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# Pseudospectra of the direct sum of linear operators in ultrametric Banach spaces

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In this paper, a characterization of essential pseudospectra of bounded linear operators on ultrametric Banach spaces over a spherically complete field was given and the notions of pseudospectra and condition pseudospectra of the direct sum of linear operators on ultrametric Banach spaces were introduced. In particular, we proved that the pseudospectra of the direct sum of bounded linear operators associated with various $\varepsilon$ are nested sets and that the intersection of all the pseudospectra of bounded linear operators is the spectrum of the direct sum of bounded linear operators in the direct sum of ultrametric Banach spaces. In addition, many results were proved about them and examples were given.

*Keywords:* Ultrametric Banach spaces, pseudospectrum, condition pseudospectrum, direct sum of operators, linear operator pencils.

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## 1 Introduction and preliminaries

In the classical theory, Trefethen and Embree [1] studied the pseudospectra of bounded linear operators on complex Banach spaces. Recently, Otkun Çevik and Ismailov [2] studied some spectral properties of the direct sum of operators in the direct sum of Hilbert spaces. Ismailov and Ipek Al introduced and studied the pseudospectra of the direct sum of operators and they established some of its properties, for more details, see [3].

In ultaremetric operator theory, the authors [4] extended and studied the concept of pseudospectra of linear operators on ultrametric Banach spaces. The condition pseudospectra of bounded linear operators on ultrametric Banach spaces were extended and studied by Ammar et al. [5]. Recently, El Amrani et al. [6] studied the notion of bounded linear operator pencils on non-Archimedean Banach spaces. The concepts of pseudospectra and condition pseudospectra of ultrametric matrices were studied by El Amrani et al. [7].

In this paper, we will extend and study the pseudospectra and the condition pseudospectra of the direct sum of bounded linear operators on ultrametric Banach spaces.

Throughout this paper, $F$ is an ultrametric Banach space over an ultrametric complete valued field $\mathbb{K}$ with a non-trivial valuation $|\cdot|, \mathcal{L}(F)$ denotes the set of all bounded linear operators on $F$ and $F^* = \mathcal{L}(F, \mathbb{K})$ is the dual space of $F$. If $S \in \mathcal{L}(F)$, $N(S)$ and $R(S)$ denote the kernel and the range of $S$ respectively, see [8]. Recall that, an unbounded linear operator $S : D(S) \subseteq F \to F$ is called closed, if for each $(x_n)_{n\in\mathbb{N}} \subset D(S)$ such that $\|x_n - x\| \to 0$ and $\|Sx_n - y\| \to 0$ as $n \to \infty$ for some $x \in F$ and $y \in F$, then $x \in D(S)$ and $y = Sx$. The collection of all closed linear operators on $F$ is denoted by $\mathcal{C}(F)$. If $S \in \mathcal{L}(F)$ and $B$ is an unbounded linear operator on $F$, then $S + B$ is closed if and only if $B$ is closed [8]. We begin with the following preliminaries.

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*Definition 1.* [8] Let $F$ be a vector space over $\mathbb{K}$. A non-negative real valued function $\|\cdot\| : F \to \mathbb{R}_+$ is an ultrametric norm if:
(i) for all $x \in F$, $\|x\| = 0$ if and only if $x = 0$,
(ii) for each $x \in F$ and $\lambda \in \mathbb{K}$, $\|\lambda x\| = |\lambda|\|x\|$,
(iii) for any $x, y \in F$, $\|x + y\| \leq \max(\|x\|, \|y\|)$.

*Definition 2.* [8] An ultrametric normed space is a pair $(F, \|\cdot\|)$ where $F$ is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is an ultrametric norm on $F$.

*Definition 3.* [8] An ultrametric Banach space is a vector space endowed with an ultrametric norm which is complete.

*Proposition 1.* [8] The direct sum of two ultrametric Banach spaces is an ultrametric Banach space.

*Definition 4.* [8] An ultrametric Banach space $F$ is said to be a free Banach space if there exists a family $(x_i)_{i\in I}$ of elements of $F$ indexed by a set $I$ such that each element $x \in F$ can be written uniquely like a pointwise convergent series defined by $x = \sum\limits_{i\in I} \lambda_i x_i$ and $\|x\| = \sup\limits_{i\in I} |\lambda_i|\|x_i\|$.

The family $(x_i)_{i\in I}$ is then called an orthogonal basis for $F$. If, for all $i \in I, \|x_i\| = 1$, then $(x_i)_{i\in I}$ is called an orthonormal basis of $F$.

*Definition 5.* [8] Let $F$ be an ultrametric Banach space over $\mathbb{K}$ and let $A \in \mathcal{L}(F)$. The resolvent set $\rho(A)$ of $A$ on $F$ is defined by

$$\rho(A) = \{\lambda \in \mathbb{K} : (A - \lambda I)^{-1} \in \mathcal{L}(F)\}.$$

The spectrum $\sigma(A)$ of $A$ on $F$ is given by $\mathbb{K}\backslash\rho(A)$.

*Example 1.* [8] Let $F$ be an ultrametric free Banach space with an orthogonal basis $(e_i)_{i\in\mathbb{N}}$. Consider $A$ on $F$ defined by for all $n \in \mathbb{N}, Ae_n = \lambda_n e_n$ whose domain is

$$D(A) = \{x = (x_n)_{n\in\mathbb{N}} \in F : \lim_{n\to\infty} |\lambda_n||x_n|\|e_n\| = 0\}.$$

If $x \in D(A)$, then one can see that

$$Ax = \sum_{n=0}^{\infty} \lambda_n x_n e_n.$$

*Proposition 2.* [8] Consider the diagonal operator $A$ given above. Then

$$\rho(A) = \{\lambda \in \mathbb{K} : \lambda \neq \lambda_n \text{ for all } n \in \mathbb{N}\}.$$

*Proposition 3.* [8] The diagonal operator $A : D(A) \subset F \to F$ given above is closed.

*Definition 6.* [8] Let $A \in \mathcal{L}(F)$. $A$ is called an operator of finite rank, if $R(A)$ is a finite-dimensional subspace of $F$.

*Definition 7.* [8] Let $F$ be an ultrametric Banach space and let $A \in \mathcal{L}(F)$. $A$ is said to be completely continuous, if there exists a sequence of finite rank linear operators $(A_n)_{n\in\mathbb{N}}$ such that $\|A_n - A\| \to 0$ as $n \to \infty$.

The collection of all completely continuous linear operators on $F$ is denoted by $\mathcal{C}_c(F)$.
Ingleton [9] proved the following theorem.

*Theorem 1.* [9] Suppose that $\mathbb{K}$ is spherically complete. Let $F$ be an ultrametric Banach space over $\mathbb{K}$. For all $x \in F\backslash\{0\}$, there exists $x^* \in F^*$ such that $x^*(x) = 1$ and $\|x^*\| = \|x\|^{-1}$.

From Lemma 4.11 and Lemma 4.13 of [10], we have:

*Lemma 1.* Let $F$ be an ultrametric normed space over a spherically complete field $\mathbb{K}$. If $f_1^*, \cdots, f_n^*$ are linearly independent vectors in $F^*$, then there are vectors $f_1, \cdots, f_n$ in $F$ such that

$$f_j^*(f_k) = \delta_{j,k} = \begin{cases} 1, \text{ if } j = k; \\ 0, \text{ if } j \neq k. \end{cases} \quad 1 \leq j, k \leq n. \tag{1}$$

Moreover, if $f_1, \cdots, f_n$ are linearly independent vectors in $F$, then there are vectors $f_1^*, \cdots, f_n^*$ in $F^*$ such that (1) holds.

*Definition 8.* [11] We say that $A \in \mathcal{L}(F)$ has an index when both $\alpha(A) = \dim N(A)$ and $\beta(A) = \dim \Big(F/R(A)\Big)$ are finite. In this case, the index of the linear operator $A$ is defined by $ind(A) = \alpha(A) - \beta(A)$.

*Definition 9.* [11] Let $A \in \mathcal{L}(F)$. $A$ is said to be an upper semi-Fredholm operator, if

$$\alpha(A) \text{ is finite and } R(A) \text{ is closed.}$$

The set of all upper semi-Fredholm operators on $F$ is denoted by $\Phi_+(F)$.

*Definition 10.* [11] Let $A \in \mathcal{L}(F)$. $A$ is said to be a lower semi-Fredholm operator, if

$$\beta(A) \text{ is finite.}$$

The set of all lower semi-Fredholm operators on $F$ is denoted by $\Phi_-(F)$.

The set of all Fredholm operators on $F$ is defined by

$$\Phi(F) = \Phi_+(F) \cap \Phi_-(F).$$

*Lemma 2.* [12] Let $F$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. If $S \in \Phi(F)$ and $C \in \mathcal{C}_c(F)$, then $S + C \in \Phi(F)$.

*Lemma 3.* [5] Let $F$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. If $S \in \Phi(F)$, then for all $C \in \mathcal{C}_c(F)$, $S + C \in \Phi(F)$ and $ind(S + C) = ind(S)$.

*Definition 11.* [4] Let $F$ be an ultrametric Banach space over $\mathbb{K}$, let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(S)$ of a bounded linear operator $S$ on $F$ is defined by

$$\sigma_\varepsilon(S) = \sigma(S) \cup \{\lambda \in \mathbb{K} : \|(S - \lambda I)^{-1}\| > \varepsilon^{-1}\}.$$

The pseudoresolvent $\rho_\varepsilon(S)$ of a bounded linear operator $S$ on $F$ is defined by

$$\rho_\varepsilon(S) = \rho(S) \cap \{\lambda \in \mathbb{K} : \|(S - \lambda I)^{-1}\| \leq \varepsilon^{-1}\},$$

by convention $\|(S - \lambda I)^{-1}\| = \infty$ if and only if $\lambda \in \sigma(S)$.

*Theorem 2.* [4] Let $F$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ and let $S \in \mathcal{L}(F)$. Then

$$\sigma_\varepsilon(S) = \bigcup_{D \in \mathcal{L}(F) : \|D\| < \varepsilon} \sigma(S + D).$$

*Theorem 3.* [5] Let $F$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ and let $S \in \mathcal{L}(F)$. Then

$$\sigma_e(S) = \bigcap_{K \in \mathcal{C}_c(F)} \sigma(S + K).$$

*Definition 12.* [8] Let $F$ be an ultrametric Banach space over $\mathbb{K}$ and let $S, B \in \mathcal{L}(F)$. The resolvent set $\rho(S,B)$ of a bounded linear operator pencil $(S,B)$ of the form $S - \lambda B$ on $F$ is defined by

$$\rho(S,B) = \{\lambda \in \mathbb{K} : (S - \lambda B)^{-1} \in \mathcal{L}(F)\}.$$

The spectrum $\sigma(S,B)$ of a bounded linear operator pencil $(S,B)$ of the form $S - \lambda B$ on $F$ is given by $\mathbb{K} \backslash \rho(S,B)$.

*Definition 13.* [6] Let $F$ be an ultrametric Banach space over $\mathbb{K}$, let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(S,B)$ of a bounded linear operator pencil $(S,B)$ of the form $S - \lambda B$ on $F$ is defined by

$$\sigma_\varepsilon(S,B) = \sigma(S,B) \cup \{\lambda \in \mathbb{K} : \|(S - \lambda B)^{-1}\| > \varepsilon^{-1}\}.$$

The pseudoresolvent $\rho_\varepsilon(S,B)$ of a bounded linear operator pencil $(S,B)$ of the form $S - \lambda B$ on $F$ is defined by

$$\rho_\varepsilon(S,B) = \rho(S,B) \cap \{\lambda \in \mathbb{K} : \|(S - \lambda B)^{-1}\| \leq \varepsilon^{-1}\},$$

by convention $\|(S - \lambda B)^{-1}\| = \infty$ if and only if $\lambda \in \sigma(S,B)$.

*Proposition 4.* [13] Let $F$ be an ultrametric Banach space over $\mathbb{K}$, let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$, we have:

(i) $\sigma(S,B) = \bigcap\limits_{\varepsilon > 0} \sigma_\varepsilon(S,B)$.

(ii) For any $\varepsilon_1$ and $\varepsilon_2$ such that $0 < \varepsilon_1 < \varepsilon_2$, $\sigma(S,B) \subset \sigma_{\varepsilon_1}(S,B) \subset \sigma_{\varepsilon_2}(S,B)$.

*Theorem 4.* [13] Let $F$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|F\| \subseteq |\mathbb{K}|$, let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$. Then

$$\sigma_\varepsilon(S,B) = \bigcup_{C \in \mathcal{L}(F): \|C\| < \varepsilon} \sigma(S + C, B).$$

Now, we characterize the essential pseudospectra of bounded linear operator pencils in ultrametric Banach spaces over a spherically complete field $\mathbb{K}$.

*Definition 14.* [14] Let $F$ be an ultrametric Banach space over $\mathbb{K}$, let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$. The essential pseudospectrum $\sigma_{e,\varepsilon}(S,B)$ of a bounded linear operator pencil $(S,B)$ of the form $S - \lambda B$ on $F$ is defined by

$$\sigma_{e,\varepsilon}(S,B) = \mathbb{K} \backslash \{\lambda \in \mathbb{K} : S + C - \lambda B \in \Phi_0(F) \text{ for all } C \in \mathcal{L}(F) \text{ such that } \|C\| < \varepsilon\},$$

where $\Phi_0(F)$ is the set of all unbounded Fredholm operators on $F$ of index 0.

We continue by recalling the following statements.

*Theorem 5.* [14] Let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$. Then,

$$\sigma_{e,\varepsilon}(S,B) = \bigcup_{C \in \mathcal{L}(F): \|C\| < \varepsilon} \sigma_e(S + C, B).$$

*Theorem 6.* [13] Let $F$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. Let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$. Then,

$$\sigma_{e,\varepsilon}(S,B) = \sigma_{e,\varepsilon}(S + K, B) \quad \text{for all} \ \ K \in \mathcal{C}_c(F).$$

*Remark 1.* [13] From Theorem 6, it follows that the essential pseudospectrum of bounded linear operator pencils is invariant under perturbation of all completely continuous linear operators on ultrametric Banach spaces over a spherically complete field $\mathbb{K}$.

*Theorem 7.* [15] Let $F$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. Let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$. Then

$$\sigma_e(S,B) = \bigcap_{K \in \mathcal{C}_c(F)} \sigma(S+K,B).$$

From Example 1 of [16], we conclude the following example.

*Example 2.* Let $F$ be an ultrametric free Banach space over $\mathbb{K}$ with an orthogonal basis $(e_i)_{i\in\mathbb{N}}$. Let $S \in \mathcal{C}(F)$, $B \in \mathcal{L}(F)$ be two diagonal operators such that $B$ is invertible defined by for all $i \in \mathbb{N}, Se_i = \lambda_i e_i$ and $Be_i = \mu_i e_i$ where for all $i \in \mathbb{N}, \lambda_i, \mu_i \in \mathbb{K}$ such that $\lim\limits_{i\to\infty} |\lambda_i| = \infty$ and $\sup\limits_{i\in\mathbb{N}} |\mu_i|$ is finite, then

$$\sigma(S,B) = \{\lambda_i \mu_i^{-1}, i \in \mathbb{N}\}$$

and for all $\lambda \in \rho(S,B)$, we have

$$\begin{aligned} \|(S-\lambda B)^{-1}\| &= \sup_{i\in\mathbb{N}} \frac{\|(S-\lambda B)^{-1}e_i\|}{\|e_i\|} \\ &= \sup_{i\in\mathbb{N}} \left|\frac{1}{\lambda_i - \lambda\mu_i}\right| \\ &= \frac{1}{\inf\limits_{i\in\mathbb{N}} |\lambda_i - \lambda\mu_i|}. \end{aligned}$$

Thus

$$\left\{\lambda \in \mathbb{K} : \|(A-\lambda B)^{-1}\| > \frac{1}{\varepsilon}\right\} = \left\{\lambda \in \mathbb{K} : \inf_{i\in\mathbb{N}} |\lambda_i - \lambda\mu_i| < \varepsilon\right\}.$$

Hence

$$\sigma_\varepsilon(S,B) = \{\lambda_i\mu_i^{-1}, i \in \mathbb{N}\} \cup \left\{\lambda \in \mathbb{K} : \inf_{i\in\mathbb{N}} |\lambda_i - \lambda\mu_i| < \varepsilon\right\}.$$

For more details on pseudospectra and condition pseudospectra of linear operators on ultrametric Banach spaces, we refer to [4, 5, 7].

## 2 *Main Results*

We begin with the following theorem.

*Theorem 8.* Let $F$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. Then $\lambda \notin \bigcap\limits_{K\in\mathcal{C}_c(F)} \sigma_\varepsilon(S+K)$ if and only if for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$, $S+C-\lambda I \in \Phi(F)$ and $ind(S+C-\lambda I) = 0$.

*Proof.* Let $\lambda \notin \bigcap\limits_{K\in\mathcal{C}_c(F)} \sigma_\varepsilon(S+K)$, then there exists $K \in \mathcal{C}_c(F)$ such that $\lambda \notin \sigma_\varepsilon(S+K)$. By Theorem 2, there is $K \in \mathcal{C}_c(F)$ such that for all $C \in \mathcal{L}(F)$ with $\|C\| < \varepsilon$, $\lambda \in \rho(S+K+C)$, hence for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$,

$$S+K+C-\lambda I \in \Phi(F)$$

and

$$ind(S+K+C-\lambda I) = 0.$$

The operator $S+C-\lambda I$ can be written in the form

$$S+C-\lambda I = S+C+K-\lambda I-K.$$

Since $K \in \mathcal{C}_c(F)$, by Lemmas 2 and 3, we have for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$,

$$S + C - \lambda I \in \Phi(F)$$

and

$$ind(S + C - \lambda I) = 0.$$

Conversely, let $\lambda \in \mathbb{K}$ and for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$, we have $S + C - \lambda I \in \Phi(F)$ and $ind(S + C - \lambda I) = 0$. Put $\alpha(S + C - \lambda I) = \beta(S + C - \lambda I) = n$. Let $\{x_1, \cdots, x_n\}$ being the basis for $N(S + C - \lambda I)$ and $\{y_1^*, \cdots, y_n^*\}$ being the basis for $R(S + C - \lambda I)^{\perp}$. By Lemma 1, there are functionals $x_1^*, \cdots, x_n^*$ in $F^*$ $\big(F^*$ is the dual space of $F\big)$ and elements $y_1, \cdots, y_n$ in $F$ such that

$$x_j^*(x_k) = \delta_{j,k} \; and \; y_j^*(y_k) = \delta_{j,k}, \; 1 \leq j, \; k \leq n,$$

where $\delta_{j,k} = 0$, if $j \neq k$ and $\delta_{j,k} = 1$, if $j = k$. Consider the operator $K$ defined on $F$ by

$$\begin{aligned} K: \quad F &\to \quad F \\ x &\mapsto \quad \sum_{i=1}^{n} x_i^*(x) y_i. \end{aligned}$$

It is easy to see that $K$ is a linear operator and $D(K) = F$. In fact, for all $x \in F$,

$$\begin{aligned} \|Kx\| &= \|\sum_{i=1}^{n} x_i^*(x) y_i\| \\ &\leq \max_{1 \leq i \leq n} \|x_i^*(x) y_i\| \\ &\leq \max_{1 \leq i \leq n} (\|x_i^*\| \|y_i\|) \|x\|. \end{aligned}$$

Moreover, $R(K)$ is contained in a finite-dimensional subspace of $F$. So, $K$ is a finite rank operator, then $K$ is completely continuous. We show that for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$, we have

$$N(S + C - \lambda I) \cap N(K) = \{0\} \tag{2}$$

and

$$R(S + C - \lambda I) \cap R(K) = \{0\}. \tag{3}$$

Let $x \in N(S + C - \lambda I) \cap N(K)$, hence $x \in N(S + C - \lambda I)$ and $x \in N(K)$. If $x \in N(S + C - \lambda I)$, then

$$x = \sum_{i=1}^{n} \alpha_i x_i \text{ with } \alpha_1, \cdots, \alpha_n \in \mathbb{K}.$$

Then for all $1 \leq j \leq n$, $x_j^*(x) = \sum_{i=1}^{n} \alpha_i \delta_{i,j} = \alpha_j$. If $x \in N(K)$, hence $Kx = 0$, so

$$\sum_{j=1}^{n} x_j^*(x) y_j = 0.$$

Therefore, we have for all $1 \leq j \leq n$, $x_j^*(x) = 0$. Hence $x = 0$. Consequently,

$$N(S + C - \lambda I) \cap N(K) = \{0\}.$$

Let $y \in R(S+C-\lambda I) \cap R(K)$, then $y \in R(S+C-\lambda I)$ and $y \in R(K)$. Let $y \in R(K)$, we have

$$y = \sum_{i=1}^{n} \alpha_i y_i \text{ with } \alpha_1, \cdots, \alpha_n \in \mathbb{K}.$$

Then for all $1 \leq j \leq n, y_j^*(y) = \sum_{i=1}^{n} \alpha_i \delta_{i,j} = \alpha_j$. Moreover, if $y \in R(S+C-\lambda I)$, hence for all $1 \leq j \leq n$, $y_j^*(y) = 0$. Thus $y = 0$. Therefore,

$$R(S+C-\lambda I) \cap R(K) = \{0\}.$$

Since $K$ is a compact operator. By Lemmas 2 and 3, $S+C-\lambda I+K \in \Phi(F)$ and $ind(S+C+K-\lambda I) = 0$. Thus

$$\alpha(S+C+K-\lambda I) = \beta(S+C+K-\lambda I). \tag{4}$$

If $x \in N(S+C+K-\lambda I)$, then $(S+C-\lambda I)x = -Kx$ in $R(S+C-\lambda I) \cap R(K)$. It follows from (3) that $(S+C-\lambda I)x = -Kx = 0$, hence $x \in N(S+C-\lambda I) \cap N(K)$ and from (2), we have $x = 0$. Thus $\alpha(S+K+C-\lambda I) = 0$, it follows from (4), $R(S+C+K-\lambda I) = X$. Consequently, $S-\lambda I+K+C$ is invertible and from Theorem 2, we conclude that $\lambda \notin \bigcap_{K \in \mathcal{C}_c(F)} \sigma_\varepsilon(S+K)$.

Let $(X_i)_{1 \leq i \leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$. The space $X = \oplus_{i=1}^n X_i$ endowed by for all $i \in \{1, \cdots, n\}$, $x_i \in X_i$, $\|x_1 \oplus x_2 \oplus \cdots \oplus x_n\| = \max_{i \in \{1, \cdots, n\}} \|x_i\|$ is an ultrametric Banach space over $\mathbb{K}$ [8]. One can see that if for all $i \in \{1, \cdots, n\}$, $A_i \in \mathcal{L}(X_i)$, then $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \in \mathcal{L}(X)$. We introduce the following definition.

*Definition 15.* Let $(X_i)_{1 \leq i \leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$ and let $A_i \in \mathcal{L}(X_i)$. The spectrum $\sigma(A)$ of $A$ on $\oplus_{i=1}^n X_i$ is given by

$$\sigma(A) = \{\lambda \in \mathbb{K} : A - \lambda I \text{ is not invertible in } \mathcal{L}(\oplus_{i=1}^n X_i)\},$$

where $I$ denotes the identity operator of $\oplus_{i=1}^n X_i$ and $A = \oplus_{i=1}^n A_i$. The resolvent set of $A$ on $\oplus_{i=1}^n X_i$ is defined by

$$\rho(A) = \{\lambda \in \mathbb{K} : (A - \lambda I)^{-1} \in \mathcal{L}(\oplus_{i=1}^n X_i)\}.$$

For $i = 2$, we have the following proposition.

*Proposition 5.* Let $X, Y$ be two ultrametric Banach spaces over $\mathbb{K}$. Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$. The spectrum of $A \oplus B \in \mathcal{L}(X \oplus Y)$ is given by

$$\sigma(A \oplus B) = \sigma(A) \cup \sigma(B).$$

*Proof.* Let $\lambda \in \sigma(A \oplus B)$, then $(A \oplus B) - (I_X \oplus I_Y)$ is not invertible, hence $A - \lambda I_X$ is not invertible in $\mathcal{L}(X)$ or $B - \lambda I_Y$ is not invertible in $\mathcal{L}(Y)$, thus $\lambda \in \sigma(A) \cup \sigma(B)$. Hence $\sigma(A \oplus B) \subseteq \sigma(A) \cup \sigma(B)$. Similarly, we obtain that $\sigma(A) \cup \sigma(B) \subseteq \sigma(A \oplus B)$. Consequently, $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$.

More generally, one can see that.

*Proposition 6.* Let $(X_i)_{1 \leq i \leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$ and let $A_i \in \mathcal{L}(X_i)$. Set $A = \oplus_{i=1}^n A_i \in \mathcal{L}(\oplus_{i=1}^n X_i)$. Then

$$\sigma(A) = \bigcup_{i=1}^{n} \sigma(A_i)$$

and

$$\rho(A) = \bigcap_{i=1}^{n} \rho(A_i).$$

Now, we define the pseudospectrum of $A$ where $A = \oplus_{i=1}^{n} A_i$ and for all $i \in \{1, \cdots, n\}, A_i \in \mathcal{L}(X_i)$ on the ultrametric Banach space $\oplus_{i=1}^{n} X_i$. We have the following definition.

*Definition 16.* Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(\oplus_{i=1}^{n} A_i)$ of $\oplus_{i=1}^{n} A_i$ on $\oplus_{i=1}^{n} X_i$ is given by

$$\sigma_\varepsilon(\oplus_{i=1}^{n} A_i) = \sigma(\oplus_{i=1}^{n} A_i) \cup \{\lambda \in \bigcap_{i=1}^{n} \rho(A_i) : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| > \varepsilon^{-1}\}.$$

*Remark 2.* One can see that $\sigma_\varepsilon(\oplus_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} \sigma_\varepsilon(A_i)$.

*Proposition 7.* Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, then

(i) $\sigma(\oplus_{i=1}^{n} A_i) = \bigcap_{\varepsilon>0} \sigma_\varepsilon(\oplus_{i=1}^{n} A_i)$.

(ii) If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma(\oplus_{i=1}^{n} A_i) \subset \sigma_{\varepsilon_1}(\oplus_{i=1}^{n} A_i) \subset \sigma_{\varepsilon_2}(\oplus_{i=1}^{n} A_i)$.

*Proof.* (i) By Definition 16, for each $\varepsilon > 0$, $\sigma(\oplus_{i=1}^{n} A_i) \subset \sigma_\varepsilon(\oplus_{i=1}^{n} A_i)$, then $\sigma(\oplus_{i=1}^{n} A_i) \subset \bigcap_{\varepsilon>0} \sigma_\varepsilon(\oplus_{i=1}^{n} A_i)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \sigma_\varepsilon(\oplus_{i=1}^{n} A_i)$, since

$$\bigcap_{\varepsilon>0} \sigma_\varepsilon(\oplus_{i=1}^{n} A_i) = \sigma(\oplus_{i=1}^{n} A_i) \cup \bigcap_{\varepsilon>0} \{\lambda \in \bigcap_{i=1}^{n} \rho(A_i) : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| > \varepsilon^{-1}\}$$

and $\bigcap_{\varepsilon>0} \{\lambda \in \bigcap_{i=1}^{n} \rho(A_i) : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| > \varepsilon^{-1}\} = \emptyset$ because of for all $i \in \{1, \cdots, n\}$, $(A_i - \lambda I)^{-1}$ are bounded linear operators. Thus $\lambda \in \sigma(\oplus_{i=1}^{n} A_i)$.

(ii) For $0 < \varepsilon_1 < \varepsilon_2$. Let $\lambda \in \sigma_{\varepsilon_1}(\oplus_{i=1}^{n} A_i)$, consequently, $\sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| > \varepsilon_1^{-1} > \varepsilon_2^{-1}$, hence $\lambda \in \sigma_{\varepsilon_2}(\oplus_{i=1}^{n} A_i)$.

Let $A \in \mathcal{L}(X)$, set $r(A) = \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}}$. We have the following lemmas.

*Lemma 4.* Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, then $r(\oplus_{i=1}^{n} A_i) = \sup_{i\in\{1,\cdots,n\}} r(A_i)$. Furthermore $\sup_{\lambda\in\sigma(\oplus_{i=1}^{n} A_i)} |\lambda| \leq \sup_{\lambda\in\sigma_\varepsilon(\oplus_{i=1}^{n} A_i)} |\lambda|$.

*Proof.* Since for all $k \in \mathbb{N}$, $(A_1 \oplus \cdots \oplus A_n)^k = A_1^k \oplus \cdots \oplus A_n^k$. Thus

$$\begin{aligned} r(\oplus_{i=1}^{n} A_i) &= \lim_{k\to\infty} \|(A_1 \oplus \cdots \oplus A_n)^k\|^{\frac{1}{k}} \\ &= \lim_{k\to\infty} \|A_1^k \oplus \cdots \oplus A_n^k\|^{\frac{1}{k}} \\ &= \lim_{k\to\infty} \sup_{i\in\{1,\cdots,n\}} \|A_i^k\|^{\frac{1}{k}} \\ &= \sup_{i\in\{1,\cdots,n\}} \lim_{k\to\infty} \|A_i^k\|^{\frac{1}{k}} \\ &= \sup_{i\in\{1,\cdots,n\}} r(A_i). \end{aligned}$$

Since $\sigma(A) \subseteq \sigma_\varepsilon(A)$, then $\sup_{\lambda\in\sigma(\oplus_{i=1}^{n} A_i)} |\lambda| \leq \sup_{\lambda\in\sigma_\varepsilon(\oplus_{i=1}^{n} A_i)} |\lambda|$.

Set $r_\varepsilon(\oplus_{i=1}^n A_i) = \sup\limits_{\lambda\in\sigma_\varepsilon(\oplus_{i=1}^n A_i)} |\lambda|$, we have the following:

*Lemma 5.* Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, then $r_\varepsilon(\oplus_{i=1}^n A_i) = \sup\limits_{i\in\{1,\cdots,n\}} r_\varepsilon(A_i)$.

*Proof.* From Remark 2, $\sigma_\varepsilon(\oplus_{i=1}^n A_i) = \bigcup\limits_{i=1}^n \sigma_\varepsilon(A_i)$. One can see that $r_\varepsilon(\oplus_{i=1}^n A_i) = \sup\limits_{i\in\{1,\cdots,n\}} r_\varepsilon(A_i)$.

We have the following examples.

*Example 3.* Consider $(A_k)_{1\leq k\leq n}$ defined on $\mathbb{K}^2$ by

$$A_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \mu_k \end{pmatrix},$$

where $\lambda_k, \mu_k \in \mathbb{K}$ for all $k \in \{1,\cdots,n\}$ and $n \in \mathbb{N}$ is fixed. Then $\sigma(\oplus_{k=1}^n A_k) = \bigcup\limits_{k=1}^n \{\lambda_k, \mu_k\}$ and

$$\sigma_\varepsilon(\oplus_{k=1}^n A_k) = \bigcup_{k=1}^n \{\lambda_k, \mu_k\} \cup \{\lambda \in \mathbb{K} : \sup_{1\leq k\leq n} \|(\lambda I - A_k)^{-1}\| > \frac{1}{\varepsilon}\}.$$

*Example 4.* Let $F$ be an ultrametric free Banach space over $\mathbb{K}$ with an orthogonal basis $(e_m)_{m\in\mathbb{N}}$. Let $(A_k)_{1\leq k\leq n}$ be defined on $F$ by for all $x \in F$ and for each $k \in \{1,\cdots,n\}$, $A_k x = \lambda_k x$. Set $A = \oplus_{k=1}^n A_k$. One can see that

$$\sigma(A) = \bigcup_{k=1}^n \{\lambda_k\}$$

and for all $k \in \{1,\cdots,n\}$ and for each $\lambda \in \rho(A_k)$, $\|(\lambda - A_k)^{-1}\| = \frac{1}{|\lambda-\lambda_k|}$. Hence $\sigma_\varepsilon(A_k) = \{\lambda_k\} \cup B(\lambda_k, \varepsilon)$. Consequently,

$$\sigma_\varepsilon(A) = \bigcup_{k=1}^n \{\lambda_k\} \cup \bigcup_{k=1}^n B(\lambda_k, \varepsilon).$$

We introduce the following definition.

*Definition 17.* Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$. The condition pseudospectrum $\Lambda_\varepsilon(\oplus_{i=1}^n A_i)$ of $\oplus_{i=1}^n A_i$ on $\oplus_{i=1}^n X_i$ is defined by

$$\Lambda_\varepsilon(\oplus_{i=1}^n A_i) = \sigma(\oplus_{i=1}^n A_i) \cup \{\lambda \in \mathbb{K} : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| > \varepsilon^{-1}\},$$

with the convention $\sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| = \infty$ if $\lambda \in \sigma(\oplus_{i=1}^n A_i)$.

*Remark 3.* It is easy to see that $\bigcup\limits_{i=1}^n \Lambda_\varepsilon(A_i) \subset \Lambda_\varepsilon(\oplus_{i=1}^n A_i)$.

*Proposition 8.* Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, then

(i) $\sigma(\oplus_{i=1}^n A_i) = \bigcap\limits_{\varepsilon>0} \Lambda_\varepsilon(\oplus_{i=1}^n A_i)$.

(ii) If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma(\oplus_{i=1}^n A_i) \subset \Lambda_{\varepsilon_1}(\oplus_{i=1}^n A_i) \subset \Lambda_{\varepsilon_2}(\oplus_{i=1}^n A_i)$.

*Proof.* (i) By Definition 17, for each $\varepsilon > 0$, $\sigma(\oplus_{i=1}^n A_i) \subset \Lambda_\varepsilon(\oplus_{i=1}^n A_i)$. Conversely, if $\lambda \in \bigcap\limits_{\varepsilon>0} \Lambda_\varepsilon(\oplus_{i=1}^n A_i)$ and $\lambda \notin \sigma(\oplus_{i=1}^n A_i)$. Using $\lim\limits_{\varepsilon\to\to 0} \sup\limits_{i\in\{1,\cdots,n\}} \|(A_i-\lambda I)\| \sup\limits_{i\in\{1,\cdots,n\}} \|(A_i-\lambda I)^{-1}\| = \infty$, we get a contradiction.

(ii) For $0 < \varepsilon_1 < \varepsilon_2$. If $\lambda \in \Lambda_{\varepsilon_1}(\oplus_{i=1}^n A_i)$, thus for all $\sup\limits_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)\| \sup\limits_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| > \varepsilon_1^{-1} > \varepsilon_2^{-1}$, then $\lambda \in \Lambda_{\varepsilon_2}(\oplus_{i=1}^n A_i)$.

Let $A \in \mathcal{L}(X)$, set $r(A) = \lim\limits_{k\to\infty} \|A^k\|^{\frac{1}{k}}$. We have the following lemmas.

*Lemma 6.* Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, then $\sup\limits_{\lambda\in\sigma(\oplus_{i=1}^n A_i)} |\lambda| \leq \sup\limits_{\lambda\in\Lambda_\varepsilon(\oplus_{i=1}^n A_i)} |\lambda|$.

*Proof.* Since $\sigma(A) \subseteq \Lambda_\varepsilon(A)$, then $\sup\limits_{\lambda\in\sigma(\oplus_{i=1}^n A_i)} |\lambda| \leq \sup\limits_{\lambda\in\Lambda_\varepsilon(\oplus_{i=1}^n A_i)} |\lambda|$.

We introduce a new definition of the condition pseudospectrum of $\oplus_{i=1}^n A_i$ as follows.

*Definition 18.* Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$ and let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$. The condition pseudospectrum $\Lambda'_\varepsilon(\oplus_{i=1}^n A_i)$ of $\oplus_{i=1}^n A_i$ on $\oplus_{i=1}^n X_i$ is

$$\Lambda'_\varepsilon(\oplus_{i=1}^n A_i) = \sigma(\oplus_{i=1}^n A_i) \cup \{\lambda \in \mathbb{K} : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)\|\|(A_i - \lambda I)^{-1}\| > \varepsilon^{-1}\}.$$

*Remark 4.* (i) It is easy to see that $\Lambda'_\varepsilon(\oplus_{i=1}^n A_i) = \bigcup\limits_{i=1}^n \Lambda'_\varepsilon(A_i)$.

(ii) $\sigma(\oplus_{i=1}^n A_i) = \bigcap\limits_{\varepsilon>0} \Lambda'_\varepsilon(\oplus_{i=1}^n A_i)$.

(iii) If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma(\oplus_{i=1}^n A_i) \subset \Lambda'_{\varepsilon_1}(\oplus_{i=1}^n A_i) \subset \Lambda'_{\varepsilon_2}(\oplus_{i=1}^n A_i)$.

(iv) For all $\varepsilon > 0$, we have $\sup\limits_{\lambda\in\sigma(\oplus_{i=1}^n A_i)} |\lambda| \leq \sup\limits_{\lambda\in\Lambda'_\varepsilon(\oplus_{i=1}^n A_i)} |\lambda|$.

(v) The condition pseudospectum $\Lambda'_\varepsilon(\oplus_{i=1}^n A_i)$ of $\oplus_{i=1}^n A_i$ gives nice properties than $\Lambda_\varepsilon(\oplus_{i=1}^n A_i)$.

We finish with the following example.

*Example 5.* Consider $(A_k)_{1\leq k\leq n}$ defined on $\mathbb{K}^2$ by

$$A_k = \begin{pmatrix} 0 & \lambda_k \\ \lambda_k & 0 \end{pmatrix}$$

for all $k \in \{1, \cdots, n\}$ and $n \in \mathbb{N}$ is fixed. Then $\sigma(\oplus_{k=1}^n A_k) = \bigcup\limits_{k=1}^n \{-\lambda_k, \lambda_k\}$ and

$$\Lambda_\varepsilon(\oplus_{k=1}^n A_k) = \bigcup_{k=1}^n \{-\lambda_k, \lambda_k\} \cup \{\lambda \in \mathbb{K} : \sup_{1\leq k\leq n} \|\lambda I - A_k\| \sup_{1\leq k\leq n} \|(\lambda I - A_k)^{-1}\| > \frac{1}{\varepsilon}\}$$

where for all $k \in \{1, \cdots, n\}$, $\|\lambda - A_k\| = \max\{|\lambda|, |\lambda_k|\}$ and for all $\lambda \in \rho(A_k)$, $\|(\lambda I - A_k)^{-1}\| = \max\left\{\frac{|\lambda|}{|\lambda^2-\lambda_k^2|}, \frac{|\lambda_k|}{|\lambda^2-\lambda_k^2|}\right\}$.

*Conflict of Interest*

The author declares that there are no conflict of interest.

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