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Hessian measures in the class of m-convex $(m-cv)$ functions

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The theory of m-convex $(m-cv)$ functions is a new direction in the real geometry. In this work, by using the connection $m-cv$ functions with strongly m-subharmonic (sh_m) functions and using well-known and rich properties of sh_m functions, we show a number of important properties of the class of $m-cv$ functions, in particular, we study Hessians $H^k(u)$, $k=1,2,...,n-m+1$, in the class of bounded $m-cv$ functions.

Keywords: Convex function, m-convex function, Strongly m-subharmonic function, Borel measures, Hessians.

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Introduction

It is well known that m-convex functions are a real analogue in $\mathbb{R}^n$ strongly m-subharmonic (sh_m) functions in the complex space $\mathbb{C}^n$. Let us recall the definition of the class sh_m of functions, which at this time has become the subject of research by many authors (Z. Błocki [1], S. Dinew and S. Kolodziej [2–4], S. Li [5], H.C. Lu [6, 7], H.C. Lu and V.D. Nguyen [8], A. Sadullaev and his students [9–11], etc.).

A twice differentiable function $u(z)\in C^2(D),\ D\subset\mathbb{C}^n$, is said to be strongly m-subharmonic, if at each point of the domain D it holds inequalities

$$(dd^cu)^k\wedge\beta^{n-k}\geq 0,\quad k=1,2,...,n-m+1,$$

where $\beta=dd^c\|z\|^2$ is the standard volume form in $\mathbb{C}^n$.

It's clear that $psh=sh_1\subset sh_2\subset ...\subset sh_n=sh$. Operators $(dd^cu)^k\wedge\beta^{n-k}$ are closely related to the Hessians. For a twice differentiable function $u\in C^2(D)$, the second-order differential $dd^cu=\frac{i}{2}\sum\limits_{j,t}\frac{\partial^2u}{\partial z_j\partial\bar z_t}dz_j\wedge d\bar z_t$ (at a fixed point $o\in D$) is a Hermitian quadratic form. After a suitable unitary coordinate transform, it is reduced to the diagonal form $dd^cu=\frac{i}{2}\left[\lambda_1dz_1\wedge d\bar z_1+...+\lambda_ndz_n\wedge d\bar z_n\right]$, where $\lambda_1,...,\lambda_n$ are the eigenvalues of the Hermitian matrix $\left(\frac{\partial^2u}{\partial z_j\partial\bar z_t}\right)$, which are real: $\lambda=(\lambda_1,...,\lambda_n)\in\mathbb{R}^n$. Note that the unitary transformation does not change the differential form. $\beta=dd^c\|z\|^2$. Therefore, it is not difficult to see that

$$(dd^cu)^k\wedge\beta^{n-k}=k!(n-k)!H_o^k(u)\beta^n,$$

where $H_o^k(u)=\sum\limits_{1\leq j_1<...<j_k\leq n}\lambda_{j_1}...\lambda_{j_k}$ is the Hessian of dimension k of the vector $\lambda=\lambda(u)\in\mathbb{R}^n$.

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Consequently, the twice differentiable function $u(z) \in C^2(D),\ \ D \subset \mathbb{C}^n$, is strongly m-subharmonic, if at each point $o \in D$ the next inequalities hold

$$H^k(u) = H_o^k(u) \geq 0, \quad k = 1, 2, ..., n - m + 1. \tag{1}$$

The following theorem is important

Theorem 1. (see [1]). For any twice differentiable $sh_m \cap C^2(D)$ functions $v_1, ..., v_k \in sh_m(D) \cap C^2(D)$, $1 \leq k \leq n - m + 1$, the relation

$$dd^c v_1 \wedge ... \wedge dd^c v_k \wedge \beta^{m-1} \geq 0$$

is valid. In particular, for $u \in sh_m(D) \cap C^2(D)$ and for any $v_1, ..., v_{n-m} \in sh_m(D) \cap C^2(D)$ it holds

$$dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \geq 0. \tag{2}$$

The last property has dual character: if a twice differentiable function u, it satisfies (2) for all $v_1, ..., v_{n-m} \in sh_m(D) \cap C^2(D)$, then the function u is certainly sh_m function. Moreover, the class of second-order polynomials of the form is sufficient here (see [1, 2])

$$v_j = \sum_{k=1}^{n} c_{j,k} \left| z_k \right|^2 \in sh_m\left(\mathbb{C}^n\right), c_{j,k} \in \mathbb{R} \text{ is } const. \tag{3}$$

Theorem 1 allows us to define sh_m functions in the class L^1_{loc}.

Definition 1. A function $u \in L^1_{loc}(D)$ is called sh_m in the domain $D \subset \mathbb{C}^n$, if it is upper semi-continuous and for any twice differentiable sh_m functions $v_1, ..., v_{n-m}$ of the form (3), the current $dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1}$ defined as

$$\left[dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1}\right](\omega) =$$
$$= \int u dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \ \ \omega \in F^{0,0}$$

is positive, $\displaystyle\int u\, dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geq 0, \ \ \forall \omega \in F^{0,0}, \ \ \omega \geq 0.$

1 m-convex functions and associated measures

In this section, similarly to (1), we define Hessians $H^k(u),\ \ k = 1, 2, ..., n - m + 1$, in the class of bounded m-convex functions as Borel measures. This method of defining $H^k(u)$ as a measure belongs to A. Sadullaev.

Let $D \subset \mathbb{R}^n$ and $u(x) \in C^2(D)$. Then matrix $\left(\frac{\partial^2 u}{\partial x_j \partial x_t}\right)$ is orthogonal, $\frac{\partial^2 u}{\partial x_j \partial x_t} = \frac{\partial^2 u}{\partial x_t \partial x_j}$. Therefore, after a suitable orthonormal transformation, it is transformed into a diagonal form,

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_t}\right) \to \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix},$$

where $\lambda_j = \lambda_j(x) \in \mathbb{R}$ are the eigenvalues of the matrix $\left(\frac{\partial^2 u}{\partial x_j \partial x_t}\right)$. Let

$$H^k(u) = H^k\left(\lambda\right) = \sum_{1 \leq j_1 < ... < j_k \leq n} \lambda_{j_1} ... \lambda_{j_k}$$

be Hessian of the dimension k of the vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$.

Definition 2. A twice differentiable function $u \in C^2(D)$ is called m-convex in $D \subset \mathbb{R}^n$, $u \in m - cv(D)$, if its eigenvalue vector $\lambda = \lambda(x) = (\lambda_1(x), \lambda_2(x), ..., \lambda_n(x))$ satisfies the conditions

$$m - cv \cap C^2(D) = \left\{ H^k(u) = H^k(\lambda(x)) \geq 0,\ \forall x \in D,\ \ k = 1, ..., n - m + 1 \right\}.$$

Potential theory of $m - cv$ functions is poorly-studied and is a new direction in the theory of real geometry. However, when $m = 1$, this class $1 - cv \cap C^2(D) = \{H^1(\lambda) \geq 0\} = \{\lambda_1 \geq 0, \lambda_2 \geq 0, ..., \lambda_n \geq 0\}$ coincides with the convex functions in $\mathbb{R}^n$, and when $m = n$, the class $n - cv \cap C^2(D) = \{\lambda_1 + \lambda_2 + ... + \lambda_n \geq 0\}$ coincides with the class of subharmonic functions in $\mathbb{R}^n$, $cv = 1 - cv \subset 2 - cv \subset ... \subset n - cv = sh$. The class of convex functions is well studied A. Aleksandrov [12], I. Bakelman [13], A. Pogorelov [14], A. Artykbaev [15] and others. When $m > 1$ this class has been studied in a series of works N. Trudinger, H. Wang, N. Ivochkina and other mathematicians (see [16–22].

Principal difficulties in the theory of $m - cv$ are the introduction of the class $m - cv \cap L^1_{loc}$, i.e. definition $m - cv(D)$ of functions in the class of upper semicontinuous, locally integrable or bounded functions and the definition of Hessians $H^k(u),\ u \in m - cv \cap L^1_{loc}$. So for $m = n$ (the case of subharmonic functions) in the class of upper semicontinuous, locally integrable functions $u(x) \in n - cv(D)$ are defined as a distribution and the Laplace operator $\Delta u = dd^c u \wedge \beta^{n-1}$ is a Borel measure.

To define operators $(dd^c u)^k \wedge \beta^{n-k} \geq 0,\ \ k = 1, 2, ..., n - m + 1$ for the function $u(z) \in sh_m(D)$ in a domain $D \subset \mathbb{C}^n$ the function $u(z)$ must be locally bounded, i.e. $u(z) \in L^\infty_{loc}(D)$. In this case, the operators $(dd^c u)^k \wedge \beta^{n-k} \geq 0,\ \ k = 1, 2, ..., n - m + 1$ are also positive Borel measures (see [10]).

In this work, by using the connection of $m - cv$ functions with strongly m-subharmonic functions and using well-known and rich properties sh_m of functions, we show a number of important properties of the class of $m - cv$ functions, in particular, of the Hessians $H^k(u),\ \ k = 1, 2, ..., n - m + 1$, in the class of bounded $m - cv$ functions.

We embed $\mathbb{R}^n_x$ into $\mathbb{C}^n$, by $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y$ $(z = x + iy)$, as a real $n-$dimensional subspace of the complex space $\mathbb{C}^n$.

Proposition 1. (see [23]). A twice differentiable function $u(x) \in C^2(D),\ \ D \subset \mathbb{R}^n_x$, is $m - cv$ in D, if and only if a function $u^c(z) = u^c(x + iy) = u(x)$ that does not depend on variables $y \in \mathbb{R}^n_y$, is sh_m in the domain $D \times \mathbb{R}^n_y$.

Proof. We establish a connection between the Hessians $H^k(u)$ and $H^k(u^c)$. We have,

$$\frac{\partial u^c}{\partial z_j} = \frac{1}{2}\left[\frac{\partial u^c}{\partial x_j} - \frac{\partial u^c}{\partial y_j}\right] = \frac{1}{2}\frac{\partial u^c}{\partial x_j};$$

$$\frac{\partial^2 u^c}{\partial z_j \partial \bar{z}_t} = \frac{1}{2}\frac{\partial}{\partial \bar{z}_t}\left[\frac{\partial u^c}{\partial x_j}\right] = \frac{1}{4}\left[\frac{\partial^2 u^c}{\partial x_j \partial x_t} + \frac{\partial^2 u^c}{\partial x_j \partial y_t}\right] = \frac{1}{4}\frac{\partial^2 u^c}{\partial x_j \partial x_t}.$$

Thus, $\left(\frac{\partial^2 u^c}{\partial z_j \partial \bar{z}_t}\right) = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x_j \partial x_t}\right)$ and therefore, $H^k(u) = 4^k H^k(u^c)$, that is the proof of the proposition.

Let now $u(x)$ be an upper semicontinuous function in the domain $D \subset \mathbb{R}^n_x$. Then $u^c(z)$ also will be upper semicontinuous function in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$.

Definition 3. An upper semicontinuous function $u(x)$ in a domain $D \subset \mathbb{R}^n_x$ is called m-convex in D, if the corresponding function $u^c(z)$ is strongly m-subharmonic, $u^c(z) \in sh_m\left(D \times \mathbb{R}^n_y\right)$.

Let $u(x)$ be a locally bounded m-convex function in the domain $D \subset \mathbb{R}^n_x$. Then $u^c(z)$ will be also locally bounded, strongly m-subharmonic function in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$. Therefore, the operators

$$(dd^c u^c)^k \wedge \beta^{n-k}, \quad k = 1, 2, ..., n - m + 1$$

are defined as Borel measures in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z,\ \ \mu_k = (dd^c u^c)^k \wedge \beta^{n-k}$.

Since for a twice differentiable function $(dd^c u^c)^k \wedge \beta^{n-k} = k!(n-k)! H^k(u^c)\beta^n$, then for a bounded, strongly m-subharmonic function in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$, it is natural to determine its Hessians, equating them to the measure

$$H^k\left(u^c\right) = \frac{\mu_k}{k!(n-k)!} = \frac{1}{k!(n-k)!}\left(dd^c u^c\right)^k \wedge \beta^{n-k}.$$

We can now define Hessians $H^k,\ \ k = 1, 2, ..., n - m + 1$ in the class of locally bounded, m-convex domain $D \subset \mathbb{R}^n_x$ functions.

Definition 4. Let a function $u(x)$ be locally bounded and m-convex in the domain $D \subset \mathbb{R}^n_x$. Let us define Borel measures in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$,

$$\mu_k = (dd^c u^c)^k \wedge \beta^{n-k}, k = 1, 2, ..., n - m + 1.$$

Since $u^c \in sh_m(D \times \mathbb{R}^n_y)$ does not depend on $y \in \mathbb{R}^n_y$, then for any Borel sets $E_x \subset D$, $E_y \subset \mathbb{R}^n_y$, the measures $\frac{4^k}{mesE_y}\mu_k(E_x \times E_y)$ do not depend on the set $E_y \subset \mathbb{R}^n_y$, i.e. $\frac{4^k}{mesE_y}\mu_k\left(E_x \times E_y\right) = \nu_k(E_x)$. The Borel measures

$$\nu_k : \quad \nu_k\left(E_x\right) = \frac{4^k}{mesE_y}\mu_k\left(E_x \times E_y\right), \quad k = 1, 2, ..., n - m + 1,$$

we call by Hessians $H^k,\ k = 1, 2, ..., n-m+1$, for a locally bounded, m-convex function $u(x) \in m - cv(D)$ in the domain $D \subset \mathbb{R}^n_x$.

For twice differentiable function $u(x) \in m - cv(D) \cap C^2(D)$ the Hessians are ordinary functions, however, for a non-twice differentiable, bounded semicontinuous function $u(x) \in m - cv(D) \cap L^\infty(D)$, the Hessians $H^k,\ \ k = 1, 2, ..., n - m + 1$ are positive Borel measures.

Using Theorem 1 and Preposition 1 (see also Definition 3) $m - cv$ functions are defined as

Definition 5. A function $u(x) \in L^1_{loc}(D)$ is called m-convex function in the domain $D \subset \mathbb{R}^n_x$, $u(x) \in m - cv(D)$, if it is upper semicontinuous and for any twice differentiable $m - cv(D)$ functions $v_1, ..., v_{n-m}$, the current $dd^c u^c \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1}$ defined as

$$\left[dd^c u^c \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1}\right](\omega) =$$

$$= \int u^c dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \wedge dd^c\omega, \omega \in F^{0,0}\left(D \times \mathbb{R}^n_y\right)$$

is positive.

2 General definitions of m-convex functions

In various works (see, for example, [18, 19]) m-convex functions in the class of bounded upper semicontinuous $m - cv(D)$ functions define using the “viscosity” definition: an upper semicontinuous function $u(x)$ is called $m - cv(D)$, $u(x) \in m - cv(D)$, if any quadratic polynomial $q(x)$ for which the difference $u(x) - q(x)$ achieves a local maximum only at a finite number of points $x^1, ..., x^q \in D$, is $m - cv(D)$, $q(x) \in m - cv(D)$.

The following important proposition belongs to Trudinger-Wang [19]

Lemma 1. A semicontinuous function $u(x)$ is in $m - cv(D)$, if for each domain $G \subset\subset D$ and each function $v(x) \in C^2(D) : H_m(v) \leq 0$ from $u|_{\partial G} \leq v|_{\partial G} \Rightarrow u_G \leq v|_G$.

Lemma 2. A semicontinuous function $u(x)$ is in $m - cv(D)$, if and only if for any domain $G \subset\subset D$ there exists $u_j(x) \in C^2(G) \cap m - cv(G) :\ \ u_j(x) \downarrow u(x)$.

Lemma 3. If $m < \frac{n}{2}+1$, then $m-cv(D) \subset C^{0,\gamma} = Lip_{\gamma}$, where $\gamma = 2 - \frac{n}{n-m+1}$, $0 < \gamma \leq 1$.

Corollary 1. If $m < \frac{n}{2}+1$, then $u(x) \in m-cv(D)$ continuous.

For our purpose, it is convenient to use the Trudenger-Wang's definition based on Lemma 2:

Definition 6. An upper semicontinuous function $u(x)$ is called m-convex $m-cv(D)$, if for any domain $G \subset\subset D$ there exists a sequence of functions $u_j(x) \in C^2(G) \cap m-cv(G) : u_j(x) \downarrow u(x)$.

In fact, the two main ones, Definition 3 and Definition 6, are equivalent.

Theorem 2. A function $u(x)$ is $m-cv(D)$ in the sense of Definition 3, if and only if it is $m-cv(D)$ in the sense of Definition 6.

Proof. Let the function $u(x)$ have a monotonically decreasing sequence of functions $u_j(x) \in m-cv(G)$: $u_j(x) \downarrow u(x)$. Let us put $\mathbb{R}^n_x$ in $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y$ $(z = x+iy)$, and construct a monotonically decreasing sequence $u^c_j(z) = u_j(x) \in sh_m\left(G \times \mathbb{R}^n_y\right)$. Then $\lim_{j\to\infty} u^c_j(z) = u^c(z) \in sh_m\left(G \times \mathbb{R}^n_y\right)$ and $u(x) = u^c(x)$ is $m-cv(G)$.

On the other side, let the function $u(x)$ be such that $u^c(z) = u(x) \in sh_m\left(D \times \mathbb{R}^n_y\right)$. Let us construct a standard approximation $u^c_j(z) = u^c \circ K_{\frac{1}{j}}(w-z)$, $j = 1,2,....$ (see [10]). For any compact domain $G \subset\subset D$, starting from a certain number $j \geq j_0$, they are defined, infinitely smooth functions $u^c_j(z) \in sh_m(G)$: $u^c_j(z) \downarrow u^c(z)$. Moreover, it is easy to see that $u^c_j(z)$ do not depend on $y \in \mathbb{R}^n_y$. Therefore, $u^c_j(x) = u_j(x) \downarrow u(x), u_j(x) \in m-cv(G) \cap C^{\infty}(G)$.

3 Example (fundamental solution)

$$\chi_m(x,0) = \begin{cases} |x|^{2-\frac{n}{n-m+1}} & if \quad m < \frac{n}{2}+1, \\ \ln|x| & if \quad m = \frac{n}{2}+1, \\ -|x|^{2-\frac{n}{n-m+1}} & if \quad m > \frac{n}{2}+1. \end{cases}$$

Thus, when $m < \frac{n}{2}+1$, the fundamental solution is bounded and Lipschitz, when $m \geq \frac{n}{2}+1$, it is equal $-\infty$ at the point $x=0$. Note that at $m=n$, i.e. for the subharmonic case it coincides with fundamental solution of the Laplass operator Δ.

4 Weakly convergence of m-convex functions

We will continue our study of Borel measures

$$\left\{H^k(u) \geq 0, \ \forall x \in D, \ k = 1,2,...,n-m+1\right\}$$

in the class $u(x) \in m-cv(D) \cap L^{\infty}_{loc}(D)$.

Theorem 3. If $u(x) \in m-cv(D) \cap L^{\infty}_{loc}(D)$ and $u_j(x) \in m-cv(D)$ are sequences of monotonically decreasing functions, converging to $u(x), u_j(x) \downarrow u(x)$, then there is weakly convergence of measures $H^k(u_j) \mapsto H^k(u)$, $k = 1,2,...,n-m+1$.

Proof. Let us continue the functions $u(x)$, $u_j(x)$ from $D \subset \mathbb{R}^n_x$ to $D \times \mathbb{R}^n_y$, as sh_m- functions $u^c(z)$, $u^c_j(z) \in sh_m\left(D \times \mathbb{R}^n_y\right)$. Then $u^c(z) \in sh_m\left(D \times \mathbb{R}^n_y\right) \cap L^{\infty}_{loc}\left(D \times \mathbb{R}^n_y\right)$ and $u^c_j(z) \downarrow u^c(z)$. According to Theorem Sadullaev-Abdullaev (see. [10]), Borel measures

$$H^k(u^c_j) = \frac{\mu^k}{k!(n-k)!} = \frac{1}{k!(n-k)!}\left(dd^c u^c_j\right)^k \wedge \beta^{n-k}$$

weakly converges: $H^k(u_j^c) \mapsto H^k(u^c)$, $k = 1, 2, ..., n - m + 1$. This implies weakly convergence $H^k(u_j) \mapsto H^k(u)$, $k = 1, 2, ..., n - m + 1$.

As is known, if $\{u_\alpha(z)\} \subset sh_m\left(D \times \mathbb{R}^n_y\right)$, $D \times \mathbb{R}^n_y \subset \mathbb{C}^n$, a family of uniformly bounded, strongly m-subharmonic functions, then for any compact set $K \subset\subset D$ there exists a constant $C(K)$, such that the integral averages

$$\int\limits_K (dd^c u_\alpha)^k \wedge \beta^{n-k} \leq C\left(K\right), \quad k = 1, 2, ..., n - m + 1$$

(see. [10]). From this it follows that the Hessians

$$H^k(u_\alpha) = \frac{1}{k!(n-k)!}(dd^c u_\alpha)^k \wedge \beta^{n-k},$$

which are Borel measures, are uniformly bounded on average on compact subsets of the domain D. This fact, discovered by Chern-Levine-Nirenberg [24] for a class of psh functions, then it played a main role in the construction of the theory of potential in the class psh and sh_m functions.

Here we will prove a similar fact for Hessians $H^k(u)$, $k = 1, 2, ..., n{-}m{+}1$, in the class of $m{-}cv(D)$, $D \subset \mathbb{R}^n$, functions. At the same time, we note that, if in a class $sh_m\left(D \times \mathbb{R}^n_y\right)$, $D \times \mathbb{R}^n_y \subset \mathbb{C}^n$, the proof is based on differential forms and Stokes' Theorem, then for the estimate $H^k(u)$, $k = 1, 2, ..., n{-}m{+}1$, in the class of $m - cv(D)$, $D \subset \mathbb{R}^n$, we do not have this technique.

Theorem 4. If $\{u_\alpha(x)\} \subset m - cv(D)$, $D \subset \mathbb{R}^n_x$, is a family of locally uniformly bounded m-convex functions, then the family of measures $\left\{H^k(u_\alpha)\right\}$, $k = 1, 2, ..., n - m + 1$, in Hessians are uniformly bounded on average on compact subsets of the domain D. In other words, for any compact set $K \subset\subset D$ there is a constant $C(K)$ that is upper bound for integral averages

$$\int\limits_K H^k(u_\alpha) \leq C(K), \quad k = 1, 2, ..., n - m + 1.$$

Proof. Let us use Proposition 1 and Definition 3. We put $\mathbb{R}^n_x$ in $\mathbb{C}^n$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\,\mathbb{R}^n_y$ $(z = x + iy)$, as a real $n-$dimensional subspace of a complex space $\mathbb{C}^n$ and construct a family of locally uniformly bounded functions. $\{u^c_\alpha(z)\} \subset sh_m\left(D \times \mathbb{R}^n_y\right)$. For this family Borel measures $\left\{H^k(u^c_\alpha)\right\}$, $k = 1, 2, ..., n - m + 1$ is uniformly bounded on average on compact subsets of the domain. $D \times \mathbb{R}^n_y$. From the definition of measures $\left\{H^k(u_\alpha)\right\}$ in Hessians it follows that the family of measures $\left\{H^k(u_\alpha)\right\}$, $k = 1, 2, ..., n - m + 1$ is uniformly bounded on average on compact subsets of the domain D.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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