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Well-posedness of a periodic boundary value problem for the system of hyperbolic equations with delayed argument

The periodic boundary value problem for the system of hyperbolic equations with delayed argument is considered. By method of introduction a new functions the investigated problem reduce to an equivalent problem, consisting the family of periodic boundary value problem for a system of differential equations with delayed argument and integral relations. Relationship of periodic boundary value problem for the system of hyperbolic equations with delayed argument with the family of periodic boundary value problems for the system of ordinary differential equations with delayed argument is established. Algorithms for finding solutions of the equivalent problem are constructed and their convergence is proved. Sufficient and necessary conditions of well-posedness of periodic boundary value problem for the system of hyperbolic equations with delayed argument are obtained.

Keywords: periodic boundary value problem, system of hyperbolic equations, delayed argument, family of periodic boundary value problems, system of differential equations with delayed argument, algorithm, unique solvability, well-posedness.

Introduction

Numerous problems of application such as problems of population dynamics, management of technical systems, the problem of physics, mathematical economics, ecology and etc., variational problems related to the regulatory process, the optimal control problem with delay systems leads to boundary value problems for a differential equations with deviating argument [1]. One of the rapidly growing field of the theory of differential equations with deviating argument is the theory of boundary value problems for a various classes of the differential equations with delayed argument [2, 3]. Mathematical modeling of the various processes in physics, chemistry, mechanics, biology leads to the hyperbolic differential equations with delayed argument [1], which in combination with periodic boundary conditions allow us to describe important classes of models. To investigate the questions of solvability of these classes of problems there have been applied the methods of the qualitative theory of differential equations, Riemann's method, the method of monotone iteration, asymptotic methods, the method of upper and lower solutions, numerical-analytical method and others. On their base, there have been obtained the solvability conditions for the considered problems and suggested the ways of finding solutions. Study of qualitative properties of nonlocal boundary value problems for the differential equations of hyperbolic type with delayed argument, as well as the conditions of solvability and finding solutions us associated with many problems, such as: the complexity of considered objects, the impossibility of constructing of analytical solution, the lack of universal methods of solving, difficulties with adaptation of known methods, etc. Note that the periodic boundary value problems for hyperbolic equations with delayed argument are widely used in various applications. Nevertheless, the problem of finding effective features of unique solvability of periodic boundary value problems for hyperbolic equations with delayed argument still holds actual today.

In this paper we investigate of the questions existence and uniqueness of solution to the periodic boundary value problem for the system of hyperbolic equations with delayed argument. Periodic boundary value problems for the system of hyperbolic equations with delayed argument will be reduced to the family of periodic boundary value problems for the system of ordinary differential equations with delayed argument and the integral relations. We establish a connection between conditions of the solvability to the periodic boundary value problem for the system of hyperbolic equations with delayed argument and the solvability of the family of periodic boundary value problems for the system of ordinary differential equations with delay argument.

We consider the periodic boundary value problem for the system of the hyperbolic equations second order with delay argument on the domain $\Omega_\tau = [-\tau, T] \times [0, \omega]$

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} = A(t,x)\frac{\partial u(t,x)}{\partial x} + A_0(t,x)\frac{\partial u(t-\tau,x)}{\partial x} + B(t,x)\frac{\partial u(t,x)}{\partial t} + C(t,x)u(t,x) + f(t,x);$$

$$(t,x) \in [0,T] \times [0,\omega], \tag{1}$$

$$\frac{\partial u(z,x)}{\partial x} = diag\Big[\frac{\partial u(0,x)}{\partial x}\Big] \cdot \varphi(z), \quad z \in [-\tau, 0], \qquad x \in [0,\omega]; \tag{2}$$

$$u(0,x) = u(T,x), \qquad x \in [0,\omega]; \tag{3}$$

$$u(t,0) = \psi(t), \qquad t \in [-\tau, T], \tag{4}$$

where $u(t,x) = col(u_1(t,x), u_2(t,x), ..., u_n(t,x))$ is unknown function, the $(n \times n)$ matrices $A(t,x)$; $A_0(t,x)$, $B(t,x)$, $C(t,x)$ and n vector-function $f(t,x)$ are continuous on $\Omega = [0,T] \times [0,\omega]$; the n vector-function $\varphi(t)$ is continuously differentiable and given on the initial set $[-\tau, 0]$ such that $\varphi_i(0) = 1$, $i = \overline{1,n}$, $\tau > 0$ is constant delay, the n vector-function $\psi(t)$ is continuously differentiable on $[-\tau, T]$, and the compatibility condition is valid: $\psi(0) = \psi(T)$.

Let $C(\Omega_\tau, R^n)$ be the space of continuous on Ω_τ vector functions $u(t,x)$ with the norm

$$||u||_0 = \max_{(t,x)\in\Omega_\tau} ||u(t,x)||, \quad ||u(t,x)|| = \max_{i=\overline{1,n}} |u_i(t,x)|;$$

$C([0,\omega], R^n)$ be a space of continuous on $[0,\omega]$ vector functions $\varphi(x)$ with the norm

$$||\varphi||_{0,1} = \max_{x\in[0,\omega]} ||\varphi(x)||;$$

$C^1([-\tau, T], R^n)$ be a space of continuously differentiable on $[-\tau, T]$ vector functions $\psi(t)$ with the norm $||\psi||_{1,0} = \max\Big(\max_{t\in[-\tau,T]} ||\psi(t)||, \max_{t\in[-\tau,T]} ||\dot{\psi}(t)||\Big)$;

$$\Omega_0 = \{(t,x) : t = 0, 0 \le x \le \omega\}.$$

The function $u(t,x) \in C(\Omega_\tau, R^n)$, that has partial derivatives $\frac{\partial u(t,x)}{\partial x} \in C(\Omega_\tau, R^n)$, $\frac{\partial u(t,x)}{\partial t} \in C(\Omega_\tau \backslash \Omega_0, R^n)$, $\frac{\partial^2 u(t,x)}{\partial t \partial x} \in C(\Omega_\tau \backslash \Omega_0, R^n)$ is called a *classical solution* to periodic boundary value problem (1)–(4) if it satisfies system (1) for all $(t,x) \in \Omega$ and the condition (2) in the initial set $[-\tau, 0]$, the boundary conditions (3), (4).

1 Problem (1)–(4) and its relationship with the family of periodic boundary value problems for ordinary differential equations with delayed argument

In [4] a nonlocal boundary value problem with an integral condition in time was studied for the system of hyperbolic equations. By introducing new functions the problem was reduced to the equivalent problem consisting of a family of boundary value problems with an integral condition for ordinary differential equations and integral relations. Family of boundary value problems with integral conditions for ordinary differential equations was solved by the parametrization method. Necessary and sufficient conditions for the well-posedness of nonlocal boundary value problem with an integral condition for a system of hyperbolic equations were set. The results of [4] will be developed to the periodic boundary value problems for differential equations of hyperbolic type with delayed argument. There will be established the conditions of unique solvability of the considered problem.

We introduce a new unknown functions $v(t,x) = \frac{\partial u(t,x)}{\partial x}$ and $w(t,x) = \frac{\partial u(t,x)}{\partial t}$ and reduce problem (1)–(4) to the equivalent problem

$$\frac{\partial v(t,x)}{\partial t} = A(t,x)v(t,x) + A_0(t,x)v(t-\tau,x) + F(t,x,w(t,x),u(t,x)), \qquad (t,x) \in \Omega; \tag{1.1}$$

$$v(z,x) = diag[v(0,x)] \cdot \varphi(z), \quad z \in [-\tau,0], \qquad x \in [0,\omega]; \tag{1.2}$$

$$v(0,x) = v(T,x), \qquad x \in [0,\omega]; \tag{1.3}$$

$$u(t,x) = \psi(t) + \int_0^x v(t,\xi)d\xi, \qquad w(t,x) = \dot{\psi}(t) + \int_0^x \frac{\partial v(t,\xi)}{\partial t} d\xi, \tag{1.4}$$

where $\quad F(t,x,w(t,x),u(t,x)) = B(t,x)w(t,x) + C(t,x)u(t,x) + f(t,x)$.
In the problem (1.1)–(1.4) the condition $u(t,0) = \psi(t)$ is taken into account in relation (1.4).

A triple $\{v(t,x), u(t,x), w(t,x)\}$ of functions is called a *solution* to problem (1.1)–(1.4) if the function $v(t,x)$ belonging to $C(\Omega_\tau, R^n)$ has a continuous derivative with respect to t on $\Omega_\tau \backslash \Omega_0$ and satisfies the one-parameter family of periodic boundary value problems for ordinary differential equations with delayed argument (1.1)–(1.3), where the functions $u(t,x)$ and $w(t,x)$ are connected with $v(t,x)$ and $\frac{\partial v(t,x)}{\partial t}$ by the integral relations (1.4).

Let $u^*(t,x)$ be a classical solution of problem (1)–(4). Then the triple $\{v^*(t,x),\ u^*(t,x),\ w^*(t,x)\}$, where $v^*(t,x) = \frac{\partial u^*(t,x)}{\partial x}$, $w^*(t,x) = \frac{\partial u^*(t,x)}{\partial t}$, is a solution to problem (1.1)–(1.4). Conversely, if a triple $\{\widetilde{v}(t,x), \widetilde{u}(t,x), \widetilde{w}(t,x)\}$ is a solution to problem (1.1)–(1.4), then $\widetilde{u}(t,x)$ is a classical solution to problem (1)–(4).

For fixed $w(t,x)$, $u(t,x)$ in problem (1.1)–(1.4) it is necessary to find a solution to a one-parameter family of periodic boundary value problems for system of ordinary differential equations with delayed argument.

Consider the family of periodic boundary value problems for system of ordinary differential equations with delayed argument

$$\frac{\partial v(t,x)}{\partial t} = A(t,x)v(t,x) + A_0(t,x)v(t-\tau,x) + F(t,x), \qquad t \in [0,T], \quad x \in [0,\omega], \quad v \in R^n; \tag{1.5}$$

$$v(z,x) = diag[v(0,x)] \cdot \varphi(z), \quad z \in [-\tau,0], \qquad x \in [0,\omega]; \tag{1.6}$$

$$v(0,x) = v(T,x), \qquad x \in [0,\omega], \tag{1.7}$$

where the n vector function $F(t,x)$ is continuous on Ω_τ.

Continuous function $v : \Omega_\tau \to R^n$ that has a continuous derivative with respect to t on $\Omega_\tau \backslash \Omega_0$ is called a solution to the family periodic boundary value problems with delayed argument (1.5)–(1.7) if it satisfies system (1.5) for all $(t,x) \in \Omega$ and has the values $v(0,x)$, $v(T,x)$ on the lines $t = 0$, $t = T$ and the equalities (1.6), (1.7) are valid for all $x \in [0,\omega]$, respectively.

For fixed $x \in [0,\omega]$ problem (1.5)–(1.7) is a linear periodic boundary value problem for the system of ordinary differential equations with delayed argument [5]. Suppose a variable x is changed on $[0,\omega]$; then we obtain a family of periodic boundary value problems for ordinary differential equations with delayed argument. In the [6] was investigated the family of periodical boundary value problems for the system of differential equations with delayed argument. Algorithms for finding solutions of the considered problem are constructed and their convergence is proved. Conditions of the unique solvability to family of periodical boundary value problems for the system of differential equations with delayed argument are established in the terms of the initial data. These conditions are also given the conditions of well-posed solvability to problem (1.5)–(1.7).

Definition 1. Problem (1.5)–(1.7) is called well-posed if for arbitrary $F(t,x) \in C(\Omega, R^n)$ it has a unique solution $v(t,x)$ and for it the estimate holds

$$\max_{t\in[0,T]} ||v(t,x)|| \le K \max_{t\in[0,T]} ||F(t,x)||, \tag{1.8}$$

where the constant K independent of $F(t,x)$ and $\ x \in [0,\omega]$.

Denote by $\Omega^\eta = [0,T] \times [0,\eta]$ and $||u||_{0,\eta} = \max\limits_{(t,x)\in\Omega^\eta} ||u(t,x)||$.

Definition 2. Boundary value problem (1)-(4) is called well-posed if for arbitrary $f(t,x) \in C(\Omega, R^n)$ and $\psi(t) \in C^1([-\tau, T], R^n)$ it has a unique classical solution $u(t,x)$ and this solution satisfies the following estimate

$$\max\Big(||u||_{0,\eta}, \Big|\Big|\frac{\partial u}{\partial x}\Big|\Big|_{0,\eta}, \Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{0,\eta}\Big) \le \tilde{K} \max\Big(||f||_{0,\eta}, ||\psi||_{1,0}, \max_{x\in[0,\eta]} ||\varphi(x)||\Big),$$

where constant $\tilde{K}$ independent of $f(t,x)$ and $\psi(t)$ and $\eta \in [0,\omega]$.

Theorem 1. The boundary value problem (1)–(4) is well-posed if and only if so is problem (1.5)–(1.7). Proof of Theorem 1 is similar to the proof of Theorem 1 in [4] taking into account the properties of the delayed argument.

From Theorem 1 it follows that the well-posedness of problem (1)–(4) are equivalent to the well-posedness of problem (1.5)–(1.7).

Hereby, the problem (1)–(4) reduce to an equivalent problem, consisting the family of periodic boundary value problem for system of differential equations with delayed argument and integral relations. For constructing of algorithms of finding approximate solutions to the equivalent problem (1.1)–(1.4) are used results of paper [7].

2 Algorithm for finding approximate solutions of problem (1)–(4) and its convergence

We suppose that problem (1.5)–(1.7) is well-posed. By virtue of the equivalence of problems (1)–(4) and (1.1)–(1.4), it suffices to justify the well-posedness of problem (1.1)–(1.4). We find a solution $\{v(t,x), u(t,x), w(t,x)\}$ of problem (1.1)–(1.4) by the successive approximation method. As the initial approximation $u(t,x)$ and $w(t,x)$ we take $\psi(t)$ and $\dot{\psi}(t)$, respectively, and then find $v^{(0)}(t,x)$ from the problem

$$\frac{\partial v^{(0)}(t,x)}{\partial t} = A(t,x)v^{(0)}(t,x) + A_0(t,x)v^{(0)}(t-\tau,x)+$$

$$+B(t,x)\dot{\psi}(t) + C(t,x)\psi(t) + f(t,x), \qquad (t,x) \in \Omega; \tag{2.1}$$

$$v^{(0)}(z,x) = diag[v^{(0)}(0,x)] \cdot \varphi(z), \quad z \in [-\tau, 0], \qquad x \in [0,\omega]; \tag{2.2}$$

$$v^{(0)}(0,x) = v^{(0)}(T,x), \qquad x \in [0,\omega]. \tag{2.3}$$

By assumption, problem (2.1)–(2.3) has a unique solution $v^{(0)}(t,x)$.

Then $u^{(0)}(t,x)$ and $w^{(0)}(t,x)$ are determined from integral relations:

$$u^{(0)}(t,x) = \psi(t) + \int_0^x v^{(0)}(t,\xi)d\xi, \qquad w^{(0)}(t,x) = \dot{\psi}(t) + \int_0^x \frac{\partial v^{(0)}(t,\xi)}{\partial t}d\xi. \tag{2.4}$$

Suppose $u^{(k-1)}(t,x)$ and $w^{(k-1)}(t,x)$ are known. Then $v^{(k)}(t,x)$ can be found from the problem (1.1)–(1.3), where $w(t,x) = w^{(k-1)}(t,x)$, $u(t,x) = u^{(k-1)}(t,x)$:

$$\frac{\partial v^{(k)}(t,x)}{\partial t} = A(t,x)v^{(k)}(t,x) + A_0(t,x)v^{(k)}(t-\tau,x)+$$

$$+B(t,x)w^{(k-1)}(t,x) + C(t,x)u^{(k-1)}(t,x) + f(t,x), \qquad (t,x) \in \Omega; \tag{2.5}$$

$$v^{(k)}(z,x) = diag[v^{(k)}(0,x)] \cdot \varphi(z), \quad z \in [-\tau, 0], \qquad x \in [0,\omega]; \tag{2.6}$$

$$v^{(k)}(0,x) = v^{(k)}(T,x), \qquad x \in [0,\omega]. \tag{2.7}$$

Once $v^{(k)}(t,x)$ is found the successive approximations for $u(t,x)$ and $w(t,x)$ are found from relations (1.4):

$$u^{(k)}(t,x) = \psi(t) + \int_0^x v^{(k)}(t,\xi)d\xi, \qquad w^{(k)}(t,x) = \dot{\psi}(t) + \int_0^x \frac{\partial v^{(k)}(t,\xi)}{\partial t}d\xi, \quad k = 1, 2, ... \tag{2.8}$$

We construct the differences $\Delta v^{(k)}(t,x) = v^{(k)}(t,x) - v^{(k-1)}(t,x)$, $\Delta u^{(k)}(t,x) = u^{(k)}(t,x) - u^{(k-1)}(t,x)$, $\Delta w^{(k)}(t,x) = w^{(k)}(t,x) - w^{(k-1)}(t,x)$, and by using the well-posedness of problem (1.5)–(1.7) establish estimates

$$\max\Big\{\max_{t\in[0,T]} ||\Delta v^{(k+1)}(t,x)||, \max_{t\in[0,T]} \Big|\Big|\frac{\partial \Delta v^{(k+1)}(t,x)}{\partial t}\Big|\Big|\Big\} \le \max\Big\{K, \max_{t\in[0,T]} ||A(t,x)||K + 1\Big\}\times$$

$$\times K_1(x) \max\Big\{\max_{t\in[0,T]} ||\Delta w^{(k)}(t,x)||, \max_{t\in[0,T]} ||\Delta u^{(k)}(t,x)||\Big\}; \tag{2.9}$$

$$\max\Big\{\max_{t\in[0,T]}||\Delta w^{(k)}(t,x)||,\max_{t\in[0,T]}||\Delta u^{(k)}(t,x)||\Big\}\leq$$

$$\leq\int_0^x\max\Big\{\max_{t\in[0,T]}||\Delta v^{(k)}(t,\xi)||,\max_{t\in[0,T]}\Big|\Big|\frac{\partial\Delta v^{(k)}(t,\xi)}{\partial t}\Big|\Big|\Big\}d\xi, \tag{2.10}$$

where $K_1(x)=\max\limits_{t\in[0,T]}||B(t,x)||+\max\limits_{t\in[0,T]}||C(t,x)||$.
This implies the main inequality

$$\max\Big\{\max_{t\in[0,T]}||\Delta v^{(k+1)}(t,x)||,\max_{t\in[0,T]}\Big|\Big|\frac{\partial\Delta v^{(k+1)}(t,x)}{\partial t}\Big|\Big|\Big\}\leq\max\Big\{K,\max_{t\in[0,T]}||A(t,x)||K+1\Big\}\times$$

$$\times K_1(x)\int_0^x\max\Big\{\max_{t\in[0,T]}||\Delta v^{(k)}(t,\xi)||,\max_{t\in[0,T]}\Big|\Big|\frac{\partial\Delta v^{(k)}(t,\xi)}{\partial t}\Big|\Big|\Big\}d\xi. \tag{2.11}$$

From (2.11) it follows that the sequences $\{v^{(k)}(t,x)\}$ and $\{\frac{\partial v^{(k)}(t,x)}{\partial t}\}$ are convergent in the space $C(\Omega,R^n)$ as $k\to\infty$. Then the uniform convergence on Ω of the sequences $\{u^{(k)}(t,x)\}$ and $\{w^{(k)}(t,x)\}$ follows from the estimate (2.10).

In this case, the limit functions $v^*(t,x)$, $\frac{\partial v^*(t,x)}{\partial t}$, $u^*(t,x)$ and $w^*(t,x)$ are continuous on Ω, and the triple $\{v^*(t,x),u^*(t,x),w^*(t,x)\}$ is a solution to problem (1.1)–(1.4). For $\eta\in[0,\omega]$ by using the estimates (2.9)–(2.11), we obtain

$$\max\Big(||v^*||_{0,\eta},||u^*||_{0,\eta},||w^*||_{0,\eta}\Big)\leq\hat{K}\cdot\max\Big(||f||_{0,\eta},||\psi||_{1,0},\max_{x\in[0,\eta]}||\varphi(x)||\Big), \tag{2.12}$$

where $\hat{K}=\max\Big(e^{K_0K_1\omega}[1+\omega K_0],K[K_1(1+\omega K_0)+1]\Big)$, $K_0=\max(K,||A||_0K+1)$, $K_1=\max\limits_{x\in[0,\omega]}K_1(x)$ and they independent of f and ψ.

Now let $\{\widetilde{v}(t,x),\widetilde{u}(t,x),\widetilde{w}(t,x)\}$ be a solution to problem (1.1)–(1.4), where $f(t,x)=0$ and $\psi(t)=0$ for all $(t,x)\in\Omega$. Then the well-posedness of problem (1.5)–(1.7) together with (1.4) imply that $\widetilde{v}(t,x)=0$, $\widetilde{u}(t,x)=0$, and $\widetilde{w}(t,x)=0$ for all $(t,x)\in\Omega$. Thus it follows from the estimate (2.12) that problem (1)–(4) is well-posedness.

So, the problem (1)–(4) reduce to an equivalent problem, consisting the family of periodic boundary value problem for system of differential equations with delayed argument and integral relations. For constructing of algorithms of finding approximate solutions to the equivalent problem are used results of paper [5]. For solve of the family to the periodic boundary value problems for system of differential equations with delayed argument are used results of articles [6-7]. Algorithms of finding solutions to the families of periodic boundary value problems for differential equations with delayed argument are constructed and their convergence proved. The conditions of the solvability to the periodic boundary value problems for hyperbolic equations with delayed argument are established. These results are partially announced in the [8].

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А.Т. Асанова, Н.Б. Искакова, Н.Т. Орумбаева

Кешігулі аргументі бар гиперболалық теңдеулер жүйесі үшін периодты шеттік есептің корректілі шешілімділігі

Кешігулі аргументі бар гиперболалық теңдеулер жүйесі үшін периодты шеттік есеп қарастырылды. Жаңа функциялар енгізу әдісі арқылы зерттеліп отырған есеп кешігулі аргументі бар дифференциалдық теңдеулер жүйесі үшін периодты шеттік есептер әулеті мен интегралдық қатынастар пара-пар есепке келтірілді. Кешігулі аргументі бар гиперболалық теңдеулер жүйесі үшін периодты шеттік есеп пен кешігулі аргументі бар жай дифференциалдық теңдеулер жүйесі үшін периодты шеттік есептер әулетінің өзара байланысы орнатылған. Пара-пар есептің шешімін табу алгоритмдері құрылған және олардың жинақтылығы дәлелденген. Кешігулі аргументі бар гиперболалық теңдеулер жүйесі үшін периодты шеттік есептің корректілі шешілімділігінің жеткілікті және қажетті шарттары алынған.

Кілт сөздер: периодты шеттік есеп, гиперболалық теңдеулер жүйесі, кешігулі аргумент, периодты шеттік есептер әулеті, кешігулі аргументі бар дифференциалдық теңдеулер жүйесі, алгоритм, бірмәнді шешілімділік, корректілі шешілімділік.

А.Т. Асанова, Н.Б. Искакова, Н.Т. Орумбаева

Корректная разрешимость периодической краевой задачи для системы гиперболических уравнений с запаздывающим аргументом

Рассмотрена периодическая краевая задача для системы гиперболических уравнений с запаздывающим аргументом. Методом введения новых функций исследуемая задача сводится к эквивалентной задаче, состоящей из семейства периодических краевых задач для системы дифференциальных уравнений с запаздывающим аргументом и интегральных соотношений. Установлена взаимосвязь периодической краевой задачи для системы гиперболических уравнений с запаздывающим аргументом и семейства периодических краевых задач для системы дифференциальных уравнений с запаздывающим аргументом. Построены алгоритмы нахождения решений эквивалентной задачи и доказана их сходимость. Получены достаточные и необходимые условия корректной разрешимости периодической краевой задачи для системы гиперболических уравнений с запаздывающим аргументом.

Ключевые слова: периодическая краевая задача, система гиперболических уравнений, запаздывающий аргумент, семейство периодических краевых задач, система дифференциальных уравнений с запаздывающим аргументом, алгоритм, однозначная разрешимость, корректная разрешимость.

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On an integral equation of the problem of heat conduction with domain boundary moving by law of $\mathbf{t = x^2}$

In the article it is shown that the homogeneous Volterra integral equation of the second kind, to which the homogeneous boundary value problem of heat conduction in the degenerating domain is reduced, has a nonzero solution. The boundary of the domain moves with a variable velocity. It is shown that the norm of the integral operator acting in classes of continuous functions is equal to 1. Mellin transformation is applied to the obtained integral equation. It is proved that for certain values of the spectral parameter the eigenvalues of the integral equation will be simple.

Keywords: heat conduction, boundary value problems, a kernel, Mellin transformation, convolution theorem, eigenfunction.

In general cases, the methods of separation of variables and integral transformations are not applicable to problems in domains that are degenerated to the point, since for this type of problems methods of separating the variables and integral transformations can not be applied, as remaining within the classical methods of mathematical physics you cannot conform the solution of the heat equation to the boundary of domain. Therefore the question about the study of boundary value problems with degenerating the domain at the initial moment of time is actual. Using the method of heat potentials solving such problems is reduced to the study of singular Volterra integral equations of the second kind. Feature of these equations consists in incompressibility of the kernel and is expressed in the fact that the homogeneous equation has nonzero solutions and corresponding non-homogeneous equation can not be solved by classical methods.

1 Statement of the problem

We consider the first boundary value problem of heat conduction in the degenerating domain (domain with a moving boundary; the boundary of the domain is moving with variable speed). In the domain $G = \left\{(x;\ t)\ :\ t > 0,\ \ 0 < x < \sqrt{t}\right\}$ to find a solution to the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

satisfying the boundary conditions:

$$u(x,\ t)|_{x=0} = 0, \quad u(x,\ t)|_{x=\sqrt{t}} = 0. \tag{2}$$

Such problems in the domain $Q = \{(x;\ t): \quad t > 0, \quad 0 < x < t\}$ are investigated in works [1–3].

2 Reducing the problem to an integral equation

We are looking for a solution to boundary problem (1)–(2) as the sum of the heat potentials of the double layer [4]

$$u(x,t) = \frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{x}{(t-\tau)^{\frac{3}{2}}} \exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\ \nu\left(\tau\right) d\tau + \tag{3}$$

$$+\frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{x-\sqrt{\tau}}{(t-\tau)^{\frac{3}{2}}} \exp\left\{-\frac{(x-\sqrt{\tau})^2}{4a^2(t-\tau)}\right\}\ \psi(\tau)\, d\tau.$$

It is known that function (3) satisfies equation (1) for any $\nu(t)$ and $\psi(t)$ [4; 476–480]. Using conditions (2) and the properties of the heat potential, we obtain the integral equation:

$$\psi(t) - \int_0^t k(t,\,\tau)\;\psi(\tau)\,d\tau = 0, \tag{4}$$

where

$$k(t,\tau) = \frac{1}{2a\sqrt{\pi}}\left\{\frac{\sqrt{t}+\sqrt{\tau}}{(t-\tau)^{3/2}}\exp\left(-\frac{\left(\sqrt{t}+\sqrt{\tau}\right)^2}{4a^2(t-\tau)}\right)+\right. \tag{5}$$

$$\left.+\frac{\sqrt{t}-\sqrt{\tau}}{(t-\tau)^{3/2}}\exp\left(-\frac{\left(\sqrt{t}-\sqrt{\tau}\right)^2}{4a^2(t-\tau)}\right)\right\}.$$

In equation (4) we introduce a new function

$$\varphi(t) = \sqrt{t}\,\psi(t) \in M(R_+),\text{ where } M(R_+) = L_\infty(R_+)\bigcap C(R_+);$$

and consider integral equation (4) with a real parameter λ

$$\varphi(t) - \lambda\int_0^t K(t,\tau)\varphi(\tau) = 0, \tag{6}$$

where

$$K(t,\tau) = \sqrt{t/\tau}\,k(t,\tau).$$

The kernel $K(t,\tau)$ has the following properties

1) $K(t,\tau) \geq 0$ and is continuous on $0 < \tau \leq t$;

2) $\lim\limits_{t\to t_0}\int_{t_0}^{t} K(t,\tau)d\tau = 0$;

3) $\int_0^t K(t,\tau)d\tau = 1$.

The properties 1) and 2) are obvious. Let us prove the validity of property 3).

$$\frac{1}{2a\sqrt{\pi}}\int_0^t\sqrt{\frac{t}{\tau}}\,k(t,\tau)\,d\tau = \left|\tau := t\cdot\left(\frac{1-z^2}{1+z^2}\right)^2\right| =$$

$$= \frac{1}{a\sqrt{\pi}}\left[\int_0^1\exp\left\{-\frac{1}{4a^2z^2}\right\}\frac{dz}{z^2}+\int_0^1\exp\left\{-\frac{z^2}{4a^2}\right\}dz\right] =$$

$$= \frac{1}{a\sqrt{\pi}}\left[\int_1^\infty\exp\left\{-\frac{z^2}{4a^2}\right\}dz+\int_0^1\exp\left\{-\frac{z^2}{4a^2}\right\}dz\right] =$$

$$= \frac{1}{a\sqrt{\pi}}\int_0^\infty\exp\left\{-\frac{z^2}{4a^2}\right\}dz = 1.$$

Remark. From this property we conclude that when $\lambda = 1$, $\varphi(t) = C$ (where $C = \text{const}$) is the solution to equation (6).

3 Solving integral equation (6) by means of the Mellin transformation

We rewrite equation (6) in the form

$$\varphi(t) - \lambda\int_0^t\sqrt{\frac{t}{\tau}}\,k_1\left(\frac{t}{\tau}\right)\,\frac{\varphi(\tau)}{\tau}d\tau = 0, \tag{7}$$

$$k_1(\theta) = \frac{1}{2a\sqrt{\pi}} \left\{ \frac{\sqrt{\theta}+1}{\left(\sqrt{\theta}-1\right)^{3/2}} \exp\left(-\frac{\left(\sqrt{\theta}+1\right)^2}{4a^2(\theta-1)} \right) + \right.$$

$$\left. + \frac{\sqrt{\theta}-1}{(\theta-1)^{3/2}} \exp\left(-\frac{\left(\sqrt{\theta}-1\right)^2}{4a^2(\theta-1)} \right) \right\}.$$

Introducing the function $K_1(\theta)$ according to the formula

$$K_1\left(\frac{t}{\tau}\right) = \begin{cases} k_1\left(\dfrac{t}{\tau}\right), & 0 < \tau < t < +\infty; \\ 0, \ 0 < t \leq \tau < +\infty, \end{cases}$$

we rewrite the equation (7) in the form

$$\varphi(t) - \int\limits_0^{+\infty} \frac{\sqrt{t}}{\sqrt{\tau}} K_1\left(\frac{t}{\tau}\right) \frac{\varphi(\tau)}{\tau} \, d\tau = 0. \tag{8}$$

Applying Mellin transformation to the last equation and taking into account the convolution theorem [5; 385]

$$t^{\alpha} \cdot \int\limits_0^{\infty} \tau^{\beta} \cdot f_1\left(\frac{t}{\tau}\right) \cdot f_2(\tau) d\tau \div \widehat{f}_1(s+\alpha) \cdot \widehat{f}_2(s+\alpha+\beta+1),$$

we get

$$\widehat{\varphi}(s) \cdot \left[1 - \lambda \cdot \widehat{K}_1\left(s + \frac{1}{2}\right) \right] = 0,$$

where

$$\widehat{K}_1\left(s+\frac{1}{2}\right) = \frac{1}{a\sqrt{\pi}} \int\limits_0^{\infty} \left\{ \frac{|1-z| \cdot (1+z)}{1+z^2} \right\}^{-2s} \exp\left\{ -\frac{z^2}{4a^2} \right\} dz,$$

$$\widehat{\varphi}(s) = \int\limits_0^{\infty} \varphi(t) \cdot t^{s-1} dt.$$

It is known [5; 180–181], that form of the eigenfunctions of integral equation (8) is determined by the roots of the transcendental equation for the parameter s:

$$\lambda \cdot \widehat{K}_1\left(s + \frac{1}{2}\right) = 1. \tag{9}$$

Eigenfunctions

$$\varphi_k(t) = t^{s_k}$$

correspond to the real (single) roots s_k.

Following [6], we denote by Λ the set of positive numbers λ, and by Σ the set of numbers $s < -\frac{1}{2}$. It is easily seen that equation (9) establishes a one-to-one correspondence between Λ and Σ. This is obvious, since

$$\frac{d}{ds}\widehat{K}_1\left(s+\frac{1}{2}\right) =$$

$$= \frac{1}{a\sqrt{\pi}} \int\limits_0^{\infty} \left\{ \frac{|1-z^2|}{1+z^2} \right\}^{-2s} \exp\left\{ -\frac{z^2}{4a^2} \right\} \ln\left| \frac{1+z^2}{1-z^2} \right| \geq 0,$$

$$\lim_{s\to+\frac{1}{2}}\widehat{K}_1\left(s+\frac{1}{2}\right)=+\infty,\ \lim_{s\to-\infty}\widehat{K}_1\left(s+\frac{1}{2}\right)=0;\ \widehat{K}_1\left(\frac{1}{2}\right)=1.$$

Also we denote by $C_\gamma(R_+)$ Banach space of functions $\varphi(t)\in C(R_+)$, with finite norm

$$\|\varphi\|_\gamma=\max|t^\gamma\varphi(t)|,\quad \gamma=\text{const}.$$

The following theorem holds.

Theorem. For all $\lambda\in\Lambda$ there is a number s such that the functions $\varphi(t)=Ct^s$ (where $C=\text{const}$) are eigenfunctions of equation (6) in the space $C_{-s}(R_+)$. In thies space $C_{-s}(R_+)$ all eigenvalues of equation (6) are simple.

From the theorem and remark it follows that for $\lambda=1$ equation (6) has eigenfunction $\varphi(t)=C=\text{const}$, and initial equation (4) has eigenfunction $\psi(t)=\dfrac{C}{\sqrt{t}}$.

Brief abstract of this work has been published in the Materials of the workshop «Differential operators and modeling of complex systems» (7-8, April, 2017, in Almaty, Kazakhstan) [7].

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Д.М. Ахманова, М.И. Рамазанов, М.Г. Ерғалиев

Шекарасы $\mathbf{t=x^2}$ заңдылығы бойынша өзгеретін жылуөткізгіштік есебінің интегралдық теңдеуі туралы

Мақалада аймақта біртекті екінші ретті Вольтерра интегралдық теңдеуіне келтіретін жылуөткізгіштік біртекті есебінің нөлдік емес шешімінің бар екендігі көрсетілген. Аймақтың шекарасы айнымалы жылдамдықпен жылжиды. Үздіксіз функциялар кластарында әрекет ететін интегралдық оператордың нормасы бірге тең екені көрсетілген. Алынған интегралдық теңдеуге Меллин түрлендіруі қолданылды. Спектралды параметрдің анықталған мәндерінде интегралдық теңдеудің меншікті мәндері қарапайым болатыны дәлелденді.

Кілт сөздер: жылуөткізгіштік теңдеуі, шеттік есептер, өзек, Меллин түрлендіруі, конволюция теоремасы, меншікті функция.

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Об одном интегральном уравнении задачи теплопроводности с границей, движущейся по закону $\mathbf{t = x^2}$

В статье показано, что однородное интегральное уравнение Вольтерра второго рода, к которому сведена однородная краевая задача теплопроводности в вырождающейся области, имеет ненулевое решение. Граница области движется с переменной скоростью. Показано, что норма интегрального оператора, действующего в классах непрерывных функций, равна 1. К полученному интегральному уравнению применяется преобразование Меллина. Доказано, что при определенных значениях спектрального параметра собственные значения интегрального уравнения будут простыми.

Ключевые слова: уравнение теплопроводности, краевые задачи, ядро, преобразование Меллина, теорема о свертке, собственная функция.

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On the boundary value problem for the loaded parabolic equations with irregular coefficients

In the paper we consider the generalized solvability of boundary value problem for the loaded parabolic equations with irregular coefficients. Theorem on unique solvability of the boundary value problem is proved. The correctness of the theorem and the accuracy of selected functional spaces are established by obtained a priori estimates. The proof of the theorem is carried out using the theory of Sobolev spaces, the method of a priori estimates, and the Galerkin method. Along with the initial boundary value problem, the corresponding adjoint boundary value problem is investigated. To prove the solvability of the adjoint problem, we define a linear continuous form and use the duality relations.

Keywords: generalized solvability, boundary value problems, irregular coefficients, a priori estimates, unique solution.

Introduction

It is well-known that one of the central issues in the theory of boundary value problems for partial differential equations is the question on the correct choice of functional spaces. Boundary value problems for the loaded equations were studied systematically in [1, 2]. The questions of existence and uniqueness of solutions of the loaded equations in the class of continuous functions were considered in [1, 2]. In present work we develop these studies.

1 Statement of the of the boundary value problem

Suppose $\Omega \in R^n$ is bounded domain with boundary Γ, $Q = \Omega \times (0,T)$, $\Sigma = \times(0,T)$, Γ is positioned locally on one side of the domain Ω. We consider the following boundary value problem

$$D_t^1 u = \sum_{i,j=1}^{n} D_{x_i}^1 (a_{ij} D_{x_j}^1 u) +$$

$$+ \sum_{i=1}^{p} \nu_i(t) \int_{\Gamma_i} e_i(x,\xi,t) u(\xi,t) d\xi + f = 0 \ \text{ on } \ Q; \tag{1}$$

$$u(x,t) = 0 \quad \text{on } \ \Sigma; \tag{2}$$

$$u(x,0) = u_0 \quad \text{on } \ \Omega, \tag{3}$$

where $e_i \in L^\infty(0,T; L^4(\Omega \times \Gamma_i))$; $\nu_i(t) \in L_2(0,T)$, $i = 1,\ldots,p$, Γ_i; are $(n-1)$ — dimensional manifolds from $\overline{\Omega}$, $n \leq 3$ (for $n = 1, \Gamma_i$ are fixed points from $\overline{\Omega}$); Γ_i, $i = 1,\ldots,n$ together with Γ from C^2; $a_{ij} \in L^\infty(0,T; C^1(\Omega))$, $a_{ij} = a_{ji}$, $i,j = 1,\ldots,p$ for almost every $\{x,t\} \in Q$:

$$\beta_2 \sum_{i=1}^{n} \zeta_i^2 \geq \sum_{i,j=1}^{n} a_{ij} \zeta_i \zeta_j \geq \beta_1 \sum_{i=1}^{n} \zeta_i^2, \tag{4}$$

$\beta_1, \beta_2 = const > 0, \forall \zeta \in R^n,\ f \in L^2(Q), u_0 \in H_0^1(\Omega)$.

2 The theorem of existence and uniqueness of the solution

For the boundary value problem (1)–(3) we obtain a priori estimates to ensure the correctness and accuracy of the selected function spaces.

Theorem 1. Let conditions (4) hold. Then the problem (1)–(3) has a unique solution $u \in Y(0,T)$ for all $f \in L^2(Q)$ and $u_0 \in H_0^1(\Omega)$. Moreover, this solution continuously depends on the initial data, i.e. the map $\{f, u_0\} \to u$ of a direct product of the spaces $L^2(Q) \times H_0^1(\Omega)$ into the space $Y(0,T)$ is continuous, where

$$Y(0,T) = \left\{ u/u \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)),\ \frac{\partial u}{\partial t} \in L^2(Q) \right\}.$$

Proof. Now we take the inner product of equation (1) with Δu

$$\left(\frac{\partial u}{\partial t}, \Delta u\right) = \left(\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right), \Delta u\right) +$$

$$+\left(\sum_{i=1}^{p} \nu_i(t) \int\limits_{\Gamma_i} e_i(x,\xi,t) u(\xi,t) d\xi, \Delta u\right) + (f, \Delta u)\,, \tag{5}$$

where $(\varphi, \psi) = \int\limits_{\Omega} \varphi\psi\, dx$, $\|\varphi\| = [(\varphi,\varphi)]^{\frac{1}{2}}$, Δ is Laplace operator.

Further, we use the following inequality [3]

$$\left(\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right), \Delta u\right) \geq \frac{3\beta_1}{4}\|u_{xx}\|^2 - C_1 \|u_x\|^2\,, \tag{6}$$

for a.e. $t \in [0,T]$ and $\forall u \in Y(0,T)$, $|u_{xx}| = \left(\sum\limits_{i,j=1}^{n} u_{x_i} u_{x_j}\right)^{\frac{1}{2}}$, $C_1 > 0$.

By inequality (6), equality (5) implies that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H_0^1(\Omega)} + \frac{3\beta_1}{4}\|u_{xx}\|^2 - C_1\|u_x\|^2 \ \leq$$

$$\leq \sum_{i=1}^{p} \left| \left(\nu_i(t) \int\limits_{\Gamma_i} e_i(x,\xi,t)\, u(\xi,t) d\xi,\ \Delta u(t) \right) \right| + |\ (f, \Delta u)\ |\,. \tag{7}$$

Applying the Hölder inequality [3], we estimate the first term of the right hand side of (7).

$$\sum_{i=1}^{p} \left| \left(\nu_i(t) \int\limits_{\Gamma_i} e_i(x,\xi,t)\, u(\xi,t) d\xi\,, \Delta u \right) \right| \leq$$

$$\leq \sum_{i=1}^{p} |\nu_i(t)|\, \|e_i(t)\|_{L^4(\Omega\times\Gamma_i)} \sqrt[4]{meas\Omega}\sqrt{meas\Gamma_i} C_i \|u(t)\|_{H^1_{0(\Omega)}} \|\Delta u\|, \tag{8}$$

where C_i satisfies the following inequality

$$\|u(t)\|_{L^4(\Gamma_i)} \leq C_i \|u(t)\|_{H_0^1(\Omega)}\,, \tag{9}$$

for a.e. $t \in (0,T)$, $i = 1, \ldots, p$ [4].

Then the relation (7) implies

$$\frac{1}{2}\frac{d}{dt}\left\|u(t)\right\|^2_{H^1_0(\Omega)}+\frac{3\beta_1}{4}\left\|u_{xx}\right\|^2-C_1\left\|u_x\right\|^2 \le$$
$$\le C(t)\left\|u\right\|_{H^1_0(\Omega)}\left\|\Delta u\right\|+|\,(f,\Delta u)\,|,$$

where $C(t)=\sum\limits_{i=1}^{p}|\nu_i(t)|\,\|e_i(t)\|_{L^4(\Omega\times\Gamma_i)}\,C_i\sqrt[4]{\mathrm{meas}\,\Omega}\sqrt{\mathrm{meas}\,\Gamma_i}$.

Further, by using the Cauchy inequality with ε [3], we have

$$\frac{1}{2}\frac{d}{dt}\left\|u(t)\right\|^2_{H^1_0(\Omega)}+\frac{3\beta_1}{4}\left\|u_{xx}(t)\right\|^2-C_1\left\|u_x(t)\right\|^2 \le$$
$$\le\frac{C^2(t)}{2\varepsilon_1}\left\|u(t)\right\|^2_{H^1_0(\Omega)}+\frac{\varepsilon_1}{2}C_2\left\|u_{xx}(t)\right\|^2+\frac{1}{2\varepsilon_2}\left\|f(t)\right\|^2+\frac{\varepsilon_2C_2}{2}\left\|u_{xx}(t)\right\|^2,$$

where C_2 is a constant in $\|\Delta u(t)\|\le\sqrt{C_2}\,\|u_{xx}(t)\|$. By choosing $\varepsilon_1,\varepsilon_2$ from conditions $\varepsilon_1C_2+\varepsilon_2C_2\le\frac{\beta_1}{4}$, we have

$$\frac{1}{2}\frac{d}{dt}\left\|u(t)\right\|^2_{H^1_0(\Omega)}+\frac{\beta_1}{2}\left\|u_{xx}(t)\right\|^2 \le$$
$$\le K_1(t)\left\|u(t)\right\|^2_{H^1_0(\Omega)}+\frac{1}{2\varepsilon_2}\left\|f(t)\right\|^2, \tag{10}$$

where $K_1(t)=C_1+\frac{C^2(t)}{2\varepsilon_1}$. Inequality (10) implies

$$\frac{d}{dt}\left\|u(t)\right\|^2_{H^1_0(\Omega)}\le K(t)\left\|u(t)\right\|^2_{H^1_0(\Omega)}+\frac{1}{\varepsilon_2}\left\|f(t)\right\|^2, \tag{11}$$

where $K(t)=2C_1+\frac{C^2(t)}{\varepsilon_1}$.

Further, we use the Gronwall lemma [3]. Multiplying both sides of inequality (11) by $e^{-\int\limits_0^t K(\tau)d\tau}$, we transfer the first term on the right hand side to the left hand side. So we obtain

$$\frac{d}{dt}\left[\left\|u(t)\right\|^2_{H^1_0(\Omega)}e^{-\int\limits_0^t K(\tau)d\tau}\right]\le\frac{1}{\varepsilon_2}\left\|f(t)\right\|^2e^{-\int\limits_0^t K(\tau)d\tau}.$$

By integrating from 0 to t and by exploiting the fact that $u(x,0)=u_0$, we have

$$\left\|u(t)\right\|^2_{H^1_0(\Omega)}\le\left\|u_0\right\|^2_{H^1_0(\Omega)}e^{\int\limits_0^t K(\tau)d\tau}+\frac{1}{\varepsilon_2}\int\limits_0^t e^{\int\limits_\theta^t K(\tau)d\tau}\left\|f(\theta)\right\|^2d\theta.$$

Hence, it follows that

$$\left\|u\right\|^2_{L^\infty(0,T;H^1_0(\Omega))}\le const\left(\left\|u_0\right\|^2_{H^1_0(\Omega)}+\int\limits_0^T\left\|f(t)\right\|^2dt\right). \tag{12}$$

Substituting inequality (12) into the right hand side of inequality (10) by a standard way, we have

$$\left\|u\right\|_{L^2(0,T;H^2(\Omega)\bigcap H^1_0(\Omega))\bigcap L^\infty(0,T;H^1_0(\Omega))}\le$$
$$\le const\left(\left\|u_0\right\|_{H^1_0(\Omega)}+\int\limits_0^T\left\|f(t)\right\|dt\right). \tag{13}$$

By inequalities (12) and (13), and equation (1) and by

$$\nu_i(t)\int\limits_{\Gamma_i}e_i(x,\xi,t)u(\xi,t)d\xi\in L^2(Q),$$

we obtain the following estimate

$$\|u\|_{L^2(0,T;H^2(\Omega)\bigcap H_0^1(\Omega))} \leq K\left(\|u_0\|_{H_0^1(\Omega)} + \int\limits_0^T \|f(t)\|\,dt\right). \tag{14}$$

For the further study we apply the Galerkin method. Let $w_1, w_2, \ldots, w_N, \ldots$, be a basis of the space $H^2(\Omega) \cap H_0^1(\Omega)$, i.e. the elements $w_1, w_2, \ldots, w_N$ are linear independent for all $N \in \mathbb{N}$; the set of linear combinations $\sum \xi_j w_j$ is dense in $H^2(\Omega) \cap H_0^1(\Omega)$ (ξ_i are constants).

We define the approximate solution of the problem (1)–(3) in the following form

$$u_N(x,t) = \sum_{i=1}^{N} g_{iN}(t) w_i(x), \tag{15}$$

where the functions $g_{iN}(t),\ i = 1, \ldots, N$ are chosen such that the following relations hold

$$\left(\frac{\partial u_N}{\partial t}, w_j\right) + \sum_{i,j=1}^{n}\left(a_{ij}\frac{\partial u_N}{\partial x_i}, \frac{\partial w_j}{\partial x_j}\right) +$$

$$+\sum_{i=1}^{p}\left(\nu_i(t)\int\limits_{\Gamma_i} e_i u_N d\xi, w_j\right) = (f, w_j)\,,\ \ 1 \leq i \leq N, \tag{16}$$

$u_N(x,0) = u_{0N}(x) = \sum\limits_{i=1}^{n} \eta_{iN} w_i$, where $\{\eta_{iN}\}$ such that

$$\sum_{i=1}^{n} \eta_{iN} w_i \to u_0 \ \text{ in } \ H_0^1(\Omega) \ \text{ when } \ N \to \infty. \tag{17}$$

The relations (16), (17) are the Cauchy problem for systems of linear differential equations for the functions $g_{iN}(t)$.

$$W_N \frac{dg_N}{dt} + A_N(t) g_N = f_N,\ g_N(0) = \{\eta_{iN}\},$$

where

$$W_N = \|(w_i, w_j)\|\,, \quad A_N(t) = \left\|\sum_{i,j=1}^{n}\left(a_{ij}\frac{\partial w_i}{\partial x_i}, \frac{\partial w_j}{\partial x_j}\right) +\right.$$

$$\left.+\sum_{i=1}^{p}\left(\nu_i(t)\int\limits_{\Gamma_i} e_i(x,\xi,t) w_i(\xi) d\xi, w_j\right)\right\|;$$

$$g_N(t) = \{g_{iN}(t)\}\,, f_N = \{(f, w_j)\}.$$

Since W_N is Gramian matrix, and consequently $\det W_N \neq 0$, for the finite number $\forall N$, then the problem (16), (17) has an unique absolutely continuous solution.

We show, that if $N \to \infty$, then $u_N \to u$, and u is the solution of the problem (1)–(3). Due to the fact that for the approximate solution $u_N(x,t)$ of the equation (15) for each N has a priori estimate of the form (14), then from the bounded sequence $\{u_N(x,t)\}_{N=1}^{\infty}$ one can take out a subsequence $\{u_\mu(x,t)\}_{\mu=1}^{\infty}$ such that

$$u_\mu \to z \ \text{ weakly in } \ H^{2,1}(Q). \tag{18}$$

By theorem on traces and by lemma on linear continuous mapping of weakly convergent sequences [4], we have

$$u_\mu \to z \ \text{ weakly in } \ H^{\frac{3}{2},\frac{3}{4}}(\Gamma_i(t) \times (0,T)). \tag{19}$$

The relation (19) implies that

$$u_\mu \to z \ \text{ stronger in } \ L^2(\Gamma_i(t) \times (0,T)). \tag{20}$$

Let now j be arbitrary fixed number and $\mu > j$. Since the relation (15) holds for $N = \mu$, then multiplying it by function $\varphi(t) \in \Phi = \left\{\varphi/\varphi \in C^1[0,T], \varphi(T) = 0\right\}$ and integrating from 0 to T, we obtain

$$\int\limits_0^T \left[\left(\frac{\partial u_N}{\partial t}, \varphi w_j\right) + \sum_{i,j=1}^n \left(a_{ij}\frac{\partial u_N}{\partial x_j}, \varphi\frac{\partial w_j}{\partial x_i}\right) + \right.$$

$$\left. + \sum_{i=1}^p \left(\nu_i(t)\int\limits_{\Gamma_i} e_i u_N d\xi, \varphi w_j\right) - (f, \varphi w_j)\right] dt = 0. \qquad (21)$$

Then integrating by part, we have

$$\int\limits_0^T \left[-\left(u_N, \frac{\partial \varphi w_j}{\partial t}\right) + \sum_{i,j=1}^n \left(a_{ij}\frac{\partial u_N}{\partial x_j}, \varphi\,\frac{\partial w_j}{\partial x_i}\right) + \right.$$

$$\left. + \sum_{i=1}^p \left(\nu_{\,i}(t)\int\limits_{\Gamma_i} e_i u_N d\xi, \varphi w_j\right) - (f, \varphi w_j)\right] dt =$$

$$= -(u_N(x,0), \varphi(0) w_j(x)),\ \forall\ i,\ j.$$

We take as $\varphi \in D((0,T)) \subset \Phi$ the function that is infinitely differentiable and finite function. Then, by taking the limit as $\mu \to \infty$ (that is possible by relations (18), (20) we obtain

$$\int\limits_0^T \left[-\left(z, \frac{\partial \varphi w_j}{\partial t}\right) + \sum_{i,j=1}^n \left(a_{ij}\frac{\partial z}{\partial x_i}, \varphi\frac{\partial w_j}{\partial x_j}\right) + \right.$$

$$\left. + \sum_{i=1}^p \left(\nu_i(t)\int\limits_{\Gamma_i} e_i z d\xi, \varphi w_i\right) - (f, \varphi w_j)\right] dt = 0,\ \forall j, \varphi \in D((0,T)) \qquad (22)$$

and

$$\int\limits_0^T \left[-\left(z, w_j\right)\,\varphi' + \sum_{i,j=1}^n \left(a_{ij}\frac{\partial z}{\partial x_j}, \frac{\partial w_j}{\partial x_i}\right)\varphi + \right.$$

$$\left. + \sum_{i=1}^p \left(\nu_{\,i}(t)\int\limits_{\Gamma_i} e_i z d\xi, w_j\right)\varphi - (f, w_j)\,\varphi\right] dt = 0.$$

By definition of the Schwarzian derivative we have

$$\int\limits_0^T \frac{\partial}{\partial z}\left[(z, w_j) + \sum_{i,j=1}^n \left(a_{ij}\frac{\partial z}{\partial x_j}, \frac{\partial w_j}{\partial x_i}\right) + \right.$$

$$\left. + \sum_{i=1}^p \left(\nu_{\,i}(t)\int\limits_{\Gamma_i} e_i z d\xi,\, w_j\right) - (f, w_j)\right]\varphi(t) dt = 0,\ \forall j, \varphi \in D((0,T)),$$

where $\frac{\partial}{\partial t}(z, w_j) \in D'((0,T))$.

It is well known that $(F, \varphi) = 0$ for all $\varphi \in D((0,T)) \Leftrightarrow \quad F = 0 \in D'((0,T))$, consequently,

$$\left(\frac{\partial z}{\partial t}, w_j\right) + \sum_{i,j=1}^n \left(a_{ij}\frac{\partial\, z}{\partial x_j}, \frac{\partial w_j}{\partial x_i}\right) + \qquad (23)$$

$$+ \sum_{i=1}^p \left(\nu_i(t)\int\limits_{\Gamma_i} e_i z(\xi, t) d\xi, w_j\right) = (f, w_j).$$

Further, since j is arbitrary, and the set of linear combinations of elements $w_1, w_2, \ldots, w_N, \ldots$ is dense in $H^2(\Omega) \cap H_0^1(\Omega)$. Then the relation (23) implies that

$$\frac{\partial z}{\partial t} + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial z}{\partial x_j}\right) + \sum_{i=1}^{p} \nu_i(t) \int\limits_{\Gamma_i} e_i z(\xi, t) d\xi = f. \tag{24}$$

Hence $\frac{\partial z}{\partial t} \in L^2(Q)$, thus $z \in Y(0,T)$.

For final derivation one can examine as z satisfies the initial conditions. We take $\varphi \in \Phi$ such, that it is not necessary to be the finite function. Then as $N \to \infty$ we have

$$\int\limits_0^T \left[-\left(z, \frac{\partial \varphi w_j}{\partial t}\right) + \sum_{i,j=1}^{n} \left(a_{ij}\frac{\partial z}{\partial x_j}, \varphi \frac{\partial w_j}{\partial x_i}\right) + \right.$$

$$\left. + \sum_{i=1}^{p} \left(\nu_i(t) \int\limits_{\Gamma_i} e_i z(\xi, t) d\xi, \varphi w_j\right) - (f, \varphi w_j) \right] dt =$$

$$= -(u_0(x), \varphi(0) w_j(x)), \ \forall j \ \text{ and for } \varphi \in \Phi. \tag{25}$$

Integrating by part the relation (25), we have

$$\int\limits_0^T \left[\left(\frac{\partial z}{\partial t} + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial z}{\partial x_j}\right) + \right.\right.$$

$$\left.\left. + \sum_{i=1}^{p} \nu_i(t) \int\limits_{\Gamma_i} e_i z(\xi, t) d\xi, \omega_j - f \right) \varphi(t) \right] =$$

$$= -(u_0(x), w_j)\varphi(0) - (z(x,0), w_j)\varphi(0).$$

By (24), we have $-(u_0(x), w_j) - (z(x,0), w_j) = 0$, $\forall j$. Consequently, namely $u_0(x) = z(x,0)$ is the solution of the boundary value problem (1)–(3). The proof of theorem 1 is complete.

3 The adjoint problem

Let

$$r \in Y_0', \ Y_0 = \{u/u \in Y(0,T), \ u(x,0) = 0\}, \ p_1 \in H^{-1}(\Omega). \tag{26}$$

We consider the problem that is adjoint of the problem (1)–(3)

$$D_t^1 p + \sum_{i,j=1}^{n} D_{x_j}^1 \left(a_{ij} D_{x_i}^1 p\right) +$$

$$+ \sum_{i=1}^{p} \nu_i(t) \delta(x - \Gamma_i) \int\limits_{\Omega} e_i(\xi, x, t) p(\xi, t) d\xi = r(x,t) \ \text{ on } \ Q; \tag{27}$$

$$p(x,t) = 0 \quad \text{on} \ \ \Sigma, \tag{28}$$

$$p(x,T) = p_1, \tag{29}$$

where $\delta(x - \Gamma_i)$ — is the Dirac function. In order to prove solvability of the problem (27)–(29) we use the schema 2 from [4].

Under the condition of theorem 1 and by $\nu_i(t) \in L_2(0,T)$ the following operator

$$u \to \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^{p} \nu_i(t) \int\limits_{\Gamma_i} e_i u(\xi, t) d\xi$$

corresponding to the problem (1)–(3) defines the homomorphism $Y(0,T) \to L^2(Q)$ [5]. Consequently, if $\sigma(u)$ is linear continuous form above $Y(0,T)$, then by Riesz theorem [3] there exists an unique element $p(\nu) \in L^2(Q)$, such that $\forall u \in Y_0$

$$\left(p(\nu), \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^{p} \nu_i(t) \int_{\Gamma_i} e_i u(\xi,t) d\xi\right) = \sigma(u). \tag{30}$$

We define the following linear continuous form

$$\sigma(u) = \langle\langle r(x,t), u(x,t)\rangle\rangle + \langle p_1, u(x,T)\rangle\,, \tag{31}$$

where $\langle\langle\,.\,,\,.\rangle\rangle$ and $\langle\,.\,,\,.\,\rangle$ are duality relations between spaces Y_0' and Y_0 , $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, respectively.

Thus we state the following intermediate result.

Theorem 2. The problem (27)–(29) has unique solution $p(x,t) \in L^2(Q)$, that satisfies the integral identity (30)–(31) for all $\{r, p_1\}$, satisfying (26).

Let us give some of possible descriptions of the elements $r(x,t) \in Y_0'$. We have

$$r(x,t) = r_0(x,t) + \frac{d}{dt}\left(\rho(t), r_1(x,t)\right);$$

$$r_0(x,t) \in L^2(0,T;(H^2(\Omega)\cap H_0^1(\Omega))'), r_1 \in L^2(Q),$$

where $\rho(t)$ is infinitely differentiable function, defined, for example, by the following way ($0 < t_0 < \frac{1}{2}$, t_0 is fixed):

$$\rho(t) = \begin{cases} t, & 0 \le t \le t_0; \\ \text{arbitrary}, & t_0 \le t \le T - t_0; \\ T - t, & T - t_0 \le t \le T. \end{cases}$$

Indeed, the estimates, that analogous with the estimates (12)–(14), hold for boundary value problem (27)–(29). And the following more stronger result is true. Let $r = 0$, we state

Theorem 3. Let condition (4) hold. Then the problem (27)–(29) has unique solution $p \in Y(0,T)$ for all $p_1 \in H_0^1(\Omega)$. This solution continuously depends on initial data, i.e. the map of the space $H_0^1(\Omega)$ into $Y(0,T)$ is continuous.

The proof of theorem 3 is similar with the proof of theorem 1.

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Регулярлы емес коэффициентті жүктелген параболалық теңдеулер үшін бір шеттік есеп жайында

Мақалада регулярлы емес коэффициенттері бар жүктелген параболалық теңдеулер үшін шекаралық есептің жалпыланған шешімділігі қарастырылды. Шекаралық есептердің бірегей шешілетіндігі туралы теорема дәлелденді. Теореманың дұрыстығы және таңдалған функционалдық кеңістіктердің дәлдігі алынған априорлы бағалаулармен анықталды. Теореманы дәлелдеу Соболев кеңістігінің теориясын, априорлы бағалау әдісін және Галеркин әдісін қолдана отырып жүзеге асырылды. Бастапқы шекаралық есеппен қатар, сәйкес түйіндес шекаралық есеп зерттелді. Түйіндес есептің шешімділігін дәлелдеу үшін сызықты үзіліссіз форма анықталды және түйіндестік қатынастар қолданылды.

Кілт сөздер: жалпыланған шешімділік, шекаралық есептер, регулярлы емес коэффициенттер, априорлы бағалаулар, жалғыз шешім.

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Об одной краевой задаче для нагруженных параболических уравнений с нерегулярными коэффициентами

В статье рассмотрена обобщенная разрешимость краевой задачи для нагруженных параболических уравнений с нерегулярными коэффициентами. Доказана теорема о единственной разрешимости краевой задачи. Корректность теоремы и точность выбранных функциональных пространств определены полученными априорными оценками. Доказательство теоремы проводится с использованием теории пространств Соболева, метода априорных оценок и метода Галеркина. Наряду с исходной краевой задачей исследуется соответствующая ей сопряженная краевая задача. Для доказательства разрешимости сопряженной задачи задается линейная непрерывная форма и используются соотношения двойственности.

Ключевые слова: обобщенная разрешимость, краевые задачи, нерегулярные коэффициенты, априорные оценки, единственное решение.

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Estimation of randomity of generators of keys of the crypto system

The aim of the study is to develop an algorithm for constructing an integer random variable representing random components of the keys of the RSA cryptosystem, and determining the table for the distribution of its probabilities. The constructed random variable can be used to estimate the hypothesis that a given generator generates its values with the same probability with which a random variable takes these values. A hypothesis testing algorithm is provided, which is accompanied by a calculation table for a uniformly distributed random variable that takes primes from a given half-open interval.

Keywords: RSA-cipher, keys, random variable, prime number, probability distribution.

Introduction

To ensure that the practical stability of the cryptosystem is consistent with its theoretical stability, it is necessary to fulfill all the requirements for using this cryptosystem. The most important requirement is the requirement of generating by generator keys with a frequency corresponding to the calculated probability. If the random elements and groups of interrelated random elements can be represented as the corresponding discrete random variables and their probability distribution can be determined, then it is possible to evaluate the hypothesis on the feasibility of requirements for the correctness of operation to this generator. In this paper, we propose an algorithm for constructing a random variable representing interrelated elements p, q, a of key of cipher RSA, where p, q are prime numbers, a is mutually prime with the numbers $(p-1)(q-1)$. The second part provides an algorithm for testing the hypothesis that this generator generates its values with a frequency that tends in probability to the probabilities of the corresponding values of a given random variable with unlimited increase in the number of output values (generation of keys by generator with a frequency which corresponds to the calculated probability). The algorithm is accompanied by a calculation table for verifying the above hypothesis for several generators, generating prime numbers from a given half-interval and for a uniformly distributed random variable, taking the prime values from this half-interval. Note that the uniform distribution of a random variable has not values when evaluating a hypothesis. Note also that the random variable constructed in this work for the RSA cipher is not uniformly distributed, while this requirement is the main one in the works [1, 2].

1 Algorithm for constructing a random variable

Let p, q be random prime numbers from the integer half-interval $\mathfrak{G} = (0, G]$, a is a random number, $1 \leqslant a < \varphi(pq)$, mutually prime number with $\varphi(pq) = (p-1)(q-1)$, i.e. $(a, \varphi(pq)) = 1$. Three random numbers p, q, a are random elements that generate the key $k(p, q, a)$ of the cipher RSA. These random numbers are dependent: the choice of p implies the elimination of p from the set of values for q; the choice of p, q determines the possible values and the corresponding probabilities for a. We construct a discrete random variable $\xi(p, q, a)$ such that the probability $\mathcal{P}(p, q, a) = \mathcal{P}(\xi(p, q, a))(*)$. Since in each key $k(p, q, a)$ of the cipher RSA the following inequality holds $p \neq q$, and also the following equality holds $k(p, q, a) = k(q, p, a)$, it suffices to define $\xi(p, q, a)$ with the condition (*) for $p < q$. Further $\xi(p, q, a)$ must be one-to-one because a partition on a probability space of triples (p, q, a) must correspond to a partition on the number axis and vice versa.

Proposition. Let $\xi(p,q,a) = pqa^2$ be prime numbers , where p, q and $p < q < G, a < \varphi(pq), (a, \varphi(p,q)) = 1$. Then the function ξ is one-to-one.

Proof. Let $(p, q, a) \neq (p_1, q_1, a_1)$. Case 1. Let $(p, q) = (p_1, q_1)$. Then $a \neq a_1$ and hence $pqa^2 \neq p_1 q_1 a_1^2$. Case 2.1. Let $(p, q) \neq (p_1, q_1), p \neq p_1, q \neq q_1$. Then $pq \neq p_1 q_1$ and let $pq < p_1 q_1$. We assume $pqa^2 = p_1 q_1 a_1^2$, then $p_1q_1|a^2$, from here $p_1q_1|a$. But $a < pq < p_1q_1$ and hence $a = 0$, contradiction. Case 2.2. Let $(p,q) \neq (p_1,q_1)$, $p = p_1, q \neq q_1$. We assume $pqa^2 = p_1q_1a_1^2$, then $qa^2 = q_1a_1^2$, hence $q|a_1^2$ and $q^2|a_1^2$. Let k be such that $q^{2k}|a_1^2$, and $q^{2(k+1)} \nmid a_1^2$, then $q^{2k+1}|a^2$ и $q^{2(k+1)}|a^2$. From here $q^{2(k+1)}|a_1^2$. Contradiction. As the case 2.2, the case 2.3 is considered $(p,q) \neq (p_1,q_1), p \neq p_1, q = q_1$. Proposition is proved.

Description of the algorithm

Let π be the number of primes in the half-interval $\mathfrak{G}$. The algorithm consists of $\pi - 1$ stages, there are exactly so many primes $p \in \mathfrak{G}$, for which there exists $q \in \mathfrak{G}$ such that $p < q$. At each stage i the probability $\mathcal{P}(p_i)$ of choice for i-th prime number p_i is determined, it equals to $\frac{1}{\pi-1}$, and $\pi - i$ is number of sub-stages. That is on the i-th stage $\pi - i$ of sub-stages are fulfilled to determine the probabilities of the choice of numbers of the form p_iq, where $p_i < q$. For i-th prime number p_i there are exactly $\pi - i$ prime numbers q, for which $p_i < q$. Then, at each sub-stage of the stage i we define q and probabilities $\mathcal{P}(p_iq) = \frac{1}{(\pi-1)(\pi-i)}$ (see Fig. 1). Further, at each sub-stage of the stage i with a certain number p_iq, $\varphi^2(p_iq)$ steps are taken, that complete the calculation of probabilities $\mathcal{P}(\xi = p_iqa^2) = \frac{1}{(\pi-1)(\pi-i)\varphi^2(p_iq)}$, where $a \in M_{p_iq}$, $M_{p_iq} = \{a : 1 \leq a < p_iq, (a, \varphi(p_iq)) = 1\}$.

p	$\mathcal{P}(p)$	q	$\mathcal{P}(q)$	pq	$\mathcal{P}(pq)$	$\varphi(pq)$	$\varphi^2(pq)$	$\mathcal{P}(a^2),\ a \in M_{pq}$	$\mathcal{P}(pqa^2),\ a \in M_{pq}$
2	$\frac{1}{(\pi-1)}$	3	$\frac{1}{(\pi-1)}$	6	$\frac{1}{(\pi-1)^2}$	2	1	1	$\frac{1}{(\pi-1)^2}$
		...	...	...	...	...	...	...	...
		p_π	$\frac{1}{(\pi-1)}$	$2p_\pi$	$\frac{1}{(\pi-1)^2}$	$p_\pi - 1$	$\varphi(p_\pi - 1)$	$\frac{1}{\varphi(p_\pi-1)}$	$\frac{1}{\varphi(p_\pi-1)(\pi-1)^2}$
...	...	...	...	...	...	...	...	...	...
$p_{\pi-2}$	$\frac{1}{(\pi-1)}$	$p_{\pi-1}$	$\frac{1}{2}$	$p_{\pi-2}\cdot$ $\cdot p_{\pi-1}$	$\frac{1}{2(\pi-1)}$	$\varphi(p_{\pi-2}p_{\pi-1})$	$\varphi^2(p_{\pi-2}p_{\pi-1})$	$\frac{1}{\varphi^2(p_{\pi-2}p_{\pi-1})}$	$\frac{1}{\varphi^2(p_{\pi-2}p_{\pi-1})2(\pi-1)}$
		p_π	$\frac{1}{2}$	$p_{\pi-2}\cdot$ $\cdot p_\pi$	$\frac{1}{2(\pi-1)}$	$\varphi(p_{\pi-2}p_\pi)$	$\varphi^2(p_{\pi-2}p_\pi)$	$\frac{1}{\varphi^2(p_{\pi-2}p_\pi)}$	$\frac{1}{\varphi^2(p_{\pi-2}p_\pi)2(\pi-1)}$
$p_{\pi-1}$	$\frac{1}{(\pi-1)}$	p_π	1	$p_{\pi-1}\cdot$ $\cdot p_\pi$	$\frac{1}{(\pi-1)}$	$\varphi(p_{\pi-1}p_\pi)$	$\varphi^2(p_{\pi-1}p_\pi)$	$\frac{1}{\varphi^2(p_{\pi-1}p_\pi)}$	$\frac{1}{\varphi^2(p_{\pi-1}p_\pi)(\pi-1)}$

Figure 1. Calculation of the probability for distribution ξ

Thus, a discrete probability distribution for a random variable $\xi(p,q,a) = pqa^2$ will be determined. On the Figure 2 application of the algorithm for $G = 10$ is considered.

p	$\mathcal{P}(p)$	q	$\mathcal{P}(q)$	pq	$\mathcal{P}(pq)$	$\phi(pq)$	$a : a \in M_{pq}$	$a^2 : a \in M_{pq}$	$\mathcal{P}(a^2)$	$pqa^2 : a \in M_{pq}$	$\mathcal{P}(pqa^2)$
2	$\frac{1}{3}$	3	$\frac{1}{3}$	6	$\frac{1}{9}$	2	1	1	1	6	$\frac{1}{9}$
		5	$\frac{1}{3}$	10	$\frac{1}{9}$	4	1,3	1,9	$\frac{1}{2}$	$10 \cdot 1, 10 \cdot 9$ $10, 90$ (1.4)	$\frac{1}{18}$
		7	$\frac{1}{3}$	14	$\frac{1}{9}$	6	1,5	1,25	$\frac{1}{2}$	$14 \cdot 1, 14 \cdot 25$ $14, 350$ (1.5)	$\frac{1}{18}$
3	$\frac{1}{3}$	5	$\frac{1}{2}$	15	$\frac{1}{6}$	8	1,3,5,7	1,9,25,49	$\frac{1}{4}$	$15 \cdot 1, 15 \cdot 9, 15 \cdot 25, 15 \cdot 49$ 15,135,375,735 (1.6)	$\frac{1}{24}$
		7	$\frac{1}{2}$	21	$\frac{1}{6}$	12	1,5,7,11	1,25,49,121	$\frac{1}{4}$	$21 \cdot 1, 21 \cdot 25, 21 \cdot 49, 21 \cdot 121$ 21,525,1029,2541 (1.7)	$\frac{1}{24}$
5	$\frac{1}{3}$	7	1	35	$\frac{1}{3}$	24	1,5,7,11,13,17, 19,23	1,25,49,121, 169,289 361,529	$\frac{1}{8}$	$35 \cdot 1, 35 \cdot 25, 35 \cdot 49, 35 \cdot 121,$ $35\cdot169, 35\cdot289, 35\cdot361, 35\cdot529$ 35,875,1715,4235,5915, (1.8) 10115,12635,18515	$\frac{1}{24}$

Figure 2. Calculation of the probability distribution $\xi, n = 10$

On the Figure 3 a discrete probability distribution function is given, the calculation of which is given on the Figure 2.

$$\begin{bmatrix} \frac{1}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{18} & \frac{1}{24} & \frac{1}{18} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} \\ 6 & 10 & 14 & 15 & 21 & 35 & 90 & 135 & 350 & 375 & 525 & 735 & 875 & 1029 & 1715 & 2541 & 4235 & 5915 & 10115 & 12635 & 18515 \end{bmatrix}$$

Figure 3. Probability distributions $\xi, n = 10$

The first line in Figure 3 is the probabilities of values of a random variable $\xi(p,q,a) = pqa^2$, located in the second line.

2 Evaluation of the generator

Suppose that for a random discrete quantity ξ, whose values lie in the half-interval $\mathfrak{G} = (0, G]$, the probability of acceptance of each of its values is known. Let Ω be sample of size $\mathrm{Dim}\{\Omega\}$ of generator $\mathfrak{F}$ for numbers from domain of values for ξ.

Hypothesis $\mathbb{H}$: $\mathfrak{F}$ generates values with frequencies tending in probability to the probabilities of the corresponding values of the random variable ξ, with an unlimited increase in size $\mathrm{Dim}\{\Omega\}$ of sample Ω.

Checking algorithm $\mathbb{H}$.

1. Run random choice of size for sample $\mathrm{Dim}\{\Omega\}$, $100 \leq \mathrm{Dim}\{\Omega\}$ and number of partitions r, $2 \leq r \leq 100$.

2. Run random partition $\mathfrak{G}_r$ of the half-interval $\mathfrak{G}$ by a random choice $r-2$ of points in $\mathfrak{G} : \mathfrak{G}_r = (\mathfrak{g}(1) = 0, \mathfrak{g}(1), ..., \mathfrak{g}(r-1), \mathfrak{g}(r) = \mathfrak{G}$, where $g(1) < g(2) < ... < g(r-1) < g(r)$.

3. Create an array of frequencies $\mathrm{Freq}\{\Omega_r\} = (\mathrm{Freq}\{\Omega_r\}[1], ..., \mathrm{Freq}\{\Omega_r\}[r])$, where $\mathrm{Freq}\{\Omega_r\}[i]$ is the number of prime numbers in the sample Ω that fall into the half-interval $(g(i), g(i+1)]$, $i = 1, ..., r-1$.

4. Create an array of probabilities $\mathcal{P}\xi_r = (\mathcal{P}\xi_r[1], ..., \mathcal{P}\xi_r[r-1])$, where $\mathcal{P}\xi_r[i]$ is the probability of falling ξ into the half-interval $(g(i), g(i+1)]$, $i = 1, ..., r-1$.

5. Compute value χ^2 for the array $\mathrm{Freq}\{\Omega_r\}$: $\chi^2(\mathrm{Freq}\{\Omega_r\}) = \sum_{i=1}^{r-1} \dfrac{\mathrm{Freq}\{\Omega_r^2\}[i]}{\mathrm{Dim}\{\Omega\} \cdot \mathcal{P}\xi_r[i]}$.

6. Test the hypothesis $\mathbb{H}$. If for a random variable χ^2 with $r-2$ degrees of freedom the probability $\mathcal{P}_{\mathfrak{F}}(r) = \mathcal{P}(\chi^2 \geq \chi^2(\mathrm{Freq}\{\Omega_r\})$, then there is reason to reject the hypothesis $\mathbb{H}$. Since it is considered unlikely in one test to obtain an event whose probability is less than 5%. End of algorithm.

Note that the number r is recommended to be taken in such a way that one of the conditions: i) $\chi^2 \geq \chi^2(\mathrm{Freq}\{\Omega_r\} \geq 10$; ii) $\mathrm{Freq}\{\Omega[i]\} \geq 10, i = 1, ..., r-1$ is satisfied. The conditions (i), (ii) are necessary for the practical application of the χ^2-Pearson criterion [3]. Repeated application of this algorithm to evaluate this generator with similar data may strengthen or weaken the grounds for rejecting the hypothesis.

3 Examples of estimates of prime numbers generators

Let the uniformly distributed discrete random variable ξ takes as values all prime numbers in the half-interval $\mathfrak{G} = (0, G]$ with probability $\mathcal{P}(\xi = p) = \int_2^G \frac{1}{x} dx$, (asymptotic formula for the number of primes in the interval $\mathfrak{G}$) [4]. Let Ω be sample of size $\mathrm{Dim}\{\Omega\}$ for generator $\mathfrak{F}$ of primes in the interval $\mathfrak{G}$. Let us check the following hypothesis $\mathbb{H}$. $\mathbb{H}$: $\mathfrak{F}$ generates every prime number $p \in \mathfrak{G}$ with probability $\mathcal{P}(\xi = p)$. Figure 4 shows the results of the implementation of the algorithm for estimating 5 generators of prime numbers:

<table>
<tr><th>G</th><th>r</th><th>Dim</th><th>A Type</th><th>δ</th><th>$\mathcal{P}_{\mathfrak{F}}(r)$</th><th>H</th></tr>
<tr><td rowspan="2">10^3</td><td rowspan="2">6</td><td rowspan="2">100</td><td>Generator of Maple 13:
Random selection of prime numbers – Ω</td><td>< 2.343
> 1.649</td><td>$>70\%$
< 80</td><td>Do not reject</td></tr>
<tr><td>$\Delta[i] = \begin{cases} \text{random prime numbers, } \textit{if i odd} \\ p_{i2}, \textit{if i even} \end{cases}$</td><td>$< 11.668$
> 9.488</td><td>$< 5\%$
$> 2\%$</td><td>Rather Reject</td></tr>
<tr><td rowspan="2">10^4</td><td rowspan="2">6</td><td rowspan="2">100</td><td>Generator of Maple 13:
Random selection of prime numbers – Ω</td><td>> 2.195
< 3.357</td><td>$>50\%$
< 70</td><td>Do not reject</td></tr>
<tr><td>$\Delta[i] = \begin{cases} \text{random prime numbers, } \textit{if i odd} \\ p_{10\cdot t/2}, \textit{i even} \end{cases}$</td><td>$> 18.465$</td><td>$< 0.1\%$</td><td>Rather Reject</td></tr>
<tr><td rowspan="2">10^5</td><td rowspan="2">6</td><td rowspan="2">100</td><td>Generator of Maple 13:
Random selection of prime numbers – Ω</td><td>> 2.195
< 3.357</td><td>$>50\%$
< 70</td><td>Do not reject</td></tr>
<tr><td>$\Delta[i] = \begin{cases} \text{random prime numbers, } \textit{if i odd} \\ p_{10^2\cdot t/2}, \textit{if i even} \end{cases}$</td><td>$> 18.465$</td><td>$< 0.1\%$</td><td>Rather Reject</td></tr>
<tr><td rowspan="2">10^6</td><td rowspan="2">7</td><td rowspan="2">100</td><td>Generator of Maple 13:
Random selection of prime numbers – Ω</td><td>> 2.343
> 1.610</td><td>$>80\%$
< 90</td><td>Do not reject</td></tr>
<tr><td>$\Delta[i] = \begin{cases} \text{random prime numbers, } \textit{if i odd} \\ p_{10^3\cdot t/2}, \textit{if i even} \end{cases}$</td><td>$> 20.517$</td><td>$< 0.1\%$</td><td>Rather Reject</td></tr>
</table>

Figure 4. Test of hypothesis

Evaluation of the generators is performed as follows. For the generator Maple 13, which has generated a sample Ω , value $\chi^2(\text{Freq}\{\Omega_r\})$ falls into the next half-interval, the boundaries of which are determined from the table of values of the random variable χ^2, Figure 5: $1.649 < \chi^2(\text{Freq}\{\Omega_r\}) = 1,8597.. < 2.195, r = 6$, Figure 4. According to the table in [3] for the distribution function of a random variable χ^2 with $r - 2 = 4$ degrees of freedom, accidental hit of values χ^2 into the interval $[1.859.., \infty)$ is probably more than 80 % and less than in 90 % cases, Figure 4. Therefore, the hypothesis $\mathbb{H}$ is not rejected.

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Криптожүйе кілттері генераторларының кездейсок қасиетін бағалау

Осы зерттеудің мақсаты RSA криптожүйе кілттері құрылатын кездейсоқ элементтерін білдіретін бүтін мәнді кездейсоқ шаманы жасау және оның ықтималдықтарын үлестіру кестесін анықтау үшін алгоритмді құрастыру болып табылады. Құрастырылған кездейсоқ шамасының мәндерінің ықтималдықтары берілген генераторды өңдейтін, сол мәндерін ықтималдықтарымен бірдей болатынын болжамдайды. Авторлар алдын ала белгілген жартылай аралығындағы жай сандарды қабылдайтын, біркелкі үлестірілген кездейсок шама болған жағдайда, осы болжамды тексеруге арналған есептеу кестесін келтіреді.

Кілт сөздер: RSA-шифр, кілттер, кездейсоқ шама, жай сан, ықтималдық, ықтималдықтарды үлестіру.

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Оценка свойства случайности генераторов ключей криптосистемы

Целью исследования является разработка алгоритма построения целочисленной случайной величины, представляющей случайные составляющие элементы ключей криптосистемы RSA, и определение таблицы распределения её вероятностей. Построенная случайная величина используется для оценки гипотезы о том, что данный генератор вырабатывает свои значения с такой же вероятностью, с какой случайная величина принимает эти значения. Приведен алгоритм проверки гипотезы, который сопровождается таблицей расчетов для равномерно распределенной случайной величины, принимающей простые числа из заданного полуинтервала.

Ключевые слова: RSA-шифр, ключи, случайная величина, простое число, вероятность, распределение вероятностей.

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On the solvability of the first boundary value problem for the loaded equation of heat conduction

In this paper we consider the first boundary value problem for the loaded equation of heat conduction in a quarter plane. The loaded term is the trace of the fractional derivative of order ν, $0 \le \nu \le 1$ with respect to the time variable on the line $x = t$. It is shown that when $0 \le \nu \le 1$ and $\forall\lambda \in$C, then the load is a weak perturbation, that is, the studied problem has a unique solution in the class of bounded functions.

Keywords: loaded equation of heat conduction, boundary value problem, fractional derivative, Volterra integral equation.

Statement of the problem. In the domain $Q = \{x \in R_+,\ t \in R_+\}$ the following boundary value problem is considered:

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} + \lambda \cdot {}_0D_t^{\nu} u\left(x,t\right)\big|_{x=t} = f\left(x,t\right); \tag{1}$$

$$u\left(x,0\right) = 0, \quad u\left(0,t\right) = 0, \tag{2}$$

where

$$\lambda \in C,\ 0 \le \nu < 1,\ {}_0D_t^{\nu} u\left(x,t\right) = \frac{1}{\Gamma\left(1-\nu\right)} \frac{\partial}{\partial t} \int\limits_0^t \frac{1}{\left(t-\tau\right)^{\nu}} u\left(x,\tau\right) d\tau\ -$$

is the fractional Riemann-Liouville derivative of the function $u(x,t)$ with respect to a variable t of order ν,

$${}_0D_t^{\nu} \left\{ \int\limits_0^t \int\limits_0^{\infty} G\left(x,\xi,t-\tau\right) \cdot f\left(\xi,\tau\right) d\xi d\tau \right\} \in M\left(R_+\right). \tag{3}$$

Hear

$$M\left(R_+\right) = L_{\infty}\left(R_+\right) \cap C\left(R_+\right),$$

$$G\left(x,\xi,t\right) = \frac{1}{2\sqrt{\pi t}} \left\{ exp\left(-\frac{\left(x-\xi\right)^2}{4t}\right) - exp\left(-\frac{\left(x+\xi\right)^2}{4t}\right) \right\} -$$

is the Green's function of the first boundary value problem for the heat equation.

Peculiarity of this problem lies in the fact that the loaded term is the value of the fractional derivative on the line x = t. Similar problems were considered in [1, 2]. In [1] it has been shown that when $\nu = 1$ the load is a strong perturbation and for certain values of the parameter λ the corresponding homogeneous problem has nonzero solutions, that is, a non-homogeneous problem has a non-unique solution.

The purpose of this paper is to determine the nature of the load when $0 \le \nu < 1$.

Inverting the differential part in the boundary value problem (1)–(2), we have the following representation for the solution:

$$u\left(x,t\right) = -\lambda \int\limits_0^t \int\limits_0^{\infty} G\left(x,\xi,t-\tau\right) \cdot \left[{}_0D_t^{\nu} u\left(\xi,\tau\right)\right]\big|_{\xi=\tau} d\xi d\tau +$$

$$+ \int\limits_0^t \int\limits_0^{\infty} G\left(x,\xi,t-\tau\right) \cdot f\left(\xi,\tau\right) d\xi d\tau,$$

or

$$u(x,t) = -\lambda \int_0^t \int_0^\infty K_0(x, t-\tau\ ) \cdot \left[{}_0D_t^\nu u(\xi,\tau)\right]\big|_{\xi=\tau} d\xi d\tau +$$

$$+ \int_0^t \int_0^\infty G(x,\xi,t-\tau) \cdot f(\xi,\tau)\, d\xi d\tau, \tag{4}$$

hear

$$K_0(x, t-\tau\ ) = \int_0^\infty G(x,\xi,t-\tau)\, d\xi = erf\left(\frac{x}{2\sqrt{t-\tau}}\right).$$

Indeed

$$\int_0^\infty G(x,\xi,t-\tau)\, d\xi = \frac{1}{2\sqrt{\pi(t-\tau)}} \int_0^\infty \left\{ exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) - exp\left(-\frac{(x+\xi)^2}{4(t-\tau)}\right)\right\} d\xi =$$

$$= \left\| \eta = \frac{x \mp \xi}{2\sqrt{t-\tau}}\,, \quad z = \frac{x}{2\sqrt{t-\tau}} \right\| =$$

$$= \frac{1}{\sqrt{\pi}} \left\{ \int_{-\infty}^z exp\left(-\eta^2\right) d\eta - \int_z^\infty exp\left(-\eta^2\right) d\eta \right\} =$$

$$= \frac{2}{\sqrt{\pi}} \int_0^z exp\left(-\eta^2\right) d\eta = erf\left(\frac{x}{2\sqrt{t-\tau}}\right).$$

Substituting the value of this integral into (4), we obtain

$$u(x,t) = -\lambda \int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}}\right) \cdot \left[{}_0D_t^\nu u(\xi,\tau)\right]\Big|_{\xi=\tau} d\tau + f_1(x,t)\,, \tag{5}$$

where

$$f_1(x,t) = \int_0^t \int_0^\infty G(x,\xi,t-\tau) \cdot f(\xi,\tau)\, d\xi d\tau.$$

It follows from (5) that in order to find solutions of problem (1)–(2) it is sufficient find an expression $\left[{}_0D_t^\nu u(\xi,\tau)\right]\big|_{\xi=\tau}$.

For this, we differentiate both sides of (5) by t, then assuming $x = t$, we get

$$\left[{}_0D_t^\nu u(x,t)\right]\big|_{x=t} = -\lambda \left\{ {}_0D_t^\nu \int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}}\right) \cdot \left[{}_0D_t^\nu u(\xi,\tau)\right]\Big|_{\xi=\tau} d\tau \right\}\Bigg|_{x=t} +$$

$$+ {}_0D_t^\nu f_1(x,t)\big|_{x=t}\,, \tag{6}$$

and introducing the notation

$$\left[{}_0D_t^\nu u(x,t)\right]\big|_{x=t} = \varphi(t)\,,\ \ {}_0D_t^\nu f_1(x,t)\big|_{x=t} = f_2(t)\,;$$

from (6) we obtain the following integral equation

$$\varphi(t) + \lambda \left\{ {}_0D_t^\nu \int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}}\right) \cdot \varphi(\tau)\, d\tau \right\}\Bigg|_{x=t} = f_2(t)\,. \tag{7}$$

Next, we calculate

$$ {}_0D_t^{\nu}\int\limits_0^t erf\left(\frac{x}{2\sqrt{t-\tau}}\right)\cdot\varphi(\tau)\,d\tau= $$

$$ =\frac{1}{\Gamma(1-\nu)}\frac{\partial}{\partial t}\int\limits_0^t\frac{1}{(t-\tau)^{\nu}}\left\{\int\limits_0^{\tau}erf\left(\frac{x}{2\sqrt{\tau-\tau_1}}\right)\cdot\varphi(\tau_1)\,d\tau_1\right\}d\tau= $$

$$ =\frac{1}{\Gamma(1-\nu)}\frac{\partial}{\partial t}\int\limits_0^t\left\{\int\limits_{\tau_1}^{\tau}\frac{1}{(t-\tau)^{\nu}}\cdot erf\left(\frac{x}{2a\sqrt{\tau-\tau_1}}\right)d\tau\right\}\varphi(\tau_1)\,d\tau_1=\left\|\begin{array}{c} t-\tau=\eta\\ \tau=t-\eta\end{array}\right\|= $$

$$ =\frac{1}{\Gamma(1-\nu)}\int\limits_0^t\frac{\partial}{\partial t}\left\{\int\limits_0^{t-\tau_1}\frac{1}{\eta^{\nu}}\cdot erf\left(\frac{x}{2a\sqrt{t-\tau_1-\eta}}\right)d\eta\right\}\varphi(\tau_1)\,d\tau_1= $$

$$ =\frac{1}{\Gamma(1-\nu)}\int\limits_0^t\left\{\frac{1}{(t-\tau_1)^{\upsilon}}\cdot erf(\infty)-\int\limits_0^{t-\tau_1}\frac{1}{\eta^{\nu}}\cdot\frac{2}{\sqrt{\pi}}\cdot exp\left\{-\frac{x^2}{4a^2(t-\tau_1-\eta)}\right\}\times\right. $$

$$ \left.\times\frac{x}{4a(t-\tau_1-\eta)^{\frac{3}{2}}}d\eta\right\}\varphi(\tau_1)\,d\tau_1=\frac{1}{\Gamma(1-\nu)}\int\limits_0^t K_{\nu}(t,\tau_1,x)\,\varphi(\tau_1)\,d\tau_1, \qquad (8) $$

where the following notation is used:

$$ K_{\nu}(t,\tau,x)=\frac{1}{(t-\tau_1)^{\upsilon}}-k_{\nu}(t,\tau_1,x); \qquad (9) $$

$$ k_{\nu}(t,\tau_1,x)=\frac{x}{2a\sqrt{\pi}}\int\limits_0^{t-\tau_1}\frac{1}{\eta^{\nu}}\cdot\frac{1}{(t-\tau_1-\eta)^{\frac{3}{2}}}\cdot exp\left\{-\frac{x^2}{4a^2(t-\tau_1-\eta)}\right\}d\eta. $$

Let us find the explicit form of the function $k_{\nu}(t,\tau_1,x)$.

$$ k_{\nu}(t,\tau_1,x)=\frac{x}{2a\sqrt{\pi}}\int\limits_0^{t-\tau_1}\frac{1}{\eta^{\nu}}\cdot\frac{1}{(t-\tau_1-\eta)^{\frac{3}{2}}}\cdot exp\left\{-\frac{x^2}{4a^2(t-\tau_1-\eta)}\right\}d\eta= $$

$$ =\left\|\begin{array}{l}\eta=(t-\tau_1)(1-z)\\ d\eta=-(t-\tau_1)\,dz\end{array}\right\|= $$

$$ =\frac{x}{2a\sqrt{\pi}}\int\limits_0^1\frac{t-\tau_1}{(t-\tau_1)^{\nu}(1-z)^{\nu}}\cdot\frac{1}{(t-\tau_1)^{\frac{3}{2}}\cdot z^{\frac{3}{2}}}\cdot exp\left\{-\frac{x^2}{4a^2(t-\tau_1)}\cdot\frac{1}{z}\right\}dz= $$

$$ =\frac{x}{2a\sqrt{\pi}}\cdot\frac{1}{(t-\tau_1)^{\nu+\frac{1}{2}}}\int\limits_0^1 z^{-\frac{3}{2}}\cdot(1-z)^{-\nu}\cdot exp\left\{-\frac{x^2}{4a^2(t-\tau_1)}\cdot\frac{1}{z}\right\}dz= $$

$$ =\|[3].\,p.367,\quad 3.471\,(2)\|= $$

$$ =\Gamma(1-\nu)\cdot\frac{x}{2a\sqrt{\pi}}\cdot\frac{1}{(t-\tau_1)^{\nu+\frac{1}{2}}}\cdot\left[\frac{x^2}{4a^2(t-\tau_1)}\right]^{-\frac{3}{4}}\cdot exp\left\{-\frac{x^2}{8a^2(t-\tau_1)}\right\}\times $$

$$ \times W_{\nu-\frac{1}{4},-\frac{1}{4}}\left\{\frac{x^2}{4a^2(t-\tau_1)}\right\}= $$

$$ =\Gamma(1-\nu)\cdot\frac{\sqrt{2a}}{\sqrt{\pi x}}\cdot\frac{1}{(t-\tau_1)^{\nu-\frac{1}{4}}}\cdot exp\left\{-\frac{x^2}{8a^2(t-\tau_1)}\right\}\cdot W_{\nu-\frac{1}{4},-\frac{1}{4}}\left\{\frac{x^2}{4a^2(t-\tau_1)}\right\}, $$

hear $W_{\nu-\frac{1}{4},-\frac{1}{4}}\left\{\frac{x^2}{4a^2(t-\tau_1)}\right\}$ is Whittaker function.

Substituting the function $k_\nu(t,\tau_1,x)$ into (9), for the kernel $K_\nu(t,\tau,x)$ finally we obtain the following representation

$$K_\nu(t,\tau,x)=\frac{1}{\Gamma(1-\nu)\cdot(t-\tau_1)^\nu}-\sqrt{\frac{2a}{\pi}}\cdot\frac{1}{\sqrt{x}}\times$$

$$\times\frac{1}{(t-\tau_1)^{\nu-\frac14}}exp\left\{-\frac{x^2}{8a^2(t-\tau_1)}\right\}\cdot W_{\nu-\frac14,-\frac14}\left\{\frac{x^2}{4a^2(t-\tau_1)}\right\}. \tag{10}$$

Thus, the integral equation (7) takes the form

$$\varphi(t)-\lambda\int\limits_0^t K_\nu(t,\tau)\cdot\varphi(\tau)\,d\tau=f_2(t), \tag{11}$$

where

$$K_\nu(t,\tau)=\sqrt{\frac{2a}{\pi}}\cdot\frac{1}{\sqrt{t}\cdot(t-\tau)^{\nu-\frac14}}exp\left\{-\frac{t^2}{8a^2(t-\tau)}\right\}\cdot W_{\nu-\frac14,-\frac14}\left\{\frac{t^2}{4a^2(t-\tau)}\right\}-\frac{1}{\Gamma(1-\nu)\cdot(t-\tau)^\nu}.$$

This kernel has a weak singularity when $0\le\nu<1$, that will be shown below.

We consider some particular cases of equation (11) for specific values of $\nu:\ \nu=1,\ \ \nu=0,\ \nu=\frac12$.

We note that earlier in [1] the problem (1)–(2) was considered for the case $\nu=1$, which was reduced to an integral equation of the form

$$(1+\lambda)\,\varphi(t)-\lambda\int\limits_0^t K_1(t,\tau)\cdot\varphi(\tau)\,d\tau=f_2(t), \tag{12}$$

the kernel of this integral equation

$$K_1(t,\tau)=\frac{1}{2a\sqrt{\pi}}\cdot\frac{t}{(t-\tau)^{\frac32}}\cdot exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\}-$$

is «incompressible», since

$$\lim_{t\to0}\int\limits_0^t K_1(t,\tau)\,d\tau=1,$$

and it was shown that the corresponding homogeneous integral equation has non-zero solutions, and their number depends on $|\lambda|$.

We show that when $\nu\to1-0$, the integral equation (11) coincides with equation (12). To do this in (11) we pass to the limit $\nu\to1-0$, that is, we calculate the following limits:

$$\lim_{\nu\to1-0}\lambda\cdot\int\limits_0^t\frac{1}{\Gamma(1-\nu)\cdot(t-\tau)^\nu}\cdot\varphi(\tau)\,d\tau=$$

$$=\left\|\begin{array}{l}u=\varphi(\tau);\quad du=\varphi'(\tau)\,d\tau\\ dv=\frac{d\tau}{(t-\tau)^\nu};v=-\frac{1}{1-\nu}(t-\tau)^{1-\nu}\end{array}\right\|=$$

$$=\lim_{\nu\to1-0}\frac{1}{\Gamma(1-\nu)}\cdot\lambda\left\{-\frac{(t-\tau)^{1-\nu}}{1-\nu}\cdot\varphi(\tau)\Big|_0^t+\frac{1}{1-\nu}\int\limits_0^t(t-\tau)^{1-\nu}\cdot\varphi'(\tau)\,d\tau\right\}=$$

$$=\lim_{\nu\to1-0}\frac{\lambda}{(1-\nu)\cdot\Gamma(1-\nu)}\cdot\left\{t^{1-\nu}\cdot\varphi(0)+\int\limits_0^t(t-\tau)^{1-\nu}\cdot\varphi'(\tau)\,d\tau\right\}=$$

$$=\lambda\cdot\varphi(0)+\lambda\cdot\varphi(\tau)|_0^t=\lambda\cdot\varphi(0)+\lambda\cdot\varphi(t)-\lambda\cdot\varphi(0)=\lambda\cdot\varphi(t).$$

Now we find the limit from the first summand

$$\lim_{\nu\to 1-0} k_\nu(t,\tau) = \sqrt{\frac{2a}{\pi}} \cdot \frac{1}{\sqrt{t}\cdot(t-\tau)^{\frac{3}{4}-0}} \times$$

$$\times exp\left\{-\frac{t^2}{8a^2(t-\tau)}\right\} \cdot W_{\frac{3}{4}-0,-\frac{1}{4}}\left\{\frac{t^2}{4a^2(t-\tau)}\right\} =$$

$$= \sqrt{\frac{2a}{\pi}} \cdot \frac{1}{\sqrt{t}\cdot(t-\tau)^{\frac{3}{4}-0}} \cdot exp\left\{-\frac{t^2}{8a^2(t-\tau)}\right\} \times$$

$$\times \left\{\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(+0)} \cdot M_{\frac{3}{4},-\frac{1}{4}}\left[\frac{t^2}{4a^2(t-\tau)}\right] + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)} \cdot M_{\frac{3}{4},-\frac{1}{4}}\left[\frac{t^2}{4a^2(t-\tau)}\right]\right\}.$$

Here we use the representation of the Whittaker function $W_{\lambda,\mu}(z)$ in terms of the function $M_{\lambda,\mu}(z)$ [3; 1024(9.220)]. Further, writing down the Whittaker function $M_{\lambda,\mu}(z)$ via the Kummer function $(\alpha,\beta;z)$ we have:

$$\lim_{\nu\to 1-0} k_\nu(t,\tau) = \sqrt{\frac{2a}{\pi}} \cdot \frac{1}{\sqrt{t}\cdot(t-\tau)^{\frac{3}{4}-0}} \cdot exp\left\{-\frac{t^2}{8a^2(t-\tau)}\right\} \cdot \left[\frac{t^2}{4a^2(t-\tau)}\right]^{\frac{3}{4}} \times$$

$$\times exp\left\{-\frac{t^2}{8a^2(t-\tau)}\right\} \cdot \Phi\left(0,\ \frac{3}{2}, \frac{t^2}{4a^2(t-\tau)}\right) =$$

$$= \frac{1}{2a\sqrt{\pi}} \cdot \frac{t}{(t-\tau)^{\frac{3}{2}}} \cdot exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\}.$$

Thus, we obtain

$$(1+\lambda)\,\varphi(t) - \lambda\int_0^t \frac{1}{2a\sqrt{\pi}} \cdot \frac{t}{(t-\tau)^{\frac{3}{2}}} \cdot exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\} \cdot \varphi(\tau)\,d\tau = f_2(t)\,. \tag{13}$$

Indeed, equation (13) coincides with equation (12) [1].We further define the form of the integral equation (11) at $\nu\to 0$,

For this, we find the value $K_\nu(t,\tau)$ при $\nu\to 0$.

$$\lim_{\nu\to 0} K_\nu(t,\tau) = 1 - \sqrt{\frac{2a}{\pi}} \cdot \frac{(t-\tau)^{\frac{1}{4}}}{\sqrt{t}} \cdot exp\left\{-\frac{t^2}{8a^2(t-\tau)}\right\} \cdot W_{-\frac{1}{4},-\frac{1}{4}}\left\{\frac{t^2}{4a^2(t-\tau)}\right\} =$$

$$= \left\| W_{-\frac{1}{4},-\frac{1}{4}}(z) = [1-\mathrm{erf}(x)] \cdot \frac{\sqrt{\pi}\cdot\sqrt{x}}{e^{-\frac{x^2}{2}}} \right\| =$$

$$= 1 - \sqrt{\frac{2a}{\pi}} \cdot \frac{(t-\tau)^{\frac{1}{4}}}{\sqrt{t}} \cdot \frac{\sqrt{\pi}\cdot\sqrt{t}}{(2a)^{\frac{1}{2}}\cdot(t-\tau)^{\frac{1}{4}}} \cdot \left[1 - \mathrm{erf}\left\{\frac{t}{2a\sqrt{t-\tau}}\right\}\right] =$$

$$= 1 - \left[1 - \mathrm{erf}\left\{\frac{t}{2a\sqrt{t-\tau}}\right\}\right] = \mathrm{erf}\left\{\frac{t}{2a\sqrt{t-\tau}}\right\}.$$

Using this relation, we obtain the following integral equation

$$\varphi(t) - \lambda\int_0^t \mathrm{erf}\left\{\frac{t}{2a\sqrt{t-\tau}}\right\} \cdot \varphi(\tau)\,d\tau = f_2(t)\,, \tag{14}$$

where $\varphi(t) = u(x,t)|_{x=t}$.

This equation coincides with the integral equation (5), when $\nu\to 0$.

Now we consider the case $\nu \to \frac{1}{2}$. That is

$$\lim_{\nu\to\frac{1}{2}} K_\nu(t,\tau) = \sqrt{\frac{2a}{\pi}} \cdot \frac{1}{\sqrt{t}\cdot(t-\tau)^{\frac{1}{4}}} exp\left\{-\frac{t^2}{8a^2(t-\tau)}\right\} \cdot W_{\frac{1}{4},-\frac{1}{4}}\left\{\frac{t^2}{4a^2(t-\tau)}\right\} -$$

$$-\frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{t-\tau}}.$$

To determine the specific form of the kernel, we use the representation of the Whittaker function

$$W_{\frac{1}{4},-\frac{1}{4}}\{z\} = M_{\frac{1}{4},-\frac{1}{4}}(z) + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(+0)} \cdot M_{\frac{1}{4},\frac{1}{4}}(z) =$$

$$= z^{\frac{1}{4}} \cdot e^{-\frac{z}{2}} + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(+0)} \cdot \frac{\sqrt{\pi}}{2} \cdot z^{\frac{1}{4}} \cdot e^{-\frac{z}{2}} \cdot erfi\left\{\sqrt{z}\right\},$$

where $erfi\{\sqrt{z}\} = \int\limits_0^{\sqrt{z}} e^{\xi^2} d\xi$. Hence the kernel $K_{\frac{1}{2}}(t,\tau)$ is equal to the sum of three terms.

We compute the first summand

$$K^{(1)}_{\frac{1}{2}}(t,\tau) = \sqrt{\frac{2a}{\pi}} \cdot \frac{1}{\sqrt{t}\cdot(t-\tau)^{\frac{1}{4}}} \cdot \left(\frac{t^2}{4a^2(t-\tau)}\right)^{\frac{1}{4}} \cdot exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\} =$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{t-\tau}} \cdot exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\}.$$

For the second summand we have

$$\lim_{\nu\to\frac{1}{2}} K^{(2)}_\nu(t,\tau) = \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(+0)} \cdot \sqrt{\frac{2a}{\pi}} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{t}\cdot(t-\tau)^{\frac{1}{4}}} \cdot \left(\frac{t^2}{4a^2(t-\tau)}\right)^{\frac{1}{4}} \times$$

$$\times exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\} \cdot \int\limits_0^{\frac{t}{2a\sqrt{t-\tau}}} exp\left\{\xi^2\right\} d\xi =$$

$$= \frac{\Gamma\left(-\frac{1}{2}\right)}{2\cdot\Gamma(+0)} \cdot \frac{1}{\sqrt{t-\tau}} \cdot exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\} \cdot \int\limits_0^{\frac{t}{2a\sqrt{t-\tau}}} exp\left\{\xi^2\right\} d\xi =$$

$$= \left\| \xi = \frac{t}{2a\sqrt{t-\tau}} \cdot z \right\| =$$

$$= \frac{\Gamma\left(-\frac{1}{2}\right)}{2\cdot\Gamma(+0)} \cdot \frac{1}{\sqrt{t-\tau}} \cdot \int\limits_0^1 \frac{t}{2a\sqrt{t-\tau}} \cdot exp\left\{-\frac{t^2}{4a^2(t-\tau)} \cdot \left(1-z^2\right)\right\} dz =$$

$$= \frac{\Gamma\left(-\frac{1}{2}\right)}{2\cdot\Gamma(+0)} \cdot \frac{1}{\sqrt{t-\tau}} \cdot \int\limits_0^1 \frac{t\cdot\sqrt{1-z^2}}{2a\sqrt{t-\tau}} \cdot exp\left\{-\frac{t^2}{4a^2(t-\tau)} \cdot \left(1-z^2\right)\right\} \frac{dz}{\sqrt{1-z^2}} = 0.$$

Thus

$$\lim_{\nu\to\frac{1}{2}} K_\nu(t,\tau) == \frac{1}{\sqrt{t-\tau}} \cdot \left\{\frac{1}{\sqrt{\pi}} \cdot exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\} - \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{t-\tau}}\right\}.$$

This result can be obtained if we take directly from both sides of (5) the derivative of order $\frac{1}{2}$:

$${}_0D_t^{\frac{1}{2}} \left\{ \int\limits_0^t erf\left(\frac{x}{2\sqrt{t-\tau}}\right) \cdot \varphi(\tau)\, d\tau \right\} =$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{\partial}{\partial t} \int\limits_0^t \frac{1}{\sqrt{t-\tau}} \left\{ \int\limits_0^\tau erf\left(\frac{x}{2\sqrt{\tau-\tau_1}} \right) \cdot \varphi(\tau_1)\, d\tau_1 \right\} d\tau =$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{\partial}{\partial t} \int\limits_0^t \left\{ \int\limits_{\tau_1}^t \frac{1}{\sqrt{t-\tau}} \cdot erf\left(\frac{x}{2a\sqrt{\tau-\tau_1}} \right) d\tau \right\} \varphi(\tau_1)\, d\tau_1 = \left\| \begin{array}{l} t-\tau=\eta \\ \tau = t-\eta \end{array} \right\| =$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{\partial}{\partial t} \int\limits_0^t \left\{ \int\limits_0^{t-\tau_1} \frac{1}{\eta^{\frac{1}{2}}} \cdot erf\left(\frac{x}{2a\sqrt{t-\tau_1-\eta}} \right) d\eta \right\} \varphi(\tau_1)\, d\tau_1 =$$

$$= \frac{1}{\sqrt{\pi}} \cdot \int\limits_0^t \frac{\partial}{\partial t} \left\{ \int\limits_0^{t-\tau_1} \frac{1}{\eta^{\frac{1}{2}}} \cdot erf\left(\frac{x}{2a\sqrt{t-\tau_1-\eta}} \right) d\eta \right\} \varphi(\tau_1)\, d\tau_1 =$$

$$= \frac{1}{\sqrt{\pi}} \cdot \int\limits_0^t \left\{ \frac{1}{\sqrt{t-\tau_1}} \cdot erf(\infty) - \int\limits_0^{t-\tau_1} \frac{1}{\eta^{\frac{1}{2}}} \cdot \frac{2}{\sqrt{\pi}} \cdot exp\left\{ -\frac{x^2}{4a^2(t-\tau_1-\eta)} \right\} \times \ \frac{x}{4a(t-\tau_1-\eta)^{\frac{3}{2}}} d\eta \right\} \varphi(\tau_1)\, d\tau_1 =$$

$$= \left\| \begin{array}{l} \eta = (t-\tau_1)\cdot(1-z) \\ d\eta = -(t-\tau_1)\, dz \end{array} \right\| =$$

$$= \|[3],\ 3.471\| =$$

$$= \frac{1}{\sqrt{\pi}} \cdot \int\limits_0^t \frac{1}{\sqrt{t-\tau_1}} \left(1 - exp\left\{ -\frac{t^2}{4a^2(t-\tau_1)} \right\} \right) \varphi(\tau_1)\, d\tau_1.$$

Now we show that the kernel of the integral equation (11) has a weak singularity for $\forall\ \nu,\ \ 0 \le \nu < 1$. To do this, we calculate the integral from the kernel $k_\nu(t,\tau)$.

$$J(t,\nu) = \int\limits_0^t \sqrt{\frac{2a}{\pi}} \frac{1}{\sqrt{t}\cdot(t-\tau)^{\nu-\frac{1}{4}}} exp\left\{ -\frac{t^2}{8a^2(t-\tau)} \right\} \cdot W_{\nu-\frac{1}{4},-\frac{1}{4}} \left\{ \frac{t^2}{4a^2(t-\tau)} \right\} d\tau =$$

$$= \left\| \frac{t^2}{4a^2(t-\tau)} = y;\ \ t-\tau = \frac{t^2}{4a^2}\cdot\frac{1}{y};\ \ d\tau = \frac{t^2}{4a^2}\cdot\frac{1}{y^2}dy \right\| =$$

$$= \int\limits_{\frac{t}{4a^2}}^\infty \sqrt{\frac{2a}{\pi}} \cdot \frac{(2a)^{2\nu-\frac{1}{2}}}{t^{2\nu-\frac{1}{2}}} \cdot \frac{t^2}{(2a)^2} \cdot y^{\nu-\frac{1}{4}} \cdot y^{-2} exp\left\{ -\frac{y}{2} \right\} \cdot W_{\nu-\frac{1}{4},-\frac{1}{4}}(y)\, dy =$$

$$= \frac{(2a)^{2\nu-2}}{\sqrt{\pi}} \cdot t^{2-2\nu} \cdot \int\limits_{\frac{t}{4a^2}}^\infty y^{\left(\nu-\frac{1}{4}\right)-2} exp\left\{ -\frac{y}{2} \right\} \cdot W_{\nu-\frac{1}{4},-\frac{1}{4}}(y)\, dy =$$

$$= \|[4, 36(1.13.2(6))]\| =$$

$$= \frac{(2a)^{2\nu-2}}{\sqrt{\pi}} \cdot t^{2-2\nu} \cdot y^{\nu-\frac{5}{4}} \cdot exp\left\{ -\frac{y}{2} \right\} \cdot W_{\nu-\frac{5}{4},-\frac{1}{4}}(y) \Big|_{\frac{t}{4a^2}}^\infty =$$

$$= \sqrt{\frac{2a}{\pi}} \cdot t^{-\nu+\frac{3}{4}} \cdot exp\left\{ -\frac{t}{8a^2} \right\} \cdot W_{\nu-\frac{5}{4},-\frac{1}{4}} \left\{ \frac{t}{4a^2} \right\}.$$

Let us study the behavior of functions $W_{\nu-\frac{5}{4},-\frac{1}{4}}(z)$ at $t \to 0$, for this, we represent the function $W_{\nu-\frac{5}{4},-\frac{1}{4}}(z)$ as follows:

$$W_{\nu-\frac{5}{4},-\frac{1}{4}}(z) = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(2-\nu)} \cdot \mathrm{M}_{\nu-\frac{5}{4},-\frac{1}{4}}(z) + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-\nu\right)} \cdot M_{\nu-\frac{5}{4},\frac{1}{4}}(z) =$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2-\nu\right)} \cdot z^{\frac{1}{4}} \cdot e^{-\frac{z}{2}} \cdot \Phi\left(\frac{3}{2}-\nu, \frac{1}{2}; z\right) + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-\nu\right)} z^{\frac{1}{4}} \cdot e^{-\frac{z}{2}} \cdot \Phi\left(2-\nu, \frac{3}{2}; z\right) =$$

$$= z^{\frac{1}{4}} \cdot e^{-\frac{z}{2}} \left\{ \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2-\nu\right)} \cdot \Phi\left(\frac{3}{2}-\nu,\ \frac{1}{2}; z\right) + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-\nu\right)} z^{\frac{1}{2}} \cdot \Phi\left(2-\nu, \frac{3}{2}; z\right) \right\}.$$

From here it follows that

$$\lim_{t\to 0} \int\limits_0^t K_1\left(t,\tau\right) d\tau = 0,\ \forall\ \nu,\ 0 \le \nu < 1.$$

This means that the integral equation (11) has a unique solution $\varphi_\lambda\left(t\right),\quad \forall \lambda \in C,\ f_2\left(t\right).$ The same can be shown as follows.

The function $k_\nu\left(t,\tau\right)$ has a singularity at $\tau \to t$, we define the order of this singularity by using the asymptotics $W_{\lambda,\mu}\left(z\right) \approx z^\lambda \cdot exp\left(-\frac{z}{2}\right)$, при $|z| \to \infty$ [3, 1026; (9.227)].

We have for the case $\nu > \frac{1}{2}$

$$k_\nu\left(t,\tau\right) \approx \sqrt{\frac{2a}{\pi}} \cdot \frac{1}{\sqrt{t} \cdot \left(t-\tau\right)^{\nu-\frac{1}{4}}} \cdot \left[\frac{t^2}{4a^2 \cdot \left(t-\tau\right)}\right]^{\nu-\frac{1}{4}} \cdot exp\left\{-\frac{t^2}{4a^2\left(t-\tau\right)}\right\} =$$

$$= \frac{1}{\left(t-\tau\right)^\nu} \cdot \left[\frac{t^2}{4a^2 \cdot \left(t-\tau\right)}\right]^{\nu-\frac{1}{2}} \cdot exp\left\{-\frac{t^2}{4a^2\left(t-\tau\right)}\right\} \le$$

$$\le c\left(\nu\right) \cdot \frac{1}{\left(t-\tau\right)^\nu}.$$

Here, we use the estimate

$$x^{\nu-\frac{1}{2}} \cdot exp\left\{-x\right\} < \left(\nu-\frac{1}{2}\right)^{\nu-\frac{1}{2}} \cdot exp\left\{-\left(\nu-\frac{1}{2}\right)\right\}.$$

And when $\nu \le \frac{1}{2}$ the following inequality is used

$$\frac{1}{\left(t-\tau\right)^\nu} \cdot \left[\frac{t^2}{4a^2 \cdot \left(t-\tau\right)}\right]^{\nu-\frac{1}{2}} \cdot exp\left\{-\frac{t^2}{4a^2\left(t-\tau\right)}\right\} \le$$

$$\le \frac{1}{\left(t-\tau\right)^\nu} \cdot \frac{t^{2\nu-1}}{\left(2a\right)^{2\nu-1} \cdot \left(t-\tau\right)^{\nu-\frac{1}{2}}} =$$

$$= \frac{1}{\left(2a\right)^{2\nu-1}} \cdot \left(\frac{t-\tau}{t}\right)^{1-2\nu} \cdot \frac{1}{\sqrt{t-\tau}} \le \left(2a\right)^{1-2\nu} \cdot \frac{1}{\sqrt{t-\tau}}.$$

Now, according to (4), we write the solution to problem (1) – (2) in the form

$$u\left(x,t\right) = -\lambda \int\limits_0^t \operatorname{erf}\left\{\frac{t}{2a\sqrt{t-\tau}}\right\} \cdot \varphi_\lambda\left(\tau\right) d\tau +$$

$$+ \int\limits_0^t \int\limits_0^\infty G\left(x,\xi,t-\tau\right) \cdot f\left(\xi,\tau\right) d\xi d\tau.$$

Thus, it is shown that when $0 \le \nu < 1$ for $\forall \lambda \in C$ the loaded term in (1) – (2) is a weak perturbation.

Thus, the following theorem is valid

Theorem. The boundary value problem (1)–(2) $\forall \nu, 0 \le \nu < 1$, $\forall \lambda \in C$ has a unique solution $u_\lambda \in U$, where

$$U = \left\{ u | \left(x+\sqrt{t}\right)^{-1} \cdot u;\ u_t - u_{xx} \in M\left(Q\right), \left[{}_0D_t^\nu u\left(x,t\right)\right]\big|_{x=t} \in M\left(R_+\right) \right\}.$$

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Жүктелген жылуөткізгіштік теңдеуі үшін бірінші шеттік есептің шешімділігі туралы

Мақалада жазықтықтың бір ширегінде жүктелген жылуөткізгіштік теңдеуі үшін бірінші шеттік есебі қарастырылған. Жүктелген қосылғыш — $x = t$ сызығында уақыттық айнымалысы бойынша ν-ші ретті бөлшек туындының ізі, $0 \leq \nu < 1$. Егер $0 \leq \nu < 1$ және $\forall\lambda \in$ C болса, онда жүктеменің әлсіз ауытқу екені көрсетілген, яғни, зерттелетін есептің шенелген функциялар класында жалғыз шешімі бар.

Кілт сөздер: жүктелген жылуөткізгіштік теңдеу, шеттік есеп, бөлшек туынды, Вольтерраның интегралдық теңдеуі.

М.Т. Дженалиев, С.А. Искаков, М.И. Рамазанов, Ж.М. Тулеутаева

О разрешимости первой краевой задачи для нагруженного уравнения теплопроводности

В статье рассмотрена первая краевая задача для нагруженного уравнения теплопроводности в четверти плоскости. Нагруженное слагаемое – след дробной производной порядка ν, $0 \leq \nu \leq 1$, по временной переменной на линии $x = t$. Показано, что при $0 \leq \nu \leq 1$ и $\forall\lambda \in$C нагрузка является слабым возмущением, то есть исследуемая задача имеет единственное решение в классе ограниченных функций.

Ключевые слова: нагруженное уравнение теплопроводности, краевая задача, дробная производная, интегральное уравнение Вольтерра.

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The admissibility and similarity of Jonsson theories

This work is related to the concepts of admissibility, interpretability, syntactic similarity and semantic similarity of Jonsson's theories.This new concept generalize the notion of syntactic and semantic similarities for Jonsson theories. In the frame of study of preservation of the definable formulas subsets of Jonsson theory's semantic model regarding to considering morphisms the concepts of admissibility, interpretability, domination and similarity play a very important role. As an example, we consider the theory of Boolean algebras. These concepts are closely related to the concept of admissibility by one theory on the other. The interest of this paper is that the examples examined show the fact that admissibility, interpretability and similarity can be considered on algebras of different signatures.In this work the main definitions of notions and the basic directions of further researches are studied.

Keywords: Jonsson theory, semantic model, admissibility, interpretability, domination of Jonsson theories, syntactic and semantic similarity.

One of the classical problems of science is the question of classifying objects of study on some common features. In mathematics, the role of such objects is played sets with given relations.With the help of mathematical logic, these objects were associated with some sets of formulas of the predicate calculus's language. This relation between the syntax and semantics of a fixed language is actually the essence of Model Theory. Therefore, it is clear that finding the syntactic and semantic features of similarity can be useful in classifying objects of Model Theory.

In this paper, we compare the notions of admissibility and similarity for complete and Jonsson theories. At one time Professor T.G. Mustafin [1] in the language of pure pair and semantic triples defined the notion of admissibility, interpretability and syntactic and semantic similarity between two complete theories.In connection with the fact that any complete theory with the help of certain manipulations (enrichment of signature and morlization) can lead to a theory that will be Jonsson theory. In addition, after morlization, the structure and number of models of the newly obtained theory in a certain sense will not differ much from the structure and number of models for the original complete theory.The remark «in a certain sense» can be deciphered in the following way: in connection with the fact that when dealing with Jonsson theories we are dealing with the class of existentially closed models of the considering theory, one can note that morlization with existentially closed models of originally complete theory (if they exist) do not change the structure and the number of these models. The next fact, connected with the existence of existentially closed models of complete theories, forces us to be in the class of complete inductive theories, because it is well that any model of an inductive theory is embedded isomorphically into some existentially closed model of this theory.

T.G. Mustafin in his work [2] defined the exact concept of syntactic [2 (Def. 1)] and semantic similarity [2 (Def. 4)] of complete theories. Moreover, in the language of these definitions and corresponding concepts (eg, the hull of theory [2 (Def. 12)], the semantic property (of theory, model, element) [2 (Def. 8]), he proved that for an arbitrary complete theory there exists syntactically similar to it some theory of polygons [2, Th. 4, Th. 5].

In this paper, we give a material that explains the connection with the various concepts of similarity between Jonsson theories. By means of a generalization of some definitions from [2] and the technique of working with Jonsson theories, it was obtained that in the class of perfect $\exists$-complete Jonsson theories the concepts of the introduced similarities of Jonsson theories coincide with the corresponding concepts in general theorems in the sense of [2]. In the class of Jonsson theories this approach to the classification of the corresponding objects is acceptable, but requires certain changes in the definitions of the corresponding similarities of theories. This is due, first, to the fact that, generally speaking, the Jonsson theories are not complete, and secondly, that in the class of models of Jonsson theory, homogeneous-universal models are, generally speaking, not saturated. Thus, all of the above suggests that the redefinition of the basic attributes of such concepts as permissibility, interpretability and similarity in the class of Jonsson theories is an actual task. On the other hand, the research of Jonsson theories is interesting in and of itself, regardless of the reduction of some complete theory to the Jonsson theory. The point is that the class of Jonsson theories of fixed signature is, generally speaking, incomplete, yet inductive and with the amalgam property and the joint embedding property, which are natural for many algebraic problems.

We give the definitions from [1].

Definition 1. If A is non-empty set, G is some group of bijections (permutations) of the set A concerning superposition then the pair (A, G) is called pure pair. If (B, H) and (A, G) are pure pairs, $B \subseteq A$ and $H = \{g \upharpoonright B;\, g \in G\}$ then pure pair (B, H) is called subpair of pure pair (A, G) and we denote $(B, H) \subseteq (A, G)$.

Definition 2. We say that pure pairs (A, G) and (B, H) are isomorphic, if exists such bijection $\psi : A \to B$ that $H = \left\{ \psi g \psi^{-1};\, g \in G \right\}$. In this case we will write $\psi : (A, G) \simeq (B, H)$. If $A = B$, H is a subgroup of group G then we will use record $(B, H)[=, \leq](A, G)$. If there is such pure pair (G, F) that $(B, H) \simeq (G, F)[=, \leq](A, G)$ we will write $(B, H)[\simeq, \leq](A, G)$.

Definition 3. Let (A, G)-pure pair, P-any m-place relation on A $(i.e. P \subseteq A^m)$. Then P will called G-invariant if for any $g \in G$, $a_1 ..., a_m \in A$ takes place $< a_1 ..., a_m > \in P \Rightarrow < g(a_1) ..., g(a_m) > \in P$.

Through G_m we will denote the set of all such permutations f of set A^m or which exist such $g \in G$ that $f(< a_1 ..., a_m >) = < g(a_1), ..., g(a_m) >$ for all a_1, ..., $a_m \in A$. It is obvious that the pair$(A^m,\, G_m)$ is pure pair. If $\sim$ is G- the invariant relation of equivalence then $\sim$ induces on G group congruence which also we will denote through $\sim$ as follows:

$$g_1 \sim g_2 \;\; \Leftrightarrow \;\; g_1(a) \sim g_2(a) \text{ for all } a \in A.$$

Clearly that $(A/_{\sim},\, G/_{\sim})$ will be pure pair if to put $\tilde{g}(\tilde{a}) = \widetilde{g(a)}$ for all $g \in G$, $a \in A$ where $\tilde{a}, \tilde{g}$- coset classes on A and G accordingly.

Definition 4. Pair (B, H) is called derivative pure pair from pure pair (A, G) if exists such $n < \omega$ and G_n – the invariant relation of equivalence $\sim$ on A that $(B, H) \subseteq (A^n /_{\sim},\, G^n /_{\sim})$. We say that pure pair (A, G) dominates over pure pair (C, F) if there is such derivative pure pair (B, H) from (A, G) that $(B, H)[=, \leq](C, F)$.

Family of subsets (relations) $p_i \subseteq A^{n_i}$, $i \in I$, we name complete family on pure pair (A, G) if G is group of all automorphisms algebraic system $< A;\; P_i,\; i \in I >$(i.e $G = Aut\,(< A;\, P_i,\, i \in I >)$.

Let τ- some way allowing for any pure pair (A, G) to allocate some system of G- invariant relations on A (so-called τ- relations). We say that the algebraic system $< C;\; R_i,\; i \in I >$- is interpreted in pure pair (A, G) if exist such $n < \omega$, $P \subseteq A^n$ and surjection $\psi : P \to C$ that complete prototypes R_i $i \in I$, and equality relations are τ- relations on A. We say that pure pair (C, F), τ is interpreted in (A, G) if exist such complete family of relations $R_i \subseteq C^{n_i}$, $i \in I$, on (C, F) that $< C;\; R_i,\; i \in I >$, τ – is interpreted (A, G).

Definition 5. Let, $\mathfrak{A}$, $\mathfrak{B}$ are arbitrary mathematical structures in the Bourbaki's sense on sets A, B accordingly, $G = Aut(\mathfrak{A})$, $G = Aut(\mathfrak{B})$. We say that:

1) $\mathfrak{A}$ admits $\mathfrak{B}$ (or $\mathfrak{B}$ it is admissible $\mathfrak{A}$), if $(A, G)[\simeq, \leq](B, H)$;

2) $\mathfrak{A}$ relatively admits $\mathfrak{B}$, if (A, G) dominates over (B, H).

Remark 1. If $\mathfrak{A}$ admits $\mathfrak{B}$, then $\mathfrak{A}$ relatively admits $\mathfrak{B}$.

If K is the class of mathematical structures then we say that $\mathfrak{A}$ admits (relatively admits) K if and only if when $\mathfrak{A}$ admits (relatively admits) some structure from K. If the class K is defined by system of axioms Σ then expression «$\mathfrak{A}$ admits Σ» («$\mathfrak{A}$ relatively admits Σ») means the same, as «$\mathfrak{A}$ admits K» («$\mathfrak{A}$ relatively admits K, accordingly»).

If T is the complete theory of algebraic systems of the first order language then we say that T (relatively) admits a class K of mathematical structures (or system of axioms Σ) if the monster-model $\mathfrak{C}$ of theory T (relatively) admits K (Σ accordingly).

One of the important concepts of Model Theory is the concept of definability (interpretability) of one algebraic system in another. It is said that the algebraic system $\mathfrak{B} =< B, R_i, i \in I >$ is definable on $\mathfrak{A} =< A, P_j, j \in J >$,if exists such formular relations Φ_i $i \in I$ in language $\mathfrak{A}$ that $< A;\; \Phi_i,\; i \in I >$ is isomorphic $< B;\; R_i,\; i \in I >$. In the course of the development of model theory, this notion was generalized, and the most general (at present) definition can be formulated as follows. If $\mathfrak{A}$ algebraic system $n < \omega$ $B \subseteq A^n$, λ is the cardinal then B is called τ_λ-subset if exists such n-type $p(x_1, ..., x_n)$ over $\emptyset$ of language of system A, such $|p(x_1, ..., x_n)| < \lambda$ and B consists of all n of A^n, realising $p(x_1, ..., x_n)$ in $\mathfrak{A}$. Obviously, τ_λ-subsets are invariant relatively automorphisms. Therefore, τ_λ it can be considered as a way of isolating a certain class of invariant subsets of algebraic systems. If $\mathfrak{A}$, $\mathfrak{B}$ - algebraic systems $G = Aut($ A$)$, then we say that $\mathfrak{B}$ is τ_λ — interpreted in $\mathfrak{A}$, if $\mathfrak{B}$, is τ_λ — interpreted in pure pair (A, G). If $\lambda = \omega$ then usually instead of τ_λ — interpretability says formally (or elementarily) interpreted (definable).

We redefine the notion of a pure triple in the frame of the study of Jonsson theories. First we recall the original definition of a pure triple from [1].

Definition 6. A triple $(A,\, G,\, N)$ is called a pure triple, if:

1) $(A,\, G)$ is pure pair;

2) N — some class of subsets A such that $g\,(M) \in N$ for all $M \in N$ $g \in G$, where $g\,(M) = \{g(a) \, : \, a \in M\}$.

If $(A,\, G,\, N)$, $(A',\, G',\, N')$ — pure triples, $\varphi \, : \, A \to A'$ bijection then φ is called as exact similarity (isomorphism), if:

a) $G' = \{\varphi\, g\, \varphi' \, : \, g \in G\}$;

б) $N' = \{\, \varphi\,(M) \, : \, M \in N\}$.

If $(A,\, G,\, N)$ — a pure triple, $\sim$ - G — the invariant relation of equivalence on A then $\sim$ is called a congruence on $(A,\, G,\, N)$, if $\forall a \in A, \forall M \in N \; (a \in M \Rightarrow \tilde{a} \in M)$.

If$(A,\, G,\, N)$, $(A',\, G',\, N')$ — pure triples, $\varphi : A \to A'$- surjection then φ is called a compression and φ^{-1} is inflating, if:

1) The relation $\widetilde{\varphi}$ is congruence, where $a\widetilde{\varphi}b \Longleftrightarrow \varphi(a) = \varphi(b)$, $a, b \in A$;

2) The mapping $\psi : A' \to A/_{\sim}$ is exact similarity, where $\psi\,(a') = \{a \in A \, : \, \varphi\,(a) = a'\}\,, a' \in A'$.

Let T is a complete theory, $\mathfrak{C}$ is its monster-model, $G = Aut(\,\mathfrak{C})$, N- a class of all elementary submodels $\mathfrak{C}$, smaller it on capacity. Then a pure triple $(\mathfrak{C},\, G,\, N)$ will be called a semantic triple of theory T. Complete theories T_1, T_2 will be called semantically exactly similar if their semantic triple are exactly similar.

Remark 2. The notion of exact similarity of theories does not depend on the choice of monster -models and the semantic model.

Definition 7 [2]. Let T_1 and T_2 are complete theories. We will speak, as T_1 and T_2 are syntactically similar, if $f : F(T_1) \to F(T_2)$ exists bijection such that

1) restriction f to $F_n(T_1)$ is isomorphism of Boolean algebras $F_n(T_1)$ and $F_n(T_1)$, $n < \omega$;

2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in F_{n+1}(T)$,$n < \omega$;

3) $f(v_1 = v_2) = (v_1 = v_2)$.

Let T is an arbitrary Jonsson theory, then $E(T) = \bigcup_{n<\omega} E_n(T)$, where $E_n(T)$ is a lattice of $\exists$-formulas with n free variables, T^* is a center of Jonsson theory T, i.e. $T^* = Th(C)$, where C is semantic model of Jonsson T theory in the sense of [3].

Definition 8 [4]. Let T_1 and T_2 are arbitrary Jonsson theories. We say, that T_1 and T_2 are Jonsson's syntactically similar, if exists a bijection $f : E(T_1) \longrightarrow E(T_2)$ such that

1) restriction f to $E_n(T_1)$ is isomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;

2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;

3) $f(v_1 = v_2) = (v_1 = v_2)$.

Definition 9 [4]. The pure triple $\langle C, AutC, SubC \rangle$ is called the Jonsson semantic triple, where C is semantic model of T, $AutC$ is the automorphism group C, $SubC$ is a class of all subsets of the carrier C, which are carriers of the corresponding is existentially-closed submodels of C.

Definition 10 [4]. Two Jonsson theories T_1 and T_2 are called Jonsson's semantically similar if their Jonsson semantic triples are isomorphic as pure triples.

The main result of this paper is the following result, related to the above definitions.

Theorem 1. Let T_1 and T_2 are $\exists$ — complete perfect Jonsson theories. Then following conditions are equivalent:

1) T_1 and T_2 are Jonsson's syntactically similar;

2) T_1^* and T_2^* are syntactically similar (in the sense of [2]).

The following two facts are necessary for the proof.

Fact 1. For any Jonsson theory T following conditions are equivalent

1) T is perfect;

2) T^* is the model complete.

Proof: follows from the fact that the perfectness of the Jonsson theory T is equivalent to that T^* is the model companion of the theory T [5 (the proposition 3.4)], [6].

Fact 2. For any complete for $\exists$ - sentences of Jonsson theory T the following conditions are equivalent:

1) T^* is model complete;

2) for each $n < \omega$, $E_n(T)$ is Boolean algebra, where $E_n(T)$ is a lattice of $\exists$ — formulas with n free variables.

Proof: 1) $\Rightarrow$ 2) Let T^* is the model complete $\Rightarrow E_n(T^*)$ is Boolean algebra, since T^* is complete theory (elementary theory of semantic model), but $E_n(T) \subseteq E_n(T^*)$, since $T \subseteq (T^*)$.

We have 2 cases:

1) T — complete, $T = T^*$ then $\Rightarrow T$ is model $\Rightarrow E_n(T)$-complete is Boolean algebra;

2) So $T \subset T^* \Leftrightarrow T^* = Th(C)$ where is C semantic model of T, then $\forall \varphi \in T \Rightarrow \varphi \in T^*$.

Since T is complete for $\exists$-sentences,then all the $\exists$-sentences deducing from T, belong to T^*. There are no other $\exists$-sentences in T^*, since T is complete for $\exists$-sentences T^* is a complete theory. Since $E_n(T^*)$ is Boolean algebra, then in it there are complements for any φ $\exists$-sentense. In general this φ will not be $\exists$-sentence, as if $\varphi \in \Sigma$, then $\neg\varphi \in \Pi$ (Σ-set of $\exists$-sentences and Π is the set of $\forall$-sentences), but T^* is model-complete $\Leftrightarrow \forall\psi \in T$, $\exists\theta \in T^* : \psi \equiv \theta$, $\theta \in \Sigma$. But we know, that $\theta \in T^* \Leftrightarrow \theta \in T \Rightarrow$ 1) $1, 0 \in E_n(T)$; 2) $\varphi \in E_n(T) \Rightarrow \neg\varphi \in E_n(T)$; 3) $\forall\varphi \in E_n(T) \neg\neg\varphi = \varphi \Rightarrow E_n(T)$ is the Boolean algebra.

2) $\Rightarrow$ 1) $E_n(T)$ is the Boolean algebra $\Rightarrow T$ is model complete, but $T \subset T^* = Th(C)$. Let $A \in ModT \Rightarrow A$ isomorphically embedded in C, since C is semantic model. By virtue of that is model complete, $\Rightarrow$ this embedding is elementary.

Let C it not saturated, then $\exists X \subset C$, $|X| < |C|$, $\exists p \in S_1(X)$: it is incorrect, that $(C, x)_{x \in X} \models p$, but $p \cup T$ — jointly, $p \cup T^*$ jointly, therefore $\exists m \notin C$: m realizes p, then, $\exists M \models T^*$ that, $m \in M$, is an elementary extension of C of that power $\Rightarrow$ there is a semantic model C' , which is $|M|^+$- saturated and elementary extension of M of power $2^{|M|}$.

But any two semantic models are elementarily equivalent to each other, in particular $C \equiv C'$. We received contradiction since in C' is p realised. Consequently, our assumption, that C is unsaturated, wrong $\Rightarrow T$ is perfect $\Rightarrow T^*$ is model complete.

We now proceed directly to the proof of the theorem.

Let's show 1) $\Rightarrow$ 2). We have that for each is $n < \omega, E_n(T_1)$ isomorphic to $E_n(T_2)$. Let this isomorphism be realized by f_{1n} . By virtue of condition of the theorem and facts 1 and 2, for each $n < \omega, E_n(T_1)$ and $E_n(T_2)$ are the Boolean algebras.

But by virtue of perfection T_1 and $T_2 \Rightarrow T_1^*$ and T_2^* are model-complete by virtue of fact 1, therefore for each $n < \omega$, for any formula $\varphi(\overline{x})$ from $F_n(T_1^*)$ there is a formula $\psi(\overline{x})$ from $E_n(T_1^*)$ such, that $T_1^* \models \varphi \leftrightarrow \psi$. Due to the fact that the theory T_1 is $\exists$ — complete and $E_n(T_1) \subseteq E_n(T_1^*)$,(since $T_1 \subseteq T_1^*$), it follows, that $E_n(T_1) = E_n(T_1^*)$. Due to the fact that the theory T_2 is $\exists$ — complete and $E_n(T_2) \subseteq E_n(T_2^*)$, (since $T_2 \subseteq T_2^*$), it follows, that $E_n(T_2) = E_n(T_2^*)$.

For each, $n < \omega$, for each $\varphi_1(\overline{x})$, from $F_n(T_1^*)$ we set the following mapping between $F_n(T_1^*)$ and $F_n(T_2^*)$:$f_{2n}(\varphi_1(\overline{x})) = f_{1n}(\psi_1(\overline{x}))$, where $T_1^* \models \psi_1 \leftrightarrow \varphi_1$, $\psi_1 \in E_n(T_1)$.

It is easy to see that, by virtue of properties f_{1n} and the above f_{2n} is a bijection defining an isomorphism between $F_n(T_1^*)$ and $F_n(T_2^*)$. Hence T_1^* and T_2^* are syntactically similar.

We will show 2) $\Rightarrow$ 1). It is trivial, as $F_n(T_1^*)$ is isomorphic $F_n(T_2^*)$ for each $n < \omega$, and the theorem and facts 1 and 2, this isomorphism extends to all subalgebras..

The following result is known from [2].

Proposition 1. If the theories T_1 and T_2 are syntactically similar, then T_1 and T_2 semantically similar, the converse is not true.

In connection with this, we can formulate the following:

Lemma 1. Any two cosemantic Jonsson theories are Jonsson's semantically similar.

The proof follows from the definition.

Lemma 2. If two perfect $\exists$-complete Jonsson theories are Jonsson's syntactically similar, then they are Jonsson's semantically similar.

Proof. Follows from Theorem 1 and Proposition 1.

All the definitions and concepts that are not defined here, connected with the Jonsson theories, can be found in [4].

Returning to the concepts of admissibility, interpretability, and similarity from [1], we recall that theories T_1 and T_2 are called τ — similar if there are satisfying models $M_1 = T_1$, $M_2 = T_2$, such that $Th(M_1, m)_{m \in M_1}$ and $Th(M_2, m)_{m \in M_2}$ ε are similar.

The following result is correct:

Proposition 2 [1]. 1) If T_1 and T_2 ε - similar, $M_i \in N_i, M_1 \to M_2$ then $Th(M_1 m)_{m \in M_1}$ and $Th(M_2, m)_{m \in M_2}$, ε are similar.

The «τ — similarity» relation is an equivalence relation.

It is absolutely analogous, as in the case of complete theories, that all the above concepts can be redefined, namely, admissibility, interpretability, dominance, τ — similar to the case of Jonsson theories. Further, all results and their corollaries on τ — similarity in the sense of [1] can be correlated to the study of the syntactic and semantic similarity of Jonsson theories in the above. We note that the syntactic similarity of two Jonsson theories, defined above preserves all types of morphisms proposed for the study of Jonsson theories in the frame of the study of Jonsson analog of τ — similarity for complete theories. In conclusion, we note that this topic also has a promising continuation in the frame of the study of Jonsson sets in the given Jonsson theories [7-9].

We would like to give some examples of syntactic similarity of certain algebraic examples. Fot this, we recall the basic definitions associated with these examples following denotions from B. Poizat [10].

A Boolean ring is an associative ring with identity, in which $x^2 = x$ for any x is called a Boolean ring; then, we have also $(\mathrm{x} + \mathrm{y})^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$ and besides$(x + y)^2 = x + y$; therefore $xy + yx = 0$ for an arbitrary x, y; $x^2 + x^2 = 0$ means $x + x = 0$, for any x or $x = -x$; hence the Boolean ring has characteristic 2 and, since $xy = -yx = yx$, it is commutative.

To axiomatize this concept, we introduce a language containing two symbols of constants 0 and 1, two symbols of binary relations + and $\cdot$.

We write down some universal axioms expressing that A is the Boolean ring , without forgetting thus $0 \neq 1$. In the Boolean ring we will define two binary operations $\wedge$ and $\vee$, and unary operation $\neg$ as follows: $x \wedge y = x \cdot y$; $x \vee y = x + y + xy$; $\neg x = 1 + x$.

It is easy to verify that the following are true for all x, y and z:

(de Morgan's laws or duality): $\neg(\neg x) = x$, $\neg (x \wedge y) = \neg x \vee \neg y \neg (x \vee y) = \neg x \wedge \neg y$;

$x \vee x = x \wedge x = x$;

(associativity $\wedge$): $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;

(associativity $\vee$): $(x \vee y) \vee z = x \vee (y \vee z)$;

(distributivity $\wedge$ over $\vee$):$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

(distributivity $\vee$ over $\wedge$): $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;

(commutativity$\wedge$ over $\vee$): $x \wedge y = y \wedge x$,$x \vee y = y \vee x$;

$x \wedge \neg x = 0, x \vee \neg x = 1$;

$x \wedge 0 = 0, x \vee 0 = \mathrm{x}, x \wedge 1 = x, x \vee 1 = 1$;

$0 \neq 1, \neg 0 = 1, \neg 1 = 0$.

A structure in language $> 0, 1, \neg, \wedge, \vee$ satisfying to these universal axioms is called a Boolean algebra.

Fact 3 [10]. In each Boolean ring one can interpret a certain Boolean algebra.

Proof. With the Boolean ring A we have connected some Boolean algebra $b\,(A)$; the converse is also true: $x \cdot y = x \wedge y$ $x + y = (x \vee y) \wedge (\neg x \vee \neg y)$ then we receive the Boolean ring $a\,(B)$; and besides $a\,(b\,(A)) = A$ $b\,(a\,(B)) = B$. Thus we see that, up to a language, the Boolean ring and Boolean algebras have the same structures, the Boolean ring canonically is transformed into a Boolean algebra and vice versa, transformations in both directions are carried out using quantifier-free formulas

The following example connects Boolean algebras with abelian groups.

Fact 4 [11]. In each Boolean algebra one can interpret an Abelian group.

Proof. In Boolean algebra A we suppose $a + b = (a \wedge b') \vee (a' \wedge b)$.

$[A, +]$ is Abelian group and in which each not unit element has an order 2.

The element 0 is group unit in G, and each element x is reciprocal to itself: $x + x = 0$ for all $\mathrm{x} \in \mathrm{A}$.

We state the obtained results.

Let's denote through T_{BA}, T_{BR}, T_{AG} accordingly theories in their signatures (they are different) of Boolean algebras, Boolean rings, and Abelian groups.

Lemma 3. T_{BA}, T_{BR}, T_{AG} are examples of Jonsson theories.

Proof. T_{BA} and T_{BR} from [4], T_{AG} from [12].

Theorem 2. Theories T_{BA} and T_{BR} are syntactically similar, and mutually interpreted among themselves, as for complete theories and for Jonsson theories.

Proof. Follows from the fact 3.

Theorem 3. Theory T_{BA} is interpreted in theory T_{AG}, as for complete theories and for the Jonsson theories.

Proof. Follows from fact 2 and Theorem 2.

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Йонсондық теориялардың рұқсаттылылығы және ұқсастылығы

Мақала йонсондық теорияларды рұқсаттылылық, интерпретациялануы, синтаксистік және семантикалық ұқсастылық ұғымдарымен байланысты. Айтылған жаңа ұғым йонсондық теориялар үшін синтаксис және семантикалық ұқсастық ұғымын жалпылайды. Рұқсаттылылық, интерпретациялау, басымдылық және ұқсастық ұғымдары қарастырып отырған морфизмдерге қатысты йонсондық теориялардың семантикалық модельдің анықталған формулалық ішкі жиындарды сақтауының зерттеу барысында өте маңызды рөлін атқарады. Негізгі мысал ретінде буль алгебралар теориясы қарастырылды. Бұл ұғымдар бір теорияның басқа теориямен рұқсаттылылығы ұғымымен байланысты. Бұл жұмыстың қызығушылығы осында қарастырылған мысалдар рұқсаттылылық, интерпретациялау және ұқсастылықты әртүрлі сигнатурадағы алгебраларда зерттелуі мүмкін екенін көрсетті. Мақалада ұғымдардың негізгі анықтамалары және әріректегі зерттеулердің негізгі бағыттары берілген.

Кілт сөздер: йонсондық теория, семантикалық модель, йонсондық теориялардың рұқсаттылылығы, интерпретациялануы, басымдылығы, синтаксистік және семантикалық ұқсастылық.

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Допустимость и подобие йонсоновских теорий

Статья связана с понятиями допустимости, интерпретации, синтаксического и семантического подобия для йонсоновских теорий. В рамках изучения сохранения определимых формульных подмножеств семантической модели йонсоновской теории относительно рассматриваемых морфизмов понятия допустимости, интерпретируемости, доминируемости и подобия играют очень важную роль. В качестве примера рассмотрена теория булевых алгебр. Эти понятия связаны с допустимостью одной теории с другой. Одним из интересных моментов этой работы является то, что представленные примеры допустимости, интерпретации, подобия изучены в различных сигнатурах. В статье даны ключевые определения понятий и основные направления дальнейших исследований.

Ключевые слова: йонсоновская теория, семантическая модель, допустимость, интерпретируемость, доминируемость йонсоновских теорий, синтаксическое и семантическое подобие.

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Criterion for the cosemanticness of the Abelian groups in the enriched signature

In the present paper we give a criterion of the cosemanticness relative to the Jonsson spectrum of the model in the class of Abelian groups with a distinguished predicate. This paper is devoted to the study of model-theoretic questions of Abelian groups in the frame of the study of Jonsson theories. Indeed, the paper shows that Abelian groups with the additional condition of the distinguished predicate satisfy conditions of Jonssonness and also the perfectness in the sense of Jonsson theory. It is well known that classical examples from algebra such as fields of fixed characteristic, groups, abelian groups, different classes of rings, Boolean algebras, polygons are examples of algebras whose theories satisfy conditions of Jonssonness. The study of the model-theoretic properties of Jonsson theories in the class of abelian groups is a very urgent problem both in the Model Theory itself and in an universal algebra. The Jonsson theories form a rather wide subclass of the class of all inductive theories. But considered Jonsson theories in general are not complete. The classical Model Theory mainly deals with complete theories and in case of the study of Jonsson theories, there is a deficit of a technical apparatus, which at the present time is developed for studying the model-theoretic properties of complete theories. Therefore, the finding of analogues of such technique for the study of Jonsson theories has practical significance in thegiven research topic. In this paper the signature for one-place predicate was extended. The elements realizing this predicate form an existentially closed submodel of the considering Jonsson theory's some model. In the final analysis, we obtain the main result of this article as a refinement of the well-known W. Szmielew's theorem on the elementary classification of Abelian groups in the frame of the study of Jonsson theories, thereby the generalization of the well-known question of elementary pairs for complete theories was obtained. Also we obtained the Jonsson analogue for the joint embeddability of two models, or in another way the Schröder-Bernstein properties in the frame of the study of the Johnson pairs of Abelian groups' theory.

Keywords: Jonsson theory, model companion, existentially closed model, perfectness, cosemanticness, Jonsson spectrum, Jonsson pair.

This paper is concerned with the study of certain model-theoretic properties of Jonsson theories in the class of Abelian groups. The class of Jonsson theories is wide enough and it is a natural subclass of the class of all inductive theories. The definition of the Jonsson theory is quite natural. Many theories of a well-known and classical algebras are essentially examples of Jonsson theories. Examples include the following algebras: groups, Abelian groups, fields of fixed characteristic, Boolean algebras, many classes of rings, polygons. As a rule, the considered Jonsson theories are not complete and since the classical Model Theory deals mainly with complete theories, in the case of the Jonsson theories there is a deficit of the technical apparatus, which is accordingly developed at the present time for studying the model-theoretic properties of complete theories. Therefore, the finding of analogues of this technique and accordingly, concepts for the study of Jonsson theories, is a very urgent problem.

In the paper [1] was considered the problem related to the concept of cosemantic and Schröder-Bernstein properties for Jonsson theory of Abelian groups. The concept of cosemanticness is a generalization of the concept of elementary equivalence, which is used as an important tool in the study of complete theories. The Schröder-Bernstein property is also related to models of some fixed complete theory. That is, like the property of cosemanticness, this property is a semantic concept, in contrast to the properties of the theory, which we attribute syntactic properties to. By virtue of the theorem about completeness, the duality of syntax and semantics allows us to seek new connections between the model-theoretic properties of theories and their classes of models. In the Jonsson case, because of the incompleteness, it is impossible to directly use this duality, and in this case we resort to the so-called semantic method, the essence of which is the «transfer» of elementary properties of the first order of the elements of Jonsson theory's center to the theory itself. It turned out that the theory of Abelian groups is an example of the perfect Jonsson theory. In this connection, it was possible to find a criterion for the cosemanticness of Abelian groups [1] and Jonsson analog of the Schröder-Bernstein property.

We note how the study of Jonsson theories differs fundamentally from the study of complete theories. The classification of a fixed complete theory and its class of models with respect to certain syntactic and semantic conditions is one of the most important tasks of the classical Model Theory. In Model Theory itself, as noted in the review article by H.J. Keisler, «Foundations of Model Theory» in the reference book, ed. J. Barwise [2], historically there were two directions. In [2] they are called «western» and «eastern» Model Theory, these names are conditional, they are related to the geographical location of the founders of the Model Theory. A. Robinson lived on the east coast of the USA and A. Tarski lived on the western coast.

«Western» Model Theory develops in the traditions of Skolem and Tarski. It was more motivated by problems in number theory, analysis and set theory, and it uses all formulas of first-order logic. In particular, various types of elementary morphisms are considered as morphisms in the «western» theory of models. «Eastern» Model Theory develops in the traditions of Maltsev and Robinson. It was motivated by problems in abstract algebra, where the formulas of theories usually have at most two blocks of quantifiers. It emphasizes the set of quantifier-free formulas and existential formulas. In the «eastern» Model Theory, as a rule, homomorphisms and isomorphisms are considered as morphisms.

Thus, we can see that when dealing with the model-theoretic attributes of the «eastern» Model Theory, we tend to deal with incomplete theories and morphisms between their models, which maximally preserve the properties of Boolean combinations of atomic formulas. As a model at the study of this type of theories, as a rule, we consider a subclass of class of all models of the considering theory, namely, the class of its existentially closed models.

In this paper we extend the signature by a one-place predicate, the essence of which is that the elements realizing this predicate form an existentially closed submodel of some model of considering Jonsson theory. Thus, we can say that we turn to the situation when the considering problem generalizes the problem of elementary pairs.

The Jonsson theories satisfy natural conditions, such as inductive, the joint embedding property and amalgam [2 (def. 6.1, p. 80)]. T. G. Mustafin in the work [3] generalized Jonsson theories and found a connection between the complete theories, Jonsson theories and the generalized Jonsson theories. In the work [4] Yeshkeyev A.R. was continued the study of Jonsson theories concerning the various model-theoretic properties of their companions, including J-stability. In particular, in the frame of the study of Jonsson theories was redefined an important notion such as forcing, which was earlier defined by S. Shelah [5] and is one of the main tools of modern technique of Model Theory in the classification of complete theories. Further A.R.Yeshkeyev were defined new classes of positive Jonsson theories and in the paper [6] were obtained positive Jonsson analogues of F. Weispfenning's work [7] for the positive lattice of the existential formulas of considered theory. The concept of positive Jonsson theories was first considered in the paper [8] and this concept, in a certain sense, was introduced after the appearance of series of I. Ben-Yaakov's works [9-10], as both concepts of theory's positivity from [8] and [9] coincide for the minimal fragment of considered theory. This implies, in particular, non-triviality (not just a generalization of the Jonssonness for generalization) of the concept of positivity in the sense of [8], because, for example, such an important class of mathematical structures as metric spaces is not Jonsson class, but is positive Jonsson in the sense of the works [9-10] and, in particular, in the sense of [8] for the minimal fragment. It should be noted that there are various regular ways of transition from an arbitrary theory to Jonsson theory, which preserves the original class of existentially closed models. One of these methods is the Morleisation of theory [2 (Theorems 2.18, 2.19, p. 63-64, Theorem 6.8′, p. 83)]. Thus, the study of model-theoretic properties of Jonsson theories is an actual problem, both in Model Theory and in universal algebra, and the questions concerning the study of Jonsson theories are exactly the essence of the problem of «eastern» Model Theory.

The study of the model-theoretic properties of complete theories of Abelian groups is a large subsection of model-theoretic algebra. Many classical results have been obtained in this field of research, in particular, the complete classification of Abelian groups up to elementary equivalence was carried out in the work of Polish mathematician W. Szmielew [11].

The following references to the relevant sources will allow the reader to obtain exhaustive information on this classification [11-15].

The concept of an elementary pair was first determined by B. Poizat in [16]. The following stages in the development of study of this concept were noted in works of E. Bouscaren and B. Poizat [17-19]. Further, the history of studying this concept is related to the work [20]. T.G. Mustafin in this paper considered new concepts of stability and showed that one of them in the particular case is the case of an elementary pair. Also to this period are the works of T. Nurmagambetov and B. Poizat [21]. In the future there are papers by E.A. Palyutin [22, 23]. In these papers E.A. Palyutin clarifies the concept of T^*-stability introduced by T.G. Mustafin in [20] with the help of concept of E^*-stability, and in a fairly wide class of primitive-normal theories consider the model-theoretic properties of elementary pairs. Recall that all these works were done in the frame of study of complete theories. It is clear that the transition to generally speaking incomplete theories on the example of Jonsson theories and, in particular, the theory of Abelian groups, is uniquely an actual continuation of above studies.

All the indefinite concepts and results associated with them in this article on Jonsson theories can be found in [24].

In brief we give the main generally accepted notation associated with Abelian groups.

If A is an arbitrary Abelian group, n is an integer, then $nA = \{na : a \in A\}$. It is not difficult to see that both nA and $A[p] = \{a \in A : pa = 0\}$ for a simple p form subgroups of the group A. We say that $n \neq 0$ divides the element a from A if $a = nb$ for some $b \in B$. If there exists $m > 0$ such that $ma = 0$, then the smallest such m is called the order of the element a. Thus, nA is the set of elements of A that are divisible by n and $A[p]$ is the set of all elements of A of order p. The subgroup $(nA)[p]$ is usually denoted by $nA[p]$. The set $T(A)$ of all elements of A of finite order is called the periodic part of A. It is clear that $T(A)$ is a subgroup of A and the factor group $A/_{T(A)}$ is torsion-free, that is group that does not have nonzero elements of finite order. If every element of A has order equal to p^n for some $n \geqslant 1$, then the periodic group A is called a p-group. A group A is called a group of bounded order if $nA = [0]$ for some natural n.

A group A is said to be divisible if for any $a \in A$ and any $n \in \mathbb{Z}\backslash\{0\}$ there exists $b \in A$ such that $a = nb$. If B is a divisible group and at the same time a subgroup of A, then it is called a divisible subgroup of A. A group that does not contain non-zero divisible subgroups is said to be reduced. A subgroup B of a group A is said to be servant if $nA \cap B = nB$ for all $n \in \mathbb{Z}$. We say that a subgroup B of a group A is distinguished in it by a direct summand if there exists a subgroup C of the group A for which $A = B \oplus C$.

A group A is said to be algebraically compact if it is distinguished by a direct summand in every group that contains A as a pure subgroup. These groups have a number of interesting properties. For example, the group A is algebraically compact $\Leftrightarrow$ in it any compatible countable set of equations from any number of unknowns with constants in A is solvable. It is easy to see that ω_1^+-saturated groups are always algebraically compact. The structure of algebraically compact groups is well studied. The following examples of Abelian groups are canonical in the study of their elementary theories.

1. Q – the additive group of rational numbers, called the complete rational group.
2. $Z_p = \left\{\frac{m}{n} : m, n \in Z, (n,p) = 1\right\}$.
3. Z_{p^n} – cyclic group of order p^n.
4. Z_{p^∞} – the multiplicative group of all the roots of equations $x^{p^n} = 1$, $n = 1, 2, ...$, from the field of complex numbers, called a quasicyclic group of type p^∞, where p is prime number.

Remark. The group Z_{p^∞} can be defined as the additive group generated by elements $c_1, c_2, ..., c_n, ...$, where $pc_1 = 0$, $pc_2 = c_1$, ..., $pc_{n+1} = c_n$.

Let A be a model of signature of Abelian groups, where $\sigma_{\text{AG}} = \langle +, -, 0\rangle$.

A formula of the form $\exists x_1 ... \exists x_n \varphi$, where φ is the conjunction of atomic formulas, is called positively primitive (p.p. formula). P.p. formulas express the solvability of finite systems of linear equations of the form $m_1x_1 + m_2x_2 + ... + m_kx_k = 0$. It is not difficult to show that p.p. formulas are closed with respect to the conjunction and suspension of the existence quantifier. One can directly verify that the truth of p.p. formulas is preserved under extensions, cartesian products and homomorphisms of Abelian groups.

The following facts are well known.

Theorem 1. Let A be an arbitrary Abelian group. Each formula of signature σ_{AG} is equivalent regarding $Th(A)$ of the Boolean combination of p.p. formulas.

Sentence 1. Q and Z_{p^∞} are divisible groups.

Theorem 2. Every divisible periodic Abelian group G is a direct sum of quasicyclic groups (possibly on different prime numbers).

It is known that any group A can be decomposed as follows:

$$A = A_d \oplus A_r,$$

where A_d is the single maximal divisible subgroup of A, A_r is a reduced subgroup, i.e. group without non-zero divisible subgroups. Algebraically compact groups are constructed in a certain way from the indecomposable groups Z_{p^∞}, Z_{p^n}, Z and Q, where p is a prime number.

The divisible group A_d has the following decomposition:

$$A_d \cong \oplus_p Z_{p^\infty}^{(\gamma_p)} \oplus Q^{(\delta)},$$

where γ_p (p is a prime number) and δ are arbitrary cardinal numbers, $\oplus_p$ means a direct summation over all simple p, $Z_{p^\infty}^{(\gamma_p)}$ is the direct sum of γ_p -copies of quasicyclic groups Z_{p^∞}, and $Q^{(\delta)}$ is the direct sum of δ-copies of the additive group of rational numbers.

Recall that two models $\mathfrak{A}$ and $\mathfrak{B}$ of the same language $\mathfrak{L}$ of the first order are called *elementarily equivalent* if the same the first-order sentence of language $\mathfrak{L}$ are realized in the above models.

We define the Szmielew's invariants. In future we assume that $\infty < \kappa$ for any cardinal $\kappa \geq \omega$.

For any abelian group A and any simple p:

- $U(p,n;A) = \min\left\{\omega,\ \dim_p\left(p^nA[p]\big/_{p^{n+1}A[p]}\right)\right\}$;
- $T_f(p;A) = \min\left\{\omega,\ \inf\limits_n \dim_p\left(p^nA\big/_{p^{n+1}A}\right)\right\}$;
- $D(p;A) = \min\left\{\omega,\ \inf\limits_n \dim_p\left(p^nA[p]\right)\right\}$;
- $Exp(A) = \begin{cases} 0, & \text{if the group } A \text{ is of bounded order;} \\ \infty, & \text{otherwise,} \end{cases}$

where $\dim_p$ is the dimension of the corresponding vector space over the field $Z\big/_{pZ}$.

This is the elementary Szmielew's invariants.

We denote by $W(A)$ the ordered sequence of elementary Szmielew's invariants of group A:

$$W(A) = \langle\langle\langle U(p,n;A) : n \in \omega\rangle,\ T_f(p;A),\ D(p;A) : p = 2,3,...\rangle,\ Exp(A)\rangle .$$

Theorem 3. Let A, B be two arbitrary groups. Then the following conditions are equivalent:

1. $A \equiv B$.
2. W(A)=W(B).

We define the standard group of Szmielew A^0 for any group A:

$$A^0 = \oplus_{p,n} Z_{p^n}^{(\alpha^0_{p_n})} \oplus \oplus_p Z_p^{(\beta^0_p)} \oplus \oplus_p Z_{p^\infty}^{(\gamma^0_p)} \oplus Q^{(\delta^0)},$$

where $\alpha^0_{p_n} = \min\left(U(p,n-1;A),\omega\right)$, $\beta^0_p = \min\left(T_f(p;A),\omega\right)$, $\gamma^0_p = \min\left(D(p;A),\omega\right)$, $\delta^0 = \min\left(Exp(A),1\right)$.

Theorem 4. $A \equiv A^0$.

Theorem 5. Any Abelian group can be embedded as a subgroup of a divisible group.

We give some well-known definitions of concepts and results related to the Jonsson theories, which are necessary for studying Abelian groups in the frameof the Jonssonness.

Definition 1. The theory T is called Jonsson if:

1) T has an infinite model;

2) T is inductive, i.e. T is equivalent to the set $\forall\exists$-propositions;

3) T has the joint embedding property (JEP), that is, any two models $\mathfrak{A} \models T$ and $\mathfrak{B} \models T$ are isomorphically embedded in a certain model $\mathfrak{C} \models T$;

4) T has the property of amalgamation (AP), that is, if for any $\mathfrak{A},\mathfrak{B},\mathfrak{C} \models T$ such that $f_1 : \mathfrak{A} \to \mathfrak{B}$, $f_2 : \mathfrak{A} \to \mathfrak{C}$ are isomorphic embeddings, exist $\mathfrak{D} \models T$ and isomorphic embeddings $g_1 : \mathfrak{B} \to \mathfrak{D}$, $g_2 : \mathfrak{C} \to \mathfrak{D}$ such that $g_1 f_1 = g_2 f_2$.

Let's define the semantic model. This model plays an important role as a semantic invariant. Such model always exists for any Jonsson theory. In future we will use the so-called semantic method [24] in the study of Jonsson Abelian groups. The essence of this method consists in translating the elementary properties of a fixed complete theory (the center of Jonsson theory) to Jonsson theory itself.

Initially, the concept of semantic model assumed another concept of homogeneity, but to prove the existence of a semantic model it was necessary to add to the axiom of the theory of sets ZF the axiom of the existence of a strongly inaccessible cardinal. To eliminate this axiom, it was necessary to change the definition of homogeneity of the semantic model to an acceptable variant. This was done in the work [25] Y.T. Mustafin. The concept of a universal model does not change. Recall it.

Definition 2. Let $\kappa \geq \omega$. The model $\mathfrak{M}$ of theory T is said to be κ-universal for T if every model T of cardinality is strictly less than κ is isomorphically embedded in $\mathfrak{M}$.

The following definition of κ-homogeneity for the model was introduced in [25].

Definition 3 [25]. Let $\kappa \geq \omega$. The model $\mathfrak{M}$ of theory T is said to be κ-homogeneous for T if for any two models $\mathfrak{A}$ and $\mathfrak{A}_1$ of T, which are submodels of $\mathfrak{M}$, the cardinality is strictly less than κ, and the isomorphism $f : \mathfrak{A} \to \mathfrak{A}_1$, for each extension $\mathfrak{B}$ of the model $\mathfrak{A}$, which is a submodel of $\mathfrak{M}$ and a model T of cardinality strictly less than κ, there exists an extension $\mathfrak{B}_1$ of the model $\mathfrak{A}_1$, which is a submodel of $\mathfrak{M}$, and an isomorphism $g : \mathfrak{B} \to \mathfrak{B}_1$ that extends f.

A homogeneous-universal model for T is a κ-homogeneous-universal model for T of cardinality κ, where $\kappa \geq \omega$.

Theorem 6 [25]. Each Jonsson theory T has a κ^+-homogeneous-universal model of power 2^κ. Conversely, if T is inductive, has an infinite model, and has a ω^+-homogeneous-universal model, then T is Jonsson theory.

Theorem 7 [25]. Let T be Jonsson theory. Two models $\mathfrak{M}$ and $\mathfrak{M}_1$ κ-homogeneous-universal for T are elementary eivalent.

Definition 4 [25]. The semantic model C_T of Jonsson theory T is the ω^+-homogeneous-universal model of theory T.

Sentence 2 [25]. Any two semantic models of Jonsson theory T are elementarily equivalent to each other.

Definition 5 [24]. The semantic completion (center) of Jonsson theory T is the elementary theory T^* of the semantic model C_T of theory T, that is, $T^* = Th(C_T)$.

Let T be some Jonsson theory of fixed signature σ and $\operatorname{Mod} T$ the class of all models of theory T. Consider an arbitrary model A from $\operatorname{Mod} T$. We define the following notion by means of which we are going to distinguish the models of Jonsson theory. Let $JSp(A) = \{T \mid T$ be Jonsson theory in the language σ and $A \in \operatorname{Mod} T\}$ and call $JSp(A)$ Jonsson spectrum of model A.

The following definitions 6, 7 belong to T. G. Mustafin.

Definition 6 [24]. We say that Jonsson theory T_1 is cosemantic to Jonsson theory T_2 ($T_1 \bowtie T_2$) if $C_{T_1} = C_{T_2}$, where C_{T_i} is the semantic model of T_i, $i = 1, 2$.

The relation of the cosemanticness on the set of theories is an equivalence relation. Then $JSp(A)/_{\bowtie}$ is the factor-set of Jonsson spectrum of model A with respect to $\bowtie$.

Definition 7 [24]. Jonsson theory of T is called perfect if every semantic model of T is a saturated model of T^*.

Definition 8 [2]. The theory T is called model-complete if for any models $\mathfrak{A}$ and $\mathfrak{B}$ of T, any subsystem $\mathfrak{A} \subseteq \mathfrak{B}$ is an elementary subsystem $\mathfrak{B}$. Equivalently, every isomorphic embedding is an elementary embedding.

Theorem 8 [2]. The theory T is model-complete if and only if theory $T \cup D(\mathfrak{M})$ is complete for any model $\mathfrak{M}$ of theory T.

Definition 9 [2]. Let T, T^* be some L-theories. The theory T^* is called a model completion of theory T if:

(a) T and T^* are mutually model joint, i.e. any model of theory T is embedded in the model of T^* and vice versa;

(b) T^* is model complete theory;

(c) if $\mathfrak{M} \models T$, then $T^* \cup \operatorname{Diagram}(\mathfrak{M})$ is complete theory.

The theory T^* is called a model companion of T if conditions (a) and (b) are satisfied.

Theorem 9 [2]. The theory T has no more than one model companion.

Theorem 10 [2, p. 68, table 1)]. The theory of algebraically closed Abelian groups is a model complement to the theory of Abelian groups.

Theorem 11 [24]. Let T be an arbitrary Jonsson theory, then the following conditions are equivalent:

1) the theory T is perfect;

2) T^* is a model companion of the theory T.

Let E_T be the class of all existentially closed models of theory T.

Theorem 12 [24]. If Jonsson theory T is perfect, then $E_T = \operatorname{Mod} T^*$, where $T^* = Th(C_T)$.

Let T be Jonsson theory, $S^J(X)$ be the set of all existential complete n-types over X that are compatible with T for any finite n, where $X \subset C$.

Definition 10 [24]. We say that Jonsson theory T J-λ-stable if for any T-existentially closed model $\mathfrak{A}$ for any subset X of A, $|X| \leq \lambda \Rightarrow |S^J(X)| \leq \lambda$.

Theorem 13. Let T be a perfect Jonsson theory complete for $\exists$-propositions, $\lambda \geq \omega$. Then the following conditions are equivalent:

1) T is J-λ-stable;

2) T^* is λ-stable, where T^* is center of Jonsson theory T.

Proof. Follow from Theorem 2.1 from [26].

We consider the language L_P obtained by adding the one-place predicate $P(x)$ to the language L. Denote by T_P the theory obtained by adding to T axioms, which state that the interpretation of P is also a model of theory T. We can say that P is interpreted as an existentially closed substructure, i.e. for each quantifier formula φ in L the following is true: $(\forall \bar{x})\,[P(\bar{x}) \wedge (\exists \bar{y})\varphi(\bar{x},\bar{y}) \rightarrow (\exists \bar{z})P(\bar{z}) \wedge \varphi(\bar{x},\bar{z})]$, where $P(\bar{x})$ means $P(x_1) \wedge \ldots \wedge P(x_n)$.

The model of theory T_P is called Jonsson pair (J-pair) of models of T. We denote this pair (N, M), where M is the interpretation of the predicate $P(\bar{x})$. In this pair we call N a large model, and M a small model.

We denote by $T_{P_{\mathrm{AG}}}$ the theory of Jonsson pairs of Abelian groups' theory. Let M be the class of models of $T_{P_{\mathrm{AG}}}$.

The following lemmas are necessary to prove the sentence 3.

Lemma 1. If $\mathfrak{A}, \mathfrak{B} \in M$; $\mathfrak{C} \in M$ and $\mathfrak{C} \subseteq \mathfrak{A}$, $\mathfrak{C} \subseteq \mathfrak{B}$; $|\mathfrak{C}| = |\mathfrak{A}| \cap |\mathfrak{B}|$, then there exists a system $\mathfrak{D} \in M$ such that $\mathfrak{A} \subseteq \mathfrak{D}$, $\mathfrak{B} \subseteq \mathfrak{D}$.

Proof. Let A and B be Abelian groups and $A \cap B = C$, $C \subseteq A$, $C \subseteq B$. Let $A \underset{C}{\times} B$ be the free product of groups A and B with the amalgamated subgroup C. Then $A \subseteq A \underset{C}{\times} B$ and $B \subseteq A \underset{C}{\times} B$.

Lemma 2. If M is an abstract and satisfies the lemma 1, then M satisfies the amalgamation property (AP).

Proof. The proof can be extracted from Theorems A, B of the paper [27], but we need only take into account the new definition of homogeneity.

Sentence 3. The theory $T_{P_{\mathrm{AG}}}$ is the perfect Jonsson theory.

Proof. We first show that $T_{P_{\mathrm{AG}}}$ is Jonsson theory. $T_{P_{\mathrm{AG}}}$ has an infinite model. It is inductive, because the union of an increasing chain of Abelian groups is Abelian group. That is the conditions (1) and (2) from the definition 1 are satisfied.

If A and B are two J-pairs of theory $T_{P_{\mathrm{AG}}}$, then their direct product $A \times B$ is Abelian group. The set of elements $< a, e^B >$, where $a \in A$, e^B is the unit element of B, is a subgroup of $A \times B$ isomorphic to A. Similarly, the set of elements $< e^A, b >$, where $b \in B$, e^A is the unit element of A, is a subgroup of $A \times B$ isomorphic to B. Thus condition (3) is satisfied.

Let us verify the satisfaction of condition (4). As the class of Abelian groups is abstract that is is closed with respect to isomorphisms, then according to the lemma 1 and the lemma 2, the class of Abelian groups has amalgamation property (AP). Thus $T_{P_{\mathrm{AG}}}$ is Jonsson theory.

Perfection follows from Sentence 3 of [1], since due to the perfection of theory of Abelian groups, the semantic model of this theory is saturated in its power. Consider the semantic model (N, M) of theory $T_{P_{\mathrm{AG}}}$. Let the realization of predicate P by the small model M be an existentially closed submodel of the large model N. All types over this model are realized in the large model N by virtue of Sentence 3 from [1]. There are no other new types, thus, the J-pair (N, M) is saturated in its power.

The next notion was considered by J. Goodrick in [28] and there he denoted it as a Schröder-Bernstein (SB) property.

Definition 11. The theory T admits the property SB if for any two mutually elementary embeddable models of the theory T it follows that they are isomorphic.

But J. Goodrick notes that this property was first considered for ω-stable theories by T.A. Nurmagambetov. In the works [29, 30]. In particular, T.A. Nurmagambetov was obtained the following result with respect to the property SB.

Theorem 1.2 from [29] If T is ω-stable theory, then T has an SB property if and only if T of bounded dimension.

J. Goodrick in the paper [28] received a description of the SB property for a classifiable (superstable, with NDOP and NOTOP) theory with a limited dimension. In particular, in work [31] J.Goodrick and M.Laskovsky described the property SB for weakly minimal theories.

Later J.Goodrick [32] found necessary and sufficient conditions for the theory of abelian groups to admit the SB property. Namely, he proved the following theorem:

Theorem 3.8 from [32] If G is Abelian group, then the following conditions are equivalent:

1. $Th(G, +)$ has the Schröder-Bernstein property.
2. $Th(G, +)$ is ω-stable.
3. G is the direct sum of a divisible group and a group with torsion of a bounded exponent.
4. $Th(G, +)$ is superstable, and if $(\overline{G}, +) \equiv (G, +)$ is saturated, then every map in $Aut(\overline{G}/\overline{G}^0)$ is unipotent.

We redefined this concept for Jonsson theories and is denoted as JSB, namely: Jonsson theory T has JSB property if for any two existentially closed models $\mathfrak{A}$ and $\mathfrak{B}$ of theory T from the fact that they are mutually isomorphically embedded into each other it follows that they are isomorphic.

The following result is Jonsson analog of Theorem 3.8 from the paper [32] in the enriched language, namely:

Theorem 14. Let $T_{P_{\rm AG}}$ be the theory of Jonsson pairs of Abelian groups' theory, then the following conditions are equivalent:

1) $T_{P_{\rm AG}}$ is J-ω-stable;
2) $T^*_{P_{\rm AG}}$ is ω-stable;
3) $T_{P_{\rm AG}}$ has the JSB property.

Proof. By the sentence 3, the equivalence of items (1) and (2) follows from theorem 13, since in theorem 13 we use $\exists$-completeness and then, by the theorem 1, we can apply the theorem 13.

We will show from (3) in (2). By virtue of the sentence 3, the theory of Jonsson pairs of Abelian groups' theory is a perfect Jonsson theory. Then by virtue of the criterion of perfectness 11 and theorem 12 we have that $E_{T_{P_{\rm A}}} = \operatorname{Mod} T^*_{P_{\rm AG}}$. Consequently, any model of the center of theory $T_{P_{\rm AG}}$ is existentially closed. By the theorem 11 $T^*_{P_{\rm AG}}$ is the center of theory $T_{P_{\rm AG}}$ is a model companion of theory $T_{P_{\rm AG}}$. Hence $T^*_{P_{\rm AG}}$ is the model complete theory and any embedding is elementary. It remains to apply Theorem 3.8 from [32].

We will show from (2) in (3). Let the center $T^*_{P_{\rm AG}}$ be ω -stable. Since $T_{P_{\rm AG}} \subset T^*_P$ it follows that $\operatorname{Mod} T^*_{P_{\rm AG}} \subset \operatorname{Mod} T_{P_{\rm AG}}$. In view of Theorem 3.8 from [32] $T^*_{P_{\rm AG}}$ admits SB. But by virtue of Theorem 11 and the perfectness of theory $T_{P_{\rm AG}}$ $\operatorname{Mod} T^*_{P_{\rm AG}} = E_{T_{P_{\rm AG}}}$. And this completes the proof, because the Josnson property of JSB is defined only for models from $E_{T_{P_{\rm AG}}}$.

The following result concerning Jonsson Abelian groups is an analog of W. Szmielew's theorem on the elementary classification of Abelian groups. We know that for any Abelian group G there exists a standard group G^0 such that $G \equiv G^0$, and in this case

$$G^0 = \oplus_{p,n} Z_{p^n}^{(\alpha^0_{p^n})} \oplus \oplus_p Z_p^{(\beta^0_p)} \oplus \oplus_p Z_{p^\infty}^{(\gamma^0_p)} \oplus Q^{(\delta^0)}.$$

Denote by $JSp(A)$ the Jonsson spectrum of Abelian group A, where

$$JSp(A)=\{T_{P_{\rm AG}} \,|\, T_{P_{\rm A\Gamma}} \text{ is Jonsson theory in the language } \sigma_{P_{\rm A\Gamma}} \text{ and } A \in \operatorname{Mod} T_{P_{\rm A\Gamma}}\}.$$

The following result gives a description of semantic model of Abelian groups' Jonsson theory.

Theorem 15. Let $T_{P_{\rm AG}}$ be Jonsson theory of Abelian groups in the language $\sigma_{P_{\rm AG}}$, then its center $C_{T_{P_{\rm AG}}} \in E_{T_{P_{\rm AG}}}$, while $C_{T_{P_{\rm AG}}}$ is a divisible group and its standard Szmielew group is representable as $\oplus_p \mathbb{Z}_{p^\infty}^{(\alpha_p)} \oplus \mathbb{Q}^{(\beta)}$, where $\alpha_p, \beta \in \omega^+$, $2^\omega = \left|C_{T_{P_{\rm AG}}}\right|$.

Proof. It follows from the theorems 5 and 2 and the fact that any Jonsson theory has a semantic model that is a ω^+-homogeneous-universal model.

Sentence 4. There exists a continuum of imperfect subclasses of the class of all Abelian groups.

Proof. In [33] this sentence was proved using the old definition of semantic model. For a new definition 4 of semantic model, we can repeat the proof from [33] considering only the power estimates of semantic model. To prove this fact, it suffices to show that not elementary equivalent semantic models of imperfect Jonsson theories of Abelian groups will be a continuum. From Theorem 15 the semantic model of any Abelian group will be the direct sum of the corresponding number of groups' copies of two kinds: $\mathbb{Z}_{p^\infty}$ and Q. In the imperfect case, only Q can be absent since Q can not be a universal model. The number of copies of $\mathbb{Z}_{p^\infty}$ can be any subset of ω.

We call the pair $(\alpha_p, \beta)^A_{C_{[T_{P_{\rm AG}}]}}$ Jonsson invariant of Abelian group A if the standard group of Szmielew's group A is representable in the form $\oplus_p \mathbb{Z}_{p^\infty}^{(\alpha_p)} \oplus \mathbb{Q}^{(\beta)}$, where $C_{[T_{P_{\rm AG}}]}$ is semantic model of $[T_{P_{\rm AG}}] \in JSp(A)/_{\bowtie}$.

We give the following definitions of concepts, which are specified in the frame of the study of Jonsson theories, the definition of elementary equivalence for complete theories.

Let A and B be models of the same signature.

Definition 12. We will say that the model A is Jonsson elementary equivalent to the model B ($A \underset{J}{\equiv} B$) if $JSp(A) = JSp(B)$.

Lemma 3. $\forall A, B \in \operatorname{Mod} \sigma_{P_{\rm AG}}$ $JSp(A) \cap JSp(B) \neq \emptyset$.

Proof. This is true because at least $T_{P_{\rm AG}} \in JSp(A) \cap JSp(B)$.

Definition 13. We say that the model A is JSp-cosemantic to model B ($A \underset{JSp}{\bowtie} B$), if

$$JSp(A)/_{\bowtie} = JSp(B)/_{\bowtie}.$$

Lemma 4. $A \underset{JSp}{\bowtie} B \Leftrightarrow JSp(A) \cap JSp(B) = JSp(A) \cup JSp(B)$.

Proof. It follows from the definition.

It is easy to understand that the concepts introduced in the definitions 12 and 13 generalize the notion of elementary equivalence. In the following lemma, by virtue of the sentence 3, we can note that the following is true:

Lemma 5. Let A and B be arbitrary abelian groups, then

$$A \equiv B \Rightarrow A \underset{J}{\equiv} B \Rightarrow A \underset{JSp}{\bowtie} B.$$

Proof. It follows from the definition.

The following result is the Jonsson analog of a well-known W. Szmielew's theorem on the elementary classification of Abelian groups and is a corollary of Theorem 15 and Lemma 5.

We define the following set $\left\{(\alpha_p,\beta)^A_{C_{[T_{P_{\mathrm{AG}}}]}} : [T_{P_{\mathrm{AG}}}] \in JSp(A)/_{\bowtie}, \text{ for all simple } p\right\}$ as Jonsson invariant of the factor-set $JSp(A)/_{\bowtie}$ and denote it by $JInv(JSp(A)/_{\bowtie})$.

Theorem 16. Let $A, B \in \operatorname{Mod}\sigma_{P_{\mathrm{AG}}}$, $A = \langle M_1, N_1\rangle$, $B = \langle M_2, N_2\rangle$, then the following conditions are equivalent:

1) $A \underset{JSp}{\bowtie} B$;

2) $JInv(JSp(A)/_{\bowtie}) = JInv(JSp(B)/_{\bowtie})$.

Proof. From (2) to (1). If (2) is satisfied, this means that the standard Szmielew's groups for A and B coincide, then by lemma 5 it follows that $A \underset{JSp}{\bowtie} B$, i.e. (1) is satisfied.

Suppose that (1) holds, then $JSp(A)/_{\bowtie} = JSp(B)/_{\bowtie}$. Assume the contrary, i.e.

$$JInv(JSp(A)/_{\bowtie}) \neq JInv(JSp(B)/_{\bowtie}).$$

Then there exists

$$(\alpha_p,\beta)^A_{C_{[T_{P_{\mathrm{AG}}}]}} \in JInv(JSp(A)/_{\bowtie}) \text{ and } (\alpha_p,\beta)^A_{C_{[T_{P_{\mathrm{AG}}}]}} \notin JInv(JSp(B)/_{\bowtie}).$$

Therefore, for each class $[T'_{P_{\mathrm{AG}}}] \in JSp(B)/_{\bowtie}$ we have $(\alpha_p,\beta)^A_{C_{[T_{P_{\mathrm{AG}}}]}} \neq (\alpha_p,\beta)^B_{C_{[T'_{P_{\mathrm{AG}}}]}}$, i.e. there is not a single Jonsson theory of the group B, the semantic model of which is $C_{[T_{P_{\mathrm{AG}}}]}$. But it is known that any Jonsson theory is uniquely determined by its semantic model ([3], Theorem 2.2 at $\alpha = 0$). It follows that there is Jonsson theory $T_{P_{\mathrm{AG}}}$, which is determined by Jonsson invariant $(\alpha_p,\beta)^A_{C_{[T_{P_{\mathrm{AG}}}]}}$ and $T_{P_{\mathrm{AG}}} \notin JSp(B)$. And this contradicts condition (1), so our assumption is incorrect.

The main result of this paper is the content of theorem 16. In this theorem we obtained Jonsson analogue in the extended signature by the one-place predicate of a well-known theorem of Polish mathematician W. Szmielew on the elementary classification of abelian groups. Interpretation of the predicate symbol in Jonsson pair $\langle M, N\rangle$ is an existentially closed submodel of M in the large model$N$. Such a statement of the problem is a generalized Jonsson generalization of a well-known problem on elementary pairs for the complete theories [17]. On the other hand, in the frame of the study of stability in the sense of [20] and [22] in connection with theorem 14 presented in this paper, there is an obvious relationship. Thus, the enrichment of signature with a single predicate can be considered in the frame of the classification of Jonsson theories and their classes of existentially closed models. As the obtained results (Theorems 14 and 16) show, the model-theoretic properties of Jonsson pair are closely related to the model-theoretic properties of the center of the considering Jonsson theory. Since the center is a complete theory and originally Jonsson theory (the theory of abelian groups) is an example of perfect Jonsson theory, we can conclude that in the case of enriching of this theory's language by the one-place predicate, the basic properties obtained in the work [1] are also preserved, as before enrichment.

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Байытылған сигнатурада абельдік группалардың косемантикалық критериі

Мақалада белгіленген предикаты бар абельдік группалар класында модельдің йонсондық спектріне қатысты косемантикалық критерийі ұсынылды. Йонсондық теориялардың модельдерін зерттеу аясында абельдік группалардың модельді-теоретикалық сұрақтары зерттелді. Жұмыста йонсондылық шарты абельдік группалардың белгіленген предикаттың қосымша шартын қанағаттандырады, сонымен қатар йонсондық теория мағынасында кемелділігі көрсетілген. Алгебрадан классикалық мысалдары болып келетін бекітілген сипаттамамен өрістер, группалар, абельдік группалар, сақинаның әртүрлі кластары, бульдік алгебра, полигондар алгебраның және йонсондық шартты қанағаттандыратын теорияның мысалдары болып табылады. Йонсондық теориялар барлық индуктивті теориялардың кластарының жеткілікті кең ішкі класын құрайды. Бірақ, қарастырып отырған йонсондық теориялар жалпы айтқанда, толық емес болып табылады. Классикалық модельдер теориясы негізінен толық теориямен жұмыс жасайды, ал казіргі уақытта толық теориялардың модельді-теоретикалық қасиеттерін зерттеу үшін дамыған, бірақ йонсондық теорияларды зерттеу барысында техникалық аппараттың жетіспеулігі орын алады. Сондықтан йонсондық теорияларды зерттеу техникасысының аналогын табу бұл зерттеу тақырыбында практикалық маңыздылығы бар. Осы мақалада сигнатура бірорынды предикатқа кеңейтілді. Осы предикатты құрайтын элементтер қарастырып отырған йонсондық теорияның кейбір модельдерінің экзистенционалды-тұйық ішкі моделін құрайды. Сонымен біз осы мақаланың басты нәтижесін В. Шмелеваның йонсондық теорияларын зерттеу шеңберінде абельдік группалардың элементарлық классификациясы бойынша танымал теоремасы ретінде нақтылап, осылайша толық теориялар үшін элементарлық қосарлары жайлы белгілі сұрақтың жалпылауын аламыз. Сонымен қатар екі модельдің үйлесімді енуінің йонсондық аналогі, баскаша айтқанда, абельдік группалар теорияларының йонсондық қосарлары зерттеу аясында Шрёдер-Бернштейн қасиеттері алынды.

Кілт сөздер: йонсондық теория, модельді компаньон, экзистенционалды-тұйық модель, кемелділік, косеманттылық, йонсондық спектр, йонсондық қосар.

А.Р. Ешкеев, М.Т. Касыметова, О.И. Ульбрихт

Критерий косемантичности абелевых групп в обогащённой сигнатуре

В статье рассмотрен критерий косемантичности относительно йонсоновского спектра модели в классе абелевых групп с выделенным предикатом. Изучены некоторые теоретико-модельные свойства абелевых групп в рамках их исследования, как моделей йонсоновских теорий. Нами получено, что абелевы группы с дополнительным условием выделенного предиката удовлетворяют условиям йонсоновости, а также совершенности в смысле йонсоновской теории. Известно, что классические примеры из алгебры, такие как поля фиксированной характеристики, группы, абелевы группы, различные классы колец, булевы алгебры, полигоны, являются примерами алгебр, теории которых удовлетворяют условиям йонсоновости. Изучение теоретико-модельных свойств йонсоновских теорий в классе абелевых групп является весьма актуальной задачей как в самой теории моделей, так и в универсальной алгебре. Йонсоновские теории образуют достаточно широкий подкласс класса всех индуктивных теорий. Но рассматриваемые йонсоновские теории, вообще говоря, не являются полными. Классическая теория моделей в основном имеет дело с полными теориями, а в случае изучения йонсоновских теорий существует дефицит технического аппарата, который в данное время развит для изучения теоретико-модельных свойств полных теорий. Поэтому нахождение аналогов такой техники для изучения йонсоновских теорий имеет практическую значимость в данной теме исследования. Авторами была расширена сигнатура на один одноместный предикат. Элементы, реализующие этот предикат, образуют экзистенциально-замкнутую подмодель некоторой модели рассматриваемой йонсоновской теории. В конечном итоге получен основной результат данной статьи как уточнение хорошо известной теоремы В. Шмелёвой об элементарной классификации абелевых групп в рамках изучения йонсоновских теорий. Тем самым найдены обобщение известного вопроса об элементарных парах для полных теорий, а также йонсоновский аналог для совместной вложимости двух моделей, или, по-другому, свойства Шрёдера-Бернштейна, в рамках изучения йонсоновских пар теории абелевых групп.

Ключевые слова: йонсоновская теория, модельный компаньон, экзистенциально-замкнутая модель, совершенность, косемантичность, йонсоновский спектр, йонсоновская пара.

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Maximum likelihood estimates of some probability model of discrete distributions

In this work the new multivariate discrete probability model of distribution of random sums with unobserved components is proposed.The maximum likelihood estimates for this model are determined in the case that all the elements of the sample implementation, namely the observed sums of unobserved components have only singular partition. In the case, that some element of the sample implementation has more then one partition, it is not possible to establish the maximum likelihood estimates.

Keywords: probability, multivariate distributions, maximum likelihood estimates.

1 Introduction

Models of probability distributions are a powerful and effective tool for studying diverse objects, systems and processes in various areas of human activity. In recent years, a significant number of probabilistic models have been developed.

Nevertheless, many unresolved problems remain, when it is possible to observe only the sums of components that can not be detected as a result of observations. For example, due to of the increasing use of digital media, there are failures of noise immunity, explained by randomly overlapping in one frequency band [1-3]. The use of probabilistic and static methods, namely, probabilistic modeling, allows us to present an approach to reducing the failures of noise immunity in information of digital media.

Also, an exclusive relevant example of the use of such a model is the advertising industry, where it is necessary to link the distribution of consumer interests with relevant advertising in various sources. Similar problems are very common in meteorology and in other areas. The probabilistic models describing such situations was considered in [4-6], where unbiased estimates were presented using the Rao-Blackwell-Kolmogorov method. Unlike the works [4-7], for these distributions in this paper we consider the maximum likelihood estimates and the conditions for the existence of their analytic derivation.

As is known, the maximum likelihood method is one of the most effective methods in terms of ensuring a minimum variance of the estimated parameters of probability distributions [5; 229]. The method is rigorous in the mathematical (probability-theoretic) plan. And its application is especially justified when there are both uncorrelated and correlated measurements in the processed information [8].

The strong consistency, the asymptotic unbiasedness, the asymptotic normality, asymptotic and the efficiency of maximum likelihood estimates provides their advantages in applied problems. Therefore the maximum likelihood estimates determined by the needs of practice, especially when using a large sample size.

The efficiency of the second order distinguishes this method of estimation among other asymptotically effective ones [9]. Invariance of maximum likelihood estimation ensures successful application of this method when estimating functions of distribution's parameters [10].

2 The model of multivariate discrete probability distribution

Consider a following probabilistic model by the example of the urn scheme with balls. Suppose that the urn contains balls, and each ball in the urn is marked by some value of number from the set of the random numbers $L_1, L_2, \ldots, L_d$. Let's the elements of the vector $\mathbf{p} = (p_1, p_2, \ldots, p_d)$ are the probabilities of retrieval from the urn of a ball marked by corresponding numbers $L_1, L_2, \ldots, L_d$ and

$$\sum_{\alpha=1}^{d} p_\alpha = 1.$$

By successive extraction of n balls from the urn with return we have the following situation. After the successive removal of n balls from the urn with the return, it is not known exactly what the balls were taken out of the urn. Only value u is known, which is the sum of the values of the numbers on the extracted balls from the urn. To study this situation, it is necessary to construct a probability distribution u. It is obvious that u is the realization of some random variable U.

Let's say, that V_u represents the number of possible combinations $r_{1_{v_u}}L_1, r_{2_{v_u}}L_2, \ldots, r_{d_{v_u}}L_d$, which in the sum formed the number u, where $r_{1_{v_u}}, r_{2_{v_u}}, \ldots, r_{d_{v_u}}$ determine the possible number of removed balls that are marked with the corresponding the random numbers $L_1, L_2, \ldots, L_d$. In other words, V_u is the number of partitions of the u into parts $L_1, L_2, \ldots, L_d$ [11; 1].

The following assertion follows from the results of [4-6]. The probability that the random variable U will take the value u, is

$$P(U=u)=\sum_{v_U=1}^{V_u} n!\prod_{\alpha=1}^{d}\frac{p_\alpha^{r_{\alpha_{v_u}}}}{r_{\alpha_{v_u}}!}. \quad (1)$$

3 Formulation of the problem

Obviously, in practice, the elements of the vector $\mathbf{p}$ are not known. Consequently, formula (1) does not find actual application. In this connection, it becomes necessary to determine the probability estimate (1).

It is also $L_1, L_2, \ldots,, L_d$, which give the sum u, are not known.

Let's $\mathbf{x}=(x_1,\ldots,x_k)$ can be interpreted as a realization of a sample $\mathbf{X}=\{X_1,...,X_k\}$ with size k, whose elements have distribution (1). We denote vector $\mathbf{r}_{v_\beta}=(r_{1_{v_\beta}},\ldots,r_{d_{v_\beta}})$, which defines v_β - th solution of equation

$$\begin{cases} \sum\limits_{\alpha=1}^{d} L_\alpha r_{\alpha_{v_\beta}}=\mathbf{x}_\beta; \\ \sum\limits_{\alpha=1}^{d} r_{\alpha_{v_\beta}}=n, \end{cases} \quad (2)$$

where $v_\beta=1,\ldots,V_\beta, V_\beta$ is the number of partitions of the x_β on the $L_1, L_2, \ldots,, L_d$. Using the $L_1, L_2, \ldots,, L_d$, and the realization of simple $\mathbf{x}$ in the system of equations (2) we define for each $\beta=1,\ldots,k$ the number of partitions V_β of the sum $\mathbf{x}_\beta$ on $L_1, L_2, \ldots,, L_d$, and vectors $\mathbf{r}_{1_\beta}, \ldots, \mathbf{r}_{V_\beta}$.

Suppose that for each $j=1,\ldots,\mu$, where

$$\mu=\prod_{\beta=1}^{k} V_\beta,$$

there is a vector $\mathbf{z}_j=(z_{1_j},\ldots,z_{d_j})$, defined as

$$\mathbf{z}_j=\sum_{\beta=1}^{k}\mathbf{r}_{v_\beta}, \quad (3)$$

and the indices on the right and left side are linked one-to-one correspondence, which is not unique. For example, this line can be described by the following form

$$j=v_1+(v_2-1)V_1+(v_3-1)V_1V_2+\ldots+(v_k-1)\prod_{\beta=1}^{k} V_\beta. \quad (4)$$

Also, it can be represented as

$$j=v_k+(v_{k-1}-1)V_k+(v_{k-2}-1)V_kV_{k-1}+\ldots+(v_1)\prod_{\beta=2}^{k} V_\beta.$$

That is, if used (4), then (3) can be represented as the following systems of equations

$$\mathbf{z}_1=\mathbf{r}_{1_1}+\mathbf{r}_{1_2}+\mathbf{r}_{1_3}+\ldots+\mathbf{r}_{1_k};$$

$$\mathbf{z}_2=\mathbf{r}_{2_1}+\mathbf{r}_{1_2}+\mathbf{r}_{1_3}+\ldots+\mathbf{r}_{1_k};$$

$$\dots$$

$$\mathbf{z}_{V_1} = \mathbf{r}_{1_{V_1}} + \mathbf{r}_{1_2} + \mathbf{r}_{1_3} + \ldots + \mathbf{r}_{1_k};$$

$$\mathbf{z}_{V_1+1} = \mathbf{r}_{1_1} + \mathbf{r}_{2_2} + \mathbf{r}_{1_3} + \ldots + \mathbf{r}_{1_k};$$

$$\mathbf{z}_{V_1+2} = \mathbf{r}_{2_1} + \mathbf{r}_{2_2} + \mathbf{r}_{1_3} + \ldots + \mathbf{r}_{1_k};$$

$$\dots$$

$$\mathbf{z}_{\mu} = \mathbf{r}_{V_1} + \mathbf{r}_{V_2} + \mathbf{r}_{V_3} + \ldots + \mathbf{r}_{V_k}.$$

The following Lemma allows to determine which vectors $\mathbf{r}_{v_1}, \mathbf{r}_{v_2}, \ldots, \mathbf{r}_{v_k}$ form the vector $\mathbf{z}_j$. Suppose that for some real value a the value $\langle a \rangle$ determines the integer part of a.

Lemma. If the indices in the right and left sides of the equation (3) are interconnected in form (4), then

$$v_k = \left\langle \frac{j-1}{\prod_{i=1}^{k-1} V_i} \right\rangle + 1;$$

$$v_{k-1} = \left\langle \frac{j - (v_k - 1)\prod_{i=1}^{k-1} V_i - 1}{\prod_{i=1}^{k-2} V_i} \right\rangle + 1;$$

$$v_{k-2} = \left\langle \frac{j - (v_{k-1} - 1)\prod_{i=1}^{k-2} V_i - (v_k - 1)\prod_{i=1}^{k-1} V_i - 1}{\prod_{i=1}^{k-3} V_i} \right\rangle + 1;$$

$$\dots$$

$$v_2 = \left\langle \frac{j - (v_3 - 1)V_1 V_2 - (v_4 - 1)V_1 V_2 V_3 - \ldots - (v_k - 1)\prod_{i=1}^{k-1} V_i - 1}{\prod_{i=1}^{k-3} V_i} \right\rangle + 1;$$

$$v_1 = j - (v_2 - 1)V_1 - (v_3 - 1)V_1 V_2 - (v_4 - 1)V_1 V_2 V_3 - \ldots - (v_k - 1)\prod_{i=1}^{k-1} V_i.$$

Proof. From (4) it follows that

$$v_k = \frac{j - c}{\prod_{i=1}^{k-1} V_i} + 1, \tag{5}$$

where

$$c = v_1 + (v_2 - 1)V_1 + (v_3 - 1)V_1 V_2 + \ldots + (v_{k-1} - 1)\prod_{i=1}^{k-2} V_i.$$

It is obvious that the latter can be represented as follows

$$c = v_{k-1}\prod_{i=1}^{k-2} V_i - \left[(V_{k-2} - v_{k-2})\prod_{i=1}^{k-3} V_i (V_{k-3} - v_{k-3})\prod_{i=1}^{k-4} V_i + \ldots + V_1 - v_1 \right].$$

Since

$$(V_{k-2} - v_{k-2})\prod_{i=1}^{k-3} V_i (V_{k-3} - v_{k-3})\prod_{i=1}^{k-4} V_i + \ldots + V_1 - v_1 \geq 0,$$

then

$$c \leq (v_{k-1} - 1) \prod_{i=1}^{k-2} V_i \leq \prod_{i=1}^{k-1} V_i$$

or

$$\frac{c-1}{\prod\limits_{i=1}^{k-1} V_i} < 1. \qquad (6)$$

By the fact that (5), then we obtain receive

$$v_k = \frac{j-1}{\prod\limits_{i=1}^{k-1} V_i} - \frac{c-1}{\prod\limits_{i=1}^{k-1} V_i} + 1.$$

Since we have (6) and v_k is non-negative integer, then

$$v_k = \left\langle \frac{j-1}{\prod\limits_{i=1}^{k-1} V_i} \right\rangle + 1.$$

The same way as v_k has been determined in the last formula, v_{k-1}, v_{k-2}, ..., v_1 are determined. Lemma is proved.

4 Construction of maximum likelihood estimates for the distribution parameters of the model investigated

Find maximum likelihood estimates for the parameters $p_1, \ldots, p_d$ of distribution (1). The likelihood function of distribution (1) has form

$$L(\mathbf{x};\mathbf{p}) = \prod_{\beta=1}^{k} P(U = x_i) = \prod_{\beta=1}^{k} \sum_{v_\beta=1}^{V_\beta} n! \prod_{\alpha=1}^{d} \frac{p_\alpha^{r_{\alpha v_\beta}}}{r_{\alpha_{v_\beta}}!},$$

which can present in form

$$L(\mathbf{x};\mathbf{p}) = \prod_{\beta=1}^{k} P(U = x_i) = \prod_{\beta=1}^{k} \frac{n! \sum\limits_{v_\beta=1}^{V_\beta} \prod\limits_{\alpha=1}^{d} \frac{p_\alpha^{r_{\alpha v_\beta}}}{r_{\alpha_{v_\beta}}!}}{\left(\sum\limits_{\alpha=1}^{d} p_\alpha\right)^{nV_\beta}}.$$

Accordingly, the log-likelihood for the parameters $p_1, \ldots, p_d$ of distribution (1) is

$$\ln L(\mathbf{x};\mathbf{p}) = k \ln n! + \sum_{\beta=1}^{k} \ln \sum_{v_\beta=1}^{V_\beta} \prod_{\alpha=1}^{d} \frac{p_\alpha^{r_{\alpha v_\beta}}}{r_{\alpha_{v_\beta}}!} - n\eta \ln \left(\sum_{\alpha=1}^{d} p_\alpha\right)^{nV_\beta},$$

where $\eta = \sum\limits_{\beta=1}^{k} V_\beta$.

It follows that for any $\alpha^* = 1, \ldots, d$ we have

$$\frac{\partial \ln L(\mathbf{x};\mathbf{p})}{\partial p_{\alpha^*}} = \sum_{\beta=1}^{k} \frac{\sum\limits_{v_\beta=1}^{V_\beta} \frac{p_{\alpha^*}^{r_{\alpha^* v_\beta}-1}}{\left(r_{\alpha^* v_\beta}-1\right)!} \prod\limits_{\substack{\alpha = 1 \\ \alpha \neq \alpha^*}}^{d} \frac{p_\alpha^{r_{\alpha v_\beta}}}{r_{\alpha_{v_\beta}}!}}{\sum\limits_{v_\beta=1}^{V_\beta} \prod\limits_{\alpha=1}^{d} \frac{p_\alpha^{r_{\alpha v_\beta}}}{r_{\alpha_{v_\beta}}!}} - n\eta;$$

or

$$\frac{\partial \ln L(\mathbf{x};\mathbf{p})}{\partial p_{\alpha^*}} = \sum_{\beta=1}^{k}\sum_{v_\beta=1}^{V_\beta} \frac{r_{\alpha^* v_\beta}}{p_{\alpha^*}\Lambda_{v_\beta}} - n\eta, \tag{7}$$

where for $\beta = 1,\ldots,k$, $v_\beta = 1,\ldots,V_\beta$

$$\Lambda_{v_\beta} = 1 + \sum_{\substack{w_\beta = 1 \\ w_\beta \neq v_\beta}}^{V_\beta} \prod_{\alpha=1}^{d} \frac{r_{\alpha_{v_\beta}}!}{r_{\alpha_{w_\beta}}!} p_\alpha^{r_{\alpha_{w_\beta}} - r_{\alpha_{v_\beta}}}. \tag{8}$$

As it is known, the maximum likelihood estimations $\hat{\mathbf{p}} = (\hat{p}_1,\ldots,\hat{p}_d)$ for vector of parameters $\mathbf{p} = (p_1,\ldots,p_d)$ satisfy the following condition for any $\alpha = 1,\ \ldots,\ d$

$$\left.\frac{\partial \ln L(\mathbf{x};\mathbf{p})}{\partial p_\alpha}\right|_{\mathbf{p}=\hat{\mathbf{p}}} = 0. \tag{9}$$

It follows that $\ln L(\mathbf{x},\mathbf{p})$ reaches a local maximum at the point $\hat{\mathbf{p}} = (\hat{p}_1,\ldots,\hat{p}_d)$, for any $\alpha_1,\alpha_2 = 1,\ldots,s$, $s = 2,\ldots,d$ carried the following conditions

$$\begin{cases} \det \left\| \frac{\partial^2 \ln L(\mathbf{x},\mathbf{p})}{\partial p_{\alpha_1}\partial p_{\alpha_2}} \right\|_{s\times s} > 0, & \text{if } s \text{ is even;} \\ \det \left\| \frac{\partial^2 \ln L(\mathbf{x},\mathbf{p})}{\partial p_{\alpha_1}\partial p_{\alpha_2}} \right\|_{s\times s} < 0, & \text{otherwise.} \end{cases} \tag{10}$$

From (7) and(9) it follows that for $\alpha = 1,\ldots,d$

$$\hat{p}_\alpha = \frac{\sum\limits_{\beta=1}^{k}\sum\limits_{v_\beta=1}^{V_\beta} \frac{r_{\alpha_{v_\beta}}}{\Lambda_{v_\beta}}}{n\eta}. \tag{11}$$

Since

$$\sum_{\alpha=1}^{d} \hat{p}_\alpha = 1,$$

then in conformity with (11)

$$\sum_{\alpha=1}^{d} \hat{p}_\alpha = \frac{\sum\limits_{\alpha=1}^{d}\sum\limits_{\beta=1}^{k}\sum\limits_{v_\beta=1}^{V_\beta} \frac{r_{\alpha_{v_\beta}}}{\Lambda_{v_\beta}}}{n\eta} = 1.$$

Hence, we have

$$\sum_{\alpha=1}^{d}\sum_{\beta=1}^{k}\sum_{v_\beta=1}^{V_\beta} \frac{r_{\alpha_{v_\beta}}}{\Lambda_{v_\beta}} = n\eta. \tag{12}$$

From (8) it is evident that for any $\beta = 1,\ldots,k$, $v_\beta = 1,\ldots,V_\beta$ $\Lambda_{v_\beta} \geq 1$. And $\Lambda_{v_\beta} = 1$, if $V_\beta = 1$, otherwise $\Lambda_{v_\beta} > 1$. Thus, we have

$$\sum_{\alpha=1}^{d}\sum_{\beta=1}^{k}\sum_{v_\beta=1}^{V_\beta} \frac{r_{\alpha_{v_\beta}}}{\Lambda_{v_\beta}} \leq \sum_{\alpha=1}^{d}\sum_{\beta=1}^{k}\sum_{v_\beta=1}^{V_\beta} r_{\alpha_{v_\beta}} = n\eta.$$

That is

$$\sum_{\alpha=1}^{d}\sum_{\beta=1}^{k}\sum_{v_\beta=1}^{V_\beta} \frac{r_{\alpha_{v_\beta}}}{\Lambda_{v_\beta}} \neq n\eta,$$

if for some $\beta = 1,\ldots,k$ $\Lambda_{v_\beta} > 1$. So (12) is satisfied if for all $\beta = 1,\ldots,k$ $V_\beta = 1$. Consequently, the construction of maximum likelihood estimates for the distribution parameters $\mathbf{p} = (p_1,\ldots,p_d)$ of this model (1) is possible only when all elements of realization of sample have no more than one partition on the submitted $L_1, L_2, \ldots, L_d$.

In other words, if for all $\beta = 1, \ldots, k$ $V_\beta = 1$, then from (8) it implies $\Lambda_{v_\beta} = 1$, and by (11) for $\alpha = 1, \ldots, d$ we have

$$\hat{p}_\alpha = \frac{\sum\limits_{\beta=1}^{k} \sum\limits_{v_\beta=1}^{V_\beta} r_{\alpha_{v_\beta}}}{n\eta} = \frac{\sum\limits_{\beta=1}^{k} r_{\alpha_1}}{nk}.$$

That is

$$\hat{p}_\alpha = \frac{\sum\limits_{\beta=1}^{k} r_{\alpha_1}}{nk}. \tag{13}$$

From (7) and (13) it's following that for any α, α_1, $\alpha_2 = 1, \ldots, d$ if $\alpha_1 \neq \alpha_2$

$$\left.\frac{\partial^2 \ln L(\mathbf{x}, \mathbf{p})}{\partial p_{\alpha_1} \partial p_{\alpha_1}}\right|_{\mathbf{p}=\hat{\mathbf{p}}} = 0$$

and

$$\left.\frac{\partial^2 \ln L(\mathbf{x}, \mathbf{p})}{\partial p_\alpha^2}\right|_{\mathbf{p}=\hat{\mathbf{p}}} = -\sum_{\beta=1}^{k} \frac{r_{\alpha_1}}{\hat{p}_\alpha^2} < 0.$$

So for any $s = 2, \ldots, d$ we have the following

$$\det \left\| \frac{\partial^2 \ln L(\mathbf{x}, \mathbf{p})}{\partial p_{\alpha_1} \partial p_{\alpha_2}} \right\|_{s \times s} = \prod_{\alpha=1}^{s} \left.\frac{\partial^2 \ln L(\mathbf{x}, \mathbf{p})}{\partial p_\alpha^2}\right|_{\mathbf{p}=\hat{\mathbf{p}}}$$

or

$$\det \left\| \frac{\partial^2 \ln L(\mathbf{x}, \mathbf{p})}{\partial p_{\alpha_1} \partial p_{\alpha_2}} \right\|_{s \times s} = \prod_{\alpha=1}^{s} \left(\frac{r_{\alpha_1}}{\hat{p}_\alpha^2} \right).$$

Consequently, if for all $\beta = 1, \ldots, k$ $V_\beta = 1$, then the elements of the vector

$$\hat{\mathbf{p}} = (\hat{p}_1, \ldots, \hat{p}_d),$$

defined in (12) satisfy (8)–(9) and they are the maximum likelihood estimates for the parameters $\mathbf{p} = (p_1, \ldots, p_d)$ of disrtibution (1). Thus, the following Theorem holds.

Theorem. If all elements of realization of sample $\mathbf{x} = (x_1, \ldots, x_k)$ of distibution (1) have no more than one partition on the on the submitted $L_1, L_2, \ldots, L_d$, then there are maximum likelihood estimates for the parameters of the distribution (1), which is defined in (12).

Consequence. If some element from realization of sample $\mathbf{x} = (x_1, \ldots, x_k)$ of distibution (1) have more than one partition on the on the submitted $L_1, L_2, \ldots, L_d$, then there not are maximum likelihood estimates for the parameters of the distribution (1).

Thus, it can not always possible to construct maximum likelihood estimators for the parameters of the distribution (1).

5 Conclusion

The analysis conducted in this paper studies allows us to formulate the following conclusion: Found that for this model maximum likelihood estimates exist if all the elements of observations have no more than one part by partition.

As is known, in practice, often some element of the implementation of the sample has more than one partition. That is, the method for determining the likelihood estimation is not actually applicable to this model. Of course, it is possible to use modified likelihood estimates by means of the apparatus of numerical methods, for example, to solve the system of maximum likelihood equations by the iterative method [12] or directly maximize the likelihood function of the type [13].

Obviously, the application of numerical methods generates numerous problems. Namely, the convergence of iterative methods requires justification [14; 202], the likelihood function can has a several local maxima [15], the choice of the moment of termination of calculations in connection with the achievement of the required accuracy requires justification [16], also the accuracy of the computation depends to the sample size [17].

Thus, if any element of the sampling of the given distribution model has more than one decomposition, then when finding the maximum likelihood estimates, we have a number of computational problems that call into question the practicality of using maximum likelihood estimates.

There is no need to absolutize the maximum likelihood estimates. In addition to these, there are other types of estimates that have good asymptotic properties. An example is the most suitable unbiased estimates presented in [4-6].

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Дискретті үлестірімдердің бір ықтималдық моделінің шындыққа ұқсас максималды бағалары

Мақалада бақыланатын компоненттерімен берілген кездейсоқ қосындылар үлестірімінің жаңа көпөлшемді дискретті ықтималдық моделі ұсынылған. Осы модель үшін шындыққа ұқсас бағалары анықталған, оның ішінде егер таңдаманы жүзеге асырудағы барлық элементтері, дәл осы бақыланбайтын компоненттердің бақыланатын қосындылары тек жалғыз бөліктеуге ие болса. Сонымен қатар таңдаманы жүзеге асырудағы элемент жалғыз бөліктеуге ие болмаса, максималды шындыққа ұқсас бағаларды алу мүмкін емес.

Кілт сөздер: ықтималдық, көпөлшемді үлестірім, максималды шындыққа ұқсас бағалар.

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Оценки максимального правдоподобия одной вероятностной модели дискретных распределений

В статье представлена новая многомерная дискретная вероятностная модель распределения случайных сумм с ненаблюдаемыми компонентами. Определены оценки максимального правдоподобия для этой модели в том случае, если все элементы реализации выборки, а именно наблюдаемые суммы ненаблюдаемых компонентов, имеют только единственные разбиения. В случае если какой-нибудь элемент реализации выборки имеет не единственное разбиение, то оценки максимального правдоподобия невозможно установить.

Ключевые слова: вероятность, многомерные распределения, оценки максимального правдоподобия.

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On generic structures preserving elementary equivalence and elementary embeddability

We consider criteria for elementary equivalence and elementary embeddability for generic structures. They use classical characterizations for the general case. The criterion for elementary equivalence is based on the well known Fraïssé–Taimanov–Ehrenfeucht overturning method. The criterion for elementary embeddability uses the known Tarski–Vaught test.

Keywords: generic structure, elementary equivalence, Fraïssé–Taimanov–Ehrenfeucht's method, elementary embeddability, Tarski-Vaught test.

We consider criteria for elementary equivalence and elementary embeddability for generic structures [1-6]. They are based on classical characterizations for the general case. The criterion for elementary equivalence uses the well known Fraïssé–Taimanov–Ehrenfeucht overturning method [7-12]. The criterion for elementary embeddability is based on the Tarski-Vaught test [11, 13].

1 Preliminaries

We consider collections of sentence and formulas in first order logic over a language Σ. Thus, as usual, $\vdash$ means proof from no hypotheses deducing $\vdash\varphi$ for a formula φ of language Σ, which may contain function symbols and constants. If deducing φ, hypotheses in a set Φ of formulas can be used, we write $\Phi\vdash\varphi$. Usually Σ will be fixed in context and not mentioned explicitly.

Below we write $X,Y,Z,\ldots$ for finite sets of variables, and denote by $A,B,C,\ldots$ finite sets of elements, as well as finite sets in structures, or else the structures with finite universes themselves.

In diagrams, $A,B,C,\ldots$ denote finite sets of constant symbols disjoint from the constant symbols in Σ and $\Sigma(A)$ is the vocabulary with the constants from A adjoined. $\Phi(A),\Psi(B),\mathrm{X}(C)$ stand for Σ-*diagrams* (of sets A, B, C), that is, *consistent sets of* $\Sigma(A)$-, $\Sigma(B)$-, $\Sigma(C)$-*sentences*, respectively.

Below we assume that for any considered diagram $\Phi(A)$, if a_1,a_2 are distinct elements in A then $\neg(a_1\approx a_2)\in\Phi(A)$. This means that if c is a constant symbol in Σ, then there is at most one element $a\in A$ such that $(a\approx c)\in\Phi(A)$.

If $\Phi(A)$ is a diagram and B is a set, we denote by $\Phi(A)|_B$ the set $\{\varphi(\bar{a})\in\Phi(A)\mid\bar{a}\in B\}$. Similarly, for a language Σ, we denote by $\Phi(A)|_\Sigma$ the restriction of $\Phi(A)$ to the set of formulas in the language Σ.

Definition [1-6]. We denote by $[\Phi(A)]^A_B$ the diagram $\Phi(B)$ obtained by replacing a subset $A'\subseteq A$ by a set $B'\subseteq B$ of constants disjoint from Σ and with $|A'|=|B'|$, where $A\setminus A'=B\setminus B'$. Similarly we call the consistent set of formulas denoted by $[\Phi(A)]^A_X$ the type $\Phi(X)$ if it is the result of a bijective substitution into $\Phi(A)$ of variables of X for the constants in A. In this case, we say that $\Phi(B)$ is a *copy* of $\Phi(A)$ and a *representative* of $\Phi(X)$. We also denote the diagram $\Phi(A)$ by $[\Phi(X)]^X_A$.

Remark. If the vocabulary contains functional symbols then diagrams $\Phi(A)$ containing equalities and inequalities of terms can generate both finite and infinite structures. The same effect is observed for purely predicate vocabularies if it is written in $\Phi(A)$ that the model for $\Phi(A)$ should be infinite. For instance, diagrams containing axioms for finitely axiomatizable theories have this property.

By the definition, for any diagram $\Phi(A)$, each constant symbol in Σ appears in some formula of $\Phi(A)$. Thus, $\Phi(A)$ can be considered as $\Phi(A\cup K)$, where K is the set of constant symbols in Σ.

We now give conditions on a partial ordering of a collection of diagrams which suffice for it to determine a structure. We modify some of the conditions for structures by d to signify they are conditions on diagrams not structures.

Definition [1-6]. Let Σ be a vocabulary. We say that $(\mathbf{D}_0;\leqslant)$ (or $\mathbf{D}_0$) is *generic*, or *generative*, if $\mathbf{D}_0$ is a class of Σ-diagrams of finite sets so that $\mathbf{D}_0$ is partially ordered by a binary relation $\leqslant$ such that $\leqslant$ is preserved by bijective substitutions, i. e., if $\Phi(A)\leqslant\Psi(B)$, and $A'\subseteq B'$ such that $[\Phi(A)]^A_{A'}=\Phi(A')$ and $[\Psi(B)]^B_{B'}=\Psi(B')$ are defined, then $[\Phi(A)]^A_{A'}$, $[\Psi(B)]^B_{B'}$ are in $\mathbf{D}_0$ and $[\Phi(A)]^A_{A'}\leqslant[\Psi(B)]^B_{B'}$. [1] Furthermore:

(i) if $\Phi(A)\in\mathbf{D}_0$ then for any quantifier free formula $\varphi(\bar{x})$ and any tuple $\bar{a}\in A$ either $\varphi(\bar{a})\in\Phi(A)$ or $\neg\varphi(\bar{a})\in\Phi(A)$;

(ii) if $\Phi\leqslant\Psi$ then $\Phi\subseteq\Psi$;[2]

(iii) if $\Phi\leqslant \mathrm{X}$, $\Psi\in\mathbf{D}_0$, and $\Phi\subseteq\Psi\subseteq \mathrm{X}$, then $\Phi\leqslant\Psi$;

(iv) some diagram $\Phi_0(\emptyset)$ is the least element of the system $(\mathbf{D}_0;\leqslant)$, and $\mathbf{D}_0\setminus\{\Phi_0(\emptyset)\}$ is nonempty;

(v) (the *d-amalgamation property*) for any diagrams $\Phi(A)$, $\Psi(B)$, $\mathrm{X}(C)\in\mathbf{D}_0$, if there exist injections $f_0\colon A\to B$ and $g_0\colon A\to C$ with $[\Phi(A)]^A_{f_0(A)}\leqslant\Psi(B)$ and $[\Phi(A)]^A_{g_0(A)}\leqslant \mathrm{X}(C)$, then there are a diagram $\Theta(D)\in\mathbf{D}_0$ and injections $f_1\colon B\to D$ and $g_1\colon C\to D$ for which $[\Psi(B)]^B_{f_1(B)}\leqslant\Theta(D)$, $[\mathrm{X}(C)]^C_{g_1(C)}\leqslant\Theta(D)$ and $f_0\circ f_1=g_0\circ g_1$; the diagram $\Theta(D)$ is called the *amalgam* of $\Psi(B)$ and $\mathrm{X}(C)$ over the diagram $\Phi(A)$ and witnessed by the four maps (f_0,g_0,f_1,g_1);

(vi) (the *local realizability property*) if $\Phi(A)\in\mathbf{D}_0$ and $\Phi(A)\vdash\exists x\,\varphi(x)$, then there are a diagram $\Psi(B)\in\mathbf{D}_0$, $\Phi(A)\leqslant\Psi(B)$, and an element $b\in B$ for which $\Psi(B)\vdash\varphi(b)$;

(vii) (the *d-uniqueness property*) for any diagrams $\Phi(A),\Psi(B)\in\mathbf{D}_0$ if $A\subseteq B$ and the set $\Phi(A)\cup\Psi(B)$ is consistent then $\Phi(A)=\{\varphi(\bar{b})\in\Psi(B)\mid\bar{b}\in A\}$.

A diagram Φ is called a *strong subdiagram* of a diagram Ψ if $\Phi\leqslant\Psi$.

A diagram $\Phi(A)$ is said to be (*strongly*) *embeddable* in a diagram $\Psi(B)$ if there is an injection $f\colon A\to B$ such that $[\Phi(A)]^A_{f(A)}\subseteq\Psi(B)$ ($[\Phi(A)]^A_{f(A)}\leqslant\Psi(B)$). The injection f, in this instance, is called a (*strong*) *embedding* of diagram $\Phi(A)$ in diagram $\Psi(B)$ and is denoted by $f\colon\Phi(A)\to\Psi(B)$. A diagram $\Phi(A)$ is said to be (*strongly*) *embeddable* in a structure $\mathcal{M}$ if $\Phi(A)$ is (strongly) embeddable in some diagram $\Psi(B)$, where $\mathcal{M}\models\Psi(B)$. The corresponding embedding $f\colon\Phi(A)\to\Psi(B)$, in this case, is called a (*strong*) *embedding* of diagram $\Phi(A)$ in structure $\mathcal{M}$ and is denoted by $f\colon\Phi(A)\to\mathcal{M}$.

Let $\mathbf{D}_0$ be a class of diagrams, $\mathbf{P}_0$ be a class of structures of some language, and $\mathcal{M}$ be a structure in $\mathbf{P}_0$. The class $\mathbf{D}_0$ is *cofinal* in the structure $\mathcal{M}$ if for each finite set $A\subseteq M$, there are a finite set B, $A\subseteq B\subseteq M$, and a diagram $\Phi(B)\in\mathbf{D}_0$ such that $\mathcal{M}\models\Phi(B)$. The class $\mathbf{D}_0$ is *cofinal* in $\mathbf{P}_0$ if $\mathbf{D}_0$ is cofinal in every structure of $\mathbf{P}_0$. We denote by $\mathbf{K}(\mathbf{D}_0)$ the class of all structures $\mathcal{M}$ with the condition that $\mathbf{D}_0$ is cofinal in $\mathcal{M}$, and by $\mathbf{P}$ a subclass of $\mathbf{K}(\mathbf{D}_0)$ such that each diagram $\Phi\in\mathbf{D}_0$ is true in some structure in $\mathbf{P}$.

Now we extend the relation $\leqslant$ from the generative class $(\mathbf{D}_0;\leqslant)$ to a class of subsets of structures in the class $\mathbf{K}(\mathbf{D}_0)$.

Let $\mathcal{M}$ be a structure in $\mathbf{K}(\mathbf{D}_0)$, A and B be finite sets in $\mathcal{M}$ with $A\subseteq B$. We call A a *strong subset* of the set B (in the structure $\mathcal{M}$), and write $A\leqslant B$, if there exist diagrams $\Phi(A),\Psi(B)\in\mathbf{D}_0$, for which $\Phi(A)\leqslant\Psi(B)$ and $\mathcal{M}\models\Psi(B)$.

A finite set A is called a *strong subset* of a set $M_0\subseteq M$ (in the structure $\mathcal{M}$), where $A\subseteq M_0$, if $A\leqslant B$ for any finite set B such that $A\subseteq B\subseteq M_0$ and $\Phi(A)\subseteq\Psi(B)$ for some diagrams $\Phi(A),\Psi(B)\in\mathbf{D}_0$ with $\mathcal{M}\models\Psi(B)$. If A is a strong subset of M_0 then, as above, we write $A\leqslant M_0$. If $A\leqslant M$ in $\mathcal{M}$ then we refer to A as a *self-sufficient set* (in $\mathcal{M}$).

Notice that, by the d-uniqueness property, the diagrams $\Phi(A)$ and $\Psi(B)$ specified in the definition of strong subsets are defined uniquely. A diagram $\Phi(A)\in\mathbf{D}_0$, corresponding to a self-sufficient set A in $\mathcal{M}$, is said to be a *self-sufficient diagram* (in $\mathcal{M}$).

Definition [1-6]. class $(\mathbf{D}_0;\leqslant)$ possesses the *joint embedding property* (JEP) if for any diagrams $\Phi(A)$, $\Psi(B)\in\mathbf{D}_0$, there is a diagram $\mathrm{X}(C)\in\mathbf{D}_0$ such that $\Phi(A)$ and $\Psi(B)$ are strongly embeddable in $\mathrm{X}(C)$.

Clearly, every generative class has JEP since JEP means the d-amalgamation property over the empty set.

Definition [1-6]. A structure $\mathcal{M}\in\mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{D}_0;\leqslant)$, or is *finitely generated over* Σ, if any finite set $A\subseteq M$ is contained in some finite self-sufficient set in $\mathcal{M}$, i. e., there is a finite set B with $A\subseteq B\subseteq M$ and $\Psi(B)\in\mathbf{D}_0$ such that $\mathcal{M}\models\Psi(B)$ and $\Psi(B)\leqslant \mathrm{X}(C)$ for any $\mathrm{X}(C)\in\mathbf{D}_0$ with $\mathcal{M}\models \mathrm{X}(C)$ and $\Psi(B)\subseteq \mathrm{X}(C)$. A class $\mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{D}_0;\leqslant)$, or is *finitely generated over* Σ, if each structure in $\mathbf{P}$ has finite closures (with respect to $(\mathbf{D}_0;\leqslant)$).

[1] Note that $\mathbf{D}_0$ is closed under bijective substitutions since $\leqslant$ is preserved by bijective substitutions and $\leqslant$ is reflexive.

[2] Note that $\Phi(A)\leqslant\Psi(B)$ implies $A\subseteq B$, since if $a\in A$ then $(a\approx a)\in\Phi(A)$, so $\Phi(A)\leqslant\Psi(B)$ implies $\Phi(A)\subseteq\Psi(B)$ and we have $(a\approx a)\in\Psi(B)$, whence $a\in B$.

Clearly, an at most countable structure $\mathcal{M}$ has finite closures with respect to $(\mathbf{D}_0;\leqslant)$ if and only if $M=\bigcup\limits_{i\in\omega} A_i$ for some self-sufficient sets A_i with $A_i\leqslant A_{i+1}$, $i\in\omega$.

Note that the finite closure property is defined modulo Σ and does not correlate with the cardinalities of algebraic closures. For instance, if Σ contains infinitely many constant symbols then $\mathrm{acl}(A)$ is always infinite whereas a finite set A can or can not be extended to a self-sufficient set.

Besides, for the finite closures of sets A we consider finite self-sufficient extensions B in a given structure $\mathcal{M}$ with respect to $(\mathbf{D}_0;\leqslant)$ only and B can be both a universe of a substructure of $\mathcal{M}$ or not. Moreover, it is permitted that corresponding diagrams $\Psi(B)$ can have only finite, finite and infinite, or only infinite models.

Thus, for instance, a finitely axiomatizable theory without finite models and with a generative class $(\mathbf{D}_0;\subseteq)$, containing diagrams for all finite sets and with axioms in diagrams, has identical finite closures whereas each diagram in $\mathbf{D}_0$ has only infinite models.

Definition [1-6]. A structure $\mathcal{M}\in\mathbf{K}(\mathbf{D}_0)$ is $(\mathbf{D}_0;\leqslant)$-*generic*, or a *generic limit for the class* $(\mathbf{D}_0;\leqslant)$ and denoted by $\mathrm{glim}(\mathbf{D}_0;\leqslant)$, if it satisfies the following conditions:

(a) $\mathcal{M}$ has finite closures with respect to $\mathbf{D}_0$;

(b) if $A\subseteq M$ is a finite set, $\Phi(A),\Psi(B)\in\mathbf{D}_0$, $\mathcal{M}\models\Phi(A)$ and $\Phi(A)\leqslant\Psi(B)$, then there exists a set $B'\leqslant M$ such that $A\subseteq B'$ and $\mathcal{M}\models\Psi(B')$.

Clearly, uncountable $(\mathbf{D}_0;\leqslant)$-generic structures can be non-isomorphic. Indeed, for instance, all infinite structures in the empty language are generic for a given generative class although these structures are non-isomorphic for distinct cardinalities. But, as the following theorem shows, they are isomorphic for at most countable cases.

Theorem 1.1 [1-6]. *For any generative class* $(\mathbf{D}_0;\leqslant)$ *with at most countably many diagrams whose copies form* $\mathbf{D}_0$*, there exists at most countable* $(\mathbf{D}_0;\leqslant)$*-generic structure, unique up to isomorphism.*

Theorem 1.2 [1-6]. *Every* ω*-homogeneous structure* $\mathcal{M}$ *is* $(\mathbf{D}_0;\leqslant)$*-generic for some generative class* $(\mathbf{D}_0;\leqslant)$*.*

Thus any first-order theory has a generic model and therefore can be represented by it.

2 Elementary equivalence and elementary embeddability

Recall that structures $\mathcal{M}_1$ and $\mathcal{M}_2$ in a language Σ are *elementarily equivalent* (denoted by $\mathcal{M}_1\equiv\mathcal{M}_2$) if for any sentence φ in the language Σ, $\mathcal{M}_1\models\varphi$ if and only if $\mathcal{M}_2\models\varphi$.

Definition [11]. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be structures in a language Σ. An injective map $f\colon X\to M_2$, where $X\subseteq M_1$, is a *partial isomorphism of* $\mathcal{M}_1$ *into* $\mathcal{M}_2$ if for every elements $a_1,\ldots,a_n\in X$ the following conditions hold:

1) for any functional symbol $F^{(n)}\in\Sigma$ and correspondent operations $F_{\mathcal{M}_1}$ and $F_{\mathcal{M}_2}$ in $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively,

$$f(F_{\mathcal{M}_1}(a_1,\ldots,a_n))=F_{\mathcal{M}_2}(f(a_1),\ldots,f(a_n));$$

2) for any predicate symbol $P^{(n)}\in\Sigma$ and correspondent predicates $P_{\mathcal{M}_1}$ and $P_{\mathcal{M}_2}$ in $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively,

$$(a_1,\ldots,a_n)\in P_{\mathcal{M}_1}\Leftrightarrow(f(a_1),\ldots,f(a_n))\in P_{\mathcal{M}_2}.$$

A partial isomorphism $f\colon X\to M_2$ is called *finite* if the set X is finite.

The set of finite partial isomorphisms of $\mathcal{M}_1$ into $\mathcal{M}_2$ is denoted by $P(\mathcal{M}_1,\mathcal{M}_2)$.

The following well-known theorem uses the Fraïssé–Taimanov–Ehrenfeucht overturning method [7-10]. It is broadly used, in particular, in [12].

Theorem 2.1 [11]. *Let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be structures in a language* Σ*. The following conditions are equivalent:*

(1) *the structures* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are elementarily equivalent;*

(2) *for any* $n\in\omega$ *and any finite language* $\Sigma_0\subseteq\Sigma$ *there are nonempty sets* $Z_1(\Sigma_0,n),\ldots,Z_n(\Sigma_0,n)$ *of finite partial isomorphisms of* $\mathcal{M}_1|_{\Sigma_0}$ *into* $\mathcal{M}_2|_{\Sigma_0}$ *such that for any* $f\in Z_i(\Sigma_0,n)$, $1\le i<n$*, and for any* $a\in M_1$, $b\in M_2$ *there are* $g_1,g_2\in Z_{i+1}(\Sigma_0,n)$*, for which* $a\in\delta_{g_1}$, $b\in\rho_{g_2}$ *and* $f\subseteq g_1\cap g_2$*.*

Notice that considering $(\mathbf{D}_i;\leqslant_i)$-generic structures $\mathcal{M}_i$ in a language Σ, $i=1,2$, we take elements for extensions $g_1,g_2\in Z_{i+1}(\Sigma_0,n)$ in diagrams $\Phi(A)$ and $\Psi(B)$ in generative classes satisfying $\mathcal{M}_1\models\Phi(A)$ and $\mathcal{M}_2\models\Psi(B)$. Moreover, since the sets A and B are finite, we can replace addition of elements a and b by addition of self-sufficient sets A and B. Finite partial isomorphisms $f\colon X\to M_2$ with $X=A$ or $\rho_f=B$ are called *coordinated* with given generative classes, *coordinated generic*, or simply *generic*.

The set of generic finite partial isomorphisms of $\mathcal{M}_1$ into $\mathcal{M}_2$ is denoted by $\mathrm{PG}(\mathcal{M}_1,\mathcal{M}_2)$.

We have $\mathrm{PG}(\mathcal{M}_1,\mathcal{M}_2)\subseteq P(\mathcal{M}_1,\mathcal{M}_2)$ and each partial isomorphism in $P(\mathcal{M}_1,\mathcal{M}_2)$ is extensible till a partial isomorphism in $\mathrm{PG}(\mathcal{M}_1,\mathcal{M}_2)$. Thus, for generic structures in Theorem 2.1 it suffices to consider generic finite partial isomorphisms in $\mathrm{PG}(\mathcal{M}_1,\mathcal{M}_2)$, with their restrictions, and a modification of that theorem holds allowing syntactically, in terms of generative classes, characterize the elementary equivalence for generic structures. Below we consider that generic modification, whose proof can be easily obtained from the proof of [11, Theorem 5.1.1].

Theorem 2.2 Let $\mathcal{M}_i$ be $(\mathbf{D}_i;\leqslant_i)$-generic structures in a language Σ, $i=1,2$. The following conditions are equivalent:

(1) *the structures $\mathcal{M}_1$ and $\mathcal{M}_2$ are elementarily equivalent;*

(2) *for any $n\in\omega$ and any finite language $\Sigma_0\subseteq\Sigma$ there are nonempty sets $Z_1(\Sigma_0,n),\ldots,Z_n(\Sigma_0,n)$ of restrictions of generic finite partial isomorphisms of $\mathcal{M}_1|_{\Sigma_0}$ into $\mathcal{M}_2|_{\Sigma_0}$ such that the following condition holds:*

$(*)$ *for any $f\in Z_i(\Sigma_0,n)$, $1\le i<n$, and for any $a\in M_1$, $b\in M_2$ there are $g_1,g_2\in Z_{i+1}(\Sigma_0,n)$, for which $a\in\delta_{g_1}$, $b\in\rho_{g_2}$ and $f\subseteq g_1\cap g_2$.*

Remark 2.3. Following Theorem 2.2 and adding for any $f\in Z_i(\Sigma_0,n)$ and for any $a\in M_1$, $b\in M_2$ all elements in some self-sufficient sets $A\supset\delta_f\cup\{a\}$ and $B\supset\rho_f\cup\{b\}$ we can consider sequences $Z_1(\Sigma_0,n),\ldots,Z_n(\Sigma_0,n)$ of nonempty families of generic finite partial isomorphisms with the property of sequential extensions by $g_1,g_2\in Z_{i+1}(\Sigma_0,n)$ with $a\in\delta_{g_1}$, $b\in\rho_{g_2}$ and $f\subseteq g_1\cap g_2$.

Proposition 2.4. If $\mathcal{M}_i$ are elementarily equivalent $(\mathbf{D}_i;\leqslant_i)$-generic structures, $i=1,2$, then the classes $(\mathbf{D}_i;\leqslant_i)$ can be extended, with some extensions of their diagrams, till a common generative class $(\mathbf{D}_0;\leqslant)$.

Proof. Since $\mathcal{M}_1\equiv\mathcal{M}_2$, complete diagrams $\Phi^*(A)$ for finite sets in $\mathcal{M}_1$ and in $\mathcal{M}_2$ can be collected for a homogeneous model $\mathcal{M}$ of the theory $\mathrm{Th}(\mathcal{M}_1)=\mathrm{Th}(\mathcal{M}_2)$ realizing the complete types $\Phi^*(X)$. The complete diagrams for $\mathcal{M}$ form the required generative class $(\mathbf{D}_0;\subseteq)$. $\square$

Proposition 2.4 immediately implies

Corollary 2.5. Any elementarily equivalent generic structures are isomorphic to some restrictions of a common generic structure.

Since any countable structure has a countable homogeneous elementary extension and homogeneous structures are generic, Corollary 2.5 has the following modification:

Corollary 2.6. Any elementarily equivalent countable structures are isomorphic to some restrictions of a common (countable) generic structure.

Recall that a substructure $\mathcal{M}_1=\langle M_1;\Sigma\rangle$ of $\mathcal{M}_2=\langle M_2;\Sigma\rangle$ is called an *elementary substructure* (denoted by $\mathcal{M}_1\preccurlyeq\mathcal{M}_2$), if for any formula $\varphi(x_1,\ldots,x_n)$ in the language Σ and for any elements $a_1,\ldots,a_n\in M_1$ the condition $\mathcal{M}_1\models\varphi(a_1,\ldots,a_n)$ is equivalent to $\mathcal{M}_2\models\varphi(a_1,\ldots,a_n)$. Here the structure $\mathcal{M}_2$ is an *elementary extension* of $\mathcal{M}_1$. If $M_1\neq M_2$, we write $\mathcal{M}_1\prec\mathcal{M}_2$ instead of $\mathcal{M}_1\preccurlyeq\mathcal{M}_2$. If $\mathcal{M}_1\subseteq\mathcal{M}_2$ and the condition $\mathcal{M}_1\preccurlyeq\mathcal{M}_2$ ($\mathcal{M}_1\prec\mathcal{M}_2$) does not hold, we write $\mathcal{M}_1\not\preccurlyeq\mathcal{M}_2$ (respectively $\mathcal{M}_1\not\prec\mathcal{M}_2$).

The following well-known *Tarski–Vaught test* [11, 13] is used for the checking that a substructure is an elementary one.

Theorem 2.7. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be structures in a language Σ, $\mathcal{M}_1\subseteq\mathcal{M}_2$. The following conditions are equivalent:

(1) $\mathcal{M}_1\preccurlyeq\mathcal{M}_2$;

(2) *for any formula $\varphi(x_0,x_1,\ldots,x_n)$ in the language Σ and for any elements $a_1,\ldots,a_n\in M_1$, if $\mathcal{M}_2\models\exists x_0\,\varphi(x_0,a_1,\ldots,a_n)$ then there is an element $a_0\in M_1$ such that $\mathcal{M}_2\models\varphi(a_0,a_1,\ldots,a_n)$.*

In the following theorem, we obviously modify Theorem 2.7 for generic cases.

Theorem 2.8. Let $\mathcal{M}_1$ be a $(\mathbf{D}_1;\leqslant_1)$-generic structures in a language Σ, $\mathcal{M}_1\subseteq\mathcal{M}_2$. The following conditions are equivalent:

(1) $\mathcal{M}_1\preccurlyeq\mathcal{M}_2$;

(2) *for any formula $\varphi(x_0,x_1,\ldots,x_n)$ in the language Σ and for any elements $a_1,\ldots,a_n$ forming a self-sufficient set $A\leqslant_1 M_1$, if $\mathcal{M}_2\models\exists x_0\,\varphi(x_0,a_1,\ldots,a_n)$ then there is an element $a_0\in M_1$ in a self-sufficient set $B\leqslant_1 M_1$ such that $A\leqslant_1 B$ and $\mathcal{M}_2\models\varphi(a_0,a_1,\ldots,a_n)$.*

Remark 2.9. If in Theorem 2.8 the diagrams $\Phi(A),\Psi(B)\in\mathbf{D}_1$, for the sets A and B, force the complete types $\mathrm{tp}(A)$, $\mathrm{tp}(B)$, respectively, we take formulas $\exists x_0\,\varphi(x_0,a_1,\ldots,a_n)$ and $\varphi(a_0,a_1,\ldots,a_n)$ which are forced by $\Phi(A)$ and $\Psi(B)$, respectively.

Recall that an *elementary embedding* of a structure $\mathcal{M}_1$ into a structure $\mathcal{M}_2$ of the same language Σ is a map $f\colon M_1\to M_2$ such that for every Σ-formula $\varphi(x_1,\ldots,x_n)$ and all elements $a_1,\ldots,a_n$ of M_1, $\mathcal{M}_1\models\varphi(a_1,\ldots,a_n)$ if and only if $\mathcal{M}_2\models\varphi(f(a_1),\ldots,f(a_n))$. In such a case, f is really an embedding denoted by $f\colon\mathcal{M}_1\to\mathcal{M}_2$ and for $\mathcal{M}_1$ and $\mathcal{M}_2$ we say that $\mathcal{M}_1$ is *elementarily embeddable* into $\mathcal{M}_2$.

Similarly to Theorems 2.7 and 2.8, the following theorems characterize the elementary embeddability in general case and for generic structures, respectively.

Theorem 2.10. Let $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be structures in a language* Σ, $f\colon \mathcal{M}_1 \to \mathcal{M}_2$ *be an embedding. The following conditions are equivalent:*

(1) *the embedding* f *is elementary*;

(2) *for any formula* $\varphi(x_0, x_1, \ldots, x_n)$ *in the language* Σ *and for any elements* $a_1, \ldots, a_n \in M_1$, *if* $\mathcal{M}_2 \models \exists x_0\, \varphi(x_0, f(a_1), \ldots, f(a_n))$ *then there is an element* $a_0 \in M_1$ *such that* $\mathcal{M}_2 \models \varphi(f(a_0), f(a_1), \ldots, f(a_n))$.

Theorem 2.11. Let $\mathcal{M}_1$ *be a* $(\mathbf{D}_1; \leqslant_1)$*-generic structure in a language* Σ, *and* $f\colon \mathcal{M}_1 \to \mathcal{M}_2$ *be an embedding. The following conditions are equivalent:*

(1) *the embedding* f *is elementary*;

(2) *for any formula* $\varphi(x_0, x_1, \ldots, x_n)$ *in the language* Σ *and for any elements* $a_1, \ldots, a_n$ *forming a self-sufficient set* $A \leqslant_1 M_1$, *if* $\mathcal{M}_2 \models \exists x_0\, \varphi(x_0, f(a_1), \ldots, f(a_n))$ *then there is an element* $a_0 \in M_1$ *in a self-sufficient set* $B \leqslant_1 M_1$ *such that* $A \leqslant_1 B$ *and* $\mathcal{M}_2 \models \varphi(f(a_0), f(a_1), \ldots, f(a_n))$.

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С.В. Судоплатов

Элементарлық енгізілуді және элементарлық эквиваленттілікті сақтайтын генерикалық құрылымдар туралы

Мақалада генерикалық құрылымдар үшін элементарлық эквиваленттілік және элементарлық енгізілу критерийлері қарастырылды. Олар үшін жалпы жағдайдағы классикалық сипаттама қолданылды. Элементарлы эквивалентті критерийі Фраиссе-Тайманов-Эренфойхтың жақсы танымал «ауыстыру» әдісінде негізделген. Элементарлық енгізілу критерийінде Тарский-Воottың танымал тесті пайдаланылды.

Кілт сөздер: генерикалық құрылымдар, элементарлық эквиваленттілік, Фраиссе-Тайманов-Эренфойхтың әдісі, элементарлық енгізілу, Тарский-Воот тесті.

С.В. Судоплатов

О генерических структурах, сохраняющих элементарную эквивалентность и элементарную вложимость

В статье рассмотрены критерии элементарной эквивалентности и элементарной вложимости для генерических структур, которые используют классические характеризации для общего случая. Критерий элементарной эквивалентности базирован на хорошо известном методе «перекидывания» Фраиссе-Тайманова-Эренфойхта, а критерий элементарной вложимости — на известном тесте Тарского-Воота.

Ключевые слова: генерические структуры, элементарная вложимость, метод Фраиссе-Тайманова-Эренфойхта, тест Тарского-Воота.

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Мұнай қабаты параметрлерін нақтылау бойынша алынған сүзгілеу теориясының бір кері есебі туралы

Мақалада нақты деректерді пайдалану кезінде нақты уақыт режимінде өнімді қабаттың коллекторлық қасиеттерін дәлірек анықтауға мүмкіндік беретіндей сүзгілеу теориясының кері есебі зерттелді. Бұл мақсатта ұңғымалар бойынша сұйық дебитінің, газ және қабат қысымының уақытқа тәуелділігінің қажетті шарттары анықталды және нақты мәліметтер беріліп, белгілі бір кен орнының технологиялық көрсеткіштері қалпына келтірілді. Тура және кері есептер шығарылды, оларды шешудің алгоритмі ұсынылды. Сулы қабаттың облыс шеттерінде жеткізу, жеңілдету және сүзгілеу-сыйымдылық параметрлерінің шарттары қалпына келтірілгенде ұңғымалардағы J сәйкессіздік функционалының азаюы қалпына келтіру және сәйкестендіру есептерінің дәйекті шешімдерін алудың жеткілікті шарты болып табылатындығы көрсетілді. Атырау облысы кен орындарында алынған мәліметтер негізінде сандық тәжірибелер жүргізілді.

Кілт сөздер: кері есеп, сүзгілеу теориясы, коллекторлық қасиеттері, диффузия теңдеуі, параметрлер, функционал, градиент.

Қарастырылып отырған жағдайдағы сүзгілеу теориясының кері есебінің қойылуы сұйықты кеуекті ортада сүзгілеу кезінде тиімді қабаттың агрегаттық жай-күйінің өзгеруі әсерінен туындап отыр. Сәйкесті айырмашылықтар сулы қабат құрылымының ерекшеліктеріне және оның коллекторлық қасиеттерінің өзгерістеріне байланысты. Сондықтан қарастырылып отырған кері есептің қойылуы және шешілуі барысында қысым және ұңғымалардың сүзгілеу параметрлері туралы ақпаратты қолдануға болады. Гидрогеологиялық зерттеулер нәтижесі табиғи сүзгілеу ағыны тағы да бір ерекшелігімен сипатталатындығын көрсетеді. Сүзгілеу ағыны болуы әсерінен ерітілген түрде көмірсутекті және көмірсутекті емес компоненттерге қарай судың белгілі бір бөлінуі болады. Геологиялық ұзақ уақыт барысында бұл компоненттер мұнай немесе газ қалындыларынан қалыптасып, сулы қабатқа ауысқан (өткен) белгілі бір анықталған индикаторлар ретінде қабылдануы мүмкін. Бұл индикаторлар концентрациясы сулы қабат құрылымының ерекшеліктеріне және оның коллекторлық қасиеттерінің өзгерістеріне байланысты алаң бойынша өзгеріп отырады. Мұндай жекелеген компоненттердің көшуінің стационарлық емес процестері конвективті диффузия теңдеуімен сипатталады [1]. Жазық сүзгілеу ағыны жағдайында конвективті диффузия теңдеуі мына түрде болады [2]:

$$m\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\lambda_1\nu_1^2+\lambda_2\nu_2^2}{\sqrt{\nu_1^2+\nu_2^2}}\frac{dc}{dx}\right)+\frac{\partial}{\partial y}\left(\frac{\lambda_1\nu_2^2+\lambda_2\nu_1^2}{\sqrt{\nu_1^2+\nu_2^2}}\frac{dc}{dy}\right)-\frac{\partial}{\partial x}(\nu_1 c), \tag{1}$$

мұндағы m — кеуектілік коэффициенті; c — қарастырылып отырған компонент концентрациясы; λ_1, λ_2 — ортаның таралуының сәйкесінше қума және көлденең параметрлері, тұрақтылар (ұзындық өлшемді); ν_1, ν_2 — сәйкесінше x және y осьтері бойынша сүзгілеу жылдамдығының компоненттері.

(1) теңдеу үшін табиғи шекаралық

$$\frac{\partial c}{\partial n} = 0; \quad (x,y)\in\mathit{\Gamma}_2,\mathit{\Gamma}_3,\mathit{\Gamma}_4; \quad c=1, \quad (x,y)\in\mathit{\Gamma}_1 \tag{2}$$

және бастапқы шарттар беріледі

$$c(x,y)=0; \qquad t=0. \tag{3}$$

Сонда уақыттың әр түрлі кезеңіндегі, соның ішінде есептеулер жүргізу кезеңіндегі сулы қабаттың барлық ауданы бойынша өріс концентрациясын анықтаудың тікелей шекаралық есебі алынады. ν_1, ν_2 сүзгілеу жылдамдықтары аудан бойынша қысымның таралуына тәуелді болады, ендеше, (1)–(3) есебінің шешімін табу үшін мұнай немесе газ қалындылары бар, біртекті емес (коллекторлық қасиеттеріне сәйкес) сулы қабаттағы P^* келтірілген қысымға қатысты эллипстік типтегі дифференциалдық теңдеумен өрнектелетін суды сүзгілеу есебінің шешімінің болуы қажетті

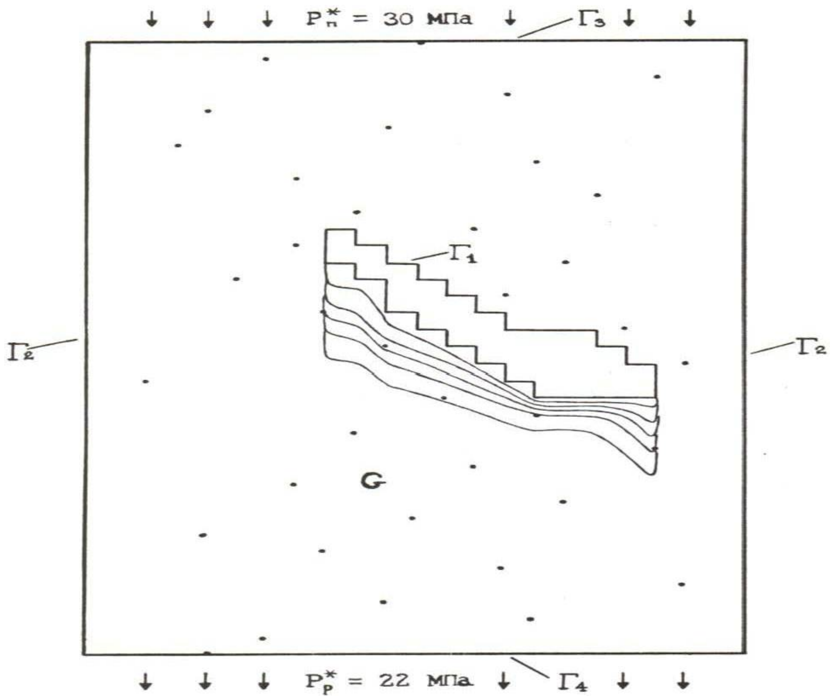
$$\frac{\partial}{\partial x}\left[k(x,y)\cdot h(x,y)\frac{\partial P^*}{\partial x}\right]+\frac{\partial}{\partial y}\left[k(x,y)\cdot h(x,y)\frac{\partial P^*}{\partial y}\right]=0. \tag{4}$$

(4) теңдеуді интегралдау қысымға қатысты келесі шекаралық шарттарға сәйкес іске асырылады

$$\frac{\partial P^*}{\partial n}=0, \qquad (x,y)\in\Gamma_1,\Gamma_2, \tag{5}$$

$$P^*=P_n^*=\text{const}, \quad (x,y)\in\Gamma_3; \quad P^*=P_p^*=\text{const}, \quad (x,y)\in\Gamma_4. \tag{6}$$

Сонымен, мұнай немесе газ шөгінділері бар, сулы қабаттағы нақты сүзгілеу ағыны үшін (4)–(6) тікелей шекаралық есебі орындалады, мұндағы $P^*=P\pm\rho_B gl$; P — координаталары x және y болатын нүктедегі қысым; ρ_B — судың тығыздығы; g — еркін түсу үдеуі; l — координаталары x және y болатын берілген нүктеден келтірілген жазықтыққа дейінгі вертикаль арақашықтық; n — G-ға қатысты сыртқы нормаль; k — қабаттың су өткізгіштік коэффициенті; h — қабаттың қалыңдығы; Γ_1 — мұнай немесе газ қалыңдыларының шекарасы; Γ_2 — сулы қабаттың су өткізбейтін шекаралары; Γ_3 — коректендіру аймағының контуры; Γ_4 — жеңілдету аймағының контуры. Қарастырылып отырған аймақ 1-суретте көрсетілген.



1-сурет. Коллекторлық қасиеттеріне сәйкес біртекті емес сулы қабаттағы барлау ұңғымаларын орналастыру сызбасы және изобар картасы

P^* қысымның c — қарастырылып отырған компонент концентрациясына кері тәуелділігі жоқ, олай болса, (4)–(6) және (1)–(3) есептері екі дербес тікелей шекаралық есептерге бөлінеді. Сонымен, қабаттың берілген коллекторлық қасиеттеріне k және h, λ_1, λ_2 шекаралық шарттарға сәйкес теңдеулерді шешу сулы қабаттың барлық ауданы бойынша келтірілген P^* қысымның таралуын анықтауға мүмкіндік береді [3–5]. Одан әрі алдыңғы есептегі P^* функциясын пайдалана отырып, сулы қабаттың k, h, m берілген коллекторлық қасиеттеріне (3) бастапқы шартқа және (2) шекаралық шартқа сәйкес (1) теңдеуді шешу c — концентрациясының сулы қабаттың барлық ауданы бойынша таралуын анықтауға мүмкіндік береді.

Кері есептің қойылуы. Сулы қабаттың барлық ауданы бойынша сол күнге сәйкес $[0,T]$ уақыт аралығының соңында $P^*(x,y)$ және $C(x,y,t)$ нақты мәндері белгілі деп есептейміз. T геологиялық уақыт мезетінде ұңғымалардағы қабат геометриясы және оның коллекторлық қасиеттерінің жуық мәндері туралы P^* және C мәліметтер белгілі болсын. Сулы қабаттың барлық нүктелеріндегі коллекторлық (сүзгілеу және сыйымдылық) қасиеттерін, сонымен қатар қолда бар деректерге негізделген жеткізу және түсіру алаңдарының шеттеріндегі шарттарды анықтау қажет. Кері есепті тиімдендіру есебі ретінде шығарамыз. J функционалын J_1 және J_2 қосындысы түрінде тұрғызамыз. Бұл функционалдың қабаттың коллекторлық қасиеттеріне тәуелді болатындығы түсінікті

$$J\left\{P_n^*, P_p^*, b, \lambda_1, \lambda_2\right\} = J_1\left(P_n^*, P_p^*, b\right) + J_2\left(m, \lambda_1, \lambda_2\right) =$$

$$= \sum_{i=1}^{N}\left[\left(P^*_{pac_i} - P^*_{\phi_i}\right)^2 + w_i\left(b_{pac_i} - b_{\phi_i}\right)^2\right] + \int_0^T \sum_{i=1}^{N} \gamma_i \left[C_{pac_i} - C_{\phi_i}\right]^2 dt. \quad (7)$$

Кері есепті мына түрде тұжырымдауға болады: (7) функционалды минималдайтын P_n^*, P_p^*, b, m, λ_1, λ_2 мәндерін табу керек. Есепті шешу нәтижесінде енгізілген шарттар мен сүзгілеудің қабылданған математикалық моделі тұрғысынан келтірілген қысымның есептік және нақты мәндерінің, сүзгілеу параметрлері мен концентрацияларының сәйкес келуін қамтамасыз ететін геологиялық модельмен эквиваленттті болатын кейбір қабаттың коллекторлық қасиеттері анықталады. Функционалды минималдау үшін итерациялық градиентті әдісті қолданамыз, мысалы, қабаттың әртүрлі нүктелеріндегі кеуектілік коэффициентін анықтау үшін келесі рекурренттік қатынасты аламыз:

$$\overset{(s+1)}{m} = \overset{(s)}{m} - \overset{(s)}{\lambda_m} \frac{\partial \overset{(s)}{J}}{\partial m}, \quad (8)$$

мұндағы s — итерация номері.

Осыған аналогиялық түрде барлық басқа ізделінді параметрлер үшін де градиентті ережелер жазылады. P_n^*, P_p^*, b, m, λ_1, λ_2 шамалары мәндері үшін нөлдік жақындату ретінде геологиялық карталар, ұңғымалар мен қабаттарды геофизикалық және гидродинамикалық зерттеулер нәтижелері пайдаланылуы мүмкін. Градиентті ережелерді (рәсімдерді) іске асыру үшін функционалды туындыларды есептеу керек.

Есепті шешу алгоритмі келесі кезеңдерден тұрады:

1. Сулы қабаттың газ немесе мұнай шөгінділері белгіленген нақты конфигурациясы немесе G жиынтығы сұлба G^1 аймағымен жуықталады. Пьезометрикалық ұңғымалардың орналасқан орындарының координаталары мен олардағы қабат қысымын өлшеу нәтижелері, сонымен қатар осы ұңғымалардағы есептеулер жүргізу немесе уақыттың әртүрлі мезеттеріндегі (судың табиғи сүзгілеудің ағынын жасанды жолмен алу барысында) индикатор компоненттерінің концентрацияларын өлшеулер туралы мәліметтер белгілі деп есептеледі. Қабаттың геологиялық моделін P_n^*, P_p^*, b, m, λ_1, λ_2 нөлдік жақындату ретінде геологиялық-геофизикалық мәліметтер негізінде тұрғызылған карталар, сонымен қатар негізгі талдау нәтижелері пайдаланылады.

2. (4)–(6) тікелей шекаралық есеп шекті айырма әдісімен шығарылады, және интегралдау облысының барлық ауданы бойынша $P^*_{i,j}$ келтірілген қабат қысымының есептік таралуы анықталады.

3. $J = \int\limits_G \sum\limits_{i=1}^{N}\left[P^*_{pac_i} - P^*_{\phi_ш}\right]^2 \delta dG + \int\limits_G \sum\limits_{i=1}^{N} w_i\left[b_{pac_i} - b_{\phi_i}\right]^2 \delta dG$ формуласы бойынша қысым және сүзгілеу параметрлері өлшенетін нүктелердегі сәйкессіздік анықталады және J_1 функционалының мәні есептеледі.

4. Алдыңғы (4)–(6) есебіндегі $P^*_{i,j}$ функциясын пайдалана отырып, (1)–(3) тікелей шектік есебі шығарылады. (1)–(3) есебін шығару нәтижесінде әртүрлі уақытша қабаттардағы, сонымен қатар T уақыт мезетіндегі $C_{i,j}$ концентрациясының есептік таралуы белгіленеді.

5. $\varepsilon_i = \int\limits_G C(x,y,T)dG - C\phi_i$ формуласымен концентрациялар өлшенетін нүктелердегі сәйкессіздік анықталып, J_2 функционалының мәні есептеледі, мұндағы $\int\limits_G C(x,y,T)dG = Cppa_i(T) - T$ уақыт мезетіндегі i-ші ұңғыманың орналасқан орнына сәйкес келетін есептік концентрация; $\varepsilon_i - T$ уақыт мезетіндегі i-ші ұңғымадағы есептік және нақты концентрациялар арасындағы сәйкессіздік. Бұдан соң, егер $\left|\overset{(s)}{J} - \overset{(s-1)}{J}\right| \leqslant \varepsilon$ шарты орындалса, $J = J_1 + J_2$ функционалының мәні анықталады, мұндағы s – итерация номері; ε – алдын ала берілген модельдің қателік шамасы орындалады, кері есептің шешімін табу аяқталады.

6. Қарама-қарсы жағдайда табылған сәйкессіз шамалармен түйіндес шектік есеп шығарылады

$$\operatorname{div} b\nabla U - 2\sum_{i=1}^{N}(P^*_{pac_i} - P^*_{\phi_i})\delta = 0; \tag{9}$$

$$\frac{\partial U}{\partial n} = 0, \qquad (x,y)\in \Gamma_1, \Gamma_2; \tag{10}$$

$$U = 0, \qquad (x,y)\in \Gamma_3, \Gamma_4, \tag{11}$$

және $U_{i,j}$ түйіндес функцияның таралуы анықталады. Аналогиялық түрде табылған сәйкессіз шамалармен түйіндес дифференциалдық теңдеу шешіледі

$$\frac{\partial}{\partial x}\left(\frac{\lambda_1\nu_1^2 + \lambda_2\nu_2^2}{\sqrt{\nu_1^2+\nu_2^2}}\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\lambda_1\nu_2^2 + \lambda_2\nu_1^2}{\sqrt{\nu_1^2+\nu_2^2}}\frac{\partial c}{\partial x}\right) - \frac{\partial U}{\partial y} + \frac{\partial}{\partial x}(\nu_1 U) + m\frac{\partial U}{\partial t} = 2\sum_{i=1}^{n}\gamma_i\varepsilon_i\delta, \tag{12}$$

келесі шекаралық және бастапқы шарттарға сәйкес

$$\frac{\partial U}{\partial n} = 0, \quad (x,y)\in \Gamma_2, \Gamma_3, \Gamma_4; \quad U = 0, \quad (x,y)\in \Gamma_1; \tag{13}$$

$$U(x,y,t) = 0, \qquad t = T. \tag{14}$$

$[0,T]$ уақыт кесіндісінде уақыттың әртүрлі мезетіндегі $U^k_{i,j}$ түйіндес функциясының таралуы анықталады.

7. Жуықтап интегралдау формулалары бойынша функционалдық туындылар есептеледі

$$\frac{\partial J}{\partial P^*_n} = \sum_{j=1}^{N_y} b_{i,j}\frac{U_{i+1,j} - U_{i,j}}{\Delta x}\Delta y, \quad i = 1; \tag{15}$$

$$\frac{\partial J}{\partial P^*_p} = \sum_{j=1}^{N_y} b_{i,j}\frac{U_{i+1,j} - U_{i,j}}{\Delta x}\Delta y, \quad i = N_x; \tag{16}$$

$$\frac{\partial J}{\partial b} = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left[\frac{U_{i+1,j} - U_{i,j}}{\Delta x}\frac{P^*_{i+1,j} - P^*_{i,j}}{\Delta x} + \frac{U_{i,j+1} - U_{i,j}}{\Delta y}\frac{P^*_{i,j+1} - P^*_{i,j}}{\Delta y} + EP_{i,j}b_{i,j}\right]\Delta x\Delta y. \tag{17}$$

$$\frac{\partial J_2}{\partial m} = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\sum_{k=1}^{N_{t-1}} U^k_{i,j}\frac{C^{k+1}_{i,j} - C^k_{i,j}}{\Delta t}\Delta x\Delta y\Delta t; \tag{18}$$

$$\begin{aligned}\frac{\partial J_2}{\partial \lambda_1} = \sum_{i=1}^{N_{x-1}}\sum_{j=1}^{N_y}\sum_{k=1}^{N_t}\Bigg[&\frac{\nu_1{}^2_{i,j}}{\sqrt{\nu_1{}^2_{i,j} + \nu_2{}^2_{i,j}}}\frac{C^k_{i+1,j} - C^k_{i,j}}{\Delta x}\frac{U^k_{i+1,j} - U^k_{i,j}}{\Delta x} + \\ &+\frac{\nu_2{}^2_{i,j}}{\sqrt{\nu_1{}^2_{i,j} + \nu_2{}^2_{i,j}}}\frac{C^k_{i,j+1} - C^k_{i,j}}{\Delta y}\frac{U^k_{i,j+1} - C^k_{i,j}}{\Delta y}\Bigg]\Delta x\Delta y\Delta t;\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{\partial J_2}{\partial \lambda_2} = \sum_{i=1}^{N_x}\sum_{j=1}^{N_{y-1}}\sum_{k=1}^{N_t}\Bigg[&\frac{\nu_2{}^2_{i,j}}{\sqrt{\nu_1{}^2_{i,j} + \nu_2{}^2_{i,j}}}\frac{C^k_{i+1,j} - C^k_{i,j}}{\Delta x}\frac{U^k_{i+1,j} - U^k_{i,j}}{\Delta x} + \\ &+\frac{\nu_1{}^2_{i,j}}{\sqrt{\nu_1{}^2_{i,j} + \nu_2{}^2_{i,j}}}\frac{C^k_{i,j+1} - C^k_{i,j}}{\Delta y}\frac{U^k_{i,j+1} - C^k_{i,j}}{\Delta y}\Bigg]\Delta x\Delta y\Delta t.\end{aligned} \tag{20}$$

8. Градиент ережелерінің параметрлері есептеледі

$$\lambda_{P^*_n}, \quad \lambda_{P^*_p}, \quad \lambda_b, \quad \lambda_m, \quad \lambda_{\lambda_1}, \quad \lambda_{\lambda_2}.$$

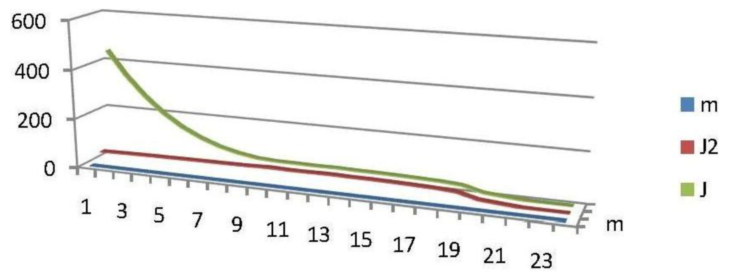
9. Градиент ережелерін пайдаланып, сулы қабаттың ізделінді параметрлері анықталады, басқаша айтқанда, $\overset{(s+1)}{P^*_n}$, $\overset{(s+1)}{P^*_p}$, $\overset{(s+1)}{b}$, $\overset{(s+1)}{m}$, $\overset{(s+1)}{\lambda_1}$ және $\overset{(s+1)}{\lambda_2}$ шамалары үшін кезекті жуықтаулар табылады және 2 пунктке ауысу жүргізіледі.

Кесте

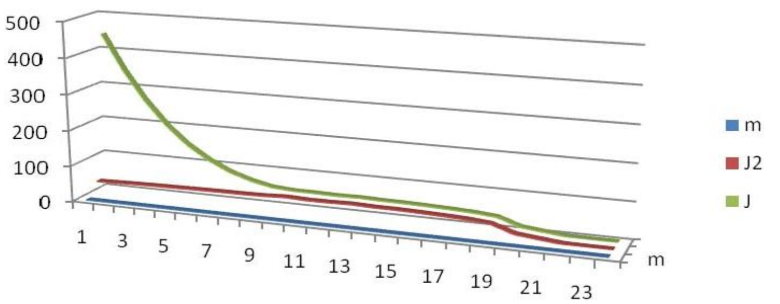
Кері есепті шешу барысындағы итерация санына байланысты J_2 және J функционалдары бойынша жинақтылық және сыйымдылық параметрлерінің нақтыға жинақтылығы

Итер. номері	m	J_2	J
1	1,0000	38,1372	444,83
2	0,9645	39,5578	345,80
3	0,9309	40,9089	265,39
4	0,8991	41,9977	200,36
5	0,8689	42,6758	149,64
6	0,8399	43,5912	112,10
7	0,8119	44,2864	83,79
8	0,7850	44,5489	64,94
9	0,7582	45,3890	52,46
10	0,7331	48,2365	48,40
11	0,7161	45,8002	48,07
12	0,6857	46,1512	46,76
13	0,6527	47,4741	47,66
14	0,6222	45,4419	45,87
15	0,5731	45,1268	45,33
16	0,5111	43,7017	43,81
17	0,4210	41,6976	41,73
18	0,3723	39,1174	39,14
19	0,2994	33,9184	33,94
20	0,1697	13,3323	13,35
21	0,1434	6,9516	6,97
22	0,1132	0,8680	0,89
23	0,0963	0,1754	0,20
24	0,1072	0,2418	0,26

Итер. номері	m	J_2	J
1	1,5000	44,6142	451,10
2	1,4211	45,1973	351,42
3	1,3438	45,7435	270,22
4	1,2680	46,1575	204,52
5	1,1925	46,4474	153,21
6	1,1157	46,6270	115,15
7	1,0360	46,7591	86,25
8	0,9518	46,5029	66,89
9	0,08562	46,5466	53,62
10	0,7504	48,3765	48,55
11	0,6838	45,3221	47,60
12	0,5808	44,3479	44,96
13	0,4709	44,1643	44,35
14	0,3586	35,9937	36,43
15	0,1466	5,7585	5,97
16	0,1212	1,1974	1,30
17	0,1086	0,4233	0,45
18	0,0993	0,0470	0,07
19	0,1029	0,0344	0,06
20	0,1005	0,0094	0,03
21	0,1014	0,0113	0,03
22	0,1012	0,0102	0,03
23	0,1013	0,0107	0,03



2-сурет. Кестенің 1-қатары бойынша кері есепті шешу барысындағы итерация санына байланысты J_2 және J функционалдары бойынша жинақтылық



3-сурет. Кестенің 2-қатары бойынша кері есепті шешу барысындағы итерация санына байланысты J_2 және J функционалдары бойынша сыйымдылық параметрлерінің нақтыға жинақтылығы

Минималданатын функционалдың мәнінің соңғы және соңғының алдындағы итерацияларда берілген ε қателігінен өзгерісі болмаған жағдайда ғана есептеулер аяқталады. Жеткізу және түсіру алаңдарының шеттеріндегі шарттар қалпына келтірілгенде және сулы қабаттың сүзгілеу-сыйымдылық параметрлерін идентификациялағанда ұңғымалардағы J сәйкессіздік функционалының азаюы қалпына келтіру және идентификациялалау есептерінің ақиқат шешімдерін алудың жеткілікті шарты болатындығы көрсетілді (кестені қара.). Нақты кен орны мәліметтері негізінде сандық тәжірибелер жүргізілді (2, 3-сур.).

Әдебиеттер тізімі

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Об одной обратной задаче теории фильтрации по уточнению параметров нефтяного пласта

В статье исследованы обратные задачи теории фильтрации, которые в режиме реального времени позволяют уточнить коллекторские свойства продуктивного пласта при использовании фактических данных. Для этих целей определены необходимые условия зависимости от времени дебитов жидкости, газа и пластовых давлений по скважинам и восстановлены технологические показатели конкретного месторождения с реальными данными. Решены прямая и обратная задачи, приведен алгоритм их решения. Показано, что при восстановлении условий на контурах областей питания и разгрузки и идентификации фильтрационно-ёмкостных параметров водоносного пласта уменьшение функционала невязки J на скважинах является достаточным условием для получения достоверных решений задач восстановления и идентификации. Проведены численные эксперименты с данными, полученными на месторождениях Атырауской области.

Ключевые слова: обратная задача, теория фильтрации, коллекторские свойства, уравнение диффузии, параметры, функционал, градиент.

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On an inverse problem of the filtration theory by the refinement of parameters of an oil layer

In the article the inverse problems of the filtration theory, that in real time allows to specify the reservoir properties of the productive formation by using the actual data. For these purposes, the necessary conditions for the dependence of liquid, gas and reservoir pressures on the wells have been determined on time, and the technological performance of a particular field with real data has been restored. The direct and inverse problems are solved, the algorithm for their solution is given. It is shown that, when conditions are restored on the contours of the feeding areas and unloading and the identification of the filtration-capacitance parameters of the aquifer, the decrease in the residual J function at the wells is a sufficient condition for obtaining reliable solutions for recovery and identification problems. Numerical experiments were carried out with the data obtained in the fields of Atyrau region.

Keywords: inverse problem, filtration theory, reservoir properties, diffusion equation, parameters, functional, gradient.

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Математическое моделирование и численное исследование зависимости термонапряженного состояния стержня от коэффициента теплообмена при наличии температуры постоянной интенсивности

В статье на основе энергетических принципов разработана математическая модель термонапряженно-деформированного состояния стержня из жаропрочного сплава. Энергетический принцип ориентирован на минимизацию потенциальной энергии упругих деформаций с применением метода квадратичного конечного элемента с тремя узлами. Стержень ограниченной длины и жестко защемлен с обоих концов. Боковая поверхность участков $(0 \leqslant x \leqslant L/3)$ и $(2L/3 \leqslant x \leqslant L)$ стержня теплоизолированная. Через площадь поперечных сечений обоих концов данного стержня происходит теплообмен с окружающими их средами. На серединном участке стержня $(L/3 \leqslant x \leqslant 2L/3)$ дана температура постоянной интенсивности $T = \mathrm{const} = 800\,^\circ\mathrm{C}$. Исследовано влияние коэффициента теплообмена на термонапряженное состояние стержня из жаропрочного сплава АНВ-300 при наличии температуры постоянной интенсивности, и приведены численные результаты исследования. Исследования проведены при разных значениях коэффициента теплообмена. В результате установлено, что при увеличении значения h_0 — коэффициента теплообмена возрастает амплитуда перемещений против направления оси Ox; координата сечения, амплитуда перемещения которого будет наибольшей, увеличивается; амплитуда перемещения по направлению оси Ox уменьшается; максимальное и среднее значения термоупругого напряжения σ уменьшаются.

Ключевые слова: метод конечных элементов, температура, теплоизоляция, теплообмен, коэффициент теплообмена, потенциальная энергия упругих деформаций, математическая модель.

Рассмотрим стержень ограниченной длины L (см), изготовленный из жаропрочного сплава АНВ-300. Площадь поперечного сечения стержня F (см^2) постоянна по всей его длине. Коэффициент теплового расширения материала стержня $\alpha(T)$ $(1/\,^\circ\mathrm{C})$ зависит от поля распределения температуры. Коэффициент теплопроводности материала стержня — K_{xx} (Вт/(см$\cdot^\circ$C)), модуль упругости — E (кГ/см^2). Оба конца рассматриваемого стержня жестко защемлены. Поэтому при наличии источников тепла из-за теплового расширения во внутренних сечениях стержня появляется напряженно-деформированное состояние. При этом составляющие деформации и напряжения будут соответственно упругие $(\varepsilon_x, \sigma_x)$, температурные $(\varepsilon_T, \sigma_T)$ и термоупругие (ε, σ). Если известно поле распределения температуры и коэффициента теплового расширения по длине стержня, то выражение функционала, которое характеризует потенциальную энергию упругих деформаций рассматриваемого стержня, имеет следующий вид:

$$\varPi = \int\limits_V \frac{\sigma_x}{2}\varepsilon_x\, dV - \int\limits_V \alpha\left(T(x)\right)\cdot E\cdot T(x)\cdot \varepsilon_x\, dV, \qquad (1)$$

где V — объем стержня; $\varepsilon_x = \frac{\partial u}{\partial x}$ — составляющая упругой деформации; $u = u(x)$ — поле упругих перемещений по длине стержня; $\sigma_x = E\cdot\varepsilon_x$ — упругая составляющая напряжения; $\alpha = \alpha\left(T(x)\right)$ — закон распределения коэффициента теплового расширения по длине стержня; E — модуль упругости материала стержня; $T = T(x)$ — закон распределения температуры по длине стержня [1, 2].

Учитывая, что рассматриваемый процесс является установившимся в пределах каждого дискретного элемента поля распределения температуры, коэффициент теплового расширения и упругого перемещения аппроксимируем полным полиномом второго порядка. Для этого в стержне выделим любой участок длиной Δx (см) — это так называемый дискретный элемент. Поле распределения температуры в пределах этого дискретного элемента будем рассматривать отдельно, как на рисунке 1.

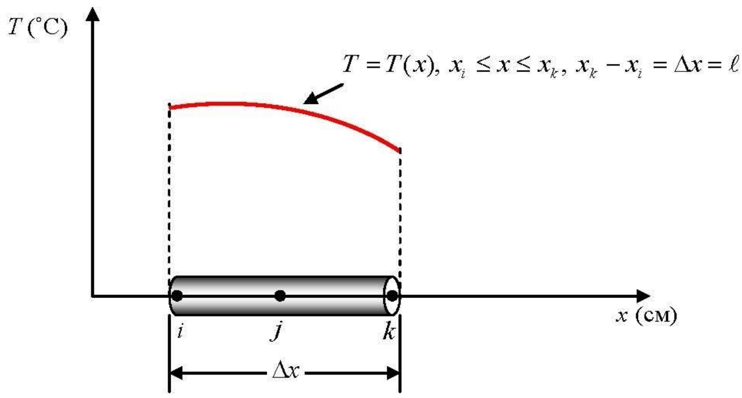

Рисунок 1. Поле распределения температуры на отрезке стержня

На рассматриваемом элементе длиной Δx возьмем сечения i, j и k с координатами $x = x_i$, $x = x_j$ и $x = x_k$. Тогда в пределах рассматриваемого дискретного элемента поле распределения температуры $T = T(x)$ можно представить как кривую второго порядка, проходящую через три точки $x = x_i$, $x = x_j$ и $x = x_k$.

$$T(x) = a + bx + cx^2, \quad x_i \leqslant x \leqslant x_k, \quad a,\ b,\ c - \text{const}. \tag{2}$$

Считая, что значения температуры в узлах $x = x_i$, $x = x_j$ и $x = x_k$ будут $T(x_i) = T_i$, $T(x_j) = T_j$, $T(x_k) = T_k$, из (2) имеем

$$\begin{cases} a + bx_i + cx_i^2 = T_i; \\ a + bx_j + cx_j^2 = T_j; \\ a + bx_k + cx_k^2 = T_k. \end{cases} \tag{3}$$

Учитывая, что $x_i = 0$, $x_j = \frac{\ell}{2}$, $x_k = \ell$, и, решая последнюю систему уравнений, находим значения констант a, b, c. Подставляя их в выражение (2), после упрощения получим, что [1, 2]

$$T(x) = \varphi_i(x) \cdot T_i + \varphi_j(x) \cdot T_j + \varphi_k(x) \cdot T_k \quad 0 \leqslant x \leqslant \ell, \tag{4}$$

где

$$\varphi_i(x) = \frac{\ell^2 - 3\ell x + 2x^2}{\ell^2}; \quad \varphi_j(x) = \frac{4(\ell x - x^2)}{\ell^2}; \quad \varphi_k(x) = \frac{2x^2 - \ell x}{\ell^2}, \quad 0 \leqslant x \leqslant \ell. \tag{5}$$

Эти функции являются функциями формы для квадратичного конечного элемента с тремя узлами. Следует отметить, что они имеют следующие свойства:

$$1.\ \begin{cases} \varphi_i(x) = 1 \\ \varphi_j(x) = 0 \ \text{ при } x = x_i; \\ \varphi_k(x) = 0 \end{cases} \quad \begin{cases} \varphi_i(x) = 0 \\ \varphi_j(x) = 1 \ \text{ при } \ x = x_j; \\ \varphi_k(x) = 0 \end{cases} \quad \begin{cases} \varphi_i(x) = 0 \\ \varphi_j(x) = 0 \ \text{ при } \ x = x_k. \\ \varphi_k(x) = 1 \end{cases} \tag{6}$$

2. Для любого x, принадлежащего интервалу $x_i \leqslant x \leqslant x_k$,

$$\varphi_i(x) + \varphi_j(x) + \varphi_k(x) = 1. \tag{7}$$

3. Для любой точки интервала $0 \leqslant x \leqslant \ell$, т.е. в пределах каждого дискретного элемента,

$$\frac{\partial \varphi_i(x)}{\partial x} + \frac{\partial \varphi_j(x)}{\partial x} + \frac{\partial \varphi_k(x)}{\partial x} = 0. \tag{8}$$

Пользуясь соотношениями (5), можно доказать эти свойства.

Приведенные свойства функции форм обеспечат непрерывность искомых функций при переходе от одного элемента к следующему. По аналогии, поле распределения упругих перемещений $u(x)$ и коэффициента теплового расширения материала стержня $\alpha\,(T(x))$ в интервале $x_i \leqslant x \leqslant x_k$ также можно представить в виде

$$u(x) = \varphi_i(x) \cdot u_i + \varphi_j(x) \cdot u_j + \varphi_k(x) \cdot u_k; \tag{9}$$

$$\alpha\left(T(x)\right) = \varphi_i(x) \cdot \alpha_i + \varphi_j(x) \cdot \alpha_j + \varphi_k(x) \cdot \alpha_k, \quad x \in (x_i \leqslant x \leqslant x_k), \tag{10}$$

где u_i, u_j, u_k — перемещения сечений по координате, которые являются координатами соответственно узлов i, j, k; α_1, α_2, α_3 — const, $\varphi_i(x)$, $\varphi_j(x)$, $\varphi_k(x)$ и функциями формы для дискретного квадратичного элемента с тремя узлами [2, 3].

Тогда выражение функционала, характеризующее потенциальную энергию упругой деформации при наличии источников тепла для одного дискретного элемента, имеет следующий вид:

$$\Pi_i = \int\limits_{V_i} \frac{E}{2}\left(\frac{\partial u}{\partial x}\right)^2 dV - \int\limits_{V_i} \alpha\left(T(x)\right) \cdot E \cdot T(x) \cdot \left(\frac{\partial u}{\partial x}\right) dV =$$

$$= \int\limits_{V_i} \frac{E}{2}\left(\frac{\partial \varphi_i}{\partial x} u_i + \frac{\partial \varphi_j}{\partial x} u_j + \frac{\partial \varphi_k}{\partial x} u_k\right)^2 dV - \int\limits_{V_i} E\left(\varphi_i\alpha_i + \varphi_j\alpha_j + \varphi_k\alpha_k\right)\left(\varphi_i T_i + \varphi_j T_j + \varphi_k T_k\right) \times$$

$$\times \left(\frac{\partial \varphi_i}{\partial x} u_i + \frac{\partial \varphi_j}{\partial x} u_j + \frac{\partial \varphi_k}{\partial x} u_k\right) dV, \tag{11}$$

где V_i — объем рассматриваемого дискретного квадратичного элемента с тремя узлами.

Тогда выражение функционала, которое характеризует потенциальную энергию упругой деформации всего стержня при наличии источников тепла, выглядит следующим образом:

$$\Pi = \sum_{i=1}^{\text{ЧДЭ}} \Pi_i, \tag{12}$$

где ЧДЭ — число дискретных элементов в стержне.

Рассмотрим каждый интеграл в выражении (11) по отдельности:

1) $$\int\limits_{V_i} \frac{E}{2}\left(\frac{\partial \varphi_i}{\partial x} u_i + \frac{\partial \varphi_j}{\partial x} u_j + \frac{\partial \varphi_k}{\partial x} u_k\right)^2 dV = \frac{EF}{2}\int\limits_0^{\ell}\left(\frac{\partial \varphi_i}{\partial x} u_i + \frac{\partial \varphi_j}{\partial x} u_j + \frac{\partial \varphi_k}{\partial x} u_k\right)^2 dx.$$

Здесь F — площадь поперечного сечения элемента стержня, которая постоянна по его длине. Из (5), находя выражение для частных производных $\frac{\partial \varphi_i}{\partial x}$, $\frac{\partial \varphi_j}{\partial x}$ и $\frac{\partial \varphi_j}{\partial x}$

$$\frac{\partial \varphi_i(x)}{\partial x} = \frac{1}{\ell^2}(-3\ell + 4x); \quad \frac{\partial \varphi_j(x)}{\partial x} = \frac{4}{\ell^2}(\ell - 2x); \quad \frac{\partial \varphi_k(x)}{\partial x} = \frac{1}{\ell^2}(4x - \ell)$$

и, подставляя их в последнее выражение, получим, что

$$\frac{EF}{2}\int\limits_0^{\ell}\left(\frac{\partial \varphi_i}{\partial x} u_i + \frac{\partial \varphi_j}{\partial x} u_j + \frac{\partial \varphi_k}{\partial x} u_k\right)^2 dx =$$

$$= \frac{EF}{2}\left[\frac{7}{3\ell} u_i^2 - \frac{16}{3\ell} u_i u_j + \frac{2}{3\ell} u_i u_k + \frac{16}{3\ell} u_j^2 - \frac{16}{3\ell} u_j u_k + \frac{7}{3\ell} u_k^2\right]. \tag{13}$$

Дальше переходим к интегрированию второго интеграла в выражении (11)

2) $$\int\limits_{V_i} \alpha\left(T(x)\right) \cdot E \cdot T(x) \cdot \left(\frac{\partial u}{\partial x}\right) dV =$$

$$= \int\limits_{V_i} E\left(\varphi_i\alpha_i + \varphi_j\alpha_j + \varphi_k\alpha_k\right)\left(\varphi_i T_i + \varphi_j T_j + \varphi_k T_k\right)\left(\tfrac{\partial \varphi_i}{\partial x} u_i + \tfrac{\partial \varphi_j}{\partial x} u_j + \tfrac{\partial \varphi_k}{\partial x} u_k\right) dV =$$

$$= EF\int\limits_0^{\ell}\left[\varphi_i^2\alpha_i T_i + \varphi_i\varphi_j\alpha_i T_j + \varphi_i\varphi_k\alpha_i T_k + \varphi_i\varphi_j\alpha_j T_i + \varphi_j^2\alpha_j T_j + \varphi_j\varphi_k\alpha_j T_k + \varphi_i\varphi_k\alpha_k T_i + \right.$$

$$+ \varphi_j\varphi_k\alpha_k T_j + \varphi_k^2\alpha_k T_k\big] \cdot \left[\tfrac{-3\ell+4x}{\ell^2}u_i + \tfrac{4\ell-8x}{\ell^2}u_j + \tfrac{4x-\ell}{\ell^2}u_k\right] dx =$$

$$= EF\left\{\left[-\tfrac{1}{3}\alpha_i T_i - \tfrac{1}{5}\alpha_i T_j + \tfrac{1}{30}\alpha_i T_k - \tfrac{1}{5}\alpha_j T_i - \tfrac{8}{15}\alpha_j T_j + \tfrac{1}{15}\alpha_j T_k + \tfrac{1}{30}\alpha_k T_i + \right.\right.$$

$$+ \left.\tfrac{1}{15}\alpha_k T_j + \tfrac{1}{15}\alpha_k T_k\right] u_i + \left[\tfrac{2}{5}\alpha_i T_i + \tfrac{4}{15}\alpha_i T_j + 0 + \tfrac{4}{15}\alpha_j T_i + 0 - \tfrac{4}{15}\alpha_j T_k + 0 - \right.$$

$$\left. - \tfrac{4}{15}\alpha_k T_j - \tfrac{2}{15}\alpha_k T_k\right] u_j + \left[-\tfrac{1}{15}\alpha_i T_i - \tfrac{1}{15}\alpha_i T_j - \tfrac{1}{30}\alpha_i T_k - \tfrac{1}{15}\alpha_j T_i + \tfrac{8}{15}\alpha_j T_j + \right.$$

$$\left.\left. + \tfrac{1}{5}\alpha_j T_k - \tfrac{1}{30}\alpha_k T_i + \tfrac{1}{5}\alpha_k T_j + \tfrac{1}{3}\alpha_k T_k\right] u_k\right\}. \quad (14)$$

Теперь, подставляя (13) и (14) в (11), получим интегрированный вид выражения функционала, который характеризует потенциальную энергию упругой деформации дискретного элемента при наличии поля температуры [2, 3]:

$$\Pi_i = \frac{EF}{2}\left[\frac{7}{3\ell}u_i^2 - \frac{16}{3\ell}u_i u_j + \frac{2}{3\ell}u_i u_k + \frac{16}{3\ell}u_j^2 - \frac{16}{3\ell}u_j u_k + \frac{7}{3\ell}u_k^2\right] -$$

$$-EF\left\{\left[-\frac{1}{3}\alpha_i T_i - \frac{1}{5}\alpha_i T_j + \frac{1}{30}\alpha_i T_k - \frac{1}{5}\alpha_j T_i - \frac{8}{15}\alpha_j T_j + \frac{1}{15}\alpha_j T_k + \frac{1}{30}\alpha_k T_i + \right.\right.$$

$$+ \left.\frac{1}{15}\alpha_k T_j + \frac{1}{15}\alpha_k T_k\right] u_i + \left[\frac{2}{5}\alpha_i T_i + \frac{4}{15}\alpha_i T_j + 0 + \frac{4}{15}\alpha_j T_i + 0 - \frac{4}{15}\alpha_j T_k + 0 - \right.$$

$$\left. - \frac{4}{15}\alpha_k T_j - \frac{2}{15}\alpha_k T_k\right] u_j + \left[-\frac{1}{15}\alpha_i T_i - \frac{1}{15}\alpha_i T_j - \frac{1}{30}\alpha_i T_k - \frac{1}{15}\alpha_j T_i + \frac{8}{15}\alpha_j T_j + \right.$$

$$\left.\left. + \frac{1}{5}\alpha_j T_k - \frac{1}{30}\alpha_k T_i + \frac{1}{5}\alpha_k T_j + \frac{1}{3}\alpha_k T_k\right] u_k\right\}. \quad (15)$$

Далее, минимизируя последний функционал по узловым значениям упругого перемещения, получим математическую модель термонапряженного состояния дискретного элемента в виде разрешающих систем линейных алгебраических уравнений относительно перемещения узлов элемента:

1) $$\frac{\partial \Pi}{\partial u_i} = 0; \Rightarrow \frac{EF}{2}\left[\frac{14}{3\ell}u_i - \frac{16}{3\ell}u_j + \frac{2}{3\ell}u_k\right] - EF\left[-\frac{1}{3}\alpha_i T_i - \frac{1}{5}\alpha_i T_j + \frac{1}{30}\alpha_i T_k - \frac{1}{5}\alpha_j T_i - \right.$$
$$\left. - \frac{8}{15}\alpha_j T_j + \frac{1}{15}\alpha_j T_k + \frac{1}{30}\alpha_k T_i + \frac{1}{15}\alpha_k T_j + \frac{1}{15}\alpha_k T_k\right] = 0. \quad (16)$$

2) $$\frac{\partial \Pi}{\partial u_j} = 0; \Rightarrow \frac{EF}{2}\left[-\frac{16}{3\ell}u_i + \frac{32}{3\ell}u_j - \frac{16}{3\ell}u_k\right] - EF\left[\frac{2}{5}\alpha_i T_i + \frac{4}{15}\alpha_i T_j + \frac{4}{15}\alpha_j T_i - \frac{4}{15}\alpha_j T_k - \right.$$
$$\left. - \frac{4}{15}\alpha_k T_j - \frac{2}{5}\alpha_k T_k\right] = 0. \quad (17)$$

3) $$\frac{\partial \Pi}{\partial u_k} = 0; \Rightarrow \frac{EF}{2}\left[\frac{2}{3\ell}u_i - \frac{16}{3\ell}u_j + \frac{14}{3\ell}u_k\right] - EF\left[-\frac{1}{15}\alpha_i T_i - \frac{1}{15}\alpha_i T_j - \frac{1}{30}\alpha_i T_k - \frac{1}{15}\alpha_j T_i + \right.$$
$$\left. + \frac{8}{15}\alpha_j T_j + \frac{1}{5}\alpha_j T_k - \frac{1}{30}\alpha_k T_i + \frac{1}{5}\alpha_k T_j + \frac{1}{3}\alpha_k T_k\right] = 0. \quad (18)$$

Здесь следует отметить, что эти уравнения получены для узлов одного дискретного элемента. Так как мы дискретизируем рассматриваемый стержень множеством квадратичных элементов с тремя узлами, то для каждого элемента должно быть записано выражение функционала потенциальной энергии упругой деформации с учетом поля температур. Тогда общее выражение потенциальной энергии для рассматриваемого стержня в целом имеет вид (12). Общее число узлов будет равно: $2\times$чэ $+ 1$. В общем случае математической моделью термонапряженного состояния рассматриваемого стержня, защемленного с обоих концов, является следующая система линейных алгебраических $2\times$чэ $+ 1$ уравнений:

$$\frac{\partial \Pi}{\partial u_i} = 0, \quad (19)$$

где $i = 2 \div (2\times \text{чэ} + 1)$.

Из-за жесткого защемления двух концов стержня

$$u_1 = u_{2\times\text{чэ}+1} = 0. \tag{20}$$

Решая систему (19), находим узловые значения упругих перемещений. Значения упругого компонента деформации в первой половине элемента определяются следующим образом:

$$\varepsilon_x^I = \frac{\partial u}{\partial x}\left(x = \frac{x_j - x_i}{2}\right) = \frac{\partial u_i\left(x = \frac{x_j - x_i}{2}\right)}{\partial x} u_i + \frac{\partial u_j\left(x = \frac{x_j - x_i}{2}\right)}{\partial x} u_j + \frac{\partial u_k\left(x = \frac{x_j - x_i}{2}\right)}{\partial x} u_k. \tag{21}$$

Аналогично для второй половины элемента имеем

$$\varepsilon_x^{II} = \frac{\partial u}{\partial x}\left(x = \frac{x_k - x_j}{2}\right) = \frac{\partial u_i\left(x = \frac{x_k - x_j}{2}\right)}{\partial x} u_i + \frac{\partial u_j\left(x = \frac{x_k - x_j}{2}\right)}{\partial x} u_j + \frac{\partial u_k\left(x = \frac{x_k - x_j}{2}\right)}{\partial x} u_k. \tag{22}$$

Соответственно, по закону Гука, значения упругого компонента напряжения определяются следующим образом:

$$\sigma_x^I = E \cdot \varepsilon_x^I; \quad \sigma_x^{II} = E \cdot \varepsilon_x^{II}. \tag{23}$$

Значения температурного составляющего деформации и напряжения определяются следующим образом:

$$\varepsilon_T^I = -\alpha\left(x = \frac{x_j - x_i}{2}\right) \cdot T\left(x = \frac{x_j - x_i}{2}\right); \quad \varepsilon_T^{II} = -\alpha\left(x = \frac{x_k - x_j}{2}\right) \cdot T\left(x = \frac{x_k - x_j}{2}\right); \tag{24}$$

$$\sigma_T^I = E \cdot \varepsilon_T^I; \quad \sigma_T^{II} = E \cdot \varepsilon_T^{II}. \tag{25}$$

При известных ε_x^I; ε_x^{II}; ε_T^I; ε_T^{II}; σ_x^I; σ_x^{II}; σ_T^I; σ_T^{II} определяются значения термоупругих составляющих деформаций и напряжений [1, 3]:

$$\varepsilon^I = \varepsilon_x^I + \varepsilon_T^I; \quad \varepsilon^{II} = \varepsilon_x^{II} + \varepsilon_T^{II}; \tag{26}$$

$$\sigma^I = \sigma_x^I + \sigma_T^I; \quad \sigma^{II} = \sigma_x^{II} + \sigma_T^{II}. \tag{27}$$

Применяя полученные математические модели, рассмотрим численное моделирование термонапряженного состояния стержня из жаропрочного сплава АНВ-300 при одновременном наличии теплообмена, теплоизоляции и температуры постоянной интенсивности $T = \text{const} = 800\,^\circ$C.

Рассмотрим горизонтальный стержень из жаропрочного сплава АНВ-300, координатную ось OX направим слева вправо (рис. 2).

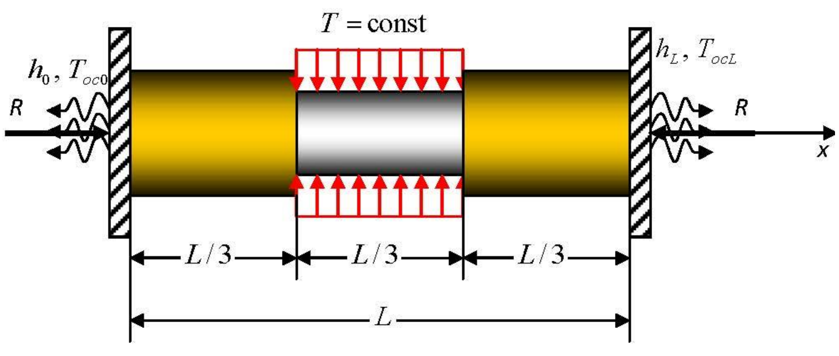

Рисунок 2. Расчетная схема задачи

Через площади поперечных сечений обоих концов происходит теплообмен с окружающими их средами. Коэффициент теплообмена и температура окружающих сред для левого конца h_0 $\left(\text{Вт}/(\text{см}^2 \cdot {}^\circ\text{C})\right)$ и T_{oc0} (°C). Аналогично для правого конца — h_L $\left(\text{Вт}/(\text{см}^2 \cdot {}^\circ\text{C})\right)$ и T_{ocL} (°C).

Боковую поверхность участков $(0 \leqslant x \leqslant L/3)$ и $(2L/3 \leqslant x \leqslant L)$ стержня считаем теплоизолированной. На участке $(L/3 \leqslant x \leqslant 2L/3)$ стержня дана температура постоянной интенсивности $T = \text{const} = 800\,^\circ$C. При наличии приведенных выше источников тепла и частичной теплоизоляции стержень старается расшириться. Но из-за защемления обоих концов появляются сжимающие усилия R. В связи с этим и из-за неоднородного поля температуры во внутренних сечениях стержня возникает неоднородное поле напряжения. Составляющие деформации будут ε_x, ε_T, ε, а напряжений — σ_x, σ_T, σ. Требуется определить поле перемещений $u = u(x)$, упругую деформацию ε_x, температурную деформацию ε_T, термоупругую деформацию ε, а также упругие, температурные и термоупругие напряжения σ_x, σ_T и σ. Функционал, характеризующий потенциальную энергию упругих деформаций при наличии поля температур, минимизируется по узловым перемещениям

$$\frac{\partial \Pi}{\partial u_i} = 0, \quad i = 2 \div 2n. \tag{28}$$

Решая последнюю систему, находим узловые значения перемещения. Далее по (21–27) определяются значения составляющих ε_x, ε_T, ε, σ_x, σ_T, σ в заданных сечениях стержня. За исходные данные принимаем следующие [3]:

$$L = 30\,(\text{см}); \quad r = 1\,(\text{см}); \quad F = \pi \cdot r^2 = \pi\,(\text{см}^2); \quad K_{xx} = 100\ \big(\text{Вт}/\,(\text{см} \cdot {}^\circ\text{C})\big);$$

$$h_L = 10\ \big(\text{Вт}/\,\big(\text{см}^2 \cdot {}^\circ\text{C}\big)\big); \quad T_{oc0} = T_{ocL} = 40\,({}^\circ\text{C}); \quad \text{T} = \text{const} = 800\,({}^\circ\text{C}).$$

При фиксированных значениях варьируем значение коэффициента теплообмена $h_0 = 7{,}5;\ 10;\ 15;\ 30\ \big(\text{Вт}/(\text{см}^2 \cdot {}^\circ\text{C})\big)$.

Сначала принимаем $h_0 = 7{,}5\ \big(\text{Вт}/(\text{см}^2 \cdot {}^\circ\text{C})\big)$. Узловые значения перемещения приведены в таблице 1. Соответствующее поле перемещений по длине стержня изображено на рисунке 3. Из этого рисунка видно, что сечения стержня, находящиеся на участке $0 \leqslant x \leqslant 14$ (см), перемещаются против направления оси Ox. В это время остальные сечения, находящиеся на участке $14 \leqslant x \leqslant 28$ (см), перемещаются по направлению оси Ox. При этом наибольшее перемещение против направления оси Ox соответствует сечению стержня $x = 6{,}6$ (см). Значение перемещения этого сечения равно $u_{66} = -0{,}0113$ (см). Наибольшее перемещение по направлению оси Ox соответствует сечению $x = 23{,}1$ (см), значение перемещения которого равно $u_{232} = 0{,}015$ (см). Сечение стержня с координатой $x = 14{,}05$ (см) не перемещается.

Таблица 1

Узловые значения перемещений при $T = 800\,({}^\circ\text{C})$; $h_0 = 7{,}5\ \big(\text{Вт}/(\text{см}^2 \cdot {}^\circ\text{C})\big)$

Узл. точки	Узл. знач. перемещений u (см)	Узл. точки	Узл. знач. перемещений u (см)	Узл. точки	Узл. знач. перемещений u (см)	Узл. точки	Узл. знач. перемещений u (см)	Узл. точки	Узл. знач. перемещений u (см)
1	0,0000000000	...	...	...	...	...	...	...	...
2	-0,0003374609	65	-0,0113061676	139	-0,0002715203	231	0,0150037799	299	0,0008018468
3	-0,0006670340	66	-0,0113075751	140	-0,0000738218	232	0,0150073841	300	0,0004052590
...	...	67	-0,0112980940	141	0,0001238766	233	0,0149983362	301	0,0000000000
		...	...	...	...	...	...		

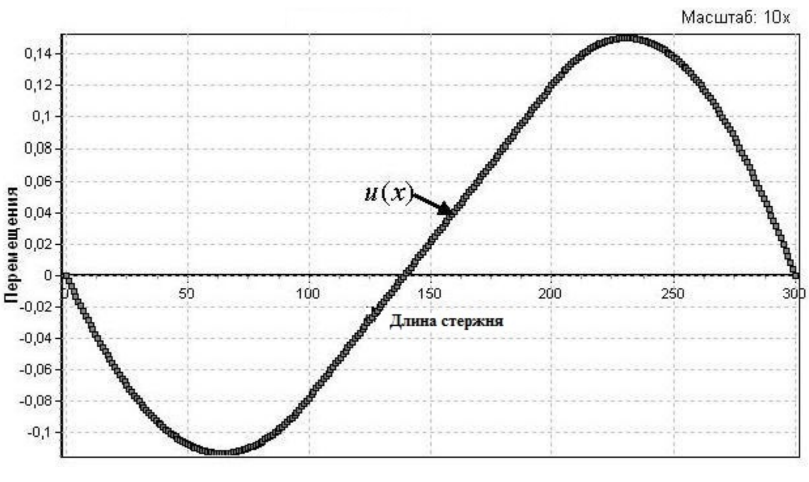

Рисунок 3. Поле распределения упругих перемещений по длине стержня

Поля распределения компонентов деформации ε_x, ε_T и ε, а также соответствующих напряжений σ_x, σ_T, σ приводятся на рисунках 4 и 5.

Из рисунка 4 видно, что характер упругого компонента деформации ε_x на участке стержня $0 \leqslant x \leqslant 6{,}65$ (см) является сжимающим. Далее на участке $6{,}65 < x \leqslant 23{,}25$ (см) ε_x имеет растягивающий характер. На остальном участке $23{,}25 < x \leqslant 30$ (см) стержня она имеет также сжимающий характер. В то время поведение температурного компонента деформаций ε_T является всюду сжимающим. Также всюду является сжимающей термоупругая составляющая деформации $\varepsilon = \varepsilon_x + \varepsilon_T$. Кроме того, что поля распределений ε_x и ε являются симметричными относительно прямой $\varepsilon = -0,0000014\,x - 0,007654$.

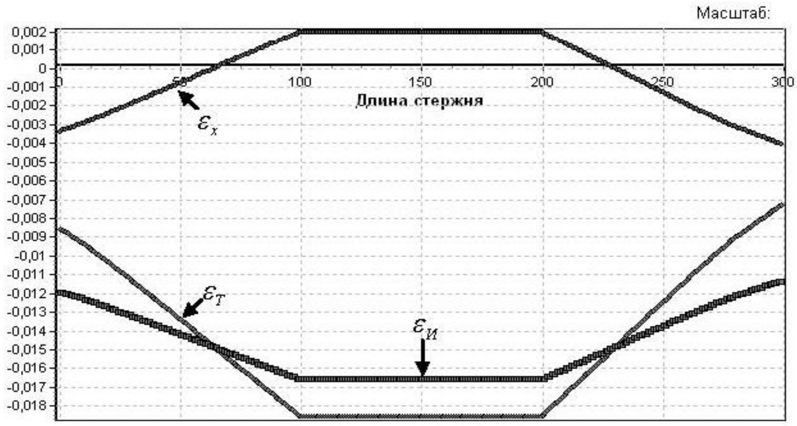

Рисунок 4. Поле распределения деформаций по длине стержня

Как видно из рисунка 5, поведение составляющих напряжений σ_x, σ_T и σ будет аналогичным, как и ε_x, ε_T, ε. При этом наибольшее термоупругое напряжение, имеющее сжимающий характер, наблюдается на участке $10{,}5 \leqslant x \leqslant 19{,}95$ (см) стержня, и его значение на этом участке будет равно $\sigma = -33166{,}03$ (кГ/см^2). На участке $0 \leqslant x < 10,5$ (см) стержня σ растет монотонно от $\sigma = -23869,66$ (кГ/см^2) в сечении $x = 0,05$ (см) до $\sigma = -33066,48$ (кГ/см^2) в сечении $x = 9,95$ (см). Из рисунка 5 также видно, что σ_x и σ являются симметричными относительно горизонтальной оси симметрии. Здесь следует отметить, что если $\alpha = \text{const}$, то $\sigma = \text{const}$ по всей длине стержня.

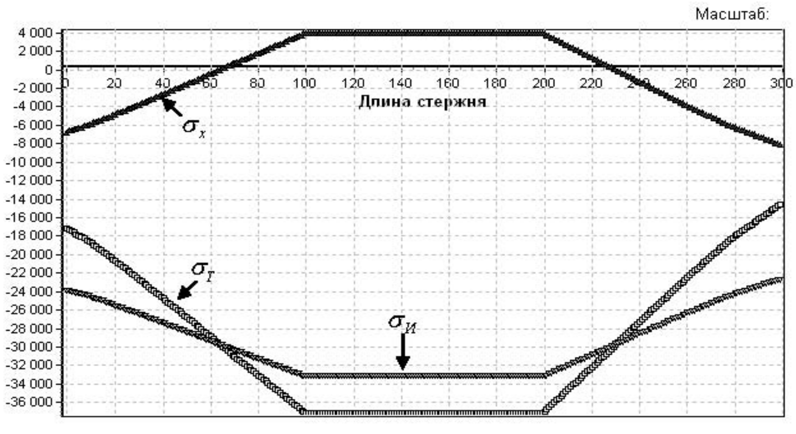

Рисунок 5. Поле распределения напряжений по длине стержня

Из полученных численных результатов видно, что в сечениях стержня, где большие температуры, значение термоупругого составляюшего напряжения σ будет большим. В связи с этим по всей длине стержня поле распределения σ не будет горизонтальной прямой. В это время на участке $19{,}95 \leqslant x < 30$ (см) поведение σ будет сжимающим, но его значение монотонно понижается от $\sigma = -33166,03$ (кГ/см^2) до $\sigma = -22690,009$ (кГ/см^2) соответственно.

Аналогично проведем численное исследование термонапряженного состояния данного стержня при разных значениях коэффициента теплообмена и по результатам проведенных численных экспериментов построим таблицу 2 [3].

Таблица 2

Влияние коэффициента теплообмена h_0 (Вт/(см$^2 \cdot {}^\circ$C)) на термонапряженно-деформированное состояние исследуемого стержня

№ п/п	$h_0\left(\frac{\text{Вт}}{\text{см}^2\cdot{}^\circ\text{C}}\right)$	$u_{\min}$ (см)	Координата соответ-го сечения (см)	$u_{\max}$ (см)	Координата соответ-го сечения (см)	$\sigma_{\max}\left(\frac{\text{кГ}}{\text{см}^2}\right)$	$\sigma_{\text{ср}}\left(\frac{\text{кГ}}{\text{см}^2}\right)$	Координата соответ-го точки, где $u=0$ (см)
1	7,5	0,0113	$x = 6{,}6$	0,015	$x = 23{,}1$	-33166	-29706,25	$x = 14{,}05$
2	10	0,01405	$x = 6{,}7$	0,01405	$x = 23{,}3$	-32885,9	-29181	$x = 15$
3	15	0,0177	$x = 7{,}1$	0,01282	$x = 23{,}5$	-32514,3	-28484,3	$x = 16{,}295$
4	30	0,0229	$x = 7{,}3$	0,011188	$x = 23{,}9$	-31991	-27503	$x = 17{,}695$

Анализируя данную таблицу, можно сделать следующие выводы:

– при увеличении значения h_0 возрастает амплитуда перемещений против направления оси Ox;

– при увеличении значения h_0 координата сечения, амплитуда перемещения которого будет наибольшей, увеличивается;

– при увеличении значения h_0 амплитуда перемещения по направлению оси Ox уменьшается;

– при увеличении значения h_0 максимальное и среднее значения термоупругого напряжения σ уменьшаются.

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Температура тұрақты болған жағдайда стерженнің термокернеулік күйінің жылу алмасу коэффициентінен тәуелдігін сандық зерттеу және математикалық модельдеу

Мақалада энергетикалық қағида негізінде қызуға төзімді құймадан жасалған стерженнің жылукернеулік-деформациялық жағдайының математикалық моделі жасақталды. Энергетикалық принцип үш түйінді квадраттық шекті элементтер әдісін қолданумен ұштастырылған серпімді деформацияның әлеуеттік энергиясын минималдауға негізделген. Стерженнің шекті ұзындығы бар және екі шеті мықты бекітілген. Оның ($0 \leqslant x \leqslant L/3$) және ($2L/3 \leqslant x \leqslant L$) бөліктері жылудан оқшауланған. Стерженнің екі жағының көлденең қимасының ауданы арқылы оларды қоршаған ортамен жылу алмасу жүреді, ал оның орта ($L/3 \leqslant x \leqslant 2L/3$) бөлігіне $T = \text{const} = 800\,{}^\circ$C болатын тұрақты температура түсірілген. Жылу алмасу коэффициентінің тұрақты температура әсеріндегі АНВ-300 қызуға төзімді құймасынан жасалған стерженнің жылукернеулік жағдайына әсері зерттеліп, зерттеудің сандық нәтижелері келтірілді. Зерттеулер жылу алмасу коэффициентінің әр түрлі мәндерінде жүргізілді. Нәтижесінде, h_0 – жылу алмасу коэффициентінің мәнін арттырғанда орын ауысудың (жылжудың) амплитудасы Ox осіне қарсы бағытта өсетіндігі; орын ауысудың амплитудасы ең үлкен болатын қима координатасы артатындығы; Ox осінің бағыты бойынша орын ауысудың амплитудасы азаятындығы; σ жылу серпімділік кернеудің ең үлкен және орташа мәндері азаятындығы анықталды.

Кілт сөздер: шекті элементтер әдісі, температура, жылудан оқшауланған, жылу алмасу, жылу алмасу коэффициенті, серпімді деформацияның потенциалдық энергиясы, математикалық модель.

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Mathematical modeling and computational investigation of the dependence of the thermal stressed state of the rod on the heat transfer coefficient in the presence of a temperature of constant intensity

In the article on the basis of energy principles, a mathematical model of thermal stressed-deformed state of a rod from a heat-resistant alloy. The energy principle is focused on minimizing the potential energy of elastic deformations in combination, the application of the method of a quadratic finite element with three nodes. Rod of limited length and rigidly pinched by two ends. The lateral surface of the rod sections ($0 \leqslant x \leqslant L/3$) and ($2L/3 \leqslant x \leqslant L$) is thermally insulated. Through the cross-sectional area of both ends of the rod, heat exchange takes place with their environment. The temperature of $T = \text{const} = 800\,^{\circ}\text{C}$ constant intensity is given on the middle section of rod ($L/3 \leqslant x \leqslant 2L/3$). Investigated effects of heat transfer coefficient on the thermally stressed state of the core of the high-temperature alloy ANV-300 in the presence of a temperature of constant intensity and numerical research results are represented. Research were conducted for different values of the heat transfer coefficient. As a result, it was found that with an increase in the value of the h_0 – coefficient of heat transfer the amplitude of displacements increases against the direction of the axis Ox; coordinate of section, displacement amplitude, which will be the largest, increases; the amplitude of the interchange of the axis Ox direction is reduced; the maximum and average values of the thermoelastic stress σ decrease.

Keywords: finite element method, temperature, heat insulation, heat transfer, heat transfer coefficient, potential energy of elastic deformations, mathematical model.

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Optimization of contents of two-component polydispersed filler by applying the mathematical design of experiment in forming composites for transport repairing

The influence of two-component polydispersed filler on the elasticity modulus in flexure and impact resilience of the developed epoxy-polyester composite is analyzed. Regression equation by applying the mathematical design of experiment is found and dependence of output parameters on chosen variable factors is determined. It is proved that introduction of two-component polydispersed filler (in contents of mica – q = 20. . . 30 pts.wt., nitride boron – q = 40. . . 60 pts.wt.) in composite allows to increase significantly parameters of elasticity modulus in flexure to E = 7.2. . . 7.6 hPa with slight decrease of impact resilience to W = 4.6. . . 4.8 kJ/m^2. Theoretical analysis of the calculation results of functional relationships is carried out and mathematical model that properly describes behavior of the investigated material is found. Optimal content of main and additional fillers in composite for each property is established using found regression equations. It is determined that composite material in contents of mica q = 20. . . 30 pts.wt. and of nitride boron q = 40. . . 60 pts.wt. has optimal parameters.

Keywords: composite, epoxy-polyester matrix, two-component polydispersed filler, method of mathematical design of experiment, regression equation.

Introduction

Development of multicomponent system, which is a polymeric composite material (PCM), is a complex and long-lasting process. The composite with necessary properties can be formed by changing the content of single components, that is, changing the composition of multicomponent system. At the same time, to determine the structure of interrelationships of object parameters (properties) and to find quantitative constraint equation of outcome indexes and outcome parameters it is necessary to conduct set of experiments that requires high material costs. Therefore, an important task is to get necessary data with a minimum number of experiments. In order to optimize the results of multicomponent systems research, the mathematical design of the experiment is used for the required accuracy of results. Mathematical model receiving allows predicting material properties. Forecasting allows taking into account all factors that affect the functioning of heterogeneous systems, and provides the formation of technical object with high efficiency and quality [1, 2]. It is important to note that the use of the created bidisperse two-component composite material with improved properties in the complex during manufacturing and repairing the elements of marine transport (protective coatings for ship hulls, parts of friction units for ship machinery, etc.) allow significantly improve their operational characteristics.

During experimental studies different disperse fillers are added to binder to increase PCM outcome parameters [3, 4]. The issues of optimizing the composition and structure of highly filled epoxy composites containing additives of various dispersion are highlighted in works [5-8]. It should be noted that epoxy-polyester matrixes filled with dispersed fillers insufficiently investigated. The influence of disperse fillers (mica and nitride boron hexagonal) on the developed matrix (each component separately) is predetermined. The purpose of chosen disperse particles is to improve the PCM tribological properties. The multifactorial interaction of two-component polydispersed filler in complex with respect to composites properties remains unexplored. The method of simultaneous variation of several parameters in order to study their impact and the impact of interaction on the composites performance characteristics is used precisely in the integrated approach to establish the optimal content of mica and NB. During active experiment mathematical design exclude implementation of a large number of experiments and significantly reduces the timing of receiving the result.

Aim of work – to optimize the content of two-component polydispersed filler to improve the performance characteristics of composite material for its use in repairing the working elements of marine transport using the method of mathematical planning of the experiment.

Results and discussion

Design of experiment allows building a research strategy based on a sequence of clear and logically deliberated operations. Received mathematical model reflects the interconnection of the physical and mechanical properties of composites (elasticity modulus in flexure and impact resilience) from the content of bidispersed filler and gives an opportunity to study its influence on composition outcome parameters. The nature of changes of elasticity modulus in flexure and impact resilience as a result of addition of different amounts of main and additional fillers (mica and nitride boron hexagonal, respectively) is investigated. Particles dispersion according to granulometric analysis: mica – 20...40 micron, NB – 8...10 micron. For standardization, as well as for simplification of calculations, each component (filler) is encoded by conditional units taking into account variations (Table 1).

Table 1

Levels of variables on conditional and natural scale

Components	Factor	Average level, q, pts.wt.	Variation step, Δq, pts.wt.	Values of variables (pts.wt.), that corresponding to conditional units		
				-1	0	+1
Main filler – mica	x_1	30	10	20	30	40
Additional filler – nitride boron hexagonal	x_2	40	20	20	40	60

According to the experiment planning scheme 9 experiments (N = 9) were conducted, each of which was repeated three times (p = 3) in order to exclude system errors (Table 2). In order that planning matrix to be orthogonal [9], the corrected values of E'level were entered, which were calculated by the formula

$$x_i' = (x_i)^2 - \frac{\sum_{u-1}^{N} x_{iu}^2}{N}. \tag{1}$$

Table 2

Scheme of experiment planning

No of experiment (u)	x_0	x_1	x_2	$E_3 = E_1^2 - d$	$E_4 = E_2^2 - d$	E_1E_2
1	2	3	4	5	6	7
1	1	-1	-1	0.33	0.33	+1
2	1	+1	-1	0.33	0.33	-1
3	1	-1	+1	0.33	0.33	-1
4	1	+1	+1	0.33	0.33	+1
5	1	0	0	-0.67	-0.67	0
6	1	+1	0	0.33	-0.67	0

Table continuation

1	2	3	4	5	6	7
7	1	-1	0	0.33	-0.67	0
8	1	0	+1	-0.67	0.33	0
9	1	0	-1	-0.67	0.33	0
$\sum_{u-1}^{N} x_{iu}^2$	9	6	6	2	2	4

The expanded matrix of planning of complete factor experiment (CFE) and its results are shown in Table 3.

Table 3

The results of investigation of the elasticity modulus in flexure and impact resilience PCM

No of experiment	Content of components, q, pts.wt.		Elasticity modulus in flexure, E, hPa	Impact resilience, W, kJ/m^2
	x_1	x_2	y_1	y_2
1	20	20	5.6	4.1
2	40	20	5.2	3.9
3	20	60	7.6	4.8
4	40	60	6.2	4.5
5	30	40	7.2	4.6
6	40	40	5.8	4.2
7	20	40	6.8	4.3
8	30	60	6.9	4.4
9	30	20	5.4	4.3

The mathematical model $y = f\ (x_1,\ x_2)$ was formed as a regression equation

$$y = b_0 + b_1x_1 + b_2x_2 + b_{11}x_1^2 + b_{22}x_2^2 + b_{12}x_1x_2. \tag{2}$$

The regression coefficients were determined by the formula:

$$b_{\mathrm{i}} = \frac{\sum_{u=1}^{N} x_i y_i}{\sum_{u-1}^{N} x_{iu}^2}. \tag{3}$$

Received coefficients of regression equation are given in Table 4.

Table 4

The coefficients of regression equation

b_0	b_1	b_2	b_{11}	b_{22}	b_{12}
6.80	-0.47	0.75	-0.3	-0.45	-0.25

As a result, in the analysis of the elasticity modulus in flexure, the following regression equation was determined:

$$y = 6.8 - 0.47x_1 + 0.75E_2 - 0.3E_1^2 - 0.45E_2^2 - 0.25E_1E_2.$$

For the statistical processing of experiment results, a test of reproducibility of experiments by the Cochran test was conducted:

$$G = \frac{S_{umax}^2}{\sum_{u=1}^{m} S_{ui}^2} \leq G_{(0,05;f_1;f_2)}, \tag{4}$$

where S_{ui}^2 — dispersion of experiment results on combinations of few factor levels for m=3; m — number of parallel experiments; $S^2{}_{umax}$ — the highest dispersion in design line.

Dispersions of adequacy were determined by the formula:

$$S_{ui}^2 = \frac{\sum_{i=1}^{m} (y_i - \overline{y_i})^2}{m-1}, \tag{5}$$

where y_{im} — value, received from each parallel experiment; $\overline{y_i}$ — average value y, received in parallel experiments.

Mean square error was determined by formula:

$$\sigma^2\{y\} = \frac{\sum_{i=1}^{N=9}\sigma^2\{y\}_i}{N(m-1)}, \tag{6}$$

where $\sigma^2\{y\}_i = \sum_{i=1}^{m=3}(y_i - \overline{y_i})^2$;

$$\sigma^2\{y_{av}\} = \frac{0^2\{C\}}{N}, \quad S^2_{b_0} = \frac{S^2_0}{N}. \tag{7}$$

Dispersion values are shown in Table 5.

Table 5

Values of dispersions of adequacy (S^2_{ui}) and mean square error ($\sigma^2\{y\}_i$)

No	The dispersions of adequacy		The mean square error	
	conditional designation	value	conditional designation	value
1	S^2_{u1}	0.03	$\sigma^2\{y\}_1$	0.06
2	S^2_{u2}	0.04	$\sigma^2\{y\}_2$	0.08
3	S^2_{u3}	0.03	$\sigma^2\{y\}_3$	0.06
4	S^2_{u4}	0.01	$\sigma^2\{y\}_4$	0.02
5	S^2_{u5}	0.03	$\sigma^2\{y\}_5$	0.06
6	S^2_{u6}	0.01	$\sigma^2\{y\}_6$	0.02
7	S^2_{u7}	0.04	$\sigma^2\{y\}_7$	0.08
8	S^2_{u8}	0.03	$\sigma^2\{y\}_8$	0.06
9	S^2_{u9}	0.01	$\sigma^2\{y\}_9$	0.02

n this case:

$$\sum_{i=1}^{N} S^2_{ui} = 0.23;$$

$$\sigma^2\{y\} = S^2_0 = 0.026.$$

Then the calculated value of the Cochran test at the 5% level of significance:

$$G_{calc} = \frac{S^2_{u_{\max}}}{\sum_{i=1}^{N} S^2_{ui}}; \tag{8}$$

$$G_{calc} = \frac{0,04}{0,23} = 0.174.$$

Testing the experiment results by the Cochran test [9] for a fixed probability $\alpha = 0.05$ confirmed the reproducibility of the experiments. Dispersion of experiment results on combinations of few factor levels: $S^2_{u\max} = 0.04$. Calculated value of Cochran test: $G_{calc} = 0.174$.

Table value of Cochran test: $G_{tab} = 0.478$.

So, the requirement is fulfilled (7):

$$G_{calc} = 0.174 \leq G_{tab} = 0.478.$$

Subsequently, the coefficients significance of regression equation was determined by analyzing the results according to the experimental design (Table 6).

Table 6

The experimental results of study of the elasticity modulus in flexure of materials

No of experiment	Elasticity modulus in flexure, E, hPa			Average value, E, hPa
	1	2	3	
1	5.4	5.7	5.7	5.6
2	5.0	5.4	5.2	5.2
3	7.4	7.7	7.7	7.6
4	6.3	6.1	6.2	6.2
5	7.1	7.4	7.1	7.2
6	5.7	5.9	5.8	5.8
7	6.6	7.0	6.8	6.8
8	6.7	7.0	7.0	6.9
9	5.3	5.5	5.4	5.4

Then the dispersions of regression coefficients (Table 7) were determined by the formula:

$$S_{b_i}^2 = \frac{S_0^2}{\sum_{u-1}^{N} x_{iu}^2}. \qquad (9)$$

The significance of the regression coefficients was determined by the Student's test [10, 11]. Here with the table (t) and calculated criterion (t) of Student's test (Table 7) were determined.

Depending on freeness: f = N (n - 1) = 9 (3 - 1) = 18 the Student's test value were calculated, which is $t = 2.1$.

Calculated values of Student's test (t) and coefficients significance were determined: $t_0, t_1, t_2, t_{11}, t_{22}, t_{12} > t_T$. Moreover:

$$t_{i@} = \frac{|b_i|}{S_{b_i}}. \qquad (10)$$

Table 7

Dispersion of coefficients of regression (S_b^2) and calculated values of Student's criterion (t)

No	Dispersion of coefficients of regression		Calculated values of Student's criterion	
	conditional designation	value	conditional designation	value
1	$S_{b_0}^2$	0.003	t_0	123.19
2	$S_{b_1}^2$	0.004	t_1	7.15
3	$S_{b_2}^2$	0.004	t_2	11.49
4	$S_{b_{11}}^2$	0.013	t_{11}	2.65
5	$S_{b_{22}}^2$	0.013	t_{22}	3.98
6	$S_{b_{12}}^2$	0.006	t_{12}	3.10

Calculated values of Student's criterion $t_0, t_1, t_2, t_{11}, t_{22}, t_{12}$ are larger than t_T, so it was considered that all coefficients of the regression equation are significant. As a result of rejection of insignificant coefficients, the following regression equation was received: $y = 6.8 - 0.47x_1 + 0.75E_2 - 0.3E_1^2 - 0.45E_2^2 - 0.25E_1E_2$.

The adequacy of the model was checked by Fisher test [3, 7]:

$$F_c = \frac{S_{u\max}^2}{S_y^2} \leq F_{(0.05;f_{0d};f_y)}, \qquad (11)$$

where $S_{umax}^2 = 0.04$ — calculated value of dispersion of adequacy (Table 5);

$$S_y^2 = \frac{\sum_{i=1}^{N} S_{ui}^2}{N}, \qquad (12)$$

$S_y^2 = 0.026$ — mean square error. So: $F_c = 1.565$.

$F_{(0.05;f_{0d};f_u)}$- table value of Fisher test in 5% significance level $f_1 = N-(k+1) = 9-(6+1) = 2$, $f_2 = N(n-1) = 9(3-1) = 18$). So: $F_{(t)} = 3.55$ [10, 11].

Calculated value of Fisher test is less than table one, so the requirement (10)is fulfilled. It is possible to assume that equation adequately characterizes the composition.

Interpretation process of received mathematical model, as a rule, is not just determination of factors influence. A simple comparison of absolute value of linear coefficients does not determine the relative degree factors influence, since there are also quadratic squared terms and paired interactions. In a detailed analysis of the received adequate model, it is necessary to take into account the fact that for a quadratic model the degree of factor influence on the change of output value is not constant.

Dependencies that connect normalized and natural values of the variables are as follows:

$$x_i = \frac{q_i - q_{i0}}{\Delta q_i}, \tag{13}$$

where q_i — value of i experiment factor; q_{i0} — value of zero level; Δq_i — variation interval [9].

Substituting these values in accordance with the formula (13) into the regression equation and transforming it, we receive the following regression equation with the natural values of the variables:

$$E = 0.86 + 0.178q_1 + 0.16125q_2 - 0.003q_1^2 - 0.001125q_2^2 - 0.00125q_1q_2.$$

Given equation in natural values allows only predicting the output value for any point in the middle of range of factor variations. However, with its help it is possible to construct graphs of dependence of output value (elasticity modulus in flexure of composites) from any factor (or two factors). Geometric interpretation of the response surface is shown on Figure 1-3.

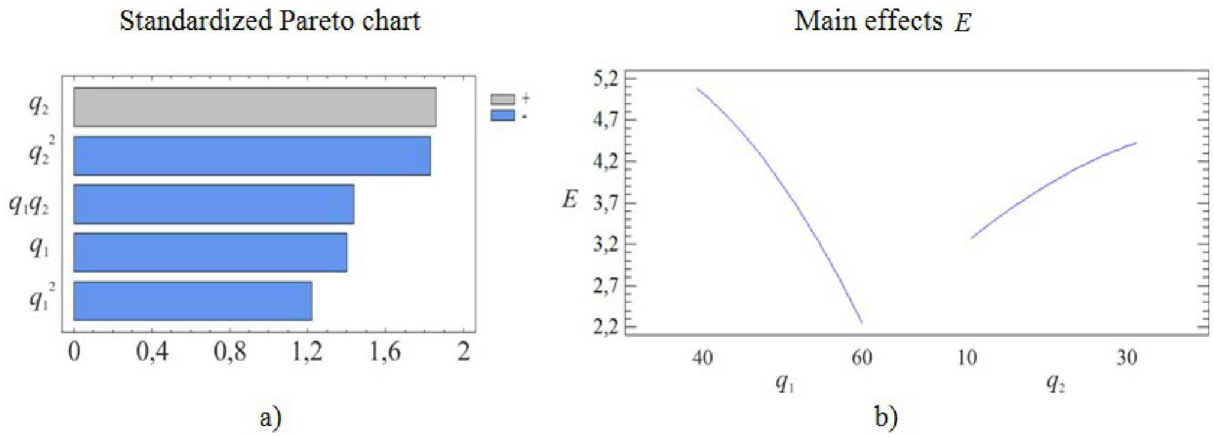

Figure 1. Standardized Pareto chart (a) and main effects (b)

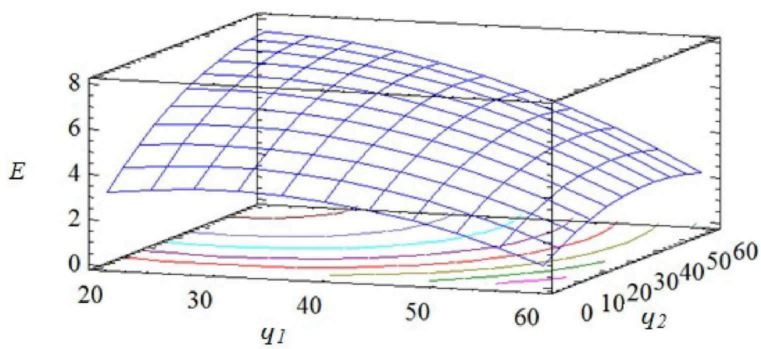

Figure 2. Estimated surface $E = f(q_1, q_2)$

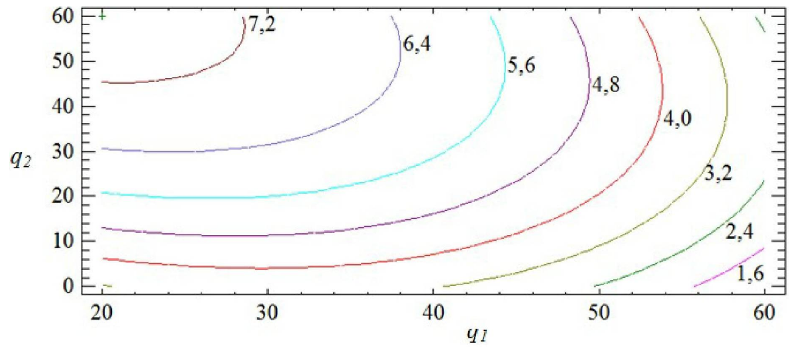

Figure 3. Contours of estimated response surface

Based on experimental studies it is set that both factors are significant. It should be noted that the effect of the additional filler content on the parameters of elasticity modulus in flexure is higher in comparison with the main one (according to Pareto chart). Analyzing the calculated response surface, it is determined that the optimum parameters of elasticity modulus in flexure have developed epoxy-polyester composite with two-component poly dispersed filler with the following content of particles: mica – 20...30 pts.wt., NB – 40...60 pts.wt. ($E = 7.2 \ldots 7.6hPa$).

Similarly to the above calculations scheme, the composition formula was optimized according to the viscosity index. The encoding of natural components values and the experimental design scheme are chosen according to Table 1 and Table 2.

In the process of study results analysis of composites impact resilience, the following values of the regression coefficients were received (Table 8).

Table 8

The coefficients of regression equation

b_0	b_1	b_2	b_{11}	b_{22}	b_{12}
4.42	-0.18	0.32	0.02	-0.18	0.05

As a result, the following regression equation was found:

$$y = 4.42 - 0.18x_1 + 0.32E_2 + 0.02E_1^2 - 0.18E_2^2 + 0.05E_1E_2.$$

For statistical processing of experiment results, a test of experiments reproducibility was conducted according to the Cochran test [9].

Dispersions values that were calculated by formula (5-7) are shown in a Table 9.

Table 9

Values of dispersions of adequacy (S_{ui}^2) and mean square error ($\sigma^2\{y\}_i$)

No	The dispersions of adequacy		The mean square errors	
	conditional designation	value	conditional designation	value
1	S_{u1}^2	0.01	$\sigma^2\{y\}_1$	0.02
2	S_{u2}^2	0.04	$\sigma^2\{y\}_2$	0.08
3	S_{u3}^2	0.03	$\sigma^2\{y\}_3$	0.06
4	S_{u4}^2	0.03	$\sigma^2\{y\}_4$	0.06
5	S_{u5}^2	0.04	$\sigma^2\{y\}_5$	0.08
6	S_{u6}^2	0.01	$\sigma^2\{y\}_6$	0.02
7	S_{u7}^2	0.01	$\sigma^2\{y\}_7$	0.02
8	S_{u8}^2	0.03	$\sigma^2\{y\}_8$	0.06
9	S_{u9}^2	0.03	$\sigma^2\{y\}_9$	0.06

Moreover:

$$\sum_{i=1}^{N} S_{ui}^2 = 0.23;$$

$$\sigma^2\{y\} = S_0^2 = 0.026.$$

Calculated value of the Cochran test at the 5% significance level was determined by formula (8):

$$G_c = \frac{0.04}{0.23} = 0.174.$$

Testing the experiment results by the Cochran test [10, 11] for a fixed probability $\alpha = 0.05$ confirmed the experiments reproducibility. Dispersion characterizing dispersal of the experiments results in combination of few factor levels: $S_{u\max}^2 = 0.04$. Calculated value of Cochran test: $G_{calc} = 0.174$.

Table value of Cochran test: $G_{tab} = 0.478$.

So, the requirement is fulfilled:

$$G_{calc} = 0.174 \le G_{tab} = 0.478.$$

At the next stage, the coefficients significance of regression equation is determined, analyzing the results according to the experimental design (Table 10).

Table 10

The experimental results of study of the impact resilience of CM

No of experiment	Impact resilience, W', kJ/m^2			Average value, W', kJ/m^2
	1	2	3	
1	4.3	4.1	4.2	4.2
2	3.9	3.5	3.7	3.7
3	4.7	4.7	5.0	4.8
4	4.6	4.3	4.6	4.5
5	4.4	4.8	4.6	4.6
6	4.1	4.2	4.3	4.2
7	4.4	4.5	4.6	4.5
8	4.5	4.2	4.5	4.4
9	3.8	4.1	3.8	3.9

Subsequently, dispersion of regression coefficients is determined by formulas (9-10). The significance of regression coefficients is determined according to Student's criterion, which table value is $t_T = 2.1$ [10,11]. Calculated values of Student's criterion are shown in Table 11.

Table 11

Dispersion of coefficients of regression (S_b^2) and calculated values of Student's criterion (t)

No	Dispersion of coefficients of regression		Calculated values of Student's criterion	
	conditional designation	value	conditional designation	value
1	$S_{b_0}^2$	0.003	t_0	81.81
2	$S_{b_1}^2$	0.004	t_1	2.81
3	$S_{b_2}^2$	0.004	t_2	4.85
4	$S_{b_{11}}^2$	0.013	t_{11}	0.15
5	$S_{b_{22}}^2$	0.013	t_{22}	1.62
6	$S_{b_{12}}^2$	0.006	t_{12}	0.60

Calculated values of Student's criterion t_0, t_1, t_2 are larger than t, so it is considered that coefficients b_0, b_1, b_2 of regression equation are significant. Calculated values t_{11}, t_{22}, t_{12} are smaller than t_T, so coefficients b_{11}, b_{22}, b_{12} are insignificant. As a result, the following regression equation is received:

$$y = 4.42 - 0.18x_1 + 0.32E_2.$$

The adequacy of the model was checked by Fisher's test [10, 11].

Calculated value of adequacy dispersion: $S^2_{u\,\max} = 0.04$ (Table 9).

The mean square error: $S^2_y = 0.026$. So: $F = 1.565$. $F_{(0{,}05;f_W;f_u)}$ — table value of Fisher's test in 5% significance level ($F_{(t)} = 2.77$) [10, 11].

Calculated value of Fisher's test is smaller than table on, so requirement (11) is fulfilled. Consequently, the equation adequately shows the composition formula.

After transformations in accordance with formula (13), the following regression equation with the natural values of variables was received:

$$W' = 4.32 - 0.018q_1 + 0.016q_2.$$

Geometric interpretation of response surface is shown on Figure 4-6.

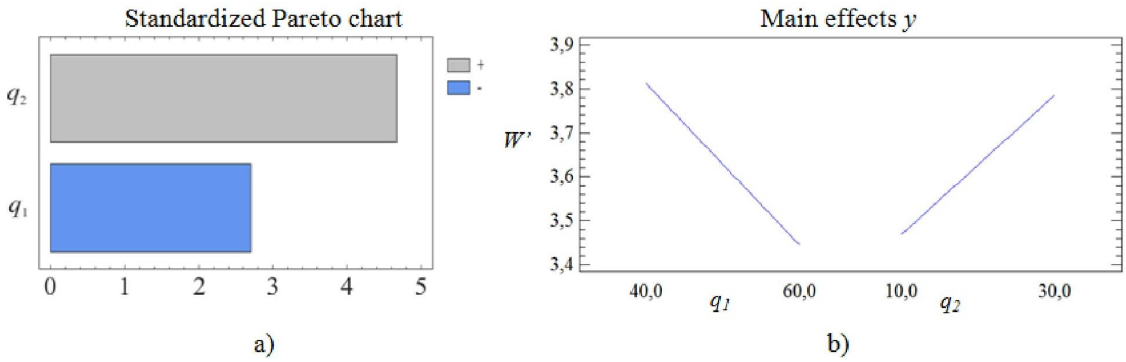

Figure 4. Standardized Pareto chart (a) and main effects (b)

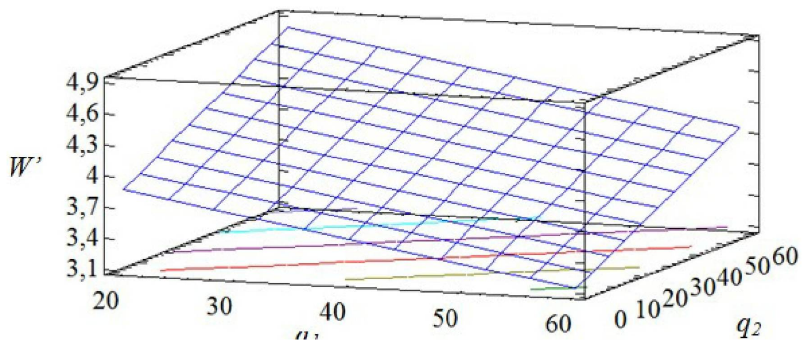

Figure 5. Estimated surface $W' = f(q_1, q_2)$

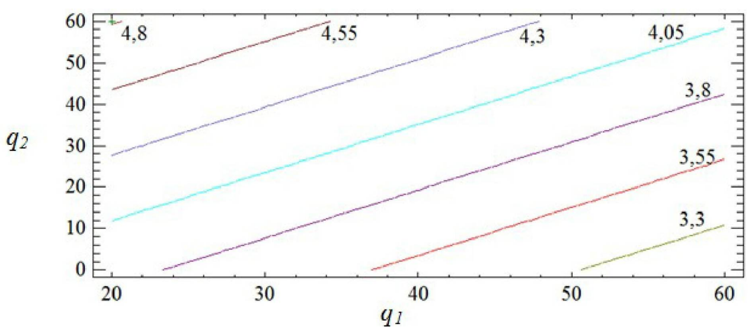

Figure 6. Contours of estimated response surface

Received results indicate that both factors of regression equation are significant. However, the output parameters of the composite are influenced only by linear dependencies of these factors. In the process of analysis, it was determined that the impact resilience values show maximum values for the fillers contents: mica – 20...30 pts.wt., nitride boron hexagonal – 40...60 pts.wt. (W' = 4,6. . . 4,8 кДж/м^2). With further increase of particles in content the impact resilience degradation was observed. In our opinion, this happens due to aggregation of fillers in polymer matrix, which negatively affects the physical and mechanical properties of the material. Therefore it is advisable to add two-component polydisperse filler with the aforementioned content into modified epoxy-polyester matrix to improve performance in the repair of marine transport elements.

Conclusions

The analysis of set of experiments results in mathematical design of experiment showed that in regression dependences linear effects have more significant effect than interaction effects. This is especially noticeable in regression dependence in study of the composite material impact resilience. Received results are confirmed by Pareto charts and response surfaces. The optimum content of two-component polydispersed filler was set: mica is 20...30 pts.wt., nitride boron hexagonal is 40...60 pts.wt. The introduction of two-component polydispersed filler into the composite on the basis of epoxy-polyester binder can significantly increase values of elasticity modulus in flexure of composites to E = 7.2...7.6 hPa with a slight decrease of impact resilience to W' = 4.6...4.8 kJ/m^2. Received results allow getting materials with improved parameters of physical and mechanical properties. Due to the use of developed CM for protective coatings for ship hulls, repair of friction units of ship machinery, etc. it is possible to increase term between their overhauls and improve the performance characteristics in general.

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Көлік құралдарын жөндеуге арналған композиттерді дайындауда эксперименттің математикалық дизайнын қолдану арқылы екікомпонентті полидисперсті толтырғыштың мазмұнын оңтайландыру

Иілген және соққы тұтқырлық өңделген эпоксидті-полиэфир композит үшін бикомпонентті полидисперсті толтырғыштың серіппелік модуліне әсері талданған. Экспериментті математикалық жоспарлау әдісімен регрессия теңдеуі және таңдалған айнымалы факторлардан шағатын параметрлерден тәуелділігі алынған. Құрамға (слюда - q = 20. . . 30 мас. сағ.; NB - q = 40. . . 60 мас. сағ.) бикомпонентті полидисперсті толтырғышты копозитті енгізу, иілуі болмашы E = 7.2. . . 7.6 hPa дейин соққы тұтқырлығын азайтуға, болмашы W ’= 4,6 . . . 4,8 кДж / м^2 дейин соққы серпімділігін азайтуға серпімділік модулінің көрсеткіштерін едәуір көтеруге мүмкіндік беретіні дәлелденген. Зерттелетін материалдың өзгерісін сипаттайтын функционалдық тәуелділік есептеулері нәтижелерінің теориялық талдауы жүргізілген және математикалық моделі алынған. Шыққан регрессия теңдеулерінің көмегімен әр қасиет үшін композиттегі негізгі және қосымша толтырғыштардың оңтайлы құрамы алынған. Құрамы q = 20...30 мас. сағ. слюда және q = 40. . . 60 мас. сағ. бор нитриды болатын композитті материал жақсартылған көрсеткіштермен сипатталатыны көрсетілген.

Кілт сөздер: композициялық, эпоксид-полиэфирлі матрица, екікомпонентті полидисперстік толтырғыш, математикалық эксперименттерді жоспарлау әдісі, регрессиялық теңдеу.

Н.В. Браило, А.В. Букетов, С.В. Якущенко, А.А. Сапронов, Л. Дулебова

Оптимизация содержания двухкомпонентного полидисперсного наполнителя путем применения математического планирования эксперимента при получении композитов для ремонта транспортных средств

Проанализировано влияние двухкомпонентного полидисперсного наполнителя на модуль упругости при изгибе и ударную вязкость разработанного эпоксидно-полиэфирного композита. Методом математического планирования эксперимента получены уравнения регрессии и установлена зависимость выходных параметров от выбранных переменных факторов. Доказано, что введение в состав композита двухкомпонентного полидисперсного наполнителя при содержании (слюда - q = 20. . . 30 мас.ч., NB - q = 40. . . 60 мас.ч.) позволяет значительно повысить показатели модуля упругости при изгибе до E = 7.2. . . 7.6 hPa при незначительном снижении ударной вязкости до W’ = 4,6. . . 4,8 кДж/м^2. Проведен теоретический анализ результатов расчета функциональных зависимостей, и получена математическая модель, которая адекватно описывает поведение исследуемого материала. С помощью полученных уравнений регрессии установлено оптимальное содержание основного и дополнительного наполнителей в композите для каждого свойства. Доказано, что улучшенными показателями характеризуется композитный материал по содержанию q = 20. . . 30 мас.ч. слюды и q = 40. . . 60 мас.ч. нитрид бора.

Ключевые слова: композит, эпоксидно-полиэфирная матрица, двухкомпонентный полидисперсный наполнитель, метод математического планирования эксперимента, уравнение регрессии.

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Dynamics of auger working body of a multifunctional conveyor

A mathematical model of bending oscillations of a multifunctional conveyor working body with consideration of the angular velocity of its rotation and the motion along its outdoor medium is developed. Based on this model, the analytical relations were defined that describe the laws of changing the defining parameters of the working body oscillations for both non-resonant and resonant cases. The amplitude of the transition through the resonance is found to depend greatly on the relative quantity of medium motion and the rate of transition through the resonance.

Keywords: dynamic modeling, amplitude, resonance, conveyor, determining parameters, speed of rotation.

Introduction

The auger working body of the multifunctional conveyor in the operating mode undergoes considerable dynamic loads due to a simultaneous effect of longitudinal compressive force (transmitted from moving regulating blocks), external driving moment, and the forces of interaction with the processed medium. The indicated power factors cause complex oscillations in the body, that is a combination of torsion, longitudinal, and bending ones. The transverse oscillations to a certain extent reduce the service life of the auger, and also create additional dynamic loads on a «fixed» cone trough. Moreover, the processed medium when moving at a certain speed relative to the auger working body causes an additional dynamic effect. This effect is largely manifested for its bending oscillations. In addition, the heterogeneous inclusions always occur in the processed medium or in the medium being transported. Due to the rotational movement of the working body, they «block» it up in separate points, thereby causing additional power actions in them. On the other hand, based on these factors, the mathematical model of the relative motion dynamics of the system 'elastic body - medium moving flow' acquires a qualitatively new consideration [1-3], for which, in the general case, the known analytical methods of studying the systems with distributed parameters cannot be applied [4]. The first two types of oscillations (in some cases) contribute to the technological process improvement, in particular the adhesion prevention of the processed medium, its additional densification, and the structure perfection) [5-10]. The application of numerical simulation methods does not lead to the desired results due to the complexity of systems, in particular the most dangerous resonant oscillations of the working body. In this paper, the main attention is paid to the development of approximate analytical methods for studying bending oscillations of the multifunctional conveyor working body, which rotates with a constant angular velocity around the longitudinal axis; and a continuous flow of the processed medium moves along it. Resonant and non-resonant cases are considered as well.

Material and method

The auger compensating multifunctional conveyor is made in the form of a casing (1) with both front (2) and rear (3) supports (Fig. 1). Their height can be changed in order to transport materials horizontally and at an angle. The fixed cone trough (4) is firmly fixed to these two supports in several variants - the solid one and with the system of through-holes in its lower part made in a known way. Inside the fixed conical body metricconverterProductID4, a4, a conical screw working element (5) with variable steps is installed in the bearing units with the possibility of turning and axial displacement. The bearing bushes 5 are welded to the left (2) and the right (3) supports; in their central openings, the movable cylindrical blocks (6) are set. The bearings are rigidly installed in the middle of blocks; the internal holes of bearing are in contact with the shaft (7) ends of the conical auger (5) with the possibility of axial displacement.

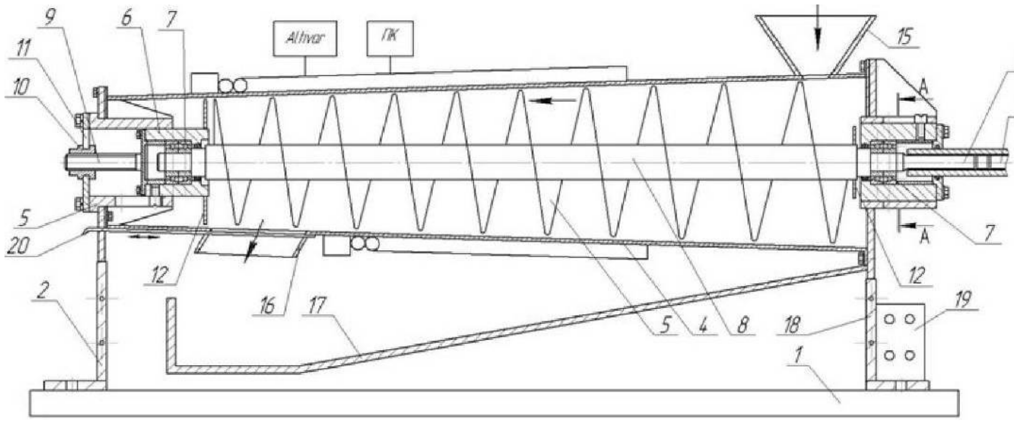


Figure 1. Schematic construction of auger compensating multifunction conveyor

The ends of movable blocks (6) are covered with lids. A screw (9) is rigidly welded to the left lid (8) of the cylindrical block (6). The screw is in contact with the adjusting nut 10, which is firmly mounted in the axial groove of the lid (11) of the movable cylinder block metricconverterProductID6. In6. In addition, the stopper (13) are firmly fixed to the left and right ends of the shaft (7) of the cylindrical movable block (6). They prevent the movement of transport mass beyond the screw working body.

Similarly, at the right end of the screw cone working body, the structure is the same, but the difference is that the ends of the shaft (7) are made with an elongated cut (14), to which the drive shaft (15) of the conveyor is connected. In the loading zone, the boot pipe (16) is installed, and in the unloading zone, the discharge nozzle (17) is installed as well. To fix the screw conical working body metric converter ProductID5, in5, in the moving blocks (6), the axial slots (19) are made with screw fixing elements (20). The system is controlled from the console (21).

Depending on operations, a stationary pipe is chosen. Thus, let us consider the operation of the conveyor for squeezing the juice. For this purpose, the inclination angle should be set by reducing the height of the left support in a known way. The stationary pipe is chosen with through-holes and the corresponding design of a conical screw working body with an appropriate gap between the holes and the stationary pipe; the gap is set with the help of an adjustor nut. The materials, from which the juice will be squeezed, are loaded into the boot pipe. A conveyor is started and the appropriate modes of operation are set on the control panel. The squeezed juice flows into a container and a trench through a through-hole system, then it is taken away. The pulp is collected through a discharge nozzle into a separate container with the help of a sliding shutter.

A multifunctional conveyor screw working body rotates with a constant angular velocity ω; the continuous flow of the processed medium moves along it at a relative speed V. Thus, the challenge is to determine the influence of external and internal factors on the transverse oscillations of the working body.

Basic assumptions about the object under study:

– the auger working body is the elastic body, which is symmetric relative to the longitudinal axis; its material satisfies the nonlinear technical law of elasticity [11] - $\sigma = E\left(\varepsilon_1 + \varepsilon\varepsilon_1^3\right)$ (ε_1 - relative deformation, the parameter ε characterizes the deviation of its elastic properties from the linear law; it is considered small in comparison with the elastic modulus E);

– its inertia moment relative to the longitudinal axis OX is $I\left(x\right)$; its mass per unit length $m_1\left(x\right)$ rotates with a constant angular velocity around the longitudinal axis inclined to the horizon at an angle α; $u\left(x,t\right)$ - transverse displacement of its neutral axis with the coordinate x at any given time t (OX axis is deduced from the upper bearing along the undeformed axis of the screw working body); the deplanation of the normal cross-section is absent.

The processed medium is not an elastic solid body [12-14] with a mass per unit length $m\left(x\right)$. Mathematical model of the object under study: based on the above [11,15], the differential equation of bending oscillations of the multifunctional conveyor working body, along which the continuous flow of the processed medium moves, is deduced

$$(m_1+m)\,(\frac{\partial^2 u(x,t)}{\partial t^2}+\omega^2 u(x,t))+2mV\frac{\partial^2 u(x,t)}{\partial t\partial x}+(mV^2+N)\frac{\partial^2 u(x,t)}{\partial x^2}+\lambda\left(\frac{\partial u(x,t)}{\partial t}\right)^s+$$

$$+\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(EI(x)\left(\frac{\partial^2 u(x,t)}{\partial x^2}+\varepsilon\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^3\right)\right)\right]=H\sin^{2q}(\omega t+\varphi_0)\,\delta(x-x_0)\,, \tag{1}$$

where N — pressure forces at the auger end; $\lambda,\ s,\ x_0,\ \varphi_0$; H — steels; $\delta(x-x_0)$ — Delta Dirac's function. Dependence $\lambda\left(\frac{\partial u(x,t)}{\partial t}\right)^s$ describes the resistance force to the working body motion; and the ratio $H\sin^{2q}(\omega t+\vartheta)\,\delta(x-x_0)$ describes the interaction force of the heterogeneous inclusion and the working body ($q=1,2,...$; x_0 indicates the location of the heterogeneous inclusion, and q - its form).

To describe the dynamic process of the system under consideration, the boundary conditions are added to the equation (1) that for simplicity are assumed in the form

$$u(x,t)|_{x=0;l}=\frac{\partial^2 u(x,t)}{\partial x^2}|_{x=0;l}, \tag{2}$$

where l — the distance between the upper and lower bearings of the working body.

To solve this problem, for the above-mentioned mathematical model of the dynamics process, it is necessary to develop the basic analytical dependences for determining the law of changing the defining parameters of the oscillation process, depending on the external and internal factors of the system.

To solve the first part of the problem, we will make additional physically based assumptions about force factors: a) the maximum values of the resistance forces and the interaction force of the inhomogeneous inclusion with the working body is a small value in comparison with the maximum value of the second or fourth terms of the left-hand side of the equation (1); b) the motion quantity of the processed medium in the relative motion (in relation to the working body) is a limited quantity; c) inertia moment of the working body is a slowly varying function; and its change along the length is neglected. Thus, the differential equation (1) is deduced:

$$\frac{\partial^2 u(x,t)}{\partial t^2}+\omega^2 u(x,t))+\frac{N}{m_1+m}\frac{\partial^2 u(x,t)}{\partial x^2}+\frac{EI}{m_1+m}\frac{\partial^4 u(x,t)}{\partial x^4}=$$

$$=\frac{1}{m_1+m}\Bigg(H\sin^{2q}(\omega t+\varphi_0)\,\delta(x-x_0)-\lambda\left(\frac{\partial u(x,t)}{\partial t}\right)^s-$$

$$-mV^2\frac{\partial^2 u(x,t)}{\partial x^2}-2mV\frac{\partial^2 u(x,t)}{\partial t\partial x}-\varepsilon EI\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^3\Bigg). \tag{3}$$

Based on the above limitations for the system under study, the maximum value of the right-hand side of the equation is small in comparison with the maximum value of the terms of its left-hand side. The latter, in turn, is the basis for solving the boundary problem (3), (2) of combining the Bubnov-Galerkin method [9] and the Van der Paul method [15]. The first one-frequency approximation in modes close to the principal oscillation mode can be presented as $u(x,t)=\sin\frac{\pi}{l}xT(t)$, where $T(t)$ is an unknown function; this function is the solution of the common quasi-linear differential equation

$$\ddot{T}(t)+\Omega^2 T(t)=$$

$$=-\frac{\bar{\lambda}}{m+m_1}\dot{T}^s+\frac{mV^2}{(m+m_1)}\left(\frac{\pi}{\ell}\right)^2 T+\frac{3\varepsilon EI}{32}\left(\frac{\pi}{\ell}\right)^6 T^3+\frac{H}{\ell\,(m+m_1)}\sin\frac{\pi x_0}{\ell}\sin^{2q}(\omega t+\varphi_0)\,, \tag{4}$$

where $\Omega^2=\frac{EI}{(m+m_1)}\left(\frac{\pi}{\ell}\right)^4-\omega^2-\frac{N}{(m+m_1)}\left(\frac{\pi}{\ell}\right)^2$, $\bar{\lambda}=\lambda\frac{\Gamma(1+s/2)}{2\Gamma(1.5+s/2)}$.

According to the principle of one-frequency oscillations in nonlinear systems [16], the dynamic process in the mode close to the first mode of the «dynamic equilibrium» of the object under study is analyzed. This process is considered the most important in view of practical use.

Results

The right-hand side of the equation (4) is periodic in time with the period $\frac{\pi}{\omega}$. Then, the resonance in the system can occur in case of a certain correlation between the frequency of the own (unperturbed) oscillations of the working body and the angular velocity of its rotation, and, therefore, a periodic perturbation. Thus, both resonant $\Omega \approx 2\omega$ and non-resonant $\Omega \neq 2\omega$ cases should be considered to solve the equation (4). In this paper, only the case of the main resonance is considered; the consideration of the combination or fractional resonance does not constitute a fundamental difference. Based on the above, the main resonance in the auger working body occurs at the next angular velocity of its rotation (Fig. 2)

$$\omega = \frac{\pi}{\sqrt{5}} \left(\frac{EI}{(m+m_1)} \left(\frac{\pi}{\ell}\right)^2 - \frac{N}{(m+m_1)} \right)^{\frac{1}{2}}. \tag{5}$$

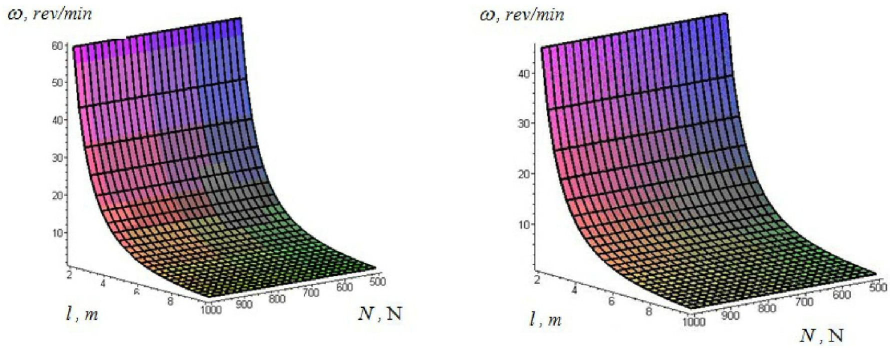


Figure 2. Dependence of the angular velocity of the working body rotation on its length and compressive force

The oscillations amplitudes, for the non-resonant case, do not depend on the external perturbation [15, 16] and are determined by the ratio

$$\frac{da}{dt} = \frac{1}{\pi\Omega} \int_0^{2\pi} \int_0^{2\pi} \{F(a,\psi,\varphi)\} \cos\psi d\varphi d\psi, \tag{6}$$

where $F(a,\psi,\varphi)$ corresponds to the value of the right-hand side of the equation (4), and $T = a\cos\psi$, $\psi = \Omega t + \psi_0$. Thus, for perturbed motion, the laws of changes in the amplitude of non-resonant oscillations are deduced

$$\frac{da}{dt} = \frac{\lambda}{\sqrt{\pi}\Omega} \frac{\Gamma(1+s/2)}{2\Gamma(1.5+s/2)} a^s. \tag{7}$$

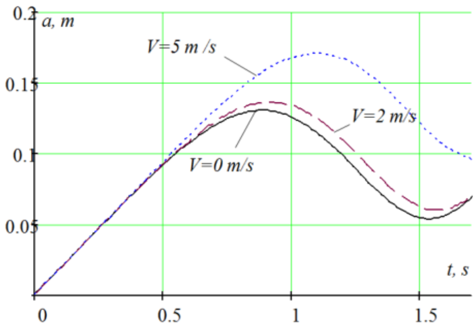
A resonant case is more difficult to study, and at the same time it is more important in view of practical use. For this case, the amplitude of the transition through resonance depends to a large extent on the difference between the phases of the own and forced oscillations, that is, the parameter $\gamma = \psi - \varphi$, $\varphi = \omega t$. The relations that describe the laws of changing the basic parameters of the working body during the transition through the main resonance are developed:

$$\frac{da}{dt} = \frac{\lambda}{\pi\Omega} \frac{^2(1+\mathrm{s}/2)}{^2(1.5+\mathrm{s}/2)} a^s - \frac{H}{(m+m_1)(\Omega+\omega)} \frac{2q}{2+q} \frac{\Gamma((2q+1)/2)}{\Gamma(1+q/2)} \sin\gamma \sin\frac{\pi x_0}{\ell};$$

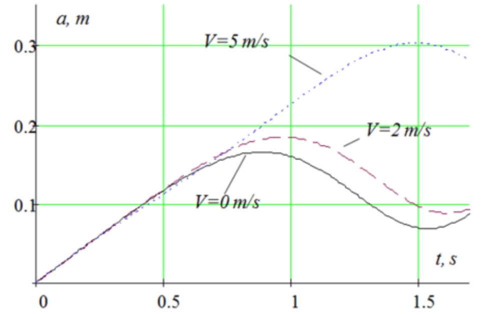
$$\frac{d\gamma}{dt} = \Omega - \omega/2 - \frac{mV^2}{2(m+m_1)\Omega} + \frac{0.07\varepsilon EI\pi}{(m+m_1)} a^2 +$$

$$+ \frac{H}{(m+m_1)(\Omega+\omega)} \frac{2q}{2+q} \frac{\Gamma((2q+1)/2)}{\Gamma(1+q/2)} \cos\gamma \sin\frac{\pi x_0}{\ell}. \tag{8}$$

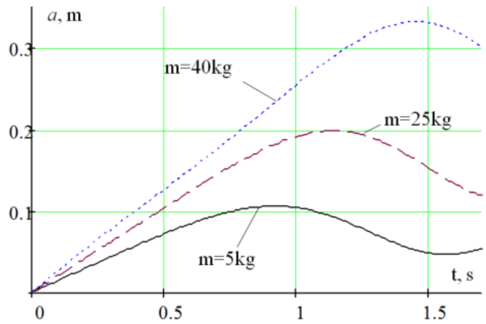
In Figure 3, the dependences of the change in time of the transverse oscillations amplitude at different values of the system parameters are represented.



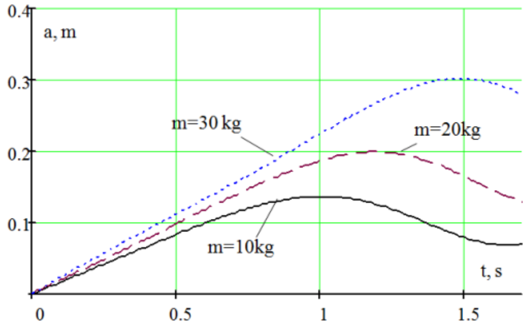
$N = 100, l = 6, m = 15m_1 = 10, \omega = 8.432\Omega = 16.988$



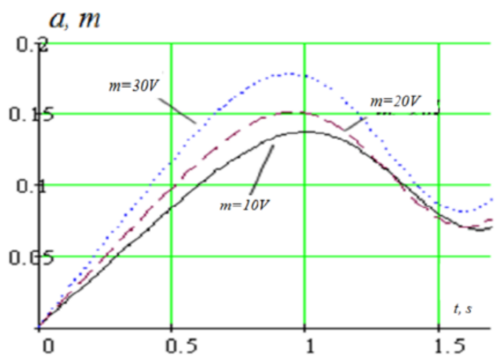
$N = 100, l = 6, m = 30m_1 = 10, \omega = 6.6632\Omega = 13.14$



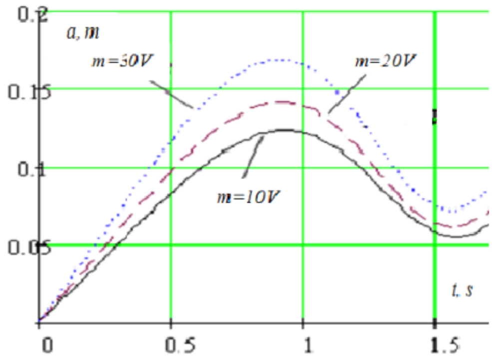
$N = 100, l = 6, m_1 = 10, V = 4, \omega = 10.886, \omega = 7.126,$
$\omega = 5.962\Omega = 21.931; \Omega = 15.335; \Omega = 12.012$



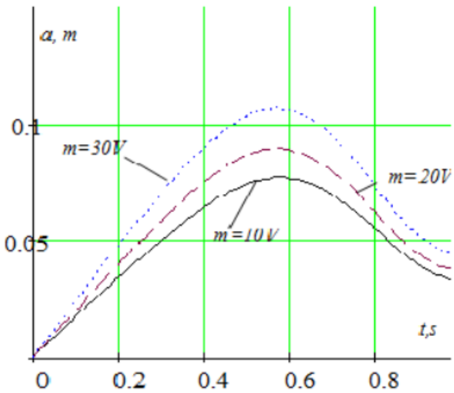
$N = 100, l = 6, m_1 = 10, V = 5, \omega = 9.4279;$
$\omega = 7.697; \omega = 6.766$



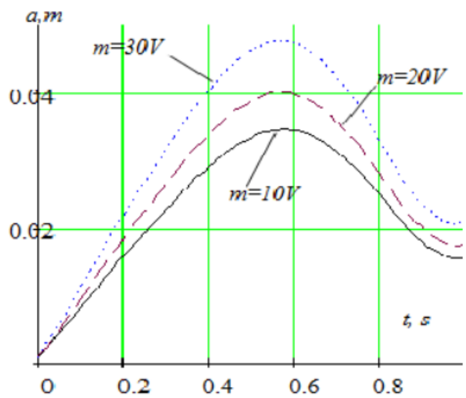
$N = 100, l = 6, m = 50, m_1 10\omega = 9.427,$
$\omega = 7.697, \omega = 6.666$



$N = 100, l = 6, mV = 30m_1 = 10\omega = 9.427,$
$\omega = 7.697, \omega = 6.666$



$N = 100, l = 6, mV = 30, m_1 = 10, \omega = 9.427,$
$\omega = 7.697, \omega = 6.666$



$N = 100, l = 4, mV = 30, m_1 = 10,$
$\omega = 21.22, \omega = 17.331, \omega = 15.011$

Figure 3. Changes in time of the amplitude of the multifunctional conveyor working body during the transition through the main resonance

Conclusions

The developed analytical and graphic dependences prove that for the resonant case:

– the amplitude of the transition through the main resonance is greater for larger velocity values of the processed medium relative motion. An increase in the relative velocity from *2* to *5 m/s* at the parameters $m = 15$ kg/m, $m_1 = 10$ kg/m, causes an increase in the amplitude of the transition through the main resonance by 17 %, and at the parameters $m = 30$ kg/m, $m_1 = 10$ kg/m, by 54 %.

– an increase in the mass per unit length of the processed medium at the constant relative velocity of its motion causes an increase in the amplitude of the transition through resonance. An increase in the mass per unit length from *25 kg/m* to *40 kg/m* at a relative velocity of its motion *4 m/s* causes an increase in the amplitude of the transition through the resonance by 61 %.

– in case of identical quantities of the processed medium relative motion, the amplitude of the transition through the main resonance is greater provided that the relative velocities of the motion are smaller

– the transition rate through the main resonance largely affects the magnitude of the resonance amplitude; and the amplitude is smaller for larger rates of transition through resonance.

In the non-resonant case, the attenuation rate of the transverse oscillations amplitude is greater for larger quantities of medium motion. The basic idea of the above methodology can be applied to the case of many inhomogeneous inclusions, as well as for torsional oscillations of the working body.

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Б.Гевко, О.Ляшук, М.Сокил, Е.Скиба, А.Маруныч, Д.Шматко

Көпфункционалды конвейердің жұмыс элементінің динамикасы

Көп функционалдық транспортердің жұмыс қаруының иілген тербелісінің математикалық моделі алынған, мұнда транспортер айналымының бұрыштық жылдамдығы мен сыртқы ортаның оның бойымен жылжыуы ескерілген. Алынған нәтиженің негізінде резонасты және бейрезонансты жағдайлары үшін оның тербелісін анықтайтын параметрлердің өзгеру заңдылықтарын айқындайтын аналитикалық қатынастар алынған. Резонанс арқылы өту амплитудасы ортаның жылжу санына және резонанс арқылы өту жылдамдығына тәуелді.

Кілт сөздер: динамикалық модельдеу, амплитуда, резонанс, транспортер, параметрлерді айқындау, айналу жылдамдығы.

Б.Гевко, О.Ляшук, М.Сокил, Е.Скиба, А.Маруныч, Д.Шматко

Динамика шнекового рабочего органа многофункционального транспортера

Получена математическая модель изгибных колебаний рабочего органа многофункционального транспортера с учетом угловой скорости его вращения и движения вдоль него наружной среды. На ее базе получены аналитические соотношения, описывающие законы изменения определяющих параметров его колебаний как для нерезонансного, так и для резонансного случаев. Установлено, что амплитуда перехода через резонанс в значительной степени зависит от относительного количества движения среды и скорости перехода через резонанс.

Ключевые слова: динамическое моделирование, амплитуда, резонанс, транспортер, определяющие параметры, скорость вращения.

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On mathematical and analytical methods for solving problems on vibrations of membranes and plates

The problems about determination of the frequencies and forms of natural vibrations of plates and shells lead to the necessity of partial differential equations integration. The well-researched cases are those where it is possible to separate the variables. In particular, these include the vibrations of a rectangular plate hinged on opposite sides, umbrella and fan vibration of circular axisymmetric plates and vibrations of cylindrical shells, closed or hinged along generating curves. In this work, the vibration of a flat homogeneous membrane is investigated for the general case of boundary conditions.

Keywords: boundary value problem, membrane, plate, vibrations, spectrum problem, orthonormal system of functions, deflection function.

Film and membrane structures are highlighted among the thin-walled structures that combine lightness with high strength. Such thin-walled structures (films, membranes, coatings, etc.) find application in all branches of production and daily life [1].

To create new films, membranes and coatings with the specified performance and durability, one needs to investigate how the time and temperature, mechanical (including vibrational), chemical and other exposures may cause destructive processes in the material structure. Therefore, the necessary quality of films, membranes and coatings are usually provided by calculating the impact of these effects on the strength and the characteristics necessary for exploitation of the material [2].

Consider a flat homogeneous rectangular membrane fixed at the edges, with sides b and c in the plane OXY,$0 \leqslant x \leqslant b, 0 \leqslant y \leqslant c$. We denote the deflection function of the membrane, that is, its deviation from the equilibrium position at the point (x, y) at the time t, by $u(x, y, t)$. Let us consider the process when the vibrations of the membrane are caused by a given initial deviation and a given initial velocity [3].

To find the function $u(x, y, t)$ we have the following boundary value problem: to find the solution of the partial differential equation describing the process of the membrane vibrations,

$$u_{tt} = a^2(u_{xx} + u_{yy}), \tag{1}$$

where $a^2 = \frac{T}{\rho}$, T is the membrane tension; ρ is the density of a membrane, in the region $0 < x < b$, $0 < y < c$, $t > 0$,

under the initial conditions

$$u(x, y, 0) = \varphi(x, y); \tag{2}$$

$$u_t(x, y, 0) = \psi(x, y). \tag{3}$$

and boundary conditions

$$\alpha_1 u(0, y, t) + \beta_1 u_x(0, y, t) = 0, \;\; \alpha_2 u(b, y, t) + \beta_2 u_x(b, y, t) = 0; \tag{4}$$

$$\gamma_1 u(x, 0, t) + \theta_1 u_y(x, 0, t) = 0, \;\; \gamma_2 u(x, c, t) + \theta_2 u_y(x, c, t) = 0, \tag{5}$$

where φ and ψ are the given functions; α_i, β_i, γ_i, θ_i are the given numbers, and $\alpha_i^2 + \beta_i^2 \neq 0$, $\gamma_i^2 + \theta_i^2 \neq 0$; $i = 1, 2$.

We seek the solution of problem (1)–(5) by the method of variables separation as a function in a form [4]

$$u(x, y, t) = \nu(x, y) \cdot T(t). \tag{6}$$

This function (6) is not identically equal to zero. Dividing the variables, we obtain an equation for the function $T(t)$

$$T'' + a^2\sigma T = 0, \tag{7}$$

and for the function $\nu(x, y)$ we get the following boundary value problem

$$\nu_{xx} + \nu_{yy} + \sigma\nu = 0, \tag{8}$$

$$\alpha_1\nu(0, y) + \beta_1\nu_x(0, y) = 0, \ \ \alpha_2\nu(b, y) + \beta_2\nu_x(b, y) = 0, \tag{9}$$

$$\gamma_1\nu(x, 0) + \theta_1\nu_y(x, 0) = 0, \ \ \gamma_2\nu(x, c) + \theta_2\nu_y(x, c) = 0, \tag{10}$$

where σ is a constant of variables separation. For the ease of calculations we take σ with a minus sign, without assuming anything about its sign. The boundary conditions (9), (10) are obtained by the direct substitution of (6) in (4), (5).

To solve problem (8) - (10) we again apply the method of variables separation. We seek a solution of this problem in the form of a function $\nu(x, y) = X(x) \cdot Y(y)$, which is not identically equal to zero. To define functions $X(x)$ and $Y(y)$ from (8) - (10) we obtain one-dimensional spectral problems

$$\begin{cases} X'' + \eta \cdot X = 0, \\ \alpha_1 X(0) + \beta_1 X'(0) = 0, \\ \alpha_2 X(b) + \beta_2 X'(b) = 0, \end{cases} \quad \begin{cases} Y'' + \tau \cdot X = 0, \\ \gamma_1 Y(0) + \theta_1 Y'(0) = 0, \\ \gamma_2 Y(c) + \theta_2 Y'(c) = 0, \end{cases} \tag{11}$$

where η is constant variables separation, and $\tau = \sigma - \eta$ [5].

Remark 1. By direct calculation, we determine that the spectral problem for an equation with a parameter ν

$$\begin{cases} Z'' + \nu \cdot Z = 0; \\ h_1 Z(0) + g_1 Z'(0) = 0; \\ h_2 Z(l) + g_2 Z'(l) = 0, \end{cases} \tag{12}$$

where $Z = Z(z)$; $0 < z < l$; h_i, g_i $(i = 1, 2)$ are the given numbers, and $h_i^2 + g_i^2 \neq 0$, $i = 1, 2$, has non-trivial solutions in the following cases:

1) $\nu = 0$ when the condition holds

$$g_1 h_2 - h_1(h_2 l + g_2) = 0; \tag{13}$$

2) $\nu > 0$.

By remark 1, the spectral problems (11) have eigenvalues and eigenfunctions if $\eta = \tau = 0$ when the condition (13) holds for the corresponding parameters, and if $\eta > 0$, $\tau > 0$.

We introduce the notations $\eta = \lambda^2$, $\tau = \mu^2$ in (11). Solving spectral problems (11), we receive that the eigenvalues $\lambda_1, \ldots, \lambda_n, \ldots$ and $\mu_1, \ \ldots, \ \mu_m, \ \ldots$ of these problems are the roots of the following equations, respectively

$$\operatorname{tg}\lambda b = \frac{(\alpha_2\beta_1 - \alpha_1\beta_2)\lambda}{\alpha_1\alpha_2 + \beta_1\beta_2\lambda^2}, \ \ \operatorname{tg}\mu c = \frac{(\gamma_2\theta_1 - \gamma_1\theta_2)\mu}{\gamma_1\gamma_2 + \theta_1\theta_2\mu^2},$$

and the eigenfunctions are functions in the form

$$X_n(x) = A_n(\beta_1\lambda_n\cos\lambda_n x - \alpha_1\sin\lambda_n x), \ \ Y_m(y) = B_m(\theta_1\mu_m\cos\mu_m y - \gamma_1\sin\mu_m y),$$

where A_n, B_m are constants.

Since $\sigma = \tau + \eta = \lambda^2 + \mu^2$, we obtain that the eigenvalues $\sigma_{n,m} = \lambda_n^2 + \mu_m^2$ correspond to eigenfunctions

$$\nu_{nm}(x, y) = A_{nm}(\beta_1\lambda_n\cos\lambda_n x - \alpha_1\sin\lambda_n x)(\theta_1\mu_m\cos\mu_m y - \gamma_1\sin\mu_m y), \tag{14}$$

where $A_{nm} = A_n \cdot B_m$ is a constant. We choose $A_{nm} = A_n \cdot B_m$ so that the norm of the function ν_{nm} with a weight of unit was equal to one, that is, we orthonormalize the functions ν_{nm}

$$\int_0^b\int_0^c \nu_{nm}^2\,dxdy = A_{nm}^2\int_0^b(\beta_1\lambda_n\cos\lambda_n x - \alpha_1\sin\lambda_n x)^2\,dx\cdot\int_0^c(\theta_1\mu_m\cos\mu_m y - \gamma_1\sin\mu_m y)\,dy = 1,$$

$$A_{nm} = \frac{1}{\sqrt{\int_0^b(\beta_1\lambda_n\cos\lambda_n x - \alpha_1\sin\lambda_n x)^2\,dx\cdot\int_0^c(\theta_1\mu_m\cos\mu_m y - \gamma_1\sin\mu_m y)\,dy}}. \tag{15}$$

Calculation of the coefficients A_{nm} in the general case, by the formula (15), is laborious and inexpedient. It is much more convenient and more rational to calculate the coefficients A_{nm} in each case of the boundary conditions of the spectral problem, than to use the cumbersome and hard-to-remember formula obtained in calculations of the integrals in (15).

Remark 2. We investigate the spectral problem (12) by introducing the notation $\nu = p^2$. Under different boundary conditions, from (12) one can obtain nine spectral problems. We carry out the calculations for the case of boundary conditions of the third kind. Obviously, the remaining particular cases of problem (12) are investigated in a similar way with more elementary calculations. The general solution of the equation in (12) is $Z(z) = A\cos pz + +B\sin pz$. Its substitution into boundary conditions of the third kind of the problem (12) gives us a system of equations

$$\begin{cases} B = \dfrac{h}{p}A, \\ A(-p\sin pl + h\cos pl + \dfrac{h^2}{p}\sin pl) = 0. \end{cases}$$

To obtain non-trivial solutions, from condition $A \neq 0$ we receive that p_k are the roots of the equation

$$\operatorname{ctg} pl = \frac{1}{2}\left(\frac{p}{h} - \frac{h}{p}\right), \tag{16}$$

and the eigenfunctions of the problem are the following functions

$$Z_k(z) = \widetilde{A_k}\Big(\cos p_k z + \frac{h}{p_k}\sin p_k z\Big) = A_k(p_k\cos p_k z + h\sin p_k z). \tag{17}$$

We normalize the functions (17), taking into account the relation (16) in the calculations,

$$\int_0^l Z_k^2(z)\,dz = \frac{A_k^2}{2}\int_0^l [p_k^2(1+\cos 2p_k z) + 2p_k h\sin 2p_k z + h^2(1-\cos 2p_k z)]\,dz =$$

$$= \frac{A_k^2}{2}\Big[p_k^2\Big(l + \frac{1}{2p_k}\sin 2p_k l\Big) + h(1-\cos 2p_k l) + h^2\Big(l - \frac{1}{2p_k}\sin 2p_k l\Big)\Big] =$$

$$= \frac{A_k^2}{2}\Big[p_k^2\Big(l + \frac{1}{2p_k}\cdot\frac{4p_k h(p_k^2-h^2)}{(p_k^2+h^2)^2}\Big) + h\Big(1 - \frac{(p_k^2-h^2)^2 - 4p_k^2h^2}{(p_k^2+h^2)^2}\Big) +$$

$$+ h^2\Big(l - \frac{1}{2p_k}\cdot\frac{4p_k h(p_k^2-h^2)}{(p_k^2-h^2)^2}\Big)\Big] =$$

$$= \frac{A_k^2}{2(p_k^2+h^2)^2}\Big[p_k^2 l(p_k^2+h^2)^2 + 2p_k^2h(p_k^2-h^2) + h(p_k^2+h^2)^2 - h(p_k^2-h^2)^2 +$$

$$+ 4p_k^2h^3 + h^2 l(p_k^2+h^2)^2 - 2h^3(p_k^2-h^2)\Big] =$$

$$= \frac{A_k^2}{2(p_k^2+h^2)^2}\Big[(p_k^2+h^2)^2(l\cdot(p_k^2+h^2)+h) + h(2p_k^4 - 2p_k^2h^2 - p_k^4 + 2p_k^2h^2 -$$

$$- h^4 + 4p_k^2h^2 - 2h^2p_k^2 + 2h^4)\Big] =$$

$$= \frac{A_k^2}{2(p_k^2+h^2)^2}\Big[(p_k^2+h^2)^2\Big(\Big[l\cdot(p_k^2+h^2)+h\Big] + h(p_k^4 + 2p_k^2h^2 + h^4)\Big)\Big] =$$

$$= \frac{A_k^2}{2(p_k^2+h^2)^2}(p_k^2+h^2)^2\Big[l\cdot((p_k^2+h^2)+2h)\Big] = A_k^2\cdot\frac{l(p_k^2+h^2)+2h}{2} = 1.$$

As a result, we obtain

$$A_k = \sqrt{\frac{2}{(p_k^2+h^2)+2h}},\quad Z_k(z) = A_k(p_k\cos p_k z + h\sin p_k z),\quad k = 1, 2, ...$$

We return to the original problem (1) - (5). We have from (14)

$$\nu_{nm}(x,y) = A_{nm}(\beta_1\lambda_n \cos\lambda_n x - \alpha_1 \sin\lambda_n x)(\theta_1\mu_m \cos\mu_m y - \gamma_1 \sin\mu_m y),$$

where the coefficients A_{nm} are calculated as in remark 2 in each particular case of boundary conditions.

Let us find the general solution of equation (7) for $\sigma_{mn} = \lambda_n^2 + \mu_m^2$

$$T_{nm}(t) = C_{nm}\cos a\sqrt{\sigma_{nm}}t + D_{nm}\sin a\sqrt{\sigma_{nm}}t,$$

where C_{nm}, D_{nm} are arbitrary constants. Returning to the original problem (1) - (5), we obtain that the particular solutions according to (6) will have the form

$$u_{nm}(x,y,t) = \nu_{nm}(x,y)\cdot T_{nm}(t) = \nu_{nm}(x,y)(C_{nm}\cos a\sqrt{\sigma_{nm}}t + D_{nm}\sin a\sqrt{\sigma_{nm}}t).$$

By the principle of superposition, the general solution of equation (1) with the boundary conditions (4), (5) has the form

$$u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(C_{nm}\cos a\sqrt{\sigma_{nm}}t + D_{nm}\sin a\sqrt{\sigma_{nm}}t)\cdot\nu_{nm}(x,y). \tag{18}$$

Using the initial conditions (2), (3), relation (18) and the property of orthonormality of functions ν_{nm}, we find the values of the constants C_{nm} and D_{nm}

$$u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\nu_{nm}(x,y) = \varphi(x,y), \;\; C_{nm} = \int_0^b\int_0^c \varphi(x,y)\nu_{nm}(x,y)\,dxdy.$$

$$u_t(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}D_{nm}a\sqrt{\sigma_{nm}}\nu_{nm}(x,y) = \psi(x,y), \;\; D_{nm} = \frac{1}{a\sqrt{\sigma_{nm}}}\int_0^b\int_0^c \psi(x,y)\nu_{nm}(x,y)\,dxdy.$$

Hence, we obtain the solution of problem (1) - (5) in the analytical form

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(C_{nm}\cos a\sqrt{\sigma_{nm}}t + D_{nm}\sin a\sqrt{\sigma_{nm}}t)\cdot\nu_{nm}(x,y),$$

where

$$\nu_{nm}(x,y) = A_{nm}(\beta_1\lambda_n \cos\lambda_n x - \alpha_1 \sin\lambda_n x)(\theta_1\mu_m \cos\mu_m y - \gamma_1 \sin\mu_m y),$$

$\sigma_{mn} = \lambda_n^2 + \mu_m^2$; $\lambda_1, ..., \lambda_n, ...$ are the roots of the equation $\operatorname{tg}\lambda b = \frac{(\alpha_2\beta_1 - \alpha_1\beta_2)\lambda}{\alpha_1\alpha_2 + \beta_1\beta_2\lambda^2}$; $\mu_1, ..., \mu_m, ...$ are the roots of the equation $\operatorname{tg}\mu c = \frac{(\gamma_2\theta_1 - \gamma_1\theta_2)\mu}{\gamma_1\gamma_2 + \theta_1\theta_2\mu^2}$,

$$A_{nm} = \frac{1}{\sqrt{\int_0^b(\beta_1\lambda_n\cos\lambda_n x - \alpha_1\sin\lambda_n x)^2\,dx\cdot\int_0^c(\theta_1\mu_m\cos\mu_m y - \gamma_1\sin\mu_m y)\,dy}},$$

$$C_{nm} = \int_0^b\int_0^c \varphi(x,y)\nu_{nm}(x,y)\,dxdy, \;\; D_{nm} = \frac{1}{a\sqrt{\sigma_{nm}}}\int_0^b\int_0^c \psi(x,y)\nu_{nm}(x,y)\,dxdy.$$

Thus, function $u(x,y,t)$ is found in the general case of boundary conditions. This functiondescribes the deviation of the membrane from the equilibrium position.

Statement and analytical methods of solving the problem on plate vibrations

The Germain-Lagrange equation describing small transverse vibrations of an elastic isotropic plate $|x| < a$, $|y| < b$ of constant thickness h, has the form

$$D\triangle\triangle w + \rho\frac{\partial^2\omega}{\partial t^2} = 0, \quad \triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Here $w(x,y,t)$ is the transverse bending of the middle plane of the plate; $\triangle$ is the two-dimensional Laplace operator; $D = Eh^3/(12(1-\nu^2))$ is bending stiffness of the plate; ν is Poisson's coefficient; E is Young's modulus; ρ is the specific density per unit area ofthe plate; t is time.

The problem of determining the natural frequencies and vibrations types of a plate with free edges is reduced to determining the deflection $W(x, y)$ (here and below the harmonic factor $e^{-i\omega t}$ is omitted) and the frequencies $k^4 = \rho h\omega^2/D$ from the homogeneous boundary value problem

$$\triangle\triangle W - k^4 W = 0 \tag{19}$$

with boundary conditions for $x = \pm a$:

$$\frac{\partial^2 W}{\partial x^2} + \nu\frac{\partial^2 W}{\partial y^2} = 0,$$

$$\frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial y^2 \partial x} = 0, \tag{20}$$

and for $y = \pm b$:

$$\frac{\partial^2 W}{\partial y^2} + \nu\frac{\partial^2 W}{\partial x^2} = 0,$$

$$\frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial y^2 \partial x} = 0. \tag{21}$$

At present, two analytical approaches to the solution of the boundary value problem (19)–(21) have become most widely used. These approaches are the Ritz method and the superposition method. In his classic paper, Ritz pointed out that this boundary value problem is equivalent to finding the minimum value of the integral

$$J = \int_{-a}^{a}\int_{-b}^{b}\Big[\Big(\frac{\partial^2 W}{\partial x^2}\Big)^2 + \Big(\frac{\partial^2 W}{\partial y^2}\Big)^2 + 2\nu\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + 2(1-\nu)\Big(\frac{\partial^2 W}{\partial x\partial y}\Big)^2\Big]\,dxdy \tag{22}$$

provided that

$$\int_{-a}^{a}\int_{-b}^{b} W^2\,dxdy = A = const. \tag{23}$$

For the particular case of a square plate $a = 1$, $b = 1$, Ritz chose the representation

$$W_S = \sum_{m=0}^{S}\sum_{n=0}^{S} A_{nm} u_m(x)\nu_n(y), \tag{24}$$

in which $u_m(x)$ and $\nu_n(y)$ are eigenfunctions of bending vibrations of an elastic rod $\xi < 1$ with both free edges. In other words, these functions are solutions of the homogeneous equation

$$\frac{d^4 u}{d\xi^4} = k^4 u \tag{25}$$

with homogeneous boundary conditions

$$\frac{d^2 u}{d\xi^2} = 0, \qquad \frac{d^3 u}{d\xi^3} = 0, \qquad \xi = \pm 1. \tag{26}$$

The normalization is chosen so that for the solution number m the following ratio holds

$$\int_{-1}^{1} u_m^2\,d\xi = 1.$$

Rayleigh studied the solution of the homogeneous boundary value problem (25), (26) in detail and showed that the required functions should be chosen as follows

• for even m

$$u_m(\xi) = \frac{chk_m \cos k_m\xi + \cos k_m chk_m\xi}{\sqrt{ch^2 k_m + \cos^2 k_m}},$$

where k_m is a root of the equation $\operatorname{tg} k_m + thk_m = 0$,

- for odd m

$$u_m(\xi) = \frac{shk_m \sin k_m\xi + \sin k_m shk_m\xi}{\sqrt{sh^2 k_m - \sin^2 k_m}},$$

where k_m is a root of the equation $tgk_m - thk_m = 0$.

It was assumed that

$$u_0(\xi) = \frac{1}{\sqrt{2}}, \qquad k_0 = 0,$$

$$u_1(\xi) = \sqrt{\frac{3}{2}}\xi, \qquad k_1 = 0.$$

When an indefinite Lagrange multiplier $\lambda = k^4$ is used, the integral J is minimized and the condition (23) holds, the substitution of the expansion (24) into an quadratic with respect A_{mn} expression (22) leads to a homogeneous system of linear algebraic equations with respect to A_{mn}. Hence λ is defined in the standard way as a value that reduces to zero the determinant of this linear system. Moreover, all the boundary conditions (20) and (21) are satisfied identically.

For the Poisson's coefficient $\nu = 0.225$ (glass, as in the experiments of Khladny) and for the case of antisymmetric vibrations with respect to the diagonals of the square $y = \pm x$ (in this case $A_{mn} = A_{mn}$) Ritz took $s = 5$ in the representation (24), manually calculated all the necessary integrals and obtained a homogeneous system of linear algebraic equations of the sixth order. He also managed to find the first two roots of the determinant of this system.Further, a bold assumption was made that the vibrationstypes of a plate are determined only by the main summand $u_m(x)\nu_n(y) \pm u_n(x)\nu_m(y)$. In articles Hladni the table is given for calculating the first 35 eigenfrequencies and their comparison with the experimental data of Khladni. Ritz also cited figures of nodal lines for vibrations types at natural frequencies corresponding to all four types of symmetry with respect to the x and y axes. He took $s = 5$ everywhere, represented the complete expression (24) with numerical coefficients, and emphasized the defining contribution of the principal terms. It was truly a titanic work, given the lack of a computer. A different approach, which is traditionally called the superposition method,represents a general solution of the differential equation (19) as the sum of solutions for bands $|x| \leqslant a$, $|y| \leqslant b$ in the form of trigonometric series. The solution is chosen in such a way as to satisfy the second boundary conditions in (20), (21) identically and to have enough arbitrariness to meet the remaining two conditions.

There are four types of symmetry of the plate deflection: the function $W_S(x, y)$ is even relative to x and y; the function $W_{SA}(x, y)$ is even relative to x and odd relative to y; the function $W_{AS}(x, y)$ is odd relative to x and even relative to y; the function $W_A(x, y)$ is odd relative to x and y. Using the standard method of variables separation, the solutions of equation (19) can be written in the form

$$W_S = \frac{bx_0^S}{k}\Big(\frac{\cos ky}{\sin kb} - \frac{chky}{shkb}\Big) + \frac{ay_0^S}{k}\Big(\frac{\cos kx}{\sin ka} - \frac{chkx}{shka}\Big) + b\sum_{n=1}^{\infty}(-1)^{n+1}x_n^S A(y, b, \alpha_n)\cos\alpha_n x +$$
$$+\alpha\sum_{n=1}^{\infty}(-1)^{n+1}y_n^S A(x, b, \beta_n)\cos\beta_n y, \tag{27}$$

$$W_{SA} = \frac{bx_0^{Sa}}{k^2}\Big(\frac{\sin ky}{\cos kb} + \frac{shky}{chkb}\Big) + b\sum_{n=1}^{\infty}(-1)^n x_n^{Sa} B(y, b, \alpha_n)\cos\alpha_n x - \alpha\sum_{n=1}^{\infty}(-1)^n y_n^{Sa} A(x, a, \delta_n)\sin\delta_n y; \tag{28}$$

$$W_A = b\sum_{n=1}^{\infty}(-1)^{n+1}x_n^a B(y, b, \gamma_n)\sin\gamma_n x + \sum_{n=1}^{\infty}(-1)^{n+1}y_n^a B(x, a, \delta_n)\sin\delta_n y, \tag{29}$$

where designations are thus introduced

$$\alpha_n = \frac{\pi n}{a}; \qquad \beta_n = \frac{\pi n}{b}; \qquad \gamma_n = \frac{\pi(2n-1)}{2a}; \qquad \delta_n = \frac{\pi(2n-1)}{2b};$$

$$A(z, h, \xi) = \frac{\xi^2 + k^2 - (2-\nu)\xi^2}{\sqrt{\xi^2 - k^2}}\frac{ch\sqrt{\xi^2 - k^2}z}{sh\sqrt{\xi^2 - k^2}h} - \frac{\xi^2 - k^2 - (2-\nu)\xi^2}{\sqrt{\xi^2 + k^2}}\frac{ch\sqrt{\xi^2 + k^2}z}{sh\sqrt{\xi^2 + k^2}h};$$

$$B(z, h, \xi) = \frac{\xi^2 + k^2 - (2-\nu)\xi^2}{\sqrt{\xi^2 - k^2}}\frac{sh\sqrt{\xi^2 - k^2}z}{ch\sqrt{\xi^2 - k^2}h} - \frac{\xi^2 - k^2 - (2-\nu)\xi^2}{\sqrt{\xi^2 + k^2}}\frac{sh\sqrt{\xi^2 + k^2}z}{ch\sqrt{\xi^2 + k^2}h}.$$

The expression for W_{AS} is not written out because, due to the symmetry of the problem all the corresponding eigenvalues and forms are constructed by the solution of SA with the substitution $x \leftrightarrow y$ and $a \leftrightarrow b$:

$$k_{SA}(a/b) = k_{AS}(b/a),$$
$$W_{SA}(x, y, a, b) = W_{AS}(y, x, b, a).$$

Note that for this reason, in the case of a square, the eigenvalues for the cases AS and SA coincide.

Substitution of solutions (27) - (29) into the first of the boundary conditions (20), (21) with the subsequent expansion of incoming functions into trigonometric series on the basis of formulas

$$\frac{chpz}{shph} = \frac{1}{ph} + \frac{2p}{h} \sum_{m=1}^{\infty} \frac{(-1)^m \cos \xi_m z}{\xi_m^2 + p^2}, \qquad \xi_m = \frac{m\pi}{h},$$

$$\frac{shpz}{chph} = \frac{2p}{h} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \sin \eta_m z}{\eta_m^2 + p^2}, \qquad \eta_m = \frac{(2m-1)\pi}{2h}$$

allows us to obtain homogeneous infinite systems of linear algebraic equations with respect to unknown coefficients from the equality under basic functions [6].

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Г.А. Есенбаева, Д.Н. Есбаева, Д. Бауыржанқызы, Д.А. Нұргали

Мембраналар мен пластиналардың тербелісі бойынша мәселелерді шешудің математикалық және аналитикалық әдістері

Пластиналар мен қабықшалардың табиғи тербелістерінің жиіліктері мен формулаларын анықтау мәселелері жартылай дифференциалдық теңдеулерді интегралдаудың қажеттілігіне әкелді. Ең жақсы зерттелген жағдай — бұл айнымалыларды бөліп алуға болатын жағдай. Олардың ішінде, атап айтқанда, қарсы жағында бекітілген тік бұрышты пластинаның тербелісі, дөңгелек осьсимметрлік плиталардағы қолшатыр және желдеткіш тербелістер, цилиндр қабықшаларының тербелісі, жабық немесе генераторларға бекітілген. Мақалада шекаралық жағдайлардың жалпы жағдайында жазық біркелкі мембрананың тербелісі зерттелді.

Кілт сөздер: шекаралық есептер, мембрана, пластина, тербеліс, спектрлік есеп, ортонормалды функциялар жүйесі, иілу функциясы.

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О математических и аналитических методах решения задач о колебании мембран и пластин

Задачи об определении частот и форм собственных колебаний пластин и оболочек приводят к необходимости интегрирования дифференциальных уравнений в частных производных. Наиболее хорошо изучены те случаи, когда оказывается возможным разделение переменных. К ним относятся, в частности, колебания прямоугольной пластины, шарнирно-опертой по противолежащим сторонам, зонтичные и веерные колебания круглых осесимметричных пластин, колебания цилиндрических оболочек, замкнутых или шарнирно-закрепленных вдоль образующих. В статье проведено исследование колебания плоской однородной мембраны для общего случая граничных условий.

Ключевые слова: краевая задача, мембрана, пластина, колебания, спектральная задача, ортонормированная система функций, функция прогиба.

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Research and development of deployable solar tracking system in SolidWorks for multiple tasks

Nowadays, we have a problem of uninterrupted automatic operation of unmanned ground vehicles (UGV). The appliance of alternative types of energy, in particular solar, as a system of main or additional power source in automatic mode, can be carried out in the body of a ground robot. Within the limited space in the trunk, it is necessary to place the solar panels and actuators densely as possible, and also effectively locate the plane of the unfolded panels in relation to the sunrays. For these reasons, a deployable solar tracking system was developed and analyzed. There were made calculations of the number of necessary solar energy, cells and panels for the effective charging of the UGV in automatic mode, construction of a 3D model of the device in SolidWorks, selection of the design of the deploying mechanism based on the motion analysis and simulation of the model, identification of location of the charging system within the existing UGV. The further stages of research are determined.

Keywords: solar, tracking, SolidWorks, deployable, automative system, ugv, portable, alternative energy, 3D modelling.

Continuous automatic functioning of the UGV can not be performed for a long time without timely charging of the batteries. One of the solutions to the problem is using of solar energy, which makes it possible to receive the necessary energy in an automatic mode, no tight binding to a certain place of charging, the amount of energy received is limited only by the capacity of the battery and the state of the environment. Among the imperfections, it can be noted an increase of the UGV's weight, the dependence on weather conditions and lower rate of charging of the storage batteries. Sufficient charging efficiency with the solar panels in a given limited space of the UGV design can be compensated by the introduction of deploying system. Therefore, the coverage area of the unfolded solar panels is increasing several times in comparison with the occupied area in the trunk.

UGV automatically executes deploying of the solar system at a definite threshold of battery charge, at sufficient solar illumination and in a safe state of the environment. After a full disclosure, the system begins tracking of the best position of the solar panels, in which the sunrays fall at right angles to the surface. Tracking panels in two axes will increase the efficiency of the charging by 81.68 percents [1]. In the case of battery is fully charged or unfavorable conditions are occurred, the system folds the panels into the UGV's body.

According to the analyzed characteristics of different types of mono and polycrystalline solar cells, one of the optimal solutions is to use Maxeon C60 with nominal parameters, shown in Table 1, and I-V curve presented in Figure 1.

Table 1

Electrical characteristics of Maxeon C60 solar cell at 25 ℃

Pinpp(Wp)	Eff.(%)	Vmpp(V)	Impp(A)	Voc(V)	Isc(A)
3.38	22.1	0.577	5.87	0.684	6.26

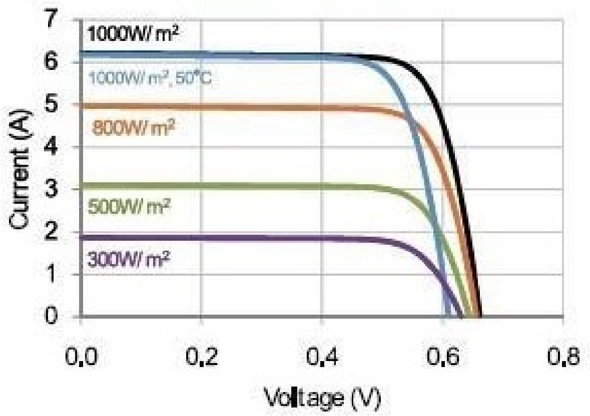

Figure 1. Maxeon C60 I-V curve

The future system must satisfy the following conditions:

– Dimensions compactness;

– Maximum simplicity of the design;

– Resistance to changes of weather conditions;

– The most effective placement of Maxeon C60 solar cells in a box measuring 420x250x800 mm.

The most suitable technology is a Smart Flower [2] for the first three criteria. To satisfy the fourth criteria, a slightly different design of the panels must be chosen. The most effective shape of the panels for the arrangement of Maxeon C60 solar cells is a square or rectangle, with the sides that are multiple to 125 x 125 mm cell dimensions. The average output voltage of one cell is 0.5V, in cloudy days 0.25V. For a stable power supplying of a 12V system, even in overcast weather, it will be required about 36-40 solar cells. The most optimal solution is the deploying mechanism [3] with four rectangular panels with dimensions of 260x640 mm, on which 10 solar cells can be arranged. With a relatively simple design and actuators, the coverage area of the solar panels increases four times. Of course, the origami method [4] can execute magnifications of more than 20 times, but to deploy such small area, it is necessary to use very thin panels that, even in conditions of slight windiness, have weak stability.

SolidWorks software has been used to create and calculate the motion of the deployable tracking solar system model (Fig. 2-5).

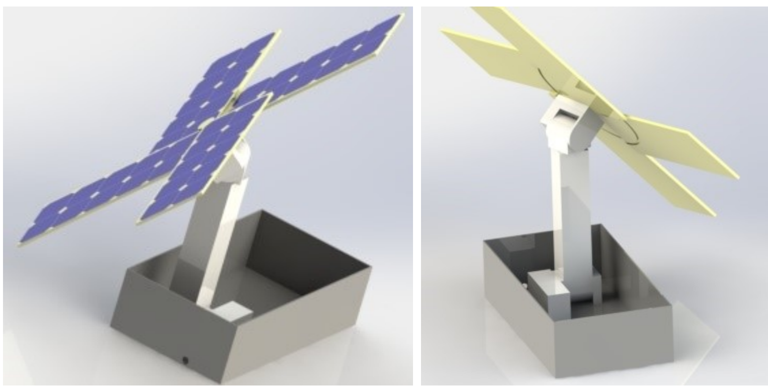

Figure 2. Fully deployed system

To store the system in the body of the UGV, there were made several structural changes, because the occupied area has the highest priority. Compactness in length can be obtained by reducing the support stand, but in this case we have to raise it in a pair with the movement of tracker axes. This compensation is necessary because of the collision when lifting only one support stand. Such change decreases the length of the box 2 times.

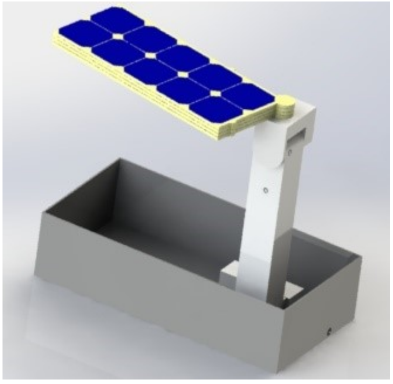

Figure 3. Support stand in unfolded position

The center of the clutch with the shaft located in the corner of the panel, so the folded panels should be placed at angle of 12.72 degrees to effectively use the space in the box (Fig.4).

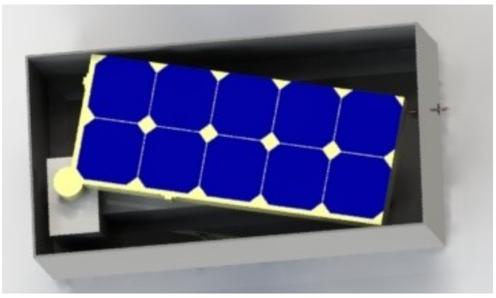

Figure 4. Solar system in folded position

As a result, we receive a box with dimensions of 410x220x750 mm, which is quite satisfactory for the dimensions of the available UGV's trunk.

For deploying system, it has been used compact DC motors with high RPM in pair with gearboxes based on cylindrical and bevel gears. The wiring of the motors passes through the centers of the shafts of each execution block. More detailed placement of motors, wiring and actuators is shown in Figure 5.

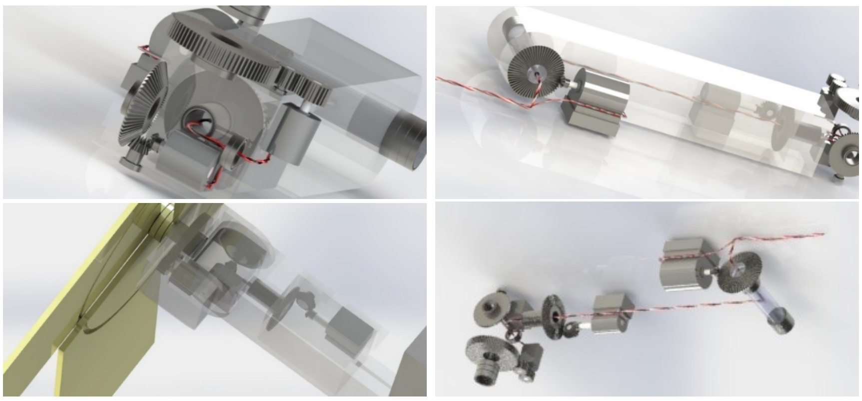

Figure 5. Location of motors, wiring and actuators

In result, developed solar system [5] has a number of advantages: portable, compact, energy efficient, ecological clean, autonomous. A system of this kind can easily be applied into various areas of obtaining an energy. In the future, it is planned to implement the system on an existing UGV.

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Н.А. Есмағамбет, А.С. Омаров

Әртүрлі тапсырмаларды орындау үшін SolidWorks-те күн шуақтарымен түсіндірілетін бақылау жүйесін зерттеу және дамыту

Қазіргі уақытта адамсыз жер үсті көлігінің (АЖҮК) үзіліссіз автоматты түрде жұмыс істеу мәселесі көтерілуде. Энергияның балама түрлерің пайдалануы, атап айтқанда, күн, негізгі немесе қосымша нәр беруші жүйелер ретінде автоматты түрде жердегі роботтың корпусында жүзеге асыруға болады. Шектеулі кеңістіктен корпуста барынша ықшамды күн тақтасын және атқарушы механизмдерді орналастыру қажет, тағы да күн сәулелеріне қатысты ашық панельдердің жазықтығын тиімді орналастыру керек. Осы мақсатта күн шуақтарымен түсіндірілетін бақылау жүйесі талданды және әзірленді. АЖҮК автоматты түрде тиімді зарядтау үшін күн энергиясын, жасушалар, тақталардың қажетті саны есептелген. Түсіндірілетін механизмдердің құрылымдарына таңдау жасалды. Аппараттың 3Д-моделі құрылды. Қажетті конструкциялардың қозғалысты талдау негізі және модельдің симуляциясына есептеу жүргізілді. Шектеулі кеңістіктең қазіргі бар АЖҮК зарядтау жүйесін орналасқан жерін білдіреді. Бұдан әрі зерттеудің кезеңдері анықталды.

Кілт сөздер: күндік, бақылау, SolidWorks, автоматты жүйесі, АЖҮК, портативті, баламалы энергия, 3Д-модельдеу.

Н.А. Есмагамбет, А.С. Омаров

Исследование и разработка развертываемой солнечной следящей системы в SolidWorks для выполнения различных задач

В настоящее время проблема бесперебойной автоматической работы беспилотных наземных аппаратов (БПНА) особенно остра. Использование альтернативных видов энергии, в частности солнечной, как системы основного или дополнительного источника питания в автоматическом режиме можно осуществить в корпусе наземного робота. В рамках ограниченного пространства в корпусе необходимо максимально компактно разместить солнечные панели и исполнительные механизмы, а также плоскость раскрытых панелей относительно солнечных лучей. Для этого разработана и проанализирована развертываемая солнечная следящая система. Рассчитано количество необходимой солнечной энергии, ячеек, панелей для эффективного заряда БПНА в автоматическом режиме. Осуществлен выбор конструкции развертываемого механизма. Построена 3Д-модель аппарата. На основе анализа движения и симуляции модели произведен расчет необходимой конструкции. Обозначено расположение системы дозарядки в рамках существующего БПНА. Определены дальнейшие этапы исследования.

Ключевые слова: солнечная, следящая, SolidWorks, развертываемая, автоматическая система, БПНА, портативная, альтернативная энергия, 3Д-моделирование.

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Calculation and modeling tracked mobile robot

Introduction and consideration of basic device of an unmanned ground vehicle (UGV). Determination of the most significant characteristics for exploitation of UGV allows us to formulate the main task: this is the most effective overall design. Analysis of design in the SolidWorks software package. This work in its structure relates to mechanical problems associated with the task of mechatronics. This area of research in mechatronics today is an actual and rapidly developing discipline. More precisely we can attribute the tasks discussed in this article to the description of modular unmanned ground robot with manual control.

Keywords: ground vehicle, caterpillar platform, calculation of electric motor, SolidWorks, hull, layout.9.

Introduction

Mechanisms and machines are an inseparable part of human life. The development of this field has led to the creation of a complex and organized system of mechanisms interaction between themselves and environment. For example, mechatronics, that was established in the 30-ies of twentieth century. Mechatronics combines mechanics, electronics, designing (the main tool is the CAD system (computer-aided design)), programming. Mechatronics allows us to answer the question: which construct of ground design is the most optimal and effective for exploitation under different environmental conditions. The main research tool is the CAD software SolidWorks. The SolidWorks software package opens up wide opportunities for research. SolidWorks allows us to design three-dimensional objects, collecting them in an assembly of a certain construction. By means of extra addittions such as SolidWorks Motion or SolidWorks Simulation, we can analyse design in a software-recreated reality that is as close as possible to the required conditions.

Layout

The layout of the UGV is a functionally conditioned location of motor-transmission installation, power supply, chassis, UGV systems and auxiliary equipment [1]. The layout can be general and private. General layout determines relative positioning of robot's elements, i.e. outlines how device will look. Private layout defines the device and appearance of individual robot's elements. The main task of the layout is to obtain the most efficient design for given conditions. The best scheme of general layout of the ground vehicle has several features:

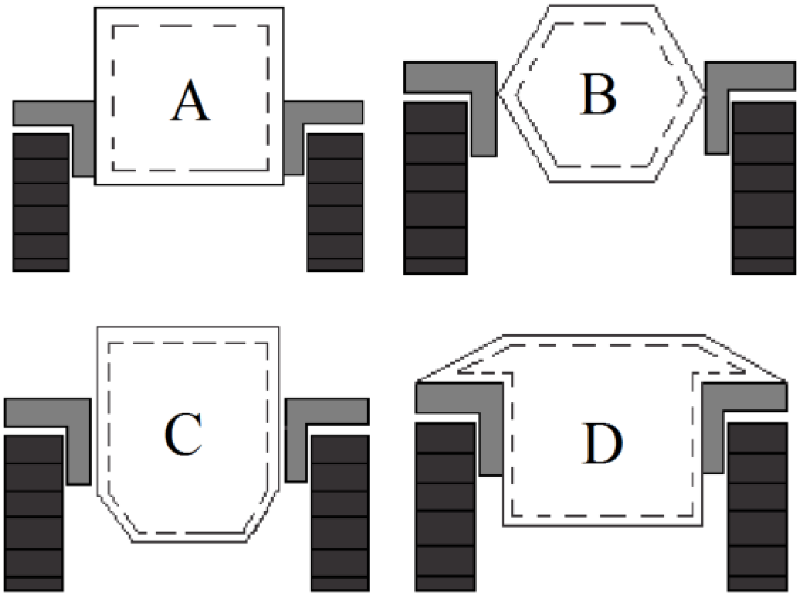
– minimum unused internal volumes;

– the shortest way of transferring energy from engine to moving parts of machine.

Generally, ground vehicle on a caterpillar platform basically have same design, i.e. body, caterpillar prop, which consists of a driving wheel, track rollers, sloth, balancers and, optionally, supporting rollers, suspension, transmission and complementary modules, motors and mechanical connections between the elements.

Hull

The hull [2] is the basic component of caterpillar platform. Optimally selected body provides the most favourable location of propulsion unit, internal components and additional modules, as well as convenient access to the components. There are various forms of device hull, each of which has its own characteristics (Picture 1).



Picture 1. Forms of the cross sections of hulls

The selection of material usually comes from relations price-quality (strength) -mass and from purposes of appointment. Used materials can be divided into groups: metallic, non-metallic and combined. From metal materials mainly used steel, aluminium. And from non-metallic - wood, fiberglass, plastic, HPL - panel, etc.

In the existing design were used HPL panels that were made of wood fibers that were pressed at high temperature and pressure. Choice of this material was the most optimal by criteria of price and weight (see Chart).

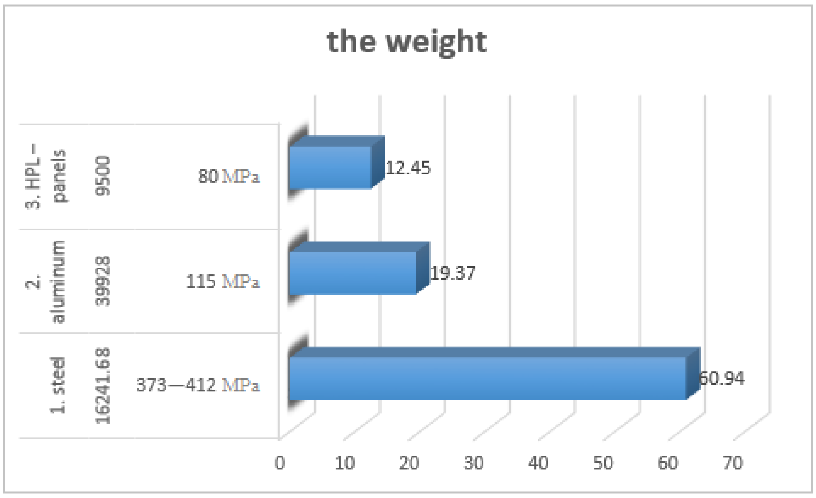


Chart. Comparative chart by price criteria - quality (strength) - weight

Transmission and engine installation

Transmission (powertrain) - a set of mechanisms that match engine with its propulsion, as well as systems that ensure work of mechanisms. Transmission is responsible for changing traction effort, speed and direction of motion.

The main requirements for transmission:

– Providing high traction and speed;

– Simplicity and easeness of management;

– High reliability of work over a long period of exploitation;
– Small mass and overall dimensions of mechanisms;
– Simplicity of production;
– Convenience in maintenance during exploitation and repair.

In ground mobile platforms mechanical transmissions are predominantly used. On such mobile bases, the transmission elements are: a reducer, a chain connecting motor to shaft on which driving wheel is located, a system of asterisks connecting chain to shaft.

The main component leading robot into motion is the drive. It includes an engine and devices that regulate engine. Drives must provide a constant engine speed with variable load, a constant torque on motor shaft, high precision of the components system included in the engine and much more. The drive significantly determines structure, parameters and technological parameters of the robot. Main characteristics of the drive are: power, velocity and performance of system, sensitivity to the control signal.

Drives are systematized according to type of engine, their number, availability and type of transfer devices. At present time are used electric, hydraulic, pneumatic and other drives.

The main characteristics of electric motor are:

Electrical:

– Voltage Range (Recommended) is permissible range of supply voltages. The higher supply voltage, the greater motor power and velocity of rotation. But if a certain voltage value is exceeded, motor will fail.

– Voltage (Nominal) is optimum voltage at which motor is able to rotate quickly and not overheat while doing that.

– No-load current is current that motor consumes at idle.

– Stall current is current that motor consumes when shaft is blocked.

Mechanical:

– Shaft diameter is diameter of moving axis of the engine making movement.

– Velocity without load is velocity of rotation of motor shaft (number of revolutions per minute) at idle.

– Moment of force (torque) is a vector physical quantity that characterizes rotational action of force on a solid body.

Motor has one of the significant characteristics, by quantity of which we can determine whether engine can pull the robot, this is the moment of force. The engines for terrestrial vehicle were chosen according to the power and thrust criteria. External DC collector motor, with integrated reducer, on permanent magnets, is designed for connection any single-speed bicycles to the rear wheel, as well as for conversion to electric traction of other vehicles: scooters, skateboards, wheelchairs, etc.

Specifications:

– Voltage (nominal) 24 volts.
– Power (nominal): 250 watts.
– Maximum power: 270 watts.
– Maximum efficiency of the wheel motor: 82.5 %.
– Maximum rpm: 360.
– The maximum velocity of an electric bike with an installed external DC collector motor depends on the diameter of rim and is usually 18-25 km/h.
– Gear ratios: 9:62.
– 4 brushes 5.5x6 mm.
– Sprocket on 12-tooth.
– Hull: aluminium alloy.
– Overall dimensions: 130 x 102 x 134 mm.
– Weight: 2.2 kg.
– Length of power wires: 0.5 m.
– operational temperature: -25 C^0/ + 45 C^0.

So, we calculate power and moment of force of robot motor with a weight of 70 kg, maximum speed of which is 2 m/s and the radius of driving wheels is 0,075 m.

First we need to calculate the acceleration:

$$v^2 = v_0^2 + 2 \cdot a \cdot s \Rightarrow a = \frac{v^2 - v_0^2}{2s}, \quad (1)$$

where s is distance overcome by robot, a is acceleration, v_0 is initial velocity:

$$a = \frac{2-0}{2 \cdot 5} = \frac{2}{10} = 0.2 m/s^2. \tag{2}$$

Using the basic equation of rotational motion's dynamics, we find the moment of force:

$$M = I \cdot \epsilon, I = \frac{m \cdot g \cdot r^2}{2} \Rightarrow M = \frac{m \cdot g \cdot r \cdot a}{2}, \epsilon = \frac{a}{r}. \tag{3}$$

I is moment of inertia, ϵ is angular acceleration, g is acceleration of gravity (10 m/s), r is wheel radius, m is the mass of entire robot:

$$M = \frac{70 \cdot 10 \cdot 0.075 \cdot 0.2}{2} = 5.25 N/m. \tag{4}$$

The resulting torque is distributed between robot's motors, and it must be divided into the gear ratio of transmission used.

Power required by engine is determined by formula:

$$P = M \cdot \omega, \tag{5}$$

ω is angular velocity.

$$\omega = \frac{v}{r} \Rightarrow P = M \cdot \frac{v}{r} \Rightarrow P = 5.25 \cdot \frac{2}{0.075} = 140\, W. \tag{6}$$

Mechanical connections and parts of the device

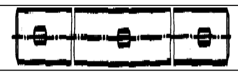
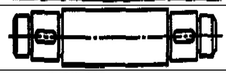
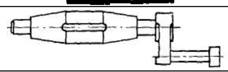
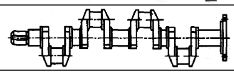
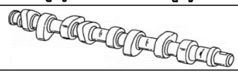
Mechanical connections and parts include: split and integral connections, shafts, various fastenings and supports, bearings and etc [3].

Connections are process of assembling a product from parts, assemblies, aggregates into a unit, holistic mechanism. Connections can be separable, allowing disassembly of elements with subsequent restoration and all-in-one, allowing analysis only under full or partial destruction.

The shaft Table is a part of mechanism that supports rotating components of mechanism and transmitting torque. The axis is a detail of mechanism that does not directly participate in transmission of motion.

Table

Common types of shafts

Sign of shaft	Scheme
Smooth	
Stepped	
Crank	
Kneed	
Cam	

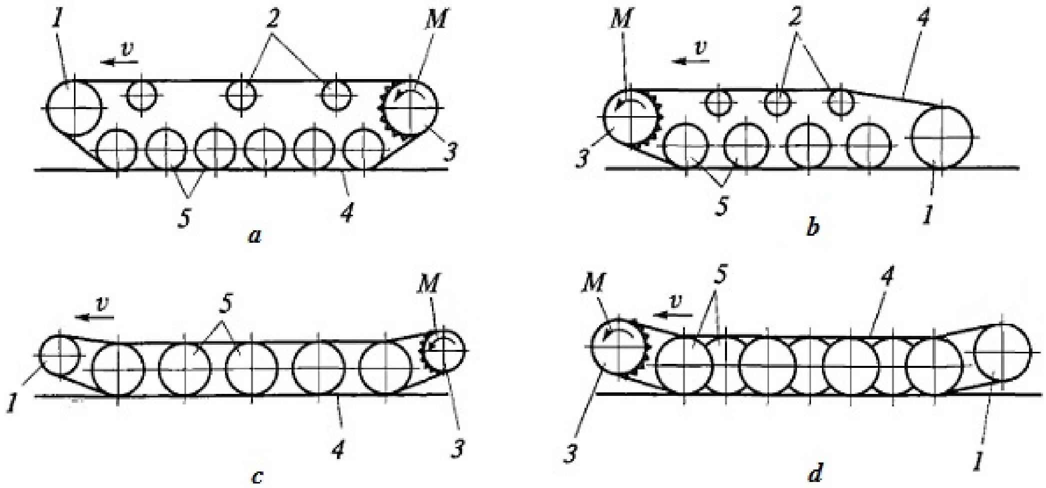
Chassis

Chassis of platform is divided into a caterpillar prop and a suspension. The term «undercarriage» can also be used.

Movement of the tank realises by means of a caterpillar mover consisting of caterpillars and associated devices: footing and supporting rollers, driving and guiding (idler) wheel, a device for pulling caterpillars. The suspension is composed of components that connect platform hull to axes of its support rollers. Suspension includes springs, balancers and shock absorbers.

The caterpillar mover is a propulsion device, in which a pulling force is created by rewinding the belt. Mover is a set of devices interacting with the external environment and creating an external traction force. In general case, caterpillar mover consists of a driving wheel, footing rollers, supporting wheels, guiding wheels (idler) and a caterpillar belt.

Under the action of torque M, the driving wheels begin to rewind caterpillar belt, which create rail tracks for machine and with the help of footing rollers the carrier system of machine moves along them (Picture 2). The next component of chassis is the suspension.



1 — is idler; *2* — are supporting wheels (rollers); *3* — is the driving wheel;
4 — is caterpillar tape; *5* — are footing rollers; *v* — is velocity; *M* — is the torque

Picture 2. Scheme of caterpillar propulsors with rear (a, b) and front (c, d) arrangement of the driving wheel

Suspension is a collection of units, parts and mechanisms that connect robot body on a caterpillar platform by means of axis of footing rollers. In general case each suspension's assembly includes an elastic element (spring), a shock absorber (damper), and a balancer.

Static stroke of roller is movement of footing roller vertically from position of fully unloaded elastic element to the position of its loading under weight of the robot.

Dynamic stroke of roller is movement of footing roller vertically from static position to the stop in the limiter of roller.

Full stroke of roller is movement of footing roller vertically from position of fully unloaded footing element to the stop in the limiter of roller, is defined as the sum of static and dynamic strokes of roller.

A number of the following requirements are put forward to the suspension:

– ensuring maximum smoothness of the stroke under various road and ground conditions;

– have high reliability at various loads;

– the most convenient maintenance and repair, easy to install and dismantle.

In addition to mechanical part of the robot there is electronic one, which includes power and control. Thanks to battery, the robot becomes «alive». And can perform its functions for some time.

So, what is a battery? The battery (Lat. Collector) is a device that stores energy for its further use. It is a device inside of which occurs a chemical reaction, during which an electric charge is produced.

Let's highlight the main characteristics of batteries:

– Construction (hull);

– electrochemical system;

– voltage;

– capacity;

– internal resistance;

– self-discharge current;

– life time.

Next significant part of the mobile robot is information management system (IMS) [4]. The IMS pursues a number of aims, such that:

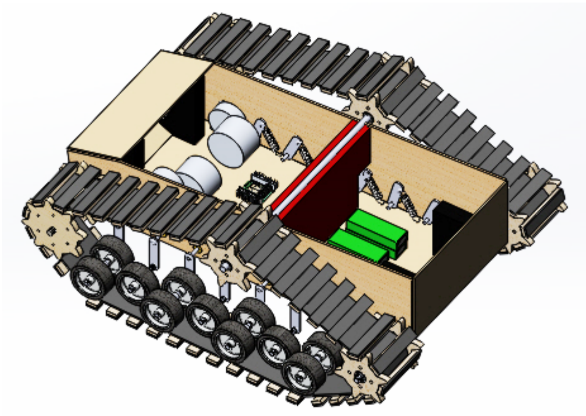
a) Transformation and perception of information about environment, as well as about the robot itself;

b) Based on the control program, command signals from control panel and information about state of the robot and environment, development of the laws of executive device's motion;

c) Transmission of signals to actuators and mechanisms of the executive system for purpose of robot's interaction with environment. In the base of the information management system is the management system (MS), information and measurement system (IMS) and communication system (CS). Determination of robot functions' number mostly depends from its information management system, such as flexibility, positioning accuracy, speed.

Modeling mobile robot in CAD SolidWorks

SolidWorks is said to be software complex CAD for works automation of an industrial enterprise at stages of design and technological preparation of production. It provides products development of any complexity and purpose. It works in a Microsoft Windows environment (Picture 3).



Picture 3. Building a ground robot model

The very first step before creating a model is choosing the plane: right, front or top. Then, we can make a sketch of the future 3D model, which is stretching out into solid body by «Piglet» tool. Sketch consists of tools such as - circle, segments, splines, arrays, rectangle, line and much more.

In order to build a model of the hull, we must recreate model of each its parts, which in turn combined into assemblies. And assemblies develop by combining into even greater, a complex assembly.

We created the hull model in several stages. First, we constructed separate components, such as armor of the hull, consisting of a front top plate, a front middle and lower armored plates, a bottom, a rear armor plate and side armor plates, driving wheels, shafts, sloths, rockers, support wheels. Which, subsequently, we combined into one complex assembly

We interconnected the armored plates when using tools such as «Object Conversion», «Object Shift», «Move Objects».

Hull's omplex assembly requires careful work, powerful support equipment, as well as software that is able to provide stable operation of the program SolidWorks.

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Н.А. Есмағамбет, А.Н. Нурпеисова

Шынжыр табан платформасындағы ұтқыр роботты есептеу және модельдеу

Мақалада пилотсыз жерүсті аппаратының (ПЖА) негізгі құрылғылары қарастырылған. Ең маңызды белгілері анықталып, ПЖА пайдалану үшін басты міндет қалыптастыруға мүмкіндік береді, яғни барынша тиімді жалпы құрылым табуға. Авторлармен конструкциялар SolidWorks бағдарламалық кешенінде талданды. Бұл жұмыс құрылымы бойынша мехатрониканың міндеттерімен байланысты механика мәселелеріне жатады. Берілген зерттеу сапасы бүгінгі таңдауы мехатроникадағы өзекті және жылдам дамылып келе жатқан пән болып есептеледі. Дәлірек айтқанда, осы мақалада қаралған мәселелерді қолмен басқарылатын модульді жерүсті роботтың сипаттамасына жатқызуға болады.

Кілт сөздер: жерүсті аппарат, шынжыр табанды алаңы, электрқозғалтқыш есептеу, SolidWorks, симуляция, құрастыру.

Н.А. Есмағамбет, А.Н. Нурпеисова

Расчёт и моделирование мобильного робота на гусеничной платформе

В статье рассмотрено основное устройство беспилотного наземного аппарата (БПНА). Определены наиболее значимые признаки для эксплуатации БПНА, позволяющие сформулировать главную задачу – максимально эффективную общую конструкцию. Авторами проанализирована конструкция в программном комплексе SolidWorks. Данная работа по своей структуре имеет отношение к проблемам механики, связанным с задачами мехатроники. Данная область исследований в мехатронике на сегодняшний день является актуальной и быстро развивающейся дисциплиной. Более точно задачи, рассмотренные в данной статье, можно отнести к описанию модульного беспилотного наземного робота с ручным управлением.

Ключевые слова: наземный аппарат, гусеничная платформа, расчёт электродвигателя, SolidWorks, симуляция, компоновка.

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Mathematical layout model of coupling with tangentially located ropes

The article deals with the results of mathematical modeling of layout coupling with tangentially located ropes. Arrangement considered limiting geometric nature that must be considered when designing joints. These conditions take into account the possibility of tightening fasteners, bushings opportunity neighborhood inner coupling halves, the possibility of relative rotation of the coupling halves, the lack of interference of the outer and inner sleeves of the coupling halves, the absence of interference of adjacent ropes and sleeves inner halfcoupling. As a result of research obtained mathematical expressions to be used in the calculation of engineering couplings checking the basic conditions of their geometric existence. These have been tested according to the design of the coupling, and the calculation results are checked by comparing them with the results of construction and found a match. The results can be used in the design of the coupling with the end of direct installation of ropes tangential location.

Keywords: mathematical model, layout, coupling, rope, load, gap, torque.

Introduction

Most of the parts and components of machines are complex structures, and their successful design is possible only after a thorough study of the processes occurring in these units during their operation, and the development of models for justifying their parameters based on new knowledge. At the same time, for many assembly units, rational design is the result of a series of calculations performed by the method of successive approximation, an example of which are the layout calculations e.g. when «implementing» gears in predetermined dimensions, etc. [1]. The elastic couplings [2, 3], in particular those with rope elements, for which the design and verification calculation methods have not been developed are not an exception, therefore further development of such calculation methods and construction of appropriate mathematical models is an important task.

Analysis of previous studies on this issue, highlighting the unsolved part of the problem

The authors of previous works [4, 5] proposed a new design of the coupling with an end-type installation of ropes of tangential location. The coupling (Fig. 1) consists of two half-couplings - the inner (1) and the outer one, which are connected by elastic elements (10), represented by ropes, each of being fixed by one end metricconverterProductID14 in (14) in the pin (8), installed in the outer half-coupling (11), and the other end metricconverterProductID15 in (15) in the pin (2) installed in the inner half coupling (1). The pins (8) and (2) are passed into the axial holes (7) of the sleeves (6) and the holes (9) of the flanges (5) of the half-couplings (11) and (1) and are tightened with the nuts (4) which are mounted on their threaded ends (3). The ropes (10) are passed into the transverse grooves (12) of the sleeves (6) and the grooves (13) of the pins (8) and (2). The pins (8) and (2) can be installed in their half-couplings (11) and (1) at different diameters which exclude interference of adjacent sleeves of the outer and inner half-couplings. This feature makes it possible for the coupling to perform safety functions, so that during overloading and pulling the ropes from the sleeves, the impact of the sleeves fixed on the half-couplings (11) and (1) is eliminated and the driving half-coupling can continue to rotate freely.

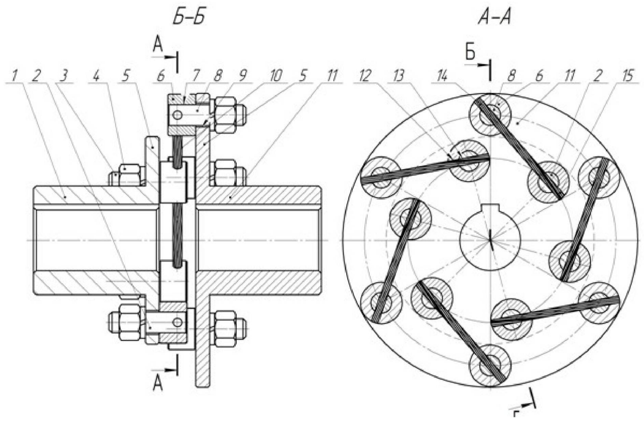

Figure 1. Scheme of the coupling with end-type installation of tangentially located straight ropes

As a result of theoretical studies, the basic geometric constraints are determined and five conditions for the geometric existence of couplings of the proposed design are formulated: the possibility of tightening fixture elements, the possibility of «proximity» of the sleeves of the inner half-coupling, the possibility of relative rotation of the half-couplings, the absence of interference between the sleeves of the outer and inner half-couplings, the absence of interference between the ropes and adjacent sleeves of the inner half-coupling. These geometric constraints also affect the loading of the coupling parts, but formulas for testing the last two conditions at the design stage and the mathematical model of the actual layout are not obtained, which complicates the process of designing couplings. Therefore, the purpose of this work is the theoretical justification of the features of geometric and power calculations, the design of couplings with end-type rope installation and obtaining a mathematical model that will simplify the work of the engineer during the layout of the couplings.

Statement of the main material

The design scheme of the coupling with end-type installation of tangentially located ropes is shown in Figure 2. In its half-couplings at different diameters, the outer D_1 and the inner D_2, sleeves (1) and (2) are fixed, in which the ropes (3) are fixed, due to their tension the rotation from the driving half-coupling to the driven one is transmitted. The main initial data for checking the specified conditions for the existence of the coupling, in addition to the diameters of the sleeves location, are the diameters of the sleeves d_1 and the ropes d_2 , as well as the angle of the installation displacement of the half-couplings ξ - the angle between the radii on which the sleeves of the outer and inner half-couplings that hold one rope are fixed. This angle can be adjusted when mounting the coupling to the required limits. The value of the angle ξ also affects the force parameters of the coupling, therefore, due to the rational choice of the angle ξ, it is necessary to ensure minimum loading of the coupling parts and its geometric existence within the limits of the indicated constraints.

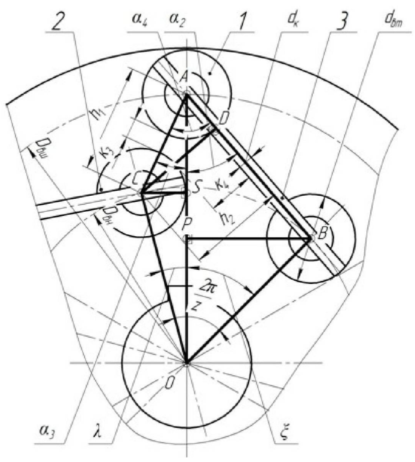

Figure 2. Design scheme of the coupling with end-type installation of tangentially located ropes

In order to obtain formulas that will test the two conditions mentioned above-the absence of interference between the sleeves of the outer and inner half-couplings, as well as the absence of interference between the ropes and adjacent sleeves of the inner half-coupling, it is necessary to consider a number of simple drawings shown in Figure 2. Thus, the first of above mentioned conditions is the condition that there is no interference between the sleeve of the outer half-coupling and the corresponding sleeve of the inner half-coupling, provided there is a gap k_3 between them. The condition is tested to ensure the shock-free operation of the coupling and is described by the expression (1). Thus, the provision of this condition actually reduces itself to the calculation of the distance h_1 between the axes A and C of the adjacent sleeves of the outer and inner half-couplings.

$$k_3 = h_1 - d_1, \quad k_3 \geq [\Delta_r], \tag{1}$$

where $[\Delta_r]$ is expected radial misalignment, at which the coupling will be operable.

The second of these conditions - the absence of interference between the ropes and adjacent sleeves of the inner half-coupling, is met when there is a gap k_4 between them. The condition is tested to ensure the shock-free operation of the coupling and is described by the expression (2). Thus, the verification of the provision of this condition reduces itself to determining the distance h_2 between the axis C of the sleeve of the inner half-coupling and the axis AB of the rope.

$$k_4 = h_2 - 0.5(d_1 + d_2), \quad k_4(2...4)mm. \tag{2}$$

To achieve this goal, let us first consider a rectangular triangle OPB for which we can write as follows:

$$PB = OB \times \sin\xi = \frac{D_2}{2}\sin\xi; \tag{3}$$

$$OP = OB \times \cos\xi = \frac{D_2}{2}\cos\xi. \tag{4}$$

It follows from the rectangular triangle APB:

$$tg\alpha_2 = \frac{PB}{AP} = \frac{PB}{OA - OP} = \frac{0,5D_2\sin\xi}{0,5D_1 - 0,5D_2\cos\xi} = \frac{D_1\sin\xi}{D_1 - D_2\cos\xi} = X; \tag{5}$$

$$\lambda = \frac{2\pi}{z} - \xi. \tag{6}$$

It follows from the rectangular triangle OSC:

$$CS = OC \times \sin\lambda = \frac{D_2}{2}\sin\lambda; \tag{7}$$

$$CS = OC \times \cos\lambda = \frac{D_2}{2}\cos\lambda. \tag{8}$$

From the rectangular triangle ASC we can derive as follows:

$$tg\alpha_3 = \frac{CS}{AS} = \frac{CS}{OA - OS} = \frac{0,5D_2\sin\lambda}{0,5D_1 - 0,5D_2\cos\lambda} = \frac{D_2\sin\lambda}{D_1 - D_2\cos\lambda} = Y; \tag{9}$$

$$\cos\alpha_3 = \frac{AS}{CA} = \frac{OA - OS}{CA} = \frac{0,5D_1 - 0,5D_2\cos\lambda}{CA}. \tag{10}$$

We introduce the substitution (11)

$$\cos\alpha_3 = \frac{1}{\sqrt{1 + tg^2\alpha_3}} = \frac{1}{\sqrt{1 + Y^2}}, \tag{11}$$

then

$$h_1 = CA = \frac{AS}{\cos\alpha_3} = (0,5D_1 - 0,5D_2\cos\lambda) \times \sqrt{1 + tg^2\alpha_3} = 0,5(D_1 - D_2\cos\lambda) \times \sqrt{1 + Y^2}. \tag{12}$$

From the rectangular triangle CDA, we can derive as follows:

$$\alpha_4 = \alpha_2 + \alpha_3. \tag{13}$$

We introduce the substitutions (14) и (15):

$$tg\alpha_4 = \frac{tg\alpha_2 + tg\alpha_3}{1 - tg\alpha_2 \times tg\alpha_3} = \frac{X+Y}{1 - X \times Y} = Z; \tag{14}$$

$$\sin\alpha_4 = \frac{tg\alpha_4}{\sqrt{1+tg^2\alpha_4}} = \frac{Z}{\sqrt{1+Z^2}}. \tag{15}$$

Then

$$h_2 = CD = CA\sin\alpha_4 = CA\frac{tg\alpha_4}{\sqrt{1+tg^2\alpha_4}} = h_1\frac{Z}{\sqrt{1+Z^2}}, \tag{16}$$

or finally

$$h_2 = \frac{0,5(D_1 - D_2\cos\lambda) \times \sqrt{1+Y^2} \times Z}{\sqrt{1+Z^2}}. \tag{17}$$

The obtained formulas (3)–(15) made it possible to derive the dependences (12) and (17), which can be used in the design and layout of couplings in order to test the possibility of their geometric existence according to the conditions mentioned above (1) and (2).

To illustrate the possibilities of applying the obtained formulas, let us consider an example of the layout simulation for the following coupling data: the torque transmitted by the coupling is T = 500 Nm, the diameters of the sleeves in the half-coupling $D_1 = 145\ mm$, $D_2 = 110\ mm$, number of ropes z = 8 pcs. In the design calculation, the angle of the installation displacement of the half couplings, which ensures the minimum tension of the ropes, was calculated from the previously [6] obtained formula (18).

$$\xi_F = \arccos\frac{D_1}{D_2} = \arccos\frac{110}{145} = 40,66^0. \tag{18}$$

After calculating the tension of the ropes according to the formula (19), the ropes were selected with a dimmer $d_2 = 3.8mm$ GOST 2688, with a breaking force of 8400 N, marking group 1170 MPa, the diameter of the sleeves is taken to be $d_1 = 24mm$.

$$F_H = \frac{4T\sqrt{0,25(D_1^2 + D_2^2) - 0,5D_1D_2\cos\xi}}{zD_1D_2\sin\xi}. \tag{19}$$

The possibility of the coupling layout with accepted and calculated parameters is illustrated by the graph (Fig. 3). With the use of the obtained formulas (1)–(19), the graph shows the influence of the angle of the mounting displacement of the half-couplings ξ for the gaps k_3 and k_4, as well as the tension of the ropes F_H. Other conditions for the geometric existence of the coupling are met for any ξ, therefore they are not shown on the graph.

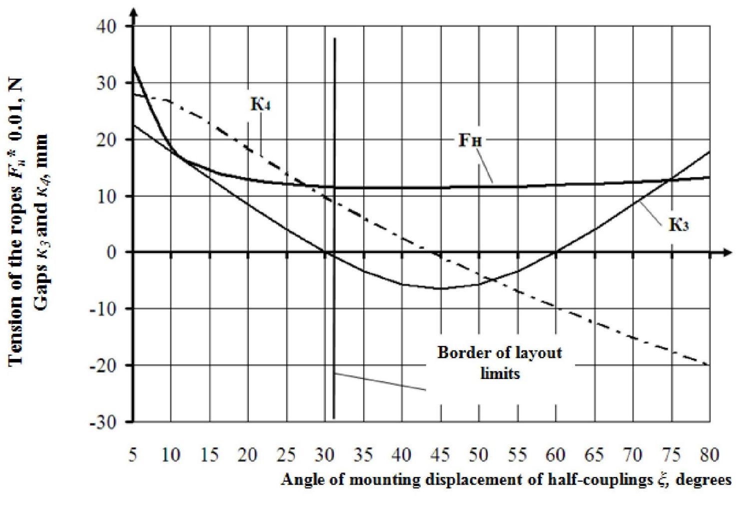

Figure 3. Graph of the mutual influence of design and force parameters of the coupling

The analysis of the graphs presented in Fig. 3 makes it possible to state that the interference between the adjacent sleeves, as well as that between the ropes and the sleeves according to the conditions (1) and (2) is absent at angles $\xi = 32^0$. The limiting layout parameter is the design gap k_3, which only at the angle value $\xi < 32^0$, has the value which is greater than zero (this position corresponds to the vertical line of the border of the layout limits on the graph).

Conclusions. To set in the model coupling the value of the mounting displacement angle ξ, calculated according to the formula (18), which is optimal from the point of view of ensuring minimum rope tension, is not possible due to interference between the adjacent sleeves of the outer and inner half-couplings. Therefore, for a successful layout, based on the graph data (Fig. 3), it is possible to reduce the angle of the mounting displacement of the half-couplings e.g. to a value of 25^0. With this change, the load of the coupling parts will change insignificantly, since the force of the rope tension when the angle ξ changes from 40.66^0 to 25^0, which is acceptable from the viewpoint of the layout, will increase by about 6%. Consequently, the research performed and the model obtained make it possible to simplify the coupling layout characterized by the compactness and the possibility of calculation automation.

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Арқандары тангенциалды орналасқан муфта бөлшектерін құрастырудың математикалық моделі

Мақалада тангенс бойынша орналасқан беттік құрылымды канат пен муфта құрылымының математикалық модельдеу нәтижесі ұсынылған. Зерттеу нәтижесінде осы әсердің ықпалы ашылған және олардың геометриялық бар болуының негізгі шарттарын тексеру барысында муфталарды жобалап есептеп қолдану үшін математикалық өрнектер алынған. Сыртқы және ішкі жартылай муфталардың тығындарының интерференцияларының болмайтыны, сонымен қатар ішкі жартылай муфталардың аралық тығындылар мен канаттардың интерференцияларының болмайтын жағдайы қарастырылған. Муфтаны жобалаудан кейін тәуелділіктер алынған, сондай-ақ есептеу нәтижелері құру нәтижелерімен салыстырылып дұрыс есептелгеніне көз жеткізілген. Алынған нәтижелер тангенс бойынша орналасқан беттік құрылымды түзу канатты муфталарды жобалау үшін қолданымын таба алады.

Кілт сөздер: математикалық модель, құрастыру, сыртқы және ішкі жартылай муфталар, канат, жүк, саңылау, мезет.

В.А. Проценко, М.В. Бабий, В.А. Настасенко, О.Ю. Клементьева

Математическая модель компоновки муфты с канатами тангенциального расположения

В статье представлены результаты математического моделирования компоновки муфт с торцевой установкой канатов тангенциального расположения. В результате выполнения исследований раскрыто это влияние и получены математические выражения для использования при проектировочном расчете муфт и проверке основных условий их геометрического существования. Рассмотрены случаи отсутствия интерференции втулок внешней и внутренней полумуфт, а также отсутствия интерференции канатов и смежных втулок внутренней полумуфты. Полученные зависимости апробированы при проектировании муфты, а результаты расчета по ним сравнены с результатами построения и показали совпадение. Данные результаты могут быть использованы при проектировании муфт с торцевой установкой прямых канатов тангенциального расположения.

Ключевые слова: математическая модель, компоновка, внешняя и внутренняя полумуфты, канат, нагрузки, зазор, момент.

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