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Spectrum and resolvent of multi-channel systems with internal energies and common boundary conditions

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In the article the spectrum and resolvent of the so-called multichannel systems with nonzero internal energies were investigated. The spectrum and resolvent of multichannel Sturm-Liouville systems with non-zero internal energies m_i^2 and general boundary conditions were investigated. These systems describe the propagation of partial waves in the theory of quantum physics. The importance of studying the spectral characteristics of these systems is presented in the well-known books of the theory of quantum physics. The finiteness of the number of eigenvalues was proved, the multiplicity of positive eigenvalues was investigated, and as well as the resolvent kernel of the system was found.

Keywords: operator, eigenvalues, edge problem, Wronskian, transformation operator, asymptotics, continuous spectrum, resolvent, multi-channel systems, internal energy, quantum physics.

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Introduction

The spectrum and resolvent of $0 = m_1^2 \leq m_2^2 \leq \ldots \leq m_n^2 = m^2$ multilayer systems with non-zero internal energies are studied in the present article. These kinds of systems are described by differential equations

$$-y_i^{''} + \sum_{j=1}^{n} q_{ij}(x)y_j + m_i^2 y_i = \lambda^2 y_i,\ i = \overline{1,n},\ 0 \leq x < \infty$$

and boundary conditions

$$y_i^{'}(0) - \sum_{j=1}^{n} h_{ij} y_j(0) = 0,\ i = \overline{1,n}.$$

This system can be rewritten in the next form

$$-y^{''} + Q(x)y + My = \lambda^2 y,\ 0 \leq x < +\infty, \quad (1)$$

$$y^{'}(0) - Hy(0) = 0, \quad (2)$$

where $Q(x) = \{q_{ij}(x)\}$ $(i,j = \overline{1,n},\ 0 \leq x < +\infty)$ is a semi-continuous matrix-function, $M = \{\delta_{ij} m_i^2\}_1^n$ is diagonally constant matrix, H is a self-constructed constant matrix, $y(x)$ is a column vector-function. Assume that the Euclidean norm of the matrix function $Q(x)$ satisfies the following condition

$$\int_0^{+\infty} x e^{mx} \|Q(x)\| dx < +\infty. \quad (3)$$

Boundary value problem (1)-(2) occurs in the theory of dispersion multichannel particles with nonzero inner energy $m_i^2, i = \overline{1,n}$ and it describes the spread of partial waves.

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Notice that for the condition $y(0) = 0$ this problem was investigated in the works [1], [2]. In recent years, many studies have been conducted on this issue [3–7].

Let us consider the diagonal matrix

$$K(\lambda) = \left(\lambda^2 I - M\right)^{\frac{1}{2}} = \{\delta_{ij} K_j(\lambda)\}, K_j(\lambda) = \sqrt{\lambda^2 - m_j^2},$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

Here we define branches so that at $\operatorname{Im} \lambda K_j(\lambda) > 0$, and at $\operatorname{Im} \lambda = 0$, $K_j(\lambda) = \lim_{\varepsilon \to +0} k_j(\lambda + i\varepsilon)$. Hence, if $\operatorname{Im} \lambda > 0$, we have

$$K_j(\lambda) = \begin{cases} \sqrt{m_j^2 - \lambda^2}, \ if \ |\lambda| \leq m_j, \\ \lambda \sqrt{1 - \frac{m_j^2}{\lambda^2}} \ if \ |\lambda| \geq m_j. \end{cases} \tag{4}$$

Note that for $|\lambda| \leq m_j$ the functions $K_j(\lambda)$ are even, and for $|\lambda| \geq m_j$ they are odd. It is not difficult to verify that, when $\operatorname{Im} \lambda > 0$, $i < j$

$$0 \leq \operatorname{Im} K_j(\lambda) - \operatorname{Im} K_i(\lambda) \leq \sqrt{m_j^2 - m_i^2} \leq m.$$

Therefore,

$$\|\exp(iK(\lambda)x)\| = \exp(-\operatorname{Im} \lambda x), \ \|\exp(-iK(\lambda)x)\| = \exp(\operatorname{Im} K_n(\lambda)x).$$

Denote by $L_2((0,\infty); E_n)$ the Hilbert space of the column vector-functions $y(x) = \{y_i(x)\}, i = \overline{1,n}$ of the quadratic integrable on the semiaxis $(0, +\infty)$ of all the components, in which the inner product is defined by the formula

$$\langle y, z \rangle = \sum_{k=1}^{n} y_k(x) \overline{z}_{k(x)} dx = \int_0^{+\infty} z^*(x) y(x) dx.$$

Denote by $[y(x), z(x)]$ the Wronskian of the differentiable matrix functions which is defined by the formula, where the transposition of the matrix means $y(x)$ and $z(x)$,

$$[y(x), z(x)] = \widetilde{y}(x) z'(x) - \widetilde{y}'(x) \ z(x),$$

where $\widetilde{y}$ is the transpose of the matrix y.

1 *Solutions at* $F(x,\lambda), \Phi(x,\lambda)$ *and connections between them at* $\operatorname{Im} \lambda = 0, \ |\lambda| > m$

Consider the matrix equation

$$-y'' + Q(x)yMy = \lambda^2 y, \ 0 \leq x < +\infty. \tag{5}$$

In the case of $Q(x) = 0$, this equation has a solution

$$e(x,\lambda) = \exp(iK(\lambda)x) = \left\{\delta_{\alpha j} e^{ikj(\lambda)x}\right\}, \ \alpha, j = \overline{1,n}.$$

In [1] it is proved that if the condition (3) is satisfied, then the equation (5) will have analytic solution λ in the upper half-scope $\operatorname{Im} \lambda > 0$ and the continuous up to the real axis $\operatorname{Im} \lambda = 0$, and the solution is $F(x,\lambda)$ satisfying the condition

$$\lim_{x \to +\infty} e(x,\lambda) F(x,\lambda) = I,$$

and there is a core $K(x,\lambda)$ of conversion operator that is

$$F(x,\lambda)=e(x,\lambda)+\int_x^{+\infty}K(x,t)\,e(t,\lambda)\,dt. \tag{6}$$

Further, the following assessments are valid (see [2]):

$$\|F(x,\lambda)-\exp(iK(x))\|\le C_1\exp(-(m+\operatorname{Im}\lambda)x),$$

$$\|F(x,\lambda)-\exp(iK(\lambda)x)\|\le C_2\,\left\|K^{-1}(\lambda)\right\|\exp(-(m+\operatorname{Im}\lambda)x),$$

where $C_{1,}C_2$ are constants.

Let us denote by $\Phi(x,\lambda)$ the solution of differential equation (5) satisfying initial conditions

$$\Phi(0,\lambda)=I,\Phi'(0,\lambda)=H. \tag{7}$$

It is known that the solution $\Phi(x,\lambda)$ of the boundary value problem (5)–(7) is an integer function of the parameter λ. Obviously, $\Phi(x,\lambda)$ is an even function of parameter λ.

If $\operatorname{Im}\lambda=0,\ |\lambda|>m$, then solutions $F(x,\lambda)$ and $F(x,-\lambda)$ of equation (5) are linearly independent (see [2]) and

$$[F(x,\lambda),F(x,-\lambda)]=2iK(\lambda).$$

Then, there are $A(\lambda),B(\lambda)$ matrices which independent of x such that

$$\Phi(x,\lambda)=F(x,\lambda)A(\lambda)+F(x,-\lambda)B(\lambda) \tag{8}$$

at $\operatorname{Im}\lambda=0,\ |\lambda|>m$. Hence,

$$[F(x,\lambda),\Phi(x,\lambda)]=-2iK(\lambda)B(\lambda),$$

$$[F(x,-\lambda),\Phi(x,\lambda)]=2iK(\lambda)A(\lambda).$$

Since

$$[F(x,\lambda),\Phi(x,\lambda)]=\widetilde{F}(0,\lambda)\Phi'(0,\lambda)-\widetilde{F'}(0,\lambda)\Phi(0,\lambda)=\widetilde{F}(0,\lambda)H-\widetilde{F'}(0,\lambda),$$

we get

$$A(\lambda)=\frac{1}{2i}K^{-1}(\lambda)-\widetilde{W}(-\lambda),\ B(\lambda)=-\frac{1}{2i}K^{-1}(\lambda)-\widetilde{W}(\lambda),$$

where

$$W(\lambda)=HF(0,\lambda)-F'(0,\lambda). \tag{9}$$

Substituting these expressions for matrices $A(\lambda)$ and $B(\lambda)$ into formula (8), we obtain

$$\Phi(x,\lambda)=\frac{1}{2i}(F(x,\lambda)K^{-1}(\lambda)\widetilde{W}(-\lambda)-F(x,-\lambda)K^{-1}(\lambda)\widetilde{W}(\lambda)) \tag{10}$$

for the case $\operatorname{Im}\lambda=0,\ |\lambda|>m$.

Lemma 1. If $\operatorname{Im}\lambda=0,\ |\lambda|>m$ then the matrix $W(\lambda)$ is non-singular.

Proof. From the condition $\Phi(0,\lambda)=0$ and equality (10), it follows that

$$F(0,\lambda)\ K^{-1}(\lambda)\widetilde{W}(-\lambda)-F(0,-\lambda)K^{-1}(\lambda)\widetilde{W}(\lambda)=2iI. \tag{11}$$

Due to (4) we have $e(x,-\lambda)=\overline{e(x,\lambda)}$. Similarly, from the equations (6) and (9) we have $W(-\lambda)=\overline{W(-\lambda)}$. Assume that there is a vector $\vec{a}$, such that $\widetilde{W}(\lambda)\vec{a}=0$. Then, $\widetilde{W}(-\lambda)\vec{a}=0$

and from (11) it follows that $2i\ I\ \vec{a} = 0,\ \vec{a} = 0$. The matrix is consistent $\widetilde{W}(\lambda)$ and it means that $W(\lambda)$ is not singular.

From formula (10) and boundary conditions (7) it follows that

$$F(0,\lambda) K^{-1}(\lambda) \widetilde{W}(-\lambda) - F(0,-\lambda) K^{-1}(\lambda) \widetilde{W}(\lambda) = 2iI,$$

$$F'(0,\lambda) K^{-1}(\lambda) \widetilde{W}(-\lambda) - F'(0,-\lambda) K^{-1}(\lambda) \widetilde{W}(\lambda) = 2i.$$

Multiplying the first equation by H and subtracting the second equation, we obtain the next formula

$$W(\lambda) K^{-1}(\lambda) \widetilde{W}(-\lambda) = W(-\lambda) K^{-1}(\lambda) \widetilde{W}(\lambda). \tag{12}$$

Multiplying equation (12) by $W^{-1}(\lambda)$ from the left and by $\widetilde{W^{-1}}(-\lambda)$ from the right, we get the formula

$$W^{-1}(-\lambda) W(\lambda) K^{-1}(\lambda) = K^{-1}(\lambda) \widetilde{W}(\lambda) \widetilde{W^{-1}}(-\lambda). \tag{13}$$

Under conditions $\operatorname{Im}\lambda = 0, |\lambda| > m$, by using equations (10) and (13), the solution $\Phi(x,\lambda)$ is presented as follows

$$\Phi(x,\lambda) = \tfrac{1}{2i}(F(x,\lambda) K^{-1}(\lambda) - F(x,-\lambda) K^{-1}(\lambda) \widetilde{W}(\lambda) \widetilde{W^{-1}}(-\lambda))\widetilde{W}(-\lambda)$$

$$= \tfrac{1}{2i}\left(F(x,\lambda) K^{-1}(\lambda) - F(x,-\lambda) W^{-1}(-\lambda) W(\lambda) K^{-1}(\lambda)\right) \widetilde{W}(-\lambda),$$

or

$$\Phi(x,\lambda) = \frac{1}{2i}\left(F(x,\lambda) - F(x,-\lambda) W^{-1}(-\lambda) W(\lambda)\right) K^{-1}(\lambda) \widetilde{W}(-\lambda).$$

In the future, we will need asymptotic behavior of solution $\Phi(x,\lambda)$, when $\lambda \to \infty$ in the case $\operatorname{Im}\lambda \geq 0$. Denote by $E(x,\lambda)$, the solution of equation (5) whose asymptotic, when $\lambda \to \infty$. In the case $\operatorname{Im}\lambda \geq 0$ it is derived by (see [2])

$$E(x,\lambda) = \exp(-iK(x)\lambda)(I + O(1)).$$

Since

$$[F(x,\lambda), E(x,\lambda)] = \lim_{x\to\infty}[c] = -2iK(\lambda),$$

the solutions $F(x,\lambda)$ and $E(x,\lambda)$ are linearly independent in the case $\operatorname{Im}\lambda \geq 0,\ \lambda \neq m_j^2,\ j = \overline{1,n}$. Thus it is easy to get

$$\Phi(x,\lambda) = F(x,\lambda) A + E(x,\lambda) B, \tag{14}$$

where

$$A = \frac{1}{2i} K^{-1}(\lambda) \left(\widetilde{E}(0,\lambda) H - \widetilde{E}'(0,\lambda)\right),\ B = \frac{1}{2i} K^{-1}(\lambda)\ \widetilde{W}(\lambda)$$

with

$$\widetilde{E}(0,\lambda) H - \widetilde{E}'(0,\lambda) = \{C_{ij}(\lambda)\},\ i,j = \overline{1,n},\ \widetilde{W}(\lambda) = \{w_{ij}(\lambda)\},\ i,j = \overline{1,n}.$$

Using asymptotic formula (14) for solutions $F(x,\lambda)$ and $E(x,\lambda)$ when $x \to \infty$, we obtain the following asymptotic formulas

$$\varphi_{\alpha j}(x,\lambda) = \frac{1}{2iK_\alpha(\lambda)}\left(C_{j\alpha}(\lambda) e^{iK_\alpha(\lambda)x} - w_{j\alpha}(\lambda) e^{-iK_\alpha(\lambda)x}\right) + O(1) \tag{15}$$

for elements of the matrix $\Phi(x,\lambda) = \{\varphi_{\alpha j}(x,\lambda)\}$, when $x \to \infty$.

2 *On the resolvent of problem (1),(2)*

In the space $L_2(0,\infty;E_n)$, the boundary value problem (1), (2) defines the differential operator by

$$l(y) = -y'' + Q(x)y + My,\ 0 \le x < +\infty$$

for y which satisfies the condition

$$y'(0) - Hy(0) = 0.$$

The domain D_L of the operator L contains vector-functions $y(x) \in L_2(0,\infty;E_n)$, satisfying the following conditions:

1. $y'(x)$ exists and absolutely continuous at finite interval $[0,a]$,
2. $y'(0) - Hy(0) = 0$,
3. $l(y) \in L_2((0,\infty);E_n)$.

It is not difficult to verify that the operator L self adjoint. Let us define the core $R_z(x,t)$ of resolvent $R_z = (L - ZI)^{-1}$. Let us solve the boundary value problem

$$-y'' + Q(x)y + My = \lambda^2 y + f(x), \tag{16}$$

$$y'(0) - Hy(0) = 0, \tag{17}$$

where $f(x)$ is an arbitrary vector functions in $L_2((0,\infty);E_n)$.

If

$$[F(x,\lambda), \Phi(x,\lambda)] = [F(x,\lambda), \Phi(x,\lambda)]_{x=0} = \widetilde{F}(0,\lambda)H - \widetilde{F}'(0,\lambda) = \widetilde{W}(\lambda),$$

then the solutions of the homogeneous equation (1) are linearly independent in the case $W(\lambda) \neq 0$. We seek the solution $y(x,\lambda)$ of the problem (16), (17) in the form

$$y(x,\lambda) = F(x,\lambda)C_1(x,\lambda) + \Phi(x,\lambda)C_2(x,\lambda), \tag{18}$$

where $C_1(x,\lambda)$ and $C_2(x,\lambda)$ are some vector-functions. Applying constant variation method, we get a system of equations

$$F(x,\lambda)C_1'(x,\lambda) + \Phi(x,\lambda)C_2'(x,\lambda) = 0,$$

$$F'(x,\lambda)C_1'(x,\lambda) + \Phi'(x,\lambda)C_2'(x,\lambda) = -f(x).$$

Solving it, we have

$$C_1'(x,\lambda) = W(\lambda)\widetilde{\Phi}(x,\lambda)f(x), \tag{19}$$

$$C_2'(x,\lambda) = -\widetilde{W^{-1}}(\lambda)\widetilde{F}(x,\lambda)f(x). \tag{20}$$

From the asympotic equation (15) it follows that the elements of the matrix function $\Phi(x,\lambda)$ do not belong to space $L_2(0,\infty)$. Consequently, from condition $y(x,\lambda) \in L_2((0,\infty);E_n)$ and equality (20) follows that

$$C_2(+\infty,\lambda) = \lim_{x\to+\infty} C_2(x,\lambda) = 0.$$

Since

$$y'(x,\lambda) = F'(x,\lambda)C_1(x,\lambda) + \Phi'(x,\lambda)C_2(x,\lambda),$$

by using condition (19), we have

$$\left[F'(0,\lambda) - HF(0,\lambda)\right]C_1(0,\lambda) = 0.$$

So $W(\lambda)C_1(0,\lambda) = 0$. Since $detW(\lambda) \neq 0$, we obtain $C_1(x,\lambda) = 0$. Now, by integrating equality (19) from 0 to x, we get

$$C_1(x,\lambda) = W^{-1}(\lambda)\int\limits_0^x \widetilde{\Phi}(t,\lambda) f(t)dt.$$

By using (20), we get

$$C_2(x,\lambda) = W^{-1}(\lambda)\int\limits_x^{+\infty} \widetilde{F}(t,\lambda) f(t)dt.$$

Substituting these expressions in equality (18), we have

$$y(x,\lambda) = R_Z f = \int\limits_0^{+\infty} R_z(x,\lambda) f(t)dt, Z = \lambda^2,$$

where

$$R_z(x,\lambda) = \begin{cases} F(x,\lambda) W^{-1}(\lambda)\widetilde{\Phi}(t,\lambda), t \le x, \\ \\ \Phi(x,\lambda)\widetilde{W^{-1}}(\lambda)\widetilde{F}(t,\lambda), t \ge x. \end{cases} \tag{21}$$

This is the resolvent of operator L.

3 Spectrum of boundary value problem (1),(2)

Since boundary value problem (1),(2) is self-adjoint, from the expression (21) of the kernel R_z(x,t) the resolvents follow that the eigenvalues of the problem are squares of the scalar function $\omega(\lambda) = detw(\lambda)$ and has no other eigenvalues. Since the eigenvalues of the problem (1),(2) are real, the function, $\omega(\lambda) = detw(\lambda)$ can only be zeros on the real and the imaginary axis of the complex plane.

Theorem 1. The boundary value problem (1), (2) has

a) only the finite number of simple negative eigenvalues $-\varkappa_1^2, -\varkappa_{2,}^2 \ldots -\varkappa_q^2$,
b) the finite number of positive eigenvalues $\lambda_1^2, \lambda_{2,}^2 \ldots \lambda_r^2$ from the interval $\left[0, m^2\right]$, the multiplicity of eigenvalue λ_j^2 from the interval $\left(m_p^2, m_{p+1}^2\right)$, $p = \overline{1,n}$, is not greater than $n-p$ and coincides with the rank of the matrix $W(\lambda, j)$.

Boundary value problem (1), (2) does not have its own values $Z > m^2$ and the continuous spectrum fills the semi-axiss.

Proof. Let $\lambda_{k,}^2 = -\varkappa_k^2$ be the eigenvalue of problem (1), (2) , i.e. $\omega(\varkappa_k) = detW(i\varkappa_k) = 0$. Then, it has a vector such that

$$W(i\varkappa_k)\overrightarrow{a}^{(k)} = HF(0,i\varkappa_k)\overrightarrow{a}^{(k)} - F'(0,i\varkappa_k)\overrightarrow{a}^{(k)} = 0.$$

From that it follows that the vector function $y_k(x) = F(x,i\varkappa_k)\overrightarrow{a}^{(k)}$ is a solution of problem (1)-(2). On the other hand, the elements of the matrix function $F(i\varkappa_k)$ belong to space $L_2(0,+\infty)$. Therefore, the vector-function $y_k(x)$ is the eigenfunction of the edge problem (1)-(2) corresponding to the eigenvalue $-\varkappa_k^2$. Without loss of generality, we will assume that the first component of $\overrightarrow{a}^{(k)}$equals to one and $m_i \neq m_j$ for $i \neq j$. From the asymptotic solution $F(x,\lambda)$, when $x \to +\infty$, it follows that

$$y_k(x) = e^{-\varkappa_k x}\omega_k(x), \lim_{x\to+\infty}\omega_k(x) = (1,0,\ldots,0) \tag{22}$$

uniformly along k.

Denoting by δ an exact lower bound of distances between two neighboring negative eigenvalues and we will prove that $\delta > 0$. Let us $\delta = 0$. Then, we can isolate the sequence of negative eigenvalues $\left\{-\varkappa_k^2\right\}$ and $\left\{-\widehat{\varkappa}_k^2\right\}$ such that $\lim\limits_{k\to+\infty}(\widehat{\varkappa}_k - \varkappa_k) = 0, \widehat{H}_k > H_k \geq 0$. Asymptotics $F(x,\lambda)$, implies that the set of zeros of the function $\omega(\lambda)$ is bounded. So $max_k\{\varkappa_k\} < A$. Later

$$\int_x^{+\infty} y_k(t)\,\widehat{y}_k(t)\,dt = \int_x^{+\infty} \widetilde{\omega}_k(t)\,\widehat{\omega}_k(t)\,e^{-(\varkappa_k+\widehat{\varkappa}_k)}dt, \tag{23}$$

where $\widehat{y}_k(x) = F(x, i\widehat{\varkappa}_k)\overrightarrow{b}^{(k)}$ the eigenfunction of the boundary problem (1),(2) is corresponding to the eigenvalue $-\widehat{\varkappa}_k^2$ and $\widehat{y}_k(x) = e^{-\widehat{\varkappa}_k x}\, x\,\widehat{\omega}_k(x)$. The condition (21) implies that, if $x > x_o$ is sufficient uniformly along k, then $\omega_k(x)\,\widehat{\omega}_k(x) > \frac{1}{2}$. Now, from (21) it follows that

$$\int_x^{+\infty} y_k(t)\,\widehat{y}_k(t)\,dt > \frac{1}{2}\int_{x_0}^{+\infty} e^{-(\varkappa_k+\widehat{\varkappa}_k)}dt = \frac{e^{-(\varkappa_k+\widehat{\varkappa}_k)}x_0}{2(\varkappa_k+\widehat{\varkappa}_k)} > \frac{e^{-Ax_0}}{4A}.$$

Since the boundary value problem (1)-(2) is self-adjoint, the vector-functions $y_k(x)$ and $\widehat{y}_k(x)$. Moreover

$0 = \int_x^{+\infty} y_k(t)\,\widehat{y}_k(t)\,dt = \int_0^{x_0}((y_k(t) - \widehat{y}_k(t))\widehat{y}_k(t)\,dt + \int_0^{x_0} y_k(t)\,\widehat{y}_k(t)\,dt + \int_x^{+\infty} y_k(t)\,\widehat{y}_k(t)\,dt.$
Passing to limit to the limit $k \to +\infty$, we find

$$0 = \lim_{k\to+\infty}\int_0^{x_0} y_k(t)\,\widehat{y}_k(t)\,dt + \lim_{k\to+\infty}\int_{x_0}^{+\infty} y_k(t)\,\widehat{y}_k(t)\,dt.$$

Thus,

$$\lim_{k\to+\infty}\int_{x_0}^{+\infty} y_k(t)\,,\widehat{y}_k(t)\,dt \leq 0.$$

Inequalities (22) and (23) lead to contradiction. Hence, $\delta > 0$, and it means, that the number of negative eigenvalues are finite. Now, let $\lambda^2 \in (m_p^2, m_{p+1}^2)$ be the eigenvalue of problem (1), (2). The corresponding eigenfunction has the form $\varphi(\lambda) = F(x,\lambda)\overrightarrow{a}$, $\overrightarrow{a} \neq a$. For $\lambda^2 \in (m_p^2, m_{p+1}^2)$ from formula (4) it follows that

$$ik_j(\lambda) = \begin{cases} i\lambda\sqrt{1 - \frac{m_j^2}{\lambda^2}}, if \quad j = 1, 2, \ldots, p, \\ -\sqrt{m_j^2 - \lambda^2}, if \quad j = p+1, \ldots, n. \end{cases}$$

Therefore, the elements of the first columns of the matrix function $F(x,\lambda)$ do not belong to space $L_2(0,\infty)$ and elements of the last $n-p$ columns belong to the space $L_2(0,+\infty)$. It is, because the eigenfunctions $\varphi(x)\,\epsilon L_2((0,+\infty); E_n)$ of the first p coordinates of the vector $\overrightarrow{a} = (a_1, a_2, \ldots, a_n)$ are zero, i.e. $a_1 = a_2 = \ldots = a_p = 0$. On the other hand, the eigenvector function $\varphi(x) = F(x,\lambda)\overrightarrow{a}$ satisfies the condition (2). Therefore, we have

$$a_{p+1}\omega_{j(p+1)}(\lambda) + \ldots + a_n\omega_{jn}(\lambda) = 0, j = 1, 2, \ldots, n,$$

where at least one of the numbers $a_{p+1}, \ldots, a_n$ is not zero. Therefore, the last $n-r$ columns of the matrix $W(\lambda) = \{\omega_{ij}(\lambda)\}_1^n$ are linearly independent, and therefore the multiplicity of eigenvalues $\lambda^2\epsilon(m_p^2, m_{p+1}^2)$ coincides with the rank of the matrix $W(\lambda)$. The finiteness of the number of eigenvalues from the interval $\left[0, m^2\right]$ is proved similarly to the case of negative eigenvalues. According to $\lambda^2\epsilon(m_1^2, +\infty)$ by Lemma 1 we get $detW(\lambda) \neq 0$. (1)-(2) does not have eigenvalues $(m_1^2, +\infty)$ from the interval. In the complex plane Z, the cut along the positive part of the real axis is a feature of the matrix function $W(\sqrt{z})$ and means the resolvent R_z by the formula (21). Hence, the half-axis $[0, +\infty]$ is the continuous spectrum of the boundary value problem (1)-(2).

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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