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On a necessary condition for belonging of a function to periodic generalized Nikol'sky-Besov-Morrey space in terms of strong summability of Fourier series

This paper is dedicated to the investigation of strong summability in the generalized Morrey spaces. First, we study boundedness of the Hardy-Littlewood maximal function on generalized Morrey spaces. We find a necessary condition for belonging of a function to the periodic generalized Nikol'skij-Besov-Morrey spaces.

Key words: Morrey space, maximal function, Fourier transform, Nikol'skij-Besov-Morrey space.

First we recall the definition of periodic Morrey spaces. As usual, $B(x,r)$ denotes the open ball with center in x and radius $r>0$.

By $\mathbb{T}^d$ we denote the d-dimensional torus as usually represented by

$$\mathbb{T}^d = \{x| \quad x\in\mathbb{R}^d, \quad -\pi\le x_j\le\pi, \quad j=1,...,d\},$$

where opposite sides are identified.

Definition 1. Let $0<p\le\infty$ and $0\le\lambda\le 1/p$.

We say that a function $f:\ \mathbb{R}\to\mathbb{C}$, 2π-periodic in each component, belongs to the periodic Morrey space $M_p^\lambda(\mathbb{T}^d)$ if $f\in L_p(B(x,r))$ for all $x\in\mathbb{R}^d$ and all $r>0$ and the following expression is finite

$$\|\, f\,\|_{M_p^\lambda(\mathbb{T}^d)} := \sup_{x\in\mathbb{R}^d}\ \sup_{0<r\le 2\sqrt{d}\pi} |B(x,r)|^{-\lambda}\, \|f\|_{L_p(B(x,r))}\,. \tag{1}$$

Obviously we have $M_p^0(\mathbb{T}^d) = L_p(\mathbb{T}^d)$ and $M_p^{1/p}(\mathbb{T}^d) = L_\infty(\mathbb{T}^d)$ in the sense of equivalent norms.

By periodicity it will be enough to restrict the supremum in (1) to $x\in[-\pi,\pi]^d$.

Generalized Morrey spaces have been introduced independently by Mizuhara [1] and by Nakai [2]. Here the parameter r^λ is replaced by a function $\varphi:(0,\infty)\to(0,\infty)$.

Definition 2. Let $0\le p<\infty$ and let $\varphi:(0,\infty)\to(0,\infty)$.

Then the generalized periodic Morrey space $M_p^\varphi(\mathbb{T}^d)$ is the collection of all functions $f:\mathbb{R}^d\to\mathbb{C}$, 2π-periodic in each component, such that $f\in L_p(B(x,r))$ for all $x\in\mathbb{R}^d$ and all $r>0$ and

$$\|f\|_{M_p^\varphi(\mathbb{T}^d)} := \sup_{x\in[-\pi,\pi]^d}\ \sup_{0<r\le 2\sqrt{d}\pi} \varphi(r)\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|^p dy\Big)^{\frac{1}{p}} < \infty\,.$$

Clearly, if $\varphi(r) := |B(0,r)|^{-\lambda+\frac{1}{p}}$, $r>0$, then we have coincidence $M_p^\varphi(\mathbb{T}^d) = M_p^\lambda(\mathbb{T}^d)$, in particular, if $\varphi(r) := |B(0,r)|^{\frac{1}{p}}$, $r>0$, then $M_p^\varphi(\mathbb{T}^d) = L_p(\mathbb{T}^d)$.

In the definition of generalized Morrey space we assume, that $\varphi\in\mathcal{G}_p$.

Definition 3. Let $0<p<\infty$. Then $\varphi:(0,\infty)\to(0,\infty)$ belongs to the class $\mathcal{G}_p$, if φ is essentially nondecreasing and there exist positive constants C' such that the inequalities $t_2^{-\frac{d}{p}}\varphi(t_2)\le C'\, t_1^{-\frac{d}{p}}\varphi(t_1)$ hold for all $0<t_1\le t_2<\infty$.

Furthermore, $\varphi \in \mathcal{G}_p$ for some p implies that φ satisfies the doubling condition, that is, there exists a constant $C > 0$ such that $\frac{1}{C} \le \frac{\varphi(r)}{\varphi(s)} \le C$, holds for all r, s such that $\frac{1}{2} \le \frac{r}{s} \le 2$.

As usual, the symbol $C, C_1, C_2, ...$ denote positive constants which depend only on the fixed parameters s, p, q and λ and probably on auxiliary functions.

In the following statement we give boundedness of the Hardy–Littlewood maximal function on generalized periodic Morrey space $M_p^\varphi(\mathbb{T}^d)$.

Let f be a complex-valued locally Lebesgue-integrable function on $\mathbb{R}^d$. Then the Hardy-Littlewood maximal function is given by $Mf(x) := \sup_Q \frac{1}{|Q|} \int_Q |f(y)|\, dy, \qquad x \in \mathbb{R}^d$, where the supremum is taken over all cubes Q centered at x with sides parallel to the coordinate axes.

We have the following statements.

Theorem 1. Let $1 < p < \infty$ and $\varphi \in \mathcal{G}_p$. Then the Hardy-Littlewood maximal function M is bounded from $M_p^\varphi(\mathbb{T}^d)$ to $M_p^\varphi(\mathbb{T}^d)$. There exists a constant $C > 0$ such that

$$\left\| Mf_j \,|M_p^\varphi(\mathbb{T}^d)\right\| \le C \left\| f_j \,|M_p^\varphi(\mathbb{T}^d)\right\|$$

holds for all $(f_j)_j$ of measurable functions.

Proof of theorem 1. Let $1 < p < \infty$ and $\varphi \in G_p$. We mention that the following inequality

$$\|Mf|M_p^\varphi(\mathbb{R}^d)\| \le C_1 \, \|f|M_p^\varphi(\mathbb{R}^d)\|$$

holds for all f of measurable functions. This result was proved in [2] and [3, Theorem 2.3]. To adapt this inequality to the periodic situation, we mention that for a periodic integrable function f we have

$$Mf(x) \le C_2 \, M\tilde{f}(x), \quad x \in [-\pi, \pi]^d,$$

where

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if} \quad x \in [-\pi, \pi]^d \\ 0, & \text{if} \quad x \notin [-\pi, \pi]^d \end{cases} ,$$

see in [4, 3.2.4 formula (7)]. This implies that the inequality $\|Mf_j|M_p^\varphi(\mathbb{T}^d)\| \le C_1 C_2 \, \|f_j|M_p^\varphi(\mathbb{T}^d)\|$ is valid for all measurable functions $(f_j)_j$.

Now we give the definition of the generalized periodic Nikol'skij-Besov-Morrey space $\mathcal{N}^s_{\varphi,p,q}(\mathbb{T}^d)$.

Let $D(\mathbb{T}^d)$ be the collection of all complex-valued infinitely differentiable functions on $\mathbb{T}^d$. By $D'(\mathbb{T}^d)$ we denote the topological dual of $D(\mathbb{T}^d)$.

We mention, that $\mathbb{Z}^d = \{k : \quad k = (k_1, k_2, ..., k_d) \in \mathbb{R}^d, \quad k_j - \text{integer}\}$. We associate to $f \in D'(\mathbb{T}^d)$ it's Fourier coefficients by

$$c_k(f) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x)\, e^{-i\langle k, x\rangle}\, dx, \qquad k \in \mathbb{Z}^d,$$

where $\langle k, x\rangle$ - scalar product. Every distribution $f \in D'(\mathbb{T}^d)$ can be represented by its Fourier series

$$f = \sum_{k\in\mathbb{Z}^d} c_k(f)\, e^{ikx} \quad (\text{convergence in} \quad D'(\mathbb{T}^d)).$$

Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be a function such that

$$\rho(x) := 1 \qquad \text{if} \quad |x| \le 1 \qquad \text{and} \qquad \rho(x) := 0, \qquad \text{if} \quad |x| \ge \frac{3}{2}\,. \tag{2}$$

Then, with $\phi_0 := \rho$, we define

$$\phi(x) := \phi_0(x/2) - \phi_0(x) \qquad \text{and} \qquad \phi_j(x) := \phi(2^{-j+1}x)\,, \quad j \in \mathbb{N}\,. \tag{3}$$

This implies $\sum_{j=0}^{\infty}\phi_j(x)=1$ for all $x\in\mathbb{R}^d$. We shall call $(\phi_j)_{j=0}^{\infty}$ a smooth dyadic decomposition of unity.

Definition 4. Let $(\phi_j)_j$ be a smooth dyadic decomposition of unity as defined in (2), (3).

Let $s\in\mathbb{R}$, $0<p\le\infty$, $0<q\le\infty$ and $\varphi\in\mathcal{G}_p$. Then generalized periodic Nikol'skij–Besov–Morrey space $\mathcal{N}^s_{\varphi,p,q}(\mathbb{T}^d)$ is defined as the set of all $f\in D'(\mathbb{T}^d)$ such that

$$\|f|\mathcal{N}^s_{\varphi,p,q}(\mathbb{T}^d)\|:=\Big(\sum_{j=0}^{\infty}2^{jsq}\Big\|\sum_{k\in\mathbb{Z}^d}\phi_j(k)c_k(f)e^{ikx}|M_p^{\varphi}(\mathbb{T}^d)\Big\|^q\Big)^{1/q}<\infty.$$

Remark.

(i) Taking $\varphi(r):=|B(0,r)|^{\frac{1}{p}}$, $r>0$, we are back in the case of classical periodic Nikol'skij-Besov spaces, i.e., we have $\mathcal{N}^s_{\varphi,p,q}(\mathbb{T}^d)=B^s_{p,q}(\mathbb{T}^d)$. These spaces are studied systematically in the monographs [5] (Nikol'skij-Besov spaces).

(ii) The spaces $\mathcal{N}^s_{\varphi,p,q}(\mathbb{T}^d)$ will be called generalized periodic Nikol'skij-Besov-Morrey spaces. They represent the Nikol'skij-Besov scale built on the generalized Morrey space $M_p^{\varphi}(\mathbb{T}^d)$. Kozono, Yamazaki in 1994 and later on Mazzucato [6] have been the first who investigated spaces of this type in the nonperiodic context. Nikol'skij-Besov-Morrey spaces with respect to generalized Morrey spaces have been studied by Nakamura, Noi and Sawano [7].

Our main aim in this work is to find necessary condition for belonging a function to generalized periodic Nikol'skij-Besov-Morrey space.

Let ψ be a real-valued function defined on $\mathbb{R}^d$. We assume $\psi(0)=1$. For a distribution $f\in D'(\mathbb{T}^d)$ we introduce associated means $M_N^{\psi}f(x)$, $N=1,2,\ldots$, by

$$M_N^{\psi}f(x):=\sum_{k\in\mathbb{Z}^d}\psi\Big(\frac{k}{N}\Big)\,c_k(f)\,e^{ikx},\qquad x\in\mathbb{R}^d,\quad N=1,2,\ldots.\tag{4}$$

Of course, (4) makes sense if ψ has compact support or if $f\in L_1(\mathbb{T}^d)$ and $\sum_{k\in\mathbb{Z}^d}\Big|\psi(k/N)|<\infty$. Then the $M_N^{\psi}f(x)$ are trigonometric polynomials.

If $0<p<\infty$, $0<q\le\infty$ and $s>0$, we ask which smoothness properties of a function $f\in L_1(\mathbb{T}^d)\cap M_p^{\varphi}(\mathbb{T}^d)$ are implied by the condition

$$\Big(\sum_{N=1}^{\infty}N^{sq-1}\Big|f(x)-M_N^{\psi}f(x)\Big|M_p^{\varphi}(\mathbb{T}^d)\Big|^q\Big)^{1/q}<\infty.\tag{5}$$

For a space $L_p(\mathbb{T}^d)$ such a theorem was proved by H.-J. Schmeisser and W. Sickel [8], in case $\varphi(r):=|B(0,r)|^{-\lambda+1/p}$ and $M_N^{\psi}f(x)=S_Nf(x)$ was proved in [9].

To attack the described problem we need the following lemma which is taken from [8].
The symbol $\mathcal{F}$ refers to the Fourier transform, $\mathcal{F}^{-1}$ to its inverse transformation.

Lemma 1. (see in [8]). Let $0<u\le 1$, $\frac{d}{u}<\lambda<\infty$ and let C_1,C_2 constants. Let

$$|(\mathcal{F}^{-1}\eta)(\xi)|\le C_1\,(1+|\xi|)^{-\lambda}\tag{6}$$

and $supp\eta\subset\{y|\quad|y|\le 2^K\}$, where $K=0,1,2,...$ is fixed. If $f_j(x)=\sum_{|k|=C_2 2^j}a_ke^{ikx}$, $j=0,1,2,...$, then there exists a positive constant C_2 independent of j, K such that

$$|\sum_{k\in\mathbb{Z}^d}\eta(2^{-j}k)c_k(f)e^{ikx}|^u\le C_2\,2^{Kd(1-u)}(M|f_j|^u)(x),\tag{7}$$

holds for all $x\in\mathbb{T}^d$.

Theorem 2. Let $\psi(\xi)$ be a continuous real-valued function on $\mathbb{R}^d$ with $\psi(0) = 1$ and having compact support. Let $0 < p < \infty$, $0 < q < \infty$, $\varphi \in \mathcal{G}_p$, and let λ be a real number satisfying $\lambda > \frac{d}{\min(1,p)}$. Let $0 < \sigma < \infty$ be such that

$$\sup_{1\leq\tau\leq 2} |\mathcal{F}^{-1}[|\xi|^{-\sigma}(\psi(\xi) - \psi(\tau\xi))](y)| \leq {}_1(1 + |y|)^{-\lambda}, \tag{8}$$

for all $y \in \mathbb{R}^d$ If $0 < s < \sigma$, then there exists a positive constant C_2 such that

$$\Big(\sum_{N=1}^{\infty} N^{sq-1}\|(f - M_N^{\psi} f)(x)|M_p^{\varphi}(\mathbb{T}^d)\|^q\Big)^{1/q} \leq C_2\|f|\mathcal{N}_{\varphi,p,q}(\mathbb{T}^d)\|, \tag{9}$$

holds for all $f \in \mathcal{N}^s_{\varphi,p,q}(\mathbb{T}^d) \cap M_p^{\varphi}(\mathbb{T}^d)$.

Proof of Theorem 2.

Step 1. Let $f \in \mathcal{N}^s_{\varphi,p,q}(\mathbb{T}^d) \cap M_p^{\varphi}(\mathbb{T}^d)$. Then we have

$$\sum_{N=1}^{\infty} N^{sq-1}\|(f - M_N^{\psi} f)(x)|M_p^{\varphi}(\mathbb{T}^d)\|^q =$$

$$= \sum_{j=0}^{\infty}\Big(\sum_{N=2^j}^{2^{j+1}-1} N^{sq-1}\|(f - M^{\psi}_{2^{j+1}} f + M^{\psi}_{2^{j+1}} f - M_N^{\psi} f)(x)|M_p^{\varphi}(\mathbb{T}^d)\|^q\Big) \leq$$

$$\leq C_3\Big(\sum_{j=0}^{\infty} 2^{jsq}\|(f - M^{\psi}_{2^{j+1}} f)(x)|M_p^{\varphi}(\mathbb{T}^d)\|^q + \tag{10}$$

$$+\sum_{j=0}^{\infty} 2^{jsq} \sup_{N=2^j,\dots,2^{j+1}-1} \|(M_{2^{j+1}} f - M_N^{\psi} f)(x)|M_p^{\varphi}(\mathbb{T}^d)\|^q\Big).$$

Since $s > 0$ and $f \in \mathcal{N}^s_{\varphi,p,q}(\mathbb{T}^d) \cap M_p^{\varphi}(\mathbb{T}^d)$, we have

$$(f - M^{\psi}_{2^{j+1}} f)(x) = \sum_{l=j+1}^{\infty} (M^{\psi}_{2^{l+1}} f - M^{\psi}_{2^l} f)(x), \tag{11}$$

where the series on the right-hand side converges a.e. Let $0 < u < \min(1, q)$. Then, considering monotonicity of space l_u, i.e., $l_u \subset l_1$, by (11), applying generalized Minkowski inequality for sums with respect $q/u > 1$, we have

$$\sum_{j=0}^{\infty} 2^{jsq}\|(f - M^{\psi}_{2^{j+1}} f)(x)|M_p^{\varphi}(\mathbb{T}^d)\|^q \leq$$

$$\leq \sum_{j=0}^{\infty}\Big(\sum_{l=1}^{\infty} 2^{jsu}\|(M^{\psi}_{2^{j+l+1}} f - M^{\psi}_{2^{l+j}} f)(x)|M_p^{\varphi}(\mathbb{T}^d)\|^u\Big)^{q/u} \leq$$

$$\leq \Big(\sum_{l=1}^{\infty} 2^{-lsu}\Big(\sum_{j=0}^{\infty} 2^{(j+l)sq}\|(M^{\psi}_{2^{l+j+1}} f - M^{\psi}_{2^{l+j}} f)(x)|M_p^{\varphi}(\mathbb{T}^d)\|^q\Big)^{u/q}\Big)^{q/u} \leq \tag{12}$$

$$\leq C_4 \sum_{j=1}^{\infty} 2^{jsq}\|(M^{\psi}_{2^{j+1}} - M^{\psi}_{2^j} f)(x)|M_p^{\varphi}(\mathbb{T}^d)\|^q.$$

(11) and (12) lead to

$$\sum_{N=1}^{\infty} N^{sq-1}\|(f-M_N^{\psi}f)(x)|M_p^{\varphi}(\mathbb{T}^d)\|^q \le C_5 \le$$

$$\le C_5\sum_{j=0}^{\infty}2^{jsq}\sup_{N=2^j,\dots,2^{j+1}-1}\|(M_{2^{j+1}}^{\psi}f-M_N^{\psi}f)\|M_p^{\varphi}(\mathbb{T}^d)|^q. \tag{13}$$

Step 2. Let $\rho\in S(\mathbb{R}^d)$ such that $supp\rho\subset\{\xi|\ 1/2\le|\xi|\le 2\}$ and

$$\sum_{j=0}^{\infty}\rho(2^{-j}\xi)=1,\quad \text{if}\quad |\xi|\ge 1.$$

We put

$$\eta_\tau(\xi):=\psi(\xi)-\psi(\tau\xi),\quad 1\le\tau\le 2.$$

If $j=0,1,2,...,$ and $N=2^j,...,2^{j+1}-1$ then it is easy to see that

$$|(M_{2^{j+1}}^{\psi}f-M_N^{\psi}f)(x)|\le\sup_{1\le\tau\le 2}|\sum_{k\in\mathbb{Z}^d}\eta_\tau(2^{-j-1}k)c_k(f)e^{ikx}|. \tag{14}$$

The function ψ has compact support. Hence, there exists a natural number L such that

$$\eta_\tau(2^{-j-1}k)=\sum_{l=0}^{j+L}\eta_\tau(2^{-j-1}k)\rho(2^{-l}k)=\sum_{l=-\infty}^{L}\eta_\tau(2^{-j-1}k)\rho_{l+j}(k), \tag{15}$$

where $\rho_l=0$, $l=-1,-2,...,$ and $\rho_l(\cdot):=\rho(2^{-l}\cdot)$ if $l=0,1,2,....$

Let $\sigma>0$. We put following function

$$f_l(x):=\sum_{k\in\mathbb{Z}^d}|2^{-l-1}k|^{\sigma}\rho_l(k)c_k(f)e^{ikx},\quad l=0,\pm1,.... \tag{16}$$

Combining (15) and (16) we obtain

$$|\sum_{k\in\mathbb{Z}^d}\eta_\tau(2^{-j-1}k)c_k(f)e^{ikx}|\le\sum_{l=-\infty}^{L}2^{l\sigma}|\sum_{k\in\mathbb{Z}^d}|2^{-j-1}k|^{-\sigma}\eta_\tau(2^{-j-1}k)c_k(f_{j+l})(k)e^{ikx}|, \tag{17}$$

for $j=0,1,2,....$ We can choose a positive number u, such that $0<u<\min(1,p,q)$ and $\lambda>n/u$. Then it follows from (8), and applying Lemma 1, with $|\xi|^{-\sigma}\eta_\tau(\xi)$ instead of $\eta(\xi)$, we have

$$|\sum_{k\in\mathbb{Z}^d}\eta_\tau(2^{-j-1}k)c_k(f)e^{ikx}|\le C_6\sum_{l=-\infty}^{L}2^{l\sigma}(M|f_{j+l}|^u)^{1/u}(x). \tag{18}$$

Here the constant C_6 depends on L. However, it is independent of τ,j,x and f. Hence, as consequence of (13),(14),(15) we obtain

$$\sum_{N=1}^{\infty}N^{sq-1}\|(f-M_N^{\psi}f)(x)|M_p^{\varphi}(\mathbb{T}^d)\|^q\le C_7\sum_{j=0}^{\infty}2^{jsq}\Big\|\sum_{l=-\infty}^{L}2^{l\sigma}(M|f_{j+l}|^u)^{1/u}(x)|M_p^{\varphi}(\mathbb{T}^d)\Big\|^q. \tag{19}$$

Step 3. Let $s<\sigma$. Let $q\le 1$. Then

$$\sum_{j=0}^{\infty}2^{jsq}\Big(\sum_{l=-\infty}^{L}2^{l\sigma}(M|f_{j+l}|^u)^{1/u}(x)\Big)^q\le$$

$$\leq \sum_{l=-\infty}^{L} 2^{l(\sigma-s)q} \sum_{j=0}^{\infty}(M|2^{(j+l)s} f_{j+l}|^u)^{q/u}(x) \leq$$

$$\leq C_8 \sum_{j=0}^{\infty} (M|2^{js} f_j|^u)^{q/u}(x)\,.$$

If $q > 1$, we have

$$\Big[\sum_{j=0}^{\infty} 2^{jsq}\Big(\sum_{l=-\infty}^{L} 2^{l\sigma}(M|f_{j+l}|^u)^{1/u}(x)\Big)^q\Big]^{1/q} \leq$$

$$\leq \sum_{l=-\infty}^{L} 2^{l(\sigma-s)} \Big[\sum_{j=0}^{\infty}(M|2^{(j+l)s} f_{j+l}|^u)^{q/u}(x)\Big]^{1/q} \leq C_9 \Big[\sum_{j=0}^{\infty}(M|2^{js} f_j|^u)^{q/u}(x)\Big]^{1/q}.$$

Applying boundedness of the Hardy-Littlewood maximal function on generalized periodic Morrey space we have

$$\Big\|\Big(M|2^{js} f_j|^u(x)\Big)^{1/u}|M_p^{\varphi}(\mathbb{T}^d)\Big\| \leq C_8\, 2^{js}\|f_j|M_p^{\varphi}(\mathbb{T}^d)\|.$$

Using Theorem 1 and (19) we obtain

$$\Big(\sum_{N=1}^{\infty} N^{sq-1}\|(f - M_N^{\psi} f)(x)|M_p^{\varphi}(\mathbb{T}^d)\|^q\Big)^{1/q} \leq C_{10}\Big(\sum_{j=0}^{\infty} 2^{jsq}\|f_j(x)|M_p^{\varphi}(\mathbb{T}^d)\|^q\Big)^{1/q}, \qquad (20)$$

with a constant C_{10} independent of f.

Step 4. (16) and the elementary inequality we get

$$f_l(x) = \sum_{k\in\mathbb{Z}^d} |2^{-l-1}k|^{\sigma}\, \rho_l(k)\, (\phi_l(k) + \phi_{l-1}(k) + \phi_{l+1}(k))\, c_k(f)\, e^{ikx}.$$

For brevity we put

$$g_l(x) = \sum_{k\in\mathbb{Z}^d} \phi_l(k)\, c_k(f)\, e^{ikx}\,, \qquad x \in \mathbb{R}^d\,, \quad l \in \mathbb{N}_0\,.$$

Using Lemma 1 with $\eta(\xi) = |\xi/2|^{\sigma}\,\rho(\xi)$ and $g_j(x)$ instead of $f_j(x)$ and Theorem 1, we obtain the following estimate

$$\Big(\sum_{j=0}^{\infty} 2^{jsq}\|f_j(x)|M_p^{\varphi}(\mathbb{T}^d\|^q\Big)^{1/q} \leq C_{11}\Big(\sum_{j=0}^{\infty} 2^{jsq}\|(M|g_j|^u)^{1/u}(x)|M_p^{\varphi}(\mathbb{T}^d)\|^q\Big)^{1/q} \leq$$

$$\leq C_{12}\Big(\sum_{j=0}^{\infty} 2^{jsq}\|g_j(x)|M_p^{\varphi}(\mathbb{T}^d)\Big)^{1/q} = C_{12}\,\|f|\mathcal{N}^s_{\varphi,p,q}(\mathbb{T}^d)\|$$

for some $u < \min(1,p)$. The proof is complete.

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Ж.Ж. Байтуякова

Фурье қатарларының күшті қосындылау терминіндегі функцияның периодты жалпыланған Никольский-Бесов-Морри кеңістігіне тиісті болуының қажетті шарты туралы

Мақала периодты жалпыланған Морри қеңістігіндегі Фурье қатарларын күшті қосындылауға арналған. Алғашқыда Харди-Литтлвуд максималды функциясының шенелгендігі зерттелген. Кейін функцияның жалпыланған периодты Никольский-Бесов-Морри кеңістігіне тиісті болуының қажетті шарты табылды.

Ж.Ж. Байтуякова

О необходимом условии принадлежности функции периодическому обобщенному пространству Никольского-Бесова-Морри в терминах сильной суммируемости рядов Фурье

Статья посвящена исследованию сильной суммируемости рядов Фурье в периодических обобщенных пространствах Морри. Сначала изучена ограниченность максимальной функции Харди-Литтлвуда. Затем найдено необходимое условие для принадлежности функции периодическому обобщенному пространству Никольского-Бесова-Морри.