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Numerical solution of source identification multi-point problem of parabolic partial differential equation with Neumann type boundary condition

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We study a source identification boundary value problem for a parabolic partial differential equation with multi-point Neumann type boundary condition. Stability estimates for the solution of the overdetermined mixed BVP for multi-dimensional parabolic equation were established. The first and second order of accuracy difference schemes for the approximate solution of this problem were proposed. Stability estimates for both difference schemes were obtained. The result of numerical illustration in test example was given.

Keywords: inverse problem, source identification, parabolic equation, difference scheme, stability, nonlocal condition, boundary value problem, well-posedness, stability estimates, mixed problem.

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Introduction

Various techniques can be used to solve source identification problems (SIPs) for parabolic equations. These may include optimization algorithms, regularization methods, or numerical techniques such as finite element and finite difference methods (see [1–28] a references therein).

In papers [3, 18], SIP for abstract differential equation with self-adjoint positive definite operator A

$$\frac{dv(t)}{dt} + Av(t) = p + f(t),\ 0 < t < 1, \tag{1}$$

$$v(0) = \varphi,\ v(1) = \psi \tag{2}$$

in a Hilbert space H was investigated. In paper [3], for solution of SIP (1), (2), stability estimates in the Hölder norms were obtained.

Some applications to boundary value problems (BVPs) for partial differential equation (PDE) and approximate solutions were studied in [8, 12].

Let $s_1, \mu_1, s_2, \mu_2, ..., s_r, \mu_r$ be given numbers so that

$$\sum_{k=1}^{r} |\mu_k| < 1,\ 0 \le s_1 < s_2 < \ldots < s_r < 1 \tag{3}$$

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and elements φ, $\psi \in H$ and function $f : [0,1] \to H$ are given. In paper [15] SIP to find a pair (p, v) for equation

$$\frac{dv(t)}{dt} + Av(t) = p + f(t), 0 < t < 1,$$

with multi-point nonlocal conditions

$$v(0) = \sum_{k=1}^{r} \mu_k v(s_k) + \psi, \tag{4}$$

$$v(1) = \varphi \tag{5}$$

was studied and stability estimates for the solution were given in the following theorem.

Theorem 1. [15] Assume that conditions (3) for interior points and coefficients are valid, $\varphi \in H$, $\psi \in D(A)$, $f \in C^{\alpha}(H)$ $(\alpha \in (0,1))$ are given. Then, for the solution $(v(t), p)$ of SIP (1), (4), (5) the stability estimates

$$\|v\|_{C(H)} \le M \left[\|\varphi\|_H + \|\psi\|_H + \|f\|_{C(H)}\right]$$

and

$$\|p\|_H \le M \left[\|A\varphi\|_H + \|A\psi\|_H + \frac{1}{\alpha}\|f\|_{C^{\alpha}(H)}\right]$$

are fullfilled, where $M \in R^{+}$ does not depend on f, ψ, φ and α. Here $C(H)$, $C^{\alpha}(H)$ and $C_1^{\alpha}(H)$ are the Banach spaces of H-valued functions $u(t)$ with the corresponding norms

$$\begin{array}{l} \|u\|_{C(H)} = \max\limits_{0\le t\le 1}\|u(t)\|_H, \|u\|_{C^{\alpha}(H)} = \|u\|_{C(H)} + \sup\limits_{0\le t<t+\tau\le 1}\frac{\|u(t+\tau)\|_H - \|u(t)\|_H}{\tau^{\alpha}}, \\ \|u\|_{C_1^{\alpha}(H)} = \|u\|_{C(H)} + \sup\limits_{0\le t<t+\tau\le 1}\frac{(1-t)^{\alpha}\|u(t+\tau)\|_H - \|u(t)\|_H}{\tau^{\alpha}}, \end{array} \tag{6}$$

respectively.

1 SI parabolic problem with multi-point boundary conditions

Now, we study a source identification (SI) BVP for the multi-dimensional PDE.

Let $\Omega = (0,1)^n \subset \mathbb{R}^n$ with boundary $S = \partial\Omega$, $\overline{\Omega} = \Omega \cup S$.

Denote by $L_2(\Omega)$ and $W_2^2(\Omega)$ the Hilbert spaces of integrable functions $u(y)$, defined on Ω, equipped with the corresponding norms

$$\|u\|_{L_2(\Omega)} = \left\{\int\limits_{y\in\Omega} |u(y)|^2 \, dy_1 \dots dy_n\right\}^{\frac{1}{2}},$$

$$\|u\|_{W_2^2(\Omega)} = \left\{\int\limits_{y\in\Omega}\left(|u(y)|^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left|u_{y_i y_j}(y)\right|^2\right) dy_1 \dots dy_n\right\}^{\frac{1}{2}}.$$

Let $\varphi \in L_2(\Omega)$, $\psi \in W_2^2(\Omega)$, $f \in C^{\alpha}(L_2(\Omega))$ be given functions, and $a_i : \Omega \to R^{+}$ be known smooth function for any index $i = 1, \dots, n$.

In $[0,1] \times \Omega$, we study multi-dimensional SIP for parabolic PDE with multi-point boundary and nonlocal conditions

$$\begin{cases} v_t(t,x) - \sum\limits_{i=1}^{n}(a_i(x) v_{x_i}(t,x))_{x_i} + \sigma v(t,x) = f(t,x) + p(x), \\ x = (x_1, \dots, x_n) \in \Omega, \ 0 < t < 1, \\ \frac{\partial}{\partial \vec{n}} v(t,x) = 0, \ x \in S, \ 0 < t < 1, \\ v(0,x) = \sum\limits_{k=1}^{r}\mu_k v(s_k, x) + \psi(x), \ v(1,x) = \varphi(x), \ x \in \overline{\Omega}, \end{cases} \tag{7}$$

where $\overrightarrow{n}$ is the normal vector to Ω at corresponding boundary point.

The differential expression

$$A^x u(x) = -\sum_{i=1}^{n} (a_i(x) u_{x_i}(x))_{x_i} + \sigma u(x)$$

defines SAPD operator A^x, which acts on the Hilbert space $L_2(\Omega)$ with the domain

$$D(A^x) = \left\{ u \mid u \in W_2^2(\Omega),\ \frac{\partial u}{\partial \overrightarrow{n}}(x) = 0 \text{ on } S \right\}.$$

So, the SI BVP (7) for the multi-dimensional parabolic PDE can be replaced with the abstract problem (1), (4), (5) for $H = L_2(\Omega)$. By using stability estimates of Theorem 1, we obtain the following stability estimates for solution of BVP (7).

Theorem 2. Suppose that conditions (3) are satisfied, $\varphi, \psi \in W_2^2(\Omega)$ and $f \in C^{\alpha}(L_2(\Omega))$. Then, for the solution of multi-dimensional SIP for parabolic PDE (7), the following estimates are valid

$$\|p\|_{L_2(\Omega)} \leq M \left[\|\varphi\|_{W_2^2(\Omega)} + \|\psi\|_{W_2^2(\Omega)} + \frac{1}{\alpha} \|f\|_{C^{\alpha}(L_2(\Omega))} \right],$$

$$\|v\|_{C(L_2(\Omega))} \leq M \left[\|\varphi\|_{L_2(\Omega)} + \|\psi\|_{L_2(\Omega)} + \|f\|_{C(L_2(\Omega))} \right],$$

where positive number M is independent of f, ψ, φ and α.

2 First and second order of ADSs

We will use the set of uniform grid points

$$[0,1]_\tau = \{t_k = k\tau,\ k = 0, 1, \ldots, N,\ N\tau = 1\}.$$

To discretize problem (7) we use algorithm with two steps. Firstly, we define grid spaces

$$\widetilde{\Omega}_h = \{x = x_m = (h_1 m_1, \ldots, h_n m_n);\ m = (m_1, \ldots, m_n),$$
$$m_j = 0, \ldots, N_j,\ h_j N_j = 1,\ j = 1, \ldots, n\},$$
$$\Omega_h = \widetilde{\Omega}_h \cap \Omega,\ S_h = \widetilde{\Omega}_h \cap S.$$

Introduce difference operator A_h^x by formula

$$A_h^x v^h(x) = -\sum_{i=1}^{n} \left(a_i(x) v_{\overline{x}_i}^h(x) \right)_{x_i, j_i} + \sigma v^h(x),$$

which acts in space of grid functions $v^h(x)$ and satisfies the condition $Dv^h(x) = 0$ for all $x \in S_h$.

Applying A_h^x, we arrive at the multi-point nonlocal BVP for some infinite system of ordinary differential equations. Secondly, by using Equation (26) [15; p. 1922], we get the first order of accuracy difference scheme (ADS)

$$\begin{cases} \tau^{-1} \left(v_k^h(x) - v_{k-1}^h(x) \right) + A_h^x v_k^h(x) = f^h(t_k, x) + p^h(x),\ 1 \leq k \leq N,\ x \in \widetilde{\Omega}_h, \\ v_N^h(x) = \varphi^h(x),\ v_0^h(x) = \sum\limits_{i=1}^{r} \mu_i v_{l_i}^h(x) + \psi^h(x),\ x \in \widetilde{\Omega}_h. \end{cases} \tag{8}$$

By using Equations (37)–(39) [15; p. 1925], we get the second order of ADS

$$\begin{cases} \tau^{-1}\left(v_k^h(x)-v_{k-1}^h(x)\right)+A_h^x\left(I+\dfrac{\tau A_h^x}{2}\right)v_k^h(x) \\ =\left(I+\dfrac{\tau A_h^x}{2}\right)\left(f^h(t_{k-\frac{\tau}{2}},x)+p^h(x)\right),\ 1\le k\le N,\ x\in\widetilde{\Omega}_h, \\ v_N^h(x)=\varphi^h(x),\ x\in\widetilde{\Omega}_h, \\ v_0^h(x)=\sum\limits_{i=1}^{r}\left\{\mu_i\left(1-\rho_i\right)v_{l_i}^h(x)+\mu_i\ \rho_i v_{l_i+1}^h(x)\right\}+\psi^h(x),\ x\in\widetilde{\Omega}_h. \end{cases} \tag{9}$$

Denote by $L_{2h}=L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2=W_2^2(\widetilde{\Omega}_h)$, the spaces of the grid functions $u^h(x)=\{u(h_1m_1,\dots,h_nm_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the corresponding norms

$$\left\|u^h\right\|_{L_{2h}}=\left(\sum_{x\in\widetilde{\Omega}_h}|u^h(x)|^2h_1\cdots h_n\right)^{1/2},$$

$$\left\|u^h\right\|_{W_{2h}^2}=\left\|u^h\right\|_{L_{2h}}+\left(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^{n}\left|(u^h(x))_{x_r\overline{x_r},m_r}\right|^2h_1\cdots h_n\right)^{1/2},$$

and by $\mathcal{C}_\tau(L_{2h})=\mathcal{C}([0,1]_\tau,L_{2h})$, the Banach space of L_{2h}-valued grid functions $u^\tau=\{u_k\}_1^N$ with the suitable norm $\|u^\tau\|_{\mathcal{C}_\tau(L_{2h})}=\max\limits_{1\le k\le N}\|u_k\|_{L_{2h}}$.

Let $\mathcal{C}^\alpha(L_{2h})=\mathcal{C}^\alpha([0,1]_\tau,L_{2h})$ and $\mathcal{C}_\tau^\alpha(L_{2h})=\mathcal{C}_\tau^\alpha([0,1]_\tau,L_{2h})$ be correspondingly Hölder and weighted Hölder spaces with the corresponding norms defined by (6) for $H=L_{2h}$.

Theorem 3. Suppose that τ and $|h|=\sqrt{h_1^2+\cdots+h_n^2}$ are sufficiently small positive numbers, $\varphi^h\in L_{2h}$, $\psi^h\in W_{2h}^2$ and $\left\{f_k^h\right\}_1^N\in\mathcal{C}_\tau^\alpha(L_{2h})$. Then, for the solution of difference schemes (DSs) (8) and (9), the following stability estimates hold

$$\left\|p^h\right\|_{\mathcal{C}_\tau(L_{2h})}\le M\ \left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{W_{2h}}+\frac{1}{\alpha}\left\|\left\{f_k^h\right\}_1^N\right\|_{\mathcal{C}_\tau^\alpha(L_{2h})}\right],$$

$$\left\|\left\{v_k^h\right\}_1^N\right\|_{\mathcal{C}_\tau(L_{2h})}\le M\ \left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^N\right\|_{\mathcal{C}_\tau(L_{2h})}\right],$$

where M is independent of $\left\{f_k^h\right\}_1^N$, $\psi^h(x)$, $\varphi^h(x)$ and τ.

The proof of Theorem 3 based on Theorems 3.1 and 3.2 of paper [15] on stability estimate for solutions of corresponding DSs for approximate solution of abstract SIP (1), (4), (5) and the theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

3 Numerical analysis

For test example, we consider the SIP

$$\begin{cases} v_t(t,x)-(3+2\cos x)v_{xx}\ (t,x)+2\sin x\cdot v_x(t,x)+\ v(t,x)=f(t,x)+p(x), \\ 0<x<\pi,\ 0<t<1, \\ v(1,x)=\varphi\ (x), \\ v(0,x)=v(\frac{1}{3},x)+\psi(x),\ 0\le x\le\pi, \\ v_x(t,0)=0,\ v_x(t,\pi)=0,\ 0\le t\le 1 \end{cases} \tag{10}$$

for one-dimensional parabolic PDE. Here

$$f(t,x) = \left(e^{-t} - e^{-1}\right)(3\cos x + 2\cos 2x) - e^{-1}\cos x,\ 0 < x < \pi,\ 0 < t < 1,$$
$$\varphi(x) = \cos x,\ \psi(x) = \left(1 - e^{-\frac{1}{3}}\right)\cos x,\ 0 \le x \le \pi.$$

It is easy to check that the pair $\left(e^{-1}(4\cos x + 2\cos 2x), e^{-t}\cos x\right)$ is the exact solution of problem (10).

An algorithm of finding the solution of problem (10) contains three stages. In the first stage, we find the solution of auxiliary BVP

$$\begin{cases} u_t(t,x) - (3 + 2\cos x)u_{xx}(t,x) + 2\sin x\cdot u_x(t,x) + u(t,x) \\ = \ (3 + 2\cos x)\cos x - 2\sin x\cdot \sin x + \cos x + f(t,x),\ 0 < x < \pi,\ 0 < t < 1, \\ u(1,x) - u(\frac{1}{3},x) = \psi(x),\ 0 \le x \le \pi, \\ u_x(t,0) = 0,\ u_x(t,\pi) = 0,\ 0 \le t \le 1. \end{cases} \tag{11}$$

Then, in the second stage, we find $p(x)$ by

$$p(x) = -(3 + 2\cos x)u_{xx}(1,x) + 2\sin x\cdot u_x(1,x) + u(1,x).$$

In the third stage, we put $p(x)$ in the right side of equation (10) and solve that problem for $v(t,x)$.

Introduce the set of grid points

$$[0,1]_\tau \times [0,\pi]_h = \{(t_k, x_n) \mid t_k = k\tau,\ k = 1,\dots,N-1,\ N\tau = 1,$$
$$x_n = nh,\ n = 1,\dots,M-1,\ Mh = \pi\}.$$

We use notation $l = \left[\frac{\gamma}{\tau}\right]$ for greatest integer function of $\frac{\gamma}{\tau}$ and $\rho = \frac{\gamma}{\tau} - l$.

So, we get the first order of ADS for SIP (10)

$$\begin{cases} \frac{v_n^k - v_n^{k-1}}{\tau} - \frac{(3+2\cos x_n)\left(v_{n+1}^k - 2v_n^k + v_{n-1}^k\right)}{h^2} + \frac{\sin(x_n)\left(v_{n+1}^k - v_{n-1}^k\right)}{h} + v_n^k \\ = f(t_k, x_n) + p(x_n),\ k = 1,\dots,N,\ n = 1,\dots,M-1, \\ v_n^N = \varphi_n,\ v_n^0 - \ \rho v_n^l = \psi_n,\ n = 0,\dots,M, \\ v_0^k = v_1^k,\ v_M^k = v_{M-1}^k,\ k = 0,\dots,N. \end{cases}$$

Later, $p(x_n)$ can be obtained by

$$p(x_n) = -\ \frac{(3 + 2\cos(x_n))\left(u_{n+1}^N - 2u_n^N + u_{n-1}^N\right)}{h^2} + \frac{\sin(x_n)\left(u_{n+1}^N - u_{n-1}^N\right)}{h} + u_n^N, \tag{12}$$

where $\left\{u_n^k\right\}$ is solution of the difference problem

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{(3 + 2\cos(x_n))\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)}{h^2} \\ +\frac{\sin(x_n)\left(u_{n+1}^k - u_{n-1}^k\right)}{h} + u_n^k = f(t_k, x_n) - \frac{(3 + 2\cos(x_n))(\varphi_{n+1} - 2\varphi_n + \varphi_{n-1})}{h^2} \\ +\frac{\sin(x_n)(\varphi_{n+1} - \varphi_{n-1})}{h} + \varphi_n,\ k = 1,\dots,N,\ n = 1,\dots,M-1, \\ u_n^0 - u_n^l = \psi_n,\ n = 0,\dots,M, \\ u_0^k - u_M^k = 0,\ u_M^k - u_{M-1}^k = 0,\ k = 0,\dots,N, \end{cases} \tag{13}$$

which is the first order of ADS for approximate solution of the nonlocal BVP (11).

For computational reasons it is convenient to write (13) in the following matrix form

$$\begin{gathered} A_n u_{n+1} + B_n u_n + C_n u_{n-1} = I\theta_n,\ n = 1,\dots,M-1, \\ u_0 = u_1,\ u_M = u_{M-1}. \end{gathered} \tag{14}$$

Here, θ_n is column vector, A_n, B_n, C_n, I are square matrices with $(N+1)$ rows and columns:

$$A_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ a_n R & & & \vdots \\ & & & 0 \end{bmatrix}, \quad C_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ c_n R & & & \vdots \\ & & & 0 \end{bmatrix},$$

$$B_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & -1 & \cdots & 0 & 0 & 0 & 0 \\ b_n & d & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_n & d & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & b_n & d & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & b_n & d \end{bmatrix},$$

$$\begin{array}{l} a_n = -(3 + 2\cos(x_n))h^{-2} + \sin(x_n)h^{-1}, d = \frac{1}{\tau}, \\ b_n = 1 + d + 2(3 + 2\cos(x_n))h^{-2}, \\ c_n = -(3 + 2\cos(x_n))h^{-2} - \sin(x_n)h^{-1}, \end{array}$$

$$\theta_n = \begin{matrix} \theta_n^0 \\ \vdots \\ \theta_n^N \end{matrix}, \quad u_{n\pm1} = \begin{bmatrix} u_{n\pm1}^0 \\ \vdots \\ u_{n\pm1}^N \end{bmatrix}_{(N+1)\times 1}, \quad u_n = \begin{bmatrix} u_n^0 \\ \vdots \\ u_n^N \end{bmatrix}_{(N+1)\times 1}.$$

R is the $N \times N$ identity matrix, as well as

$$\begin{array}{l} \theta_n^0 = \psi_n, \ n = 1, \dots, M-1, \\ \theta_n^k = f(t_k, x_n) - \frac{(3+2\cos(x_n))(\varphi_{n+1} - 2\varphi_n + \varphi)}{h^2} + \frac{\sin(x_n)(\varphi_{n+1} - \varphi_{n-1})}{h} + \varphi_n, \\ k = 1, \dots, N, \ n = 1, \dots, M-1. \end{array}$$

We search solution of (14) by reccurence formula

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \ n = M-1, \dots, 1,$$

where α_n and β_n $(n = 1, \dots, M-1)$ are column vectors with $(N+1)$ elements. For the solution of difference equation (14) we use the following formulas for α_n, β_n

$$\begin{array}{rcl} \alpha_n & = & -(B_n + C_n\alpha_{n-1})^{-1} A_n, \\ \beta_n & = & (B_n + C_n\alpha_{n-1})^{-1}(R\theta_n - C_n\beta_{n-1}), \ n = 1, \dots, M-1, \end{array}$$

where α_1 is the $(N+1) \times (N+1)$ identity matrix and β_1 is the column vector with $(N+1)$ zeros. u_M is computed by formula

$$u_M = (A_M + B_M + C_M\alpha_M)^{-1}(I\theta_M - C_M\beta_M).$$

Second, applying appropriate approximation formulas for derivatives in the nonlocal BVP (10), we get the second order of ADS in t and x

$$\begin{cases} \frac{v_n^k - v_n^{k-1}}{\tau} + \frac{q_2\left(v_{n+1}^k - v_{n-1}^k\right)}{2h} + \frac{q_3\left(v_{n+1}^k - 2v_n^k + v_{n-1}^k\right)}{h^2} \\ + \frac{\tau q_0}{2} \frac{\left(v_{n+2}^k - 3v_{n+1}^k + 3v_n^k - v_{n-1}^k\right)}{h^3} + \frac{\tau q_1}{2} \frac{\left(v_{n+2}^k - 4v_{n+1}^k + 6v_n^k - 4v_{n-1}^k + v_{n-2}^k\right)}{h^4} \\ = \theta_n^k + p(x_n) - \frac{\tau}{2} \cdot \frac{(3+2\sin x_n)\ (p(x_{n+1}) - 2p(x_n) + p(x_{n-1}))}{h^2} \\ -\frac{\tau}{2} \cdot \frac{\cos(x_n)(p(x_{n+1}) - p(x_{n-1}))}{h} + \frac{\tau p(x_n)}{2}, \\ k = 1, \dots N,\ n = 2, \dots, M-2, \\ -3v_0^k + 4v_1^k - v_2^k = 0, \ \ -3v_M^k + 4v_{M-1}^k - v_{M-2}^k = 0, \\ 10v_0^k - 15v_1^k + 6v_2^k - v_3^k = 0, \\ 10v_M^k - 15v_{M-1}^k + 6v_{M-2}^k - v_{M-3}^k = 0,\ k = 0, \dots, N,\ n = 0, \dots, M, \\ v_n^N = \varphi_n,\ v_n^0 - (1-\rho)v_n^l - \rho v_n^{l+1} = \psi(x_n),\ n = 0, \dots, M \end{cases}$$

for the approximate solution of the nonlocal BVP (10).

Later, we calculate $p(x_n)$ by using (12), where $\left\{u_n^k\right\}$ is solution of the difference problem

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} + \frac{q_2\left(u_{n+1}^k - u_{n-1}^k\right)}{2h} + \frac{q_3\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)}{h^2} + \frac{\tau}{2} \frac{q_0\left(u_{n+2}^k - 2u_{n+1}^k + 2u_{n-1}^k - u_{n-2}^k\right)}{2h^3} \\ + \frac{\tau}{2} \frac{q_1\left(u_{n+2}^k - 4u_{n+1}^k + 6u_n^k - 4u_{n-1}^k + u_{n-2}^k\right)}{h^4} = \theta_n^k, k = 1, \dots, N,\ n = 2, \dots, M-2, \\ -3u_0^k + 4u_1^k - u_2^k = 0, \ \ -3u_M^k + 4u_{M-1}^k - u_{M-2}^k = 0, \\ 10u_0^k - 15u_1^k + 6u_2^k - u_3^k = 0, \\ 10u_M^k - 15u_{M-1}^k + 6u_{M-2}^k - u_{M-3}^k = 0,\ k = 0, \dots, N, \\ u_n^0 - (1-\rho)u_n^l - \rho u_n^{l+1} = \psi(x_n),\ n = 0, \dots, M, \end{cases} \tag{15}$$

which is the second order of ADS for nonlocal BVP (11).

For computational reasons it is convenient to rewrite the system (15) in the following matrix form

$$\begin{aligned} & A_n u_{n+2} + B_n u_{n+1} + C_n u_n + D_n u_{n-1} + E_n u_{n-2} = I\theta_n,\ n = 2, \dots, M-2, \\ & -3u_0 + 4u_1 - u_2 = \vec{0}, \ \ -3u_M + 4u_{M-1} - u_{M-2} = \vec{0}, \\ & 10u_0 - 15u_1 + 6u_2 - u_3 = \vec{0},\ 10u_M - 15u_{M-1} + 6u_{M-2} - u_{M-3} = \vec{0}, \end{aligned} \tag{16}$$

where θ_n is column vector, A_n, B_n, C_n, D_n, E_n, I are $(N+1)\times(N+1)$ square matrices, R is $N\times N$ identity matrix,

$$A_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ & e_n R & & \vdots \\ & & & 0 \end{bmatrix}, \quad B_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ & y_n R & & \vdots \\ & & & 0 \end{bmatrix},$$

$$D_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ & z_n R & & \vdots \\ & & & 0 \end{bmatrix}, \quad E_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ & w_n R & & \vdots \\ & & & 0 \end{bmatrix},$$

$$C_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & -(1-\rho) & \rho & \cdots & 0 & 0 & 0 \\ r_n & d & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & r_n & d & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & r_n & d & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & r_n & d \end{bmatrix}, \quad \theta_n = \begin{bmatrix} \theta_n^0 \\ \vdots \\ \theta_n^N \end{bmatrix},$$

$$\begin{aligned}
&e_n = \frac{\tau q_0}{4h^3} + \frac{\tau q_1}{2h^4},\ y_n = \frac{q_2}{2h} + \frac{1}{h^2} q_3 - \frac{\tau q_0}{2h^3} - \frac{2\tau q_1}{h^4},\\
&r_n = 1 + \frac{1}{\tau} + \frac{\tau}{2} - \frac{2}{h^2} q_3 + \frac{3\tau q_1}{h^4},\\
&z_n = -\frac{q_2}{2h} + \frac{1}{h^2} q_3 + \frac{\tau q_0}{h^3} - \frac{2\tau q_1}{h^4},\\
&w_n = -\frac{\tau q_0}{4h^3} + \frac{\tau q_1}{2h^4},\ n = 2, \ldots, M-2.
\end{aligned}$$

We search solution of linear system equation (16) in the next form

$$\begin{aligned}
&u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1} u_{n+2} + \gamma_{n+1},\ n = M-2, \ldots, 0,\\
&u_M = D_M^{-1} \left((3I - 2\alpha_{M-2}) \gamma_{M-1} - 3\gamma_{M-2} \right),\\
&u_{M-1} = D_M^{-1} \left[(4I - \alpha_{M-2}) \gamma_{M-1} - \gamma_{M-2} \right],\\
&D_M = (\beta_{M-2} + 5I) - (4I - \alpha_{M-2}) \alpha_{M-1},
\end{aligned}$$

where

$$\gamma_0 = \gamma_1 = \overrightarrow{0},\ \alpha_0 = \frac{4}{3} I,\ \beta_0 = -\frac{1}{3} I,\ \alpha_1 = \frac{8}{5} I,\ \beta_1 = -\frac{3}{5} I$$

$$\gamma_{M-2} = \gamma_{M-3} = \overrightarrow{0},\ \alpha_{M-2} = 4I,\ \beta_{M-2} = -3I,\ \alpha_{M-3} = \frac{8}{5} I,\ \beta_{M-3} = -\frac{3}{5} I,$$

and

$$\begin{aligned}
&F_n = \left(C_n + D_n \alpha_{n-1} + E_n \beta_{n-2} + E_n \alpha_{n-2} \alpha_{n-1} \right),\ n = 2, \ldots, M-4.\\
&\alpha_n = -F_n^{-1} \left(B_n + D_n \beta_{n-1} + E_n \alpha_{n-2} \beta_{n-1} \right),\ \beta_n = -F_n^{-1} A_n,\\
&\gamma_n = -F_n^{-1} \left(I\varphi_n - D_n \gamma_{n-1} - E_n \alpha_{n-2} \gamma_{n-1} - E_n \gamma_{n-2} \right),
\end{aligned}$$

$$\begin{aligned}
&Q_{11} = -3B_{M-2} - 8C_{M-2} - 8D_{M-2}\alpha_{M-3} - 3D_{M-2}\beta_{M-3}\\
&-8E_{M-2}\alpha_{M-4}\alpha_{M-3} - 3E_{M-2}\alpha_{M-4}\beta_{M-3} - 8E_{M-2}\beta_{M-4},\\
&Q_{12} = A_{M-2} + 4B_{M-2} + 9C_{M-2} + 9D_{M-2}\alpha_{M-3} + 4D_{M-2}\beta_{M-3}\\
&+9E_{M-2}\alpha_{M-4}\alpha_{M-3} + 4E_{M-2}\alpha_{M-4}\beta_{M-3} + 9E_{M-2}\beta_{M-4},\\
&Q_{21} = A_{M-1} - 3C_{M-1} - 8D_{M-1} - E_{M-1}(8\alpha_{M-3} + 3\beta_{M-3}),\\
&Q_{22} = B_{M-1} + 4C_{M-1} + 9D_{M-1} + E_{M-1}(9\alpha_{M-3} + 4\beta_{M-3}),\\
&G_1 = I\varphi_{M-2} - D_{M-2}\gamma_{M-3} - E_{M-2}\alpha_{M-4}\gamma_{M-3} - E_{M-2}\gamma_{M-3},\\
&G_2 = I\varphi_{M-1} - E_{M-1}\gamma_{M-3},\\
&u_M = (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})^{-1}(G_1 - Q_{12}Q_{22}^{-1}G_2),\\
&u_{M-1} = Q_{22}^{-1}(G_2 - Q_{21}u_M).
\end{aligned}$$

Numerical illustration is carried out by using MATLAB program. Solutions of DSs are computed for different values of (N, M). v_n^k and u_n^k correspond to the corresponding numerical values of $v(t,x)$ and $u(t,x)$ at $(t,x) = (t_k, x_n)$ and p_n represents the numerical value of $p(x)$ at point $x = x_n$. The errors are computed by

$$\begin{aligned}
&Ev_M^N = \max_{0 \le k \le N} \left(\sum_{n=1}^{M-1} \left| v(t_k, x_n) - v_n^k \right|^2 h \right)^{\frac{1}{2}},\\
&Eu_M^N = \max_{0 \le k \le N} \left(\sum_{n=1}^{M-1} \left| u(t_k, x_n) - u_n^k \right|^2 h \right)^{\frac{1}{2}},\\
&Ep_M = \left(\sum_{n=1}^{M-1} \left| p(x_n) - p_n \right|^2 h \right)^{\frac{1}{2}}.
\end{aligned}$$

Tables 1 and 2 illustrate the errors between the exact and approximate solutions of DSs for various N and M, respectively. It can be seen from output results that the second order of ADS is more accurate than the first order of ADS. The error analysis shown in Tables 1 and 2 indicate that both DSs have correct convergence rates.

Table 1

Mesh grid absolute value of difference between exact solution and solution of first order of ADS

N=M	20	40	80	160
Ev_M^N	0.034277	0.016674	0.008483	0.004278
Ep_M	0.086716	0.043925	0.022123	0.011104
Ev_M^N	0.152320	0.075113	0.037321	0.018601

Table 2

Mesh grid absolute value of difference between exact solution and solution of second order of ADS

N=M	20	40	80	160
Ev_M^N	0.020123	0.004141	0.000920	0.000217
Ep_M	0.08946	0.024373	0.006796	0.001926
Ev_M^N	0.089678	0.018803	0.004188	0.000969

Conclusion

In this work, SIP for a multi-dimensional parabolic partial differential equation with multi-point nonlocal boundary condition was studied. Stability estimates for solution of inverse problem were obtained. Well-posedness of three SIPs for the reverse parabolic partial differential equations was established.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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