigma)=C_1(\sigma)+\sum_{j=2}^{p}C_{1j}(\sigma)\cos(\beta_j(\sigma)\tau)+C_{2j}(\sigma)\sin(\beta_j(\sigma)\tau),$$

where $p<n$, $C_1(\sigma)$, $C_{1j}(\sigma)$, $C_{2j}(\sigma)$ are differentiable arbitrary ω-periodic functions.

From Theorem 9 we see that presented solution here $x(\tau,\sigma)$ consists of line combinations $\theta_j(\sigma)=\frac{2\pi}{\beta_j(\sigma)}$-periodic in τ of functions $j=\overline{1,p}$.

Further we consider the case of Theorem 8. Let $x^j(\tau-s,\sigma)$, $j=\overline{1,n}$ – fundamental system of solutions of homogeneous equation (4) are satisfied by the conditions

$$D_e^{k-1}x^{(j)}(\tau-s,\sigma)|_{\tau=s}=\begin{cases}0, & k\neq j;\\ 1, & k=j.\end{cases}$$

Then solution $X(\tau-s,\sigma)$ of equation (4) is satisfied by the condition

$$D_e^{k-1}X(\tau-s,\sigma)|_{\tau=s}=0,\quad (k=\overline{1,n-2}),\quad D_e^{n-1}X(\tau-s,\sigma)|_{\tau=s}=1 \qquad (35)$$

according to Theorem 5 is presented in the form $X(\tau-s,\sigma)=\sum\limits_{j=1}^{n}u_j(\sigma)x(\tau-s,\sigma)$. This solution fall into sum of solutions $X(\tau-s,\sigma)=X_-(\tau-s,\sigma)+X_+(\tau-s,\sigma)$, which are satisfied by estimations:

$$|X_-(\tau-s,\sigma)|\leq\Gamma_-e^{-\gamma(\tau-s)},\quad \tau\geq s,\quad |X_+(\tau-s,\sigma)|\leq\Gamma_+e^{\gamma(\tau-s)},\quad \tau<s \qquad (36)$$

with several positive constants Γ_- and Γ_+ moreover in view of (35) have

$$D_e^{k-1}X_-(\tau-s,\sigma)|_{\tau=s}+D_e^{k-1}X_+(\tau-s,\sigma)|_{\tau=s}=0,\quad (k=\overline{1,n-2});$$

$$D_e^{n-1}X_-(\tau-s,\sigma)|_{\tau=s}+D_e^{n-1}X_+(\tau-s,\sigma)|_{\tau=s}=1. \qquad (37)$$

Further we introduce function

$$x^*(\tau,t,\sigma) = \int\limits_{-\infty}^{\tau} X_-(\tau-s,\sigma)f(s,\sigma+es,\sigma)ds - \int\limits_{\tau}^{+\infty} X_+(\tau-s,\sigma)f(s,\sigma+es,\sigma)ds. \qquad (38)$$

Obviously that integrals in correlation (38) in view of (36) are converged evenly, allowed under the integral n-time differentiality and in view of (37) is satisfied equation (1). It is easy to check that in view of (3) $x^*(\tau,t,\sigma)$ has property (θ,ω,ω)-periodicity in (τ,t,σ). Condition (36) is provided unique bounded solution (38).

Therefore, the following Theorem is proved.

Theorem 10. Under the conditions (2), (3) and $1^0 - 4^0$ equation (1) has unique (θ,ω,ω)-periodic in (τ,t,σ) solution of the form (38).

This study is adjacent to the studies [9–11].

In conclusion we shall notice that in this work at research of basic object we used properties $1^0 - 4^0$ of eigenvalues and structures of general solutions are satisfied to being considered equations.

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Негізгі диагональ бойынша дифференциалдау операторлы сызықты теңдеудің жалпы, шенелген және көппериодты шешімдері

Мақалада тәуелсіз айнымалылар кеңістігінің негізгі диагоналінің бағыты бойынша дифференциалдау операторымен және меншікті мәндерге қойылатын кейбір шарттар кезінде осы оператордың сипаттамаларында тұрақты болатын коэффициенттермен n-ші ретті сызықты теңдеудің жалпы шешімінің құрылымы анықталды. Коэффициенттермен берілген вектор-функция периодтылық және тегістік қасиеттеріне ие деп ұйғарылады, мұндағы периодтар — рационалды өлшемдес емес, оң тұрақтылар. Әуелі біртекті сызықты жүйеге ауыстыру көмегімен келтірілетін, біртекті теңдеу зерттелді. Әрі қарай осы негізде меншікті мәндер терминінде, сызықты теңдеудің барлық тәуелсіз айнымалылар бойынша периодтылығының бар болу шарттары орнатылды. Біртексіз теңдеудің көппериодты шешімінің интегралды көрінісі берілген. n-ші ретті біртексіз сызықты теңдеудің шенелген және көппериодты шешімдерінің бар және жалғыз болу шарттары орнатылды. Біртексіз теңдеудің шенелген шешімі шенелген айнымалы периодымен барлық айнымалылар бойынша периодты шешім болатындығы көрсетілген. Бұл негізгі диагональ бағыты бойынша дифференциалдау операторлы теңдеудің спецификалық ерешеліктерінің бірі.

Кілт сөздер: сызықты теңдеу, дифференциалдау оператор, көппериодты шешім, интегралды көрініс.

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Общие, ограниченные и многопериодические решения линейного уравнения с дифференциальным оператором по главной диагонали

В статье определена структура общего решения линейного уравнения n-го порядка с дифференциальным оператором по направлению главной диагонали пространства независимых переменных и коэффициентами, постоянными на характеристике этого оператора при некоторых условиях на собственные значения. Предположено, что коэффициенты и заданная вектор-функция обладают свойствами периодичности и гладкости, где периоды – рационально несоизмеримые положительные постоянные. Сначала исследовано однородное уравнение, которое с помощью замены сводится к однородной линейной системе. Далее, на этой основе, в терминах собственных значений устанавливаются условия существования периодических по всем независимым переменным (многопериодическим) решений линейного уравнения. Дано интегральное представление многопериодического решения неоднородного уравнения. Установлены условия существования и единственности ограниченного и многопериодического решения линейного неоднородного уравнения n-го порядка. Показано, что ограниченное решение неоднородного уравнения является периодическим по всем переменным решением с переменным ограниченным периодом. Это есть одна из специфических особенностей уравнений с оператором дифференцирования по направлению главной диагонали.

Ключевые слова: линейное уравнение, дифференциальный оператор, многопериодическое решение, интегральное представление.

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Sufficient conditions for the precompactness of sets in Local Morrey-type spaces

In this paper we give sufficient conditions for the pre-compactness of sets in local Morrey-type spaces $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. For $w(r) = r^{-\lambda},\ \theta = \infty,\ 0 \leq \lambda \leq \frac{n}{p}$ there follows a known result for the Morrey spaces $M_p^\lambda(\mathbb{R}^n)$. In the case $\lambda = 0$ this is the well-known Frechet-Kolmogorov theorem. The pre-compactness of sets in Morrey spaces was investigated in the works [1, 2], and in generalized Morrey spaces $M_p^{w(\cdot)}(\mathbb{R}^n)$ in the works [3, 4]. The aim of this paper is to generalize these results to the case of Local Morrey-type spaces $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. By using theorem of pre-compactness set in local Morrey-type spaces, compact of operators can be checked in this spaces, since compact operator transfers from bounded set of one space to pre-compact set of another space. In this paper, the conditions of precompactness of sets in local spaces of Morrey type are given in terms of the difference of the function $\lim_{u\to 0}\sup_{f\in S}\|f(\cdot+u)-f(\cdot)\|_{LM_{p\theta,w}} = 0$. Earlier, the necessary and sufficient conditions for precompactness of sets in local spaces of Morrey type were published in [5], which were given in terms of the mean functions $\lim\limits_{\delta\to 0^+}\sup\limits_{f\in S}\|A_\delta f - f\|_{L_p(B(0,R_2)\setminus B(0,R_1))} = 0$.

Keywords: compactness, precompact, Freche-Kolmogorov theorem, local Morrey-type spaces.

The classical Morrey space was introduced in the works of Charles Morrey [6] in 1938 in connection with the investigation of the solution of quasilinear elliptic differential equations. In recent decades questions of the boundedness and compactness of various operators in Morrey-type spaces have been actively studied ([7–9]).

Morrey spaces M_p^λ are defined as the set of all functions $f \in L_p^{loc}(\mathbb{R}^n)$, for which , $0 \leq \lambda \leq \frac{n}{p}$, $0 < p \leq \infty$, with a finite quasinorm

$$\|f\|_{M_p^\lambda(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\ r>0} r^{-\lambda}\left(\int\limits_{B(x,r)} |f(y)|^p dy\right)^{\frac{1}{p}} < \infty,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at the point x of radius $r > 0$.

Note that

$$\|f\|_{M_p^0(\mathbb{R}^n)} \equiv \|f\|_{L_p(\mathbb{R}^n)}, \|f\|_{M_p^{\frac{n}{p}}(\mathbb{R}^n)} \equiv \|f\|_{L_\infty(\mathbb{R}^n)}.$$

If $\lambda < 0,\ \lambda > \frac{n}{p}$ the space $M_p^\lambda(\mathbb{R}^n)$ is trivial, i.e. consists only of functions equivalent to zero on $\mathbb{R}^n$.

According to the well-known Freche-Kolmogorov theorem [10], the set $S \subset L_p(\mathbb{R}^n)$, where $1 \leq p < \infty$, is precompact if and only if

$$\sup_{f\in S}\|f\|_{L_p(\mathbb{R}^n)} < \infty; \tag{1}$$

$$\lim_{\delta\to 0^+}\sup_{|h|\leq\delta}\|f(\cdot+h)-f(\cdot)\|_{L_p(\mathbb{R}^n)} = 0 \tag{2}$$

and

$$\lim_{R\to\infty}\sup_{f\in S}\|f\|_{L_p(^c B(0,R))} = 0, \tag{3}$$

where $^c B(0,R)$ is the complement of a ball $B(0,R)$.

Conditions (1)–(3) are equivalent to the union of conditions (1), (3) and

$$\lim_{\delta\to 0^+}\sup_{f\in S}\|A_\delta f - f\|_{L_p(\mathbb{R}^n)} = 0, \tag{4}$$

where for any $\delta > 0$ and $f \in L_1^{loc}(\mathbb{R}^n)$

$$(A_\delta f)(x) = \frac{1}{|B(x,\delta)|} \int\limits_{B(x,\delta)} f(y)dy = \int\limits_{B(0,\delta)} \omega_\delta(y) f(x-y)dy = (\omega_\delta * f)(x),\ x \in \mathbb{R}^n,$$

where $\omega_\delta(x) = \frac{\chi_{B(0,\delta)}(x)}{|B(0,\delta)|}$, $\chi_{(A)}$ is characteristic function of the set $A \subset \mathbb{R}^n$, ${}^c A$ is the complement of the set A, $|A|$ is Lebesgue measure of the set A. Recall that condition (4) follows from condition (2), and condition (2) follows from the set of conditions (4) and (3).

Note also that if $A \subset \mathbb{R}^n$ is bounded set, then for precompactness of the set $S \subset L_p(A)$ it is necessary and sufficient that conditions (1)–(2) are satisfied, where $\mathbb{R}^n$ replaced by a set A.

The questions of the precompactness of sets in Morrey spaces were investigated in the works [1, 2, 5, 6, 11–14], and when $r^{-\lambda} \equiv w(r)$ for generalized spaces Morrey $M_p^{w(\cdot)}(\mathbb{R}^n)$ were investigated in the works [3, 4]. The questions of the precompactness of sets in Banach spaces [15].

The aim of this paper is to generalize this results to the case of general local Morrey-type spaces.

Definition. Let $0 < p, \theta \le \infty$, and let w be a nonnegative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,w(\cdot)}$ the general local Morrey-type space, the space of all functions $f \in L_p^{loc}(R^n)$ with finite quasi-norm

$$\|f\|_{LM_{p\theta,w(\cdot)}} \equiv \|f\|_{LM_{p\theta,w(\cdot)}(\mathbb{R}^n)} = \left\| w(r)\, \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)}.$$

We denote by Ω_θ the set of all functions that are nonnegative, measurable on $(0,\infty)$, not equivalent 0 and such, that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

The space $LM_{p\theta,w(\cdot)}$ is non-trivial, that is, it consists not only of functions, equivalent to 0 on $\mathbb{R}^n$, if and only if $w \in \Omega_\theta$ [14].

Theorem. Let $1 \le p \le \theta \le \infty$ and $w \in \Omega_{p\theta}$. Let $S \subset LM_{p\theta,w}(R^n)$ is satisfied:

$$\sup_{f\in S} \|f\|_{LM_{p\theta,w}} < \infty; \tag{5}$$

$$\lim_{u\to 0} \sup_{f\in S} \|f(\cdot+u) - f(\cdot)\|_{LM_{p\theta,w}} = 0 \tag{6}$$

and

$$\lim_{r\to\infty} \sup_{f\in S} \left\| f\chi_{C_{B(0,r)}} \right\|_{LM_{p\theta,w}} = 0. \tag{7}$$

Then the set S is precompact in $LM_{p\theta,w}(R^n)$.

For proof this theorem we need next statements.

For $f \in L_1^{loc}(\mathbb{R}^n)$ and $r > 0$ define

$$(M_r f)(x) = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)dy,$$

where $|\ A\ |$ means Lebega spaces $A \subset \mathbb{R}^n$.

Lemma 1. Let $1 \le p \le \theta \le \infty$, $w \in \Omega_\theta$. Then for all $f \in LM_{p\theta}^{w(\cdot)}$ and $r > 0$ next is true:

$$\|M_r f - f\|_{LM_{p\theta,w}} \le \sup_{u\in B(0,r)} \|f(\cdot+u) - f(\cdot)\|_{LM_{p\theta,w}}.$$

Proof. Let $z \in R^n$ and $\rho > 0$. Then by inequality of Gelder

$$\|M_r f - f\|_{L_p(B(z,\rho))} =$$

$$= \left(\int\limits_{B(z,\rho)} \left| \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)dy - f(x) \right|^p dx \right)^{\frac{1}{p}} =$$

$$= \left(\int\limits_{B(z,\rho)} \left| \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} (f(y) - f(x))dy \right|^p dx \right)^{\frac{1}{p}} \le$$

$$\le \left(\int\limits_{B(z,\rho)} \left(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y) - f(x)|^p dy \right) dx \right)^{\frac{1}{p}} = |y = x + u| =$$

$$= \left(\int\limits_{B(z,\rho)} \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} |f(x+u) - f(x)|^p du \right) dx \right)^{\frac{1}{p}} =$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\int\limits_{B(z,\rho)} |f(x+u) - f(x)|^p dx \right) du \right)^{\frac{1}{p}} =$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \|f(\cdot+u) - f(\cdot)\|^p_{L_p(B(z,\rho))} du \right)^{\frac{1}{p}}.$$

Next

$$\|M_r f - f\|_{LM_{p\theta,w}} = \left\| w(\rho) \|M_r f - f\|_{L_p(B(0,\rho))} \right\|_{L_\theta(0,\infty)} \le$$

$$\le \left\| w(\rho) \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \|f(\cdot+u) - f(\cdot)\|^p_{L_p(B(0,\rho))} du \right)^{\frac{1}{p}} \right\|_{L_\theta(0,\infty)} =$$

$$= \left\| \frac{1}{|B(0,r)|} \int\limits_{B(0,r)} w(\rho)^p \|f(\cdot+u) - f(\cdot)\|^p_{L_p(B(0,\rho))} du \right\|^{\frac{1}{p}}_{L_{\frac{\theta}{p}}(0,\infty)}.$$

As this $\frac{\theta}{p} \ge 1$, then using inequality of Minkovskogo for integrals, we get next

$$\|M_r f - f\|_{LM_{p\theta,w}} \le \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\int\limits_0^\infty w(\rho)^\theta \|f(\cdot+u) - f(\cdot)\|^\theta_{L_p(B(0,\rho))} d\rho \right)^{\frac{p}{\theta}} du \right)^{\frac{1}{p}} =$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \|f(\cdot+u) - f(\cdot)\|^p_{LM_{p\theta,w}} du \right)^{\frac{1}{p}} \le$$

$$\le \sup_{u \in B(0,r)} \|f(\cdot+u) - f(\cdot)\|_{LM_{p\theta,w}}.$$

Lemma 1 is proofed.

Lemma 2. Let $1 \le p \le \theta \le \infty$, $w \in \Omega_{p\theta}$. Then for all $f \in LM^{w(\cdot)}_{p\theta}$ and $r > 0$ next inequality is:

$$\|M_r f\|_{LM_{p\theta,w}} \le \|f\|_{LM_{p\theta,w}}. \qquad (8)$$

Proof. By inequality of Gelder

$$\|M_r f\|_{L_p(B(z,\rho))} = \left(\int\limits_{B(z,\rho)} \left| \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)dy \right|^p dx \right)^{\frac{1}{p}} \le$$

$$\leq \left(\int\limits_{B(z,\rho)} \left(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|^p \, dy \right) dx \right)^{\frac{1}{p}} =$$

$$= (y = x + u)$$

$$= \left(\int\limits_{B(z,\rho)} \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} |f(x+u)|^p \, du \right) dx \right)^{\frac{1}{p}} =$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\int\limits_{B(z,\rho)} |f(x+u)|^p \, dx \right) du \right)^{\frac{1}{p}} =$$

$$= (x + u = v)$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\int\limits_{B(z+u,\rho)} |f(v)|^p \, dv \right) du \right)^{\frac{1}{p}} =$$

$$= \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \|f\|^p_{L_p(B(z+u,\rho))} \, du \right)^{\frac{1}{p}} .$$

Since $\frac{\theta}{p} \geq 1$, that

$$\|M_r f\|_{LM_{p\theta,w}} \leq \sup_{z\in R^n} \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\int\limits_0^\infty w(\rho)^\theta \, \|f\|^\theta_{L_p(B(z+u,\rho))} \, d\rho \right)^{\frac{p}{\theta}} du \right)^{\frac{1}{p}} \leq$$

$$\leq \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \sup_{z\in R^n} \left(\int\limits_0^\infty w(\rho)^\theta \, \|f\|^\theta_{L_p(B(z+u,\rho))} \, d\rho \right)^{\frac{p}{\theta}} du \right)^{\frac{1}{p}} =$$

$$= \sup_{z\in R^n} \left\| \frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(w(\rho) \, \|f\|_{L_p(B(z+u,\rho))} \right)^p du \right\|^{\frac{1}{p}}_{L_{\frac{\theta}{p}}(0,\infty)} \leq$$

$$\leq \sup_{z\in R^n} \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left\| w(\rho) \, \|f\|_{L_p(B(z+u,\rho))} \right\|^p_{L_\theta(0,\infty)} du \right)^{\frac{1}{p}} \leq$$

$$\leq \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \left(\sup_{v\in R^n} \left\| w(\rho) \, \|f\|_{L_p(B(v,\rho))} \right\|_{L_\theta(0,\infty)} \right)^p dv \right)^{\frac{1}{p}} = \|f\|_{LM_{p\theta,w}}.$$

For defining $\delta > 0$ by $w_\delta(f, G)$ function f in set $G \subset R^n$:

$$w_\delta(f, G) = \sup_{\substack{x_1, x_2 \in G \\ |x_1 - x_2| \leq \delta}} |f(x_1) - f(x_2)| .$$

$$\|M_r f\|_{LM_{p\theta,w}} \leq \sup_{z\in R^n} \left\| w(\rho) \, \|M_r f\|_{L_p(B(z,\rho))} \right\|_{L_\theta(0,\infty)} \leq$$

$$\leq \sup_{z\in R^n} \left\| w(\rho) \left(\frac{1}{|B(0,r)|} \int\limits_{B(0,r)} \|f\|^p_{L_p(B(z+u,\rho))} du \right)^{\frac{1}{p}} \right\|_{L_\theta(0,\infty)} .$$

Lemma 2 is proofed.

Lemma 3. Let $1 \leq p, \theta \leq \infty$, $w \in \Omega_{p\theta}$. Then exists $r_0 > 0$ for every $0 < r \leq r_0$ exists $C_r > 0$, depending from r, n, p, θ, w

1) for every $f \in LM^{w(\cdot)}_{p\theta}$

$$\|M_r f\|_{C(R^n)} \leq C_r \|f\|_{LM_{p\theta,w}} ; \tag{9}$$

2) for every $\delta > 0$

$$w_\delta(M_r f; R^n) \leq C_r \sup_{\substack{|u|\leq\delta \\ u\in B(0,\delta)}} \|f(\cdot+u) - f(\cdot)\|_{LM_{p\theta,w}} .$$

Proof. 1. For function $w \in \Omega_{p\theta}$ no equivalent 0, then exists $r_0 > 0$ this, that $\|w\|_{L_\theta(r_0,\infty)} > 0$. Let $0 < r \leq r_0$. $x \in R^n$

$$|M_r f(x)| \leq \frac{1}{|B(x,r)|^{\frac{1}{p}}} \|f\|_{L_p(B(x,r))} .$$

Hence,

$$\|w(\rho) M_r f(x)\|_{L_\theta(r,\infty)} \leq \frac{1}{(v_n r^n)^{\frac{1}{p}}} \left\| w(\rho) \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(r,\infty)} ,$$

where v_n — volume of unit ball in R^n and

$$|M_r f(x)| \, \|w(\rho)\|_{L_\theta(r,\infty)} \leq \frac{1}{(v_n r^n)^{\frac{1}{p}}} \left\| w(\rho) \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(0,\infty)} .$$

That's why

$$\sup_{x\in R^n} |M_r f(x)| \leq C_r \sup_{x\in R^n} \left\| w(\rho) \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(0,\infty)} = C_r \|f\|_{LM_{p\theta,w}} ,$$

where $C_r = \left(\|w\|_{L_\theta(r,\infty)} (v_n r^n)^{\frac{1}{p}} \right)^{-1}$.

2. Next for every $x_1, x_2 \in B(0,r)$

$$|(M_r f)(x_1) - (M_r f)(x_2)| = \frac{1}{v_n r^n} \left| \int\limits_{B(x_1,r)} f(y)dy - \int\limits_{B(x_2,r)} f(y)dy \right| =$$

$$= (v_n r^n)^{-1} \left| \int\limits_{B(0,r)} f(z+x_1)dz - \int\limits_{B(0,r)} f(z+x_2)dz \right| \leq$$

$$\leq (v_n r^n)^{-1} \int\limits_{B(0,r)} |f(z+x_1) - f(z+x_2)| \, dz =$$

$$= (v_n r^n)^{-1} \int\limits_{B(x_2,r)} |f(s+x_1-x_2) - f(s)| \, ds \leq$$

$$\leq (v_n r^n)^{-\frac{1}{p}} \|f(\cdot+x_1-x_2) - f(\cdot)\|_{L_p(B(x_2,r))} .$$

That's why, by first step of proof

$$\sup_{\substack{x_1,x_2\in R^n \\ |x_1-x_2|\leq\delta}} |(M_r f)(x_1) - (M_r f)(x_2)| \leq C_r \sup_{\substack{x_1,x_2\in R^n \\ |x_1-x_2|\leq\delta}} \|f(\cdot+x_1-x_2) - f(\cdot)\|_{LM_{p\theta,w}} =$$

$$= C_r \sup_{\substack{|u|\leq\delta \\ u\in B(0,\delta)}} \|f(\cdot+u) - f(\cdot)\|_{LM_{p\theta,w}}.$$

Lemma 3 is proofed.

Lemma 4. Let $1 \leq p, \theta \leq \infty$, $w \in \Omega_{p\theta}$. Then exists $C > 0$, depending only from n, p, θ, w, that's for every, $r, R > 0$ and for all $f, g \in LM_{p\theta,w}$ next statement

$$\|M_r f - M_r g\|_{LM_{p\theta,w}} \leq C\left(1 + R^{\frac{n}{p}}\right)\|M_r f - M_r g\|_{C(\overline{B(0,R)})} + \sup_{u\in B(0,R)}\|f(\cdot + u) - f(\cdot)\|_{LM_{p\theta,w}} +$$

$$+ \sup_{u\in B(0,R)}\|g(\cdot + u) - g(\cdot)\|_{LM_{p\theta,w}} + \left\|f\chi_{{}^cB(0,R)}\right\|_{LM_{p\theta,w}} + \left\|g\chi_{{}^cB(0,R)}\right\|_{LM_{p\theta,w}}.$$

Proof. Indeed,

$$\|M_r f - M_r g\|_{LM_{p\theta,w}} \leq \left\|(M_r f - M_r g)\,\chi_{B(0,R)}\right\|_{LM_{p\theta,w}} + \left\|(M_r f - M_r g)\,\chi_{{}^cB(0,R)}\right\|_{LM_{p\theta,w}} = I_1 + I_2.$$

Next

$$I_1 = \sup_{x\in R^n}\left\|w(\rho)\,\|M_r f - M_r g\|_{L_p(B(x,\rho)\cap B(0,R))}\right\|_{L_\theta(0,\infty)} \leq$$

$$\leq \sup_{x\in R^n}\left\|w(\rho)\,\|M_r f - M_r g\|_{L_p(B(x,\rho)\cap B(0,R))}\right\|_{L_\theta(0,1)} +$$

$$+ \sup_{x\in R^n}\left\|w(\rho)\,\|M_r f - M_r g\|_{L_p(B(x,\rho)\cap B(0,R))}\right\|_{L_\theta(1,\infty)} \leq$$

$$\leq \|M_r f - M_r g\|_{C(\overline{B(0,R)})} \times$$

$$\times\left(\left\|w(\rho)\,(v_n\rho^n)^{\frac{1}{p}}\right\|_{L_\theta(0,1)} + \left\|w(\rho)\,(v_n R^n)^{\frac{1}{p}}\right\|_{L_\theta(1,\infty)}\right) \leq$$

$$\leq C\left(1 + R^{\frac{n}{p}}\right)\|M_r f - M_r g\|_{C(\overline{B(0,R)})},$$

where

$$C = v_n^{\frac{1}{p}}\left(\left\|w(\rho)\rho^{\frac{n}{p}}\right\|_{L_\theta(0,1)} + \|w(\rho)\|_{L_\theta(1,\infty)}\right) < \infty,$$

since $w \in \Omega_{p\theta}$.

By Lemma 1

$$I_2 \leq \|M_r f - f\|_{LM_{p\theta,w}} + \left\|(f-g)\chi_{{}^cB(0,R)}\right\|_{LM_{p\theta,w}} + \|M_r g - g\|_{LM_{p\theta,w}} \leq$$

$$\leq \sup_{u\in B(0,r)}\|f(\cdot + u) - f(\cdot)\|_{LM_{p\theta,w}} + \sup_{u\in B(0,r)}\|g(\cdot + u) - g(\cdot)\|_{LM_{p\theta,w}} +$$

$$+ \left\|f\chi_{{}^cB(0,R)}\right\|_{LM_{p\theta,w}} + \left\|g\chi_{{}^cB(0,R)}\right\|_{LM_{p\theta,w}}.$$

Lemma 4 is proofed.

Lemma 5. Let $1 \leq p, \theta \leq \infty$, $w \in \Omega_{p\theta}$. Then for every $r, R > 0$and for every $f, g \in LM_{p\theta,w}$

$$\|f - g\|_{LM_{p\theta,w}} \leq C\left(1 + R^{\frac{n}{p}}\right)\|M_r f - M_r g\|_{C(\overline{B(0,R)})} +$$

$$+2\sup_{u\in B(0,r)}\|f(\cdot + u) - f(\cdot)\|_{LM_{p\theta,w}} + 2\sup_{u\in B(0,r)}\|g(\cdot + u) - g(\cdot)\|_{LM_{p\theta,w}} +$$

$$+ \left\|f\chi_{{}^cB(0,R)}\right\|_{LM_{p\theta,w}} + \left\|g\chi_{{}^cB(0,R)}\right\|_{LM_{p\theta,w}},$$

where $C > 0$ is like in Lemma 4.

Proof. Enough to notice,that

$$\|f - g\|_{LM_{p\theta,w}} \leq \|M_r f - f\|_{LM_{p\theta,w}} + \|M_r f - M_r g\|_{LM_{p\theta,w}} + \|M_r g - g\|_{LM_{p\theta,w}}$$

and is used Lemma 1 and 4

Proof of theorem. Let$S \subset LM_{p\theta}^{w(\cdot)}$ and let conditionals be done (5)–(7).

Step 1. Let $0 < r < r_0$, where r_0 defined in Lemma 3, and $R > 0$ fixed. By using inequality (7) and conditions (1), followed, next

$$\sup_{f\in S} \|M_r f\|_{C(\overline{B(0,R)})} < \infty.$$

Apart from (8) and conditions (6), followed, next

$$\lim_{u\to 0} \sup_{f\in S} \|M_r f(\cdot + u) - M_r f(\cdot)\|_{C(\overline{B(0,R)})} = 0.$$

Hence, by theorem Askoli-Arcellas' set $S_r = \{M_r f : f \in S\}$ pre-compactness in $C(\overline{B(0,R)})$, or, is the same,set S_r completely limited ,then for all $\varepsilon > 0$ exists $m \in \mathbb{N}$, $f_1, ..., f_m \in S$ (depending from ε, r and R) that, for all $f \in S$

$$\min_{j=1,...,m} \|M_r f - M_r f_j\|_{C(\overline{B(0,R)})} < \varepsilon.$$

Step 2. Let $\{\varphi_1, ..., \varphi_m\}$ any bounded set S. By using inequality from lemma 5 for any $f \in S$ and for any $j = 1, ..., m$

$$\|f - \varphi_j\|_{LM^w_{p\theta}} \leq C(1 + R^{\frac{n}{p}}) \|M_r f - M\varphi_j\|_{C(\overline{B(0,R)})} +$$
$$+2 \sup_{u\in B(0,r)} \|f(\cdot + u) - f(\cdot)\|_{LM^{w(\cdot)}_{p\theta}} + 2 \sup_{u\in B(0,r)} \|\varphi_j(\cdot + u) - \varphi_j(\cdot)\|_{LM^{w(\cdot)}_{p\theta}} +$$
$$+ \left\|f\chi_{^cB(0,R)}\right\|_{LM^{w(\cdot)}_{p\theta}} + \left\|\varphi_j\chi_{^cB(0,R)}\right\|_{LM^{w(\cdot)}_{p\theta}} \leq$$
$$\leq C(1 + \mathbb{R}^{\frac{n}{p}}) \|M_r f - M_r\varphi_j\|_{C(\overline{B(0,R)})} + 4 \sup_{u\in B(0,r)} \sup_{g\in S} \|g(\cdot + u) - g(\cdot)\|_{LM^w_{p\theta}} + 2 \sup_{g\in S} \left\|g\chi_{^cB(0,R)}\right\|_{LM^w_{p\theta}}.$$

Hence,

$$\min_{j=1,...,m} \|f - \varphi_j\|_{LM^w_{p\theta}} \leq C(1 + \mathbb{R}^{\frac{n}{p}}) \min_{j=1,...,m} \|M_r f - M_r\varphi_j\|_{C(\overline{B(0,R)})} +$$
$$+4 \sup_{u\in B(0,r)} \sup_{g\in S} \|g(\cdot + u) - g(\cdot)\|_{LM^w_{p\theta}} + 2 \sup_{g\in S} \left\|g\chi_{^cB(0,R)}\right\|_{LM^w_{p\theta}}.$$

Step 3. Let $\varepsilon > 0$. The first , By using conditions (7), we take $R(\varepsilon) > 0$, that

$$\sup_{g\in S} \left\|g\chi_{^cB(0,R(\varepsilon))}\right\|_{LM^w_{p\theta}} < \frac{\varepsilon}{6}.$$

Next by using conditions (6), we take $r(\varepsilon)$, that

$$\sup_{u\in B(0,r(\varepsilon))} \sup_{g\in S} \|g(\cdot + u) - g(\cdot)\|_{LM^w_{p\theta}} < \frac{\varepsilon}{12}.$$

Because set pre-compactness $S_{r(\varepsilon)}$ в $C(\overline{B(0, R(\varepsilon))})$ exists $m(\varepsilon) \in \mathbb{N}$ and $f_{1,\varepsilon}, ..., f_{m(\varepsilon),\varepsilon} \in S$, than for any $f \in S$

$$\min_{j=1,...,m(\varepsilon)} \left\|M_{r(\varepsilon)} f - M_{r(\varepsilon)} f_{j,\varepsilon}\right\|_{C(\overline{B(0,R(\varepsilon))})} < \frac{\varepsilon}{3C(1 + R(\varepsilon)^{\frac{n}{p}})}.$$

By using inequality (9) с $\varphi_j = f_{j,\varepsilon}, j = 1, ..., m(\varepsilon)$, for any $f \in S$

$$\min_{j=1,...,m(\varepsilon)} \|f - f_{j,\varepsilon}\|_{LM^w_{p\theta}} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Then S pre-compactness set in $LM^w_{p\theta}$,the proofed theorem.

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Локалды Морри типтес кеңістігінде жиынның компактылы болуының жеткілікті шарттары

Мақалада локалды Морри типтес кеңістігінде $LM_{p\theta}^{w(\cdot)}(\mathbb{R}^n)$ жиындардың компактылығының жеткілікті шарттары келтірілген. Берілген теоремадан $\theta = \infty$ жағдайында жалпыланған Морри кеңістігіндегі $M_p^{w(\cdot)}$ нәтиже шығады, ал $w(r) = r^{-\lambda}$, $\theta = \infty 0 \leq \lambda \leq \frac{n}{p}$ жағдайындайы үшін Морри кеңістігінің белгілі теоремасы шығады, ал $\lambda = 0$ жағдайында бұл белгілі Фреше-Колмогоров теоремасы. Морри кеңістігіндегі жиындардың компактты болуының шарттары — [1, 2], ал жалпыланған Морри кеңістігі $M_p^{w(\cdot)}(\mathbb{R}^n)$ үшін [3, 4] жұмыстарында дәлелденген. Берілген мақаланың мақсаты осы нәтижелерді локалды Морри кеңістігі $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ үшін жиындардың компактты болуының шарттарын жалпылау болып табылады. Локалды Морри кеңістігіндегі жиындардың компактты болуының шарттарын пайдаланып, осы кеңістігіндегі операторлардың компактты болу шарттарын тексеруге болады. Себебі оператор бір кеңістіктегі шенелген жиынды келесі кеңістіктегі компактылы жиынға аударады. Авторлар локалды Морри типтес кеңістігінде жиындардың компактылығының жеткілікті шарттары функциялардың айырымы терминінде $\lim_{u\to 0}\sup_{f\in S}\|f(\cdot+u)-f(\cdot)\|_{LM_{p\theta,w}}=0$ келтірген. [5] жұмыста локалды Морри типтес кеңістігінде жиындардың компактылығының қажетті және жеткілікті шарттары алынып, олар функциялардың орта мәні терминінде $\lim_{u\to 0}\sup_{f\in S}\|f(\cdot+u)-f(\cdot)\|_{LM_{p\theta,w}}=0$ келтірілген.

Кілт сөздер: компактылық, компактылық алды, Фреше-Колмагоров теоремасы, локалды Морри типтес кеңістігі.

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Достаточные условия предкомпактности множеств в локальных пространствах типа Морри

В статье приведены достаточные условия предкомпактности множеств в локальных пространствах типа Морри $LM_{p\theta}^{w(\cdot)}(\mathbb{R}^n)$. Из доказанной теоремы в случае $\theta=\infty$ вытекает результат для обобщенного пространства $M_p^{w(\cdot)}$, а при $w(r)=r^{-\lambda},\ \theta=\infty,\ 0\le\lambda\le\frac{n}{p}$ — известный результат для пространства Морри $M_p^{\lambda}(\mathbb{R}^n)$, а в случае $\lambda=0$ – это хорошо известная теорема Фреше-Колмогорова. Условия предкомпактности множеств в пространствах Морри были доказаны в работах [1, 2], а в случае обобщенных пространств Морри $M_p^{w(\cdot)}(\mathbb{R}^n)$ — в [3, 4]. Цель статьи — обобщение результатов предкомпактности множеств для локальных пространств типа Морри $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. Используя теорему предкомпактности множеств в локальных пространствах типа Морри, можно проверить условия компактности операторов в этих пространствах, так как компактный оператор переводит ограниченное множество одного пространства в компактное множество другого пространства. В этой работе условия предкомпактности множеств в локальных пространствах типа Морри даны в терминах разности функции $\lim_{u\to 0}\sup_{f\in S}\|f(\cdot+u)-f(\cdot)\|_{LM_{p\theta,w}}=0$. Ранее [5] были опубликованы необходимые и достаточные условия предкомпактности множеств в локальных пространствах типа Морри, которые были приведены в терминах средних функций $\lim\limits_{\delta\to 0^+}\sup\limits_{f\in S}\|A_\delta f-f\|_{L_p(B(0,R_2)\setminus B(0,R_1))}=0$.

Ключевые слова: компактность, предкомпактность, теорема Фреше-Колмагорова, локальные пространства типа Морри.

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Asymptotic convergence of the solution for singularly perturbed boundary value problem with boundary jumps

The article is devoted to study of boundary value problem with boundary jumps for third order linear integro-differential equation with a small parameter at the highest derivatives, provided that additional characteristic equation's roots have opposite signs. The modified unperturbed boundary value problem is constructed. The solution of modified unperturbed problem is obtained. Initial jumps' values of the integral term and solution are defined. An estimate difference of solution for singularly perturbed and modified unperturbed boundary value problems is obtained. The convergence of solution for singularly perturbed boundary value problem to solution of modified unperturbed boundary value problem is proved.

Keywords: singular perturbation, small parameter, the boundary jump, the initial jump, boundary functions, asymptotic.

Introduction

The theory of singular perturbations has been with us, in one form or another, for a little over a century (although the term 'singular perturbation' dates from the 1940s). The subject, and the techniques associated with it, have evolved over this period as a response to the need to find approximate solutions (in an analytical form) to complex problems. Typically, such problems are expressed in terms of differential equations which contain at least one small parameter, and they can arise in many fields: fluid mechanics, particle physics and combustion processes, to name but three. The essential hallmark of a singular perturbation problem is that a simple and straightforward approximation (based on the smallness of the parameter) does not give an accurate solution throughout the domain of that solution.

Mathematical problems that make extensive use of a small parameter were probably first described by J.H. Poincare (1854–1912) as part of his investigations in celestial mechanics. (The small parameter, in this context, is usually the ratio of two masses.) Although the majority of these problems were not obviously 'singular'—and Poincare did not dwell upon this—some are; for example, one is the earth-moon-spaceship problem mentioned. Nevertheless, Poincare did lay the foundations for the technique that underpins our approach: the use of asymptotic expansions. The notion of a singular perturbation problem was first evident in the seminal work of L. Prandtl (1874–1953) on the viscous boundary layer (1904). Here, the small parameter is the inverse Reynolds number and the equations are based on the classical Navier-Stokes equation of fluid mechanics. This analysis, coupled with small-Reynolds-number approximations that were developed at about the same time (1910), prepared the ground for a century of singular perturbation work in fluid mechanics. But other fields over the century also made important contributions, for example: integration of differential equations, particularly in the context of quantum mechanics; the theory of nonlinear oscillations; control theory; the theory of semiconductors.

Theory of asymptotic integration of singularly perturbed equations has become purposefully developed starting with the works of L. Schlesinger, G.D. Birkhoff, P. Noaillon. In a further development of the main trends of the theory W. Wasow, A.H. Nayfeh, M. Nagumo, A.N. Tikhonov, M.I. Vishik, L.A. Lusternik, N.N. Bogolyubov, U.A Mitropolsky, A.B. Vasilieva and V.F. Butuzov, R.E. O'Malley, D.R. Smith, W. Eckhaus, K. W. Chang and F. A. Howes, J. Kevorkian and J.D. Cole, Sanders and F. Verhulst, E.F. Mischenko and N.X. Rozov, S.A. Lomov, K.A. Kassymov and others have made a significant contribution. For a broad class of singularly perturbed problems effective asymptotic methods to build a uniform approximation with any degree of accuracy in the small parameter were developed.

For the first time, boundary value problems with initial jumps for singularly perturbed linear ordinary differential and integro-differential equations of the second order was studied by K.A. Kassymov [1, 2]. A systematic study of boundary value problems with initial jumps Kassymov and his students began in the nineties of the last century. He developed methods for qualitative research and the construction of an asymptotic expansion of solutions of boundary value problems with initial jumps for singularly perturbed ordinary differential equations [3, 4]. General boundary-value problems for singularly perturbed ordinary differential equations of higher orders are investigated by D.N. Nurgabyl. He singled out a class of singularly perturbed boundary value problems with an initial jump and developed an algorithm for constructing and investigating the asymptotic behavior of solutions of general boundary value problems [5, 6]. K.A. Kassymov and M.K. Dauylbaev for singularly perturbed higher-order integro-differential equations studied problems of a special type, when the presence of integral terms leads to a qualitative change in the behavior of the solution [7–9].

M. K. Dauylbaev [10–12] studied boundary value problems with two boundary layers possessing the phenomena of initial jumps. The novelty of these studies is that when the small parameter tends to zero, the fast solution variable grows unlimitedly, not only at one, the so-called initial point, but also at the other end of the considered segment. Thus, a class of singularly perturbed integro-differential equations with initial jump phenomena at both ends of the given segment is singled out. He also developed a method for studying and constructing the asymptotic of the solution of the Cauchy problem with initial jump for singularly perturbed linear differential equations with impulse action [13].

Consider the singularly perturbed integro-differential equation

$$L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A_0(t)y'' + A_1(t)y' + A_2(t)y = F(t) + \int\limits_0^1 \sum_{i=0}^{1} H_i(t,x)y^{(i)}(x,\varepsilon)dx, \tag{1}$$

with integral boundary conditions

$$y(0,\varepsilon) = \alpha, \quad y'(0,\varepsilon) = \beta, \quad y(1,\varepsilon) = \gamma + \int\limits_0^1 \sum_{i=0}^{1} a_i(x)y^{(i)}(x,\varepsilon)dx, \tag{2}$$

where $\varepsilon > 0$ is a small parameter, α, β,γ are known constants independent of ε.

We will need the following assumptions:

C1) $A_i(t), i = \overline{0,2}$, $F(t)$, $a_j(x), j = 0,1$ are sufficiently smooth functions defined on the interval $[0,1]$, $H_0(t,x), H_1(t,x)$ are sufficiently smooth functions defined in the domain $D = \{0 \le t \le 1, 0 \le x \le 1\}$.

C2) The roots $\mu_i(t)$, $i = 1,2$ of «additional characteristic equation» $\mu^2 + A_0(t)\mu + A_1(t) = 0$ satisfy the following inequalities $\mu_1(t) < -\gamma_1 < 0$, $\mu_2(t) > \gamma_2 > 0$.

C3) 1 is not an eigenvalue of the kernel

$$H(t,s) = \frac{H_1(t,s)}{A_1(s)} + \int\limits_s^1 \frac{1}{A_1(s)}\left(H_0(t,x) - H_1(t,x)\frac{A_2(x)}{A_1(x)}\right)\exp\left(-\int\limits_s^x \frac{A_2(p)}{A_1(p)}dp\right)dx.$$

C4) $a_1(1) \neq 1$.

C5)$\bar{\delta} \neq 0$, where

$$\bar{\delta} = \int\limits_0^1 \frac{H_1(s,1)}{(1-a_1(1))A_1(s)y_{30}(s)}\left(y_{30}(1) - a_1(s)y_{30}(s) - \int\limits_0^1 \sum_{i=0}^{1} a_i(x)y_{30}^{(i)}(x)dx\right)ds.$$

For the solution of the boundary value problem (1),(2) are valid the following asymptotic estimations [10] as $\varepsilon \to 0$:

$$\left|y^{(q)}(t,\varepsilon)\right| \le C\left(|\alpha| + \varepsilon|\beta| + \max_{0\le t\le 1}\left|F(t) + \frac{\gamma}{1-a_1(1)}H_1(t,1)\right|\right) +$$
$$+C\varepsilon^{1-q}e^{-\gamma_1\frac{t}{\varepsilon}}\left(|\alpha| + |\beta| + \max_{0\le t\le 1}\left|F(t) + \frac{\gamma}{1-a_1(1)}H_1(t,1)\right|\right) +$$
$$+\frac{C}{\varepsilon^i}e^{-\gamma_2\frac{1-t}{\varepsilon}}\left(\left|\frac{\alpha}{1-a_1(1)}\right| + \varepsilon\left|\frac{\beta}{1-a_1(1)}\right| + \max_{0\le t\le 1}\left|F(t) + \frac{\gamma}{1-a_1(1)}H_1(t,1)\right|\right), \quad q = 0,1,2. \tag{3}$$

Consider the following modified unperturbed problem as $\varepsilon = 0$:

$$L_0\overline{y} \equiv A_1(t)\overline{y}'(t) + A_2(t)\overline{y}(t) = F(t) + \int\limits_0^1 \sum_{i=0}^{1} H_i(t,x)\overline{y}^{(i)}(x)dx + \Delta(t), \tag{4}$$

$$\overline{y}(0) = \alpha, \quad \overline{y}(1) = \gamma + \int_0^1 \sum_{i=0}^{1} a_i(x)\overline{y}^{(i)}(x)dx + \Delta_1, \tag{5}$$

where $\Delta(t)$ and Δ_1 are respectively unknown initial jumps of the integral term and the solution.

Let us denote by

$$u(t,\varepsilon) = y(t,\varepsilon) - \overline{y}(t), \Rightarrow y(t,\varepsilon) = u(t,\varepsilon) + \overline{y}(t), \tag{6}$$

where $y(t,\varepsilon)$ is a solution of singularly perturbed problem (1), (2) and $\overline{y}(t)$ is a solution of the unperturbed problem (4), (5).

Substituting (6) into (1), (2), we obtain the problem for $u(t,\varepsilon)$:

$$L_\varepsilon u \equiv \varepsilon^2 u''' + \varepsilon A_0(t)u'' + A_1(t)u' + A_2(t)u = -\Delta(t) + \varepsilon^2 \overline{y}''' - $$
$$-\varepsilon A_0(t)\overline{y}'' + \int_0^1 \sum_{i=0}^{1} H_i(t,x)u^{(i)}(x,\varepsilon)dx, \tag{7}$$

with boundary conditions

$$u(0,\varepsilon) = 0, \quad u'(0,\varepsilon) = \beta - \overline{y}'(0), \quad u(1,\varepsilon) = -\Delta_1 + \int_0^1 \sum_{i=0}^{1} a_i(x)u^{(i)}(x,\varepsilon)dx, \tag{8}$$

here $\Delta_1 = (1 - a_1(1))\Delta_0$.

The problem (7), (8) is of the same type as the problem (1), (2), applying the asymptotic estimations (3) for $u(t,\varepsilon)$, we get

$$\left|u^{(q)}(t,\varepsilon)\right| \leq C\left(\varepsilon + \varepsilon\left|\beta - \overline{y}'(0)\right| + \left|\Delta(t) + H_1(t,1)\Delta_0\right|\right) +$$
$$+C\varepsilon^{1-q}e^{-\gamma_1 \frac{t}{\varepsilon}}\left(\varepsilon + \left|\beta - \overline{y}'(0)\right| + \left|\Delta(t) + H_1(t,1)\Delta_0\right|\right) +$$
$$+\frac{C}{\varepsilon^i}e^{-\gamma_2 \frac{1-t}{\varepsilon}}\left(\varepsilon + \varepsilon\left|\frac{\beta - \overline{y}'(0)}{1 - a_1(1)}\right| + \left|\Delta(t) + H_1(t,1)\Delta_0\right|\right), \quad q = 0,1,2.$$

We choose the unknown function $\Delta(t)$ that the solution of the problem (7), (8) approach zero as $\varepsilon \to 0$, i.e. if the equality

$$\Delta(t) = -H_1(t,1)\Delta_0 \tag{9}$$

is valid, then the solution of the problem (1), (2) approaches to the modified unperturbed problem (4), (5) as $\varepsilon \to 0$. Thus, if the initial jump of the integral term $\Delta(t)$ is defined by the formula (9), then the solution of the problem (1), (2) approaches to the solution of the following modified unperturbed problem:

$$L_0\overline{y} \equiv A_1(t)\overline{y}'(t) + A_2(t)\overline{y}(t) = F(t) + \int_0^1 \sum_{i=0}^{1} H_i(t,x)\overline{y}^{(i)}(x)dx - H_1(t,1)\Delta_0; \tag{10}$$

$$\overline{y}(0) = \alpha, \quad \overline{y}(1) = \gamma + \int_0^1 \sum_{i=0}^{1} a_i(x)\overline{y}^{(i)}(x)dx + (1 - a_1(1))\Delta_0. \tag{11}$$

At first, we consider the equation (10) with condition

$$\overline{y}(0) = \alpha.$$

We seek the solution of the problem (10), (11):

$$\overline{y}(t) = \alpha \exp\left(-\int_0^t \frac{A_2(x)}{A_1(x)}dx\right) + \int_0^t \frac{\overline{z}(s)}{A_1(s)} \exp\left(-\int_s^t \frac{A_2(x)}{A_1(x)}dx\right) ds, \tag{12}$$

where

$$\overline{z}(t) = F(t) + \int_0^1 \sum_{i=0}^{1} H_i(t,x)\overline{y}^{(i)}(x)dx - H_1(t,1)\Delta_0. \tag{13}$$

Substituting (12) into the equation (13), we obtain that $\overline{z}(t)$ satisfies the following Fredholm integral equation of the second kind:

$$\overline{z}(t) = \varphi(t) + \int_0^1 H(t,s)\overline{z}(s)ds, \tag{14}$$

where

$$\varphi(t) = F(t) - H_1(t,1)\Delta_0 + \alpha \int_0^1 \left(H_0(t,x) - H_1(t,x)\frac{A_2(x)}{A_1(x)} \right) \exp\left(-\int_0^x \frac{A_2(p)}{A_1(p)}dp \right) dx; \tag{15}$$

$$H(t,s) = \frac{H_1(t,s)}{A_1(s)} + \int_s^1 \frac{1}{A_1(s)} \left(H_0(t,x) - H_1(t,x)\frac{A_2(x)}{A_1(x)} \right) \exp\left(-\int_s^x \frac{A_2(p)}{A_1(p)}dp \right) dx.$$

In view of the condition (C3), integral equation (14) has an unique solution, that can be represented in the form:

$$\overline{z}(t) = \varphi(t) + \int_0^1 R(t,s)\varphi(s)ds, \tag{16}$$

here $R(t,s)$ is a resolvent of the kernel $H(t,s)$, the function $\varphi(t)$ is defined by the formula (15). Substituting (16) into the function (12), by virtue of (15), we get the solution of the problem (10), (11):

$$\overline{y}(t) = \alpha \exp\left(-\int_0^t \frac{A_2(x)}{A_1(x)}dx \right) + \int_0^t \frac{1}{A_1(s)} \exp\left(-\int_s^t \frac{A_2(x)}{A_1(x)}dx \right) \Big[\overline{F}(s) - \overline{H}_1(s,1)\Delta_0 +$$

$$+\alpha \int_0^1 \left(\overline{H}_0(s,x) - \overline{H}_1(s,x)\frac{A_2(x)}{A_1(x)} \right) \exp\left(-\int_0^x \frac{A_2(p)}{A_1(p)}dp \right) dx \Big] ds, \tag{17}$$

where

$$\overline{F}(s) = F(s) + \int_0^1 R(s,p)F(p)dp;$$

$$\overline{H}_i(s,x) = H_i(s,x) + \int_0^1 R(s,p)H_i(p,x)dp, \quad i = 0,1.$$

To determine the initial jump Δ_0 of solution, we substitute (17) into the second condition of (11). As a result, we can find the initial jump Δ_0 of solution:

$$\Delta_0 = \frac{\overline{y}(1) - \int_0^1 \sum_{i=0}^{1} a_i(x)\overline{y}^{(i)}(x)dx - \gamma}{1 - a_1(1)}. \tag{18}$$

Theorem 1. Under the above assumptions (C1)–(C5), for the solution $y(t,\varepsilon)$ of the boundary value problem (1), (2) hold the following limiting equalities:

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = \overline{y}(t), \quad 0 \le t < 1;$$

$$\lim_{\varepsilon\to 0} y'(t,\varepsilon) = \overline{y}'(t), \quad 0 < t < 1;$$

$$\lim_{\varepsilon\to 0} y''(t,\varepsilon) = \overline{y}''(t), \quad 0 < t < 1,$$

where $\overline{y}(t)$ is a solution of the problem (10), (11) and expressed by (17), the initial jump Δ_0 is defined by the formula (18).

Example. Consider the following singularly perturbed boundary value problem with boundary jumps:

$$\varepsilon^2 y''' - \varepsilon y'' - 2y' = \delta \int\limits_0^1 y'(x,\varepsilon)dx; \tag{19}$$

$$y(0,\varepsilon) = 1,\ y'(0,\varepsilon) = 0,\ \ y(1,\varepsilon) = \int\limits_0^1 ay'(x,\varepsilon)dx,\ \ a \neq 1. \tag{20}$$

The fundamental system of solutions of the equation $\varepsilon^2 y''' - \varepsilon y'' - 2y' = 0$ have the form:

$$y_1(t,\varepsilon) = 1,\ \ y_2(t,\varepsilon) = e^{-\frac{t}{\varepsilon}},\ \ y_3(t,\varepsilon) = e^{-\frac{2}{\varepsilon}(1-t)}.$$

Let us denote by the right-hand side of the equation (19)

$$z(\varepsilon) = \delta \int\limits_0^1 y'(x,\varepsilon)dx. \tag{21}$$

Then general solution of the equation (19) has the form:

$$y(t,\varepsilon) = C_1 + C_2 e^{-\frac{t}{\varepsilon}} + C_3 e^{-\frac{2}{\varepsilon}(1-t)} - 0,5z(\varepsilon)t, \tag{22}$$

here $C_i, i = \overline{1,3}$ are unknown constants, $z(\varepsilon)$ is an unknown function.

Substituting the function (22) into the (21), we obtain the equality to define the function $z(\varepsilon)$

$$z(\varepsilon) = \frac{C_2\delta(e^{-\frac{1}{\varepsilon}} - 1) + C_3\delta(1 - e^{-\frac{2}{\varepsilon}})}{1 + 0,5\delta}.$$

Now, we determine the unknown constants $C_i, i = \overline{1,3}$ in (22), which satisfy the boundary conditions (20). Thus, we need to solve the system of algebraic equations:

$$\begin{cases} C_1 + C_2 + e^{-\frac{2}{\varepsilon}} C_3 = 1; \\ \frac{\varepsilon\delta\left(1-e^{-\frac{1}{\varepsilon}}\right)-2-\delta}{\varepsilon(2+\delta)} C_2 + \frac{2e^{-\frac{2}{\varepsilon}}(2+\delta)-\varepsilon\delta\left(1-e^{-\frac{2}{\varepsilon}}\right)}{\varepsilon(2+\delta)} C_3 = 0; \\ C_1 + \frac{2(1-a)e^{-\frac{1}{\varepsilon}}+\delta+2a}{2+\delta} C_2 + \frac{2(1-a)+(\delta+2a)e^{-\frac{2}{\varepsilon}}}{2+\delta} C_3 = 0. \end{cases}$$

The main determinant of the linear algebraic system has the form:

$$\Delta(\varepsilon) = \frac{2(1-a)}{\varepsilon(2+\delta)^2}\left[(\varepsilon\delta(1-e^{-\frac{1}{\varepsilon}}) - 2 - \delta)(1 - e^{-\frac{2}{\varepsilon}}) + (2e^{-\frac{2}{\varepsilon}}(2+\delta) - \varepsilon\delta(1-e^{-\frac{2}{\varepsilon}}))(1 - e^{-\frac{1}{\varepsilon}}\right].$$

As a result, the solutions of the system are defined by the formula:

$$C_1(\varepsilon) = \frac{(\varepsilon\delta(1-e^{-\frac{1}{\varepsilon}}) - 2 - \delta)(2(1-a) + (\delta+2a)e^{-\frac{2}{\varepsilon}})}{2(1-a)\left[(\varepsilon\delta(1-e^{-\frac{1}{\varepsilon}}) - 2 - \delta)(1 - e^{-\frac{2}{\varepsilon}}) + (2e^{-\frac{2}{\varepsilon}}(2+\delta) - \varepsilon\delta(1-e^{-\frac{2}{\varepsilon}}))(1 - e^{-\frac{1}{\varepsilon}})\right]} -$$

$$\frac{-(2e^{-\frac{2}{\varepsilon}}(2+\delta) - \varepsilon\delta(1-e^{-\frac{2}{\varepsilon}})(2(1-a)e^{-\frac{1}{\varepsilon}} + \delta + 2a))}{};$$

$$C_2(\varepsilon) = \frac{(2+\delta)(2e^{-\frac{2}{\varepsilon}}(2+\delta) - \varepsilon\delta(1-e^{-\frac{2}{\varepsilon}}))}{2(1-a)\left[(\varepsilon\delta(1-e^{-\frac{1}{\varepsilon}}) - 2 - \delta)(1 - e^{-\frac{2}{\varepsilon}}) + (2e^{-\frac{2}{\varepsilon}}(2+\delta) - \varepsilon\delta(1-e^{-\frac{2}{\varepsilon}}))(1 - e^{-\frac{1}{\varepsilon}})\right]};$$

$$C_3(\varepsilon) = \frac{(2+\delta)(2+\delta-\varepsilon\delta(1-e^{-\frac{1}{\varepsilon}}))}{2(1-a)\left[(\varepsilon\delta(1-e^{-\frac{1}{\varepsilon}})-2-\delta)(1-e^{-\frac{2}{\varepsilon}})+(2e^{-\frac{2}{\varepsilon}}(2+\delta)-\varepsilon\delta(1-e^{-\frac{2}{\varepsilon}}))(1-e^{-\frac{1}{\varepsilon}}\right]}.$$

If as $\varepsilon \to 0$, then

$$C_1(\varepsilon) \to 1, \ \ C_2(\varepsilon) \to 0, \ \ C_3(\varepsilon) \to \frac{2+\delta}{2(a-1)}.$$

Consider the following modified unperturbed problem:

$$-2\overline{y}'(t) = \int_0^1 \delta\overline{y}(x)dx - \delta\Delta_0, \tag{23}$$

$$\overline{y}(0) = 1, \ \ \overline{y}(1) = \int_0^1 a\overline{y}(x)dx + (1-a)\Delta_0,$$

here Δ_0 is called the initial jump of the solution.

Let us denote by

$$\overline{z} = \int_0^1 \delta\overline{y}(x)dx - \delta\Delta_0. \tag{24}$$

The general solution of the equation (23) is as follows

$$\overline{y}(t) = \frac{\delta\Delta_0}{2+\delta}t + C. \tag{25}$$

Substituting (25) in (24), we define the unknown function $\overline{z}$:

$$\overline{z} = -\frac{\delta\Delta_0}{1+0,5\delta}.$$

As a result, the solution of the modified unperturbed problem (23) has the form:

$$\overline{y}(t) = \frac{\delta}{2(1-a)}t + 1.$$

The initial jump Δ_0 of solution is defined by the following formula

$$\Delta_0 = \frac{2+\delta}{2(1-a)}.$$

The results can be seen to perform the following limiting equalities:

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = 1 + \frac{\delta}{2(1-a)}t \equiv \overline{y}(t), \quad 0 \le t < 1;$$

$$\lim_{\varepsilon\to 0} y'(t,\varepsilon) = \frac{\delta}{2(1-a)} \equiv \overline{y}'(t), \quad 0 < t < 1;$$

$$\lim_{\varepsilon\to 0} y''(t,\varepsilon) = 0 \equiv \overline{y}''(t), \quad 0 < t < 1.$$

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Шекаралы секірісті сингулярлы ауытқыған шеттік есебі шешімінің асимптотикалық жинақтылығы

Мақала екі үлкен туындысының алдында кіші параметрі бар үшінші ретті сингулярлы ауытқыған сызықты интегралды-дифференциалдық теңдеу үшін қосымша сипаттауыш теңдеудің түбірлерінің таңбасы қарама-қарсы болған жағдайдағы шекаралы секірісті шеттік есебін зерттеуге арналған. Өзгертілген ауытқымаған есеп құрылды, оның шешімі алынды. Сингулярлы ауытқыған шеттік есеп шешімі мен өзгертілген ауытқымаған есеп шешімінің арасындағы айырым бағаланды. Интегралдық мүшенің және шешімнің бастапқы секірістерінің шамалары анықталды. Берілген сингулярлы ауытқыған шеттік есеп шешімінің өзгертілген ауытқымаған шеттік есеп шешіміне ұмтылатыны дәлелденді.

Кілт сөздер: сингулярлы ауытқу, кіші параметр, шекаралық секіріс, бастапқы секіріс, шекаралық функциялар.

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Асимптотическая сходимость решения сингулярно возмущенной краевой задачи с граничными скачками

Статья посвящена изучению краевой задачи с граничными скачками для линейного интегродифференциального уравнения третьего порядка с малым параметром при старших производных при условии, что корни дополнительного характеристического уравнения имеют противоположные знаки. Построена модифицированная невозмущенная краевая задача. Получено решение модифицированной невозмущенной задачи. Определены значения начальных скачков интегрального члена и решения. Получена оценка разности решений сингулярно возмущенных и модифицированных невозмущенных краевых задач. Доказана сходимость решений сингулярно возмущенной краевой задачи к решению модифицированной невозмущенной краевой задачи.

Ключевые слова: сингулярное возмущение, малый параметр, граничный скачок, начальный скачок, граничные функции, асимптотика.

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Families of theories of abelian groups and their closures

In studying the structural properties of elementary theories, a relationship between theories with respect to a series of natural operators plays an important role. This relationship can be determined by placing models of given theories in various formula definable sets. Such sets include, for example, sets defined by unary predicates or equivalence relations. In this way, P-operators and E-operators arise, as well as their closures, and e-spectra, i.e. the numbers of new theories that may be generated by these operators. For E-operators, applicable to the families of theories of abelian groups, closures and generating sets, as well as their e-spectra are described. Szmielew invariants are used as a tool for the established characterization of a theory belonging to the E-closure of a family of theories of abelian groups. A series of families of theories corresponding to the sets of Szmielew invariants, properties of these families, and values of e-spectra are also described.

Keywords: family of theories, abelian group, E-operator, generating set, closure, e-spectrum.

0.1 Introduction

In a series of papers topological properties for families of theories are studied [1–6]. The notions of P-operator and E-operator were introduced allowing to study links between theories with respect to appropriate closure operators. These operators give possibilities to generate new theories by given families of ones, and, in special cases, to find minimal/least generating sets. Counting e-spectra for families of theories we get characteristics for maximal variations of theories in closed sets of theories with respect to generating sets.

We continue this investigation applying for families of theories of abelian groups and describing closures for families of theories of abelian groups with respect to the E-operator. In Sections 2 and 3 we consider basic notions and known results for families of theories and theories of abelian groups. In Section 4 we characterize the property when a theory of abelian groups belongs to E-closure of a given family of theories of abelian groups. This characterization allows to describe closed sets of theories of abelian groups with(out) least generating sets. Examples of these descriptions, for natural families of theories, are presented in Section 5.

0.2 Preliminaries

Throughout the paper we use the following terminology in [1, 2].

Let $P = (P_i)_{i\in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i\in I}$ be a family of structures such that P_i is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols P_i are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. The structure $\mathcal{A}_P \rightleftharpoons \bigcup\limits_{i\in I} \mathcal{A}_i$ expanded by the predicates P_i is the *P-union* of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_P$ is the *P-operator*. The structure $\mathcal{A}_P$ is called the *P-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}_i = (\mathcal{A}_P|_{A_i})|_{\Sigma(\mathcal{A}_i)}$, $i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$, will be also considered as P-combinations.

Clearly, all structures $\mathcal{A}' \equiv \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}'|_{P_i})|_{\Sigma(\mathcal{A}_i)}$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I}$, we write $\mathcal{A}' = \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I\cup\{\infty\}}$, where $\mathcal{A}'_\infty = \mathcal{A}'|_{\bigcap\limits_{i\in I} \overline{P_i}}$, maybe applying Morleyzation. Moreover, we write $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$ for $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure $\mathcal{A}_\infty$.

Note that if all predicates P_i are disjoint, a structure $\mathcal{A}_P$ is a P-combination and a disjoint union of structures $\mathcal{A}_i$. In this case the P-combination $\mathcal{A}_P$ is called *disjoint*. Clearly, for any disjoint P-combination $\mathcal{A}_P$, $\mathrm{Th}(\mathcal{A}_P) = \mathrm{Th}(\mathcal{A}'_P)$, where $\mathcal{A}'_P$ is obtained from $\mathcal{A}_P$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory

$T_P = \mathrm{Th}(\mathcal{A}_P)$, being *P-combination* of T_i, which is denoted by $\mathrm{Comb}_P(T_i)_{i\in I}$. In general, for non-disjoint case, the theory T_P will be also called a *P-combination* of the theories T_i, but in such a case we will keep in mind that this P-combination is constructed with respect (and depending) to the structure $\mathcal{A}_P$, or, equivalently, with respect to any/some $\mathcal{A}' \equiv \mathcal{A}_P$.

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure $\mathcal{A}_E$ being the *E-union* of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_E$ is the *E-operator*. The structure $\mathcal{A}_E$ is also called the *E-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E|_{A_i})|_{\Sigma(\mathcal{A}_i)}$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\mathrm{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its E-classes. The E-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_E = \mathrm{Th}(\mathcal{A}_E)$, being *E-combination* of T_i, which is denoted by $\mathrm{Comb}_E(T_i)_{i\in I}$ or by $\mathrm{Comb}_E(\mathcal{T})$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint P-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the *E-representability*.

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a *e-completion*, or a *e-saturation*, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called *e-complete*, or *e-saturated*, or *e-universal*, or *e-largest*.

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the *e-spectrum* of $\mathcal{A}_E$ and denoted by $e\text{-}\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-}\mathrm{Sp}(\mathcal{A}') \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the *e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by $e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$.

If $\mathcal{A}_E$ does not have E-classes $\mathcal{A}_i$, which can be removed, with all E-classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\mathrm{Th}(\mathcal{A}_E)$, then $\mathcal{A}_E$ is called *e-prime*, or *e-minimal*.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $\mathrm{TH}(\mathcal{A}')$ the set of all theories $\mathrm{Th}(\mathcal{A}_i)$ of E-classes $\mathcal{A}_i$ in $\mathcal{A}'$.

By the definition, an e-minimal structure $\mathcal{A}'$ consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then $\mathcal{A}'$ is called *e-least*.

Definition [2]. Let $\overline{\mathcal{T}}$ be the class of all complete elementary theories of relational languages. For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ we denote by $\mathrm{Cl}_E(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some E-class in $\mathcal{A}' \equiv \mathcal{A}_E$, $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, $\mathrm{Th}(\mathcal{A}_i) \in \mathcal{T}$. As usual, if $\mathcal{T} = \mathrm{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be *E-closed*.

The operator Cl_E of E-closure can be naturally extended to the classes $\mathcal{T} \subset \overline{\mathcal{T}}$ as follows: $\mathrm{Cl}_E(\mathcal{T})$ is the union of all $\mathrm{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0 \subseteq \mathcal{T}$.

For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi) \subseteq \Sigma$ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$.

Proposition 2.1 [2]. *If $\mathcal{T} \subset \overline{\mathcal{T}}$ is an infinite set and $T \in \overline{\mathcal{T}} \setminus \mathcal{T}$ then $T \in \mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an accumulation point for $\mathcal{T}$ with respect to E-closure Cl_E) if and only if for any formula $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.*

Theorem 2.2 [2]. *If $\mathcal{T}'_0$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*

(1) *$\mathcal{T}'_0$ is the least generating set for $\mathcal{T}_0$;*

(2) *$\mathcal{T}'_0$ is a minimal generating set for $\mathcal{T}_0$;*

(3) *any theory in $\mathcal{T}'_0$ is isolated by some set $(\mathcal{T}'_0)_\varphi$, i.e., for any $T \in \mathcal{T}'_0$ there is $\varphi \in T$ such that $(\mathcal{T}'_0)_\varphi = \{T\}$;*

(4) *any theory in $\mathcal{T}'_0$ is isolated by some set $(\mathcal{T}_0)_\varphi$, i.e., for any $T \in \mathcal{T}'_0$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.*

0.3 Theories of abelian groups

Let A be an abelian group. Then kA denotes its subgroup $\{ka \mid a \in A\}$ and $A[k]$ denotes the subgroup $\{a \in A \mid ka = 0\}$. It p is a prime number and $pA = \{0\}$ then $\dim A$ denotes the dimension of the group A, considered as a vector space over a field with p elements. The following numbers, for arbitrary p and n (p is prime and n is natural) are called the *Szmielew invariants* for the group A [7]:

$$\alpha_{p,n}(A) = \min\{\dim((p^nA)[p]/(p^{n+1}A)[p]), \omega\};$$

$$\beta_p(A) = \min\{\inf\{\dim((p^nA)[p] \mid n \in \omega\}, \omega\};$$

$$\gamma_p(A) = \min\{\inf\{\dim((A/A[p^n])/p(A/A[p^n])) \mid n \in \omega\}, \omega\};$$

$$\varepsilon(A) \in \{0,1\} \,\text{and}\, \varepsilon(A) = 0 \Leftrightarrow (nA = \{0\} \,\text{for some}\, n \in \omega, n \neq 0).$$

It is known [7, Theorem 8.4.10] that two abelian groups are elementary equivalent if and only if they have same Szmielew invariants. Besides, the following proposition holds.

Proposition 3.1 [7, Proposition 8.4.12]. *Let for any p and n the cardinals $\alpha_{p,n}$, β_p, $\gamma_p \le \omega$, and $\varepsilon \in \{0,1\}$ are given. Then there is an abelian group A such that the Szmielew invariants $\alpha_{p,n}(A)$, $\beta_p(A)$,$\gamma_p(A)$, and $\varepsilon(A)$ are equal to $\alpha_{p,n}$, β_p, γ_p, and ε, respectively, if and only if the following conditions hold:*

(1) *if for prime p the set $\{n \mid \alpha_{p,n} \ne 0\}$ is infinite then $\beta_p = \gamma_p = \omega$;*

(2) *if $\varepsilon = 0$ then for any prime p, $\beta_p = \gamma_p = 0$ and the set $\{\langle p,n\rangle \mid \alpha_{p,n} \ne 0\}$ is finite.*

We denote by $\mathbf{Q}$ the additive group of rational numbers, $\mathbf{Z}_{p^n}$ — the cyclic group of the order p^n, $\mathbf{Z}_{p^\infty}$ — the quasi-cyclic group of all complex roots of 1 of degrees p^n for all $n \ge 1$, R_p — the group of irreducible fractions with denominators which are mutually prime with p. The groups $\mathbf{Q}$, $\mathbf{Z}_{p^n}$, R_p, $\mathbf{Z}_{p^\infty}$ are called *basic*. Below the notations of these groups will be identified with their universes.

Since abelian groups with same Szmielew invariants have same theories, any abelian group A is elementary equivalent to a group

$$\oplus_{p,n}\mathbf{Z}_{p^n}^{(\alpha_{p,n})} \oplus \oplus_p \mathbf{Z}_{p^\infty}^{(\beta_p)} \oplus \oplus_p R_p^{(\gamma_p)} \oplus \mathbf{Q}^{(\varepsilon)},$$

where $B^{(k)}$ denotes the direct sum of k subgroups isomorphic to a group B. Thus, any theory of an abelian group has a model being a direct sum of based groups.

Recall that any complete theory of an abelian group is based by the set of positive primitive formulas [7, Lemma 8.4.5], reduced to the set of the following formulas:

$$\exists y(m_1x_1 + \ldots + m_nx_n \approx p^k y); \tag{11}$$

$$m_1x_1 + \ldots + m_nx_n \approx 0, \tag{12}$$

where $m_i \in \mathbf{Z}$, $k \in \omega$, p is a prime number [8], [7, Lemma 8.4.7]. Formulas (11) and (12) allow to witness that Szmielew invariants defines theories of abelian groups modulo Proposition 3.1.

0.4 Families of theories of abelian groups and their closures

Denote by $\overline{\mathcal{TA}}$ the set of all theories of abelian groups. Below we consider families $\mathcal{T} \subseteq \overline{\mathcal{TA}}$ and corresponding families $\mathcal{T}'$, where constants 0 are replaced by unary predicates P_0 with unique realizations 0, and operations $+$ are replaced by ternary predicates S, where $\models S(a,b,c) \Leftrightarrow a + b = c$. Clearly, each theory $T \in \mathcal{T}$ can be reconstructed by the correspondent theory $T' \in \mathcal{T}'$ and vice versa. So we can freely replace the closure $\mathrm{Cl}_E(\mathcal{T}')$ (and its elements) by the correspondent set of theories of abelian groups, denoted by $\mathrm{Cl}_E(\mathcal{T})$ (as well as by correspondent theories).

Now we fix a family $\mathcal{T}$. In view of Proposition 2.1 and the basedness by the set of formulas (11) and (12) we have the following lemmas.

Lemma 4.1. A family $\mathrm{Cl}_E(\mathcal{T})$ does not contain theories with new finite invariants $\alpha_{p,n}$, β_p, γ_p as well as invariants with new p and n.

Lemma 4.2. A family $\mathrm{Cl}_E(\mathcal{T})$ contains a theory with infinite invariant $\alpha_{p,n}$ if and only if $\mathcal{T}$ contains a theory with that infinite invariant or $\mathcal{T}$ has theories with infinitely many distinct finite invariants $\alpha_{p,n}$.

Using Proposition 3.1 and Lemma 4.2 we have.

Lemma 4.3. A family $\mathrm{Cl}_E(\mathcal{T})$ contains a theory with an infinite invariant β_p (respectively, γ_p) if and only if $\mathcal{T}$ contains a theory with that infinite invariant, or $\mathcal{T}$ has theories with infinitely many distinct finite invariants $\alpha_{p,n}$, or $\mathcal{T}$ has theories with infinitely many distinct finite invariants β_p (γ_p).

Lemma 4.4. A family $\mathrm{Cl}_E(\mathcal{T})$ contains a theory with $\varepsilon = 1$ if and only if $\mathcal{T}$ contains a theory with $\varepsilon = 1$, or $\mathcal{T}$ has theories forming infinite set $\{\langle p,n\rangle \mid \alpha_{p,n} \ne 0\}$, or $\mathcal{T}$ has theories with positive invariants β_p or γ_p.

Lemmas 4.2 – 4.4 describe approximations of new infinite Szmielew invariants by finite ones.

Applying Proposition 2.1, the basedness of theories in Section 3, and Lemmas 4.1 – 4.4 we can describe E-closures for families of theories of abelian groups.

For this aim we remember the following fact for a family $\mathcal{T}$ of language uniform theories T_I defined by sets I of nonempty predicates.

Recall [3] that a theory T in a predicate language Σ is said to be *language uniform*, or a LU-*theory* if for each arity n any substitution on the set of non-empty n-ary predicates preserves T.

Proposition 4.5 [3, Proposition 6]. *If $T_J \notin \mathcal{T}$ then $T_J \in \mathrm{Cl}_E(\mathcal{T})$ if and only if for any finite set $J_0 \subset I_0$ there are infinitely many $T_I\mathcal{T}$ with*

$$J \cap J_0 = I \cap J_0. \tag{13}$$

The equations (13) for indexes mean the *local correspondence* between $\mathcal{T}$ and T_J. Using replacements of these index sets by sequences of Szmielew invariants for theories of abelian groups we get the local correspondence for families of theories in $\overline{\mathcal{TA}}$: for a family $\mathcal{T} \subseteq \overline{\mathcal{TA}}$, a theory $T \in \overline{\mathcal{TA}} \setminus \mathcal{T}$ *locally corresponds* to $\mathcal{T}$ if replaced (13) holds modulo infinite Szmielew invariants and, besides, simultaneously the sequence of infinite Szmielew invariants for T, which are not represented in infinitely many theories in $\mathcal{T}$, is approximated by sequences of corresponding finite Szmielew invariants for theories in $\mathcal{T}$ used for replaced (13).

Theorem 4.6. If $\mathcal{T}$ is an infinite family of theories of abelian groups and $T \notin \mathcal{T}$ is a theory of an abelian group then $T \in \mathrm{Cl}_E(\mathcal{T})$ if and only if T has infinite models (i. e., T has some infinite $\alpha_{p,n}$ or some positive β_p, γ_p, ε) and locally corresponds to $\mathcal{T}$.

Proof. If $T \in \mathrm{Cl}_E(\mathcal{T})$ then T has infinite models since finite models define only finitely many positive Szmielew invariants, these invariants are exhausted by finite $\alpha_{p,n}$, and theories with these invariants are isolated. If T locally does not correspond to $\mathcal{T}$ then $T \notin \mathrm{Cl}_E(\mathcal{T})$ in view of Proposition 2.1.

Conversely if T has infinite models and locally corresponds to $\mathcal{T}$ then $\mathrm{Cl}_E(\mathcal{T})$ contains a theory with same Szmielew invariants as for T and thus $T \in \mathrm{Cl}_E(\mathcal{T})$.

0.5 Generating sets and e-spectra

Theorem 2.2, Proposition 3.1, and Theorem 4.6 allow to characterize families of theories of abelian groups with(out) least generating sets as well as to describe e-spectra for E-combinations of theories in $\overline{\mathcal{TA}}$.

Following the series of Szmielew invariants, for a theory $T \in \overline{\mathcal{TA}}$ we consider the *support* $\mathrm{Supp}(T)$ of T being the set of positive Szmielew invariants for T. Now we denote by $\mathcal{FS}$ (respectively, $\mathcal{CFS}$) the set of all theories in $\overline{\mathcal{TA}}$ having (co)finite supports. By $\mathcal{ICIS}$ we denote the set of all theories in $\overline{\mathcal{TA}}$ having infinite and co-infinite supports. By $\mathcal{F}$ we denote the set of all theories in $\overline{\mathcal{TA}}$ with finite Szmielew invariants, and by $\mathcal{INF}$ — with infinite Szmielew invariants $\alpha_{p,n} > 0$, $\beta_p > 0$, $\gamma_p > 0$.

Clearly, $\mathrm{Cl}_E(\mathcal{FS}) = \overline{\mathcal{TA}}$ and $\mathrm{Cl}_E(\mathcal{CFS}) = \overline{\mathcal{TA}}$ implying $\mathrm{Cl}_E(\mathcal{ICIS}) = \overline{\mathcal{TA}}$. Note also that $\mathrm{Cl}_E(\mathcal{F}) = \overline{\mathcal{TA}}$ whereas $\mathcal{INF}$ is E-closed.

By $\mathbf{A}$, $\mathbf{B}$, $\mathbf{\Gamma}$, $\mathbf{E}$ we denote the classes of all theories in $\overline{\mathcal{TA}}$ whose positive Szmielew invariants are exhausted by $\alpha_{p,n}$, β_p, γ_p, ε, respectively. For $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{U} \in \{\mathbf{A}, \mathbf{B}, \mathbf{\Gamma}, \mathbf{E}\}$ we denote by $\mathbf{XY}$, $\mathbf{XYZ}$, $\mathbf{XYZU}$, respectively, the set of all theories in $\overline{\mathcal{TA}}$ whose positive Szmielew invariants are exhausted by corresponding $\alpha_{p,n}$, β_p, γ_p, ε for $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{U}$.

For $\mathbf{X}$ as above and for a sequence S of some Szmielew invariants, we write $\mathbf{X}_S$ for the set of of all theories T in $\mathbf{X}$ such that Szmielew invariants for T equal to corresponding values in S. If the sequences S do not have finite positive values we denote by $\mathbf{X}^\infty$ the union of these $\mathbf{X}_S$. If for a subset P_0 of the set P of all prime numbers the sequences S do not have positive values for $p \in P \setminus P_0$ we denote by $\mathbf{X}_{P_0}$ the union of these $\mathbf{X}_S$. We write $\mathbf{X}_p$ instead X_{P_0} if P_0 is a singleton $\{p\}$.

As above we denote by $\mathbf{X}_{P_0}\mathbf{Y}_{P'_0}$, $\mathbf{X}_{P_0}\mathbf{Y}_{P'_0}\mathbf{Z}_{P''_0}$, $\mathbf{X}_{P_0}\mathbf{Y}_{P'_0}\mathbf{Z}_{P''_0}\mathbf{U}_{P'''_0}$, respectively, the set of all theories in $\overline{\mathcal{TA}}$ whose positive Szmielew invariants are exhausted by corresponding $\alpha_{p,n}$, β_p, γ_p, ε for $\mathbf{X}_{P_0}, \mathbf{Y}_{P'_0}, \mathbf{Z}_{P''_0}, \mathbf{U}_{P'''_0}$.

Remark 5.1. If S_T is a sequence of all Szmielew invariants for a theory T then $\mathbf{A}_{S_T}\mathbf{B}_{S_T}\mathbf{\Gamma}_{S_T}\mathbf{E}_{S_T} = \{T\}$. As Proposition 3.1 asserts, some Szmielew invariants can be reconstructed automatically using the rest. Therefore, for instance, if Szmielew invariants $\alpha_{p,n}$, β_p, γ_p for T imply $\varepsilon = 1$ then we have $\mathbf{A}_{S'_T}\mathbf{B}_{S'_T}\mathbf{\Gamma}_{S'_T} = \{T\}$ for the subsequence S'_T of S_T which is obtained removing ε. We have a similar effect for $\beta_p = \omega$ and $\gamma_p = \omega$ with $|\{n \mid \alpha_{p,n} \neq 0\}| = \omega$. In such a case S_T can be reconstructed both from S'_T and from S''_T which is obtained from S'_T removing considered β_p and γ_p.

In general case, subsequences S'''_T of S_T define, for combinations of $\mathbf{A}_{S'''_T}$, $\mathbf{B}_{S'''_T}$, $\mathbf{\Gamma}_{S'''_T}$, $\mathbf{E}_{S'''_T}$, the following possibilities for cardinalities: 1 (if T is uniquely defined), 2 (having, for instance, positive $\alpha_{p,n}$ only and varying ε), ω (having finitely many free positions for Szmielew invariants which can vary independently from 0 to ω), 2^ω (having countably many free positions for Szmielew invariants which can vary independently from 0 to ω).

Recall [9] that a group $\mathcal{A}$ is *divisible* if for any natural $n > 0$ and any element $a \in A$ the equation $nx = a$ has a solution in $\mathcal{A}$.

Theorem 5.2. [9]. *Any divisible subgroup $\mathcal{A}$ of abelian group $\mathcal{B}$ is a direct summand in $\mathcal{B}$.*

Theorem 5.3. [9]. *Any nonzero divisible abelian group $\mathcal{A}$ is represented a direct sum of groups isomorphic to* $\mathbf{Q}$ *or* $\mathbf{Z}_{p^\infty}$.

Recall [9, 10] that a group $\mathcal{A}$ is *bounded* if there is a positive number n such that $n\mathcal{A} = \{0\}$. Otherwise the group $\mathcal{A}$ is called *unbounded*. A group $\mathcal{A}$ is *torsion free* if all nonunit elements have infinite order.

The following proposition is implied by Proposition 3.1 and summarizes possibilities for combinations of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{\Gamma}$, $\mathbf{E}$.

Proposition 5.4. 1. $\mathbf{AB} = \mathbf{A\Gamma} = \mathbf{AB\Gamma} = \mathbf{A} \subset \mathcal{FS}$, $\mathbf{A}$ is divided into $\mathbf{A} \cap \mathcal{F}$, consisting of theories with finite models, and $\mathbf{A} \setminus \mathcal{F}$, consisting of theories with infinite bounded models.

2. $\mathbf{B} = \mathbf{\Gamma} = \mathbf{B\Gamma} = \mathbf{O}$, where $\mathbf{O}$ is a singleton consisting of the one-element group.

3. $\mathbf{AE}$ consists of theories T without β_p and γ_p in $\mathrm{Supp}(T)$ and such that sets $\{n \mid \alpha_{p,n} \neq 0\}$ are finite for each p, i.e., with bounded quotients with respect to maximal divisible subgroups.

4. $\mathbf{BE}$ consists of all theories of divisible abelian groups.

5. $\mathbf{\Gamma E}$ consists of all theories of torsion free abelian groups.

6. $\mathbf{ABE}$ consists of all theories of abelian groups $\mathcal{A}$ with bounded quotients relative to maximal divisible subgroups $\mathcal{B}$, i. e., the theories $\mathrm{Th}(\mathcal{A}/\mathcal{B})$ form the set $\mathbf{A}$.

7. $\mathbf{A\Gamma E}$ consists of all theories of abelian groups without β_p in $\mathrm{Supp}(T)$ and such that sets $\{n \mid \alpha_{p,n} \neq 0\}$ are finite for each p.

8. $\mathbf{B\Gamma E}$ consists of all theories of abelian groups such that quotients with respect to maximal divisible subgroups are torsion free.

9. $\mathbf{AB\Gamma E} = \overline{\mathcal{TA}}$.

Since theories of finite abelian groups with unbounded $\alpha_{p,n}$ are isolated and force theories with infinite $\alpha_{p,n}$, β_p, γ_p, and positive ε, as well as since these values for distinct p are independent, we have the following proposition.

Proposition 5.5. 1. *For any* $P_0 \subseteq P$, $\mathrm{Cl}_E\left(\bigcup\limits_{p\in P_0}(\mathbf{A}_p \cap \mathcal{F})\right) = \mathrm{Cl}_E\left(\bigcup\limits_{p\in P_0}\mathbf{A}_p\right) = \bigcup\limits_{p\in P_0}\mathbf{A}_p\mathbf{B}_p^\infty\mathbf{\Gamma}_p^\infty\mathbf{E}$ *with the least generating set* $\bigcup\limits_{p\in P_0}(\mathbf{A}_p \cap \mathcal{F})$.

2. *For any* $P_0 \subseteq P$, $\mathrm{Cl}_E(\mathbf{A}_{P_0} \cap \mathcal{F}) = \mathrm{Cl}_E(\mathbf{A}_{P_0}) = \mathbf{A}_{P_0}\mathbf{B}_{P_0}^\infty\mathbf{\Gamma}_{P_0}^\infty\mathbf{E}$ *with the least generating set* $\mathbf{A}_{P_0} \cap \mathcal{F}$; $\mathrm{Cl}_E\left(\bigcup\limits_{p\in P_0}(\mathbf{A}_p \cap \mathcal{F})\right)$ *is a subset of* $\mathrm{Cl}_E(\mathbf{A}_{P_0} \cap \mathcal{F})$ *which is proper if and only if* $|P_0| \geq 2$.

Taking $P_0 = P$ we have the following

Corollary 5.6. $\mathrm{Cl}_E(\mathbf{A} \cap \mathcal{F}) = \mathrm{Cl}_E(\mathbf{A}) = \mathbf{AB}^\infty\mathbf{\Gamma}^\infty\mathbf{E}$ *with the least generating set* $\mathbf{A} \cap \mathcal{F}$.

Clearly, e-$\mathrm{Sp}(T) = 0$ for any theory T being a E-combination with unique finite structure, in particular, for a finite abelian group. Now we divide $\mathbf{A}_p$ into singletons $\mathbf{A}_{\alpha_{p,n}}$ consisting of theories of abelian groups with unique positive Szmielew invariant $\alpha_{p,n}$. For a fixed p and n and an infinite union $\bigcup\limits_{\alpha_{p,n}}\mathbf{A}_{\alpha_{p,n}}$ produces a family of theories whose E-combination $T_{p,n}$ has e-$\mathrm{Sp}(T_{p,n}) = 1$ witnessed by $\mathbf{A}_{\alpha_{p,n}}$ with $\alpha_{p,n} = \omega$. Uniting the families $\bigcup\limits_{\alpha_{p,n}}\mathbf{A}_{\alpha_{p,n}}$ for $p \in P_0$ we get E-combinations T with e-$\mathrm{Sp}(T) = |P_0|$ which is obtained by additivity as in [4]. Taking finite direct sums $\oplus_{p,n}\mathbf{Z}_{p^n}^{(\alpha_{p,n})}$ we again can produce infinite $\alpha_{p,n}$ for E-closures such that these $\alpha_{p,n}$ can be independently achieved or not achieved. Thus we get 2^ω possibilities for variations of infinite $\alpha_{p,n}$ which is witnessed by some E-combinations T with e-$\mathrm{Sp}(T) = 2^\omega$. Since there are 2^ω distinct theories of abelian groups this value is maximal. Summarizing the arguments we have arbitrary admissible values of e-spectra and obtain the following

Theorem 5.7. For any $\lambda \in \omega \cup \{\omega, 2^\omega\}$ *there is an* E*-combination* T *of theories of finite abelian groups* (*in* $\mathbf{A} \cap \mathcal{F}$ *and with least generating set*) *such that* e-$\mathrm{Sp}(T) = \lambda$.

Now we define a subfamily of $\mathrm{Cl}_E(\mathbf{A})$ producing an E-combination without the least generating set. Choose an infinite set $P_0 \subseteq P$ and take a countable set $D \subset \mathcal{P}(P_0)$ such that $\langle D, \subseteq\rangle$ is a dense linearly ordered set isomorphic to $\langle \mathbf{Q}, \leq\rangle$ and without cuts (A, A') having $\bigcup A \neq \bigcap A'$. Denote by $\mathrm{Cl}_E(\mathbf{A})_D$ the family

$$\left\{\mathrm{Th}\left(\oplus_{p\in X}\mathbf{Z}_p^{(\omega)}\right) \mid X \in D\right\}.$$

Clearly, $\mathrm{Cl}_E(\mathbf{A})_D$ does not have isolated points and has 2^ω cuts producing $|\mathrm{Cl}_E(\mathrm{Cl}_E(\mathbf{A})_D)| = 2^\omega$. Moreover, for any $P_0 \subseteq P$ with $|P \setminus P_0| = \omega$ we can take continuum many infinite $P_0' \subset P$ which are disjoint from P_0 and produce continuum many theories in corresponding sets $\mathrm{Cl}_E(\mathbf{A})_{D'}$, for $D' \subset \mathcal{P}(P_0')$, and separated from $\mathrm{Cl}_E(\mathbf{A})_D$ with respect to Hausdorff topology.

Similarly we can add theories in $\mathbf{A} \cap \mathcal{F}$ with positive invariants for $P_0' \subset P$ which is disjoint from P_0 and produce the value 2^ω for e-spectrum. Again the E-closure of that extended family does not have the least generating set.

Thus, by Theorem 2.2, the following theorem holds.

Theorem 5.8. There are 2^ω *families* $\mathrm{Cl}_E(\mathbf{A})_D$ *whose* E*-closures do not have least generating sets and whose* E*-combinations* T *satisfy* $e\text{-}\mathrm{Sp}(T) = 2^\omega$.

Addind/replacing the arguments above for $\alpha_{p,n}$ with β_p and/or γ_p we get the following theorems.

Theorem 5.9. For any $\lambda \in \omega \cup \{\omega, 2^\omega\}$ *there is an* E*-combination* T *of theories in* **ВЕ** (*respectively,* **ГЕ**, **АГЕ**, **ВГЕ**, $\overline{\mathcal{TA}}$) *and with least generating set*) *such that* $e\text{-}\mathrm{Sp}(T) = \lambda$.

Theorem 5.10. There are 2^ω *families* $\mathrm{Cl}_E(\mathbf{ВЕ})_D$ (*respectively,* $\mathrm{Cl}_E(\mathbf{ГЕ})_D$, $\mathrm{Cl}_E(\mathbf{АГЕ})_D$, $\mathrm{Cl}_E(\mathbf{ВГЕ})_D$, $\mathrm{Cl}_E(\overline{\mathcal{TA}})_D$) *whose* E*-closures do not have least generating sets and whose* E*-combinations* T *satisfy* $e\text{-}\mathrm{Sp}(T) = 2^\omega$.

Clearly, Theorems 5.8 and 5.10 are witnessed by subfamilies of $\mathcal{ICIS}$.

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Абельдік группалар теориясының жиынтықтары және олардың тұйықтамасы

Элементарлы теориялардың қасиеттерінің құрылымдарын оқу барысында маңызды рөлді кәдімгі операторлар қатарына қатысты теориялардың арасындағы өзара байланыстары атқарады. Бұл өзара байланысты осы теориялардың модельдерін әртүрлі формульді анықталған жиындарға орналастыру арқылы анықтауға болады. Осындай теорияларға, мысалы, бірорынды предикатпен немесе эквивалентті қатынаспен берілген жиындар жатады. Сол себепті, P-операторлар және E-операторлар және олардың тұйықтамалары e-спектрлар, яғни осы теориялармен туындалатын жаңа теориялар саны, пайда болады. E-операторлар үшін абельдік группалар теориясының жиынтықтарына сәйкес тұйықтамалар және туындалған жиындар, сонымен бірге e-спектрлар сипатталады. Құрал ретінде қойылған сипаттамада теорияның E-тұйықтамасында осы абельдік группалар теориясының жиынтығы үшін шмелевтік инварианттары қолданылады. Бұл мақалада теориялар жиындықтарының сериялары және сәйкесінше шмелевтік инварианттарының жиынтықтары анықталады, осы жиынтықтардың қасиеттері зерттеледі, сонымен қатар e-спектрлардың мағыналары сипатталады.

Кілт сөздер: теориялар жиынтығы, абельдік группа, E-оператор, туындалған жиын, тұйықтама, e-спектр.

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Семейства теорий абелевых групп и их замыкания

При изучении структурных свойств элементарных теорий важную роль играет взаимосвязь между теориями относительно ряда естественных операторов. Эту взаимосвязь можно определять, помещая модели данных теорий в различные формульно определимые множества. К таким множествам относятся, например, множества, задаваемые одноместными предикатами или отношениями эквивалентности. Таким образом, возникают P-операторы и E-операторы, их замыкания, а также e-спектры, т.е. новые теории, которые могут порождаться данными операторами. Для E-операторов, применительно к семействам теорий абелевых групп, описываются замыкания и порождающие множества, а также их e-спектры. В качестве инструмента для установленной характеризации попадания теории в E-замыкание данного семейства теорий абелевых групп используются шмелевские инварианты. Определяются серии семейств теорий, соответствующих совокупностям шмелевских инвариантов, исследуются свойства этих семейств, а также описываются значения e-спектров.

Ключевые слова: семейство теорий, абелева группа, E-оператор, порождающее множество, замыкание, e-спектр.

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Boundary value problems for essentially-loaded parabolic equation

In this paper we investigate the first boundary value problem for essentially loaded equation of heat conduction, i.e. when laden terms are derivatives for any finite order. It is shown that if the point of load is fixed, this problem is uniquely solvable. The stated boundary problem is reduced to the Volterra integral equation of the second kind. Estimates of the kernel of the integral equation are made, which indicate a weak singularity of the kernel. It is shown that if the point of load is fixed, then the stated boundary problem is uniquely solvable.

Keywords: heat equation, boundary value problems, loaded equation, kernel, convolution theorem, eigenfunction.

Introduction

In work [1] the boundary value problems for the loaded parabolic equations are considered, and the loaded terms contain values of derivatives for only fixed points of their domain of definition. This paper argues that the corresponding boundary value problems are absolutely and uniquely solvable in the class of continuous functions, if the orders of the derivatives loaded terms are $\alpha < \frac{1}{2}$.

In works [2–5] it is shown that if the order of the derivative in the loaded term equals to the order of the differential part of the equation and the point of load moves at a constant or variable velocity, the corresponding boundary value problems are spectrally loaded,i.e. not uniquely solvable.

The purpose of this work is to show the unique solvability of the first boundary value problem for a loaded equation of heat conduction in the case, where the loaded terms are derivatives for any finite order and the point of load is fixed.

1 Statement of the problem

In the domain $Q = \{(x,t); 0 < x < l, t > 0\}$ we consider a loaded equation of heat conduction [2]:

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} + \lambda \frac{\partial^k u(x,t)}{\partial x^k}\bigg|_{x=\bar{x}} = f_0(x,t), \tag{1}$$

where $k > 0$ is an any integer number and $\bar{x}$ is fixed point, $0 < \bar{x} < l$.

Problem. Find in the domain Q a regular solution to equation (1) from the class $C(Q)$, satisfying the conditions:

$$u(x,0) = 0;\;\; u(0,t) = u(l,t) = 0. \tag{2}$$

2 Case $k = 1$

Let us consider in detail the case $k = 1$. We invert the differential operator of problem (4)–(5), considering temporarily that a loaded term is known, and we obtain:

$$u(x,t) = -\lambda \int\limits_0^t \frac{\partial u(\xi,\tau)}{\partial \tau}\bigg|_{\tau=\bar{x}} \int\limits_0^l G(x,\xi;t-\tau)d\xi d\tau + \tag{3}$$

$$+\int\limits_0^t\int\limits_0^l f_0(\xi,\tau)G(x,\xi;t-\tau)d\xi d\tau,$$

where the Green's function $G(x,\xi;t)$ has the form [6]:

$$G(x,\xi;t)=\frac{1}{2a\sqrt{\pi t}}\sum_{n=-\infty}^{\infty}\left[\exp\left\{-\frac{(x-\xi+2nl)^2}{4a^2t}\right\}-\right. \tag{4}$$

$$\left.-\exp\left\{-\frac{x+\xi+2nl}{4a^2t}\right\}\right].$$

We calculate the integral:

$$k(x,t-\tau)=\sum_{n=-\infty}^{\infty}\int\limits_0^l\frac{1}{2a\sqrt{\pi(t-\tau)}}\left[\exp\left\{-\frac{(x-\xi+2nl)^2}{4a^2(t-\tau)}\right\}-\right.$$

$$\left.-\exp\left\{-\frac{(x+\xi+2nl)^2}{4a^2(t-\tau)}\right\}\right]d\xi=$$

$$=\left\| z_1=\frac{x-\xi+2nl}{2a\sqrt{t-\tau}};\ z_2=\frac{x+\xi+2nl}{2a\sqrt{t-\tau}}\right\|=$$

$$=\frac{1}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}\left[\int\limits_{\frac{x+(2n-1)l}{2a\sqrt{t-\tau}}}^{\frac{x+2nl}{2a\sqrt{t-\tau}}}\exp\left\{-z_1^2\right\}dz_1-\int\limits_{\frac{x+2nl}{2a\sqrt{t-\tau}}}^{\frac{x+(2n+1)l}{2a\sqrt{t-\tau}}}\exp\left\{-z_2^2\right\}dz_2\right]=$$

$$=\sum_{n=-\infty}^{\infty}\left[\text{erfc}\left(\frac{x+2nl}{2a\sqrt{t-\tau}}\right)-\frac{1}{2}\text{erfc}\left(\frac{x+(2n-1)l}{2a\sqrt{t-\tau}}\right)-\right.$$

$$\left.-\frac{1}{2}\text{erfc}\left(\frac{x+(2n+1)l}{2a\sqrt{t-\tau}}\right)\right].$$

Equality (6) can be represented as:

$$u(x,t)=-\lambda\int_0^t k(x,t-\tau)\frac{\partial u(\xi,\tau)}{\partial\tau}\bigg|_{\xi=\bar{x}}d\tau+f(x,t), \tag{5}$$

where

$$f(x,t)=\int_0^t\int_0^l f_0(\xi,\tau)G(x,\xi;t-\tau)d\xi d\tau.$$

Differentiating both sides of (8) with respect to t, we assume $x=\bar{x}$, and introducing the notation

$$\varphi_1(t)=\frac{\partial u(x,t)}{\partial t}\bigg|_{x=\bar{x}},$$

we get an integral equation:

$$\varphi_1(t)+\lambda\int_0^t K_1(t,\tau,\bar{x})\varphi_1(\tau)d\tau=f_1(t), \tag{6}$$

where

$$K_1(t,\tau,\bar{x})=\frac{\partial k(t,\tau,x)}{\partial t}\bigg|_{x=\bar{x}},\quad f_1(t)=\frac{\partial f(x,t)}{\partial t}\bigg|_{x=\bar{x}}.$$

We find the explicit form of the kernel $K_1(t,\tau,\bar{x})$:

$$K_1(t,\tau,\bar{x}) =$$

$$= -\frac{2}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \frac{1}{4a(t-\tau)^{3/2}} \left[(x+2nl) \exp\left\{ -\frac{(x+2nl)^2}{4a^2(t-\tau)} \right\} - \right.$$

$$-\frac{x+(2n-1)l}{2} \exp\left\{ -\frac{(x+(2n-1)l)^2}{4a^2(t-\tau)} \right\} -$$

$$\left. -\frac{x+(2n+1)l}{2} \exp\left\{ -\frac{(x+(2n+1)l)^2}{4a^2(t-\tau)} \right\} \right] =$$

$$= K_1^{(0)}(t,\tau,\bar{x}) + K_1^{(+)}(t,\tau,\bar{x}) + K_1^{(-)}(t,\tau,\bar{x}),$$

where the designations are introduced

$$K_1^{(0)}(t,\tau,\bar{x}) = -\frac{1}{2a\sqrt{\pi}(t-\tau)^{3/2}} \left[\bar{x} \exp\left\{ -\frac{\bar{x}^2}{4a^2(t-\tau)} \right\} + \right.$$

$$\left. + \frac{l-\bar{x}}{2} \exp\left\{ -\frac{(l-\bar{x})^2}{4a^2(t-\tau)} \right\} - \frac{\bar{x}+l}{2} \exp\left\{ -\frac{(l+\bar{x})^2}{4a^2(t-\tau)} \right\} \right];$$

$$K_1^{(+)}(t,\tau,\bar{x}) =$$

$$= -\frac{1}{2a\sqrt{\pi}(t-\tau)^{3/2}} \sum_{n=1}^{\infty} \left[(\bar{x}+2nl) \exp\left\{ -\frac{(\bar{x}+2nl)^2}{4a^2(t-\tau)} \right\} - \right.$$

$$-\frac{\bar{x}+(2n-1)l}{2} \exp\left\{ -\frac{[\bar{x}+(2n-1)l]^2}{4a^2(t-\tau)} \right\} -$$

$$\left. -\frac{\bar{x}+(2n+1)l}{2} \exp\left\{ -\frac{[\bar{x}+(2n+1)l]^2}{4a^2(t-\tau)} \right\} \right];$$

$$K_1^{(-)}(t,\tau,\bar{x}) =$$

$$= \frac{1}{2a\sqrt{\pi}(t-\tau)^{3/2}} \sum_{m=1}^{\infty} \left[(2ml-\bar{x}) \exp\left\{ -\frac{2ml-(\bar{x})^2}{4a^2(t-\tau)} \right\} - \right.$$

$$-\frac{(2m+1)l-x}{2} \exp\left\{ -\frac{((2m+1)l-\bar{x})^2}{4a^2(t-\tau)} \right\} -$$

$$\left. -\frac{(2m-1)l-\bar{x}}{2} \exp\left\{ -\frac{[(2m-1)l-\bar{x}]^2}{4a^2(t-\tau)} \right\} \right].$$

We estimate the kernel $K_1(t,\tau,\bar{x})$. For this we will estimate $K_1^{(0)}$, $K_1^{(+)}$, $K_1^{(-)}$ separately, in this case we use the inequality $z \cdot \exp\{-z\} \le \exp\{-1\}$, $z > 0$:

$$|K_1^{(0)}(t,\tau,\bar{x})| \le \frac{a}{e\sqrt{\pi}} \left\{ \frac{2}{\bar{x}} + \frac{1}{l-\bar{x}} + \frac{1}{l+\bar{x}} \right\} \frac{1}{\sqrt{t-\tau}} =$$

$$= \frac{2a}{e\sqrt{\pi}} \left\{ \frac{1}{\bar{x}} + \frac{l}{l^2-\bar{x}^2} \right\} \frac{1}{\sqrt{t-\tau}}.$$

To estimate $|K_1^{(+)}(t,\tau,\bar{x})|$ and $|K_1^{(-)}(t,\tau,\bar{x})|$ the signs of the sum are replaced by the integrals in which we make replacements accordingly:

$$|K_1(t,\tau,\bar{x}) - K_1^{(0)}(t,\tau,\bar{x})| \le$$

$$\le \int_1^{\infty} |K_1^{(+)}(t,\tau,\bar{x})| dn + \int_1^{\infty} |K_1^{(-)}(t,\tau,\bar{x})| dn \le$$

$$\leq \left\| \begin{array}{l} \xi_1(n) = \dfrac{(\bar{x}+2nl)^2}{4a^2(t-\tau)}, \ \xi_2(n) = \dfrac{[\bar{x}+(2n-1)l]^2}{4a^2(t-\tau)}, \\ \xi_3(n) = \dfrac{[\bar{x}+(2n+1)l]^2}{4a^2(t-\tau)}, \ \eta_1(n) = \dfrac{(2ml-\bar{x})^2}{4a^2(t-\tau)}, \\ \eta_2(n) = \dfrac{[(2m+1)l-\bar{x}]^2}{4a^2(t-\tau)}, \ \eta_3(n) = \dfrac{[(2m-1)l-\bar{x}]^2}{4a^2(t-\tau)} \end{array} \right\| \leq$$

$$\leq \frac{1}{2l\sqrt{\pi(t-\tau)}} \left[\int_{\xi_1(1)}^{\infty} \exp\{-\xi_1\} d\xi_1 + \right.$$

$$+\frac{1}{2}\int_{\xi_2(1)}^{\infty} \exp\{-\xi_2\} d\xi_2 + \frac{1}{2}\int_{\xi_3(1)}^{\infty} \exp\{-\xi_3\} d\xi_3 +$$

$$+\int_{\eta_1(1)}^{\infty} \exp\{-\eta_1\} d\eta_1 + \frac{1}{2}\int_{\eta_2(1)}^{\infty} \exp\{-\eta_2\} d\eta_2 +$$

$$\left. +\frac{1}{2}\int_{\eta_3(1)}^{\infty} \exp\{-\eta_3\} d\eta_3 \right] = \frac{a}{4l\sqrt{\pi(t-\tau)}} [2\exp\{-\xi_1(1)\}+$$

$$+\exp\{-\xi_2(1)\} + \exp\{-\xi_3(1)\} + 2\exp\{-\eta_1(1)\} + \exp\{-\eta_2(1)\}+$$

$$+\exp\{-\eta_3(1)\}] \leq \frac{2a}{l\sqrt{\pi}} \frac{1}{\sqrt{t-\tau}} \exp\left\{-\frac{(l-\bar{x})^2}{4a^2(t-\tau)}\right\}.$$

Thus, the kernel of integral equation (9) has a weak singularity, i.e. integral equation (9) is uniquely solvable. From relation (3) it follows that boundary value problem (1)–(2) has a unique solution.

3 Case $k = 2$

Now let $k = 2$. Equality (6) in this case will have the form:

$$u(x,t) = -\lambda \int_0^t \frac{\partial^2 u(\xi,\tau)}{\partial \tau^2}\bigg|_{x=\bar{x}} \int_0^l G(x,\xi;t-\tau) d\xi d\tau + \tag{7}$$

$$+\int_0^t \int_0^l f_0(\xi,\tau) G(x,\xi;t-\tau) d\xi d\tau.$$

If we differentiate both sides of this equality with respect to t twice and we introduce the notation

$$\varphi_2(t) = \frac{\partial^2 u(x,t)}{\partial t^2}\bigg|_{x=\bar{x}},$$

we will have the following Volterra integral equation:

$$\varphi_2(t) + \lambda \int_0^t K_2(t,\tau,\bar{x}) \varphi_2(\tau) d\tau = f_2(t); \tag{8}$$

where

$$K_2(t,\tau,\bar{x}) = \frac{\partial^2 k(t,\tau,x)}{\partial t^2}\bigg|_{x=\bar{x}}, \ f_2(t) = \frac{\partial^2 f(x,t)}{\partial t^2}\bigg|_{x=\bar{x}};$$

$$f(x,t) = \int_0^t \int_0^l f_0(\xi,\tau) G(x,\xi;t-\tau) d\xi d\tau.$$

Similarly to the case $k = 1$, we estimate the kernel of integral equation (8), for that we find the explicit form of the kernel $K_2(t,\tau,\bar{x})$:

$$K_2(t,\tau,\bar{x}) = \frac{\partial^2 k(t,\tau,x)}{\partial t^2}\bigg|_{x=\bar{x}} = \frac{1}{8a\sqrt{\pi}(t-\tau)^{1/2}} \sum_{n=-\infty}^{\infty} \left[6\frac{\bar{x}+2nl}{(t-\tau)^2} \times \right.$$

$$\times \exp\left\{-\frac{(\bar{x}+2nl)^2}{4a^2(t-\tau)}\right\}-\frac{(\bar{x}+2nl)^3}{2a^2(t-\tau)^3}\exp\left\{-\frac{(\bar{x}+2nl)^2}{4a^2(t-\tau)}\right\}-$$
$$-\frac{3(\bar{x}+(2n-1)l)}{8a\sqrt{\pi}(t-\tau)^{5/2}}\exp\left\{-\frac{(\bar{x}+(2n-1)l)^2}{4a^2(t-\tau)}\right\}+$$
$$+\frac{(\bar{x}+(2n-1)l)^3}{8a^3\sqrt{\pi}(t-\tau)^{7/2}}\exp\left\{-\frac{(\bar{x}+(2n-1)l)^2}{4a^2(t-\tau)}\right\}-$$
$$-\frac{3(\bar{x}+(2n+1)l)}{8a\sqrt{\pi}(t-\tau)^{5/2}}\exp\left\{-\frac{(\bar{x}+(2n+1)l)^2}{4a^2(t-\tau)}\right\}+$$
$$+\frac{(\bar{x}+(2n+1)l)^3}{8a^3\sqrt{\pi}(t-\tau)^{7/2}}\exp\left\{-\frac{(\bar{x}+(2n+1)l)^2}{4a^2(t-\tau)}\right\}\Bigg]=$$
$$=K_2^{(0)}(t,\tau,\bar{x})+K_2^{(+)}(t,\tau,\bar{x})+K_2^{(-)}(t,\tau,\bar{x}).$$

At first we estimate the terms of this sum when $n=0$:

$$|K_2^{(0)}(t,\tau,\bar{x})|=\frac{1}{\sqrt{t-\tau}}\left[\frac{3\bar{x}}{4a\sqrt{\pi}(t-\tau)^2}\exp\left\{-\frac{\bar{x}^2}{4a^2(t-\tau)}\right\}+\right.$$
$$+\frac{\bar{x}^3}{8a^3\sqrt{\pi}(t-\tau)^3}\exp\left\{-\frac{\bar{x}^2}{4a^2(t-\tau)}\right\}+$$
$$+\frac{3(l-\bar{x})}{8a\sqrt{\pi}(t-\tau)^2}\exp\left\{-\frac{(l-\bar{x})^2}{4a^2(t-\tau)}\right\}+$$
$$+\frac{(l-\bar{x})^3}{8a^3\sqrt{\pi}(t-\tau)^3}\exp\left\{-\frac{(l-\bar{x})^2}{4a^2(t-\tau)}\right\}+$$
$$+\frac{3(l+\bar{x})}{8a\sqrt{\pi}(t-\tau)^2}\exp\left\{-\frac{(l+\bar{x})^2}{4a^2(t-\tau)}\right\}+$$
$$\left.+\frac{(l+\bar{x})^3}{8a^3\sqrt{\pi}(t-\tau)^3}\exp\left\{-\frac{(l+\bar{x})^2}{4a^2(t-\tau)}\right\}\right].$$

Further we use estimates of the following form:

$$J_1=\frac{b}{(t-\tau)^2}\exp\left\{-\frac{b^2}{4a^2(t-\tau)}\right\}\le\frac{64\,a^4}{b^3}\exp\{-2\}$$

and

$$J_2=\frac{b^3}{(t-\tau)^3}\exp\left\{-\frac{b^2}{4a^2(t-\tau)}\right\}\le\frac{1728\,a^6}{b^3}\exp\{-3\},(b>0),$$

here we take into account the validity of inequality: $(z>0)$

$$z^n e^{-z}\le n^n e^{-n},\ n=0,1,2,...$$

Thus we finally have:

$$|K_2^{(0)}(t,\tau,\bar{x})|\le C\cdot\frac{1}{\sqrt{\pi(t-\tau)}}\cdot\frac{a^3}{d^3(\bar{x})},$$

where $d(\bar{x})=\min\{\bar{x};l-\bar{x}\}$.

Similarly to the case $k=1$, we estimate $K_2^{(+)}(t,\tau,\bar{x})$ and $K_2^{(-)}(t,\tau,\bar{x})$. It is enough to consider the following integral and its estimate:

$$\frac{1}{\sqrt{\pi}(t-\tau)^{3/2}}\int\limits_1^\infty\left[b_1\frac{\bar{x}+2nl}{(t-\tau)}+b_2\frac{(\bar{x}+2nl)^3}{(t-\tau)^2}\right]\exp\left\{-\frac{(\bar{x}+2nl)^2}{4a^2(t-\tau)}\right\}dn=$$
$$=\left\|\frac{(\bar{x}+2nl)^2}{4a^2(t-\tau)}=\xi;\quad\frac{4l(\bar{x}+2nl)}{4a^2(t-\tau)}dn=d\xi\right\|=$$

$$= \frac{1}{\sqrt{\pi}(t-\tau)^{3/2}} \int_{\frac{(\bar{x}+2l)^2}{4a^2(t-\tau)}}^{\infty} \left[\frac{a^2}{l} \cdot b_1 + \frac{4a^2 \cdot a^2}{l} \cdot \xi \right] \exp\{-\xi\}\, d\xi =$$

$$= \frac{1}{l\sqrt{\pi}(t-\tau)^{3/2}} \left[\frac{a^2 b_1}{l} \exp\{-\xi\} - \frac{4a^4}{l}(\xi+1)\exp\{-\xi\} \right] \Bigg|_{\frac{(\bar{x}+2l)^2}{4a^2(t-\tau)}}^{\infty} =$$

$$= \frac{1}{l\sqrt{\pi}(t-\tau)^{3/2}} \exp\left\{ -\frac{(\bar{x}+2l)^2}{4a^2(t-\tau)} \right\} \times$$

$$\times \left[(a^2 b_1 + 4a^4) + \frac{(\bar{x}+2l)^2}{4a^2(t-\tau)} \cdot 4a^4 \right] =$$

$$= \frac{a^2 b_1 + 4a^4}{l\sqrt{\pi(t-\tau)}} \cdot \frac{1}{(t-\tau)} \exp\left\{ -\frac{(\bar{x}+2l)^2}{4a^2(t-\tau)} \right\} +$$

$$+ \frac{(\bar{x}+2l)^2 a^2}{l\sqrt{\pi(t-\tau)}} \cdot \frac{1}{(t-\tau)^2} \exp\left\{ -\frac{(\bar{x}+2l)^2}{4a^2(t-\tau)} \right\} \le$$

$$\le \frac{a^2 b_1 + 4a^4}{\sqrt{\pi(t-\tau)}} \cdot \frac{4a^2}{(\bar{x}+2l)^2} e^{-1} + \frac{(4a^2)^2 \cdot (\bar{x}+2l)^2 \cdot a^2 \cdot 4e^{-2}}{(\bar{x}+2l)^4 \cdot l\sqrt{\pi(t-\tau)}} \le$$

$$\le \frac{C}{(\bar{x}+2l)^2} \cdot \frac{1}{\sqrt{\pi(t-\tau)}}.$$

Using this estimate for $K^{(+)}$ and $K^{(-)}$, we get:

$$|K_2^{(+)}(t,\tau,\bar{x})| + |K_2^{(-)}(t,\tau,\bar{x})| \le \frac{C}{(l-\bar{x})^2} \cdot \frac{1}{\sqrt{\pi(t-\tau)}}.$$

Thus, for kernel of integral equation (8) the following estimate is valid:

$$|K_2(t,\tau,\bar{x})| \le C \frac{1}{d_2(\bar{x})} \frac{1}{\sqrt{t-\tau}},$$

where $d_2(\bar{x}) = \min\{\bar{x}, l-\bar{x}, (l-\bar{x})^2\}$.

From this estimate it follows that the Volterra integral equation (8) is uniquely solvable, and from relation (7) we obtain the unique solvability of problem (1)–(2) for the case of $k = 2$.

4 Case $\forall k \in \mathbb{N}$. *Main result*

Carrying out similar arguments for any finite value k, we can show that the kernel of the corresponding integral equation has estimate of the form:

$$|K_k(t,\tau,\bar{x})| \le \frac{C}{d_k(\bar{x})} \frac{1}{\sqrt{t-\tau}},$$

where $d_k(\bar{x}) = \min\{\bar{x}, l-\bar{x}, (l-\bar{x})^k\}$, i.e. $K_k(t,\tau,\bar{x})$ has also a weak singularity.

Thus it is proved the validity of the following theorem.

Theorem. The problem (4)–(5) is absolutely and uniquely solvable for every $f(x,t)$ such that

$$\left\{ \frac{\partial^k}{\partial t^k} \int_0^t \int_0^l G(x,\xi;t-\tau) f(\xi,\tau)\, d\xi d\tau \right\}_{x=\bar{x}} \in C(0,\infty).$$

Remark. Articles [3–5] are closer to the subject of this work.

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Айтарлықтай-жүктемелі параболалық теңдеулер үшін шекаралық есептер

Көрсетілген жұмыста айтарлықтай-жүктемелі параболалық теңдеулер үшін бірінші шекаралық есептер зерттелді және олардың жүктемелі мүшелері кез келген ақырғы реттік туындысы болып табылады. Қойылған шекаралық есеп Вольтерра теңдеуінің екінші түріне келеді. Интегралдық теңдеуінің ядросынының ерекшелігі әлсіз екенін көрсететін бағалау жасалды. Егер жүктемелі нүкте белгіленген болса, онда қойылған есеп бірмәнді шешіледі.

Кілт сөздер: жылуөткізгіштік теңдеу, шекаралық есептер, жүктемелі теңдеу, ядро, орама теоремасы, меншікті функция.

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Краевые задачи для существенно-нагруженного параболического уравнения

В статье исследована в полуполосе первая краевая задача для существенно-нагруженного уравнения теплопроводности, причем нагруженные члены являются производными любого конечного порядка. Поставленная граничная задача сведена к интегральному уравнению Вольтерра второго рода. Произведены оценки ядра интегрального уравнения, которые указывают на слабую особенность ядра. Показано, что если точка нагрузки фиксирована, то поставленная краевая задача однозначно разрешима.

Ключевые слова: уравнение теплопроводности, краевые задачи, нагруженное уравнение, ядро, теорема свертки, собственная функция.

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A solvability conditions of mixed problems for equations of parabolic type with involution

In this work the partial differential equations with involutions are considered. The mixed problems for the parabolic type equation, with constant and variable constants, corresponding to the Dirichlet type boundary conditions is investigated. The involution is contained by the second derivative with respect to the variable x, which is the difficult case for investigations. One-dimensional differential operators with involution have an infinite number of positive and negative eigenvalues. This means that the operator on the right-hand side of the equation under study is not semi-bounded. In the case of classical problems, ordinary differential operators usually appear on the right-hand side of the equations, which are semi-bounded. Therefore, the incorrectness of mixed problems for a parabolic equation with an involution is discussed in this paper. Examples are given. Sufficient conditions for the initial data are found when the problem under study has a unique solution. The representation of the solution in the form of partial sums of the Fourier series in eigenfunctions is found. The density in the space $L_2(-1,1)$ of the set of initial functions is proved everywhere, when the problem has a unique solution.

Keywords: Fourier method, mixed problem, involution, eigenfunctions, basis.

Introduction

We study the solvability of the following problem:

$$u_t(x,t) = u_{xx}(-x,t), \quad -1 \le x \le 1, \quad t \ge 0; \tag{1}$$

$$u_x(-1,t) = u_x(1,t) = 0, \quad u(x,0) = \varphi(x), \tag{2}$$

A transformation S of a function $f(x)$ from the class $L_2(-1,1)$ is said to be an involution, if $\left(S^2 f\right)(x) = f(x)$. In particular, a transformation of the form $(Sf)(x) = f(-x)$ is involution. Equation (1) is said to be an equation of parabolic type with involution. This name has nothing to do with the well-known classification of equations of mathematical physics.

A necessary condition of the existence of a solution of problem (1), (2) is the consistency of the initial data with equation (1) and boundary conditions (2). Therefore, we will require that

$$\varphi(x) \in C^2[-1,1] \text{and} \varphi'(-1) = \varphi'(1) = 0.$$

We say, that problem (1), (2) is well-posed, if 1) the solution of the problem exists, 2) the solution of the problem is unique, (3) the solution of the problem depends continuously on the initial data (is stable).

The application of the Fourier method to problem (1), (2) leads to a spectral problem with involution

$$-X''(-x) = \lambda X(x), \quad X'(-1) = X'(1) = 0. \tag{3}$$

Questions of the well-posedness of mixed problems for differential equations with involution are considered in [1–3]. In works [4, 5], inverse problems for equations with involution are considered. Spectral problems with involution were investigated in [6–15]. Mixed problems for equations of the form (1), apparently, are considered first in this paper.

The incorrectness of the mixed problem (1), (2)

It is well known [13, 14] that the spectral problem (3) is self-adjoint and has two series of eigenvalues $\lambda_{k1} = k^2\pi^2$, $\lambda_{k2} = -\left(k+\frac{1}{2}\right)^2\pi^2$. The corresponding eigenfunctions have the form $X_{k1}(x) = \cos k\pi x$, $k = 0,1,2,,,,\ ,;$ $X_{k2}(x) = \sin\left(k+\frac{1}{2}\right)\pi x$, $k = 0,1,2,,,;$ which form a complete orthonormal system in the class $L_2(-1,1)$.

We note, that the spectral problem (3) has an infinite number of negative eigenvalues, the double differentiation operator on the left-hand side of the differential equation (3) (or the operator of double differentiation with respect to x on the right-hand side of the differential equation (1) is not semi-bounded. That is the fundamental difference of the equation has studied from many different equations.

The standard method consists of the idea of representing the formal solution of the mixed problem (1), (2) in the following form of infinite series

$$u(x,t) = \sum_{\lambda_{k1}} A_k e^{-\lambda_{k1} t} \cos k\pi x + \sum_{\lambda_{k2}} B_k e^{-\lambda_{k2} t} \sin\left(k+\frac{1}{2}\right)\pi x, \tag{4}$$

where

$$A_k = \int_{-1}^{1} \varphi(x) \cos k\pi x dx, \quad B_k = \int_{-1}^{1} \varphi(x) \sin\left(k+\frac{1}{2}\right)\pi x dx. \tag{5}$$

If function $\varphi(x)$ is not infinite times differentiable, i.e. if the Fourier coefficients B_k of the function $\varphi(x)$ do not decrease with sufficient rapidity, then the second term in (4) diverges, since $\lambda_{k2} < 0$. Therefore, in the case of general initial value, the mixed problem (1), (2) may not have a solution. In the case when the solution exists but it does not have the property of stability i.e. does not depend continuously on the initial value. For example, perturbation

$$u_\delta(x,t) = \varepsilon e^{-\lambda_{k2} t} \sin\left(k+\frac{1}{2}\right)\pi x$$

does not exceed the number ε for $t = 0$, but will be greater than any preassigned number C_0 for $t = \delta$, at sufficiently small ε and δ and sufficiently large k. Thus, the mixed problem (2) in the case of parabolic type with involution (1) is not well-posed. Nevertheless, we show that the solution of the mixed problem under study exists and is unique.

The solvability classes of the mixed problem (1), (2)

First of all, let us show the uniqueness of the solution of the mixed problem.

Theorem 1. If a solution of the mixed problem (1), (2) exists, then it is unique.

Proof. Assume that the mixed problem (1), (2) exists. Any solution $u(x,t)$ of problem (1), (2), as a function of x, can be represented as a Fourier series.

$$u(x,t) = \sum_{k=0}^{\infty} T_{k1}(t) \cos k\pi x + \sum_{k=0}^{\infty} T_{k2}(t) \sin\left(k+\frac{1}{2}\right)\pi x$$

by orthonormal basis $\{X_k(x)\} = \left\{X_{k1} = cosk\pi x,\ X_{k2} = \sin\left(k+\frac{1}{2}\right)\pi x\right\}$. Since this series converges in the sense of the norm of the space $L_2(-1,1)$, then it also converges in the sense of the scalar product.
Therefore

$$T_{k1}(t) = (u(x,t), \cos k\pi x), \quad T_{k2}(t) = \left(u(x,t), \sin\left(k+\frac{1}{2}\right)\pi x\right).$$

We write these two equations in brief in the form

$$T_k(t) = (u(x,t), X_k(x)). \tag{6}$$

We multiply both sides of equation (1) by scalar product to $X_k(x)$, which gives

$$(u_t,\ X_k) = (u_{xx}(-x,t), X_k).$$

The right-hand side of the equality obtained is twice integrable by parts, and on the left side we use the rule of differentiation with respect to the parameter t under the integral sign. Taking into account the equation (3), we obtain relation $\frac{\partial}{\partial t}(u, X_k) = \lambda_{k1}(u(x,t), X_k)$. By substitution the equality (6), we obtain the Cauchy problem for an ordinary first-order differential equation

$$T'_k(t) = -\lambda_k T_k(t), \quad T_k(0) = (\varphi,\ X_k).$$

The initial condition is obtained from (6) for $t = 0$. By the uniqueness of the solution of the Cauchy problem, $T_k(t)$ is uniquely determined. This proves the uniqueness of the solution of problem (1), (2). Theorem 1 is proved.

We show classes of admissible initial functions $\varphi(x)$, for which problem (1), (2) has a solution. First we show that the series (4) is a solution of the problem (1), (2) if all the coefficients of B_k are zero.

Theorem 2. If initial function $\varphi(x)$ is even, belongs to the class $C^2[-1,1]$ and satisfies the conditions $\varphi'(-1) = \varphi'(1) = 0$, then the solution of problem (1), (2) exists, is unique and can be represented as a series (4).

Proof. If $\varphi(x)$ - even function, then all the Fourier coefficients B_k of the form (5) are equal to zero. Therefore, the series (4) takes the form

$$u(x,t) = \sum_{k=0}^{\infty} A_k e^{-k^2\pi^2 t} \cos k\pi x. \tag{7}$$

In order to prove the theorem we have to show that the series (7) converges for any $t > 0$, and it can be term-by-term differentiated once with respect to the variable t and twice with respect to the variable x. The last two operations are possible under the condition of uniform convergence of the series

$$-\sum_{k=0}^{\infty} k^2\pi^2 A_k e^{-k^2\pi^2 t} \cos k\pi x \tag{8}$$

for all $t > 0$. Uniform convergence of the series (8) is proved in the same way as in the case of a classical equation of parabolic type (see, for example, [16; 203]).
The convergence of the series (7) follows from the convergence of the majorant series

$$\sum_{k=0}^{\infty} |A_k \cos k\pi x|. \tag{9}$$

The convergence of the series (9) is proved in exactly the same way as the absolute and uniform convergence of the classical Fourier series in the trigonometric system is proved [16; 203]. Thus, the solution of problem (1), (2) exists, unique and can be represented as a series (7). The proof of the theorem is completed.

Next, let consider the problem (1), (2), where the initial function $\varphi(x)$ is a trigonometric polynomial

$$\varphi(x) = \sum_{k=0}^{N_1} a_k \cos k\pi x + \sum_{k=0}^{N_2} b_k \sin\left(k + \frac{1}{2}\right)\pi x. \tag{10}$$

We note that functions in the form of series or polynomials in eigenfunctions are used in the study of various problems. For example, in [17] (see also references in it) functions of the type (10) are used in the study of the spectral properties of loaded differential operators.

Theorem 3. If initial function $\varphi(x)$ is a trigonometric polynomial of the form (10), then the solution of problem (1), (2) exists, is unique and can be represented in the form

$$u(x,t) = \sum_{k=0}^{N_1} A_k \cos k\pi x\, e^{-k^2\pi^2 t} + \sum_{k=0}^{N_2} B_k sin\left(k + \frac{1}{2}\right)\pi x\, e^{\left(k+\frac{1}{2}\right)^2\pi^2 t},$$

where

$$A_k = \int_{-1}^{1} \varphi(x) \cos k\pi x dx, \quad k = 0, 1, 2, ..., N_1;$$

$$B_k = \int_{-1}^{1} \varphi(x) \sin\left(k + \frac{1}{2}\right)\pi x dx, \;\; k = 0, 1, 2, ..., N_2.$$

Proof. The validity of the theorem follows from the fact that the coefficients

$$A_k = 0, \quad k = N_1 + 1, ..., \quad B_k = 0, \quad k = N_2 + 1, ...$$

and from the statement of the Theorem 1.

Since the set of trigonometric polynomials in the complete orthonormal system $\{X_{k1},\, X_{k2}\}$ is everywhere dense in $L_2(-1,1)$, then from theorem 3 implies

Theorem 4. The set M of admissible initial functions is everywhere dense in $L_2(-1,1)$, if the mixed problem (1), (2) is solvable for all function from M.

A mixed problem for an equation with a variable coefficient

We consider the mixed problem (2) for an equation with a variable coefficient

$$u_t(x,t) = u_{xx}(-x,t) + q(x)\,u(x,t), \quad -1 \le x \le 1, \quad t \ge 0. \tag{11}$$

The application of the Fourier method to problem (11), (2) leads to a spectral problem with involution

$$-X''(-x) + q(x)\,X(x) = \lambda X(x), \quad X'(-1) = X'(1) = 0. \tag{12}$$

In the paper [15] it is shown that the baseness eigenfunction $\{X_k(x)\}$ of spectral problem (12) in the space $L_2(-1,1)$. If coefficient $q(x)$ is a real continuous function in the interval under consideration, then this basis is an orthonormal basis by virtue of the self-adjointness of the spectral problem (11). Therefore initial function $\varphi(x)$ can be decomposed into convergent in norm of the space $L_2(-1,1)$ Fourier series by orthonormal basis $\{X_k(x)\}$.

We have following

Theorem 5. If in equation (11) the coefficient $q(x)$ is a real continuous function and an initial function $\varphi(x)$ is a polynomial of the following form

$$u(x,t) = \sum_{\substack{\lambda_{k1}>0,\\ k=\overline{1,N}}} A_k X_{k1}(x) + \sum_{\substack{\lambda_{k2}<0,\\ k=\overline{1,M}}} B_k X_{k2}(x),$$

then the solution of problem (11), (2) exists, unique and can be represented by the following

$$u(x,t) = \sum_{\substack{\lambda_{k1}>0,\\ k=\overline{1,N}}} A_k e^{-\lambda_{k1}t} X_{k1}(x) + \sum_{\substack{\lambda_{k2}<0,\\ k=\overline{1,M}}} B_k e^{-\lambda_{k2}t} X_{k2}(x),$$

where

$$A_k = \int_{-1}^{1} \varphi(x)\,X_{k1}(x)\,dx, \quad k = 1,2,...,N;$$

$$B_k = \int_{-1}^{1} \varphi(x)\,X_{k2}(x)\,dx, \quad k = 1,2,...,M,$$

$X_{k1}(x)$, $X_{k2}(x)$ — eigenfunction, corresponding to eigenvalues $\lambda_{k1} > 0$ and $\lambda_{k2} < 0$ respectively.

By virtue of the density of the set of polynomials in the complete orthonormal system $\{X_k(x)\}$ in class $L_2(-1,1)$, for the mixed problem (11), (2) the assertion of Theorem 4 is satisfies.

In conclusion, we note that all the results formulated remain valid in the case of conditions $u(-1,t) = u(1,t) = 0, \quad u(x,0) = \varphi(x)$,

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Ә.Ә. Сәрсенбі

Инволюциясы бар параболалық түрдегі теңдеулер үшін аралас есептердің шешімділік шарттары

Мақалада инволюциясы бар параболалық түрдегі теңдеу үшін шеттік шарттары Дирихле түрінде болатын аралас есептер қарастырылды. Коэффициенттері тұрақты және айнымалы болатын теңдеулер зерттелген. Теңдеудің x айнымалысы бойынша екінші туындысында инволюция бар. Мұндай жағдайда есептерді зерттеудің өз қиындықтары бар. Инволюциясы бар бірөлшемді дифференциалды операторлардың оң және теріс таңбалы меншікті мәндері шексіз көп болады. Бұл теңдеудің оң жағындағы оператор оң анықталған емес дегенді білдіреді. Классикалық жағдайларда әдетте теңдеудің оң жағындағы операторлар оң анықталған болып келеді. Сондықтан автор инволюциясы бар параболалық түрдегі теңдеу үшін аралас есептердің қойылымы корректілі емес болатындығын талқылаған. Мысалдар келтірген. Қарастырылып отырған есептердің жалғыз шешімі бар болуын қамтамасыз ететін бастапқы функциялар үшін жеткілікті шарттар алған. Мұндай бастапқы функциялар жиыны

$L_2(-1,1)$ кеңістігінде тығыз орналасқан жиын болатындығы есептің жалғыз шешімі бар болған жағдайда көрсетілген. Шешімнің меншікті функциялар бойынша Фурье қатарының дербес қосындылары түрінде кескінделетіндігі анықталған.

Кілт сөздер: Фурье тәсілі, аралас есеп, инволюция, меншікті функциялар, базис.

А.А. Сарсенби

Условия разрешимости смешанных задач для уравнений параболического вида с инволюцией

Работа посвящена изучению смешанных задач для уравнения параболического вида с инволюцией с краевыми условиями типа Дирихле. Рассмотрены уравнения с постоянными и переменными коэффициентами. Инволюцию содержит вторая производная по переменной x. Этот случай является трудным для изучения. Одномерные дифференциальные операторы с инволюцией имеют бесконечное число положительных и отрицательных собственных значений. Это означает, что оператор в правой части изучаемого уравнения не является полуограниченным. В случае классических задач в правой части уравнений обычно стоят обыкновенные дифференциальные операторы, которые являются полуограниченными. Поэтому в статье автором показана некорректность смешанных задач для уравнения параболического вида с инволюцией. Приведены примеры. Найдены достаточные условия на начальные данные, когда изучаемая задача имеет единственное решение. Найдено представление решения в виде частичных сумм ряда Фурье по собственным функциям. Доказана всюду плотность в пространстве $L_2(-1,1)$ множества начальных функций, когда задача имеет единственное решение.

Ключевые слова: метод Фурье, смешанная задача, инволюция, собственные функции, базис.

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Traveling wave solutions for the two-dimensional Zakharov-Kuznetsov-Burgers equation

In this paper, the two-dimensional Zakharov-Kuznetsov-Burgers (ZKB) equation is investigated. The basic set of fluid equations is reduced to ZKB equation. This equation is a two-dimensional analog of the well-known Korteweg-de Vries-Burgers equation, and also is typical example of so-called dispersive equations which attract the considerable attention of both pure and applied mathematicians in the past decades. We obtain traveling wave solutions for two-dimensional Zakharov-Kuznetsov-Burgers equation by modified Kudryashov method which is a powerful method for obtaining exact solutions of integrable and non-integrable nonlinear evolution equations. Graphical representation of obtained solutions is demonstrated.

Keywords: modified Kudryashov method, Zakharov-Kuznetsov-Burgers equation, kink, nonlinear equation, traveling wave.

Introduction

The two-dimensional Zakharov-Kuznetsov-Burgers (ZKB) equation [1] in given by:

$$u_t + u_{xxx} + u_{xyy} + uu_x + \delta(u_{xx} + u_{yy}) = 0, \tag{1}$$

where $\delta = const > 0$. The equation (1) is referred as Zakharov-Kuznetsov-Burgers equation because in case of $\delta = 0$ it will be Zakharov-Kuznetsov equation. And also this equation is a two-dimensional analog of the well-known Korteweg-de Vries-Burgers (KdV) equation which includes dissipation and dispersion and has been studied by various researchers due to its applications in mechanics and physics [2]. The two-dimensional ZKB equation on a strip was studied by [3]. The author had proved the existence and uniqueness results for regular and weak solutions. In work [4] Lie symmetry analysis, nonlinear self-adjointness and conservation laws to the extended two-dimensional ZKB equation were studied. Application of the ZKB equation in dusty plasma was studied in [5].

The aim of the paper is to obtain new traveling wave solutions of the two-dimensional ZKB equation by using the modified Kudryashov method. This method is a powerful method for obtaining exact solutions of nonlinear evolution equations [6–8]. The modified Kudryashov method was applied to the generalized Kuramoto-Sivashinsky equation [9], the Kudryashov-Sinelshchikov equation [10], the generalized Fisher equation [11].

The paper is organized as follows. In Section 2, the key idea of the method is described. In Section 3, the proposed method is applied to the two-dimensional ZKB equation. We conclude this paper in Section 4.

The modified Kudryashov method

Let us present the algorithm of the modified Kudryashov method for finding exact solutions of nonliner partial differential equation (NPDE). We consider the NPDE in the following form:

$$E_1(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}, ...) = 0, \tag{2}$$

where E_1 is a polynomial of u(x, y, t) and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. Using the traveling transformation

$$u(x, y, t) = u(\xi), \quad \xi = kx + ry + \omega t, \tag{3}$$

where k, r, ω are constants, the NPDE (2) is reduced to nonlinear ordinary differential equation (ODE)

$$E_2(u, \omega u', ku', ru', \omega^2 u'', k^2 u'', r^2 u'', ...) = 0, \tag{4}$$

where prime denotes the derivation with respect to ξ. We look for exact solutions of (4) in the following form:

$$u(\xi) = \sum_{n=0}^{N} a_n Q(\xi)^n, \tag{5}$$

where $a_0, a_1, a_2, a_3, ..., a_N$ are unknown constants and $Q(\xi)$ is the following function:

$$Q(\xi) = \frac{1}{1+e^{\xi}}. \tag{6}$$

The function satisfies the first-order ordinary differential equation

$$Q_\xi = Q^2 - Q. \tag{7}$$

Equation (7) is necessary to calculate the derivatives of function $u(\xi)$. We can calculate the necessary number of derivatives of function u. For instance, we consider the general case when N is arbitrary. Differentiating (5) with respect to ξ and taking into account (7) we have

$$u' = \sum_{n=0}^{N} a_n n Q^n (Q-1); \tag{8a}$$

$$u'' = \sum_{n=0}^{N} a_n n Q^n (Q-1)[(n+1)Q - n]; \tag{8b}$$

$$u''' = \sum_{n=0}^{N} a_n n Q^n (Q-1)[(n^2+3n+2)Q^2 - (2n^2+3n+1)Q + n^2]. \tag{8c}$$

Next, substitute equations (8) in (4). Then we collect all terms with the same powers of function $Q(\xi)$ and equate the resulting expressions to zero. Finally, we obtain algebraic system of equations. Solving this system, we get values for the unknown parameters $a_0, a_1, a_2, a_3, ..., a_N$.

Traveling wave solutions for the Zakharov-Kuznetsov-Burgers equation

In this section, we will find the traveling wave solutions for the two-dimensional ZKB equation through the modified Kudryashov method. Let us consider two-dimensional ZKB equation (1). Using the traveling wave transformation of the form (3), equation (1) is converted into the following ordinary differential eqution

$$\omega u' + (k^3 + kr^2)u''' + kuu' + \delta(k^2 + r^2)u'' = 0. \tag{9}$$

Equation (9) is integrable, therefore, integrating once equation (9) with respect to ξ, we obtain

$$\omega u + (k^3 + kr^2)u'' + \frac{k}{2}u^2 + \delta(k^2 + r^2)u' + C_1 = 0, \tag{10}$$

where C_1 is constant integration. Considering the homogeneous balance between the highest order derivatives $u^{''}$ and the nonlinear terms u^2 in the equation (10), we can get $N = 2$. Then the equation (5) reduces to

$$u = a_0 + a_1 Q + a_2 Q^2, \tag{11}$$

where a_0, a_1, a_2 are constants to be determined later. Now substituting (11) into (10), and setting coefficients of the same power of Q^n equal to zero, we obtain these algebraic equations:

$$Q^4 : 6k(r^2 + k^2)a_2 + \frac{1}{2}ka_2^2 = 0; \tag{12a}$$

$$Q^3 : (2\delta - 10k)(k^2 + r^2)a_2 + 2k(k^2 + r^2)a_1 + ka_1a_2 = 0; \tag{12b}$$

$$Q^2 : (k^2 - r^2)(\delta - 3k)a_1 + ((k^2 + r^2)(4k - 2\delta) + \omega)a_2 + \frac{1}{2}ka_1^2 + ka_0a_2 = 0; \tag{12c}$$

$$Q^1 : ((k^2 + r^2)(k - \delta) + \omega)a_1 + ka_0a_1 = 0; \tag{12d}$$

$$Q^0 : C_1 + \omega a_0 + \frac{1}{2}ka_0^2 = 0. \tag{12e}$$

Solving system of equations (12) with the aid of Maple, we obtain the following results:
Case A:

$$a_0 = \frac{1}{25\delta}(6\delta^3 + 150\delta r^2 - 125\omega), \quad a_1 = 0, a_2 = -\frac{12}{25}\delta^2 - 12r^2; \tag{13}$$

$$k = \frac{\delta}{5}, C_1 = -\frac{1}{6250}[(6\delta^3 + 150\delta r^2)^2 - (125\omega)^2]. \tag{14}$$

Substituting (13)–(14) into (11) with (6), respectively, we obtain new traveling wave solution for ZKB equation (1) as follows:

$$u_1(x,y,t) = \frac{1}{25\delta}(6\delta^3 + 150\delta r^2 - 125\omega) - \left(\frac{12}{25}\delta^2 + 12r^2\right)\left(\frac{1}{1+e^{\xi}}\right)^2, \tag{15}$$

where $\xi = kx + ry + \omega t$. The graphical representation of obtained solution (15) is depicted on Figure 1.

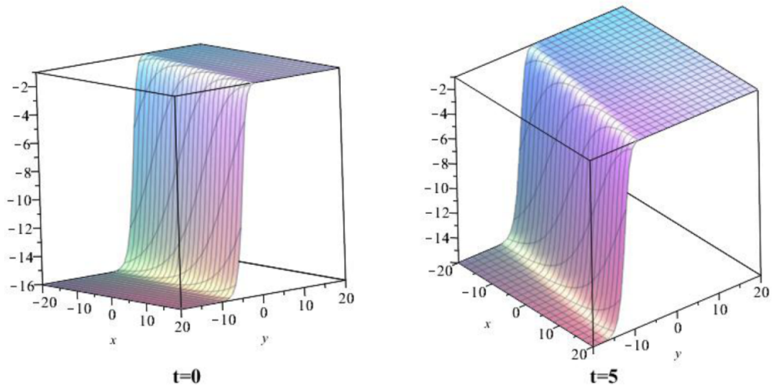

Figure 1. Dynamics the solution of $u_1(x,y,t)$ for ZKB equation when $\delta = 1, r = 1.1, \omega = 1.7$

Case B:

$$a_0 = -\frac{1}{25\delta}(6\delta^3 + 150\delta r^2 - 125\omega), \quad a_1 = \frac{24}{25}\delta^2 + 24r^2, a_2 = -\frac{12}{25}\delta^2 - 12r^2; \tag{16}$$

$$k = -\frac{\delta}{5}, C_1 = \frac{1}{6250}[(6\delta^2 + 150\delta r^2)^2 - (125\omega)^2]. \tag{17}$$

Substituting (16)–(17) into (11) with (6), respectively, we obtain new traveling wave solution for ZKB equation (1) as follows:

$$u_2(x,y,t) = -\frac{1}{25\delta}(6\delta^3 + 150\delta r^2 - 125\omega) + \left(\frac{24}{25}\delta^2 + 24r^2\right)\left(\frac{1}{1+e^{\xi}}\right) - \left(\frac{12}{25}\delta^2 + 12r^2\right)\left(\frac{1}{1+e^{\xi}}\right)^2, \tag{18}$$

where $\xi = kx + ry + \omega t$. We demonstrate solution (18) on Figure 2.

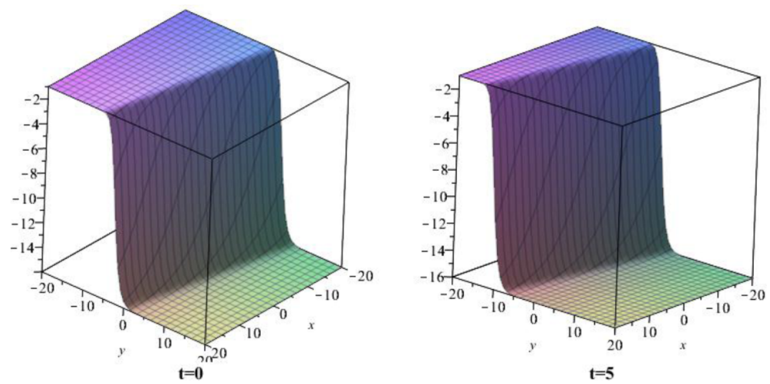

Figure 2. Dynamics the solution of $u_2(x,y,t)$ for ZKB equation when $\delta = 1, r = 1.1, \omega = 1.7$

Conclusion

In this work, we have demonstrated the efficiency of the modified Kudryashov method for finding exact solutions of the two-dimensional Zakharov-Kuznetsov-Burgers equation. We have obtained new traveling wave solution. Graphical representation of these exact solutions is presented. This method can be more successfully applied to study nonlinear evolution equations, which frequently arise in nonlinear sciences.

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Екіөлшемді Захаров-Кузнецов-Бюргерс теңдеуінің қозғалатын толқындық шешімдері

Мақалада екіөлшемді Захаров-Кузнецов-Бюргерстің (ЗКБ) теңдеуі зерттелді. Сұйықтық теңдеулерінің базалық жинағы осыған келеді. ЗКБ теңдеуі белгілі екіөлшемді Кортевег-де Фриз-Бюргерс теңдеуіне ұқсас болып табылады, сонымен қатар соңғы жылдықта іргелі және қолданбалы математиктердің елеулі назарын аударып отырған дисперсионды теңдеулердің әдеттегі мысалы. Кудряшовтың модификациялық әдісі арқылы екіөлшемді Захаров-Кузнецов-Бюргерстің теңдеуі үшін қозғалатын толқындық шешімдер алынды. Бұл әдіс интегралданатын және интегралдалданбайтын сызықты емес эволюциялық теңдеулердің нақты шешімдерін алу үшін үздік әдіс болып табылады. Алынған шешімдердің суреттері ұсынылған.

Кілт сөздер: модификацияланған Кудряшов әдісі, Захаров-Кузнецов-Бюргерс теңдеуі, кинк, сызықты емес теңдеу, қозғалатын толқын.

Г.С. Шаихова, Г.Н. Шайхова

Перемещающиеся волновые решения двумерного уравнения Захарова-Кузнецова-Бюргерса

В статье исследовано двумерное уравнение Захарова-Кузнецова-Бюргерса (ЗКБ). Базовый набор уравнений жидкости сводится к уравнению ЗКБ. Это уравнение является двумерным аналогом известного уравнения Кортевега-де Фриза-Бюргерса, а также типичным примером так называемых дисперсионных уравнений, которые привлекают значительное внимание как фундаментальных, так и прикладных математиков в последние десятилетия. Получены перемещающиеся волновые решения для двумерного уравнения Захарова-Кузнецова-Бюргерса с помощью модифицированного метода Кудряшова, который является мощным методом для получения точных решений интегрируемых и неинтегрируемых нелинейных эволюционных уравнений. Показано графическое представление полученных решений.

Ключевые слова: модифицированный метод Кудряшова, уравнения Захарова-Кузнецова-Бюргерса, кинк, нелинейное уравнение, перемещающаяся волна.

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Properties of hybrids of Jonsson theories

This work is an introduction to the study of the properties of a new concept, as a hybrid of Jonsson theories. We define the basic concepts and framework for studying the model-theoretic properties of these concepts. The main goal of this paper is to study the model-theoretic properties of companions of hybrids of Jonsson theories. The main objects of study are the Jonsson hybrids and their classes of models. In this paper, the main task is to consider the various links between such theories. In order to understand more deeply these connections and ultimately the connection with the primary theory itself, special algebraic constructions of semantic models of the considered fragments were identified and on this basis hybrids of these fragments were determined. In this paper, such algebraic constructions are called semantic hybrids.

Keywords: Jonsson theory, semantic model, hybrid, existentially prime, pregeometry, model companion.

This work refers to the model theory a branch of mathematics that is the language of mathematical logic studies the laws of a general nature of various types of mathematical structures. The most advanced ideas and concepts of the model theory were interpreted using examples from classical algebra and set theory. Subsequently, with their help, profound scientific results were obtained in various algebraic structures. It is no coincidence, because model theory was originally defined as symbiosis of universal algebra and mathematical logic.

In the modern model theory there is a conditional division into «Western» and «Eastern» themes. This convention appeared by the remark of a well-known expert in the area of model theory J. Keisler. Thus he divided the studies related to the works of A. Tarski and A. Robinson. The first author lived on the west coast of the USA, and A. Robinson respectively on the east coast. As a rule, studies related to the eastern theme are connected with first-order formulas, the prenex of which does not exceed two, and the role of morphisms that preserve these formulas and compare various models using mappings between themselves is played by either isomorphic embeddings or various homomorphisms. Moreover, the semantic aspect of such problems is determined by studying the behavior of class of existentially closed models of some fixed inductive theory.

In the class of inductive theories, a special place is occupied by Jonsson theories, i.e. theories that admit the properties of amalgam and joint embedding.

It is well known that many algebraic examples are related to Jonsson theories, for example, fields of a fixed characteristic, Boolean algebras, groups, Abelian groups, various types of rings, polygons, various types of lattices, various types of orders.

Progress in the study of Jonsson theories was achieved in the case of a perfect Jonsson theory. It turned out that such theories have a model companion as their center. For example, the center of a field of a fixed characteristic is an algebraic closed field of the same characteristic. In addition, it is necessary to note the following semantic fact that in a perfect Jonsson theory the class of existentially closed models coincides with the class of all models of its center.

Recently, experts on the model theory of the western direction attach great importance to the study of model-theoretic properties of structural problems of small models in enrichment signature. At the same time, these enrichments should retain some properties of the objects under study. As a rule, in the case of the study of the specific model-theoretic properties of the complete theory, they are very rarely transferred to the study of Jonsson theories. This is primarily due to the fact that the Jonsson theories are not complete theories. Therefore, the ability to find model-theoretic conditions, when possible, is an important task. In this regard, the introduction of new concepts and the corresponding technical apparatus is an important moment for studying the properties of Jonsson theory.

In this work are considered the model-theoretic properties of a new class of Jonsson theories, namely the theories obtained using various algebraic constructions of semantic models of two different Jonsson theories of the same language.

Earlier in the study of Jonsson theories, it was noted that due to the incompleteness of the considered theory, the requirement of $\forall\exists$ completeness, or $\exists$ completeness is a necessary condition for obtaining analogs of theorems obtained for complete theories. And also, due to the fact that in a perfect Jonsson theory the class of the center models coincides with the class of its existentially closed models, the so-called Jonsson sets were defined, i.e. special subsets of the semantic model of the considered Jonsson theory, the definable closures of which were some existentially closed submodels of this semantic model. It is well known that the set of all $\forall\exists$ consequences of the true ones in some existentially closed model form the Jonsson theory. The set of all $\forall\exists$ consequences that are true on a definable closure of a Jonsson subset forms the Jonsson theory and is called a fragment of this special subset.

In this article, the main task is to consideration the various connections between such theories. In order to understand more profound these connections and ultimately the connection with the original theory itself, and were determined special algebraic constructions of semantic models of the considered fragments and on this basis were determined hybrids of these fragments. Such algebraic constructions will be called semantic hybrids.

As an example of a semantic hybrid, we can consider the union and intersection, the Cartesian product, the direct sum, the product of filters and ultrafilters of semantic models of fragments of $\nabla - cl$-subsets of semantic model of in the considered Jonsson theory.

It is interesting to study the model-theoretic properties of various companions of a fixed hybrid. These properties of theories include almost all the classical attributes of the modern model theory, such as stability, categorical, strong minimality, model completeness, axiomatizability, interpretability, spectral issues, etc. As for the semantic aspect, there will be interesting various properties related to the concept of definable formula subsets of the semantic model of a hybrid with respect to the following concepts: atomicity, algebraic primeness, existential closure, convexity, existential primeness.

Thus, given the above, we can note that the results of this work in their content are related to the «Eastern» model theory.

We give the necessary definitions of the basic concepts of this article.

Let given a countable language of the first order.

The following definition describes one of the basic concepts of this article.

Definition 1. A theory T is Jonsson if:

1) theory T has infinite models;
2) theory T is inductive;
3) theory T has the joint embedding property (JEP);
4) theory T has the property of amalgam (AP).

Examples of Jonsson theories are:

1) group Theory;
2) theory of Abelian groups;
3) theory of fields of fixed characteristics;
4) theory of Boolean algebras;
5) theory of polygons over a fixed monoid;
6) theory of modules over a fixed ring;
7) theory of linear order.

The following definition of the universality and homogeneity of model allocates semantic invariant of any Jonsson theory, namely its semantic model. Moreover, it turned out that the saturation or non-saturation of this model significantly changes the structural properties of both the Jonsson theory itself and its class of models.

Definition 2. Let $\kappa \geq \omega$. Model M of theory T is called κ-universal for T, if each model T with the power strictly less κ isomorphically imbedded in M; κ-homogeneous for T, if for any two models A and A_1 of theory T, which are submodels of M with the power strictly less then κ and for isomorphism $f : A \to A_1$ for each extension B of model A, wich is a submodel of M and is model of T with the power strictly less then κ there is exist the extension B_1 of model A_1, which is a submodel of M and an isomorphism $g : B \to B_1$ which extends f.

Definition 3. Model C of Jonsson theory T is called semantic model, if it is ω^+-homogeneous-universal.

As can be seen from the definition of the Jonsson theory, this theory is not complete. But nevertheless, with the help of its semantic invariant (semantic model) we can always determine the center of Jonsson theory, which is a complete theory.

Definition 4. The center of Jonsson theory T is called an elementary theory of the its semantic model. Denoted through T^*, i.e. $T^* = Th(C)$.

The following two facts speak about the «good» exclusivity of the semantic model.

Fact 1 [1; 160]. Each Jonsson theory T has k^+-homogeneous-universal model of power 2^k. Conversely, if a theory T is inductive and has infinite model and ω^+-homogeneous-universal model then the theory T is a Jonsson theory.

Fact 2 [1; 160]. Let T is a Jonsson theory. Two k-homogeneous-universal models M and M_1 of T are elementary equivalent.

It is well known from the course of model theory that a saturated model is always a homogeneous-universal model, the reverse is also true. But this definition of homogeneous-universal model [2; 299] is considered as a rule in the framework in the study of complete theory. In the framework of the study of Jonsson theory, we are dealing with a particular case of the definition of a homogeneous-universal model belonging to B. Jonsson [3]. The concept of a saturated model is the same in both cases. By virtue of a more general situation of homogeneous-universality in the case of Jonsson theory, we do not have a saturation criterion in terms of homogeneous-universal as in [2; 299]. Therefore, those Jonsson theories, the semantic model of which is saturated, allocate in a special subclass of class of all Jonsson theories, and such theories are called perfect. We give a definition of perfectness of Jonsson theory.

Definition 5. Jonsson theory T is called a perfect theory, if each a semantic model of theory T is saturated model of T^*.

The first author of this article obtained a result describing the perfect Jonsson theory.

Theorem 1 [1; 158]. Let T is a Jonsson theory. Then the following conditions are equivalent:

1) theory T is perfect;

2) theory T^* is a model companion of theory T.

From the above list of Jonsson theories, the following examples 2)–4), 6), 7) are examples of a perfect Jonsson theory. But, for example, group theory is not such, due to the fact that it does not have a model companion.

Let E_T be a class of all existentially closed models of Jonsson theory T.

This class of models in general case for an arbitrary theory can be empty. But the following result [4; 367] is well known, who says that any inductive theory has a nonempty class of existentially closed models. Since the Jonsson theory is a subclass of class of inductive theories, we can say that E_T is a non-empty class.

In the case of a perfect Jonsson theory, the class of models of center of this theory coincides with E_T. This follows from the following theorem.

Theorem 2 [1; 162]. If T is a perfect Jonsson theory then $E_T = ModT^*$.

Let L is a countable language of first order. Let T is Jonsson theory in the language L and its semantic model is $\mathcal{C}$.

Let us turn to the definition of central concept of this article. Namely, the concept of a hybrid of Jonsson theories. In the beginning, we define a hybrid for two Jonsson theories, and two cases are possible. The first case is a hybrid of Jonsson theories of one signature. The second case is a hybrid of Jonsson theories of different signatures. In the first case, we are talking about a hybrid of the first type, in the second case about a hybrid of the second type. In this article we will deal only with hybrids of the first kind for two Jonsson theories, but it is easy to understand that this concept of a hybrid is generalized to an arbitrary number of Jonsson theories. Consider the case of the first type.

Let T be some Jonsson theory in a fixed language and C its semantic model.

Definition 6. Let $X \subseteq C$. We will say that a set X is $\nabla - cl$-Jonsson subset of C, if X satisfies the following conditions:

1) X is ∇-definable set (this means that there is a formula from ∇, the solution of which in the C is the set X, where $\nabla \subseteq L$, that is ∇ is a view of formula, for example $\exists, \forall, \forall\exists$ and so on.);

2) $cl(X) = M$, $M \in E_T$, where cl is some closure operator defining a pregeometry [5; 289] over C (for example $cl = acl$ or $cl = dcl$).

Lemma 1. Let T be Jonsson theory, E_T be the class of its existentially closed models. Then for any model $A \in E_T$ the theory $Th_{\forall\exists}(A)$ is a Jonsson theory.

Proof can be extract from [1, 4].

Let X_1, X_2 be ∇-cl-Jonsson subset of C, where C is semantic model of theory T.

Let $M_1 = cl(X_1)$, $M_2 = cl(X_2)$, where $M_1, M_2 \in E_T$.

$Th_{\forall\exists}(M_1) = T_1$, $Th_{\forall\exists}(M_2) = T_2$.

C_1 is semantic model of theory T_1, C_2 is semantic model of theory T_2.

We define the essence of the operation of an algebraic construction. Let $\boxdot \in \{\cup, \cap, \times, +, \oplus, \prod_F, \prod_U\}$, where $\cup$ — union, $\cap$ — -intersection; $\times$ — Cartesian product; $+$-sum and $\oplus$ — direct sum; $\prod_F$ — filtered and $\prod_U$ — ultra-production.

Let $Th_{\forall\exists}(C_1 \boxdot C_2) = H(T_1, T_2)$, where C_1 is semantic model of theory T_1, C_2 is semantic model of theory T_2.

The following definition gives a hybrid of the first type for two Jonsson theories.

Definition 7. A hybrid $H(T_1, T_2)$ of Jonsson theories T_1, T_2 is called the theory $Th_{\forall\exists}(C_1 \boxdot C_2)$, if it is Jonsson. Herewith, the algebraic construction $(C_1 \boxdot C_2)$ is called a semantic hybrid of the theories T_1, T_2.

Note the following fact:

Fact 3. In order for the theory $H(T_1, T_2)$ to be Jonsson enough to $(C_1 \boxdot C_2) \in E_T$.

Proof. This follows by Lemma 1.

Let us give an example of a semantic hybrid. Linear space is a special case of a module over a field. A well-known result from linear algebra is related to the dimension of linear space:

$dimV = dimV_1 + dimV_2 - dim(V_1 \cap V_2)$, where V is linear space, a V_1, V_2 are its subspace and $V = V_1 + V_2$.

It is easy to see that this dependence of the dimensions of these linear spaces can be interpreted in the language of R-modules, where R is field and $\nabla - cl$-sets will be considered $\forall\exists$-dcl-sets and $acl = dcl$. Moreover, V will be a semantic hybrid of V_1 and V_2, where the algebraic construction is a direct sum of subspaces, i.e. $\boxdot = \oplus$. This follows from the fact that the theory of modules is a Jonsson theory.

Thus, we note that the above definition of a hybrid of Jonsson theories and their semantic hybrid was defined in the class of fragments of some fixed Jonsson theory. Moreover, we have several parameters regarding this definition:

1) view formulas from $\nabla \subseteq L$;

2) view of closure operator cl;

3) views of algebraic constructions of semantic hybrids.

In the general case, algebraic constructions of a semantic hybrid can be non-closed with respect to the class of models of this given theory. In this connection, it is further assumed that the theory under consideration is closed with respect to the algebraic construction under consideration.

Those, it is necessary to select the following parameter:

4) the closedness of the theory under consideration with respect to an algebraic construction.

By virtue of the fact that the definition of a hybrid is sufficiently general, i.e. it depends on many parameters, we must always specify these parameters to obtain specific results. In this article, we will deal with a convex existentially prime Jonsson theory complete for $\forall\exists$ -sentences. As the closure operator, we will consider the operator dcl and such that it is equal to the algebraic closure, i.e. $acl = dcl$. As an algebraic construction for obtaining a semantic hybrid, we will consider a Cartesian product. The above parameters define a sufficiently wide class Jonsson theories, in particular linear spaces get there. The example of linear spaces was basic for us in the sense of intuition and ideas. Therefore, in order to preserve some internal ideology of linear spaces and at the same time not losing generality, we will deal with a certain subclass of class of all Jonsson theories, which contains the theory of linear spaces and also satisfies the properties of other types of algebras. For this we consider the following definitions.

Definition 8. The inductive theory T is called the existentially prime if: it has a algebraically prime model, the class of its AP (algebraically prime models) denote by AP_T; class E_T non trivial intersects with class AP_T, i.e. $AP_T \bigcap E_T \neq 0$.

Definition 9. The theory is called convex if for any its model A and for any family $\{B_i \mid i \in I\}$ of substructures of A, which are models of the theory T, the intersection $\bigcap_{i \in I} B_i$ is a model of T.

Further, the object of our study will be the class of existentially prime convex $\forall\exists$ - complete Jonsson theories.

In the study of this class of theories, we obtained the following results:

Theorem 3. Let T be perfect convex existentially prime complete for $\forall\exists$-sentences Jonsson theory. X_1, X_2 are $\forall\exists$-dcl-sets of the theory T, where M_i=$dcl(X_i) \in E_T$, $T_i = Th_{\forall\exists}(M_i)$ are also perfect convex existentially prime complete for $\forall\exists$-sentences Jonsson theories. C_1, C_2 are their semantic models, respectively. Then, if their hybrid $H(T_1, T_2)$ is a model consistent with T_i, then $H(T_i)$ is a perfect Jonsson theory for $i = 1, 2$.

Proof. Suppose the contrary. Then, since the hybrid $H(T_1, T_2)$ is a Jonsson theory and has a semantic model M, by the assumption not perfectness of this hybrid $H(T_1, T_2)$, the considered semantic model M will not be saturated in its power. And this means that there is such $X \subseteq M$ and such type $p \in S_1(X)$, which is not realized in M, more precisely in $(M, m)_{m \in X}$. By virtue of the consistency of type p, this type is realized in some elementary extension $M' \succ M$. By virtue of the Jonssonness of hybrid $H(T_1, T_2)$ and model consistency with T_i, $i = 1, 2$, there is a model $A_i \in ModT_i$, $i = 1, 2$, such that M' is a submodel of A. A in turn, is embedded in the semantic model C_i, $i \in 1, 2$, but C_i is a saturated model of the theory T_i, $i \in 1, 2$. By virtue of an isomorphic embedding, suppose f from M' to A, $f(X) \subseteq A$ and since the type of p is realized in M' it will be realized in C_i. But $C_i \in E_{T_i}$ and since T_i are existentially prime convex theories, there exists a countable model $N_i \in E_{T_i}$, in which the type p will be realized. By virtue of convexity, the model N_i will be a nuclear model, i.e. it is algebraically prime embedded in other models from $ModT_i$ exactly one time. But by virtue of the model consistency of T_i with the hybrid $H(T_i)$, N_i is isomorphically embedded in some model from $ModH(T_i)$, i.e. by the map g. Since T_i are perfect theories, their center is model-complete, i.e. any monomorphism is elementary between the models of this center. And such, by virtue of perfection, are all the models from E_{T_i}. Then the above isomorphism g will be elementary, i.e. type p is realized in a countable submodel of model M. We got a contradiction with the assumption of imperfection.

Theorem 4. Let T, T_1, T_2 satisfy the conditions of Theorem 3 and T_1, T_2 be ω-categorical. Then their hybrid $H(T_1, T_2)$ is also a ω-categorical Jonsson theory.

Proof. We note that, by virtue of the above Theorem 3, the hybrid $H(T_1, T_2)$ will be a perfect Jonsson theory. Suppose the contrary, i.e. the hybrid $H(T_i)$ is not a ω-categorical Jonsson theory. Let A and B be two countable models from $ModH(T_i)$ that are not isomorphic to each other. Then there are A' and B' countable models from $E_{H(T_i)}$ such that A is isomorphically embedded into A', and B is isomorphically embedded into B'. This follows from the fact that in any inductive theory any model is isomorphically embedded in some existentially closed model of this theory. But the theories of T_i are mutually model consistent with $H(T_i)$ by virtue of the condition of the theorem. Then A' and B' are isomorphically embedded in some countable model $D \in E_{T_i}$, but as T_i are convex theories, then the image of A' and the image of B' in the model D intersects non-empty. Let this intersection be a model R. By virtue of the above existential primeness and countable categoricity of T_i, since $R \in E_{T_i}$ it follows that in $R \models \varphi(x) \wedge \neg\varphi(x)$, where in $A' \models \varphi(x)$, and in $B' \models \neg\varphi(x)$. But this is not true, as T_i are ω-categorical by condition. Consequently, we obtain a contradiction with the assumption of non-ω-categoricity $H(T_i)$.

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Йонсондық теориялардың гибридтерінің қасиеттері

Мақаланың негізгі мақсаты — йонсондық теориялардың гибридтерінің компаньондарының модельдітеоретикалық қасиеттерін зерттеу. Негізгі нысандары йонсондық гибридтер мен олардың модельдер класы болып табылады. Осындай теориялардың әртүрлі байланыстары қарастырылды. Ол байланыстарды тереңірек түсіну және алғашқы теориямен байланыстыру үшін қаралған фрагменттің семантикалық модельдерінің арнайы алгебралық құрылымдары анықталды және осы негізде бұл фрагменттердің гибридтері белгілі болды. Жалпы мұндай алгебралық құрылымдар семантикалық гибридтер деп аталады.

Кілт сөздер: йонсондық теория, семантикалық модель, гибрид, экзистенционалды жай, модельді компаньон.

А.Р. Ешкеев, Н.М. Мусина

Свойства гибридов йонсоновских теорий

В статье основной задачей является рассмотрение различных связей между йонсоновскими теориями. Для того чтобы более глубоко понять эти связи и, в конечном итоге, связь с самой первоначальной теорией и были определены специальные алгебраические конструкции семантических моделей рассматриваемых фрагментов. Кроме того, были определены гибриды этих фрагментов, такие алгебраические конструкции называются семантическими гибридами.

Ключевые слова: йонсоновская теория, семантическая модель, гибрид, экзистенциально простая, модельный компаньон.

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Companions of fragments in admissible enrichments

In this paper, model-theoretic properties of companions of fragments of special subsets in admissible enrichments are considered. Admissible enrichments are understood as enrichments of a signature that preserve the basic syntactic properties of the Jonsson theory under consideration. The study of the properties of companions of the Jonsson theory is related to the classical problematics of studying inductive theories, which was determined at the time by one of the founders of the theory of models A. Robinson. In this paper, the main properties of companion fragments of definable subsets of the semantic model of a given Jonsson theory is categoricity and it is considered.

Keywords: Jonsson theory, semantic model,categoricity existentially prime, pregeometry, modular pregeometry, model companion.

Let us single out two directions in the development of model theory. In the well-known book H.J. Keisler they are called the western and eastern theory of models, since one of the founders of the theory of models A. Tarski lived on the west coast of the United States since 1940, and another founder A. Robinson - in the east. Western theory of models develops in the traditions of Skolem and Tarski. It was more motivated in theory, analysis and set theory, and it uses all formulas of first-order logic.

The Eastern theory of models develops in the traditions of Maltsev and Robinson. It was motivated by problems in abstract algebra, where the theory formulas usually have at most two blocks of quantifiers. It emphasizes the set of quantifier-free formulas and existential formulas. Unlike the Western theory of models, which studies complete theories, Eastern model theory generally deals with incomplete thorium. The class of incomplete theories is wide enough, so we can confine ourselves to inductive theories ($\forall\exists$-axiomatizable). In the sense of completeness of the theory considered, the maximum requirement, as a rule, is $\forall\exists$-completeness. All these conditions are satisfied by the Johnson theories. Thus, we conclude that the study of Johnson's theories refers to its essence to the problems of the «eastern» theory of models.

This article is related to one of the branches of Model Theory, and more precisely to studying of Jonsson sets. This part of Model Theory is concerned with the study of incomplete inductive theories and more precisely Jonsson theories and some of their generalizations [1]. Actually it examines the model-theoretical properties of Jonsson subsets of semantic model of some Jonsson theory. In particular the lattice of special formulas of such sets is considered.

In the study of complete theories, one of the main methods is to use the properties of a topological space. In the case of a fragment of a Jonsson set, one can consider a lattice of existential formulas, which is a sublattice of Boolean algebra. The main goal of this article is to develop the basic concepts and methods of that part of the theory of models, which will provide an opportunity for fruitful research of Jonsson theory and some of its generalizations, taking into account modern developments in model theory. Our technique is standard for studying incomplete theories. The method of research consists in translating the elementary properties of the center of Jonsson theory into the theory itself.

This article discusses the method of studying Jonsson theories, first proposed by T.G Mustafin in [2]. The basis of this method is the natural connection of the class of models of arbitrary Jonsson theory T with the class of models of the theory T^*, where $T^* = Th(C)$ and C are T — universal, T — homogeneous model of the theory T. The model C exists by the Morley-Vaught theorem [3], thereby C is semantic, and T^* is a syntactic invariant of the Jonsson theory of T. The weak point of such an approach to the study of Johnson theories is the presence in the proof of the Morley-Vaught theorem a hypothesis about the existence of a strongly unreachable cardinal. To avoid set-theoretic problems allows a change in the definition of the semantic model (T — universal, T — homogeneous model of the theory T). This was done in the work of E.T. Mustafin [4].

In the future, we will adhere to the method proposed by T. G. Musafin in [2] only with a change in the definition of the semantic model in [3]. Designations are standard as in [5]. All indefinable concepts here are considered known and can be found in [5]. We give the necessary definitions of the basic concepts of this article.

1 Preliminary Information on Jonsson theories

Consider the theory T of a countable language of first order L.

Definition 1.1. A theory T is Jonsson if:

1) theory T has infinite models;

2) theory T is inductive;

3) theory T has the joint embedding property (JEP);

4) theory T has the property of amalgam (AP).

Definition 1.2. Let $\kappa \geq \omega$. Model M of theory T is called κ-universal for T, if each model T with the power strictly less κ isomorphically imbedded in M; κ-homogeneous for T, if for any two models A and A_1 of theory T, which are submodels of M with the power strictly less then κ and for isomorphism $f : A \to A_1$ for each extension B of model A, wich is a submodel of M and is model of T with the power strictly less then κ there is exist the extension B_1 of model A_1, which is a submodel of M and an isomorphism $g : B \to B_1$ which extends f.

Definition 1.3. A homogeneous-universal for T model is called κ — homogeneous-universal for T power model κ, where $\kappa \geq \omega$.

The following sentences can be found in [6]:

Fact 1. [6; 0.1]. Each Jonsson theory T has κ^+ — homogeneous-universal model of power 2^κ. Conversely, if the theory T is inductive, has an infinite model, and has a ω^+ — homogeneous-universal model, then the theory T is a Jonsson theory.

Fact 2. [6; 0.2].

1. Let T be a Jonsson theory. Two models M and M_1 κ-homogeneous-universal for T are elementarily equivalent.

2. If there exists a model M κ-homogeneous-universal for T of cardinality κ, then it is unique up to isomorphism. In addition, the model is M κ-homogeneous, i.e. any isomorphism between two submodels A and B of a model M of cardinality is strictly less than κ, which are models of the theory T, extends to an automorphism M.

We show that in the framework of the definition of homogeneity and universality from [6] the following is true:

Definition 1.4. A model C of a Jonsson theory T is called a semantic model of the theory T, if it is ω^+-homogeneous-universal in the sense of [6].

Definition 1.5. A model $\mathcal{A}$ of the theory T is called T-existentially closed if for any model $\mathcal{B}$ of the theory T and any existential formula $\varphi(\overline{x})$ with constants from $\mathcal{A}$ is $\mathcal{A} \models \exists\overline{x}\varphi(\overline{x})$ provided that, $\mathcal{A}$ is a submodel $\mathcal{B}$ and $\mathcal{B} \models \exists\overline{x}\varphi(\overline{x})$.

In connection with the definition 1.4, the following fact is true.

Lemma 1.1. The semantic model C of the Jonsson theory T is T-existentially closed.

Proof. Let C have cardinality $\kappa \geq \omega$. Let M be some extension of C cardinality 2^{κ_1}, which exists by virtue to the fact [6; 0.1]. The model M is isomorphically embedded in M_1 by virtue of κ_1^+-universality of M_1. Then C is isomorphically embedded in M_1. Let $m \in C$ and $M \models \exists x\varphi(x, m)$. Then $M_1 \models \exists x\varphi(x, m)$. therefore, by virtue of [6; 0.2]. (1), M_1 and C are elementarily equivalent. So, $C \models \exists x\varphi(x, m)$. Thus, C - T-exsentially closed.

The following fact was considered in [6].

Fact 3. [6; 0.3]. Let T be a Jonsson theory. If T^* is model complete and $\kappa \geq \omega$, then κ is homogeneous - universal for T models κ-saturated; if T^* is not model-complete, then no model is ω^*-saturated.

In the framework of the new definition of the semantic model of the Jonsson theory we give the following definition.

Definition 1.6. A Jonsson theory of T is called perfect if each semantic model of T is ω^*-a saturated model of T^*.

Theorem 1.1 [6]. Let T is a Jonsson theory. Then the following conditions are equivalent:

– theory T is perfect;

– theory T^* is a model companion of theory T.

Consequence 1.1. Let T be a Jonsson theory. Then the following conditions are equivalent:

1. $ModT^* = E_T$.

2. $T^* = T^f$, where E_T is a class of T-existentially closed models, $T^f = Th(F_T)$, where F_T is a class of generic models T (in the sense of Robinson finite forcing).

Moreover, you can notice the following:

Remark 1.1. Perfectness of the Jonsson theory is equivalent to the model completeness T^*.

2 Countable and uncountable categoricity

The purpose of this section is to give proof of two results (Theorems 2.5 and 2.9) related to the countable and uncountable categorical nature of the center of some classes of Jonsson theories. We define the concepts and related results necessary for the proof of Theorem 2.5.

Definition 2.1. The inductive theory T is called the existentially prime if: it has a algebraically prime model, the class of its AP (algebraically prime models) denote by AP_T; class E_T non trivial intersects with class AP_T, i.e. $AP_T \bigcap E_T \neq 0$.

Definition 2.2. The theory is called convex if for any its model A and for any family $\{B_i \mid i \in I\}$ of substructures of A, which are models of the theory T, the intersection $\bigcap_{i\in I} B_i$ is a model of T.

Definition 2.3. Let X be $\Delta - cl$-Jonsson subset of semantic model of fixed Jonsson theory and let $cl : P(X) \to P(X)$ be an operator on the power set of X. We say that (X, cl) is a Jonsson pregeometry if the following conditions are satisfied.

If $A \subseteq X$, then $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$.

If $A \subseteq B \subseteq X$, then $cl(A) \subseteq cl(B)$.

(exchange) $A \subseteq X$, $a, b \in X$, and $a \in cl(A \cup \{B\})$, then $a \in cl(A), b \in cl(A \cup \{a\})$.

(finite character) If $A \subseteq X$ and $a \in cl(A)$, then there is a finite $A_0 \subseteq A$ such that a $a \in cl(A_0)$.

We say that $A \subseteq X$ is closed if $cl(A) = A$.

Definition 2.4. If (X, cl) is a Jonsson pregeometry, we say that A is Jonsson independent if $a \notin cl(A \setminus \{a\})$ for all $a \in A$ and that B is a J-basis for Y if is J-independent and $Y \subseteq acl(B)$.

Lemma 2.1. If (X, cl) is a J — pregeometry, $Y \subseteq X$, $B_1, B_2 \subseteq Y$ and each B_i is a J — basis for Y, then $\mid B_1 \mid=\mid B_2 \mid$.

We call $\mid B_i \mid$ the J-dimension of Y and write $Jdim(Y) =\mid B_i \mid$.

If$A \subseteq X$, we also consider the localization $cl_A(B) = cl(A \cup B)$.

Lemma 2.2. If (X, cl) is a J-pregeometry, then (X, cl_A) is a J-pregeometry.

If (X, cl) is a J-pregeometry, we say that $Y \subseteq X$ is J-independent over A if Y is J-independent in (X, cl_A). We let $Jdim(Y/A)$ be the J-dimension of Y in the localization (X, cl_A). We call $Jdim(Y/A)$ the J-dimension of Y over A.

Definition 2.5. We say that a J-pregeometry (X, cl) is J-geometry if $cl(\emptyset) = \emptyset$ and $cl(\{x\}) = \{x\}$ for any $x \in X$.

If (X, cl) is a J-pregeometry, then we can naturally define a J-geometry. Let $X_0 = X \setminus cl(\emptyset)$. Consider the relation $\sim$ on X_0 given by $a \sim b$ if and only if $cl(\{a\}) = cl(\{b\})$. By exchange, $\sim$ is an equivalence relation. Let $\widehat{X}$ be $X_0/\sim$. Define $\widehat{cl}$ on $\widehat{X}$ by $\widehat{cl}(A/\sim) = \{b/\sim : b \in cl(A)\}$.

Lemma 2.3. If (X, cl) is a J-pregeometry, then $(\widehat{X}, \widehat{cl})$ is a J-geometry.

Definition 2.6. Let (X, cl) be J-pregeometry. We say that (X, cl) is trivial if $cl(A) = Y_{a\in A} cl\{a\}$ for any $A \subseteq X$. We say that (X, cl) is modular if for any finite-dimensional closed $Jdim(A \cup B) = Jdim(A) + Jdim(B) -$ $-Jdim(A \cap B)$

We say that (X, cl) is locally modular if (X, cl_a) is modular for some $a \in X$.

Definition 2.7. We say that (X, cl) is modular if for any finite-dimensional closed $A, B \subseteq X$

$$dim(A \cup B) = dimA + dimB - dim(A \cap B).$$

Definition 2.8. If $X = C$, then the Jonsson theory of T is called modular.

Theorem 2.1. Let (X, cl) be J- pregeometry. The following are equivalent.

1) (X, cl) is modular;

2) if $A \subseteq X$ is closed and nonempty, $b \in X$, and $x \in cl(A, b)$, then there is $a \in A$ such that $x \in cl(a, b)$;

3) if $A, B \subseteq X$ are closed and nonempty, and $x \in cl(A, B)$, then there are $a \in A$ and $b \in B$, such that $x \in cl(a, b)$.

Definition 2.9. [7]. $\mathfrak{A}$ is called (Γ_1, Γ_2)-the atomic model of T theory, if $\mathfrak{A}$ model T and for each n, each n-k elements from $\mathfrak{A}$ satisfies some formula from Γ_1, which is complete for Γ_2 - formulas.

Theorem 2.2. If L is a countable language and T is a complete ω-categorical theory, then T has a ω-categorical model companion.

Definition 2.10. Let $X \subseteq C$. We will say that a set X is $\nabla - cl$-Jonsson subset of C, if X satisfies the following conditions:

1) X is ∇-definable set (this means that there is a formula from ∇, the solution of which in the C is the set X, where $\nabla \subseteq L$, that is ∇ is a type of formula, for example $\exists, \forall, \forall\exists$ and so on);

2) $cl(X) = M$, $M \in E_T$, where cl is some closure operator defining a pregeometry [8; 289] over C (for example $cl = acl$ or $cl = dcl$).

Definition 2.11 [7].

1. $\mathcal{A}, a_0, ..., a_{n-1} \Rightarrow_\Gamma (\mathcal{B}, b_0, ..., b_{n-1})$ means that for each formula $\varphi(x_0, x_2, ..., x_{n-1})$ of Γ, if $\mathcal{A} \models \varphi(\bar{a})$, then $\mathcal{B} \models \varphi(\bar{b})$.

2. $(\mathcal{A}, \bar{a}) \equiv_\Gamma (\mathcal{B}, \bar{b})$ means $(\mathcal{A}, \bar{a}) \Rightarrow_\Gamma (\mathcal{B}, \bar{b})$ and $(\mathcal{B}, \bar{b}) \Rightarrow_\Gamma (\mathcal{A}, \bar{a})$.

Definition 2.12.

1. $\mathcal{A}$ is called a Σ-nice-algebraically prime model of T, if $\mathcal{A}$ is a countable model of T and for each model of $\mathcal{B}$ theory of T, each $n \in \omega$ and for all $a_0, a_2, ..., a_{n-1} \in A$, $b_0, b_2, ..., b_{n-1} \in B$, if $\mathcal{A}, a_0, ..., a_{n-1} \Rightarrow_\exists (\mathcal{B}, b_0, ..., b_{n-1})$, then for each $a_n \in A$ there exists some $b_n \in B$ such that $\mathcal{A}, a_0, ..., _n \Rightarrow_\exists (\mathcal{B}, b_0, ..., b_n)$.

2. $\mathcal{A}$ is called a Σ^*-nice-algebraically prime model of the theory T, if $\mathcal{A}$ is a countable model of T and for each model $\mathcal{B}$ theory T, each $n \in \omega$ and for all $a_0, a_2, ..., a_{n-1} \in A$, $b_0, b_2, ..., b_{n-1} \in B$, if $\mathcal{A}, a_0, ..., a_{n-1} \equiv_\exists (\mathcal{B}, b_0, ..., b_{n-1})$, then for each $a_n \in A$ there exists some $b_n \in B$ such that $\mathcal{A}, a_0, ..., a_n \equiv_\exists (\mathcal{B}, b_0, ..., b_n)$.

Theorem 2.3. Let T $\forall\exists$ theory be complete for existential sentences, and let $\mathcal{A}$ be a countable model of T. Then (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3), where:

1. $\mathcal{A}$ - (Σ, Σ) is atomic;
2. $\mathcal{A}$ - Σ^*-nice;
3. $\mathcal{A}$ is existentially closed and Σ-nice.

Theorem 2.4. Let T - is a theory complete for existential sentences. Then any two countable (Σ, Σ)-atomic models of the theory T are isomorphic.

Now we consider the admissible enrichment. An enrichment is called admissible if it preserves a definability of the type in any existentially closed extension. The following $\{P\} \cup \{c\}$ - enrichment is admissible.

Let T be an arbitrary Jonsson theory in the first-order language of the signature σ. Let $\mathcal{C}$ be a semantic model of the theory T. Let $A \subset C$ be a $\nabla - cl$- a subset in T theory, where $\nabla = \forall\exists$, $cl = acl$ and at the same time $acl = dcl$. Let $\sigma_\Gamma(A) = \sigma \cup \{c_a | a \in A\} \cup \Gamma$, $\Gamma = \{P\} \cup \{c\}$. Let $T_A^C = T \cup Th_{\forall\exists}(C, a)_a \in A \cup \{P(c_a) | a \in \in A\} \cup \{P(c)\} \cup \{''P \subseteq''\}$, where $\{''P \subseteq''\}$ is an infinite number of sentences expressing the fact that the interpretation of the P symbol is existentially closed submodel in the language of the signature $\sigma_\Gamma(A)$ and this model is the definable closure of the set A. It is clear that the considered set of sentences is not necessarily a Jonsson theory, and this theory, generally speaking, is not complete. The reason for the no jonssoness is the lack of amalgam in some cases, i.e. there are counterexamples of enrichment by the predicate of the Jonsson theory, which do not allow amalgam. In the case of modularity, there is amalgam. Therefore, in the future, the considered theories are modular.Let T^* be the center of the Jonsson theory T_A^C and $T^* = Th(C')$, where C' is the semantic model of the theory T_A^C.

Below we give the results of [Theorem 2.5, 2.9] on countable and uncountable categoricity in the framework of the above enrichment of the signature of the Jonsson theory. Relatively prime Jonsson theory without enrichment, the first author had previously obtained similar results [9; 104].

Theorem 2.5. Let T_A^C be a modular convex Jonsson theory complete for $\forall\exists$-sentences. Then the following conditions are equivalent:

1. T^* ω-categorical;
2. T_A^C ω-categorical.

1) $\Rightarrow$ 2) Let T^* ω-categorical. Then, by virtue of the theorem 2.2, T^* has a ω-categorical model companion $T^{*\prime}$. With the virtue of consistency the model T_A^C и T^*, T^* и $T^{*\prime}$, $T^{*\prime}$ the model of consistency with T_A^C, therefore, $T^{*\prime}$ is a model companion of T_A^C, in particular, $T^{*\prime}$ is model complete. By virtue of the model completeness of $T^{*\prime}$ any formula in the language $T^{*\prime}$ is equivalent to some existential formula. Then, by the Robinson theorem on the uniqueness of a model companion and by the Jonsson theory 1.1 criterion of perfectness, it follows that $T^* = T^{*\prime}$. Since $T^{*\prime}$ ω-categorical, its only countable model is countably saturated and belongs to $ModT_A^C$, since $ModT^* \subseteq ModT_A^C$. By virtue of 1.1 $ModT^{*\prime} = E_T$ and in E_T there is one up to isomorphism a countable model $\mathcal{A}$, where (L, L) is atomic (in the sense of 2.2), where L is the whole language. With the virtue of convexity of T hence $\mathcal{A}$ is (Σ, Σ)-the atomic model T^*, because of its model completeness ($T^* = T^{*\prime}$. Due to $\forall\exists$-completeness of the theory T_A^C, for any n $E_n(T_A^C) = E_n(T^*)$, where $E_n(T_A^C)$ - is a lattice of existential formulas of the theory T_A^C of n free variables. Then $\mathcal{A}$ - (Σ, Σ)-atomic model of the theory T_A^C. Then, by Theorem 2 the model $\mathcal{A}$ Σ^*- a nice-model. Let $\mathcal{B} \in ModT_A^C$, $|\mathcal{B}| = \omega$. We show that $\mathcal{B}$ is a (Σ, Σ)-atomic model of the theory T_A^C. We prove by induction. By virtue of $\forall\exists$-completeness of the theory T_A^C we have the theory T_A^C $\exists$-complete, and therefore $\mathcal{A} \equiv_\exists \mathcal{B}$ (induction basis). Suppose $(\mathcal{A}, a_1, ..., a_{n-1})_{a_1,...,a_{n-1} \in A} \equiv_\exists (\mathcal{B}, f(a_1), ..., f(a_{n-1}))_{a_1,...,a_{n-1} \in A}$, where f - is an isomorphic embedding from $\mathcal{A}$ to model $\mathcal{B}$. This isomorphic embedding exists because $\mathcal{A}$, being (Σ, Σ)-atomic, is algebraically prime, that is, is isomorphically embedded in any model of the theory T_A^C. This

fact follows from [9]. And since $\mathcal{A}$ is Σ^*-a nice model of the theory of T_A^C, for any $a_n \in A$ there is such $b \in B$, such that $(\mathcal{A}, a_1, ..., a_{n-1}, a_n)_{a_1,...,a_n \in A} \equiv_\exists (\mathcal{B}, f(a_1), ..., f(a_{n-1}), b)_{a_1,...,a_n \in A}$. Let $f(a_n) = b$. Hence $\mathcal{A} \preceq_\exists \mathcal{B}$, т.е. $\mathcal{B}$ is an elementary extension with respect to existential formulas. Therefore, $\mathcal{B}$ - (Σ, Σ) is the atomic model T_A^C. Otherwise, since $\mathcal{A}$ belongs to E_T, $\mathcal{A}$ will not be (Σ, Σ)-the atomic model of T_A^C. Then, by Theorem 2.4 we have $\mathcal{B} \cong \mathcal{A}$. Since the model is $\mathcal{B}$ arbitrary, the theory T_A^C ω-categorical.

2) $\Rightarrow$ 1) Let a theory T_A^C ω-categorical. Suppose that the theory T^* not ω-categorical, then there exist non-isomorphic countable models $\mathcal{A}$ and $\mathcal{B}$ of the theory T^*. But, since $T_A^C \subseteq T^*$, then $ModT^* \subseteq ModT_A^C$. Therefore, $\mathcal{A}$ и $\mathcal{B}$ belong to $ModT_A^C$. We obtain a contradiction with the ω-categorical of T_A^C.

We define the concepts and related results necessary for the proof of Theorem 2.9 [9].

Definition 2.13. The formula $\varphi(\overline{x})$ is called a Δ-formula with respect to the theory T, if there are existential formulas $\psi_1(\overline{x})$ and $\psi_2(\overline{x})$ such that $T \models (\varphi \leftrightarrow \psi_1)$ and $T \models (\neg\varphi \leftrightarrow \psi_1)$

Definition 2.14. We will say that the theory T admits R_1, if for any existential formula $\varphi(\overline{x})$ consistency with T there is a formula $\psi(\overline{x}) \in \Delta$ consistency with T such that $T \models \psi \to \varphi$.

Definition 2.15. A countable model of the theory T is called a countably algebraic universal model, if all countable models of a given theory is isomorphically embedded into it.

Theorem 2.6. Let T be a universal theory, complete for existential sentences, having a countably algebraically universal model. Then T has an algebraically prime model, which (Σ, Δ) is atomic.

Theorem 2.7. Let T $\forall\exists$ theory be complete for existential sentences, admitting R_1. Then the following conditions are equivalent:

(1) T has an algebraically prime model;

(2) T has (Σ, Δ) is an atomic model;

(3) T has (Δ, Σ) is an atomic model;

(4) T has Δ-nice algebraically prime model;

(5) T has a unique algebraically prime model.

Definition 2.16. Let $\mathcal{A}, \mathcal{B} \in E_T$ and $\mathcal{A} \subsetneqq \mathcal{B}$. Then a $\mathcal{B}$ is called an algebraically prime model extension of $\mathcal{A}$ в E_T, if for any model $\mathcal{C} \in E_T$ from the fact that $\mathcal{A}$ isomorphically embedded in $\mathcal{C}$ in its own way, it follows that $\mathcal{B}$ is isomorphically embedded in $\mathcal{C}$.

Definition 2.17. [10]. A model $\mathcal{A}$ is called a prime own elementary extension $\mathcal{B}$, if $\mathcal{A} \supsetneqq \mathcal{B}$ and for any model $\mathcal{C}$ such that $\mathcal{C} \supsetneqq \mathcal{B}$ $\mathcal{A}$ is elementary embedded in $\mathcal{C}$.

Theorem 2.8. [11]. A complete theory T ω_1-categorical if and only if any of its countable models has a prime own elementary extension.

The following theorem describes the properties of T_A^C in the above enrichment under the following conditions.

Theorem 2.9. Let T_A^C be a modular convex Jonsson theory complete for $\exists$-sentences, for which R_1 is executed. Then the following conditions are equivalent:

1) theory of T^* ω_1-categorical;

2) any countable model in $E_{T_A^C}$ has an algebraically prime model extension in $E_{T_A^C}$.

Proof. 1) $\Rightarrow$ 2) If the theory is T^* ω_1-categorical, then it is perfect by the Morley theorem on uncountable categorical. Then, by virtue of the criterion of perfectness of the 1.1 Jonsson theory, we have that the theory T^* is model complete and $ModT^* = E_{T_A^C}$. If the theory T^* s model-complete, then any isomorphic embedding is elementary. Since T^* is a complete theory, then applying the theorem 2.8 to it, we obtain the required one.

2) $\Rightarrow$ 1) Applying the 1.1 lemma to the $\mathcal{C}$ semantic model of the T_A^C theory (it exists because T_A^C - Jonsson theory), we obtain that the model $\mathcal{C}$ is ω-universal. Its power, generally speaking, is greater than the counting one. Therefore, we consider its countable elementary submodel $\mathcal{D}$. Since T_A^C is a convex and the model $\mathcal{C}$ is existentially closed, its elementary submodel $\mathcal{D}$ is also existentially closed. Hence we have that it is countably algebraically universal. It now remains to apply Theorem 2.6, according to which the theory T_A^C has an algebraically prime model $\mathcal{A}_0$. We define by induction $\mathcal{A}_{\delta+1}$, which is an algebraically prime model extension of the model $\mathcal{A}_\delta$ and $\mathcal{A}_\lambda = \bigcup\{\mathcal{A}_\delta | \delta < \lambda\}$. Then let $\mathcal{A} = \bigcup\{\mathcal{A}_\delta | \delta < \omega_1\}$. Suppose that $\mathcal{B} \models T_A^C$ and $|\mathcal{B}| = \omega_1$. To show that $\mathcal{B} \approx \mathcal{A}$, decompose $\mathcal{B}$ into a chain $\{\mathcal{B}_\delta | \delta < \omega_1\}$ countable models. By virtue of the jonssoness theory of T_A^C this is possible. Define the function $g : \omega_1 \to \omega_1$ and the chain $\{f_\delta : \mathcal{A}_{g\delta} \to \mathcal{B} | 0 \leq \delta < \omega_1\}$ isomorphisms by induction on δ:

1. $g_0 = 0$ and $f_0 : \mathcal{A}_0 \to \mathcal{B}_0$.
2. $g_\lambda = \bigcup\{g_\delta | \delta < \lambda\}$ and $f_\lambda = \bigcup\{f_\delta | \delta < \lambda\}$.
3. $f_{\delta+1}$ equals the union of the chain $\{f_\delta^\gamma | \gamma \leq \rho\}$, which is defined by induction on γ.
4. $f_{\delta+1}^0 = f_\delta$, $f_{\delta+1}^\lambda = f_\delta = \bigcup\{f_{\delta+1}^\lambda | \delta < \lambda\}$.

5. Suppose that $f_0^\gamma : \mathcal{A}_{g\delta+\gamma} \to \mathcal{B}_{\delta+1}$. If $f_{\delta+1}^\gamma$ is a mapping on, then $\rho = \gamma$. Otherwise, by the algebraic simplicity of $\mathcal{A}_{g\delta+\gamma+1}$ you can continue $f_{\delta+1}^\gamma$ to $f_{\delta+1}^{\gamma+1} : \mathcal{A}_{g\delta+\gamma+1} \to \mathcal{B}_{\delta+1}$.

6. $g(\delta+1) = g\delta + \rho$. Clearly, $f = \bigcup\{f_\delta | \delta < \omega_1\}$ maps $\mathcal{A}$ isomorphically to $\mathcal{B}$. Now it remains to apply the theorem.

7. By virtue of the convexity of the theory T_A^C and since $\mathcal{B}$ is an arbitrary model of the theory T_A^C, and $\mathcal{A}$ is the only algebraic prime and existentially closed model By virtue of the condition and construction, then it follows that $E_{T_A^C}$ in uncountable cardinality has a single model, which means the semantic model of the theory T_A^C is saturated, i.e. the Jonsson theory of T_A^C is perfect. It follows that $ModT^* = E_{T_A^C}$. Therefore, T^*-ω_1 — categorical.

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А.Р. Ешкеев, Г.Е. Жумабекова

Байытуда рұқсаттылығы бар фрагменттердің компаньондары

Мақалада рұқсат етілген байытулардағы арнайы ішкі жиындардың фрагменттерінің модельді-теориялық қасиеттері қарастырылды. Рұқсат етілген байытулар йонсон теориясының негізгі синтаксистік қасиетін сақтайтын сигнатурадағы байыту ретінде түсіндірілді. Йонсондар теориясындағы компаньондар қасиетін зерттеу индуктивті теориялардың классикалық мәселесін зерттеумен байланысты, яғни, өз уақытында модельдер теориясының негізін қалаушылардың бірі — А. Робинсон анықтаған. Авторлар қарастырылған йонсон теориясының анықталған жиындардағы компаньон фрагменттерінің қасиеті қатаң болады деген қорытынды жасады.

Кілт сөздер: йонсондық теория, семантикалық модель, экзистенциалды жай, предгеометрия, модулярлы предгеометрия, модельді компаньон.

А.Р. Ешкеев, Г.Е. Жумабекова

Компаньоны фрагментов в допустимых обогащениях

В статье рассмотрены теоретико-модельные свойства компаньонов фрагментов специальных подмножеств в допустимых обогащениях. Под допустимыми обогащениями понимаются обогащения сигнатуры, которые сохраняют основные синтаксические свойства рассматриваемой йонсоновской теории. Изучение свойств компаньонов йонсоновской теории относится к классической проблематике изучения индуктивных теорий, которую определил в свое время один из основателей теории моделей А. Робинсон. Авторы статьи пришли к выводу, что основные свойства компаньонных фрагментов определяемых подмножеств семантической модели йонсоновской теории категоричны.

Ключевые слова: йонсоновская теория, семантическая модель, экзистенциально простой, предгеометрия, модулярная предгеометрия, модельный компаньон.

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МЕХАНИКА
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Calculation of velocities and accelerations of points of a crank mechanism

In this article, presented is an analytical method of determining linear and angular velocities and accelerations of links of a crank mechanism on the basis of a grapho-analytical method (without the construction of displacement, velocity, and acceleration diagrams) with the use of angles, which form between vectors on velocity and acceleration diagrams, as well as between links on the mechanism diagram. Pointed out is the role of analytical methods (with the help of which the research of kinematics of mechanisms can be conducted with high degree of accuracy), which has especially increased in recent years due to the fact that by having analytical expressions that link the main kinematic and structural mechanism parameters with each other, it is possible to compile a calculation program for a counting machine at any moment, and with the help of such a machine all the necessary results can be obtained. Given are the results of the implementation of the proposed calculation algorithm on computer equipment, a comparative analysis of the obtained data and an estimation of the relative error of the calculation. A justification for the suitability of the developed algorithm of calculating velocities and accelerations of points of the mechanism is provided. The proposed calculation algorithm of kinematic parameters of a crank mechanisms allows to automate the process of calculating the velocities and accelerations, significantly reduces the amount of labor needed for the calculation, ensures a high degree of accuracy.

Keywords: crank mechanism, velocity, acceleration, analytical method.

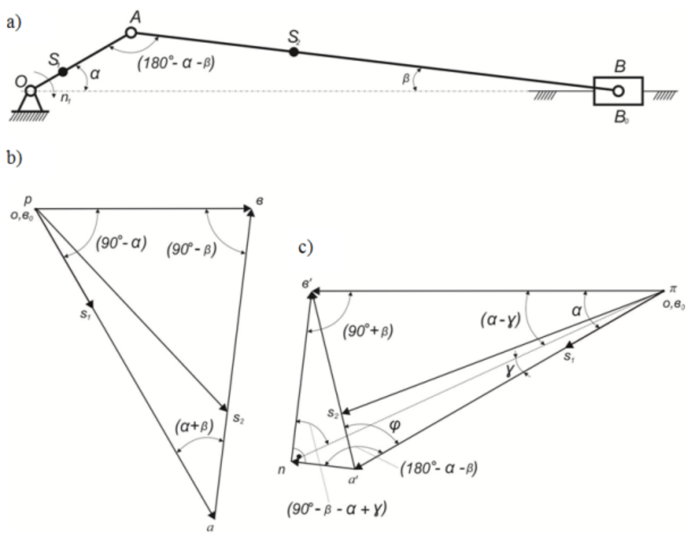
Introduction

During the process of research of the dynamics of mechanisms and machines, three problems of kinematics need to be solved: the position problem, the velocity problem, and the acceleration problem. Graphical, grapho-analytical and analytical methods exist to solve these problems. The graphical (charting method) and grapho-analytical methods (method of velocity and acceleration diagrams) of a kinematic analysis of mechanisms possess following flaws: low accuracy that is defined by the accuracy of graphical constructions, and great laboriousness. When using the graphical method, graphs of displacements, velocities and accelerations for each investigated point of the mechanism ought to be constructed. When the grapho-analytical method is utilized, several velocity and acceleration diagrams of the mechanism need to be constructed in order to define the time history of the velocity and acceleration of the points that we are interested in. To achieve this, vector equations for the velocities and accelerations of points of links, which make complex motions, are composed in advance. The solution of vector equations is executed graphically by constructing so-called velocity and acceleration diagrams, on which absolute velocities and accelerations are deposited, on a certain scale, from one point that called a pole [1].

There are no such flaws in the existing analytical methods. Given that, nevertheless, it is necessary to compose quite complex analytical dependences (formulas) and to be able to solve them with the use of computer equipment and technology, which is possible and accessible as of late. The role of analytical methods, with the help of which the research of kinematics of mechanisms can be conducted with high accuracy rate, has especially increased in recent years due to the fact that by having analytical expressions that link the main kinematic and structural mechanism parameters with each other, it is possible to compile a calculation program for a counting machine at any moment, and with the help of such a machine all the necessary results can be obtained.

Therefore, what is proposed in this work is a more simple analytical method of a kinematic analysis of a crank mechanism on the basis of the grapho-analytical method without the construction of displacement, velocity, and acceleration diagrams. Moreover, a problem of an analytical determination of linear velocities and accelerations of points of links is set, as well as a problem of angular velocities and accelerations of links under following input data: number of crank revolutions n_1 (rev/min); crank length ℓ_A (m); length of connecting rod ℓ_{AB}(m); position of the center of gravity of crank ℓ_{S_1}(м); position of the center of gravity of connecting rod ℓ_{AS_2}(м); angle α, line segment $@0 = \pi 0'$ (mm).

The key to solve the given problem with this method is to use angles that form between vectors on velocity and acceleration diagrams, as well as between links on the mechanism diagram. All the necessary angles for the calculation of velocities and accelerations are depicted in Figure 1.



a – position diagram of mechanism; *b* – velocity diagram; *c* – acceleration diagram

Figure 1. Computational scheme of a crank mechanism

From ΔOAB we find angle β, i.e. the angle between connecting rod AB and column OB

$$\beta = \arcsin\left(\frac{\ell_{OA} \cdot \sin\alpha}{\ell_{AB}}\right). \tag{1}$$

Considering the construction principle of the velocity diagram, where $pb \| OB$, $ab \perp AB$, $pa \perp OA$, we find angles Δpab on the velocity diagram.

$$\angle pba = 90^\circ - \beta;$$

$$\angle apb = 90^\circ - \alpha,$$

$$\angle pab = 180^\circ - (90^\circ - \alpha) - (90^\circ - \beta) = 180^\circ - 90^\circ + \alpha - 90^\circ + \beta = \alpha + \beta.$$

From Δpab we derive an equation

$$\frac{pa}{\sin(90^\circ - \beta)} = \frac{pb}{\sin(\alpha + \beta)} = \frac{ab}{\sin(90^\circ - \alpha)}.$$

Lengths of line segments pb and ab are therefore

$$pb = \frac{pa \cdot \sin(\alpha + \beta)}{\sin(90^\circ - \beta)}; \tag{2}$$

$$ab = \frac{pa \cdot \sin(90^\circ - \alpha)}{\sin(90^\circ + \beta)}. \tag{3}$$

We will define the position of the centers of gravity on the velocity diagram by the principle of similarity

$$ps_1 = \frac{pa \cdot \ell_{OS_1}}{\ell_{OA}}; \tag{4}$$

$$as_2 = \frac{ab \cdot \ell_{AS_2}}{\ell_{AB}}. \tag{5}$$

From Δpas_2 we find the length of line segment ps_2

$$ps_2 = \sqrt{(pa)^2 + (as_2)^2 - 2 \cdot pa \cdot as_2 \cdot |\cos(\alpha + \beta)|}. \tag{6}$$

Rotation frequency of the crank

$$\omega_1 = \frac{\pi \cdot n_1}{30}. \tag{7}$$

Velocity of point A is determined by the formula

$$V_A = \omega_1 \cdot \ell_{OA}. \tag{8}$$

Scale of velocity diagram

$$\mu_V = \frac{V_A}{pa}. \tag{9}$$

The required absolute and relative velocities of points are calculated with the use of the scale of the velocity diagram

$$V_B = pb \cdot \mu_V;\ V_{S_1} = ps_1 \cdot \mu_V;\ V_{S_2} = ps_2 \cdot \mu_V;\ V_{BA} = ab \cdot \mu_V. \tag{10}$$

The angular velocity of the connecting rod is determined by formula

$$\omega_2 = \frac{V_{BA}}{\ell_{AB}}. \tag{11}$$

The acceleration of point A is determined by formula

$$a_A = \omega_1^2 \cdot \ell_{OA}. \tag{12}$$

Scale of acceleration diagram

$$\mu_a = \frac{a_A}{\pi a'}. \tag{13}$$

Normal acceleration is determined by formula

$$a_{BA}^n = \frac{V_{BA}^2}{\ell_{AB}}. \tag{14}$$

Length of line segment $a'n$

$$a'n = \frac{a_{BA}^n}{\mu_a}. \tag{15}$$

From $\Delta \pi a'n$ we find the length of line segment πn and the unknown angle γ

$$\pi n = \sqrt{(\pi a')^2 + (a'n)^2 - 2 \cdot \pi a' \cdot a'n \cdot \cos(180^\circ - \alpha - \beta)}; \tag{16}$$

$$\gamma = \arccos\left(\frac{(\pi a')^2 + (\pi n)^2 - (a'n)^2}{2 \cdot \pi a' \cdot \pi n}\right). \tag{17}$$

Considering the construction principle of the acceleration diagram, where $\pi b' \| OB$, $\pi a' \| OA$, $nb' \perp AB$, $a'n \| AB$, we find the angles of the triangles on the acceleration diagram.

From $\Delta \pi a'b'$ we find $a'\pi b' = \alpha$.

From $\Delta \pi s_2 b'$ we find $\angle b'\pi s_2 = \alpha - \gamma$.

From $\Delta \pi nb'$ we find $\angle \pi b'n = 90^\circ + \beta$.

From $\Delta \pi a'n$ we find $\angle \pi a'n = 180^\circ - \alpha - \beta$.

From $\Delta \pi nb'$ we will determine

$\angle \pi nb' = 180^\circ - (90^\circ + \beta) - (\alpha - \gamma) = 180^\circ - 90^\circ - \beta - \alpha + \gamma = 90^\circ - \beta - \alpha + \gamma$.

From $\Delta\pi n b'$ we will obtain the equation

$$\frac{\pi n}{\sin(90^\circ+\beta)}=\frac{\pi b'}{\sin(90^\circ-\beta-\alpha+\gamma)}.$$

Hence is the length of line segment $\pi b'$

$$\pi b'=\frac{\pi n\cdot\sin(90^\circ-\beta-\alpha+\gamma)}{\sin(90^\circ+\beta)}. \tag{18}$$

From $\Delta a'\pi b'$ we find the length of line segment $a'b'$

$$a'b'=\sqrt{(\pi a')^2+(\pi b')^2-2\cdot\pi a'\cdot\pi b'\cdot\cos\alpha}. \tag{19}$$

From $\Delta a'nb'$ we find the length of line segment nb'

$$nb'=\sqrt{(a'b')^2-(a'n)^2}. \tag{20}$$

We will determine the position of the centers of gravity on the acceleration diagram by the principle of similarity

$$\pi s_1=\frac{\pi a'\cdot\ell_{OS_1}}{\ell_{OA}}, \tag{21}$$

$$a's_2=\frac{a'b'\cdot\ell_{AS_2}}{\ell_{AB}}. \tag{22}$$

From $\Delta\pi a'b'$ we find the unknown angle φ

$$\varphi=\arccos\left(\frac{(\pi a')^2+(a'b')^2-(\pi b')^2}{2\cdot\pi a'\cdot a'b'}\right). \tag{23}$$

From $\Delta\pi a's_2$ we find the length of line segment πs_2

$$\pi s_2=\sqrt{(\pi a')^2+(a's_2)^2-2\cdot\pi a'\cdot a's_2\cdot\cos\varphi}. \tag{24}$$

The required absolute and relative accelerations of points are calculated with the use of the scale of the acceleration diagram

$$a_B=\pi b'\cdot\mu_a;\ a_{S_1}=\pi s_1\cdot\mu_a;\ a_{S_2}=\pi s_2\cdot\mu_a;\ a^\tau_{BA}=nb'\cdot\mu_a;\ a_{BA}=a'b'\cdot\mu_a. \tag{25}$$

Angular acceleration ε_1 of the crank, which executes uniform motion, equals to zero.

The angular acceleration of the connecting rod is determined by formula

$$\varepsilon_2=\frac{a^\tau_{BA}}{\ell_{AB}}. \tag{26}$$

To sum up, the algorithm of the proposed analytical method of determining the velocities and accelerations of points of a crank mechanism with the use of computer equipment and technology can be expressed by the following sequence of actions:

1 Accounting of the input data: number of crank revolutions n_1 (rev/min); crank length ℓ_{OA} (m); length of connecting rod ℓ_{AB}(m); position of the center of gravity of crank ℓ_{OS_1}(m); position of the center of gravity of connecting rod ℓ_{AS_2}(m); angle α , line segment $pa=\pi a'$ (mm).
2 Calculation of the value of the angle β by formula (1).
3 Calculation of the value of the length of line segment pb by formula (2).
4 Calculation of the value of the length of line segment ab by formula (3).
5 Calculation of the value of the length of line segment ps_1 by formula (4).
6 Calculation of the value of the length of line segment as_2 by formula (5).
7 Calculation of the value of the length of line segment ps_2 by formula (6).

8 Calculation of the value of the revolution frequency of crank ω_1 by formula (7).
9 Calculation of the value of the velocity of point A by formula (8).
10 Calculation of the value of the scale of velocity diagram μ_V by formula (9).
11 Calculation of the value of the absolute and relative velocities by formula (10).
12 Calculation of the value of the angular velocity of crank ω_2 by formula (11).
13 Calculation of the value of the acceleration of point A by formula (12).
14 Calculation of the value of the scale of acceleration diagram μ_a by formula (13).
15 Calculation of the value of normal acceleration a^n_{BA} by formula (14).
16 Calculation of the value of the length of line segment $a'n$ by formula (15).
17 Calculation of the value of the length of line segment πn by formula (16).
18 Calculation of the value of angle γ by formula (17).
19 Calculation of the value of the length of line segment $\pi b'$ by formula (18).
20 Calculation of the value of the length of line segment $a'b'$ by formula (19).
21 Calculation of the value of the length of line segment nb' by formula (20).
22 Calculation of the value of the length of line segment πs_1 by formula (21).
23 Calculation of the value of the length of line segment $a's_2$ by formula (22).
24 Calculation of the value of angle φ by formula (23).
25 Calculation of the value of the length of line segment πs_2 by formula (24).
26 Calculation of the value of absolute and relative by formula (25).
27 Calculation of the value of the angular velocity of connecting rod ε_2 by formula (26).

Currently, contemporary specialized professional programs are used for engineering calculations. However, in certain problems of mechanics, it is possible to achieve the assigned goal by using an old program written in the «Basic» language [2, 3].

The results of the implementation of the abovementioned algorithm on computer equipment in the domain of «Turbo Basic» (when $n_1 = 850$rev/min; $\ell_{OA} = 0,11$ m; $\ell_{AB} = 0,462$ m; $\ell_{OS_1} = 0,0363$ m; $\ell_{AS_2} = 0,15246$ m; $\alpha = 30°$; $pa = \pi a' = 50$ mm) are shown in Figure 2.

For the sake of a comparative analysis, the required magnitudes are determined by the grapho-analytical method with the use of the КОМПАС 3DV17 computer-aided design and put in Table.

Table

Results of the calculation of velocities and accelerations

Magnitude	Unit	Value		Relative error, δrel, %
		by the proposed method in the domain of «Turbo Basic»	by the grapho-analytical method with the help of КОМПАС 3D V17	
V	m/s	9,79129695892334	9,79	0,013
V_B	m/s	5,912344455718994	5,905164	0,12
V_{S_1}	m/s	3,231127977371216	3,2274	0,11
V_{S_2}	m/s	7,722815036773682	7,694904	0,36
V_B	m/s	8,540245056152344	8,530116	0,12
ω_2	s-1	18,48537826538086	18,46345454	0,12
a	m/s2	871,5409545898438	870,72	0,094
a_B	m/s2	860,3972778320312	860,445504	0,0056
a_{S_1}	m/s2	287,6085205078125	287,3376	0,094
a_{S_2}	m/s2	841,8800659179688	841,11552	0,091
a^n_{BA}	m/s2	157,8696594238281	157,49112	0,23
a^τ_{BA}	m/s2	419,677001953125	419,512896	0,036
a_{BA}	m/s2	448,3877868652344	448,246656	0,031
ε_2	s-2	908,3917846679688	908,0365714	0,039

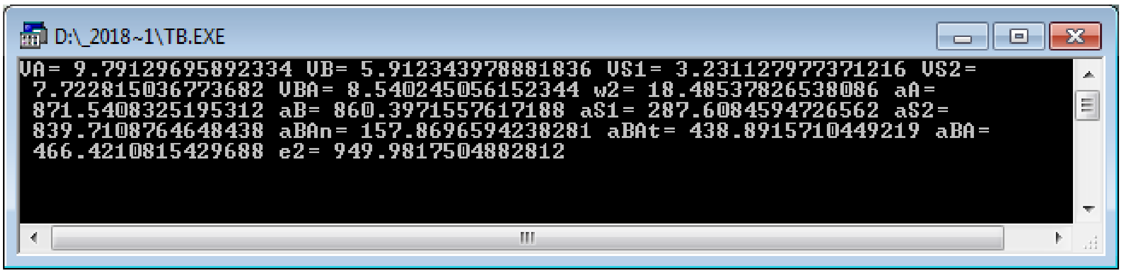


Figure 2. Results of the calculation in the domain of «Turbo Basic»

After comparing the obtained data, a conclusion can be made that the proposed calculation algorithm of kinematic parameters of a crank mechanism:

- allows to automate the process of calculating the velocities and accelerations;
- significantly reduces the amount of labor needed for the calculation;
- ensures a high degree of accuracy.

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Н.К. Бейсенов

Иінді-бұлғақты механизм нүктелерінің жылдамдықтарын және үдеулерін есептеу

Мақалада жылдамдықтар және үдеулер пландарындағы векторлардың арасында, оған қоса механизм планындағы звенолардың арасында құрылатын бұрыштарды пайдалана отырып, графоаналитикалық әдіс негізінде орындар, жылдамдықтар, үдеулер пландарын құрмай-ақ, иінді-бұлғақты механизмдегі нүктелердің сызықтық жылдамдықтары мен үдеулерін, звенолардың бұрыштық жылдамдықтары мен үдеулерін анықтаудың аналитикалық әдісі ұсынылған. Механизмдердің кинематикасын зерттеуде жоғары дәлдік дәрежесін қамтамасыз етуге мүмкіндік беретін аналитикалық әдістердің маңызы атап көрсетілген. Соңғы жылдары механизмнің кинематикалық және құрылымдық параметрлерін бір-бірімен байланыстыратын аналитикалық өрнектің көмегімен есептеу-шешу машинасы үшін есептеулер бағдарламасын жасап, барлық керекті нәтижелерді алу маңызы ерекше артты. Иінді-бұлғақты механизмнің кинематикалық параметрлерін анықтаудың ұсынылған алгоритмі жылдамдықтар мен үдеулерді есептеу үрдісін автоматтандыруға, есептеу уақытын едәуір қысқартуға, жоғары дәлдік дәрежесін қамтамасыз етуге мүмкіндік береді. Ұсынылған есептеудің алгоритмі компьютерлік техникада жүзеге асыру нәтижелерін, алынған мәліметтерді салыстыра талдап, есептеудің салыстырмалы ауытқуын бағалауға мүмкіндік береді. Механизмнің нүктелерінің жылдамдықтары мен үдеулерін есептеудің ұсынылған алгоритмінің жарамдылығы негізделген.

Кілт сөздер: иінді-бұлғақты механизм, жылдамдық, үдеу, аналитикалық әдіс.

Н.К. Бейсенов

Расчет скоростей и ускорений точек кривошипно-шатунного механизма

В статье представлен аналитический метод определения линейных скоростей и ускорений, угловых скоростей и ускорений точек звеньев кривошипно-шатунного механизма на основе графоаналитического метода без построения планов положений, скоростей, ускорений с использованием углов, образующихся между векторами на планах скоростей и ускорений, а также между звеньями на плане механизма. Отмечена роль аналитических методов, с помощью которых исследование кинематики механизмов может быть сделано с высокой степенью точности, особенно возросших в последние годы в связи с тем, что, имея аналитические выражения, связывающие между собой основные кинематические и структурные параметры механизма, можно всегда составить программу вычислений для счетно-решающей машины и с помощью машины получить все необходимые результаты. Приведены результаты реализаций предложенного алгоритма расчета на компьютерной технике, сравнительный анализ полученных данных и оценка относительной погрешности вычисления. Обоснована пригодность разработанного алгоритма расчета скоростей и ускорений точек механизма. Предложенный алгоритм вычисления кинематических параметров кривошипно-шатунного механизма позволяет автоматизировать процесс вычисления скоростей и ускорений, значительно сокращает трудоемкость расчета, обеспечивает высокую степень точности.

Ключевые слова: кривошипно-шатунный механизм, скорость, ускорение, аналитический метод.

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Optimization of components in development of polymeric coatings for restoration of transport vehicles

It is proved, that for improving the performance characteristics of vehicle parts, including their corrosion resistance and wear resistance, it is advisable to use protective polymeric composite coatings. It is shown that in order to increase the indexes of physical-mechanical and thermophysical properties in the epoxy binder, it is necessary to introduce additives: modifiers, plasticizers, dispersed and fiber fillers. The introduction of dispersed additives into the epoxy binder is actual. It this case, it is effective to use fillers of different dispersity in the complex. The influence of two-component polydispersed filler on the elasticity modulus in flexure of the developed epoxy composite is analyzed. The critical content of a two-component polydispersed filler is found by the method of mathematical planning of an experiment. a mixture of nanodispersed compounds 1 $(d = 20\dots 80\ nm) - 0.75\dots 1.0\ pts.wt.$, a mixture of discrete fibers 1 $(l = 0.5\dots 1.0\ mm$, $d = 18\dots 25\ \mu m)$ – 0.2 pts.wt. by the 100 pts.wt. of the epoxy oligomer ED-20. An introduction of the two-component polydispersed filler to the epoxy binder allows significantly to increase the values of the elasticity modulus in flexure of the protective coatings to $= 4.8\dots 5.0 hPa$. Additionally, the effect of two-component polydispersed filler on the impact resilience of the developed epoxy composite was determined. It is proved that the critical content of a two-component polydisperse filler is: a mixture of nanodispersed compounds 2 $(d = 30\dots 40\ nm) - 1.00\dots 1.25\ pts.wt.$, a mixture of discrete fibers 2 $(l = 0.5\dots 1.0\ mm, d = 18\dots 25\ \mu m) - 0.1\dots 0.2\ pts.wt.$ by the 100 pts.wt. of epoxy ED-20. An introduction of the two-component polydispersed filler to the epoxy binder allows significantly to increase the values of the impact resilience to W' = 10.0...10.2 kJ/m^2. The obtained results allow us to create polymeric coating with improved indexes of the physical and mechanical properties in complex.

Keywords: composite, epoxy matrix, two-component polydispersed filler, method of mathematical design of experiment, regression equation.

Introduction

It is known [1–4], that for the protection of metals and alloys from corrosion and to increase physical and mechanical properties, widely used protective polymer composite materials (PCM). Important for the creation of PCM with improved properties is the introduction into its composition of various chemical additives. The content of the fillers can be controlled to influence the properties of the composite material and have a PCM with predetermined properties in the end. At the same time, the process of the development of the composite material is costly and takes significant time intervals to obtain reliable experimental data. In order to optimize the composition of the polymer coating and the effectiveness of its development (decreasing the cost of materials and time of study) and obtaining of improved properties of the material in the complex, it is relevant to use the method of mathematical planning of the experiment.

Analyzing the scientific work of leading scientists in the direction of creating polymer materials [5–8], it has been established that the use of materials based on epoxy resins is effective for the protection of metal surfaces from corrosion. In the works [3–13] is proved that for the creation of PCM with improved physical and mechanical properties in the complex, it is necessary to introduce particles of fillers of different chemical composition and dispersion at the critical content. Taking into account the interaction on the phase boundary «matrix-filler», this allows to create of composite materials with improved physical and mechanical properties in the complex.

In this context, in order to reduce the number of experimental studies, it is proposed to carry out mathematical experiment planning. In the previous stage, the authors of the work studied the influence of the number of dispersed fillers on the base properties of epoxy composite material (CM). It is found the critical content of the main and additional fillers in the polymeric matrix. In particular, as the main filler, powders which are a mixture of nanodispersed compounds (MNDC) are used and are characterized by the following composition, %:

– MNDC 1: Si3N4 – 59,5; Al2O3 – 24,4; AlN – 10,1; TiN – 6,0;

– MNDC 2: Si3N4 – 85; AlF3 – 5; IH – 5; ZrH – 5.

The grains of the particles are: MNDC 1 – $d = 20 \ldots 80\ nm, MNDC\ 2 - -d = 30 \ldots 40\ nm.$

As an additional filler, a mixture of discrete fibers (MDF) from the following ingredients is used, %:

– MDF 1: viscose – 37; polyamide – 23; matka silk – 18; rong – 18; cashmere – 4;

– MDF 2: wool – 60; polyacrylonitrile (PAN) – 30; cashmere – 10.

Dimensions of the discrete fibers: $l = 0.5 \ldots 1.0\ mm, d = 18 \ldots 25\ \mu m.$

However, interesting from the practical point of view is the formation of composites with two-component filler, which, in our opinion, will allow to improve the properties of the studied CM in complex. In this context, it is expedient and necessary to use the method of mathematical planning of the experiment, which will allow to reduce the number of studies conducted and optimize the content of ingredients for obtaining CM with the maximum indexes of the selected characteristics.

Aim of work – to optimize the content of two-component polydisperse filler for the protective coatings of the transport equipment by the method of mathematical planning of the experiment.

Results and Discussion

On the first stage, for the optimization of the ingredients content into the material PCM 1 the elasticity modulus in flexion of composites at the different content of the main and additional fillers (MNDC 1 and MDF 1 in accordance) is studied. For standardization, as well as for simplification of calculations, each component (filler) is encoded by conditional units taking into account variations (Table 1).

Table 1

Levels of variables on conditional and natural scale for PCM 1

Components	Factor	Average level, q, pts.wt.	Variation step, Δq, pts.wt.	Values of variables (pts.wt.), that corresponding to conditional units		
				-1	0	+1
Main filler – MNDC 1	*x1*	0.75	0.25	0.50	0.75	1.00
Additional filler – MDF 1	*x2*	0.02	0.01	0.01	0.02	0.03

According to the experiment planning Scheme 9 experiments (N = 9) were conducted, each of which was repeated three times (p = 3) in order to exclude system errors (Table 2).

Table 2

Scheme of experiment planning

No. of exp. *(u)*	x_0	x_1	x_2	$E_3 = E_1^2 - d$	$E_4 = E_2^2 - d$	E_1E_2
1	1	-1	-1	0.33	0.33	+1
2	1	+1	-1	0.33	0.33	-1
3	1	-1	+1	0.33	0.33	-1
4	1	+1	+1	0.33	0.33	+1
5	1	0	0	-0.67	-0.67	0
6	1	+1	0	0.33	-0.67	0
7	1	-1	0	0.33	-0.67	0
8	1	0	+1	-0.67	0.33	0
9	1	0	-1	-0.67	0.33	0
$\sum_{u-1}^{N} x_{iu}^2$	9	6	6	2	2	4

In order that planning matrix to be orthogonal [9, 11, 14, 16], the corrected values of E' level were entered, which were calculated by the formula:

$$x_i' = (x_i)^2 - \frac{\sum_{u-1}^{N} x_{iu}^2}{N}. \qquad (1)$$

The expanded matrix of planning of complete factor experiment (CFE) and its results are shown in Table 2.

The mathematical model $y = f\ (x1,\ x2)$ was formed as a regression equation:

$$y = b_0 + b_1 x_1 + b_2 x_2 + b_{11} x_1^2 + b_{22} x_2^2 + b_{12} x_1 x_2. \qquad (2)$$

The regression coefficients were determined by the formula

$$b_{\mathrm{i}} = \frac{\sum_{u=1}^{N} x_i y_i}{\sum_{u-1}^{N} x_{iu}^2}. \qquad (3)$$

Received coefficients of regression equation are given in Table 3.

Table 3

The coefficients of regression equation

b0	*b1*	*b2*	*b11*	*b22*	*b12*
4.92	0.40	0.35	-0.73	-0.58	0.28

As a result, in the analysis of the elasticity modulus in flexure, the following regression equation was determined:

$$y = 4.92 + 0.40x_1 + 0.35x_2 - 0.73x_1^2 - 0.58x_2^2 + 0.28x_1x_2.$$

For the statistical processing of experiment results, a test of reproducibility of experiments by the Cochran test was conducted:

$$G = \frac{S_{u\max}^2}{\sum_{u=1}^{N} S_u^2} \le G_{(0{,}05;f_1;f_2)}, \qquad (4)$$

where S_{ui}^2– dispersion of experiment results on combinations of few factor levels for m=3; m – number of parallel experiments; *S2u max* – the highest dispersion in design line.

Dispersions of adequacy were determined by the formula

$$S_{ui}^2 = \frac{\sum_{i=1}^{m} (y_i - \overline{y_i})^2}{m-1}, \qquad (5)$$

where y_{im}– value, received from each parallel experiment; $\overline{y_i}$ – average value y, received in parallel experiments.

Mean square error was determined by formula

$$\sigma^2 \{y\} = \frac{\sum_{i=1}^{N=9} \sigma^2 \{y\}_i}{N(m-1)}, \qquad (6)$$

where $\sigma^2 \{y\}_i = \sum_{i=1}^{m=3} (y_i - \overline{y_i})^2$; $\sigma^2 \{y_{av}\} = \frac{0^2\{C\}}{N}$,

$$S_{b_0}^2 = \frac{S_0^2}{N}. \qquad (7)$$

Dispersion values are shown in Table 4.

Table 4

Values of dispersions of adequacy (S^2_{ui}) and mean square error ($\sigma^2\{y\}_i$)

No. of exp.	The dispersions of adequacy		The mean square error	
	conditional designation	value	conditional designation	value
1	S^2_{u1}	0.07	$\sigma^2\{y\}_1$	0.14
2	S^2_{u2}	0.01	$\sigma^2\{y\}_2$	0.02
3	S^2_{u3}	0.03	$\sigma^2\{y\}_3$	0.06
4	S^2_{u4}	0.09	$\sigma^2\{y\}_4$	0.18
5	S^2_{u5}	0.01	$\sigma^2\{y\}_5$	0.02
6	S^2_{u6}	0.03	$\sigma^2\{y\}_6$	0.06
7	S^2_{u7}	0.03	$\sigma^2\{y\}_7$	0.06
8	S^2_{u8}	0.03	$\sigma^2\{y\}_8$	0.06
9	S^2_{u9}	0.01	$\sigma^2\{y\}_9$	0.02

Moreover:

$$\sum_{i=1}^{N} S^2_{ui} = 0.31;$$

$$\sigma^2\{y\} = S^2_0 = 0.034.$$

Then the calculated value of the Cochran test at the 5% level of significance:

$$G_{calc} = \frac{S^2_{u_{\max}}}{\sum_{i=1}^{N} S^2_{ui}}; \qquad (8)$$

$$G_{calc} = \frac{0.09}{0.31} = 0.290.$$

Testing the experiment results by the Cochran test [14] for a fixed probability $\alpha = 0.05$ confirmed the reproducibility of the experiments. Dispersion of experiment results on combinations of few factor levels: $S^2_{u\max} = 0.09$. Calculated value of Cochran test: $Gcalc = 0.290$.

Table value of Cochran test: $Gtab = 0.478$.

That is, the condition (7) is fulfilled:

$$G_{calc} = 0.290 \leq G_{tab} = 0.478.$$

Subsequently, the coefficients significance of regression equation was determined by analyzing the results according to the experimental design (Table 5).

Table 5

The experimental results of study of the elasticity modulus in flexure of PCM 1

No. of exp.	Content of components, q, pts.wt.		Elasticity modulus in flexure, E, hPa			Average value, E, hPa
	x1	*x2*	1	2	3	
1	0.50	0.01	3.0	3.4	3.5	3.3
2	1.00	0.01	3.2	3.1	3.3	3.2
3	0.50	0.03	3.3	3.6	3.6	3.5
4	1.00	0.03	4.2	4.8	4.5	4.5
5	0.75	0.02	5.0	4.9	5.1	5.0
6	1.00	0.02	5.1	4.8	4.8	4.9
7	0.50	0.02	3.3	3.3	3.6	3.4
8	0.75	0.03	4.5	4.8	4.5	4.6
9	0.75	0.01	3.9	4.0	4.1	4.0

Then the dispersions of regression coefficients (Table 6) were determined by the formula

$$S_{b_i}^2 = \frac{S_0^2}{\sum_{u-1}^{N} x_{iu}^2}. \tag{9}$$

The significance of the regression coefficients was determined by the Student's test [14, 15]. Here with the table (*tm*) and calculated criterion (*tcalc*) of Student's test (Table 6) were determined.

Table 6

Dispersion of coefficients of regression (S_b^2) and calculated values of Student's criterion (*tcalc*)

No. of exp.	Dispersion of of exp.coefficients of regression		Calculated values of Student's criterion	
	conditional designation	value	conditional designation	value
1	$S_{b_0}^2$	0.004	*t0*	72.88
2	$S_{b_1}^2$	0.006	*t1*	5.28
3	$S_{b_2}^2$	0.006	*t2*	4.62
4	$S_{b_{11}}^2$	0.017	*t11*	5.59
5	$S_{b_{22}}^2$	0.017	*t22*	4.45
6	$S_{b_{12}}^2$	0.009	*t12*	3.0

Depending on freeness: f = N (n - 1) = 9 (3 - 1) = 18 the Student's test value was calculated, which is $tT = 2.1$.

Calculated values of Student's test (*tcalc*) and coefficients significance were determined: *t0, t1, t2, t11, t22, t12* > *tT.*

Moreover:

$$t_i = \frac{|b_i|}{S_{b_i}}. \tag{10}$$

Calculated values of Student's criterion *t0, t1, t2, t11, t22, t12* are larger than *tT*, so it was considered that all coefficients of the regression equation are significant. As a result of rejection of insignificant coefficients, the following regression equation was received:

$$y = 4.92 + 0.40x_1 + 0.35x_2 - 0.73x_1^2 - 0.58x_2^2 + 0.28x_1x_2.$$

The adequacy of the model was checked by Fisher test [16]:

$$F_{calc} = \frac{S_{u\max}^2}{S_y^2} \leq F_{(0.05;f_{0d};f_y)}, \tag{11}$$

where $S_{umax}^2 = 0.09$ – calculated value of dispersion of adequacy (Table 4):

$$S_y^2 = \frac{\sum_{i=1}^{N} S_{ui}^2}{N}, \tag{12}$$

$S2 = 0.034$ – mean square error;

So: $Fcalc = 2.65$.

$F_{(0.05;f_{0d};f_u)}$– table value of Fisher test in 5% significance level (*f1* = N - (k + 1) = 9 - (6 + 1) = 2, *f2* = N (n - 1) = 9 (3 - 1) = 18). So: $F(t) = 3.55$ [14, 15].

Calculated value of Fisher test is less than table one, so the requirement (10) is fulfilled. It is possible to assume that equation adequately characterizes the composition.

Interpretation process of received mathematical model, as a rule, is not just determination of factors influence. A simple comparison of absolute value of linear coefficients does not determine the relative degree factors influence, since there are also quadratic squared terms and paired interactions. In a detailed analysis of the received adequate model, it is necessary to take into account the fact that for a quadratic model the degree of factor influence on the change of output value is not constant.

Dependencies that connect normalized and natural values of the variables are as follows:

$$x_i = \frac{q_i - q_{i0}}{\Delta q_i}, \tag{13}$$

where q_i – value of i experiment factor; q_{i0} – value of zero level; Δq_i – variation interval [14].

Substituting these values in accordance with the formula (13) into the regression equation and transforming it, we receive the following regression equation with the natural values of the variables:

$$E = 10.93 - 12.04q_1 - 573q_2 - 11.68q_1^2 - 5800q_2^2 + 1120q_1q_2.$$

Given equation in natural values allows only predicting the output value for any point in the middle of range of factor variations. However, with its help it is possible to construct graphs of dependence of output value (elasticity modulus in flexure of composites) from any factor (or two factors). Geometric interpretation of the response surface is shown on Figures 1–3.

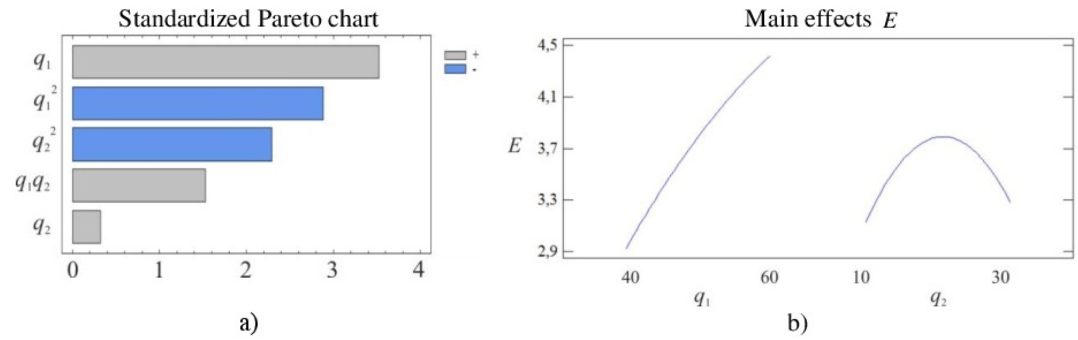

Figure 1. Standardized Pareto chart (a) and main effects (b)

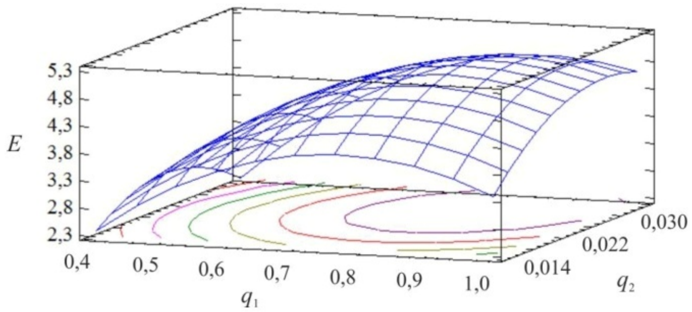

Figure 2. Estimated surface = f(q1,q2)

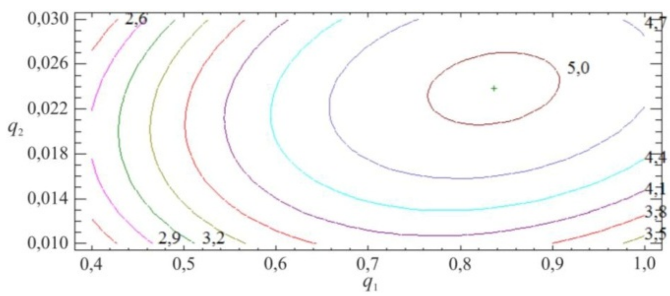

Figure 3. Contours of estimated response surface

Based on experimental studies it is set that both factors are significant. It should be noted that the effect of the additional filler content on the parameters of elasticity modulus in flexure is higher in comparison with the main one (according to Pareto chart). Analyzing the calculated response surface, it is determined that the optimum parameters of elasticity modulus in flexure have developed epoxy composite with two-component polydispersed filler with the following content of particles: MNDC 1 – 0.75. . . 1.0 pts.wt., MDF 1 – 0.2 pts.wt. (E = 4.8. . . 5.0 hPa).

On the second stage for optimization of the ingredients content in the material PCM 2 the impact resilience of the composites with different content of the main and additional fillers (MNDC 2 and MDF 2 in accordance) was studied. For standardization, as well as for simplification of calculations, each component (filler) is encoded by conditional units taking into account variations (Table 7).

Table 7

Levels of variables on conditional and natural scale for PCM 2

Components	Factor	Average level, q, pts.wt.	Variation step, Δq, pts.wt.	Values of variables (pts.wt.), that corresponding to conditional units		
				-1	0	+1
Main filler – MNDC 2	*x1*	1.00	0.25	0.75	1.00	1.25
Additional filler – MDF 2	*x2*	0.02	0.01	0.01	0.02	0.03

Similarly, to the above calculations scheme, the composition of the CM was optimized according to the values of the impact resilience. The encoding of natural components values and the experimental design scheme are chosen according to Table 2 and Table 7.

In the process of study results analysis of composites impact resilience, the following values of the regression coefficients were received (Table 8).

Table 8

The coefficients of regression equation

b0	*b1*	*b2*	*b11*	*b22*	*b12*
9.18	0.42	-0.22	-0.22	-0.12	-0.15

As a result, the following regression equation was found:

$$y = 9.18 + 0.42x_1 - 0.22x_2 - 0.22x_1^2 - 0.12x_2^2 - 0.15x_1x_2.$$

For statistical processing of experiment results, a test of experiments reproducibility was conducted according to the Cochran test [14].

Dispersions values that were calculated by formula (5-7) are shown in Table 9.

Table 9

Values of dispersions of adequacy (S_{ui}^2) and mean square error ($\sigma^2 \{y\}_i$)

No. of exp.	The dispersions of adequacy		The mean square errors	
	conditional designation	value	conditional designation	value
1	S_{u1}^2	0.010	$\sigma^2 \{y\}_1$	0.020
2	S_{u2}^2	0.010	$\sigma^2 \{y\}_2$	0.020
3	S_{u3}^2	0.010	$\sigma^2 \{y\}_3$	0.020
4	S_{u4}^2	0.030	$\sigma^2 \{y\}_4$	0.060
5	S_{u5}^2	0.010	$\sigma^2 \{y\}_5$	0.020
6	S_{u6}^2	0.040	$\sigma^2 \{y\}_6$	0.080
7	S_{u7}^2	0.040	$\sigma^2 \{y\}_7$	0.080
8	S_{u8}^2	0.010	$\sigma^2 \{y\}_8$	0.020
9	S_{u9}^2	0.010	$\sigma^2 \{y\}_9$	0.020

Moreover:

$$\sum_{i=1}^{N} S_{ui}^2 = 0.170;$$

$$\sigma^2 \{y\} = S_0^2 = 0.019.$$

Calculated value of the Cochran test at the 5% significance level was determined by formula (8):

$$G_{calc} = \frac{0.040}{0.170} = 0.235.$$

Testing the experiment results by the Cochran test [9, 14, 15] for a fixed probability $\alpha = 0.05$ confirmed the experiments reproducibility. Dispersion characterizing dispersal of the experiments results in combination of few factor levels: $S_{u\max}^2 = 0.040$. Calculated value of Cochran test: $Gcalc = 0.235$.

Table value of Cochran test: $Gtab = 0.478$.
So, the requirement is fulfilled:

$$G_{calc} = 0.235 \leq G_{tab} = 0.478.$$

At the next stage, the coefficients significance of regression equation is determined, analyzing the results according to the experimental design (Table 10).

Table 10

The experimental results of study of the impact resilience of PCM 2

No. of exp.	Content of components, q, pts.wt.		Impact resilience, W', kJ/m2			Average value W', kJ/m2
	x1	*x2*	1	2	3	
1	0.75	0.01	8.9	8.8	8.7	8.8
2	1.25	0.01	10.3	10.1	10.2	10.2
3	0.75	0.03	8.1	8.3	8.2	8.2
4	1.25	0.03	8.9	9.2	8.9	9.0
5	1.00	0.02	10.0	10.1	9.9	10.0
6	1.25	0.02	8.9	8.5	8.7	8.7
7	0.75	0.02	8.2	8.6	8.4	8.4
8	1.00	0.03	8.8	9.0	8.9	8.9
9	1.00	0.01	8.3	8.5	8.4	8.4

Subsequently, dispersion of regression coefficients is determined by formulas (9-10). The significance of regression coefficients is determined according to Student's criterion, which Table value is $tT = 2.1$ [15, 16]. Calculated values of Student's criterion are shown in Table 11.

Table 11

Dispersion of coefficients of regression (S_b^2) and calculated values of Student's criterion (*tcalc*)

No. of exp.	Dispersion of coefficients of regression		Calculated values of Student's criterion	
	conditional designation	value	conditional designation	value
1	$S_{b_0}^2$	0.002	*t0*	198.05
2	$S_{b_1}^2$	0.003	*t1*	7.43
3	$S_{b_2}^2$	0.003	*t2*	3.86
4	$S_{b_{11}}^2$	0.009	*t11*	2.23
5	$S_{b_{22}}^2$	0.009	*t22*	1.20
6	$S_{b_{12}}^2$	0.005	*t12*	2.2

Calculated values of Student's criterion *t0, t1, t2, t11, t12* are larger than tT, so it is considered that coefficients *b0, b1, b2, b11, b12* of regression equation are significant. Calculated value *t22,* is smaller than tT, so coefficients *b22,* is insignificant. As a result, the following regression equation is received:

$$y = 9.18 + 0.42x_1 - 0.22x_2 - 0.22x_1^2 - 0.15x_1x_2.$$

The adequacy of the model was checked by Fisher's test [15, 16].
Calculated value of adequacy dispersion: $S_{u\,\max}^2 = 0.04$ (Table 9).
The mean square error: $S_y^2 = 0.019$.
So: $F_{calc} = 0.475$.
$F_{(0,05;f_W;f_u)}$– table value of Fisher's test in 5% significance level ($F(t) = 2.77$) [15, 16].
Calculated value of Fisher's test is smaller than table on, so requirement (11) is fulfilled. Consequently, the equation adequately shows the composition formula.
After transformations in accordance with formula (13), the following regression equation with the natural values of variables was received:

$$W' = 3.22 + 9.92q_1 + 38q_2 - 3.52q_1^2 - 60q_1q_2.$$

Geometric interpretation of response surface is shown on Figures 4–6.

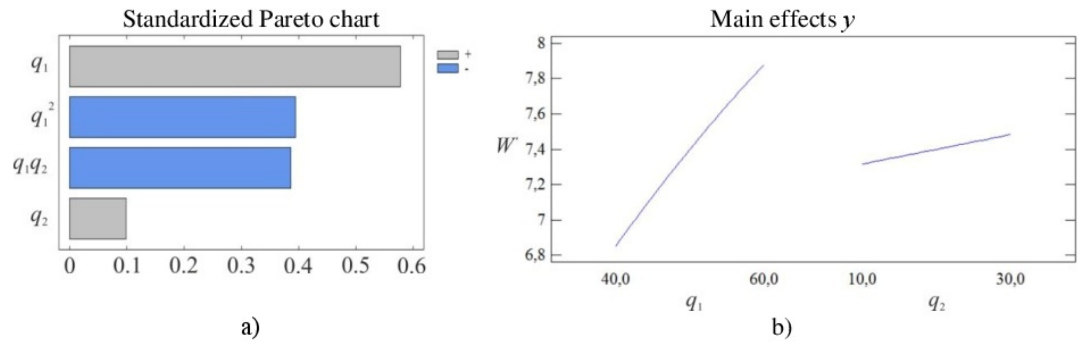

Figure 4. Standardized Pareto chart (a) and main effects (b)

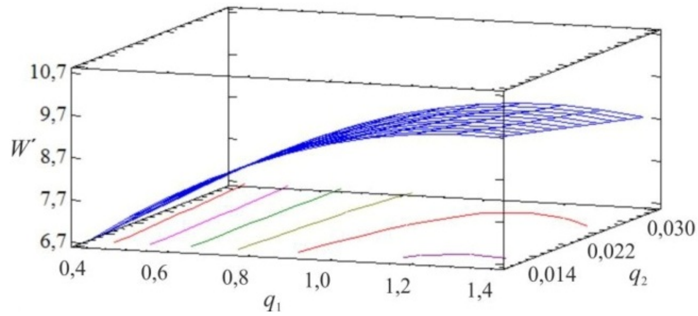

Figure 5. Estimated surface W'= f(q1,q2)

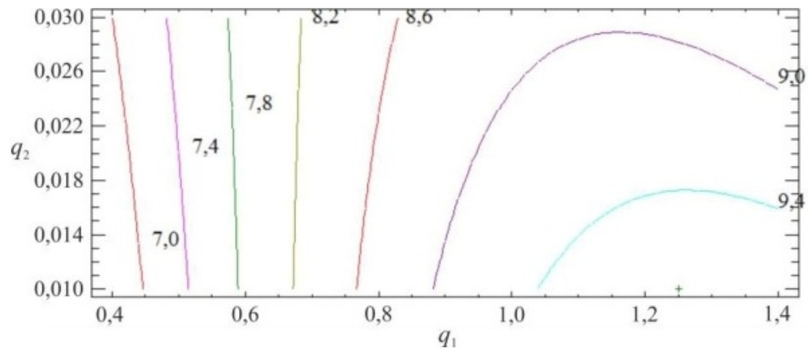

Figure 6. Contours of estimated response surface

Received results indicate that both factors of regression equation are significant. In the process of analysis, it was determined that the impact resilience values show maximum values for the fillers contents: MNDC 2 – 1.00. . . 1.25 pts.wt., MDF 2 – 0.1. . . 0.2 pts.wt. (W' = 10.0. . . 10.2 kJ/m2). With further increase the content of particles the decreasing of impact resilience was observed. Therefore, it is advisable to add two-component polydispersed filler with the aforementioned content into modified epoxy matrix to improve performance in the repair of marine transport elements.

Conclusions

The critical content of a two-component polydispersed filler is found by the method of mathematical planning of the experiment: a mixture of nanodispersed compounds 1 $(d = 20 \dots 80\ nm) - -0.75 \dots 1.0\ pts.wt.$, a mixture of discrete fibers 1 $(l = 0.5 \dots 1.0\ mm, d = 18 \dots 25\ \mu m) - -0.2\ pts.wt.bythe 100\ pts.wt.$ of the epoxy oligomer ED-20. An introduction of the two-component polydispersed filler to the epoxy binder allows significantly to increase the values of the elasticity modulus in flexure of the protective coatings to $= 4.8 \dots 5.0\ hPa$.

It is proved that in order to create a composite material with improved impact resilience, it is necessary to introduce: a mixture of nanodispersed compounds 2 ($d = 30 \ldots 40\ nm$) – $-1.00 \ldots 1.25\ pts.wt.$, a mixture of discrete fibers 2 ($l = 0.5 \ldots 1.0\ mm, d = 18 \ldots 25\ \mu m$) – $-0.1 \ldots 0.2\ pts.wt.$ by the 100 pts.wt. of epoxy ED-20. At the same time, the values of impact resilience increase to the $W' = 10.0 \ldots 10.2\ kJ/m2$. The obtained results allow us to create materials with improved indexes of the physical and mechanical properties in complex. These materials can be used in the form of protective coatings to improve the performance and repair of parts of transport equipment.

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Автокөліктерді қалпына келтіру үшін полимерлік жабындардың дамуындағы компоненттерді оңтайландыру

Автокөлік құралдарының бөліктерінің, соның ішінде олардың коррозияға қарсы қасиеттерін және тозуға төзімділігін жақсарту үшін, қорғаныш полимерлі композитті жабындарды пайдалану ұсынылады. Эпоксидті байланыстырушы физика-механикалық және жылулық қасиеттерін жақсарту үшін қоспаларды: модификаторларды, пластификаторларды, дисперсті және талшықты толтырғыштарды енгізу қажет. Эпоксидті байланыстырғышқа дисперсті қоспаларды енгізу өзекті болып табылады және кешенде әртүрлі дисперсті толтырғыштарды қолдануға тиімді. Екікомпонентті полидисперсті толтырғыштың дамыған эпоксидті композиттің иілу кезінде серпімді модульге әсері талданды. Екікомпонентті полидисперстік толтырғыштың сыни мазмұны эксперименттің математикалық жоспарлау әдісімен анықталды: нанодисперсті қосылыстардың қоспасы 1 (d = 20 ... 80 нм) - 0,75 ... 1,0 масс. Бөлшектер, дискреттік талшықтардың қоспасы 1 (л = 0,5 ... 1,0 мм, д = 18 ... 25 мкм) - 0,2 вт.ч. 100 вт.ч. эпоксидті олигомер ED-20. Екікомпонентті полидисперсті толтырғыштың эпоксидті байланыстырушы құралына кіріспе E = 4.9 ... 5.0 ГПа дейін қорғаныш қабаттардың иілгіш модулін едәуір арттыра алады. Бұдан басқа, екікомпонентті полидисперсті толтырғыштың дамыған эпоксидтік композиттің соңғыға төзімділігіне әсері белгіленді. Екікомпонентті полидисперсті толтырғыштың сыни мазмұны: нанодисперсті қосылыстардың қоспасы 2 (d = 30 ... 40 нм) - 1,0 ... 1,25 массалық бөлшектер, дискреттік талшықтардың қоспасы 2 (л = 0,5). .. 1.0 мм, d = 18 ... 25 мкм) - 0,1 ... 0,2 масс. б. 100 масс. б. эпоксидті олигомер ED-20. Екікомпонентті полидисперсті толтырғышты эпоксидті байланыстырғышқа енгізу W = 10,0 ... 10,2 кДж / м^2 дейін әсер ету күшін айтарлықтай арттыра алады. Алынған нәтижелер күрделі физика-механикалық қасиеттері бар полимерлі жабынды жасауға мүмкіндік береді.

Кілт сөздер: композит, эпоксидті матрица, екікомпонентті полидисперстік толтырғыш, эксперименттi математикалық жоспарлау әдісі, регрессия теңдеуі.

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Оптимизация компонентов при разработке полимерных покрытий для восстановления средств транспорта

Обосновано, что для повышения эксплуатационных характеристик деталей транспортных средств, в том числе их антикоррозийных свойств и износостойкости, целесообразно использовать защитные полимерные композитные покрытия. Показано, что для повышения показателей физико-механических и теплофизических свойств в эпоксидное связующее необходимо вводить добавки: модификаторы, пластификаторы, дисперсные и волокнистые наполнители. Актуальным является введение в эпоксидное связующее дисперсных добавок, причем эффективно использовать наполнители различной дисперсности в комплексе. Проанализировано влияние двухкомпонентного полидисперсного наполнителя на модуль упругости при изгибе разработанного эпоксидного композита. Методом математического планирования эксперимента установлено критическое содержание двухкомпонентного полидисперсного наполнителя: смесь нанодисперсных соединений 1 (d = 20 ... 80 нм) – 0,75 ... 1,0 масс.ч., смесь дискретных волокон 1 (l = 0,5 ... 1,0 мм, d = 18 ... 25 мкм) – 0,2 масс.ч. на 100 масс.ч. эпоксидного олигомера ЭД-20. Введение в эпоксидное связующее двухкомпонентного полидисперсного наполнителя позволяет значительно повысить показатели модуля упругости при изгибе защитных покрытий до E = 4,9 ... 5,0 ГПа. Дополнительно установлено влияние двухкомпонентного полидисперсного наполнителя на ударную вязкость разработанного эпоксидного композита. Доказано, что критическое содержание двухкомпонентного полидисперсного наполнителя: смесь нанодисперсных соединений 2 (d = 30 ... 40 нм) – 1,0 ... 1,25 масс.ч., смесь дискретных волокон 2 (l = 0,5 ... 1,0 мм, d = 18 ... 25 мкм) – 0,1 ... 0,2 масс.ч. на 100 масс.ч. эпоксидного олигомера ЭД -20. Введение в эпоксидное связующее двухкомпонентного полидисперсного наполнителя ведет к значительному повышению показателей ударной вязкости до W = 10,0 ... 10,2 кДж/м^2. Полученные результаты позволяют создать полимерное покрытие с улучшенными в комплексе показателями физико-механических свойств.

Ключевые слова: композит, эпоксидная матрица, двухкомпонентный полидисперсный наполнитель, метод математического планирования эксперимента, уравнение регрессии.

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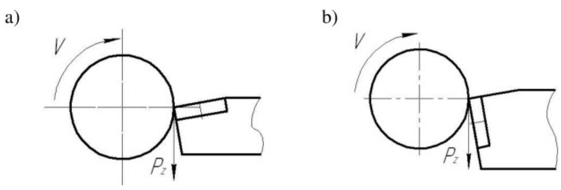
Mathematical model of cutting process of cutting tools with a side-mounted multifaceted, requiring no sharpening plates

The article presents the results of experimental research of the cutting process with cutoff tools with laterally mounted multifaceted unresharpenable plates (MUP), which allowed to confirm their efficiency and progressiveness. As a result of research and experimental data processing, for the first time, the mathematical models were obtained that adequately describe force parameters (Pz and Py) of cutting process by the proposedcutoff tools. The rational values of rake and relief angles are determined.

Keywords: assembled cutoff tool, multifaceted unresharpenable plate, mechanical mounting.

Trends in the development of modern tool production determine the widespread implementation into the machine-building industry of metal-cutting assembled tools with mechanical mounting of multifaceted unresharpenable plates (MUP). The conditions of operation of this type of tools significantly affect the way of mounting and fixing of the cutting plates. From the literature sources [1, 2] radial and tangential mounting of MUP are known (Fig. 1). In case of the radial mounting of MUP (along the front surface), the cutting force acts on a smaller overcutting of the plate, which limits the permissible feed of the instrument. With the tangential mounting of MUP (along the back surface), the force of cutting takes up larger overcutting of the plate and therefore a significant increase in feed is allowed.



a) radial b) tangential

Figure 1. Ways of MUP mounting

However, none of these mounting methods presupposes the use of MUP for assembled cutoff tools due to the large cutting area width.

The attempts to use MUP for cutoff tools at their lateral installation (along the lateral surface) are known and lateral fixing to cutter body (Fig. 2) [3]. The disadvantages of these designs are the complexity of the design of the plates, the low reliability of MUP fixing, the limitation of the depth of the grooves due to MUP basing simultaneously on the supporting and thrust surfaces - up to 6.5 mm, which limits the scope of the use of these cutting tools.

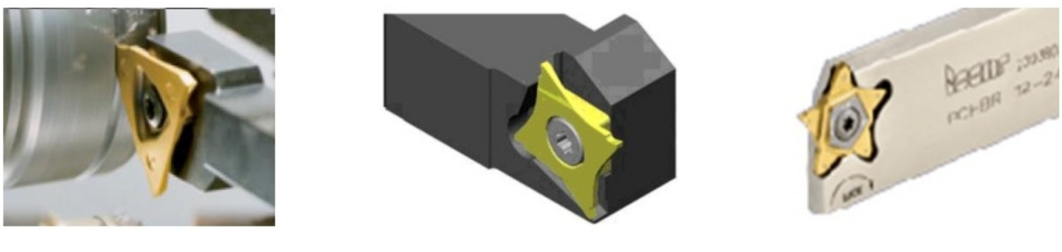


Figure 2. Groove cutters with lateral mounting of multifaceted plates

In order to eliminate the disadvantages listed above, the authors [4] for the first time proposed a new «lateral» scheme of installation of a multifaceted unresharpenable plate and on its basis a design of a cut-off tool with a laterally mounted MUP (Fig. 3, 4), consisting of a holder (1), hook (2), screw (3), and multi-faceted unresharpenable plate 4. In this design of the cutting tool, locating and fixing of MUP is carried out only on the thrust surfaces, which significantly increases the length of the cutting part and makes it possible to perform cutting of rods with a diameter of up to 24 mm.

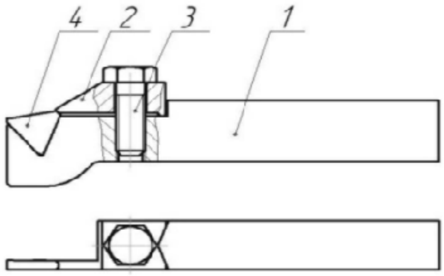


Figure 3. Design of assembled cutoff tool with lateral mounting of MUP

Figure 4. General view of cutoff tool with lateral mounting of MUP

However, the force parameters (Pz and Py) of the cutting process with the proposed cutting tools remain unexplored till now, and as a result, the rational geometrical parameters of the proposed tools (the value of rake γ and relief α angles), which is proposed to be performed in this paper.

Research results

Experimental investigations of the cutting process by cutoff tools with laterally mounted MUP were performed in laboratory conditions on a screw-cutting lathe of 1K62 model (Fig. 5).

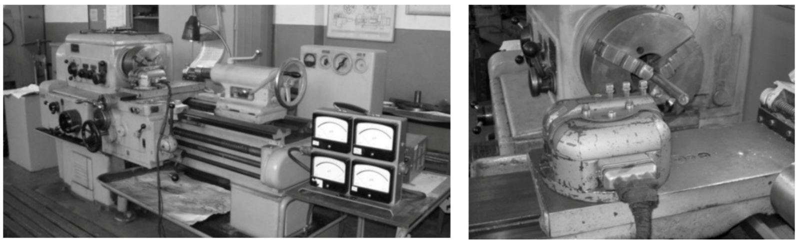


Figure 5. General view of experimental study of force parameters of cutting process with cutoff tool

The cutting forces were determined with the help of an electric universal dynamometer UDM-600 with a set of amplifying and indicating equipment. Measurements of the components of the cutting forces were carried out experimentally on two axes of coordinates: Pz — vertically; Py — perpendicularly to the axis of the workpiece. In experimental studies, cylindrical workpieces were used. The outer diameter of the workpieces was D = 20 mm. The material of processed workpieces was Steel 45.

In the first series of experiments, three experiments were carried out to measure the cutting forces at different values of cutting speed of $\upsilon 1 = 20$ m/min, $\upsilon 2 = 31.4$ m/min, $\upsilon 3 = 40$ m/min, but with a fixed value of rake angle $\gamma = -5°$ and feed $s1 = 0.07$ mm/rev. The value of the components of the cutting forces in each experiment was entered in Table 1.

In the 2nd series of experiments, 3 experiments were carried out to measure the cutting forces at different values of cutting speed $\upsilon 1 = 20$ m/min, $\upsilon 2 = 31.4$ m/min, $\upsilon 3 = 40$ m/min, but with a fixed value of rake angle $\gamma = -5°$ and feed $s4 = 0.12$ mm/rev.

In the 3rd series of experiments, 3 experiments were carried out to measure the cutting forces at different values of cutting speed $\upsilon 1 = 20$ m/min, $\upsilon 2 = 31.4$ m/min, $\upsilon 3 = 40$ m/min, but with a fixed value of rake angle $\gamma = -6°$ and feed $s2 = 0.074$ mm/rev.

In the 4th series of experiments, 3 experiments were carried out to measure the cutting forces at different values of cutting speed $\upsilon 1 = 20$ m/min, $\upsilon 2 = 31,4$ m/min, $\upsilon 3 = 40$ m/min, but with a fixed value of rake angle $\gamma = -6°$ and feed $s3 = 0.097$ mm/rev.

In the 5th series of experiments, 3 experiments were carried out to measure the cutting forces at different values of cutting speed $\upsilon 1 = 20$ m/min, $\upsilon 2 = 31,4$ m/min, $\upsilon 3 = 40$ m/min, but with a fixed value of rake angle $\gamma = -8°$ and feed $s2 = 0,074$ mm/rev.

In the 6th series of experiments, 3 experiments were carried out to measure the cutting forces at different values of cutting speed $\upsilon 1 = 20$ m/min, $\upsilon 2 = 31.4$ m/min, $\upsilon 3 = 40$ m/min, but with a fixed value of rake angle $\gamma = -8°$ and feed $s3 = 0.097$ mm/rev.

Let us derive the equation of vertical (main) Pz and radial Py of the cutting forces components for each fixed value of rake angle γ according to the experimental data presented in Table 1. In this case, the components of the cutting forces are functions of the arguments: the feed S (mm/rev) and the cutting speed V (m/min). Since the measurement of the components of the cutting forces was performed for two values of the feed S (mm/rev), the response surface can be restored as lined one.

Table 1

Results of experimental data of the research of cutting process with cutoff tools with lateral mounting of MUP

Series No.	Experiment No.	Cutting modes				Cutting force components	
		γ, degrees	S, mm/rev	n, rev/min	υ, m/min	Pz, N	Py, N
1	1	– 5	0.07	315	20	1600	650
	2	– 5	0.07	500	31.4	2000	780
	3	– 5	0.07	630	40	2225	920
2	4	– 5	0.12	315	20	1600	650
	5	– 5	0.12	500	31.4	1900	780
	6	– 5	0.12	630	40	2250	940
3	7	– 6	0.074	315	20	1500	600
	8	– 6	0.074	500	31.4	1800	720
	9	– 6	0.074	630	40	2225	920
4	10	– 6	0.097	315	20	1815	650
	11	– 6	0.097	500	31.4	1900	780
	12	– 6	0.097	630	40	2100	820
5	13	– 8	0.074	315	20	2225	920
	14	– 8	0.074	500	31.4	2400	940
	15	– 8	0.074	630	40	2600	1100
6	16	– 8	0.097	315	20	2720	1180
	17	– 8	0.097	500	31.4	2720	1180
	18	– 8	0.097	630	40	3100	1310
7	19	– 10	0.07	315	20	2400	940
	20	– 10	0.07	500	31.4	2700	1080
	21	– 10	0.07	630	40	3100	1310
8	22	– 10	0.12	315	20	3250	1380
	23	– 10	0.12	500	31.4	3330	1395
	24	– 10	0.12	630	40	3540	1415

As it is know, the line surface has the following equation:

$$P(S;V) = f(0;V) \cdot (1-w) + f(1;V) \cdot w, \tag{1}$$

where w is normalized value ($0 \leq w \leq 1$), which corresponds to the variable S, being connected with the former by the following formula:

$$w = \frac{S - S_1}{S_k - S_1}, \tag{2}$$

where S_1 is the first and S_k - the last experimental value of the feed S.

The formula (2) transforms the segment $[S_1; S_k]$ to unit segment $[0; 1]$.

The functional dependences $f(0;V)$ and $f(1;V)$ at fixed feed values S are obtained by the least square method (LSM) by setting unknown values of the coefficients a and b in the formulas:

$$f(0;V) = a_0 V^{b_0}; \quad f(1;V) = a_1 V^{b_1}. \tag{3}$$

For the realization of LSM we apply CAS Maple 15, namely the Nonlinear Fit command from the Statistics package, which performs nonlinear approximation of experimental data [5]. Applying it to experimental dependencies (*Vi; Pi*) for all cases of fixed values of rake angle γ and feed S, we obtain the analytic dependencies of the form (3), which are given in Table 2.

Table 2

Analytical dependences $\mathbf{f(0;V)}$ and $\mathbf{f(1;V)}$, reestablished by means of experimental data

γ, degrees	P_z		P_y	
	$f(0;;V)$	$f(1;;V)$	$f(0;;V)$	$f(1;;V)$
–5	$386.094 \cdot V^{0.476}$	$358.601 \cdot V^{0.493}$	$141.908 \cdot V^{0.503}$	$127.914 \cdot V^{0.536}$
–6	$260.391 \cdot V^{0.575}$	$985.635 \cdot V^{0.195}$	$87.988 \cdot V^{0.628}$	$299.075 \cdot V^{0.248}$
–8	$1139.795 \cdot V^{0.221}$	$1579.327 \cdot V^{0.174}$	$429.294 \cdot V^{0.246}$	$768.142 \cdot V^{0.138}$
–10	$788.178 \cdot V^{0.367}$	$2273.029 \cdot V^{0.117}$	$217.021 \cdot V^{0.480}$	$1242.151 \cdot V^{0.035}$

According to the formula (2) let us calculate the values of the expressions $\boldsymbol{w}$ and $\boldsymbol{(1\text{-}w)}$ for each fixed value of the rake angle γ. The results of calculations are presented in Table 3.

Table 3

Expressions of normalized variable $\boldsymbol{w}$ and $\mathbf{(1–}\boldsymbol{w}\mathbf{)}$, found from experimental data

γ, degrees	$\boldsymbol{w}$	$\mathbf{(1-}\boldsymbol{w}\mathbf{)}$
–5	$20.000S - 1.400$	$2.400-20.000S$
–6	$43.478S - 3.217$	$4.217-43.478S$
–8	$43.478S - 3.217$	$4.217-43.478S$
–10	$20.000S - 1.400$	$2.400-20.000S$

Substituting found expressions from Tables 2 and 3 in the formula (1), we obtain the approximating equations for the vertical (main) P_z (Table 4) and the radial P_y components of the cutting forces (Table 5).

Table 4

Approximation equation of the vertical (main) component of the cutting force $\mathbf{P_z}$

γ, degrees	Equation	Maximum relative error, %
–5	$\left(386.094 \cdot V^{0.476}\right)(2.400 - 20.000S) +$ $+\left(358.601 \cdot V^{0,493}\right)(20.000S - 1.400)$	3.4
–6	$\left(260.391 \cdot V^{0.575}\right)(4,217 - 43.478S) +$ $+\left(985.635 \cdot V^{0.195}\right)(43.478S - 3.217)$	6.5
–8	$\left(1139.795 \cdot V^{0.221}\right)(4.217 - 43.478S) +$ $+\left(1579.327 \cdot V^{0.174}\right)(43.478S - 3.217)$	5.8
–10	$\left(788.178 \cdot V^{0.367}\right)(2.400 - 20.000S) +$ $+\left(2273.029 \cdot V^{0.117}\right)(20.000S - 1.400)$	3.3

Table 5

Approximation equation of the radial component of the cutting force $\mathbf{P_y}$

γ, degrees	Equation	Maximum relative error, %
–5	$\left(141.908 \cdot V^{0.503}\right)(2.400 - 20.000S) +$ $+\left(127.914 \cdot V^{0.536}\right)(20.000S - 1.400)$	3.9
–6	$\left(87.988 \cdot V^{0.628}\right)(4.217 - 43.478S) +$ $+\left(299.075 \cdot V^{0.248}\right)(43.478S - 3.217)$	8.2
–8	$\left(429.294 \cdot V^{0.246}\right)(4.217 - 43.478S) +$ $+\left(768.142 \cdot V^{0.138}\right)(43.478S - 3.217)$	6.5
–10	$\left(217.021 \cdot V^{0.480}\right)(2.400 - 20.000S) +$ $+\left(1242.151 \cdot V^{0.035}\right)(20.000S - 1.400)$	5.3

Graphs of line surfaces $P_z(S;;V)$ i $P_y(S;;V)$are given in Figures 6, 7.

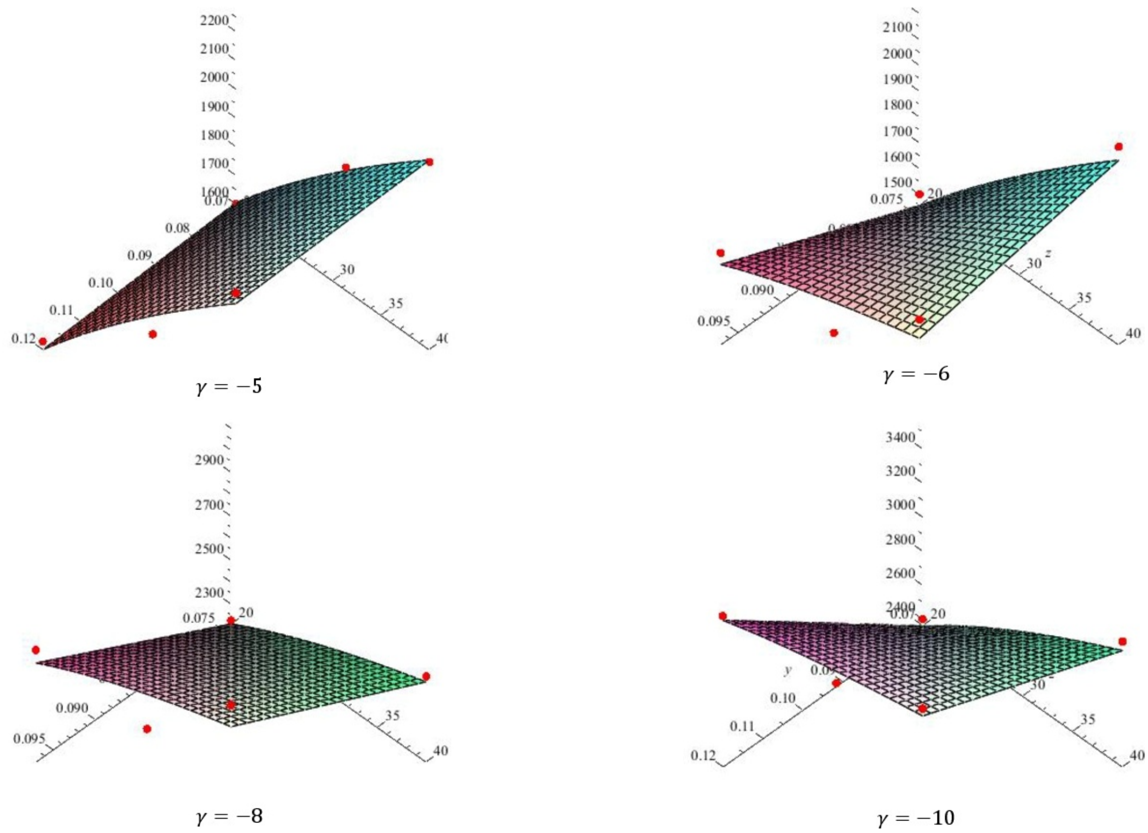


Figure 6. Graphs of line surfaces $P_z(S;V)$

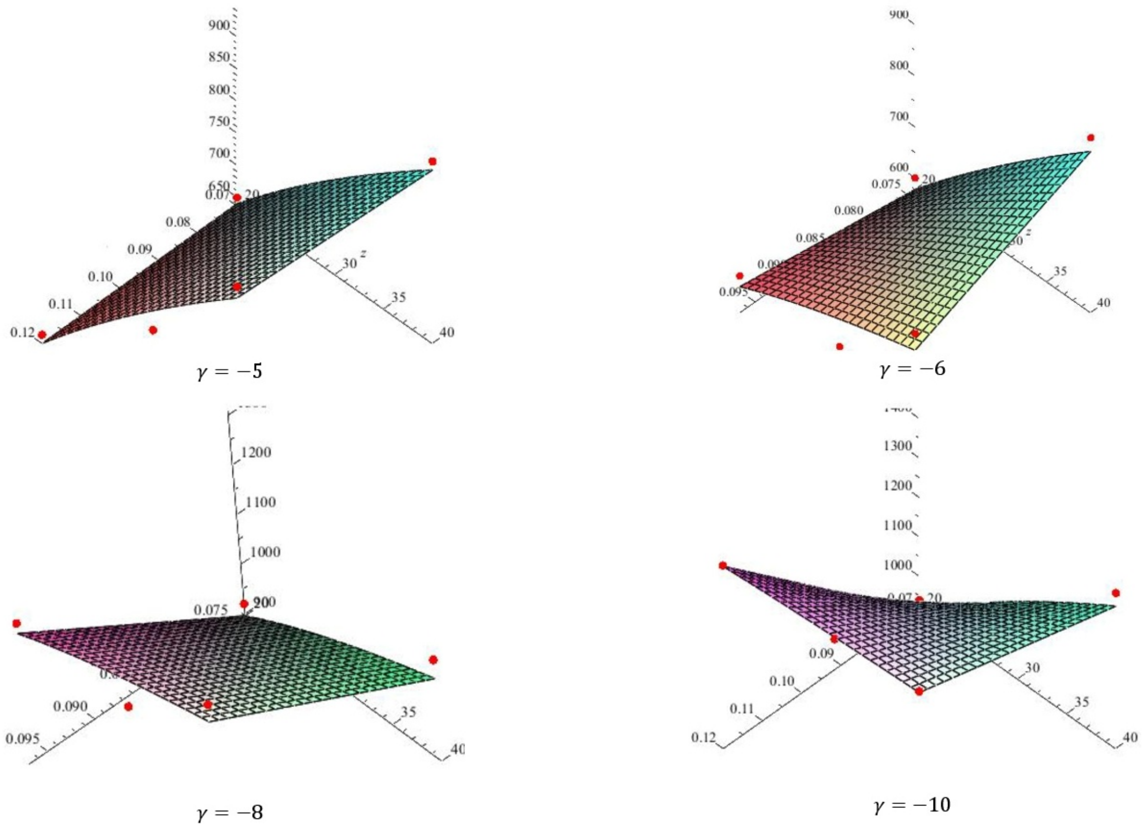


Figure 7. Graphs of line surfaces $P_y(S;V)$

Conclusions

Experimental studies of the cutting process with proposed cutting tools with laterally mounted MUP, carried out in laboratory conditions, allowed to confirm their performance. As a result of these studies and the processing of experimental data, mathematical models were first obtained that adequately describe the force parameters (Pz and Py) of the cutting process with the proposed cutting tools. It is found that it is inappropriate to perform negative rake angles $\gamma>$ -6o, since this results in a significant increase in the cutting forces and relief angles must not be α <6o, as this leads to the rubbing on the back surface. There are no other principal differences in the cutting conditions by proposed cutting tools and MUP from those previously known, which allows to consider, with high level of reliability, that all the basic modes of their operation and performance indicators are identical.

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Бүйірлі орнатылған көпқырлы қайралмайтын пластиналарды кесілген кескіштермен кесу процесінің математикалық моделі

Мақалада бүйірлі орнатылған көпқырлы қайралмайтын пластиналарды кесілген кескіштермен кесу процесінің эксперименталды зерттеулер, олардың жұмыс жасау мүмкіндігі мен прогрессивтілігін растайтын нәтижелері берілген. Зерттеулерді жүргізу және эксперименталды нәтижелерді өңдеу нәтижесінде ұсынылып отырған кескіштермен кесу процесінің (Pz және Py) күш параметрлерін сипаттайтын алғашқы рет математикалық модельдері алынған. Алдыңғы және артқы бұрыштардың рационалды мәндері келтірілген.

Кілт сөздер: жиналған кесетін кескіш, көпқырлы қайралмайтын кесетін пластина, механикалық бекіткіш.

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Математическая модель процесса резания отрезными резцами с боковой установкой многогранных неперетачиваемых пластин

В статье представлены результаты экспериментальных исследований процесса резания отрезными резцами с боковой установкой многогранных неперетачиваемых пластин, позволившие подтвердить их работоспособность и прогрессивность. В результате выполнения исследований и обработки экспериментальных данных впервые получены математические модели, адекватно описывающие силовые параметры (Pz и Py) процесса резания предлагаемыми резцами. Определены рациональные значения переднего и заднего углов.

Ключевые слова: сборный отрезной резец, многогранная неперетачиваемая режущая пластина, механическое крепление.

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Calculation of continuous beams taking into account the elastic compliance of supports

This work considers multispan structures (beams, columns) supported on intermediate elastically compliant supports. The structures are also extensively used in high-rise construction. The research is carried out by an exact analytical force method, based on the equation of five moments. The basic resolving equations are given. The work of a two-span structure with elastically compliant supports is considered in detail. The chart of flexure, the bending moments and the lateral forces are obtained depending on the compliance coefficient α, which varies within the limits of $0, .., \infty$. The analysis of the obtained results is carried out from the point of view of rational work of the operated structures.

Keywords: multi-storey frame columns, power loads, compliance of supports, vertical displacement of beams components, compliance coefficient.

Introduction

When calculating the load-bearing multi-storey frame columns of residential, public and industrial buildings and facilities from the effect of various power loads (including the effect of horizontal wind loads), it becomes necessary to take into account horizontal displacements of the column components at the levels of each of the frame floors, which can substantially change the distribution pattern of internal forces along the length of the columns [1, 2].

The columns design diagram, in the general view, in the presence of a number of intermediate supports, can be displayed as a multi-span (by the number of the building floors) continuous beam on elastic-settling supports (Fig. 1, *a*), («m» is the total number of beam supports).

Such structures are extensively used in high-rise construction; while the compliance of the intermediate supports models the flexibility of the intermediate floors. In this case, it is important to do the research on the compliance of the supports (components) to the strain-stress state of the structures (flexures, internal forces) [3]. Theoretical bases of calculations of multi-span structures (beams and frames) are considered with variable spans, bending stiffness, compliance coefficients, and external load intensity. Obtained diagram dependences can be used in practical design of load-bearing structures of multi-storey buildings.

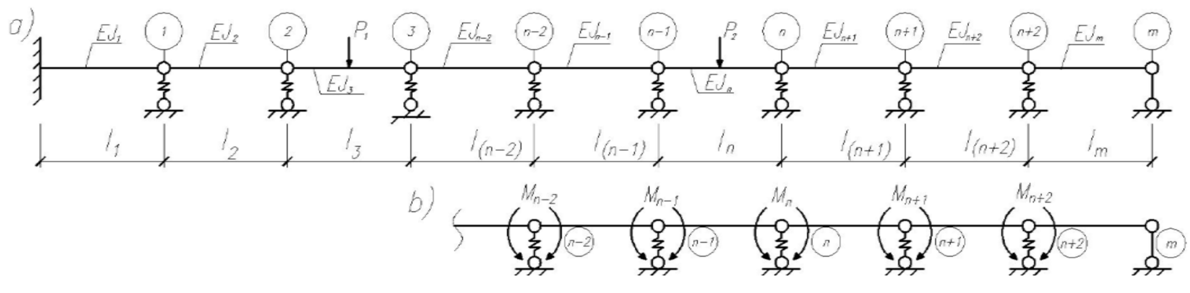

Figure 1. Design diagram (initial diagram) (a) and main system (b) of continuous beam

An elastic support is a support the movement of which is directly proportional to the support reaction arising from the external action.

In practical construction, examples of such supports can serve as long columns, cross-beams of the roadway of various types of bridges, as well as pontoons serving as supports for a floating bridge [4–6].

The elastic supports are characterized by the compliance coefficient C_i is the displacement of support caused by a single force [7].

Purpose and objectives of the research

Due to the fact that the elastic compliance of the components (posts) of building structures significantly affects the strain-stress state of the objects of research, it is important to develop methods for calculating them with a variety of structural concepts, external loading, including the compliance rate of the supports.

In this work, a complex approach has been applied to solve these problems: the difference in the spans' dimensions, bending stiffness, the intensity of uniformly distributed loading, and the variety of the supports compliance coefficients are taken into account simultaneously in the design diagrams.The resolving equations of the force method will take into account the above mentioned factors.

For the proposed theory illustration, a two-span continuous beam was studied in detail, for which dependencies of displacements and internal forces on the compliance coefficients of the 2^{nd} and 3^{rd} supports were obtained. The obtained dependency diagrams will extensively use in the practice of designing the load-bearing building structures.

Theory and calculation methods

The calculation of such structures can be made by an exact analytical force method; The main unknowns in the main system, shown in Fig. 1, b, will be support bending moments $M_1, M_2, \ldots, M_{m-1}$. In this case, the resolving canonical equation of the force method will have the form of an equation of five moments [8]

$$\delta_{n,n-2}M_{n-2} + \delta_{n,n-1}M_{n-1} + \delta_{n,n}M_n + \delta_{n,n+1}M_{n+1} + \delta_{n,n+2}M_{n+2} + \Delta_{np} = 0. \tag{1}$$

The canonical equation coefficients (1) determined by the known procedure of the force method, based on the Vereshchagin rule, will have the form [9]:

$$\begin{gathered}
\delta_{n,n-2} = \frac{C_{n-1}}{l_{n-1}l_n};\\
\delta_{n,n-1} = \frac{l_n}{6EJ_n} - \frac{C_{n-1}}{l_n}\left(\frac{1}{l_{n-1}} + \frac{1}{l_n}\right) - \frac{C_n}{l_n}\left(\frac{1}{l_n} + \frac{1}{l_{n+1}}\right);\\
\delta_{n,n} = \frac{l_n}{3EJ_n} + \frac{l_{n+1}}{3EJ_n l_{n+1}} + \frac{C_{n-1}}{l_n^2} + C_n\left(\frac{1}{l_n} + \frac{1}{l_{n+1}}\right)^2 + \frac{C_{n+1}}{l_{n+1}};\\
\delta_{n,n+1} = \frac{l_{n+1}}{6EJ_{n+1}} - \frac{C_{n+1}}{l_{n+1}}\left(\frac{1}{l_{n+1}} + \frac{1}{l_{n+2}}\right) - \frac{C_n}{l_{n+1}}\left(\frac{1}{l_n} + \frac{1}{l_{n+1}}\right);\\
\delta_{n,n+2} = \frac{C_{n+1}}{l_{n+1}l_{n+2}};\\
\Delta_{1p} = \frac{B_n^f}{EJ_n} + \frac{A_{n+1}^f}{EJ_{n+1}} + \frac{C_{n-1}}{l_n}R_{n-1} - l_n\left(\frac{1}{l_n} + \frac{1}{l_{n+1}}\right)R_n + \frac{C_{n+1}}{l_{n+1}}R_{n+1},
\end{gathered} \tag{2}$$

where $R_{n-1},\ R_n,\ R_{n+1}$ are respectively the supports reactions $n-1,\ n,\ n+1$ in the main system (Fig. 1, b); $B_n^f,\ {}_{n+1}^{f}$ are fictitious reactions of the considered support in n and $n+1$ beam spans; for example, under the action of a uniformly distributed load q_i for a span of length l_i, these reactions equals to: $B_i^f = A_i^f = \dfrac{q_i l_i^3}{24}$.

Let us introduce the following notation:

$$\alpha_i = \frac{C_i C_0 (EJ_0)}{l_0^3}, \tag{3}$$

where $C_0,\ EJ_0,\ l_0$ are respectively scaling (conventional) values of the compliance coefficient, bending stiffness, beam span; C_i is digital coefficient which varies within the limits of $C_i = 0, 1, ..., \infty$; the compliance of the corresponding support is absent at $C_i = 0$ i.e., the rigid support of the beam components takes place; the compliance of the support is very large at $C_i = \infty$, which characterizes the absence of this «i-th» support in the beam design diagram.

In case when all spans have a constant section and are equal-sized ($EJ = \text{const},\ l = \text{const}$), therein the supports have the same compliance coefficient $C_i = C_0 = C = \text{const}$, then by introducing the notation,

$$\alpha_i = \frac{6EJC}{l^3}$$

we obtain instead of (1) a simplified canonical equation of the form

$$\alpha M_{n-z} + (1-4\alpha)M_{n-z} + (4+6\alpha)M_n + (1-4\alpha)M_{n+z} + \alpha M_{n+z=} = -\frac{6EJ}{l}\Delta_{np}.$$

In this case

$$\frac{6EJ}{l}\Delta_{np} = \frac{6}{l}\left(B_n^f + A_{n+1}^f\right) + \alpha l\left(R_{n-1} - 2R_n + R_{n+1}\right).$$

For testing purposes of the proposed calculation theory and researching of the continuous beams or columns work, let us consider the structure in the form of a two-span beam with variable bending stiffness, with different spans and different compliance coefficients of supports loaded with a uniformly distributed load of different intensity on the spans (Fig. 2, *a*).

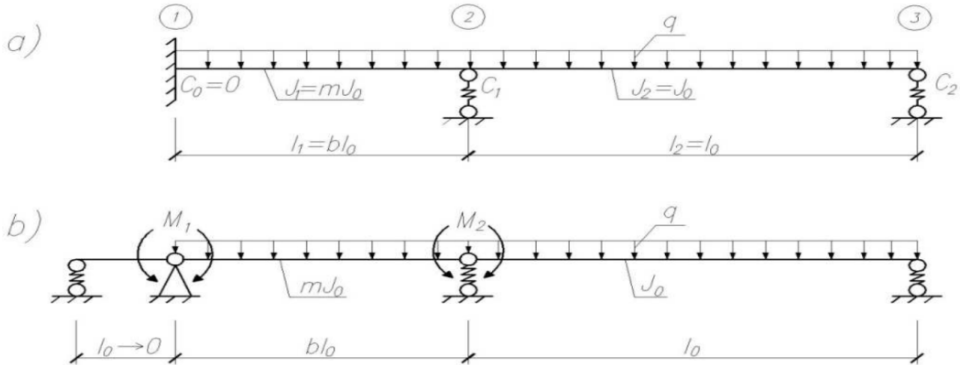

Figure 2. Design diagram of two-span continuous beam (a) and its main system (b)

System of canonical equations according to (1) (at $n = 2$) is as follows:

$$\begin{cases} \delta_{11}M_1 + \delta_{12}M_2 + \Delta_{1p} = 0, \\ \delta_{21}M_1 + \delta_{22}M_2 + \Delta_{2p} = 0. \end{cases} \tag{4}$$

According to the formula (2), we determine the coefficients of equation (4) taking into account the expression (3):

$$\delta_{11} = \frac{l_0}{EJ_0}\left(\frac{b}{3m} + \frac{C_1}{b_n^2}\alpha\right); \quad \alpha = \frac{C_0 EJ_0}{l_0^3};$$

$$\delta_{12} = \delta_{21} = \frac{l_0}{EJ_0}\left(\frac{b}{6m} - \frac{C_1(1+b)}{b^2}\alpha\right);$$

$$\delta_{22} = \frac{l_0}{EJ_0}\left[\left(\frac{b}{3m} - \frac{1}{3}\right) + \frac{\alpha}{b^2}\left(C_1\left(1+2b+b^2\right) + C_2 b^2\right)\right]; \tag{5}$$

$$\Delta_{1p} = \frac{ql_0^3}{EJ_0}\left(\frac{0.0417b^3}{m} + 0.5C_1\alpha\frac{(1+b)}{b}\right);$$

$$\Delta_{2p} = \frac{ql_0^3}{EJ_0}\left\{\left(\frac{0.0417b^3}{m} + 0.0417\right) - \left(\frac{1+b}{b}\right)\alpha\left[0.5C_1(1+b) + 0.5C_2\right]\right\}.$$

Results and Discussion

As a test task representing the proposed theory, let us consider an example of calculating of two-span continuous beam with identical spans, constant bending stiffness, and equal values of the compliance supports coefficients 2, 3 (see Fig. 3, *a*) i.e.:

$$l_1 = l_2 = l_0 = 6\text{m};\;\; C_1 = C_2 = C_0;\;\; EJ_1 = EJ_2 = EJ_0;\;\; q = \text{const} = q_0;\;\; \alpha = \frac{C_0 EJ_0}{l_0^3}.$$

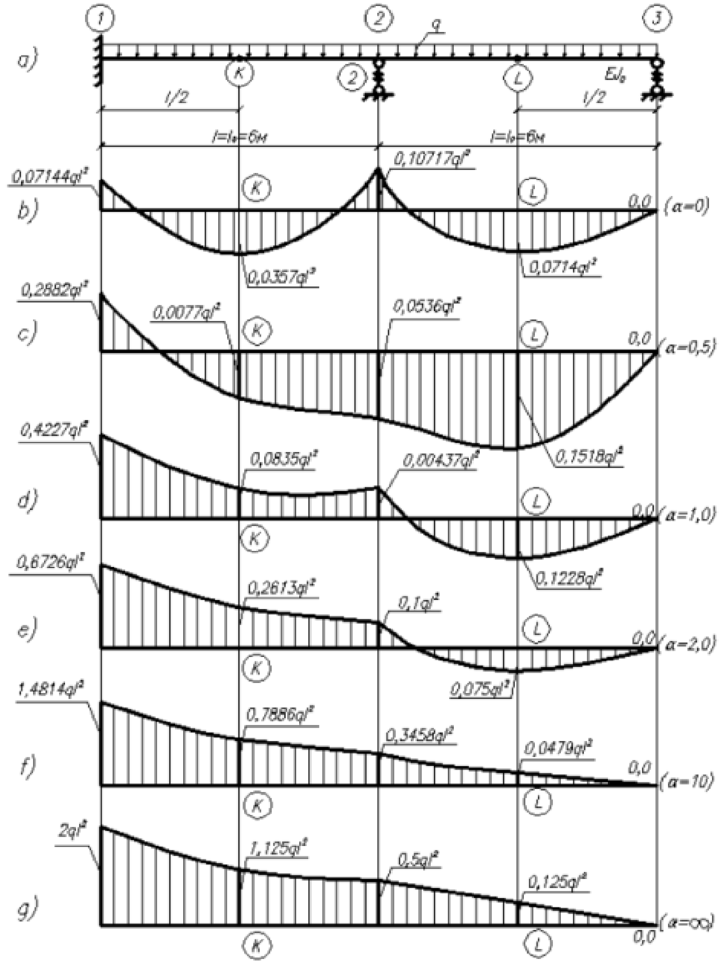

Figure 3. Bending moment diagrams under a variety of «*a*» values

According to the formula (5) we have:

$$\delta_{11} = \left(\frac{0.33+\alpha}{EJ_0}\right) l_0;\;\; \delta_{12} = \delta_{21} = \frac{l_0}{EJ_0}(0.167 - 2\alpha);\;\; \delta_{22} = \frac{l_0}{EJ_0}(0.667 + 5\alpha); \tag{6}$$

$$\Delta_{1p} = \frac{q_0 l_0^3}{EJ_0}(0.0417 + \alpha) = b_1;\;\; \Delta_{2p} = \frac{q l_0^3}{EJ_0}(0.0834 - 1.5\alpha) = b_2.$$

Solving the system of linear algebraic equations of the second order (taking into account the expressions (6), we determine the support moments (Fig. 2, b), (in general form):

$$M_1 = \frac{\left(-b_1 - \delta_{12}\left(\dfrac{b_2\delta_{11}}{\delta_{12}} - b_1\right)\right) \Big/ \left(\delta_{12} - \dfrac{\delta_{11}\delta_{22}}{\delta_{21}}\right)}{\delta_{11}};\;\; M_2 = \frac{b_2\left(\dfrac{\delta_{11}}{\delta_{21}} - b_1\right)}{\delta_{12} - \dfrac{\delta_{11}\delta_{22}}{\delta_{12}}}. \tag{7}$$

According to the expression (7), taking into account the initial data of the task, we have:

$$M_1 = k_1 q_0 l_0^2; \;\; M_2 = k_2 q_0 l_0^2, \tag{8}$$

where

$$k_1 = \frac{-1 - 75\alpha - 99\alpha^2 - 324\alpha^3}{14 + 258\alpha + 720\alpha^2 + 216\alpha^3}; \;\; k_2 = \frac{0.75 - 0.18\alpha + 18\alpha^2}{-7 - 108\alpha - 36\alpha^2}. \tag{9}$$

Table shows the results of the calculation of the continuous beam (Fig. 3, *a*), produced by the formulas (8), (9) with the following data:

$$l_1 = l_2 = l; \quad EJ_1 = EJ_2 = EJ; \quad q = \text{const}; \quad \alpha = 0.0;\, 0.5;\, 0.1;\, 1.0;\, 2.0;\, 10.0;\, \infty.$$

Figure 3 b–g shows the diagrams of bending moments at different values of the coefficients «α_i», taking into account the effect of the variable support coefficient C_0.

Figures 4–7 show, respectively, the dependence diagrams of bending moments M_i, flexures C_i, lateral forces Q_i, support reactions R_i of the beam depending on the coefficient value α.

The analysis of the dependencies in Figures 4–7 shows the following:

1. The ordinate values of moment diagram (Fig. 4) depend essentially on the value α and change their signs (from the two-digit torque value to single-digit (negative) torque values) at $\alpha \geq 2.0$.

2. The bending moments, the lateral forces, the reactions in the beam sections are increasing monotonically with a monotonous increase of α, except the support reaction at component 3, which in this case is decreasing monotonically.

3. The values M_1, M_2 take positive value in the $\alpha \geq 1.0$ values range (Fig. 4).

4. The flexures values are increasing slowly in the beam sections at $\alpha \leq 1.0$ and increasing rapidly at $\alpha \geq 2.0$ (Fig. 5), in this case the increase of elastic compliance is particularly significant.

5. The dependency diagrams on the value of lateral forces Q_1, Q_2^{left} and Q_2^{right}, Q_3 are parallel in their lineament (Fig. 6).

6. The dependency diagrams of reactions R_1 is increasing monotonically with an increase of α, therein the R_2 reaction is decreasing monotonically, and the R_3 reaction diagram has a complex lineament at $\alpha \leq 2.5$, and it is decreasing monotonically at $\alpha \geq 2.5$ (Fig. 7).

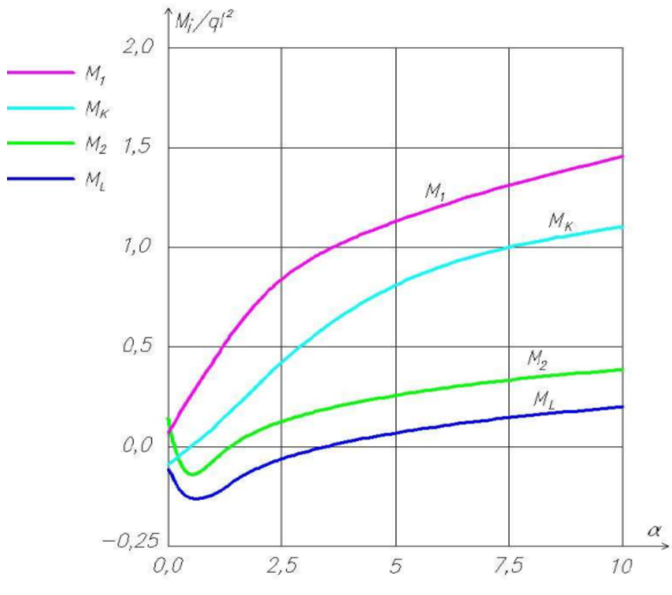

Figure 4. Dependence of the bending moment values M_K, M_L, M_1, M_2 on the coefficient of elastic compliance of the beam supports «α»

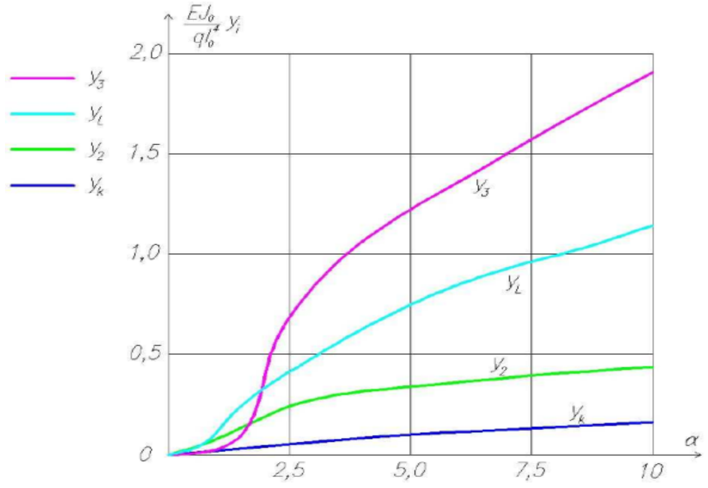

Figure 5. Dependence of flexure on «α»

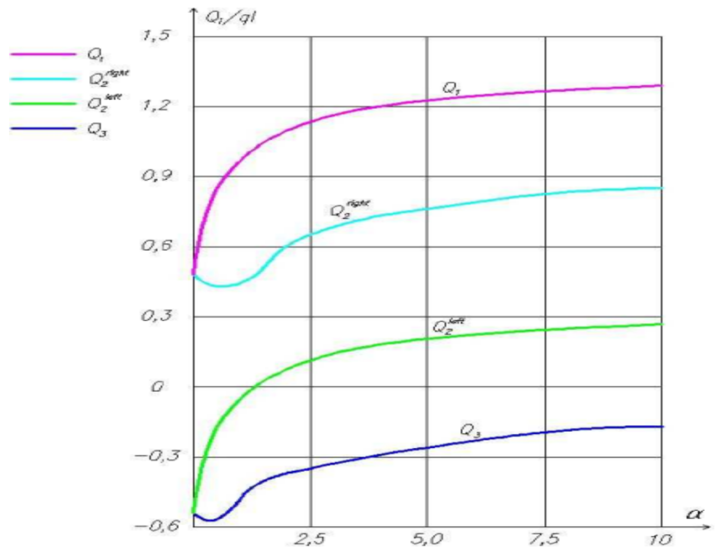

Figure 6. Dependence of lateral forces on «α»

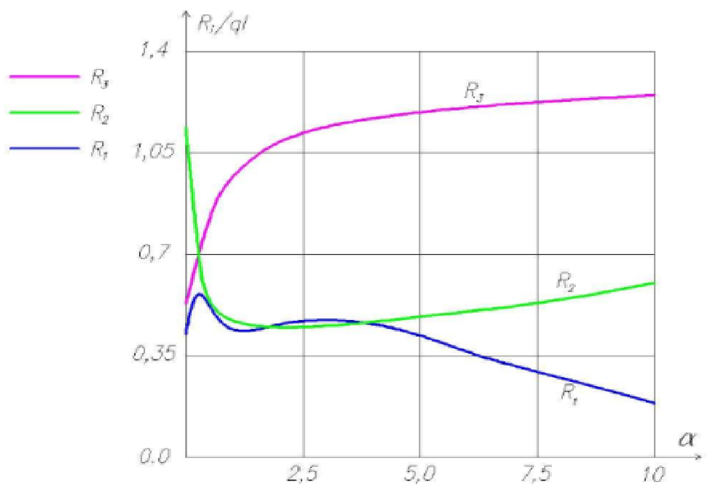

Figure 7. Dependence of support reaction on «α»

The results obtained in this work, for a two-span continuous beam based on the equation of five moments in terms of their reliability, are estimated as follows:

1) Relative comparison of results:

The bending moment diagrams (Fig. 3) with an increase of the compliance coefficient of the 2^{nd} and 3^{rd} beam supports «α» from a two-digit lineament (Fig. 3, b) change to a single-digit negative lineament (Fig. 3, g), which reflects the actual work of the structure at the action of the load uniformly distributed on the spans (Fig. 3, *a*); the indicated dependence is also present in the ordinates values of the moment diagram (see Table);

Table

Dependence of internal forces and support reactions of the beam on the value «α»

α	0.0	0.5	1.0	2.0	10.0	∞
$\frac{M_1}{ql^2}$	-0.0074	-0.2882	-0.4127	-1.6726	-1.4814	-2.0
$\frac{M_2}{ql^2}$	-0.10717	0.0536	-0.000497	-0.100	-0.3458	-0.500
$\frac{M_3}{ql^2}$	0.00	0.00	0.00	0.00	0.00	0.00
$\frac{Q_1}{ql}$	0.4643	0.8428	0.90833	1.0726	1.2583	2.0
$\frac{Q_2^{\text{left}}}{ql}$	-0.535	-0.1582	-0.0917	0.0726	0.2583	1.0
$\frac{Q_2^{\text{right}}}{ql}$	0.6072	0.4464	0.5044	0.600	0.8458	1.0
$\frac{Q_3}{ql}$	-0.3928	-0.5538	-0.4956	-0.400	-0.1542	0.0
$\frac{R_1}{ql}$	0.4643	0.8428	0.9083	1.073	1.2583	2.0
$\frac{R_2}{ql}$	1.1423	0.6046	0.5236	0,5274	0,5875	0.0
$\frac{R_3}{ql}$	0.3928	0.5536	0.4956	0.400	0.1542	0.0

2) Absolute comparison of the results:

a) At $\alpha = 0$ (the compliance of the supports is minimal and there is a rigid support of the 2^{nd} and 3^{rd} supports) (Fig. 3, b) the values of the support ordinates of the moment diagram coincide (without the presence of an error) with the calculation of a beam according to the well-known theory of three moments [7, 9];

b) At $\alpha = 0$ (the compliance of the supports is maximal and there is a single-span cantilever beam with a length «$2l$») (Fig. 3, *a*) the greatest bending moment in the support «1» is equal to ($M_1 = 2ql^2$), (Fig. 3, g) which coincides with the known value for the cantilever beams $\left[M_1 = \frac{q(2l)^2}{2} = 2ql^2\right]$.

Other results (at $\alpha \neq 0, \ \alpha \neq \infty$), given in this work are original, i.e., new in the scientific literature for today.

As for comparing the obtained results with the experimental data of other authors, we have not found similar results for today, apparently, the experiments on the subject of our research have not yet been carried out.

Along with this, it should be noted that we compared the results with analytical calculations: the relative (semantic) and absolute (the coincidence of data with the results obtained by other methods) comparison of the results (see the section «analysis of dependencies»).

Conclusion

1. In this work, the research is performed on the stress-strain state of a two-span continuous beam with elastic-settling supports for equal spans, with the same bending stiffness under the action of a uniformly distributed load of the same intensity.

2. The method of calculation is an exact analytical force method based on the five moments equation [9], [10], which is used here to calculate a variety of continuous beams (columns) with multiple spans, with different spans values, variability of bending stiffness and lateral uniformly distributed load on the spans, and also at different values of the compliance coefficients of the beams supports (Fig. 2, *a*).

3. The two-span continuous beam of constant bending stiffness, equal to spans and loads on them, as well as the same parameters of C_1, C_2^0 was performed as an illustration of the proposed theory operability (Fig. 3, *a*); the flexure dependency diagrams, the bending moments and the lateral forces on the change in the compliance coefficients C_1, C_2 were obtained for this variant (Fig. 4–7).

4. At the same time it is established that with an increase in the compliance of the supports, the lineaments of the diagrams M, Q change substantially and the beam flexures increase therein.

5. The proposed theory also makes it possible to calculate the columns of multi-storey buildings, the design diagram of which can be represented as a multi-span continuous beam. The presence of intermediate floorings, in this case, will be modeled as the elastic compliance of the components of the columns joints and flooring.

6. The proposed theory and the practical results can be used both in scientific researches, and in practice of designing of load-bearing structures of high-rise and unique buildings.

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Тіректердің серпімді икемділігін ескерумен кесілмейтін арқалықтарды есептеу

Мақалада аралық серпімді икемді тіректерге сүйенген көпаралықты конструкцияларды (арқалықтарды, бағаналарды) зерттеу орындалған. Сонымен қатар конструкциялар биік құрылыста кең қолданыс тапқан, зерттеу бес сәттің теңдеуінің негізінде дәл талдамалы күштер әдісімен келтірілген. Негізгі бұзушы теңдеулер берілген. Серпімді икемді тіректермен қосаралықты конструкцияның жұмысы толық қарастырылған. $0, .., \infty$ шегінде өзгеретін икемділік коэффициентіне α тәуелділікте иілулердің, иілу кездері мен көлденең күштерінің кестелері алынған. Пайдаланылатын конструкциялардың дұрыс жұмысы тұрғысынан алынған нәтижелерді талдау жүргізілген.

Кілт сөздер: көп қабатты рамалы қаңқалардың бағаналары, күштік жүктемелер, тіректердің икемділігі, арқалықтардың тораптардың тік ығысулары, икемділік коэффициенті.

Ж.С. Нугужинов, С.К. Ахмедиев, С.Р. Жолмагамбетов, О. Хабидолда

Расчет неразрезных балок с учетом упругой податливости опор

В статье выполнено исследование многопролетных конструкций (балок, колонн), опертых на промежуточные упруго податливые опоры. Такие конструкции находят широкое применение в высотном строительстве. Исследование проведено точным аналитическим методом сил, на основе уравнения пяти моментов. Приведены основные разрешающие уравнения. Подробно рассмотрена работа двухпролетной конструкции с упруго податливыми опорами. Получены графики прогибов, изгибающих моментов и поперечных сил в зависимости от коэффициента податливости α, который меняется в пределах 0...∞. Проведен анализ полученных результатов с точки зрения рациональной работы эксплуатируемых конструкций.

Ключевые слова: колонны многоэтажных рамных каркасов, силовые нагрузки, податливость опор, вертикальные смещения узлов балок, коэффициент податливости.

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ИНФОРМАТИКА
INFORMATICS

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Development of a mathematical model for signal processing using laboratory data

In this paper, we consider a mathematical model for the interpretation of the radarograms which obtained by GPR systems. As noted in [1–3], in addition to testing the algorithms, it is necessary to compare the calculated data of the mathematical model with the real data obtained from the GPR. One of the reasons preventing the spread of GPR technologies is the complexity of data interpretation, which requires the involvement of highly qualified specialists. In connection with this research as a mathematical model and a comparison with the real data of the GPR in an ideal layered medium, will provide a method for interpreting radarograms. We have conducted a series of experimental studies using the Loza – A georadar at the newly created laboratory ground. A distinctive feature of these studies is the choice of several localized objects in the form of iron sheets placed in an ideal layered medium, namely in clean dry sand. The choice of such an environment is necessary for testing the algorithms, the mathematical models developed by us for determining the depth of localized several objects. A series of experimental studies were conducted using georadar and a number of radarograms were obtained to study the depth of objects. A cycle of calculations was carried out to verify the conformity of the results of mathematical modeling with real georadar data. Key words: electrodynamics equation, magnetic permeability, dobeshi wavelets, medium conductivity, dielectric permeability, Maxwell equation.

Keywords: electrodynamics equation, magnetic permeability, dobeshi wavelets, medium conductivity, dielectric permeability, Maxwell equation.

1 Problem statement. Mathematical model

One of the main reasons preventing the wide spread of georadar technologies is the complexity of data interpretation, which at the present stage requires the involvement of highly qualified specialists. The way out of this situation is to create a mathematical apparatus for solving the inverse problem of radar sensing, which will minimize the operator's participation in obtaining the final result, as well as extract the maximum amount of information from georadar data. In connection with this research as a mathematical model and comparison with real data of ground-penetrating radar in the ideal layered medium, will provide the methodology of the interpretation of the GPR.

This kind of problems are related to the inverse and incorrect problems, the foundations of which were laid in the theory of the work Tikhonova, M. M. Lavrenteva, V. K. Ivanova, V. G. Romanova.

One of the main obstacles in the localization of underground facilities is the upper part of the soil lying above the desired objects. Passing through this area, the electromagnetic waves reflected from various objects interact with each other, can be amplified or, conversely, mutually reduced. One way to overcome this problem is to continue to solve the system of Maxwell's equations from the earth's surface in the direction of the location of the desired objects. The problem of continuation is one of the most difficult and incorrect problems of mathematical physics, complicated in this case by the presence of attenuation of the electromagnetic field in conducting media. Problems of continuation of solutions of equations of mathematical physics from the part of the boundary in many cases are strongly ill-posed problems in the classes of finite smoothness functions and are the first step in solving the coefficient inverse problems [4–7]. The approach of regularization of the field problem was proposed in the paper by V. Kozlov, V. G. Mazya, and V. Fomin in 1991 [8].

Consider the system of Maxwell's equations [9]:

$$\begin{cases} \varepsilon\frac{\partial E}{\partial t} - rot\,H + \sigma E + j^{cm} = 0, \\ \mu\frac{\partial H}{\partial t} + rot\,E = 0, \end{cases} \quad (x,y,z) \in R^3, \quad x \neq 0, \quad t > 0. \tag{1}$$

There are positive functions $\varepsilon(x,y,z)$, $\sigma(x,y,z)$, $\mu(x,y,z)$ and the permittivity, conductivity and magnetic permeability of the medium, respectively.

$$R^3_- = \left\{ x,y,z \in R^3, \quad x < 0\right\} \text{ — air, } R^3_+ = \left\{ x,y,z \in R^3, \quad x > 0\right\} \text{ — earth.}$$

We consider that electromagnetic oscillations up to the moment of time $t = 0$ are absent:

$$(E,H)\ |_{\ t<0} \equiv 0 j^{cm}\,|_{t<0} \equiv 0$$

and then induced by a side current $j^{cm}(x,y,z,t)$.

Let us consider one of the simplest variants of the problem, when ε, σ and μ depend only on the depth x and one horizontal variable y, and the source of the external current is a sufficiently long (infinite) cable located in the center and stretched along the z axis:

$$j^{cm}(x,y,z,t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} g(x,y)\ V(t). \tag{2}$$

Here, the function $g(x,y)$ describes the transverse dimensions of the source.

In this case, ignoring the influence of the cable ends, we conclude that only three components E_z,H_x, H_y. remain nonzero in the system of Maxwell equations.

After exclusion of the first equation of partial derivatives of the H_x component and H_y obtain regarding $E_z$the second order equation:

$$\mu\varepsilon\frac{\partial^2 E_z}{\partial t^2} + \mu\sigma\frac{\partial E_z}{\partial t} = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - g(x,y)V'(t), \tag{3}$$

to which we add the initial condition

$$E_z|_{t=0} = 0, \quad \left.\frac{\partial E_z}{\partial t}\right|_{t=0} = 0;$$

boundary conditions:

$$\left.\frac{\partial E_z}{\partial x}\right|_{x=x_1} = 0, \quad E_z|_{x=x_2} = 0,\ E_z|_{y=-y_1} = 0, \quad E_z\Big|_{y=+y_1} = 0$$

and conditions at the interface of media:

$$[E_z]_{\,x=x_1} = 0, \quad \left[\frac{\partial E_z}{\partial x}\right]_{x=x_2} = 0, [E_z]_{x=x_3} = 0, \quad \left[\frac{\partial E_z}{\partial x}\right]_{x=x_3} = 0,$$

where $\varepsilon = \varepsilon_0\varepsilon_{rel}$; $\varepsilon_0 = 8.854 \cdot 10^{-12} F/m$ dielectric constant; ε_{rel}= relative dielectric constant

$$\mu_0 = 4\pi \cdot 10^{-7} g/m\sigma\ - sm/m, V(t) = \exp\left\{-\frac{(t-t_0)^2}{t_1^2}\right\}, g(x,y) = \theta(a-x)\theta(a-y),$$

$a = 0.025\ m$ — the size of the source. $V'(t)$ — function describing the source of electromagnetic oscillations emitted by the transmitting antenna.

2 Numerical calculation

Consider a homogeneous medium at a site of 10 by 12 meters with an inclusion size of 0.6 by 0.4 m. The capacity of the pit at a depth of 0.6 m.

Parameters of homogeneous medium

Dry sand	$\varepsilon_1 = 6$	$\sigma_1 = 0.62$	$h_1 = 0.6$ m
Rectagular iron sheet thickness	$\varepsilon_2 = 1$	$\sigma_2 = 0.769 \cdot 10^7$	$h_2 = 0.005$ m
Wet sand	$\varepsilon_3 = 40$	$\sigma_3 = 0.005$	$h_3 = 0.5$ m or more

Calculation parameters: step x is equal to 0.01; step y equal to 0.01. The time step is considered from the Courant condition. The calculation time is from 0 to 60 ns. The problem is solved by a finite-difference method using an explicit scheme.

First, we consider a homogeneous medium without inclusions. Calculate field $E_z^{(1)}(x = 0, y, t)$.
Then consider the environment with inclusion and calculate the total field $E_z^{(2)}(x = 0, y, t)$. The anomalous field allows you to see the reflections from the localized object in the form of a hodograph (Fig. 1). In the case of two localized objects (Fig. 2).

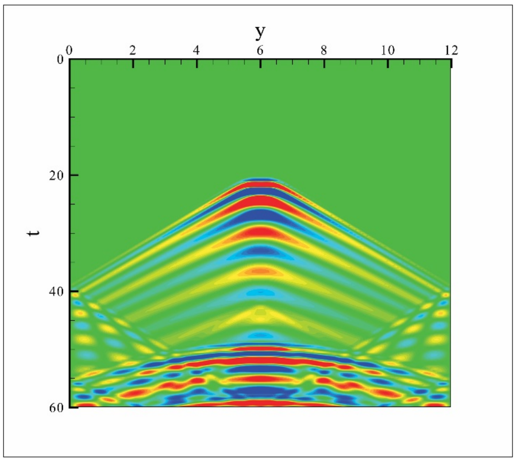

Figure 1. Anomalous field of a localized single object in a homogeneous medium

The calculated results are in good agreement with the measured GPR a GPR.

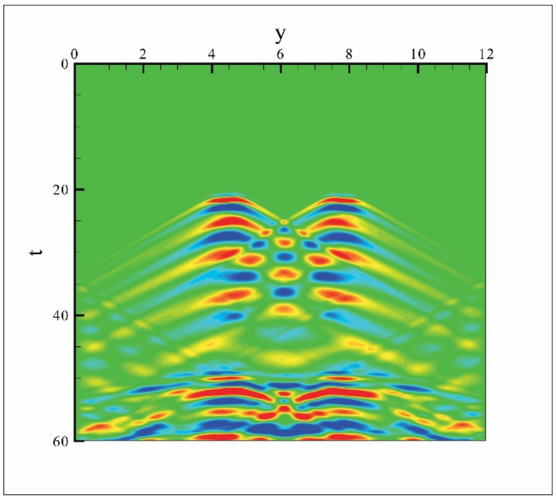

Figure 2. Anomalous field of two localized objects in a homogeneous medium

The results of the calculations are qualitatively the same as the radarogram measured by the georadar.

3 Processing of georadar signals. Cleaning the route from noise using wavelet transform

To improve the noise immunity of the georadar method, as a rule, pre-processing of experimental measurements is performed in order to isolate informative signals. The essence of the processing of georadar data is, first of all, in the allocation of a useful signal on the background of noise and noise. To distinguish useful signals, the difference between their characteristics and the corresponding characteristics of noise and interference waves is used [10, 11].

One of the ways of the primary processing of the radargram is the wavelet transform. With the help of wavelet digital signal conversion in the radar can reduce the influence of high-frequency components in the spectrum of the signal.

One of the ways of processing of the radargram is the wavelet transform. With the help of the digital signal wavelet transform in the radargram, it is possible to remove high-frequency components from the signal spectrum.

The wavelet transform of a one-dimensional signal is its representation as a generalized series or Fourier integral over a system of basis functions [12]:

$$\varphi_{ab}\left(t\right) = \left|a\right|^{-1/2} \varphi\left(\frac{t-b}{a}\right)$$

constructed from the parent (generating) wavelet $\psi(t)$, due to time - shift operations – b and time-scale changes – a. In the study, the signal is represented as a set of successive approximations of coarse(approximating) $Aj(t)$and refined(detailing) components:

$$f\left(t\right) = A_j\left(t\right) + \sum_{i=1}^{j} D_i(t)$$

with subsequent refinement by iterative method. Each step of refinement corresponds to a certain scale, that is, the level $j$of analysis (decomposition) and synthesis (reconstruction) of the signal. This representation of each component of the signal by wavelets can be considered in both the time and frequency domains. In a multiscale analysis, the signal $f(t)$ decomposes into two components:

$$f\left(t\right) = \sum_k a_k\ \varphi_k\left(t\right) + \sum_k d_k\ \psi_k\left(t\right).$$

The basis functions $\varphi(t)$ and $\psi(t)$are uniquely determined by the coefficients h_l:

$$\varphi\left(t\right) = 2\sum_l h_l \varphi(2t-l);$$

$$\psi\left(t\right) = 2\sum_l g_l \psi(2t-l).$$

In the transition from the current scale j to the next $j+1$, the number of wavelet coefficients is halved, and they are determined by the recurrence relations:

$$a_{j+1,k} = \sum_l h_{l-2,k} a_{j,k};$$

$$d_{j+1,k} = \sum_l g_{l-2,k} a_{j,k},$$

where

$$g_l = \left(-1\right)^l h_{2n-l-1}.$$

When restoring (reconstructing) a signal by its wavelet coefficients, the process proceeds from large to small scales and is described by the expression [12]:

$$a_{j-1,k} = \sum_l \left(h_{k-2l} a_{j,l} + g_{k-2l} a_{j,l}\right).$$

Daubechies wavelet DB4 were used to clear the signal from noise [13]. The Daubechies wavelets do not have analytical expressions and are determined only by the filters. In practical applications, approximating h_k and detailing g_k wavelet coefficients are used, without calculating the specific shape of the wavelets [14]. For Daubechies wavelet db4 the factors are the following: Decomposition into components of discrete Daubechies wavelets is carried out according to the formulas

$$a_i = h_0 s_{2i-1} + h_1 s_{2i} + h_2 s_{2i+1} + h_3 s_{2i+2};$$

$$d_i = g_0 s_{2i-1} + g_1 s_{2i} + g_2 s_{2i+1} + g_3 s_{2i+2} \mathit{i=1,\ 2,\dots,\ n/2\text{-}1};$$

$$a_{n/2} = h_0 s_{n-2} + h_1 s_{n-1} + h_2 s_0 + h_3 s_1;$$

$$d_{n/2} = g_0 s_{n-2} + g_1 s_{n-1} + g_2 s_0 + g_3 s_1.$$

Formula (1) is a pyramidal algorithm for calculating the wavelet coefficients of Mall [14]. These formulas digital filter h_n from the s_k signal allocates low frequencies, and the filter g_n allocates the upper frequencies. Wavelet transform the track radargram has been used on a software-controlled threshold processing of detail coefficients (thresholding). The algorithms thresholding the adaptive threshold limits established for each factor on the criterion of Stein's unbiased risk estimation (Stein's unbiased risk estimation) [15].

When restoring (reconstructing) a signal by its wavelet coefficients, the process proceeds from large to small scales and is described by formulas at each step [16]:

$$a_1 = h_2 s_{\frac{n}{2}} + h_1 s_n + h_0 s_1 + h_3 s_{\frac{n}{2}-1};$$

$$a_2 = g_0 s_{\frac{n}{2}} + g_1 s_{\frac{n}{2}-1} + g_2 s_1 + g_3 s_n;$$

$$a_i = h_2 s_{\frac{i-1}{2}} + h_1 s_{\frac{i-1}{2}+\frac{n}{2}} + h_0 s_{\frac{i-1}{2}+1} + h_3 s_{\frac{i-1}{2}+\frac{n}{2}-1} i = 3, 5, \dots n/2 - -1 (i = 4, 6, \dots n/2\ \ (odd);$$

$$a_i = g_0 s_{\frac{i-1}{2}} + g_1 s_{\frac{i-1}{2}+\frac{n}{2}-1} + g_2 s_{\frac{i-1}{2}+1} + g_3 s_{\frac{i-1}{2}+\frac{n}{2}} i = 4, 6, \dots n/2\ \ (even).$$

provided that the detailing coefficients of the previous levels are recorded in place of the signal values.

The results of the transformation track of Daubechies wavelets is presented in Figure 3.

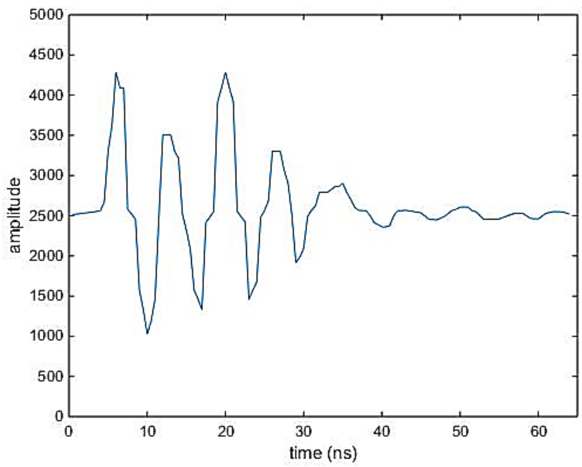

Figure 3. Chart of the radargram route after the wavelet transform

4 Testing of the model based on the device Loza - B. Experimental studies

To test the algorithm of the mathematical model to detect the depth of the objects and study its physical properties, a laboratory polygon was created, located 80 kilometers from Astana along the Kurgaldzhinskaya highway. A sand pit was chosen for the landfill, which corresponds to the model of the environment: air; clean dry sand; targets; and the underlying layer (wet sand), see figure 4 left fragment). The size of the pit: length 0.6 m.; width 0.5 m.; depth 0.65 m. the dimensions of the target - iron sheet rectangular: width 0.3 m; length 0.4 m.; thickness 0.005 m. Cm. Figure 4 (left fragment). The target is placed at a depth of 0.6 m. for georadar studies, it is necessary to mark the site where the object is located, see Figure 4 (right fragment).

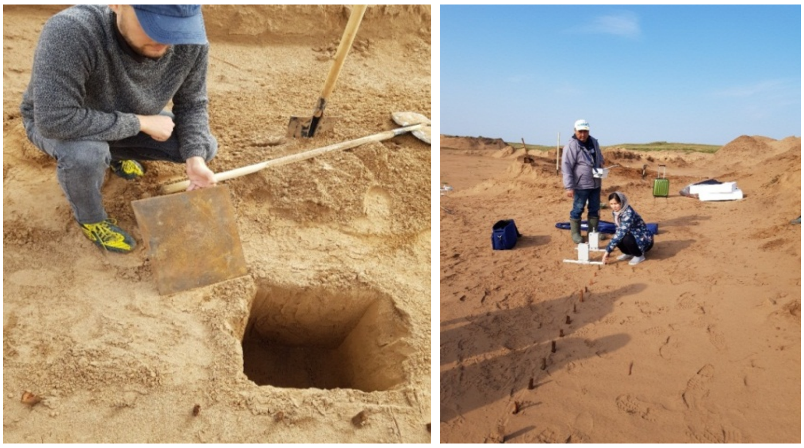

Figure 4. Metal disk

In order to test the algorithm, which will be given in the next section below, and to verify the results of numerical calculations for the detection of localized objects, a test experiment was conducted to detect a localized object using a georadar.

Georadar Loza - V was measured with the diversity of antennas. The measurements consist of 20 points, starts with 0 point and ends with 20 point. The transmitting antenna (source) is at point 0 and the receiving antenna measures at all other points. Then the transmitting antenna (source) is moved to the next point, and the receiving antenna goes through all the other points. And so continue until 20 point. See Figure 4 (right fragment), Figure 5. All data were digitized and summed. Then a comparative analysis of the measured data with the results of solving the inverse problem of detecting localized objects was carried out.

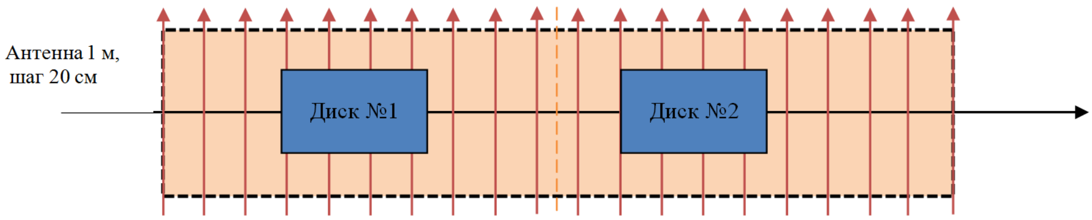

Figure 5. Scheme of the experiment with antenna diversity

Figure 6 shows the results of measurements carried out by the device in a homogeneous medium, i.e. without a target. This is necessary for us to analyze the radargrams obtained directly from the media in which the targets are placed.

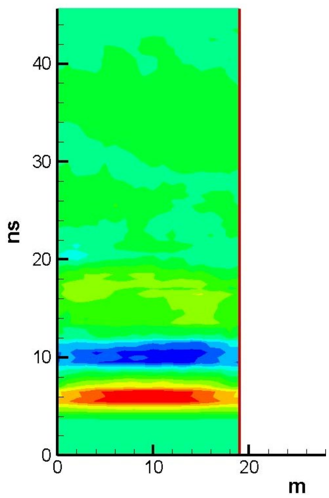

Figure 6. The radargram of a homogeneous medium

Figure 7 shows the results of measurements carried out by GPR according to the above scheme. Researched the area in which is hidden a single object – an iron sheet of a rectangular shape.

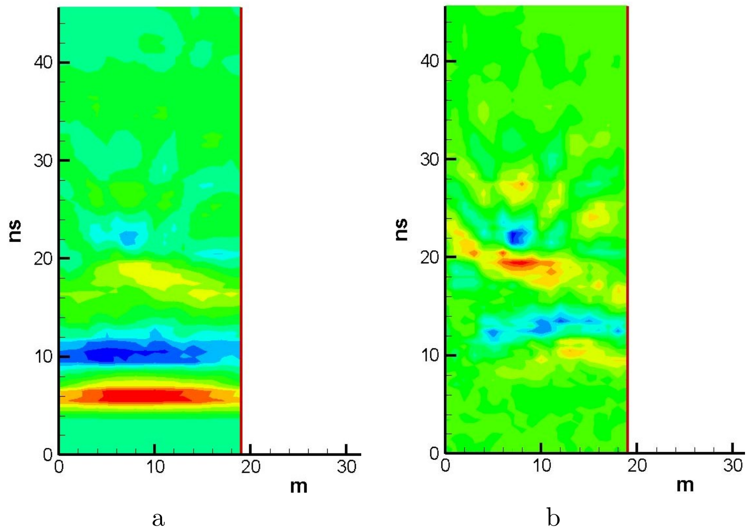

Figure 7. a) Radargram of the area with one hidden object-iron sheet; b) The difference of the radargrams received from the environment with the target and without it on the same site

Similar experimental studies were carried out on a site in which two identical objects are hidden – iron sheets at a depth of 60 cm and at a distance from each other relative to the day surface by 20 cm.the results of the studies in the form of radargrams are presented in Figure 8 (a) with targets and in Figure 8 (b), the result is the same as in the past case of the difference.

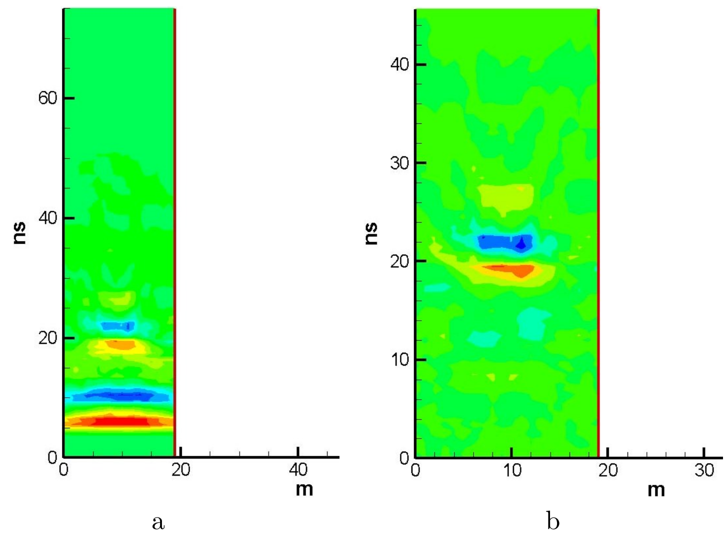

Figure 8. a) radargram of the area with hidden two identical objects-iron sheet; b) the difference of radargrams obtained from the medium with and without targets on the same site

By the type of the hodograph it is possible to assume, what layers with what parameters are in the studied environment. You can also use a hodograph to determine that a localized object is present in your environment, and you can specify its location.

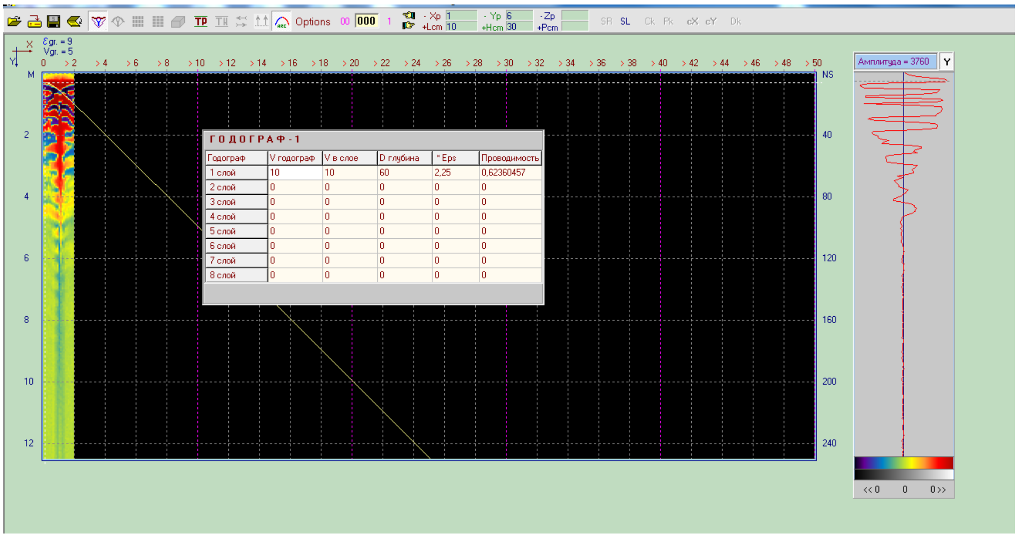

Figure 9. Radargram of the experiment

According to the hodograph, the depth of the localized object is determined, which is equal to 60 cm. (Fig. 9). In this paper, a mathematical model to determine the depth of localized objects is constructed.

A series of experimental studies, with the use of georadar Loza-B. a cycle of calculations to verify the compliance of the results of mathematical modeling of real georadar data.

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С.И. Кабанихин, Қ.Т. Искаков, Б.Б. Шолпанбаев, М.А. Шишленин, Д.Қ. Тоқсейіт

Зертханалық деректерді пайдалана отырып дабылды өңдеу үшін математикалық модельді әзірлеу

Мақалада георадиолокациялау жүйесімен алынған радар сигналдарын түсіндіру үшін математикалық модель қарастырылған. [1–3] жұмыстарда көрсетілгендей, алгоритмдерді тестілеуге қосымша математикалық модельдің есептік деректерін георадардан алынған нақты деректермен салыстыру қажет. Георадар технологияларының таралуына кедергі келтіретін себептердің бірі — жоғары білікті мамандарды тартуды талап ететін деректерді түсіндірудің күрделілігі. Осыған байланысты математикалық модельді және георадардың нақты деректерімен салыстыра отырып, идеалды деңгейдегі ортаны зерттеу радарограммаларды түсіндіру әдісін ұсынады.

Кілт сөздер: электрдинамика теңдеуі, магнит өткізгіштігі, Добеши вейвлеттері, ортаның өткізгіштігі, диэлектрлік өткізгіштігі, Максвелл теңдеуі.

С.И. Кабанихин, К.Т. Искаков, Б.Б. Шолпанбаев, М.А. Шишленин, Д.К. Токсеит

Разработка математической модели по обработке сигнала с использованием данных лабораторных исследований

В статье рассмотрена математическая модель для интерпретации радарограмм, полученных георадиолокационной системой. Как отмечено в работах [1–3], помимо апробации алгоритмов, необходимо сопоставить расчетные данные математической модели с реальными данными полученных от георадара. Одной из причин, препятствующих распространению георадарных технологий, является сложность интерпретации данных, требующая привлечения высококвалифицированных специалистов. В связи с этим исследования математической модели, а также сопоставление с реальными данными георадара в идеальной слоистой среде позволит получить методику интерпретации радарограмм.

Ключевые слова: уравнение электродинамики, магнитная проницаемость, вейвлеты Добеши, проводимость среды, диэлектрическая проницаемость, уравнение Максвелла.

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№ б.

МАТЕМАТИКА

МЕХАНИКА

ИНФОРМАТИКА

ТАҢДАМАЛЫ ТАҚЫРЫПТАРҒА ШОЛУ

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МЕХАНИКА

ИНФОРМАТИКА

ОБЗОРЫ ИЗБРАННЫХ ТЕМАТИК

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