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Source identification problems for the neutron transport equations

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In this study, the time-dependent source identification problem for the two-dimensional neutron transport equation was studied. For the approximate solution of this problem a first order of accuracy difference scheme was presented. Stability estimates for the solution of these differential and difference problems were established. Numerical results were given.

Keywords: identification problem, neutron transport equation, difference scheme, differential equation, stability inequality.

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Introduction

The neutron transport equation describes the distribution of neutrons in terms of their positions in space and time, their energies and their travel directions. The various neutron transport equations are studied by many researchers (see, [1–4] and the references given therein). Identification problems play an important role in applied sciences and engineering applications and have been investigated in various papers (see, e.g., [5–27] and the references given therein). In the present paper, we consider the time-dependent source identification problem for two dimensional neutron transport equation

$$\begin{cases} \frac{\partial u(t,x,y)}{\partial t} = \frac{\partial u(t,x,y)}{\partial x} + \frac{\partial u(t,x,y)}{\partial y} + p\left(t\right)q\left(x,y\right) + f\left(t,x,y\right), \\ t \in (0,T)\,, \ x,\, y \in (0,L)\,, \\ u\left(0,x,y\right) = \varphi\left(x,y\right), \quad x,\, y \in [0,L]\,, \\ u\left(t,0,y\right) = 0, \quad u\left(t,x,0\right) = 0, \quad t \in [0,T]\,, \ x,\, y \in [0,L]\,, \\ u(t,l,y) = \alpha\left(t,y\right), \quad t \in [0,T]\,, \quad y \in [0,L]\,, \quad l \in (0,L]\,. \end{cases} \tag{1}$$

Here, $u\left(t,x,y\right)$ and $p\left(t\right)$ are unknown functions, $f\left(t,x,y\right)$, $q\left(x,y\right)$, $\varphi\left(x,y\right)$, and $\alpha\left(t,y\right)$ are given sufficiently smooth functions and all compatibility conditions are satisfied.

In the rest of paper, the theorem on the stability of differential problem (1) is established. For the approximate solution of problem (1), a first order of accuracy difference scheme is proposed. The theorem on stability of this difference scheme is established. Some results of numerical experiment are presented.

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1 *Stability of differential equation*

To formulate our results, we introduce the Banach space $C(E) = C([0,T], E)$ of all abstract continuous functions $\phi(t)$ defined on $[0,T]$ with values in E equipped with the norm

$$\|\phi\|_{C(E)} = \max_{0\le t\le T} \|\phi(t)\|_E .$$

Let $E = C_{[0,L]\times[0,L]}$ be the space of all continuous functions $\psi(x,y)$ defined on $[0,L]\times[0,L]$ equipped with norm

$$\|\psi\|_{C_{[0,L]\times[0,L]}} = \max_{0\le x,y\le L} |\psi(x,y)|$$

and $C^{(1)}_{[0,L]\times[0,L]}$ be the space of all continuously differentiable functions $\psi(x,y)$ defined on $[0,L]\times[0,L]$ equipped with norm

$$\|\psi\|_{C^{(1)}_{[0,L]\times[0,L]}} = \|\psi\|_{C_{[0,L]\times[0,L]}} + \max_{0<x,y<L} |\psi_x(x,y)| + \max_{0<x,y<L} |\psi_y(x,y)| .$$

We introduce the positive operator A, defined by formula

$$Au = -\left(\frac{\partial u(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\right)$$

with the domain

$$D(A) = \left\{u : u, u_x, u_y \in C_{[0,L]\times[0,L]},\ u(0,y) = u(x,0) = 0,\ 0\le x,y\le L\right\}.$$

Throughout the present paper, M denotes positive constants, which may differ in time and thus are not a subject of precision. However, we will use $M(\alpha,\ \beta,\ \gamma,...)$ to stress the fact that the constant depends only on $\alpha,\ \beta,\ \gamma,...$.

We have the following theorem on the stability of problem (1):

Theorem 1. Assume that $\varphi \in C^{(1)}_{[0,L]\times[0,L]}$, $f(t,x,y)$ is a continuously differentiable function in t and continuous in x and y, and $\alpha(t,y)$ is a continuously differentiable function in t and continuous in y. Then, for the solution of problem (1) the following stability estimates hold:

$$\left\|\frac{\partial u}{\partial t}\right\|_{C\left(C_{[0,L]\times[0,L]}\right)} + \|u\|_{C\left(C^{(1)}_{[0,L]\times[0,L]}\right)} + \|p\|_{C[0,T]} \le M(q)\left[\|\varphi\|_{C^{(1)}_{[0,L]\times[0,L]}} + \right.$$

$$\left. + \|f(0,.)\|_{C_{[0,L]\times[0,L]}} + \left\|\frac{\partial f}{\partial t}\right\|_{C\left(C_{[0,L]\times[0,L]}\right)} + \|\alpha(0,\cdot)\|_{C[0,L]} + \|\alpha_t\|_{C(C[0,L])}\right].$$

Proof. We will use the following substitution

$$u(t,x,y) = w(t,x,y) + \eta(t)\, q(x,y),$$

where $\eta(t)$ is the function defined by formula

$$\eta(t) = \int_0^t p(s)\,ds,\ \eta(0) = 0. \tag{2}$$

It is clear that $w(t,x,y)$ is the solution of the following initial boundary value problem

$$\begin{cases} \frac{\partial w(t,x,y)}{\partial t} = \frac{\partial w(t,x,y)}{\partial x} + \frac{\partial w(t,x,y)}{\partial y} + \eta(t)\left(q_x(x,y) + q_y(x,y)\right) + f(t,x,y), \\ t \in (0,T),\ x,y \in (0,L), \\ w(0,x,y) = \varphi(x,y),\ x,y \in [0,L], \\ w(t,0,y) = 0,\ t \in [0,T],\ y \in [0,L], \\ w(t,x,0) = 0,\ t \in [0,T],\ x \in [0,L], \\ q(x,0) = 0,\ q(0,y) = 0,\ q(l,y) \neq 0, \\ w(t,l,y) = \alpha(t,y) - \eta(t)\, q(l,y),\ t \in [0,T],\ y \in [0,L]. \end{cases} \tag{3}$$

Applying the over determined condition $u(t,l,y) = \alpha(t,y)$ at substitution (2), we get

$$w(t,l,y) + \eta(t)\, q(l,y) = \alpha(t,y),$$

$$\eta(t) = \frac{\alpha(t,y) - w(t,l,y)}{q(l,y)}.$$

From that and $p(t) = \eta'(t)$, it follows

$$p(t) = \frac{\alpha_t(t,y) - w_t(t,l,y)}{q(l,y)}. \tag{4}$$

From identity (4) and the triangle inequality, we get the estimate

$$|p(t)| = \left|\frac{\alpha_t(t,y) - w_t(t,l,y)}{q(l,y)}\right| \le M(q)\left[|\alpha_t(t,y)| + |w_t(t,l,y)|\right] \le$$

$$\le M(q)\left[\max_{0\le t\le T} |\alpha_t(t,y)| + \max_{0\le t\le T}\max_{0\le y\le L} |w_t(t,l,y)|\right].$$

From that it follows

$$\|p\|_{C[0,T]} \le M(q)\left[\|\alpha_t\|_{C(C[0,T],\ C[0,L])} + \|w_t\|_{C(C[0,T],\ C[0,L])}\right]. \tag{5}$$

Using operator A with the domain $D(A)$ we can rewrite problem (3) in the abstract form as an initial value problem

$$\begin{cases} \frac{dw}{\partial t} + Aw = -\frac{\alpha(t,.) - w(t,l,.)}{q(l,.)} Aq + f(t), \\ w(0) = \varphi. \end{cases}$$

By the Cauchy formula, the solution can be written as

$$w(t) = e^{-tA}\varphi + \int_0^t e^{-(t-s)A}\left\{-\frac{\alpha(s,.) - w(s,l,.)}{q(l,.)} Aq + f(s)\right\} ds.$$

Taking derivative with respect to t and using Leibniz integral rule, we obtain

$$w_t(t) = -Ae^{-tA}\varphi + \left\{-\tfrac{\alpha(t,.)-w(t,l,.)}{q(l,.)}Aq + f(t)\right\} + \int\limits_0^t -Ae^{-(t-s)A}\left\{-\tfrac{\alpha(s,.)-w(s,l,.)}{q(l,.)}Aq + f(s)\right\}ds.$$

Applying the integration by parts formula, we get

$$w_t(t) = -Ae^{-tA}\varphi + e^{-tA}\left\{-\tfrac{\alpha(0,.)-w(0,l,.)}{q(l,.)}Aq + f(0)\right\} +$$

$$+\int\limits_0^t e^{-(t-s)A}\left\{-\tfrac{\alpha_s(s,.)-w_s(s,l,.)}{q(l,.)}Aq + f'(s)\right\}ds = \sum_{k=1}^{3} G_k(t),$$

where

$$G_1(t) = -Ae^{-tA}\varphi,$$

$$G_2(t) = e^{-tA}\left\{-\tfrac{\alpha(0,.)-w(0,l,.)}{q(l,.)}Aq + f(0)\right\},$$

$$G_3(t) = \int\limits_0^t e^{-(t-s)A}\left\{-\tfrac{\alpha_s(s,.)-w_s(s,l,.)}{q(l,.)}Aq + f'(s)\right\}ds.$$

Now, we estimate, $G_1,\ G_2,$ and G_3, separately. Using the triangle inequality, we obtain

$$\|w_t\|_E \le \|G_1(t)\|_E + \|G_2(t)\|_E + \|G_3(t)\|_E.$$

It is known (see [20]) that for any $t \in [0,T]$,

$$\left\|e^{-tA}\right\|_{E\to E} \le Me^{-\delta t},\ M>0,\ \delta>0. \tag{6}$$

Applying the definition of norm of the spaces E and estimate (6), we get

$$\|G_1(t)\|_E = \left\|-Ae^{-tA}\varphi\right\|_E \le \left\|e^{-tA}\right\|_{E\to E}\|A\varphi\|_E \le M_1(\delta)\|A\varphi\|_E. \tag{7}$$

Let us estimate $G_2(t)$. Using the triangle inequality, we get

$$\|G_2(t)\|_E = \left\|e^{-tA}\left\{-\frac{\alpha(0,.)-w(0,l,.)}{q(l,.)}Aq + f(0)\right\}\right\|_E \le$$

$$\le \left\|e^{-tA}\right\|_{E\to E}\left[\left[\left|\frac{\alpha(0,.)}{q(l,.)}\right| + \left|\frac{w(0,l,.)}{q(l,.)}\right|\right]\|Aq\|_E + \|f(0)\|_E\right],$$

$$\|G_2(t)\|_E \le \left\|e^{-tA}\right\|_{E\to E}\left\{\|Aq\|_{E\to E}\frac{\|\alpha(0,.)\|_E + \|w(0,l,.)\|_E}{\min\limits_{0\le y\le L}|q(l,.)|} + \|f(0)\|_E\right\}.$$

Hence,

$$\|G_2(t)\|_E \le M_2(\delta,q)\left[\|\alpha(0,.)\|_E + \|\varphi\|_E + \|f(0)\|_E\right] \tag{8}$$

for any $t,\ t\in[0,T]$.

Let us estimate $G_3(t)$. Using the triangle inequality, we get

$$\begin{aligned}\|G_3(t)\|_E &\le \int\limits_0^t \left\|e^{-(t-s)A}\right\|_{E\to E}\left\{\tfrac{\max\limits_{0\le s\le T}|\alpha_s(s,.)|_{E'} + \|w_s(s,.)\|_E}{\min\limits_{0\le y\le L}|q(l,.)|}\|Aq\|_E + \|f'(s)\|_E\right\}ds \le \\ &\le M_3(\delta,q)\int\limits_0^t\left[\max_{0\le s\le T}\|f'(s)\|_E + \max_{0\le s\le T}|\alpha_s(s,.)|\right]ds + \int\limits_0^t M_4(\delta,q)\|w_s(s)\|_E\,ds,\end{aligned} \tag{9}$$

where $E^{'} \subset E$.

Combining estimates (7), (8), and (9), we get

$$\|w_t\|_E \le M_1(\delta)\|A\varphi\|_E + M_2(\delta,q)\left[\|\alpha(0,.)\|_E + \|\varphi\|_E + \|f(0)\|_E\right] +$$

$$+M_3(\delta,q)\int_0^t \left[\max_{0\le s\le T}\|f'(s)\|_E + \max_{0\le s\le T}\|\alpha_s(s,.)\|_{E'}\right]ds + \int_0^t M_4(\delta,q)\|w_s(s)\|_E\,ds.$$

Using Grönwall's inequality, we can write

$$\|w_t\|_E \le M_5 e^{M_4(\delta,q)T},$$

where

$$M_5 = M_6(\delta,q)\left[\|A\varphi\|_E + \|\alpha(0,.)\|_E + \|\varphi\|_E + \|f(0)\|_E + \max_{0\le s\le T}\|f'(s)\|_E + \max_{0\le s\le T}|\alpha_s(s,.)|\right]. \tag{10}$$

Finally, combining estimates (10) and (5) it completes the proof of Theorem 1.

2 Stability of difference scheme

For the approximate solution of problem (1) we present the first order of accuracy difference scheme

$$\begin{cases} \frac{u_{n,m}^k - u_{n,m}^{k-1}}{\tau} = \frac{u_{n+1,m+1}^k - u_{n,m+1}^k}{h} + \frac{u_{n,m+1}^k - u_{n,m}^k}{h} + p^k q_{n,m} + f_{n,m}^k, \\ f_{n,m}^k = f(t_k, x_n, y_m),\ q_{n,m} = q(x_n,y_m),\ x_n = nh,\ y_m = mh, \\ t_k = k\tau,\ 1\le k\le N,\ \ 1\le n,m\le M-1,\ Mh = L,\ N\tau = T, \\ u_{n,m}^0 = \varphi(x_n,y_m),\ \ 0\le n,m\le M, \\ u_{0,m}^k = 0,\ u_{n,0}^k = 0,\ 0\le k\le N,\ 0\le n,m\le M, \\ u_{s,m}^k = \alpha(t_k,y_m),\ \ 0\le k\le N,\ 0\le m\le M,\ s = \left\lfloor \frac{l}{h}\right\rfloor. \end{cases} \tag{11}$$

To formulate the results on difference problem, we introduce the Banach space

$$C_\tau(E) = C([0,T]_\tau, E)$$

of all grid functions

$$\phi^\tau = \{\phi(t_k)\}_{k=0}^N$$

defined on

$$[0,T]_\tau = \{t_k:\ t_k = k\tau,\ 0\le k\le N,\ N\tau = T\}$$

with values in E equipped with the norm

$$\|\phi^\tau\|_{C_\tau(E)} = \max_{0\le k\le N}\|\phi(t_k)\|_E.$$

Let $C_h = C_{[0,L]_h\times[0,L]_h}$ and $C_h^{(1)} = C^{(1)}_{[0,L]_h\times[0,L]_h}$ be spaces of all grid functions $\psi^h = \{\psi_{n,m}\}_{m,n=1}^M$ defined on $[0,L]_h\times[0,L]_h = \{x_n = nh,\ y_m = mh,\ 0\le n,m\le M\}$ equipped with the norms

$$\left\|\psi^h\right\|_{C_h} = \max_{0\le n,m\le M}|\psi_{n,m}|,$$

$$\left\|\psi^h\right\|_{C_h^{(1)}}=\left\|\psi^h\right\|_{C_h}+\frac{1}{h}\max_{0\le n\le M}\max_{1\le m\le M}\left|\psi_{n,m}-\psi_{n,m-1}\right|+\frac{1}{h}\max_{1\le n\le M}\max_{0\le m\le M}\left|\psi_{n,m}-\psi_{n-1,m}\right|,$$

respectively.

Moreover, we introduce difference neutron transport operator A_h

$$A_h u^h=-\left\{\frac{u_{n+1,m+1}-u_{n,m+1}}{h}+\frac{u_{n,m+1}-u_{n,m}}{h}\right\}_{n,m=1}^{M-1}$$

acting in the space of grid functions $u^h=\{u_{n,m}\}_{n,m=1}^{M}$, $u_{0,m}=0,\ u_{n,0}=0,\ 0\le n,m\le M.$

Then, the following theorem on stability of problem (11) is established.

Theorem 2. For the solution of problem (11), the following stability estimates hold

$$\left\|\left\{\left\{\frac{u_{n,m}^k-u_{n,m}^{k-1}}{\tau}\right\}_{k=1}^{N}\right\}_{n,m=0}^{M}\right\|_{C_\tau(C_h)}+\left\|\left\{\left\{u_{n,m}^k\right\}_{k=1}^{N}\right\}_{n,m=0}^{M}\right\|_{C_\tau\left(C_h^{(1)}\right)}+\left\|p^\tau\right\|_{C_\tau}\le$$

$$\le M_1(q)\left[\left\|\varphi^h\right\|_{C_h^{(1)}}+\left\|f^{1,h}\right\|_{C_h}+\left\|\left\{\left\{\frac{f_{n,m}^k-f_{n,m}^{k-1}}{\tau}\right\}_{k=2}^{N}\right\}_{n,m=0}^{M}\right\|_{C_\tau(C_h)}+\right.$$

$$\left.+\left\|\left\{\left\{\frac{\alpha_{s,m}^k-\alpha_{s,m}^{k-1}}{\tau}\right\}_{k=2}^{N}\right\}_{m=0}^{M}\right\|_{C_\tau\left(C[0,L]_h\right)}+\left\|\left\{\alpha_m^1\right\}_{m=0}^{M}\right\|_{C[0,L]_h}\right].$$

Proof. For the solution of difference scheme (11), we consider substitution

$$u_{n,m}^k=\eta^k q_{n,m}+w_{n,m}^k, \tag{12}$$

where

$$q_{n,m}=q\left(x_n,y_m\right),$$

and η^k is the grid function determined by

$$\eta^k=\sum_{i=1}^{k}p_i\,\tau,\ \eta^0=0,\ p_k=\frac{\eta^k-\eta^{k-1}}{\tau},\ 0\le k\le N.$$

It is easy to see that grid function $\left\{\left\{w_{n,m}^k\right\}_{k=1}^{N}\right\}_{n,m=0}^{M}$ is the solution of difference scheme

$$\begin{cases}\frac{w_{n,m}^k-w_{n,m}^{k-1}}{\tau}=\frac{w_{n+1,m+1}^k-w_{n,m+1}^k}{h}+\frac{w_{n,m+1}^k-w_{n,m}^k}{h}\\[2mm] +\eta^k\left[\frac{q_{n+1,m+1}-q_{n,m+1}}{h}+\frac{q_{n,m+1}-q_{n,m}}{h}\right]+f\left(t_k,x_n,y_m\right),\\[2mm] f_{n,m}^k=f\left(t_k,x_n,y_m\right),\ q_{n,m}=q\left(x_n,y_m\right),\ x_n=nh,\ y_m=mh,\\[2mm] t_k=k\tau,\ 1\le k\le N,\ \ 1\le n,m\le M-1,\ Mh=L,\ N\tau=T,\\[2mm] w_{n,m}^0=\varphi\left(x_n,y_m\right),\ \ 0\le n,m\le M,\\[2mm] w_{0,m}^k=0,\ u_{n,0}^k=0,\ 0\le k\le N,\ 0\le n,m\le M,\\[2mm] w_{s,m}^k=\alpha\left(t_k,y_m\right),\ \ 0\le k\le N,\ 0\le m\le M,\ s=\left\lfloor\frac{l}{h}\right\rfloor.\end{cases} \tag{13}$$

Difference derivative of (12) can be written as

$$\frac{u_{n,m}^{k}-u_{n,m}^{k-1}}{\tau}=\frac{\eta^{k}-\eta^{k-1}}{\tau}q_{n,m}+\frac{w_{n,m}^{k}-w_{n,m}^{k-1}}{\tau}=p_{k}q_{n,m}+\frac{w_{n,m}^{k}-w_{n,m}^{k-1}}{\tau}. \tag{14}$$

Hence,

$$p_{k}=\frac{\frac{u_{n,m}^{k}-u_{n,m}^{k-1}}{\tau}-\frac{w_{n,m}^{k}-w_{n,m}^{k-1}}{\tau}}{q_{n,m}} \tag{15}$$

for n,m and k, $1\leq n,m\leq M-1$ and $1\leq k\leq N$. Applying the overdetermined condition $u_{s,m}^{k}$ in (15), we obtain that

$$p_{k}=\frac{\frac{u_{s,m}^{k}-u_{s,m}^{k-1}}{\tau}-\frac{w_{s,m}^{k}-w_{s,m}^{k-1}}{\tau}}{q_{s,m}}.$$

Using the triangle inequality, we obtain

$$\left|p_{k}\right|\leq M_{7}\left(q\right)\left[\left|\frac{u_{s,m}^{k}-u_{s,m}^{k-1}}{\tau}\right|+\left|\frac{w_{s,m}^{k}-w_{s,m}^{k-1}}{\tau}\right|\right]$$

for all $0\leq k\leq N$. From that it follows,

$$\begin{aligned}\left\|\left\{p_{k}\right\}_{k=1}^{N}\right\|_{C[0,T]_{\tau}}\leq M_{7}\left(q\right)&\left[\left\|\left\{\frac{u_{s,m}^{k}-u_{s,m}^{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_{\tau}}+\right.\\&\left.+\left\|\left\{\frac{w_{s,m}^{k}-w_{s,m}^{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C_{\tau}\left(C\left([0,L]_{h}\times[0,L]_{h},\ E\right)\right)}\right].\end{aligned} \tag{16}$$

Now using substitution (14) we get

$$\frac{u_{n,m}^{k}-u_{n,m}^{k-1}}{\tau}=\frac{w_{n,m}^{k}-w_{n,m}^{k-1}}{\tau}+p_{k}q_{n,m}.$$

Applying the triangle inequality, we obtain

$$\begin{aligned}&\left\|\left\{\frac{u_{n,m}^{k}-u_{n,m}^{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_{\tau}}\leq\left\|\left\{\frac{w_{n,m}^{k}-w_{n,m}^{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C_{\tau}\left(C\left([0,L]_{h}\times[0,L]_{h}\right)\right)}+\\&+\left\|\left\{p_{k}\right\}_{k=1}^{N}\right\|_{C[0,T]_{\tau}}\left\|\left\{\left\{q_{n,m}\right\}_{n=1}^{M}\right\}_{m=1}^{M}\right\|_{C\left([0,L]_{h}\times[0,L]_{h}\right)}\end{aligned} \tag{17}$$

for all $0\leq k\leq N$. We can rewrite difference scheme (13) in the abstract form as

$$\begin{cases}\frac{w_{h}^{k}-w_{h}^{k-1}}{\tau}+A_{h}w_{h}^{k}+\eta^{k}Aq=f^{h}\left(t_{k}\right),\\ w_{h}^{0}=\varphi^{h},\ \eta^{0}=0,\ t_{k}=k\tau,\ 1\leq k\leq N,\ N\tau=T\end{cases} \tag{18}$$

in a Banach space $C_{\tau}\left(E\right)=C\left(\left[0,T\right]_{\tau},\ E\right)$ with the positive operator A_{h} defined by

$$A_{h}u^{h}=-\left\{\frac{u_{n+1,m+1}-u_{n,m+1}}{h}+\frac{u_{n,m+1}-u_{n,m}}{h}\right\}_{n,m=1}^{M-1},$$

acting on grid functions u^h such that satisfies the condition $u^h = \{u_{n,m}\}_{n,m=1}^{M}, u_{0,m} = 0,\;\; u_{n,0} = 0$, $0 \leq n, m \leq M$.

For equation (18) we have that

$$w_h^k = Rw_h^{k-1} + R\tau \left(Aq\frac{\alpha(t_k) - w_s^k}{q_s} + f^h(t_k) \right),$$

for all k, $1 \leq k \leq N$, where $R = (I + \tau A_h)^{-1}$. By recurrence relations, we get

$$w_h^k = R^k \varphi^h + \sum_{i=1}^{k} R^{k-i+1} \tfrac{\tau}{q_s} \alpha(t_i) Aq - \sum_{i=1}^{k} R^{k-i+1} \tfrac{\tau}{q_s} w_s^i Aq + \sum_{i=1}^{k} R^{k-i+1} \tau f^h(t_i)$$

for any k, $1 \leq k \leq N$. Taking the difference derivative of both sides, we obtain that

$$\tfrac{w_h^k - w_h^{k-1}}{\tau} = \tfrac{R^k - R^{k-1}}{\tau} \varphi^h + \tfrac{1}{q_s} \alpha(t_k) Aq + \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \tfrac{1}{q_s} \alpha(t_i) Aq -$$

$$- \tfrac{1}{q_s} w_s^k Aq - \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \tfrac{1}{q_s} w_s^i Aq + f^h(t_k) + \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) f^h(t_i).$$

Applying the formula,

$$\sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) w_s^i = \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \varphi(x_s, y_m) +$$

$$+ \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \sum_{j=1}^{i} \tfrac{w_s^j - w_s^{j-1}}{\tau} \tau$$

and changing the order of summation, we get

$$\sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) w_s^i = \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \varphi(x_s, y_m) +$$

$$+ \sum_{j=1}^{k} \sum_{i=j}^{k} \left(R^{k-i+1} - R^{k-i}\right) \tfrac{w_s^j - w_s^{j-1}}{\tau} \tau.$$

Consequently, we obtain the following presentation for the solution of equation (13)

$$\tfrac{w_h^k - w_h^{k-1}}{\tau} = \tfrac{R^k - R^{k-1}}{\tau} \varphi^h + \tfrac{1}{q_s} \alpha(t_k) Aq + \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \tfrac{1}{q_s} \alpha(t_i) Aq -$$

$$- \tfrac{1}{q_s} w_s^k Aq - \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) \tfrac{1}{q_s} Aq \varphi^h(x_s, y_m) - \sum_{j=1}^{k} \sum_{i=j}^{k} \left(R^{k-i+1} - R^{k-i}\right) \tfrac{w_s^j - w_s^{j-1}}{\tau} \tau +$$

$$+ f^h(t_k) + \sum_{i=1}^{k} \left(R^{k-i+1} - R^{k-i}\right) f^h(t_i).$$

Applying the definition of norm of the spaces $C_\tau(E) = C([0, T]_\tau,\ E)$ and methods of monograph [20], we can write,

$$\left\| \left\{ \frac{w_{n,m}^{k}-w_{n,m}^{k-1}}{\tau} \right\}_{k=1}^{N} \right\|_{C_\tau\left(C\left([0,L]_h\times[0,L]_h\right)\right)} \leq M_8\left(q\right)\left[\left\|\varphi^h\right\|_{C_h^{(1)}} + \left\| \left\{ \left\{ \frac{f_{n,m}^{k}-f_{n,m}^{k-1}}{\tau} \right\}_{k=2}^{N} \right\}_{n,m=0}^{M} \right\|_{C_\tau(C_h)} + \right.$$

$$\left. + \left\|f^{1,h}\right\|_{C_h} + \left\| \left\{ \left\{ \frac{\alpha_{s,m}^{k}-\alpha_{s,m}^{k-1}}{\tau} \right\}_{k=2}^{N} \right\}_{m=0}^{M} \right\|_{C_\tau\left(C[0,L]_h\right)} + \left\| \left\{ \alpha_m^1 \right\}_{m=0}^{M} \right\|_{C[0,L]_h} \right]. \tag{19}$$

Finally, combining estimates (16), (17), and (19), it completes the proof of Theorem 2.

3 Numerical experiments

In this section, we study the numerical solution of the neutron transport identification problem with initial condition

$$\begin{cases} \frac{\partial u(t,x,y)}{\partial t} = \frac{\partial u(t,x,y)}{\partial x} + \frac{\partial u(t,x,y)}{\partial y} + p\left(t\right)\sin\pi x\sin\pi y + f\left(t,x,y\right), \\ f\left(t,x,y\right) = -e^{-2t}(3\sin\pi x\sin\pi y + \pi\cos\pi x\sin\pi y + \pi\sin\pi x\cos\pi y), \\ t\in\left(0,1\right],\ x,y\in\left(0,1\right], \\ u\left(0,x,y\right) = \sin\pi x\sin\pi y,\ x,y\in\left[0,1\right], \\ u\left(t,0,y\right) = 0,\ t\in\left[0,1\right],\ y\in\left[0,1\right], \\ u\left(t,x,0\right) = 0,\ t\in\left[0,1\right],\ x\in\left[0,1\right], \\ u(t,\frac{1}{2},y) = e^{-2t}\sin\pi y,\quad t\in\left[0,1\right],\ y\in\left[0,1\right]. \end{cases} \tag{20}$$

The exact solution of problem is $u(t,x,y) = e^{-2t}\sin\pi x\sin\pi y$ and for the control parameter $p\left(t\right) = e^{-2t}$.

For the approximate solution of problem (20), we get the following first order of accuracy difference scheme

$$\begin{cases} \frac{u_{n,m}^{k}-u_{n,m}^{k-1}}{\tau} = \frac{u_{n+1,m+1}^{k}-u_{n,m+1}^{k}}{h} + \frac{u_{n,m+1}^{k}-u_{n,m}^{k}}{h} + p^k q_{n,m} + f_{n,m}^{k}, \\ f_{n,m}^{k} = -e^{-2t_k}(3\sin\pi x_n\sin\pi y_m + \pi\cos\pi x_n\sin\pi y_m + \pi\sin\pi x_n\cos\pi y_m), \\ q_{n,m} = \sin\pi x_n\sin\pi y_m,\ x_n = nh,\ y_m = mh,\ t_k = k\tau, \\ 1\leq k\leq N,\quad 0\leq n,m\leq M-1,\ Mh=1,\quad N\tau=1, \\ u_{n,m}^{0} = \sin\pi x_n\sin\pi y_m,\ 0\leq n,m\leq M, \\ u_{0,m}^{k} = 0,\ u_{n,0}^{k} = 0,\quad 0\leq k\leq N,\ 0\leq n,m\leq M, \\ u_{s,m}^{k} = e^{-2t_k}\sin\pi y_m,\quad 0\leq k\leq N,\ 0\leq m\leq M, s = \left\lfloor \frac{M}{2} \right\rfloor. \end{cases} \tag{21}$$

For the solution of difference scheme (21), we consider the substitution

$$u_{n,m}^k = \eta^k q_{n,m} + w_{n,m}^k, \tag{22}$$

where

$$\eta^k = \sum_{i=1}^{k} p_i \ \tau, \ \eta^0 = 0, \tag{23}$$

$w_{n,m}^k$ is the solution of difference scheme

$$\begin{cases} \frac{w_{n,m}^k - w_{n,m}^{k-1}}{\tau} = \frac{w_{n+1,m+1}^k - w_{n,m+1}^k}{h} + \frac{w_{n,m+1}^k - w_{n,m}^k}{h} + \\ +\eta^k \left[\frac{q_{n+1,m+1} - q_{n,m+1}}{h} + \frac{q_{n,m+1} - q_{n,m}}{h}\right] + f_{n,m}^k, \\ 1 \le k \le N, \quad 1 \le n,m \le M, \\ w_{n,m}^0 = \sin \pi x_n \sin \pi y_m, \quad 0 \le n,m \le M, \\ w_{0,m}^k = 0, \ w_{n,0}^k = 0, \ 0 \le k \le N, \ 0 \le n,m \le M. \end{cases} \tag{24}$$

Applying (21) and formulas (22), (23), we get

$$\eta^k = \frac{u_{s,m}^k - w_{s,m}^k}{q_{s,m}} = \frac{e^{-2t_k} \sin \pi y_m - w_{s,m}^k}{q_{s,m}}, \tag{25}$$

$$p^k = \frac{1}{\tau}\left[\frac{(e^{-2t_k} - e^{-2t_{k-1}}) \sin \pi y_m - (w_{s,m}^k - w_{s,m}^{k-1})}{\sin \pi x_s \sin \pi y_m}\right] \tag{26}$$

for any k, $1 \le k \le N$.

It is easy to see that (24) and (25) can be written in the matrix form

$$A\, w^k + B\, w^{k-1} = \varphi^k, \quad 1 \le k \le N, \quad w^0 = \{\sin \pi x_n \sin \pi y_m\}_{n,m=0}^{M},$$

where

$$\begin{cases} \varphi_{n,m}^k = e^{-2t_k}\left[\frac{\sin \pi x_{n+1} \sin \pi y_{m+1} - \sin \pi x_n \sin \pi y_{m+1}}{h} + \frac{\sin \pi x_n \sin \pi y_{m+1} - \sin \pi x_n \sin \pi y_m}{h}\right] - \\ -e^{-2t_k}(3 \sin \pi x_n \sin \pi y_m + \pi \cos \pi x_n \sin \pi y_m + \pi \sin \pi x_n \cos \pi y_m), \\ 1 \le n,m \le M, \ \varphi_{0,m}^k = 0, \ \varphi_{n,0}^k = 0, \ 1 \le n,m \le M. \end{cases}$$

Here A and B are $(M+1)\times(M+1)\times(N+1)$ square matrices, w^k and φ^k are $(M+1)\times(M+1)\times 1$ column matrices. First, we obtain w^k by formula

$$w^k = -A^{-1}Bw^{k-1} + A^{-1}\varphi^k, \ 1 \le k \le N, w^0 = \{\sin \pi x_n \sin \pi y_m\}_{n,m=0}^{M}.$$

Second, applying formulas (22) and (26), we get p^k and u^k.

4 Error analysis

Now, we will give the results of the numerical analysis. In order to get the solution of (21), we used MATLAB program. The errors are computed by

$$E_M^N u = \max_{0\leq k\leq N} \max_{0\leq n,m\leq M} \left| u(t_k, x_n, y_m) - u_{n,m}^k \right|, \quad E^N p = \max_{1\leq k\leq N} \left| p(t_k) - p^k \right|$$

of the numerical solutions for different values of M and N, where $u(t_k, x_n, y_m)$ represents the exact solution, $u_{n,m}^k$ represents the numerical solution at (t_k, x_n, y_m), $p(t_k)$ represents the exact solution, and p^k represents the numerical solution at t_k. Now, let us give the obtained numerical results (Table).

Table

Error analysis of first order DS

Error	$N = M = 10$	$N = M = 20$	$N = M = 40$	$N = M = 80$
$E_M^N u$	0.1813	0.0952	0.0488	0.0247
$E^N p$	0.0698	0.0481	0.0264	0.0137

The obtained results indicate that when the numerical parameters N and M are multiplied by two, the errors in the solution for first order difference scheme (21) decrease by approximately half.

Conclusion

In this study, we consider an inverse problem related to the two-dimensional neutron transport equation with a time-dependent source control parameter. For the approximate solution of this problem, a first-order accuracy difference scheme is constructed. A finite difference scheme is presented for identifying the control parameter. Stability inequalities for the solution of this problem are established. The results of a numerical experiment are presented, and the accuracy of the solution for this inverse problem is discussed.

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Conflict of Interest

The author declare no conflict of interest.

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