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Operator-pencil treatment of multi-interval Sturm-Liouville equation with boundary-transmission conditions

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This paper is devoted to a new type of boundary-value problems for Sturm-Liouville equations defined on three disjoint intervals $(-\pi, -\pi + d), (-\pi + d, \pi - d)$ and $(\pi - d, \pi)$ together with eigenparameter dependent boundary conditions and with additional transmission conditions specified at the common end points $-\pi + d$ and $\pi - d$, where $0 < d < \pi$. The considered problem cannot be treated by known techniques within the usual framework of classical Sturm-Liouville theory. To establish some important spectral characteristics we introduced the polynomial-operator formulation of the problem. Moreover, we develop a new modification of the Rayleigh method to obtain lower bound of eigenvalues.

Keywords: boundary-value-transmission problems, eigenvalues, generalized eigenfunctions, lower bound estimation, Rayleigh's method, transmission conditions.

2020 Mathematics Subject Classification: 34L10, 34B24.

Introduction

This work is motivated by the problem of understanding the nature of the spectral characteristics of the class of boundary-value problems (BVPs) for Sturm-Liouville equations (SLEs) defined of finite number of nonintersecting intervals together with additional interaction conditions specified at the common endpoints of these intervals. Moreover, the spectral parameter appears linearly in both differential equation and boundary conditions (BCs). Such type of BVPs (the so-called many-interval boundary value transmission problems (MIBVTPs)) are encountered in solving various transfer problems of mathematical physics. For example, some MIBVTPs arise in heat transfer problems, mass transfer problems, diffraction problems, seismic behavior of the Earth's, waves in the atmosphere, etc. (see, [1–7]). Its solutions are determined by different special functions, such as Bessel functions, Chebyshev polynomials, Legendre polynomials, Hypergeometric functions etc. Important studies have been carried out recently regarding MIBVTPs [8–24].

The aim of this work is to investigate the following MIBVTP, consisting of three-interval SLE

$$- g''(x) + q(x)g(x) = \lambda r(x)g(x) \tag{1}$$

defined on three-interval $(-\pi, -\pi + d) \cup (-\pi + d, \pi - d) \cup (\pi - d, \pi)$, together with the λ-dependent BCs given by

$$\cos\varphi\, g(-\pi + d) + \sin\varphi\, g'(-\pi + d) = 0\,,\ 0 < \varphi < \pi, \tag{2}$$

$$\alpha g(\pi) - \alpha' g'(\pi) + \lambda \left(\beta g(\pi) - \beta' g'(\pi)\right) = 0 \tag{3}$$

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and with the additional transmission conditions (TCs) at the points of interaction $-\pi + d$ and $\pi - d$ given by

$$T_{-\pi+d}(g) \ = \ 0 \ , \ T_{-\pi+d}(g') \ = \ \theta_1 \ g(-\pi + d), \tag{4}$$

$$T_{\pi-d}(g) \ = \ 0 \ , \ T_{\pi-d}(g') \ = \ \theta_2 \ g(\pi - d), \tag{5}$$

where $0 < d < \pi$, $T_x(g)$ is the linear form defined by $T_x(g) \ = \ \lim_{\delta\to 0} g(x + |\delta|) - \lim_{\delta\to 0} g(x - |\delta|)$, $\alpha, \alpha^{'}, \beta, \beta^{'}, \theta_1, \theta_2$ are real numbers, $q(x)$ is a real-valued function, $q \in L_2(-\pi, \pi)$. Everywhere we shall assume that

$$\theta_3 := \begin{vmatrix} \alpha^{'} & \alpha \\ \beta^{'} & \beta \end{vmatrix} \ > \ 0.$$

To study some important spectral characteristic of the considered MIBVTP (1)-(5) we introduced a corresponding operator-polynomial in appropriate Hilbert space. Note that, MIBVTPs have been an important research in recent years [25–31].

1 Operator-pencil treatment of the problem

To study some spectral characteristics of the MIBVTP (1)–(5) we shall use the operator-pencil theory and Rayleigh theory. Let us formulate some definitions and facts, which is needed for further consideration.

Let $k \geq 0$ be an integer. The Sobolev space $W_2^k(a, b)$ is defined to be the linear space consisting of all functions $g \in L_2(a, b)$ having generalized derivatives $g', g'', ..., g^{(k)} \in L_2(a, b)$ equipped with the inner product

$$\langle g, h\rangle_{W_2^k(a,b)} := \sum_{j=0}^{k} \langle g^{(j)}, \overline{h}^{(j)}\rangle_{L_2(a,b)}$$

and corresponding norm $\|g\|^2_{W_2^k(a,b)} \ = \ \langle g, g\rangle_{W_2^k(a,b)}$. Here, $L_2(a, b)$ denotes the space of all complex-valued functions g, such that $\int_a^b |g^2(x)|dx < \infty$, equipped with the inner product

$$\langle g, h\rangle_{L_2(a,b)} := \int_a^b g(x) \ \overline{h}(x)dx.$$

Denote $\Omega_1 = (-\pi, -\pi+d)$, $\Omega_2 = (-\pi+d, \pi-d)$, $\Omega_3 = (\pi-d, \pi)$ and $\Omega = \Omega_1\cup\Omega_2\cup\Omega_3$. For investigation of the BVTP (1)–(5) we shall use the discret sum space $\oplus L_2 := L_2(\Omega_1) \oplus L_2(\Omega_2) \oplus L_2(\Omega_3)$ with the inner-product

$$\langle g, h\rangle_0 := \sum_{i=1}^{3} \int_{\Omega_i} g(x)\overline{h}(x)dx$$

and direct sum space

$$\oplus W_2^1 = \left\{ \begin{array}{c} g \in \oplus L_2 \left| g \in W_2^1(\Omega_i)(i = 1, 2, 3), g(-\pi + d + 0) = g(-\pi + d - 0), \right. \\ g(\pi - d + 0) = g(\pi - d - 0) \end{array} \right\}$$

with the inner-product

$$\langle g, h\rangle_1 := \sum_{i=1}^{3} \int_{\Omega_i} \left(g'(x)\overline{h}'(x) \ + \ g(x)\overline{h}(x)\right) dx.$$

We can show that the inner-product spaces $\oplus L_2$ and $\oplus W_2^1$ are Hilbert spaces.

In the Hilbert space $\oplus W_2^1$ we define a new inner-product by

$$\langle g,h\rangle_2 \ := \ \sum_{i=1}^{3}\int_{\Omega_i}\left\{g'(x)\overline{h}'(x)+q(x)g(x)\overline{h}(x)\right\}dx$$

with the corresponding norm $\|g\|_2^2 \ = \ \langle g,g\rangle_2$. Obviously, there are positive constants m and M, such that

$$m\ \|g\|_1<\|g\|_2<M\ \|g\|_1$$

for all $g\in\oplus W_2^1$.

Using the well-known embedding properties for Sobolev spaces (see [20]) we can show that

$$|g(x_j)|^2\le \ell\ ||g'||_0^2+\frac{2}{\ell}\ ||g||_0^2, \tag{6}$$

$$|g(\xi)| \ \ \le \ \ C(\xi)\ ||g||_2 \tag{7}$$

for any $g\in\oplus W_2^1$ where $j=1,2,3,4,\ x_1=-\pi,\ x_2=-\pi+d\mp0,\ x_3=\pi-d\mp0,\ x_4=\pi,\ \ell$ is a positive number (small enough), $\xi\in\Omega$, the constant $C(\xi)$ is independent of the function g and dependent only of ξ. Let us introduce to the consideration the Hilbert space $\mathbb{H}$, consisting of all vector-functions $\left(\ \chi(x),\chi_1\ \right)\in\oplus W_2^1\oplus\mathbb{C}\ :=\ \mathbb{H}$ equipped with the inner product

$$\langle\Gamma,\Psi\rangle_{\mathbb{H}}\ :=\ \langle\chi,\varphi\rangle_1+\chi_1\ \overline{\varphi_1},$$

where $\Gamma=(\chi,\ \chi_1)$ and $\Psi=(\varphi,\ \varphi_1)\in\mathbb{H}$.

The concept of weak eigenfunction is based on the weak solutions of the problem (1)–(5), which we shall define by the following procedure. By multiplying the differential equation (1) by the conjugate of an arbitrary $h\in\oplus W_2^1$ satisfying the conditions $h(\pi-d+0)=h(\pi-d-0)$ and $h(-\pi+d+0)=h(-\pi+d-0)$ and then integrating by parts over the intervals Ω_i $(i=1,2,3)$ we have

$$\begin{aligned}&\sum_{i=1}^{3}\int_{\Omega_i}\left\{g'(x)\overline{h}'(x)+q(x)g(x)\overline{h}(x)\right\}dx-\frac{\beta}{\beta'}g(\pi)\overline{h}(\pi)-\frac{\cos\varphi}{\sin\varphi}g(-\pi)\overline{h}(-\pi)+\\&+\ \ \theta_1g(-\pi+d)\overline{h}(-\pi+d)\ +\ \theta_2g(\pi-d)\overline{h}(\pi-d)+\frac{\kappa}{\beta'}\overline{h}(\pi)\ =\ \lambda\sum_{i=1}^{3}\int_{\Omega_i}g\overline{h}dx,\end{aligned}\tag{8}$$

and

$$\frac{g(\pi)}{\beta'}-\frac{\alpha^{'}}{\beta'}\frac{\kappa}{\theta_3}=\lambda\frac{\kappa}{\theta_3}, \tag{9}$$

where $\kappa\ :=\ \beta g(\pi)-\beta^{'}g'(\pi)$. Thus the BVTP (1)–(5) is transformed into the system of equalities (8) and (9), all terms of which are defined for the $g,h\in\oplus W_2^1$.

Definition 1. The element $\Gamma=(g(x),\ \kappa)\in\oplus W_2^1$ is said to be a weak solution of the BVTP (1)–(5) if the equations (8)-(9) are satisfied for any $h\in\oplus W_2^1$.

Let us introduce to the consideration the following bilinear forms:

$$\begin{aligned}\tau_0(g,h)\ \ :=\ \ &-\frac{\beta}{\beta'}g(\pi)\overline{h}(\pi)-\frac{\cos\varphi}{\sin\varphi}g(-\pi)\overline{h}(-\pi)+\theta_1g(-\pi+d)\overline{h}(-\pi+d)+\\&+\ \ \theta_2g(\pi-d)\overline{h}(\pi-d),\end{aligned}\tag{10}$$

$$\tau_1(g,h) \;:=\; \sum_{i=1}^{3}\int_{\Omega_i} r(x)g(x)\,\overline{h}(x)dx, \tag{11}$$

and

$$\tau_2(\kappa,h) \;:=\; \frac{\kappa}{\beta'}\overline{h}(\pi). \tag{12}$$

The reduction of identities (8)-(9) to an operator equation is based on the following result.

Theorem 1. There are bounded linear operators $S_0, S_1 : \oplus W_2^1 \to \oplus W_2^1$ and $S_2 : \mathbb{C} \to \oplus W_2^1$ such that

$$\begin{aligned}\tau_n(g,h) \;&=\; \langle S_n g, h\rangle_2 \text{ for } n=0,1 \text{ and}\\ \tau_n(\kappa,h) \;&=\; \langle S_n \kappa, h\rangle_2 \text{ for } n=2.\end{aligned} \tag{13}$$

Proof. $\tau_n(g,h),\ n=0,1,$ are linear functionals in $h \in \oplus W_2^1$ for any given $g \in \oplus W_2^1$ and that $\tau_2(\kappa,h)$ is a linear functional in $h \in \oplus W_2^1$ for any given $\kappa \in \mathbb{C}$.

Let $g \in \oplus W_2^1$ be any function. From (10)–(12), it follows immediately that

$$|\tau_0(g,h)| \leq\; C_1\left\{|g(\pi)||h(\pi)| + |g(-\pi)||h(-\pi)| + |g(-\pi+d)||h(-\pi+d)| + \right.$$

$$\left. +|g(\pi-d)||h(\pi-d)|\right\},$$

$$|\tau_1(g,h)| \leq C_2\|g\|\|h\|,$$

$$|\tau_2(\kappa,h)| \leq C_3|\kappa|\;|h(\pi)|.$$

Here and below, the symbols $C_k,$ for $k=1,2,\dots$ denote different positive constants whose exact values are not important for the proof.

The interpolation inequalities (6)-(7) imply

$$\|g\| \;\leq\; C_4\|g\|_2 \quad \text{and} \quad |g(\xi)| \;\leq\; C_5\|g\|_2 \;\text{ for any } \xi \in \Omega.$$

Hence, the functionals τ_n $(n=0,1,2)$ allow the following estimates:

$$\begin{aligned}|\tau_0(g,h)| \quad &\leq \quad C_6\;\|g\|_2\;\|h\|_2,\\ |\tau_1(g,h)| \quad &\leq \quad C_7\;\|g\|_2\;\|h\|_2,\\ |\tau_2(\kappa,h)| \quad &\leq \quad C_8\;|\kappa|\;\|h\|_2.\end{aligned}$$

Therefore, τ_n $(n=0,1,2)$ are linear continuous functionals in $h \in \oplus W_2^1$ for any given $g \in \oplus W_2^1$, $n=0,1,$ and $\kappa \in \mathbb{C}$, $n=2$, respectively. Then, the existence of linear bounded operators S_0, S_1 and S_2 follows immediately from the well-known Riesz representation theorem (see, for example, [25]).

Theorem 2. The operators $S_0, S_1 : \oplus W_2^1 \to \oplus W_2^1$ are self-adjoint and the operator S_1 is positive.

Proof. Let $g, h \in \oplus W_2^1$ be arbitrary functions. By (10) and (13), we have that

$$\langle g, S_0 h\rangle_{\oplus W_2^1} \;=\; \overline{\langle S_0 h, g\rangle}_{\oplus W_2^1} \;=\; \overline{\tau_0(h,g)} \;=\; \tau_0(g,h) = \;\langle S_0 g, h\rangle_{\oplus W_2^1}.$$

Hence, the operator S_0 is self-adjoint in $\oplus W_2^1$. The proof of the self-adjointness of S_1 is totaly similar. The positivity of S_1 follows immediately from the fact that the function $r(x)$ is positive definitely.

Theorem 3. The operators $S_i : \oplus W_2^1 \to \oplus W_2^1$ $(i=0,1),$ $S_2 : \mathbb{C} \to \oplus W_2^1$ and $S_2^* : \oplus W_2^1 \to \mathbb{C}$ are compact, where S_2^* is the adjoint of S_2.

Proof. To prove the compactness of the operator S_0 it is sufficient to show that any weakly convergent sequence $\{g_k\}(k = 1, 2, ...)$ in $\oplus W_2^1$ is transformed by S_0 into a strongly convergent sequence $\{S_0 g_k\}$ in the same space. The boundedness of S_0 implies the weakly convergence of $\{S_0 g_k\}$ to $S_0 g$ in $\oplus W_2^1$, where $g(x)$ is the weak limit of $\{g_k\}$. Since the embedding operator $J : \oplus W_2^1 \hookrightarrow \oplus L_2$ is compact [20], the sequences (g_k) and $(S_0 g_k)$ converge strongly to g and $S_0 g$ in $\oplus L_2$ respectively. In addition, since for each bounded interval $I \subset \mathbb{R}$ the embedding operator $J : W_2^1(I) \hookrightarrow C(I)$ is compact and the sequences $\{g_k\}$ and $\{S_0 g_k\}$ are bounded in $\oplus W_2^1$ it follows that these sequences converge in $C(\Omega_1) \oplus C(\Omega_2) \oplus C(\Omega_3)$.

Further, the compactness of the embedding operator $J : \oplus W_2^1 \hookrightarrow C(\Omega_1) \oplus C(\Omega_2) \oplus C(\Omega_3)$ (see, for example, [20]) implies that the sequences $\{g_k(d_i)\}$ and $\{(S_0 g_k)(d_i)\}$ converge in $\mathbb{C}$ to $g(d_i)$ and $(S_0 g)(d_i)$ $(i = 1, 2, 3, 4)$ with $d_1 = -\pi$ or $d_2 = -\pi + d \mp 0$ or $d_3 = \pi - d \mp 0$ or $d_4 = \pi$, respectively. The representations (10)–(12) and inequalities (6) imply

$$
\begin{aligned}
\| \quad & S_0(g_k - g_m)\|_2^2 = \langle S_0(g_k - g_m), S_0(g_k - g_m)\rangle_2 \; = \; \tau_0\big(g_k - g_m, S_0(g_k - g_m)\big) \\
\leq \quad & C_1 \left\{ |(g_k(\pi) - g_m(\pi))| + |(g_k(-\pi) - g_m(-\pi))| \right\} \\
+ \quad & C_1 \left\{ |(g_k(-\pi + d + 0) - g_m(-\pi + d - 0))| + |(g_k(\pi - d + 0) - g_m(\pi - d - 0))| \right\}.
\end{aligned}
$$

Therefore, $\|S_0(g_k - g_m)\|_2 \to 0$ as $k,\ m \to \infty$. Hence, the sequence $\{S_0 g_k\}$ is the Cauchy sequence in the space $\oplus W_2^1$ and therefore converges strongly in $\oplus W_2^1$. Thus the compactness of the operator S_0 is proven. The proof of the compactness of the operator S_1 is totally similar.

It is easy to show that the adjoint operator S_2^* is defined by the equality $S_2^* g = \frac{g(\pi)}{\beta'}$, from which it follows that this operator is compact. Then by virtue of well-known theorem of Functional Analysis the operator S_2 is also compact. The proof is complete.

2 Positiveness of the operator-pencil

It is evident that the BVTP (1)–(5) can be written as the operator-pencil equation in $\mathbb{H}$, given by

$$
\mathcal{A}(\lambda)\, \Gamma = 0\,, \qquad \mathcal{A}(\lambda) = \Delta - \lambda\, \Lambda, \tag{14}
$$

where the operators Δ and Λ are defined by

$$
\Delta(g, \kappa) = \left(g + S_0 g + S_2 \kappa\,, \quad S_2^* g - \frac{\alpha'}{\beta'} \frac{\kappa}{\theta_3} \right), \tag{15}
$$

$$
\Lambda(g, \kappa) = \left(S_1 g\,,\ \frac{\kappa}{\theta_3} \right), \tag{16}
$$

respectively.

Lemma 1. For all real λ_0, the operator $\mathcal{A}(-\lambda_0) = \Delta + \lambda_0\, \Lambda$ is self-adjoint in the Hilbert space $\mathbb{H}$.

Proof. Using Theorem 2, it is easy to show that the linear operators Δ and Λ are self-adjoint. Therefore, the operator-pencil $\mathcal{A}(-\lambda_0) = \Delta + \lambda_0\, \Lambda$ is also self-adjoint in the Hilbert space $\mathbb{H}$.

Lemma 2. The operator-polynomial $\mathcal{A}(-\lambda_0)$ is positive definite for sufficiently large positive values of λ_0.

Proof. Taking in view the equality

$$
\mathcal{A}(-\lambda_0)\Gamma = \Big(\; g(x) + S_0 g(x) + S_2 \kappa + \lambda_0 S_1 g(x)\;,\; S_2^* g(x) - \tfrac{\alpha'}{\beta'} \tfrac{\kappa}{\theta_3} \;+\; \lambda_0 \tfrac{\kappa}{\theta_3} \;\Big)
$$

for $\Gamma=(g(x),\kappa)$, we get

$$\begin{aligned}\langle\mathcal{A}(-\lambda_0)\Gamma,\Gamma\rangle_{\mathbb{H}} &= \langle g(x),g(x)\rangle_2+\langle S_0 g(x),g(x)\rangle_2+\langle S_2\kappa,g(x)\rangle_2+(S_2^* g(x))\overline{\kappa}- \\ &- \frac{\alpha'}{\beta'\ \theta_3}|\kappa|^2+\lambda_0\left\{\langle S_1 g(x),g(x)\rangle_2+\frac{1}{\theta_3}|\kappa|^2\right\}. \end{aligned} \tag{17}$$

Let us define the following functionals

$$P(g):=\langle g',g'\rangle_0,\quad Q(g):=\langle qg,g\rangle_0,\quad R(g):=\langle rg,g\rangle_0. \tag{18}$$

From the well-known embedding theorems for Sobolev spaces it follows easily that the inequalities

$$|g(x_j)|^2\le C_{j1}\epsilon_j P(g)+\frac{C_{j2}}{\epsilon_j}Q(g) \tag{19}$$

hold for sufficiently small positive ϵ_j, where $g\in\oplus W_2^1$ $(j=1,2,3,4)$, C_{jk} $(k=1,2)$ are positive constants; $x_1=-\pi,\ x_2=-\pi+d\mp0,\ x_3=\pi-d\mp0,\ x_4=\pi.$

Using (18) and (19) and applying the well-known Young inequality, we have the following estimates

$$\begin{aligned}\langle S_0 g(x),g(x)\rangle_2 &= -\frac{\beta}{\beta'}\ |g(\pi)|^2-\frac{\cos\varphi}{\sin\varphi}\ |g(-\pi)|^2+\theta_1|g(-\pi+d)|^2+\theta_2|g(\pi-d)|^2 \\ &\ge \left(-\frac{\cos\varphi}{\sin\varphi}C_{11}\epsilon_1+\theta_1 C_{21}\epsilon_2+\theta_2 C_{31}\epsilon_3-\frac{\beta}{\beta'}C_{41}\epsilon_4\right)P(g) \\ &+ \left(-\frac{\cos\varphi}{\sin\varphi}\frac{C_{12}}{\epsilon_1}+\theta_1\frac{C_{22}}{\epsilon_2}+\theta_2\frac{C_{32}}{\epsilon_3}-\frac{\beta}{\beta'}\frac{C_{42}}{\epsilon_4}\right)Q(g). \end{aligned} \tag{20}$$

$$\begin{aligned}\langle S_2\kappa,g(x)\rangle_2 &+ (S_2^* g(x))\overline{\kappa}=\frac{2}{\beta'}\ Re(\kappa\ \overline{g}(\pi)) \\ &\ge -\frac{1}{|\beta'|\ \gamma}\ |g(\pi)|^2\ -\frac{\gamma}{|\beta'|}|\kappa|^2 \\ &\ge -\frac{1}{|\beta'|\ \gamma}\left\{C_{41}\epsilon_4 P(g)+\frac{C_{42}}{\epsilon_4}Q(g)\right\} \\ &- \frac{\gamma}{|\beta'|}|\kappa|^2 \end{aligned} \tag{21}$$

for arbitrary $\gamma\ >0$. It is easy to see that,

$$\langle S_1 g,g\rangle_2=R(g)\ge M_1 Q(g) \tag{22}$$

for some $M_1>0$.

Taking in view the equality

$$\|g\|_2^2=P(g)+Q(g)\ ,g\in\oplus W_2^1 \tag{23}$$

and substituting (20)–(23) into (17) we have

$$\langle\mathcal{A}(-\lambda_0)\Gamma,\Gamma\rangle_{\mathbb{H}}\ \ \ge\ \ \Phi_1 P(g)\ +\ \Phi_2(\lambda_0)Q(g)\ +\Phi_3(\lambda_0)|\kappa|^2, \tag{24}$$

where

$$\begin{aligned}\Phi_1 \;&:=\; 1-\left|\frac{\cos\varphi}{\sin\varphi}\right| C_{11}\epsilon_1+\theta_1\ C_{21}\ \epsilon_2+\theta_2\ C_{31}\ \epsilon_3\\ &-\;\left(\left|\frac{\beta}{\beta'}\right|+\frac{1}{\gamma\ |\beta'|}\right) C_{41}\epsilon_4, \qquad (25)\\ \Phi_2(\lambda_0)\;&:=\;1-\left|\frac{\cos\varphi}{\sin\varphi}\right|\frac{C_{12}}{\epsilon_1}+\theta_1\frac{C_{22}}{\epsilon_2}+\theta_2\frac{C_{32}}{\epsilon_3}\\ &-\;\left(\left|\frac{\beta}{\beta'}\right|+\frac{1}{\gamma|\beta'|}\right)\frac{C_{42}}{\epsilon_4}+\lambda_0 M, \qquad (26)\\ \Phi_3(\lambda_0)\;&=\;-\left|\frac{\alpha'}{\beta'}\right|\frac{1}{\theta_3}-\frac{\gamma}{|\beta'|}+\frac{\lambda_0}{\theta_3}. \qquad (27)\end{aligned}$$

Since $\theta_3 > 0$, it is possible to choose the positive parameters $\gamma, \epsilon_1, \epsilon_2,\ \epsilon_3$ and ϵ_4 so small and the positive parameter λ_0 so large that $\Phi_1 > 0,\ \Phi_2(\lambda_0) > 0,\ \Phi_3(\lambda_0) > 0$. Now denoting

$$\Phi(\lambda_0) := \min\left(\Phi_1\ ,\ \Phi_2(\lambda_0)\ ,\ \Phi_3(\lambda_0)\right),$$

we have

$$\langle \mathcal{A}(-\lambda_0)\Gamma,\Gamma\rangle_{\mathbb{H}} \;\geq\; \Phi(\lambda_0)\ \|\Gamma\|^2_{\mathbb{H}}$$

for all $\Gamma \in \mathbb{H}$. Consequently the operator pencil $\mathcal{A}(-\lambda_0)$ is positive definite for sufficiently large $\lambda_0 > 0$. The proof is complete.

3 *Modified Rayleigh quotient and estimation of the eigenvalues*

For finding lower bound estimation for eigenvalues we shall introduce a new spectral parameter $\mu = \lambda + \lambda_0$, where λ_0 is the parameter from Lemma 2. Then the operator pencil equation $\mathcal{A}(\lambda)\ \Gamma = 0$ is transformed to the spectral problem

$$\mathcal{A}(-\lambda_0)\ \Gamma \;-\; \mu\Lambda\Gamma \;=\; 0 \qquad (28)$$

with the new spectral parameter μ. This problem can be rewritten as

$$\mu \;=\; \frac{\langle\ \left(\Delta+\lambda_0\ \Lambda\right)\ \Gamma\ ,\ \Gamma\rangle_{\mathbb{H}}}{\langle\Lambda\Gamma\ ,\ \Gamma\rangle_{\mathbb{H}}}. \qquad (29)$$

Let $h = g$ in (8). Then equation (8) is converted into the form

$$\langle g,g\rangle_2+\langle S_0 g,g\rangle_2+\langle S_2\kappa,g\rangle_2 \;=\; \lambda\langle S_1 g,g\rangle_2. \qquad (30)$$

Using (30), we have the following Rayleigh quotient

$$\mu \;=\; \frac{\langle g,g\rangle_2+\langle S_0 g,g\rangle_2+\langle S_2\kappa,g\rangle_2+(S_2^* g)\kappa-\frac{\alpha'}{\beta'\theta_3}|\kappa|^2+\lambda_0\left\{\langle S_1 g,g\rangle_2+\frac{1}{\theta_3}|\kappa|^2\right\}}{\langle S_1 g,g\rangle_2+\frac{1}{\theta_3}|\kappa|^2}. \qquad (31)$$

Using (14)–(16), (20)–(27) and (28)–(31) we have the following inequality

$$\mu \;\geq\; \frac{\Phi_1 P(g)+\Phi_2(\lambda_0)Q(g)+\lambda_0 R(g)+\Phi_3(\lambda_0)|\kappa|^2}{|\kappa|^2+\frac{1}{\theta_3}|\kappa|^2}. \qquad (32)$$

It is easy to show that there are $M_2 > 0$ and $M_3 > 0$, such that

$$R(g) \leq M_2 Q(g) \leq M_3 \|g\|^2$$

for all g.

Then from inequality (32) we get

$$\mu \geq \min\left(M_2 \Phi_2(\lambda_0) + \lambda_0 , \theta_3 \Phi_3(\lambda_0)\right).$$

Thus, we have the lower bound estimation for eigenvalues of the BVTP (1)–(5) given by

$$\lambda_k \geq -\lambda_0 + \min\left(M_2 \Phi_2(\lambda_0) + \lambda_0 , \theta_3 \Phi_3(\lambda_0)\right).$$

Conclusion

In this work, we investigated a new type of boundary value problems (BVPs) for Sturm-Liouville equations. The problem addressed in our study is different from standard Sturm-Liouville problems in the sense that the differential equation is defined on three non-overlapping intervals $(-\pi, -\pi + d)$, $(-\pi+d, \pi-d)$ and $(\pi-d, \pi)$ and the boundary conditions are included four additional conditions at the interaction points $x = -\pi + d$ and $x = \pi - d$, so-called transmission conditions. Spectral analysis, such type of multi-interval boundary value transmission problems (MIBVTPs), is much more complicated to analyze than BVPs. It is not obvious how to apply the known classical methods to such MIBVTPs. To establish some important spectral characteristics, we introduced a new type polynomial-operator formulation of the considered MIBVTP. We then proved that this polynomial-operator is self-adjoint and positive definite for sufficiently large positive values of the spectral parameter λ. Moreover, we have been developed a new modification of the Rayleigh method to obtain a lower bound for the eigenvalues.

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Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

The authors declare no conflict of interest.

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