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Existence of extremal solutions for a class of fractional integro-differential equations

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In the study the existence of solutions of a class of fractional integro-differential equations with boundary conditions was considered. The main tool, we employ, is the conventional monotone iterative technique, which is highly effective method to examine the quantitative and qualitative characteristics of various nonlinear problems. This technique produces monotone sequences whose iterations are unique solutions of the certain linear problems. These bounds converge uniformly to the maximal solutions of the given problems. Some types of coupled solutions are considered to obtain the claim of the main results under suitable conditions.

Keywords: Caputo derivative, integro-differential equation, Riemann-Liouville integral, extremal solutions, monotone iterative technique, upper and lower solutions.

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Introduction

The basis of our understanding of the world is frequently based on classical calculus, which involves the operation of derivatives and integrals on integer orders. However, many real-world phenomena exhibit memory effects and non-local interactions that cannot be fully captured by these integer-order operations. At this point in the discussion, the concept of fractional calculus presents itself as a relevant topic that should be considered [1]. Fractional calculus is a fascinating field of mathematics that extends the concepts of differentiation and integration to non-integer orders [2, 3]. This extension facilitates a more sophisticated representation of memory-dependent processes, in which the current state is affected by the entire history of the process. See [4–10] for recent works.

Fractional integro-differential equations (FIDEs) are of great importance in the area of fractional calculus. Fractional derivatives and integral terms are combined in FIDEs, making them effective tools for modeling many systems. For instance, FIDEs provide a flexible framework for modeling intricate financial systems with memory effects, such as long-range dependencies in market behavior, and non-classical diffusion processes characterized by varying anomalous diffusion rates, diverging from classical diffusion. Furthermore, FIDEs effectively capture the delayed response of viscoelastic materials to external forces, as these materials exhibit a combination of elastic and viscous properties [11–13].

Monotone iterative technique (MIT) proposes a powerful combination of theoretical and practical tools for nonlinear problems. It provides a theoretical framework to determine the existence and uniqueness of solutions for certain equations, while also offering an efficient iterative algorithm to approximate these solutions numerically, making it valuable for various applications. MIT produces a sequence of functions in which each iteration is derived by substituting the preceding one into the specified linear differential equation. The fundamental principle of MIT is the notion of monotonicity, which guarantees that the sequence is either consistently growing or consistently decreasing, hence

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gradually converging towards the solution of the given nonlinear problem. Under specific conditions, MIT guarantees that the generated sequences converges uniformly in a closed set to the unique solution of the differential equation lying between the initial lower and upper solutions (LUSs) [14]. Recently MIT was adapted for some types of fractional differential or integro-differential equations involving initial or boundary conditions. See [15–22] and the references therein.

In this work, we discuss the following FIDE with boundary conditions of the form:

$$^{C}D^{q_1}u\left(t\right)=F\left(t,u\left(t\right),I^{q_2}u\left(t\right)\right),h\left(u\left(0\right),u\left(T\right)\right)=0, \tag{1}$$

where $F\in C\left[J\times\mathbb{R}\times\mathbb{R},\mathbb{R}\right]$, $J=[0,T]$, $h\in C\left[\mathbb{R}^2,\mathbb{R}\right]$, and $0<q_2\leq q_1<1$.

It should be observed that supplementary conditions $h\left(u\left(0\right),u\left(T\right)\right)=0$ may indicate initial, boundary or other general conditions, depending on the selection of the function h. Therefore, problem (1) can be seen as a more comprehensive version of the boundary value problems that were previously mentioned.

The basic objective of the study is to utilize the MIT in order to solve the problem (1), consequently getting the extremal (minimal and maximal) solutions as the limit of the functions of sequences which converge uniformly, by considering several types of coupled lower and upper solutions (LUSs) of (1).

The remainder of this article is structured as follows: Section 1 provides a brief overview of fractional calculus and FIDEs with necessary definitions and lemmas, required for the proofs of main results. The subsequent part presents the main results including the existence and uniqueness theorem for the solution via selection of coupled LUSs. Final section offers concluding remarks and potential directions for future research.

1 *Mathematical preliminaries*

Definition 1. [3] Let $[0,T]\subset\mathbb{R}$, $Re(\theta)>0$ and $f\in L_1[0,T]$. Then the Riemann-Liouville(R-L) fractional integrals I^{θ}_{0+} of order θ is given by

$$I^{\theta}_{0+}f\left(x\right)=\frac{1}{\Gamma\left(\theta\right)}\int\limits_0^x\frac{f\left(t\right)dt}{\left(x-t\right)^{1-\theta}},\ x\in\left(0,T\right].$$

Definition 2. The Caputo derivative of order $0\leq\theta<1$ for $t\in[0,T]$, designated by ${}^{c}D_{0+}$ is given by

$${}^{c}D_{0+}f(x):=I^{1-\theta}_{0+}Df\left(x\right)=\frac{1}{\Gamma\left(1-\theta\right)}\int\limits_0^x\frac{f'\left(t\right)dt}{\left(x-t\right)^{\theta}}.$$

We offer multiple definitions regarding coupled LUSs to problem (1).

Definition 3. Let $\vartheta,\omega\in C^1[J,\mathbb{R}]$. Then ϑ and ω are said to be

(i) natural LUSs of (1) if

$$^{C}D^{q_1}\vartheta\left(t\right)\leq F\left(t,\vartheta\left(t\right),I^{q_2}\vartheta\left(t\right)\right),\quad h\left(\vartheta\left(0\right),\vartheta\left(T\right)\right)\leq 0,$$
$$^{C}D^{q_1}\omega\left(t\right)\geq F\left(t,\omega\left(t\right),I^{q_2}\omega\left(t\right)\right),\ \ h\left(\omega\left(0\right),\omega\left(T\right)\right)\geq 0;$$

(ii) coupled LUSs of type 1 of (1) if

$$^{C}D^{q_1}\vartheta\left(t\right)\leq F\left(t,\vartheta\left(t\right),I^{q_2}\omega\left(t\right)\right),\quad h\left(\vartheta\left(0\right),\vartheta\left(T\right)\right)\leq 0,$$
$$^{C}D^{q_1}\omega\left(t\right)\geq F\left(t,\omega\left(t\right),I^{q_2}\vartheta\left(t\right)\right),\ \ h\left(\omega\left(0\right),\omega\left(T\right)\right)\geq 0;$$

(iii) coupled LUSs of type 2 of (1) if

$${}^{C}D^{q_1}\vartheta(t) \leq F(t,\omega(t), I^{q_2}\vartheta(t)), \quad h(\vartheta(0),\vartheta(T)) \leq 0,$$
$${}^{C}D^{q_1}\omega(t) \geq F(t,\vartheta(t), I^{q_2}\omega(t)), \quad h(\omega(0),\omega(T)) \geq 0;$$

(iv) coupled LUSs of type 3 of (1) if

$${}^{C}D^{q_1}\vartheta(t) \leq F(t,\omega(t), I^{q_2}\omega(t)), \quad h(\vartheta(0),\vartheta(T)) \leq 0,$$
$${}^{C}D^{q_1}\omega(t) \geq F(t,\vartheta(t), I^{q_2}\vartheta(t)), \quad h(\omega(0),\omega(T)) \geq 0.$$

Definition 4. The functions ϱ and r, both belonging to the space $C^1[J,R]$, are called to be coupled minimal and maximal solutions (MMSs) of (1), if, for any coupled solutions ϑ and ω, it holds that $\varrho \leq \vartheta, \omega \leq r$.

Next result is related to the solution of a linear fractional integro-differential equation.

Lemma 1. Let $\varphi \in C^1[J,\mathbb{R}]$, $0 < q_2 \leq q_1 < 1$ and L, M be real numbers. Then, there exists a unique solution $\varphi \in C^1[J,\mathbb{R}]$ of the problem

$$ {}^{C}D^{q_1}\varphi(t) = L\varphi(t) + MI^{q_2}\varphi(t), \varphi(0) = \varphi_0, \tag{2}$$

such that

$$\varphi(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(M)^n (L)^m \binom{n+m}{m} t^{q_1(n+m)+nq_2}}{\Gamma(q_1(n+m)+nq_2+1)}\varphi_0.$$

Proof. The proof and more general form of this result can be found in [23, 24].

Lemma 2. [23] Suppose that ϑ and ω are natural LUSs of (1). Moreover following condition holds

$$F(t,u_1(t),v_1(t)) - F(t,u_2(t),v_2(t)) \leq L(u_1-u_2) + M(v_1-v_2),$$

$L, M \geq 0$, whenever $u_1 \geq u_2, v_1 \geq v_2$.

Then $\vartheta(0) \leq \omega(0)$ implies $\vartheta(t) \leq \omega(t)$ on J.

Corollary 1. ([23]) Let p belongs to the space $C^1[J,\mathbb{R}]$ and $L \geq 0$, $M \geq 0$. If the inequality

$${}^{C}D^{q_1}p(t) \leq Lp(t) + MI^{q_2}p(t), p(0) \leq 0,$$

holds, then we get $p(t) \leq 0$ on J.

Analogously, ${}^{C}D^{q_1}p(t) \geq -Lp(t) - MI^{q_2}p(t), p(0) \geq 0$ implies $p(t) \geq 0$ on J.

2 Main results

In this section, we formulate the monotone technique for the problem (1) via coupled LUSs with the aid of the method of LUSs. We construct monotone functions of sequences, whose iterations are generated by unique solutions of corresponding Caputo type fractional linear initial value problems, hence converging uniformly and monotonically to the minimal and maximal solutions of the given BVP problem (1).

In the following theorem, we first employ natural LUSs to reach the main objective.

Theorem 1. Assume that

(A_1) $\vartheta_0, \omega_0 \in C^1[J,\mathbb{R}]$ are natural LUSs of problem (1) with $\vartheta_0(t) \leq \omega_0(t)$ on J;

(A_2) $h(u,v) \in C[\mathbb{R}^2,\mathbb{R}]$ is non-increasing in the second variable and there is a positive constant M satisfying

$$h(u_1,v) - h(u_2,v) \leq M(u_1-u_2),$$

for $\vartheta_0(0) \leq u_2 \leq u_1 \leq \omega_0(0)$, $\vartheta_0(T) \leq v \leq \omega_0(T)$;

(A_3) the function $F \in C\left[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\right]$ satisfies

$$F\left(t, u_1\left(t\right), v_1\left(t\right)\right) - F\left(t, u_2\left(t\right), v_2\left(t\right)\right) \geq -L\left(u_1 - u_2\right) - M\left(v_1 - v_2\right), \tag{3}$$

where $\vartheta_0 \leq u_2 \leq u_1 \leq \omega_0$ and $\vartheta_0 \leq v_2 \leq v_1 \leq \omega_0$ and $L > 0$, $M > 0$.

Then there exist monotone sequences $\{\vartheta_n\left(t\right)\}$, $\{\omega_n\left(t\right)\}$ converging uniformly and monotonically to the functions ϱ and r on J, indicating that ϱ and r serve as minimal and maximal solutions of (1), respectively.

Proof. For any function $\mu \in C^1\left[J, \mathbb{R}\right]$, we define the linear initial value problem

$$\begin{aligned} {}^{C}D^{q_1}u\left(t\right) &= F\left(t, \mu\left(t\right), I^{q_2}\mu\left(t\right)\right) - L\left(u - \mu\right) - MI^{q_2}\left(u - \mu\right), && (4)\\ u\left(0\right) &= \mu\left(0\right) - \frac{1}{M}h\left(\mu\left(0\right), \mu\left(T\right)\right). && (5) \end{aligned}$$

where $\vartheta_0 \leq \mu \leq \omega_0$. Pay attention to the fact that the right-hand side of the equation (4) is Lipschitzian, thus unique solution exits for every μ.

Consider A as an operator, such that $A\mu = u$, which assists in the construction the sequences $\{\vartheta_n\}$ and $\{\omega_n\}$.

We have to prove that

(i) $\vartheta_0 \leq A\,\vartheta_0$ and $\omega_0 \geq A\omega_0$;

(ii) the operator A is monotone on the sector $\left[\vartheta_0, \omega_0\right] = \left\{u \in C^1\left[J, \mathbb{R}\right] : \vartheta_0 \leq u \leq \omega_0\right\}$.

To prove (i), set $A\vartheta_0 = \vartheta_1$, where ϑ_1 is the unique soluion of (4)-(5) with $\mu = \vartheta_0$. Setting $p\left(t\right) = \vartheta_1\left(t\right) - \vartheta_0\left(t\right)$ for $t \in J$, we see that

$$\begin{aligned} {}^{C}D^{q_1}p\left(t\right) &= {}^{C}D^{q_1}\vartheta_1\left(t\right) - {}^{C}D^{q_1}\vartheta_0\left(t\right) \\ &\geq F\left(t, \vartheta_0\left(t\right), I^{q_2}\vartheta_0\left(t\right)\right) - L\left(\vartheta_1 - \vartheta_0\right) - MI^{q_2}\left(\vartheta_1 - \vartheta_0\right) \\ &\quad - F\left(t, \vartheta_0\left(t\right), I^{q_2}\vartheta_0\left(t\right)\right) \\ &= -Lp\left(t\right) - MI^{q_2}p\left(t\right), \end{aligned}$$

and

$$\begin{aligned} p\left(0\right) &= \vartheta_1\left(0\right) - \vartheta_0\left(0\right) \\ &= \vartheta_0\left(0\right) - \frac{1}{M}h\left(\vartheta_0\left(0\right), \vartheta_0\left(T\right)\right) - \vartheta_0\left(0\right) \\ &\geq 0. \end{aligned}$$

This gives, from Corollary 1, $p\left(t\right) \geq 0$ on J, hence $\vartheta_0 \leq \vartheta_1$. In the similar way, one can form $p\left(t\right) = \omega_0\left(t\right) - \omega_1\left(t\right)$, where $A\omega_0 = \omega_1$. Then, we obtain

$$\begin{aligned} {}^{C}D^{q_1}p\left(t\right) &= {}^{C}D^{q_1}\omega_0\left(t\right) - {}^{C}D^{q_1}\omega_1\left(t\right) \\ &\geq F\left(t, \omega_0\left(t\right), I^{q_2}\omega_0\left(t\right)\right) - \left(F\left(t, \omega_0\left(t\right), I^{q_2}\omega_0\left(t\right)\right) - L\left(\omega_1 - \omega_0\right) - MI^{q_2}\left(\omega_1 - \omega_0\right)\right) \\ &= -Lp\left(t\right) - MI^{q_2}p\left(t\right) \end{aligned}$$

and

$$\begin{aligned} p\left(0\right) &= \omega_0\left(0\right) - \omega_1\left(0\right) \\ &= \omega_0\left(0\right) - \left(\omega_0\left(0\right) - \frac{1}{M}h\left(\omega_0\left(0\right), \omega_0\left(T\right)\right)\right) \\ &= \frac{1}{M}h\left(\omega_0\left(0\right), \omega_0\left(T\right)\right) \\ &\geq 0. \end{aligned}$$

This ensures that $p(t) \geq 0$, thus meaning $\omega_0 \geq \omega_1$ on J.

To achieve (ii), consider $\mu_1, \mu_2 \in [\vartheta_0, \omega_0]$, such that $\mu_1 \leq \mu_2$. Suppose that $A\mu_1 = u_1$ and $A\mu_2 = u_2$. Set $p(t) = u_2(t) - u_1(t)$, then

$$\begin{aligned} {}^C D^{q_1} p(t) &= {}^C D^{q_1} u_2(t) - {}^C D^{q_1} u_1(t) \\ &= F\left(t, \mu_2(t), I^{q_2}\mu_2(t)\right) - L(u_2 - \mu_2) - M I^{q_2}(u_2 - \mu_2) \\ &\quad - F\left(t, \mu_1(t), I^{q_2}\mu_1(t)\right) + L(u_1 - \mu_1) + M I^{q_2}(u_1 - \mu_1) \\ &= F\left(t, \mu_2(t), I^{q_2}\mu_2(t)\right) - F\left(t, \mu_1(t), I^{q_2}\mu_1(t)\right) + L(u_1 - \mu_1 - u_2 + \mu_2) \\ &\quad + M I^{q_2}(u_1 - \mu_1 - u_2 + \mu_2). \end{aligned}$$

Using the inequality (2), we receive

$$F\left(t, \mu_2(t), I^{q_2}\mu_2(t)\right) - F\left(t, \mu_1(t), I^{q_2}\mu_1(t)\right) \geq -L(\mu_2 - \mu_1) - M I^{q_2}(\mu_2 - \mu_1).$$

If the expression is plugged into the last inequality, we derive

$$\begin{aligned} {}^C D^{q_1} p(t) &\geq -L(\mu_2 - \mu_1) - M I^{q_2}(\mu_2 - \mu_1) + L(u_1 - \mu_1 - u_2 + \mu_2) + M I^{q_2}(u_1 - \mu_1 - u_2 + \mu_2) \\ &= -Lp(t) - M I^{q_2} p(t). \end{aligned}$$

Also we obtain

$$\begin{aligned} p(0) &= u_2(0) - u_1(0) \\ &= \mu_2(0) - \frac{1}{M} h\left(\mu_2(0), \mu_2(T)\right) - \mu_1(0) + \frac{1}{M} h\left(\mu_1(0), \mu_1(T)\right) \\ &= \mu_2(0) - \mu_1(0) + \frac{1}{M}\left(h\left(\mu_1(0), \mu_1(T)\right) - h\left(\mu_2(0), \mu_2(T)\right)\right) \\ &\geq \mu_2(0) - \mu_1(0) + \frac{1}{M}\left(h\left(\mu_1(0), \mu_2(T)\right) - h\left(\mu_2(0), \mu_2(T)\right)\right) \\ &\geq \mu_2(0) - \mu_1(0) + \frac{1}{M}(-M)\left(\mu_2(0) - \mu_1(0)\right) \\ &= 0. \end{aligned}$$

Therefore, by applying Corollary 1, we can conclude that $A\mu_2 \geq A\mu_1$.

We now define the sequences $\vartheta_n = A\vartheta_{n-1}$ and $\omega_n = A\omega_{n-1}$ for $n = 1, 2, ...$ Based on the monotonicity argument of the operator, we can infer that

$$\vartheta_0 \leq \vartheta_1 \leq ... \leq \vartheta_n \leq \omega_n \leq ... \leq \omega_1 \leq \omega_0,$$

on $[0, T]$ for all $n \in \mathbb{N}$. These functions correspond to solutions of the following linear equations:

$${}^C D^{q_1} \vartheta_{n+1}(t) = F\left(t, \vartheta_n(t), I^{q_2}\vartheta_n\right) - L(\vartheta_{n+1} - \vartheta_n) - M I^{q_2}(\vartheta_{n+1} - \vartheta_n), \tag{6}$$

$$\vartheta_{n+1}(0) = \vartheta_n(0) - \frac{1}{M} h\left(\vartheta_n(0), \vartheta_n(T)\right). \tag{7}$$

$${}^C D^{q_1} \omega_{n+1}(t) = F\left(t, \omega_n(t), I^{q_2}\omega_n\right) - L(\omega_{n+1} - \omega_n) - M I^{q_2}(\omega_{n+1} - \omega_n), \tag{8}$$

$$\omega_{n+1}(0) = \omega_n(0) - \frac{1}{M} h\left(\omega_n(0), \omega_n(T)\right). \tag{9}$$

Now we have to prove that the monotone sequences $\{\vartheta_n\}$ and $\{\omega_n\}$ converge uniformly. In order to accomplish this, we will utilize the Arzela-Ascoli's theorem once we have revealed that the sequences are equicontinuous and uniformly bounded.

Given that $\vartheta_0, \omega_0 \in C^1[J, \mathbb{R}]$ are bounded on J, a constant $K > 0$ exists, such that $|\vartheta_0(t)| \le K$ and $|\omega_0(t)| \le K$ on J. In the light of the fact that $\vartheta_0 \le \vartheta_n \le \omega_n \le \omega_0$, it can be concluded that for all $n \in N$, $|\vartheta_n(t)| \le K$ and $|\omega_n(t)| \le K$ on J. As a result, $\{\vartheta_n\}$ and $\{\omega_n\}$ are uniformly bounded on J. Our next objective is to demonstrate that $\{\vartheta_n\}$ is equicontinuous. To do so, let $0 \le t_1 \le t_2 \le T$. Then for $n > 0$,

$$|\vartheta_n(t_1) - \vartheta_n(t_2)| =$$

$$= \Bigg|\vartheta_n(0) + \frac{1}{\Gamma(q_1)} \int_0^{t_1} (t_1 - \sigma)^{q_1 - 1} \left[F(\sigma, \vartheta_{n-1}(\sigma), I^{q_2}\vartheta_{n-1}(\sigma)) - L(\vartheta_n - \vartheta_{n-1}) - MI^{q_2}(\vartheta_n - \vartheta_{n-1})\right] d\sigma$$

$$-\vartheta_n(0) - \frac{1}{\Gamma(q_1)} \int_0^{t_2} (t_2 - \sigma)^{q_1 - 1} \left[F(\sigma, \vartheta_{n-1}(\sigma), I^{q_2}\vartheta_{n-1}(\sigma)) - L(\vartheta_n - \vartheta_{n-1}) - MI^{q_2}(\vartheta_n - \vartheta_{n-1})\right] d\sigma\Bigg|$$

$$\le \frac{1}{\Gamma(q_1)} \int_0^{t_1} \left((t_1 - \sigma)^{q_1 - 1} - (t_2 - \sigma)^{q_1 - 1}\right) \left|F(\sigma, \vartheta_{n-1}(\sigma), I^{q_2}\vartheta_{n-1}(\sigma)) - L(\vartheta_n - \vartheta_{n-1}) - MI^{q_2}(\vartheta_n - \vartheta_{n-1})\right| d\sigma$$

$$+ \frac{1}{\Gamma(q_1)} \int_{t_1}^{t_2} (t_2 - \sigma)^{q_1 - 1} \left|F(\sigma, \vartheta_{n-1}(\sigma), I^{q_2}\vartheta_{n-1}(\sigma)) - L(\vartheta_n - \vartheta_{n-1}) - MI^{q_2}(\vartheta_n - \vartheta_{n-1})\right| d\sigma.$$

Since $\{\vartheta_n\}$, $\{\omega_n\}, \{I^{q_2}\vartheta_n\}$ and $\{I^{q_2}\omega_n\}$ are uniformly bounded, there exist a $K_1 > 0$, independent of n, such that

$$|F(t, \vartheta_n(t), I^{q_2}\vartheta_n(t))| \le K_1,$$

$$|F(t, \omega_n(t), I^{q_2}\omega_n(t))| \le K_1,$$

$$|I^{q_2}\vartheta_n(t)| \le K_1,$$

and

$$|I^{q_2}\omega_n(t)| \le K_1.$$

Thus, if these expressions are substituted into the inequality above, we get

$$\begin{aligned}
&|\vartheta_n(t_1) - \vartheta_n(t_2)| \\
&\le \frac{K_2}{\Gamma(q_1)} \int_0^{t_1} \left((t_1 - \sigma)^{q_1 - 1} - (t_2 - \sigma)^{q_1 - 1}\right) d\sigma + \frac{K_2}{\Gamma(q_1)} \int_{t_1}^{t_2} (t_2 - \sigma)^{q_1 - 1} d\sigma \\
&= -\frac{K_2}{q_1\Gamma(q_1)} (t_1 - \sigma)^{q_1} \Big|_{\sigma=0}^{\sigma=t_1} + \frac{K_2}{q_1\Gamma(q_1)} (t_2 - \sigma)^{q_1} \Big|_{\sigma=0}^{\sigma=t_1} - \frac{K_2}{q_1\Gamma(q_1)} (t_2 - \sigma)^{q_1} \Big|_{\sigma=t_1}^{\sigma=t_2} \\
&= \frac{K_2}{\Gamma(q_1 + 1)} t_1^{q_1} + \frac{K_2}{\Gamma(q_1 + 1)} (t_2 - t_1)^{q_1} - \frac{K_2}{\Gamma(q_1 + 1)} t_2^{q_1} + \frac{K_2}{\Gamma(q_1 + 1)} (t_2 - t_1)^{q_1} \\
&= \frac{K_2}{\Gamma(q_1 + 1)} \left[(t_1)^{q_1} - (t_2)^{q_1}\right] + \frac{2K_2}{\Gamma(q_1 + 1)} (t_2 - t_1)^{q_1} \\
&\le \frac{2K_2}{\Gamma(q_1 + 1)} (t_2 - t_1)^{q_1} \\
&= \frac{2K_2}{\Gamma(q_1 + 1)} |t_2 - t_1|^{q_1},
\end{aligned}$$

where $K_2 = K_1 + 2LK + 2MK_1$. We conclude, that for given $\epsilon > 0$, there is a $\delta(\epsilon) = \left(\frac{\epsilon\Gamma(q_1+1)}{2K_2}\right)^{\frac{1}{q_1}}$ (which merely depends on ϵ), such that $|t_2 - t_1| < \delta$ imply that $|\vartheta_n(t_1) - \vartheta_n(t_2)| < \epsilon$. Therefore $\{\vartheta_n\}$ is equicontinuous on J and so is $\{\omega_n\}$ in the similar fashion. The use of Arzela-Ascoli's theorem allows us to conclude, that there exist subsequences $\{\vartheta_{n_k}\}$ and $\{\omega_{n_k}\}$ that uniformly converge to ϱ and r respectively. Due to their monotonic nature, the entire sequences $\{\vartheta_n\}$ and $\{\omega_n\}$ converge uniformly to ϱ and r respectively on J.

We can prove that the limit functions (ϱ, r) satisfy the problem (1). To do so, we establish corresponding integral equations to (6)-(7) and (8)-(9), then take limits as $n \longrightarrow \infty$.

Finally, it is required to clarify that (r, ϱ) occurs as the maximal and minimal solutions of (1), respectively. For any given solution u of (1) such that $\vartheta_0(t) \le u(t) \le \omega_0(t)$ on J, we need to check that

$$\vartheta_0(t) \le \varrho(t) \le u(t) \le r(t) \le \omega_0(t),$$

on J. To achieve this, it is sufficient to demonstrate $\vartheta_n(t) \le u(t) \le \omega_n(t)$ on J. This fact is obvious for $n = 0$. By applying induction principle, we claim that for some $k > 0$, the inequality $\vartheta_k(t) \le u(t) \le \omega_k(t)$ on J is true. It is necessary to prove that the following relation holds:

$$\vartheta_{k+1}(t) \le u(t) \le \omega_{k+1}(t),$$

on J. Taking $p(t) = u(t) - \vartheta_{k+1}(t)$ leads to

$$\begin{aligned} {}^C D^{q_1} p(t) &= {}^C D^{q_1} u(t) - {}^C D^{q_1} \vartheta_{k+1}(t) \\ &= F(t, u(t), I^{q_2} u(t)) - [F(t, \vartheta_k(t), I^{q_2}\vartheta_k) - L(\vartheta_{k+1} - \vartheta_k) - MI^{q_2}(\vartheta_{k+1} - \vartheta_k)]. \end{aligned}$$

Since we know that $\vartheta_k(t) \le u(t)$, we can use the inequality (3) to attain

$$F(t, u(t), I^{q_2} u(t)) - F(t, \vartheta_k(t), I^{q_2}\vartheta_k) \ge -L(u - \vartheta_k) - MI^{q_2}(u - \vartheta_k).$$

By inserting the foregoing expression into the equation above, we acquire

$$\begin{aligned} {}^C D^{q_1} p(t) &\ge -L(u - \vartheta_k) - MI^{q_2}(u - \vartheta_k) + L(\vartheta_{k+1} - \vartheta_k) + MI^{q_2}(\vartheta_{k+1} - \vartheta_k) \\ &= -Lp(t) - MI^{q_2} p(t). \end{aligned}$$

Meanwhile, if we recall the characteristics of the function $h(u, v)$, we can deduce

$$\begin{aligned} p(0) &= u(0) - \vartheta_{k+1}(0) \\ &= u(0) - \frac{1}{M} h(u(0), u(T)) - \left[\vartheta_k(0) - \frac{1}{M} h(\vartheta_k(0), \vartheta_k(T))\right] \\ &= u(0) - \vartheta_k(0) - \frac{1}{M}(h(u(0), u(T)) - h(\vartheta_k(0), \vartheta_k(T))) \\ &\ge u(0) - \vartheta_k(0) - \frac{1}{M}(h(u(0), \vartheta_k(T)) - h(\vartheta_k(0), \vartheta_k(T))) \\ &\ge u(0) - \vartheta_k(0) - \frac{1}{M} M(u(0) - \vartheta_k(0)) \\ &= 0. \end{aligned}$$

Owing to Corollary 1, it directly results in $p(0) \ge 0$ on J. As a result, $\vartheta_{k+1}(t) \le u(t)$. In the same manner, we are able to demonstrate that $u(t) \le \omega_{k+1}(t)$ on J. Therefore, for all n, we get

$$\vartheta_n(t) \le u(t) \le \omega_n(t).$$

By taking the limit, as n approaches infinity, we may deduce that

$$\varrho(t) \le u(t) \le r(t),$$

on J, which establishes the validity of the proof.

Theorem 2. Along with the assumptions stated in Theorem 1, further assume that for $L > 0$, $M > 0$

$$F(t, u_1(t), v_1(t)) - F(t, u_2(t), v_2(t)) \leq L(u_1 - u_2) + M(v_1 - v_2),$$

where $\vartheta_0 \leq u_2 \leq u_1 \leq \omega_0$ and $\vartheta_0 \leq v_2 \leq v_1 \leq \omega_0$. Thereafter, a unique solution to equation (1) exists in which $\varrho = u = r$.

Proof. If we continue by keeping the fact $\varrho \leq r$ aside, let $p(t) = r(t) - \varrho(t)$. Then, it follows that

$$\begin{aligned} {}^{C}D^{q_1}p(t) &= {}^{C}D^{q_1}r(t) - {}^{C}D^{q_1}\varrho(t) \\ &= F(t, r(t), I^{q_2}r(t)) - F(t, \varrho(t), I^{q_2}\varrho(t)) \\ &\leq L(r - \varrho) + MI^{q_2}(r - \varrho) \\ &= Lp(t) + MI^{q_2}p(t), \end{aligned}$$

and

$$\begin{aligned} p(0) &= r(0) - \varrho(0) \\ &= 0. \end{aligned}$$

This facts indicate that $p(t) \leq 0$. As a consequence, we arrive at $\varrho = u = r$ meaning that the sequences approach to the same solution of (1).

In the subsequent result, we employ coupled LUSs of type 1 to derive monotone sequences that uniformly and monotonically converge to coupled MMSs of the problem (1).

Theorem 3. Assume that

(B_1) $\vartheta_0, \omega_0 \in C^1[J, \mathbb{R}]$ are coupled LUSs of type 1 of problem (1) with $\vartheta_0(t) \leq \omega_0(t)$ on J;

(B_2) (A_2) holds;

(B_3) $F(t, u, v) \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ is non-decreasing in u and is non-increasing in v and

$$F(t, u_1(t), v(t)) - F(t, u_2(t), v(t)) \geq -L(u_1 - u_2), \tag{10}$$

$$F(t, u(t), v_1(t)) - F(t, u(t), v_2(t)) \leq M(v_1 - v_2), \tag{11}$$

whenever $u_1 \geq u_2$, $v_1 \geq v_2$ and $L > 0$, $M > 0$.

Then there exist monotone sequences $\{\vartheta_n(t)\}$, $\{\omega_n(t)\}$ converging uniformly and monotonically to the functions ϱ and r on J. It is implied that ϱ and r coupled MMSs of (1), respectively.

Proof. Let $\psi, \xi \in C^1[J, \mathbb{R}]$ such that $\vartheta_0 \leq \psi \leq \omega_0$ and $\vartheta_0 \leq \xi \leq \omega_0$. We set the linear fractional integro-differential initial value problems (IVPs):

$$\begin{aligned} {}^{C}D^{q_1}u(t) &= F(t, \psi(t), I^{q_2}\omega_0(t)) - L(u - \psi), & (12) \\ u(0) &= \psi(0) - \frac{1}{M}h(\psi(0), \psi(T)), & (13) \\ {}^{C}D^{q_1}v(t) &= F(t, \omega_0(t), I^{q_2}\xi(t)) + MI^{q_2}(v - \xi), & (14) \\ v(0) &= \xi(0) - \frac{1}{M}h(\xi(0), \xi(T)). & (15) \end{aligned}$$

Define the mapping A and B by $A\psi = u$ and $B\xi = v$ and use it to construct the sequences $\{\vartheta_n\}$ and $\{\omega_n\}$. We aim to prove that

(i) $\vartheta_0 \leq A\vartheta_0$ and $\omega_0 \geq B\omega_0$;

(ii) the operators A and B are monotone on the sector $[\vartheta_0, \omega_0]$.

To prove (i), take $A\vartheta_0 = \vartheta_1$, where ϑ_1 is the unique solution of (12)-(13) with $\psi = \vartheta_0$. By letting $p(t) = \vartheta_1(t) - \vartheta_0(t)$, we see that

$$\begin{aligned}
{}^C D^{q_1} p(t) &= {}^C D^{q_1} \vartheta_1(t) - {}^C D^{q_1} \vartheta_0(t) \\
&\geq F(t, \vartheta_0(t), I^{q_2}\omega_0(t)) - L(\vartheta_1 - \vartheta_0) - F(t, \vartheta_0(t), I^{q_2}\omega_0(t)) \\
&= -Lp(t),
\end{aligned}$$

and

$$\begin{aligned}
p(0) &= \vartheta_1(0) - \vartheta_0(0) \\
&= \vartheta_0(0) - \frac{1}{M} h(\vartheta_0(0), \vartheta_0(T)) - \vartheta_0(0) \\
&\geq 0.
\end{aligned}$$

According to Corollary 1, it appears that $p(t) \geq 0$, which implies $\vartheta_0(t) \leq \vartheta_1(t)$ on J. Similarly, let $B\omega_0 = \omega_1$, where ω_1 is the unique solution of (14)-(15) with $\xi = \omega_0$. Setting $p(t) = \omega_1(t) - \omega_0(t)$, we get

$$\begin{aligned}
{}^C D^{q_1} p(t) &= {}^C D^{q_1} \omega_1(t) - {}^C D^{q_1} \omega_0(t) \\
&\leq F(t, \omega_0(t), I^{q_2}\omega_0(t)) + MI^{q_2}(\omega_1 - \omega_0) - F(t, \omega_0(t), I^{q_2}\vartheta_0(t)) \\
&= F(t, \omega_0(t), I^{q_2}\omega_0(t)) - F(t, \omega_0(t), I^{q_2}\vartheta_0(t)) + MI^{q_2}(\omega_1 - \omega_0) \\
&\leq F(t, \omega_0(t), I^{q_2}\vartheta_0(t)) - F(t, \omega_0(t), I^{q_2}\vartheta_0(t)) + MI^{q_2}(\omega_1 - \omega_0) \\
&= MI^{q_2} p(t),
\end{aligned}$$

and

$$\begin{aligned}
p(0) &= \omega_1(0) - \omega_0(0) \\
&= \omega_0(0) - \frac{1}{M} h(\omega_0(0), \omega_0(T)) - \omega_0(0) \\
&= -\frac{1}{M} h(\omega_0(0), \omega_0(T)) \\
&\leq 0.
\end{aligned}$$

By utilizing Corollary 1 yields $\omega_0(t) \geq \omega_1(t)$ on J.

To prove (ii), let $\psi_1, \psi_2 \in [\vartheta_0, \omega_0]$, such that $\psi_1 \leq \psi_2$ and put $A\psi_1 = u_1$ and $A\psi_2 = u_2$. It is enough to define $p(t) = u_2(t) - u_1(t)$ in a manner that

$$\begin{aligned}
{}^C D^{q_1} p(t) &= {}^C D^{q_1} u_2(t) - {}^C D^{q_1} u_1(t) \\
&= F(t, \psi_2(t), I^{q_2}\omega_0(t)) - L(u_2 - \psi_2) - F(t, \psi_1(t), I^{q_2}\omega_0(t)) + L(u_1 - \psi_1) \\
&= F(t, \psi_2(t), I^{q_2}\omega_0(t)) - F(t, \psi_1(t), I^{q_2}\omega_0(t)) + L(u_1 - \psi_1 - u_2 + \psi_2).
\end{aligned}$$

Using the inequality (10) and recalling the fact that $\psi_1 \leq \psi_2$, we may deduce that

$$F(t, \psi_2(t), I^{q_2}\omega_0(t)) - F(t, \psi_1(t), I^{q_2}\omega_0(t)) \geq -L(\psi_2 - \psi_1).$$

Substituting that expression into previous equation yields

$$\begin{aligned}
{}^C D^{q_1} p(t) &= F(t, \psi_2(t), I^{q_2}\omega_0(t)) - F(t, \psi_1(t), I^{q_2}\omega_0(t)) + L(u_1 - \psi_1 - u_2 + \psi_2) \\
&\geq -L(\psi_2 - \psi_1) + L(u_1 - \psi_1 - u_2 + \psi_2) \\
&= -L(\psi_2 - \psi_1 - u_1 + \psi_1 + u_2 - \psi_2) \\
&= -L(u_2 - u_1) \\
&= -Lp(t),
\end{aligned}$$

and

$$\begin{aligned} p(0) &= u_2(0) - u_1(0) \\ &= \psi_2(0) - \frac{1}{M} h(\psi_2(0), \psi_2(T)) - \psi_1(0) + \frac{1}{M} h(\psi_1(0), \psi_1(T)) \\ &= \psi_2(0) - \psi_1(0) + \frac{1}{M}(h(\psi_1(0), \psi_1(T)) - h(\psi_2(0), \psi_2(T))) \\ &\geq \psi_2(0) - \psi_1(0) + \frac{1}{M}(h(\psi_1(0), \psi_2(T)) - h(\psi_2(0), \psi_2(T))) \\ &\geq \psi_2(0) - \psi_1(0) + \frac{1}{M}(-M)(\psi_2(0) - \psi_1(0)) \\ &= 0. \end{aligned}$$

It follows that $A\psi_2 \leq A\psi_1$, whenever $\psi_1 \leq \psi_2$ on J.

Similarly, assume that $\xi_1, \xi_2 \in [\vartheta_0, \omega_0]$ such that $\xi_1 \leq \xi_2$. Let $B\xi_1 = v_1$, $B\xi_2 = v_2$ and set $p(t) = v_2(t) - v_1(t)$, so that

$$\begin{aligned} {}^C D^{q_1} p(t) &= {}^C D^{q_1} v_2(t) - {}^C D^{q_1} v_1(t) \\ &= F(t, \omega_0(t), I^{q_2}\xi_2(t)) + MI^{q_2}(v_2 - \xi_2) - F(t, \omega_0(t), I^{q_2}\xi_1(t)) - MI^{q_2}(v_1 - \xi_1). \end{aligned}$$

Furthermore, utilizing the inequality (11), we have

$$F(t, \omega_0(t), I^{q_2}\xi_2(t)) - F(t, \omega_0(t), I^{q_2}\xi_1(t)) \leq MI^{q_2}(\xi_2 - \xi_1).$$

When the last phrase is included into previous relation, it gives

$$\begin{aligned} {}^C D^{q_1} p(t) &\leq MI^{q_2}(\xi_2 - \xi_1) + MI^{q_2}(v_2 - \xi_2) - MI^{q_2}(v_1 - \xi_1) \\ &= MI^{q_2} p(t). \end{aligned}$$

We can figure out that $p(0) \leq 0$ implies $p(t) \leq 0$, based on the implications outlined in Corollary 1.

At this point, one may specify the sequences $\vartheta_n = A\vartheta_{n-1}$ and $\omega_n = B\omega_{n-1}$ for $n = 1, 2, \ldots$. In this case, the monotone sequences $\{\vartheta_n\}$ and $\{\omega_n\}$ can be represented by the following iterative schemes.

$$\begin{aligned} {}^C D^{q_1} \vartheta_{n+1}(t) &= F(t, \vartheta_n(t), I^{q_2}\omega_n) - L(\vartheta_{n+1} - \vartheta_n), &(16)\\ \vartheta_{n+1}(0) &= \vartheta_n(0) - \frac{1}{M} h(\vartheta_n(0), \vartheta_n(T)). &(17)\\ {}^C D^{q_1} \omega_{n+1}(t) &= F(t, \omega_n(t), I^{q_2}\omega_n) + MI^{q_2}(\omega_{n+1} - \omega_n), &(18)\\ \omega_{n+1}(0) &= \omega_n(0) - \frac{1}{M} h(\omega_n(0), \omega_n(T)). &(19) \end{aligned}$$

Suppose that u is an arbitrary solution to the problem (1) satisfying $\vartheta_0(t) \leq u(t) \leq \omega_0(t)$. Then we must demonstrate that $\vartheta_n(t) \leq u(t) \leq \omega_n(t)$ for $n \in \mathbb{N}$. The proof is clear for $n = 0$. Assume that for some k, $\vartheta_k(t) \leq u(t) \leq \omega_k(t)$ is true on J. Thus, we prove the validity of the subsequent relationship

$$\vartheta_{k+1}(t) \leq u(t) \leq \omega_{k+1}(t)$$

on J. In order to verify this, we implement $p(t) = u(t) - \vartheta_{k+1}(t)$ and, have

$$\begin{aligned} {}^C D^{q_1} p(t) &= {}^C D^{q_1} u(t) - {}^C D^{q_1} \vartheta_{k+1}(t) \\ &= F(t, u(t), I^{q_2} u(t)) - [F(t, \vartheta_k(t), I^{q_2}\omega_k) - L(\vartheta_{k+1} - \vartheta_k)] \\ &\geq -L(u - \vartheta_k) + L(\vartheta_{k+1} - \vartheta_k) \\ &= -Lp(t). \end{aligned}$$

Reviewing the fundamental characteristics of the function g, we get

$$\begin{aligned} p(0) &= u(0) - \vartheta_{k+1}(0) \\ &= u(0) - \left[\vartheta_k(0) - \frac{1}{M}h(\vartheta_k(0), \vartheta_k(T))\right] - \frac{1}{M}h(u(0), u(T)) \\ &= u(0) - \vartheta_k(0) + \frac{1}{M}(h(\vartheta_k(0), \vartheta_k(T)) - h(u(0), u(T))) \\ &\geq u(0) - \vartheta_k(0) + \frac{1}{M}(h(\vartheta_k(0), u(T)) - h(u(0), u(T))) \\ &\geq u(0) - \vartheta_k(0) + \frac{1}{M}(-M)(u(0) - \vartheta_k(0)) \\ &= 0. \end{aligned}$$

Following Corollary 1, we see that $\vartheta_{k+1}(t) \leq u(t)$ on J. By using a similar approach, we can show that $u(t) \leq \omega_{k+1}(t)$ on J. This result in for all n,

$$\vartheta_0 \leq \vartheta_1 \leq ... \leq \vartheta_n \leq u \leq \omega_n \leq ... \leq \omega_1 \leq \omega_0.$$

By employing standard techniques as in the preceding result, we reveal that the sequences $\{\vartheta_n\}$ and $\{\omega_n\}$ converge uniformly and monotonically to the functions ϱ and r, respectively. To prove that ϱ and r are coupled solutions of the main problem, one can establish the corresponding Volterra integral equations to the problems (16–19) and then taking limits as $n \to \infty$, that is,

$${}^C D^{q_1} \varrho(t) = F(t, \varrho(t), I^{q_2} r(t)), \quad h(\varrho(0), \varrho(T)) = 0,$$

and

$${}^C D^{q_1} r(t) = F(t, r(t), I^{q_2} \varrho(t)), \quad h(r(0), r(T)) = 0.$$

Finally, we need to demonstrate that (ϱ, r) are coupled MMSs of (1), respectively. Let u be any solution of (1) such that $\vartheta_0(t) \leq u(t) \leq \omega_0(t)$ on J. After proving the inequality $\vartheta_n(t) \leq u(t) \leq \omega_n(t)$ with the same approach as before and considering the limit as $n \longrightarrow \infty$, we receive $\varrho(t) \leq u(t) \leq r(t)$, which concludes the proof.

Theorem 4. In addition to conditions of Theorem 3, suppose also

$$F(t, u_1(t), v(t)) - F(t, u_2(t), v(t)) \leq L(u_1 - u_2),$$

$$F(t, u(t), v_1(t)) - F(t, u(t), v_2(t)) \geq -M(v_1 - v_2),$$

whenever $u_1 \geq u_2$, $v_1 \geq v_2$ and $L > 0$, $M > 0$. Then we have unique solution of (1) such that $\varrho = u = r$.

If we utilize coupled LUSs of type 2 of (1), we get also monotone sequences that converge uniformly and monotonically to the extremal solutions of (1) that we state as the next result.

In order to prevent repetition, we shall omit the details of the proofs for the subsequent results.

Theorem 5. Suppose that

(C_1) $\vartheta_0, \omega_0 \in C^1[J, \mathbb{R}]$ are coupled LUSs of type 2 of problem (1) with $\vartheta_0(t) \leq \omega_0(t)$ on J;

(C_2) (A_2) holds;

(C_3) $F(t,u,v) \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ is non-increasing in u and non-decreasing in v, moreover

$$F(t,u_1(t),v(t)) - F(t,u_2(t),v(t)) \leq L(u_1 - u_2),$$

$$F(t,u(t),v_1(t)) - F(t,u(t),v_2(t)) \geq -M(v_1 - v_2),$$

where $u_1 \geq u_2$, $v_1 \geq v_2$ and $L > 0$, $M > 0$.

Then there exist two sequences $\{\vartheta_n(t)\}$, $\{\omega_n(t)\}$ such that $\lim_{n\longrightarrow\infty} \omega_n = r$, $\lim_{n\longrightarrow\infty} \vartheta_n = \varrho$ uniformly and monotonically on J and that (ϱ, r) are coupled MMSs of (1).

Remark 1. Observe that coupled LUSs of type 1 together with increasing and decreasing properties of F in Theorem 3 result in the natural ULSs and coupled ULSs of type 3 separately, hence both yield the coupled LUSs of type 2 at the end. The analogous approach for coupled LUSs of type 2 is true and this can be stated in the opposite manner.

In the following theorem, we take coupled LUSs of type 3 and find the similar conclusion as in Theorem 1.

Theorem 6. Let the following conditions hold:

(D_1) $\vartheta_0, \omega_0 \in C^1[J,\mathbb{R}]$ are coupled LUSs of type 3 of (1) with $\vartheta_0(t) \leq \omega_0(t)$ on J;

(D_2) (A_2) holds;

(D_3) the function $F(t,u,v) \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ is non-increasing in both u, v for each $t \in J$ and

$$F(t,u_1(t),v_1(t)) - F(t,u_2(t),v_2(t)) \leq L(u_1 - u_2) + M(v_1 - v_2),$$

where $\vartheta_0 \leq u_2 \leq u_1 \leq \omega_0$ and $\vartheta_0 \leq v_2 \leq v_1 \leq \omega_0$ and $L > 0$, $M > 0$.

Then we obtain the sequences $\{\vartheta_n(t)\}$, $\{\omega_n(t)\}$ such that $\vartheta_n \longrightarrow \varrho$ and $\omega_n \longrightarrow r$ as $n \longrightarrow \infty$ uniformly and monotonically on J and ϱ and r are the MMSs of (1), respectively.

Remark 2. Note that the assumption (D_1) with the non-increasing property of $F(t,u,v)$ in both u and v for each $t \in J$ implies the natural LUSs of (1) for the functions ϑ_0, ω_0.

3 *Conclusion*

We have considered the boundary value problem of a Caputo fractional integro-differential equation to analyze the existence and uniqueness of the problem. We employ the monotone iterative technique generating monotone sequences that converge uniformly to the extremal solutions of the main problem. It would be valuable to explore extensions and refinements of the monotone iterative technique for solving more general classes of FIDEs, as well as to investigate its applicability to practical problems arising in real-world applications. Additionally, the development of computational algorithms based on the theoretical results could lead to the implementation of efficient numerical solvers for FIDEs with boundary conditions.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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