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Absolutely stable difference scheme for the delay partial differential equation with involution and Robin boundary condition

A. Ashyralyev[1,2,3], S. Ibrahim[4,*], E. Hincal[4]

[1] *Department of Mathematics, Bahcesehir University Istanbul, Turkey;*
[2] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[3] *Peoples' Friendship University of Russia (RUDN University), Moscow, Russian Federation;*
[4] *Department of Mathematics, Near East University, Nicosia, TRNC, Mersin 10, Turkey*
(E-mail: allaberen.ashyralyev@bau.edu.tr, ibrahim.suleiman@neu.edu.tr, evren.hincal@neu.edu.tr)

This paper examines the initial value problem for a third-order delay partial differential equation with involution and Robin boundary condition. We construct a first-order accurate difference scheme to obtain the numerical solution for this equation. Illustrative numerical results are provided.

Keywords: numerical algorithm, involution, Robin boundary condition, third order partial differential equations, delay.

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Introduction

Over the years, nonlocal and local boundary value problems (BVPs) for third-order partial differential equations (PDEs) have gone through extensive investigations (see, for instance, [1–9]). Time delay (TD) is a common phenomenon in various engineering projects. The theory as well as applications of delay nonlinear and linear third-order ordinary differential and difference equations having delay terms have been explored in numerous works (see, for instance, [10–16]).

The stability of the third order partial delay differential equation (PDDE) having involution and Dirichlet condition was investigated in [17]. Nevertheless, the third order PDDE with involution and Robin condition (IRC) is not studied before. Therefore, the main motivation for this paper is to study the stability of the third order partial delay differential and difference equations with IRC.

1 Differential problem stability

In $[0,\infty)\times(-\rho,\rho)$, the initial BVP for the TD third order PDE with IRC.

$$\begin{cases} \frac{\partial^3 u(\zeta,y)}{\partial\zeta^3}-\left(\delta(y)u_{\zeta y}(\zeta,y)\right)_y+\beta\left(\delta(-y)u_{\zeta,-y}(\zeta,-y)\right)_{-y} \\ =-b\left(-\delta(y)u_y(\zeta-w,y)\right)_y+\beta\left(\delta(-y)u_{-y}(\zeta,-y)\right)_{-y} \\ +\Phi(\zeta,y),\ 0<t<\infty,(-\rho,\rho), \\ \\ u(\zeta,y)=g(\zeta,y),-w\le\zeta\le 0,y\in[-\rho,\rho], \\ \\ \alpha_1 u(\zeta,-\rho)-\gamma_1 u_y(\zeta,-\rho)=0,\alpha_2 u(\zeta,\rho)+\gamma_2 u_y(\zeta,\rho)=0,\ \ 0\le\zeta<\infty \end{cases} \tag{1}$$

*Corresponding author. *E-mail: ibrahim.suleiman@neu.edu.tr*

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is studied. In this study, we make the assumption that $w > 0$, $\overline{\delta} \geq \delta(y) = \delta(-y) \geq \underline{\delta} > 0$, $y \in (-t, t)$ and $\underline{\delta} - \overline{\delta}|\beta| \geq 0$, α_1, $\alpha_2, \gamma_1, \gamma_2$ are non negative constants.

We examine the Hilbert space $L_2[-\rho, \rho]$ consisting of all square integrable functions defined on $[-\rho, \rho]$, equipped with the norm

$$\| \Phi \|_{L_2[-\rho,\rho]} = \left(\int_{-\rho}^{\rho} |\Phi(y)|^2 dy \right)^{\frac{1}{2}}.$$

A unique solution $u(\zeta, y)$ is possessed by problem (1) for the smooth functions $\delta(y)$, $y \in (-t, t)$, $g(\zeta, y), -w \leq \zeta \leq 0$, $y \in [-\rho, \rho], \Phi(\zeta, y)$, $0 < \zeta < \infty$, $y \in (-\rho, \rho)$, and $b \in R^1$, provided the compatibility conditions are met.

Theorem 1. The following stability estimates hold for the solutions of problem (1):

$$\max_{0 \leq \zeta \leq nw} \|v_{\zeta\zeta}(\zeta, \cdot)\|_{W_2^1(-\rho,\rho)}, \max_{0 \leq \zeta \leq nw} \|v_{\zeta}(\zeta, \cdot)\|_{W_2^2(-\rho,\rho)}, \max_{0 \leq \zeta \leq nw} \|v(\zeta, \cdot)\|_{W_2^3(-\rho,\rho)}$$

$$\leq M_2 \left[(2 + |b|\, w)^n a_0 + \sum_{i=1}^{n} (2 + |b|\, w)^{n-i} \int_{(i-1)\omega}^{iw} \|\Phi(s, \cdot)\|_{W_2^1(-\rho,\rho)} ds \right],$$

$$a_0 = \max \left\{ \max_{-w \leq \zeta \leq 0} \|g_{\zeta\zeta}(\zeta, \cdot)\|_{W_2^1(-\rho,\rho)}, \right.$$

$$\left. \max_{-w \leq \zeta \leq 0} \|g(\zeta, \cdot)\|_{W_2^3(-\rho,\rho)} \right\}.$$

Here, the Sobolev spaces W_2^k for $k = 1, 2, 3$ consist of all square integrable functions $\psi(y)$ defined on $[-\rho, \rho]$, each equipped with their respective norms

$$\|\psi\|_{W_2^k(-\rho,\rho)} = \left(\int_{-\rho}^{\rho} \sum_{i=0}^{k} \left(\psi_{\underbrace{y \cdots y}_{i \text{ time}}}(y) \right)^2 dy \right)^{\frac{1}{2}}.$$

Note that M_2 does not depend on $g(t, y)$ and $\Phi(\zeta, y)$.

Proof. With this we are able to change problem (1) to the following initial value problem

$$\begin{cases} \frac{d^3 v(\zeta)}{d\zeta^3} + A \frac{dv(\zeta)}{d\zeta} = bAv(\zeta - w) + \Phi(\zeta), \ 0 < \zeta < \infty, \\ v(\zeta) = g(\zeta), -w \leq \zeta \leq 0 \end{cases} \tag{2}$$

in $H = L_2[-\rho, \rho]$ which happens to be Hilbert space having a self-adjoint positive definite operator (SAPDO) A that is given by the formula below:

$$Au(y) = -(\delta(y)u_y(y))_y + \beta(\delta(-y)u_{-y}(-y))_{-y}, \tag{3}$$

having domain

$$D(A) = \{u(y) : u(y),\ u_y(y),\ (\delta(y)u_y)_y \in L_2[-\rho, \rho], \alpha_1 u(-\rho) - \gamma_1 u_y(-\rho) = 0, \alpha_2 u(\rho) + \gamma_2 u_y(\rho) = 0\}.$$

Theorem 1's proof relies on the positive definiteness as well as the self-adjointness of the space operator A as specified by equation (3), as well as the results presented in paper [18]. Additionally, the proof incorporates the theorem on the stability of the solution to problem (2).

Theorem 2. [19] The following estimate applies to the solution of problem (2):

$$\max_{0\le\zeta\le nw}\left\|A^{\frac{1}{2}}\frac{d^2v(\zeta)}{d\zeta^2}\right\|_H,\ \max_{0\le\zeta\le nw}\left\|A\frac{d\zeta(\zeta)}{d\zeta}\right\|_H,\frac{1}{2}\max_{0\le\zeta\le nw}\left\|A^{\frac{3}{2}}v(\zeta)\right\|_H$$

$$\le (2+|b|\,w)^n a_0+\int_0^{nw}\left\|A^{\frac{1}{2}}\Phi(s)\right\|_H ds, n=1,2,...,$$

where

$$a_0=\max\left\{\max_{-w\le\zeta\le 0}\left\|A^{\frac{1}{2}}\frac{d^2g(\zeta)}{d\zeta^2}\right\|_H,\max_{-w\le\zeta\le 0}\left\|A\frac{dg(\zeta)}{d\zeta}\right\|_H,\max_{-w\le\zeta\le 0}\left\|A^{\frac{3}{2}}g(t)\right\|_H\right\}.$$

Stability of the difference scheme

For the approximate solution of problem (1), we study the stable difference scheme (DS). Problem (1) discretization is conducted in two stages.

Firstly, the spatial discretization is executed. The equation below defines the grid space:

$$[-t,t]_h=\left\{y=y_n \mid y_n=nh,\ -\Gamma\le n\le\Gamma,\ \Gamma h=t\right\}.$$

We present the Hilbert space $L_{2h}=L_2([-t,t]_h)$ of the grid functions $\varphi^h(y)=\{\varphi^n\}_{-\Gamma}^{\Gamma}$ defined on $[-t,t]_h$, endowed with the norm

$$\left\|\varphi^h\right\|_{L_{2h}}=\left(\sum_{y\in[-t,t]_h}\left|\varphi^h(y)\right|^2 h\right)^{1/2}.$$

We associate the difference operator A_h^y with the differential operator A that is defined by equation (3), using the following expression

$$A_h^y\varphi^h(y)=\left\{-\left(\delta(y)\varphi_{\bar{y}}^n\right)_y-\beta\left(\delta(-y)\varphi_{\bar{y}}^{-n}\right)_y\right\}_{-\Gamma+1}^{\Gamma-1}, \tag{4}$$

that acts in the space of grid functions $\varphi^h(y)=\{\varphi^n\}_{-\Gamma}^{\Gamma}$ and meeting the requirements

$$\alpha_1 h\varphi^{-\Gamma}-\gamma_1(\varphi^{-\Gamma}-\varphi^{-\Gamma+1})=0,\ \alpha_2 h\varphi^{\Gamma}+\gamma_2(\varphi^{\Gamma}-\varphi^{\Gamma-1})=0.$$

Here

$$\varphi_{\bar{y}}^n=\frac{\varphi^n-\varphi^{n-1}}{h},\ -\Gamma+1\le n\le\Gamma,\quad \varphi_y^n=\frac{\varphi^{n+1}-\varphi^n}{h},\ -\Gamma\le n\le\Gamma-1.$$

It is properly-established that A_h^y, as defined by equation (4) is a SAPDO in L_{2h}. By making use of A_h^y, the initial discretization step leads to the problem that follows:

$$\begin{cases}\dfrac{\partial^3u^h(\zeta,y)}{\partial\zeta^3}+A_h^yu^h(\zeta,y)=-bA_h^yu^h(\zeta-w,y)\\ +\Phi^h(\zeta,y),\ y\in[-t,t]_h,\ 0<\zeta<\infty,\\ u^h(\zeta,y)=g^h(\zeta,y),-w\le\zeta\le 0,\ y\in[-t,t]_h,\ -w<\zeta<0.\end{cases} \tag{5}$$

Secondly, problem (5) is replaced with the following first order of accuracy DS

$$\begin{cases} \frac{u_{k+2}^h(y)-3u_{k+1}^h(y)+3u_k^h(y)-u_{k-1}^h(y)}{\eta^3} + A_h^y \frac{u_{k+2}^h(y)-u_{k+1}^h(y)}{\eta} \\ = bA_h^y u_{k-\mathrm{M}}^h(y) + \Phi_k^h(y), \Phi_k^h(y) = \Phi^h(\zeta_k, y), k \geq 1, \quad y \in [-t,t]_h, \\ u_k^h(y) = g^h(\zeta_k, y), -\mathrm{M} \leq k \leq 0, \\ (\Upsilon_h + \eta^2 A_h^y)\frac{u_1^h(y)-u_0^h(y)}{\eta} = g_\zeta^h(0,y), \\ (\Upsilon_h + \eta^2 A_h^y)\frac{u_2^h(y)-2u_1^h(y)+u_0^h(y)}{\eta^2} = g_{\zeta\zeta}^h(0,y), y \in [-t,t]_h, \\ (\Upsilon_h + \eta^2 A_h^y)\frac{u_{m\mathrm{M}+1}^h(y)-u_{m\mathrm{M}}^h(y)}{\eta} = \frac{u_{m\mathrm{M}}^h(y)-u_{m\mathrm{M}-1}^h(y)}{\eta}, y \in [-t,t]_h, \\ (\Upsilon_h + \eta^2 A_h^y)\frac{u_{m\mathrm{M}+2}^h(y)-2u_{m\mathrm{M}+1}^h(y)+u_{m\mathrm{M}}^h(y)}{\eta^2} \\ = \frac{u_{m\mathrm{M}}^h(y)-2u_{m\mathrm{M}-1}^h(y)+u_{m\mathrm{M}-2}^h(y)}{\eta^2}, y \in [-t,t]_h, m = 1,2,..., \end{cases} \tag{6}$$

here $\eta = 1/\mathrm{M}$ and $\zeta_k = k\eta$, $-\mathrm{M} \leq k < \infty$.

Theorem 3. Let h and η be values that are small enough. The following estimates hold for the solution of DS (6):

$$\max_{0\leq k\leq (m+1)\mathrm{M}-2} \left\| \frac{u_{k+2}^h - 2u_{k+1}^h + u_k^h}{\eta^2} \right\|_{W_{2h}^1}, \max_{1\leq k\leq (m+1)\mathrm{M}} \left\| \frac{u_k^h - u_{k-1}^h}{\eta} \right\|_{W_{2h}^2},$$

$$\max_{0\leq k\leq (m+1)\mathrm{M}} \|u_k^h\|_{W_{2h}^3} \leq \chi_1 \Big[(2+\eta|b|(\mathrm{M}-2))^m b_0^h$$

$$+ \sum_{i=1}^{m} (2+\eta|b|(\mathrm{M}-2))^{m-i} \eta \sum_{s=(i-1)\mathrm{M}+1}^{i\mathrm{M}} \|\Phi(\zeta_s)\|_{W_{2h}^1} \Big], m = 0,1,...,$$

$$b_0^h = \max\left\{ \max_{-\mathrm{M}\leq k\leq 0} \|g_{\zeta\zeta}^h(\zeta_k)\|_{W_{2h}^1}, \max_{-\mathrm{M}\leq k\leq 0} \|g_\zeta^h(\zeta_k)\|_{W_{2h}^2}, \max_{-\mathrm{M}\leq k\leq 0} \|g^h(\zeta_k)\|_{W_{2h}^3} \right\}.$$

Here, W_{2h}^1, W_{2h}^2 and W_{2h}^3 represent spaces of all mesh functions $\psi^h(\zeta)$ defined on the interval $[-\rho,\rho]_h$ having the specific norm

$$\left\|\psi^h\right\|_{W_{2h}^k} = \left(\sum_{y\in[-\rho,\rho]} \sum_{i=0}^{k} \left(\psi^h_{\underbrace{y\cdots y}_{i\ \text{time}}}(y) \right)^2 h^k \right)^{\frac{1}{2}}.$$

Note that χ_1 does not depend on η, h, $g^h(t_k)$, and $\Phi_k^h(y)$.

Proof. DS (6) can be written in abstract form

$$\begin{cases} \frac{u_{k+2}^h-3u_{k+1}^h+3u_k^h-u_{k-1}^h}{\eta^3}+A_h\frac{u_{k+2}^h-u_{k+1}^h}{\eta}=bA_hu_{k-\mathrm{M}}^h+\Phi_k^h, k\geq 1, \\ u_k^h=g_k^h, -\mathrm{M}\leq k\leq 0, \\ (\Upsilon_h+\eta^2A_h)\frac{u_1^h-u_0^h}{\eta}=g_\zeta^h(0), (\Upsilon_h+\eta^2A_h)\frac{u_2^h-2u_1^h+u_0^h}{\eta^2}=g_{\zeta\zeta}^h(0), \\ (\Upsilon_h+\eta^2A_h)\frac{u_{m\mathrm{M}+2}^h-2u_{m\mathrm{M}+1}^h+u_{m\mathrm{M}}^h}{\eta^2}=\frac{u_{m\mathrm{M}}^h-2u_{m\mathrm{M}-1}^h+u_{m\mathrm{M}-2}^h}{\eta^2}, \\ (\Upsilon_h+\eta^2A_h)\frac{u_{m\mathrm{M}+1}^h-u_{m\mathrm{M}}^h}{\eta}=\frac{u_{m\mathrm{M}}^h-u_{m\mathrm{M}-1}^h}{\eta}, m=1,2,... \end{cases} \tag{7}$$

in L_{2h} which is a Hilbert space with SAPDO $A_h=A_h^y$ that is defined using the formula (4). Where, $g_k^h=g_k^h(y)$, $\Phi_k^h=\Phi_k^h(y)$ and $u_k^h=u_k^h(y)$ are known and unknown abstract mesh functions that are defined on $[-\rho,\rho]_h$ with the values in $H=L_{2h}$. Consequently, Theorem 2 proof relies on the theorem 4 below as well as the self-adjointness and positive definiteness of the space operator A_h (4) [20].

Theorem 4. [21] The following estimate holds for the solution of DS (7):

$$\frac{1}{2}\max_{0\leq k\leq (m+1)\mathrm{M}-2}\left\|A_h^{\frac{1}{2}}\frac{u_{k+2}^h-2u_{k+1}^h+u_k^h}{\eta^2}\right\|_H, \max_{1\leq k\leq (m+1)\mathrm{M}}\left\|A_h\frac{u_k^h-u_{k-1}^h}{\eta}\right\|_H,$$

$$\max_{0\leq k\leq (m+1)\mathrm{M}}\|A_h^{\frac{3}{2}}u_k^h\|_H\leq \chi_1\left[(2+\eta|b|(\mathrm{M}-2))^m b_0^h\right.$$

$$\left.+\sum_{i=1}^{m}(2+\eta|b|(\mathrm{M}-2))^{m-i}\eta\sum_{s=(i-1)\mathrm{M}+1}^{i\mathrm{M}}\|A_H^{\frac{1}{2}}\Phi(\zeta_s)\|_H\right], m=0,1,...,$$

where $b_0=\max\left\{\max_{-\mathrm{M}\leq k\leq 0}\|A_h^{\frac{1}{2}}g''(\zeta_k)\|_H,\ \max_{-\mathrm{M}\leq k\leq 0}\|A_hg_\zeta^h(\zeta_k)\|_H,\ \max_{-\mathrm{M}\leq k\leq 0}\|A_h^{\frac{3}{2}}g^h(\zeta_k)\|_H\right\}$.

2 *Numerical algorithm for the third order delay partial differential equation*

We give the algorithm for numerically solving the initial BVP for third order delay PDE having involution and Robin boundary condition

$$\begin{cases} \frac{\partial^3u(\zeta,y)}{\partial\zeta^3}-\frac{\partial^3u(\zeta,y)}{\partial\zeta\partial y^2}+8\frac{\partial u(\zeta,y)}{\partial\zeta}-\frac{1}{8}\frac{\partial^3u(\zeta,-y)}{\partial\zeta\partial y^2}+\frac{\partial u(\zeta,-y)}{\partial\zeta} \\ =-0.1(-\frac{\partial^2u(\zeta-1,y)}{\partial y^2}+8u(\zeta-1,y)) \\ -35e^{-2\zeta}\cos 2y+1.2e^{-2(\zeta-1)}\cos 2y, \\ 0<\zeta<\infty,\ -\pi<y<\pi, \\ u(\zeta,y)=e^{-2\zeta}\cos 2y,\ -1\leq\zeta\leq 0,\ -\pi\leq y\leq\pi, \\ u(\zeta,-\pi)-e^{-2\zeta}=28u_y(\zeta,-\pi), -u(\zeta,\pi)+e^{-2\zeta}=28u_y(\zeta,\pi),\ 0\leq\zeta<\infty. \end{cases} \tag{8}$$

The exact solution of problem (8) is $u(\zeta,y)=e^{-2\zeta}\cos 2y, -\pi\leq y\leq\pi, -1\leq\zeta<\infty$. We use the set of grid points $[-1,\infty)_\eta\times[-\pi,\pi]_h=\{(\zeta_k,y_n):\zeta_k=k\eta,-\mathrm{M}\leq k,\mathrm{M}\eta=1,y_n=nh,\ -\Gamma\leq n\leq\Gamma,$

$\Gamma h=\pi\}$, for the approximate solutions of the problem (8), we get the first order of accuracy DS in t

$$\begin{cases}\frac{u_n^{k+2}-3u_n^{k+1}+3u_n^{k}-u_n^{k-1}}{\eta^3}-\frac{u_{n+1}^{k+2}-u_{n+1}^{k+1}-2(u_n^{k+2}-u_n^{k+1})+u_{n-1}^{k+2}-u_{n-1}^{k+1}}{\eta h^2}\\ +8\frac{u_n^{k+2}-u_n^{k+1}}{\eta}-\frac{1}{8}\frac{u_{-n+1}^{k+2}-u_{-n+1}^{k+1}-2(u_{-n}^{k+2}-u_{-n}^{k+1})+u_{-n-1}^{k+2}-u_{-n-1}^{k+1}}{\eta h^2}\\ +\frac{u_{-n}^{k+2}-u_{-n}^{k+1}}{\eta}=-(0.1)\left(-\frac{u_{n+1}^{k-\mathrm{M}}-2u_n^{k-\mathrm{M}}+u_{n-1}^{k-\mathrm{M}}}{h^2}+8u_n^{k-\mathrm{M}}\right)\\ -35e^{-2\zeta_k}\cos 2y_n+1.2e^{-2(\zeta_{k-\mathrm{M}})}\cos 2y_n,\\ t_k=k\eta,\ m\mathrm{M}+1\le k\le (m+1)\mathrm{M}-2,\\ m=0,1,...,\ -\Gamma+1\le n\le \Gamma-1,\\ \mathrm{M}\eta=1,\ y_n=nh,\ -\Gamma+1\le n\le\Gamma-1,\ \Gamma h=\pi,\\ u_n^0=\cos(2nh),\\ \frac{u_n^1-u_n^0}{\eta}+\eta(-\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}+8u_n^1)\\ +\eta(\frac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2}-8u_n^0)=-2\cos(2nh),\\ \frac{u_n^2-2u_n^1+u_n^0}{\eta^2}+(-\frac{u_{n+1}^2-2u_n^2+u_{n-1}^2}{h^2}+8u_n^2)\\ +2(\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}-8u_n^1)\\ +(-\frac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2}+8u_n^0)=4\cos(2nh),\ -\Gamma+1\le n\le\Gamma-1,\\ \frac{u_n^{m\mathrm{M}+1}-u_n^{m\mathrm{M}}}{\eta}+\eta(-\frac{u_{n+1}^{m\mathrm{M}+1}-2u_n^{m\mathrm{M}+1}+u_{n-1}^{m\mathrm{M}+1}}{h^2}+8u_n^{m\mathrm{M}+1})\\ +\eta(\frac{u_{n+1}^{m\mathrm{M}}-2u_n^{m\mathrm{M}}+u_{n-1}^{m\mathrm{M}}}{h^2}-8u_n^{m\mathrm{M}})=\frac{u_n^{m\mathrm{M}}-u_n^{m\mathrm{M}-1}}{\eta},\\ \frac{u_n^{m\mathrm{M}+2}-2u_n^{m\mathrm{M}+1}+u_n^{m\mathrm{M}}}{\eta^2}+(-\frac{u_{n+1}^{m\mathrm{M}+2}-2u_n^{m\mathrm{M}+2}+u_{n-1}^{m\mathrm{M}+2}}{h^2}+8u_n^{m\mathrm{M}+2})\\ +2(\frac{u_{n+1}^{m\mathrm{M}+1}-2u_n^{m\mathrm{M}+1}+u_{n-1}^{m\mathrm{M}+1}}{h^2}-8u_n^{m\mathrm{M}+1})+(-\frac{u_{n+1}^{m\mathrm{M}}-2u_n^{m\mathrm{M}}+u_{n-1}^{m\mathrm{M}}}{h^2}+8u_n^{m\mathrm{M}})\\ =\frac{u_n^{m\mathrm{M}}-2u_n^{m\mathrm{M}-1}+u_n^{m\mathrm{M}-2}}{\eta^2},\ -\Gamma+1\le n\le\Gamma-1,\ m=1,2,\dots,\\ u_{-\Gamma}^k-e^{-2\zeta_k}=\frac{28}{h}\left(u_{-\Gamma+1}^k-u_{-\Gamma}^k\right),\\ -u_{\Gamma}^k+e^{-2\zeta_k}=\frac{28}{h}\left(u_{\Gamma}^k-u_{\Gamma-1}^k\right),\ 0\le k<\infty,\\ m\mathrm{M}\le k\le(m+1)\mathrm{M},\ m=1,2,\dots\end{cases}\qquad(9)$$

We rewrite (9) in the matrix form as in the following:

$$
\begin{cases}
\Delta\chi^{k+2}+\Theta\chi^{k+1}+\Lambda\chi^{k}+\Omega\chi^{k-1}=\varphi(\chi^{k-\mathrm{M}}),\\
k=1,2,3,\dots\\
\chi^{0}=\begin{bmatrix}\cos\left(2(-\Gamma)h\right)\\ \cos\left(2(-\Gamma+1)h\right)\\ \vdots\\ \cos\left(2(\Gamma-1)h\right)\\ \cos\left(2(\Gamma)h\right)\end{bmatrix},\\
\chi^{1}=LH\chi^{0},\\
\chi^{2}=YP\chi^{1}+YQ\chi^{0},\\
\chi^{m\mathrm{M}+1}=LJ\chi^{m\mathrm{M}}+LW\chi^{m\mathrm{M}-1},\\
\chi^{m\mathrm{M}+2}=YP\chi^{m\mathrm{M}+1}+YX\chi^{m\mathrm{M}}+YS\chi^{m\mathrm{M}-1}\\
+YZ\chi^{m\mathrm{M}-2},\\
m=1,2,\dots,
\end{cases}
$$

where $\Delta,\Theta,\Lambda,\Omega,F,H,J,P,Q,S,V,W,X$ and Z are $(2\Gamma+1)\times(2\Gamma+1)$ matrices, $\varphi(\chi^{k-\mathrm{M}})$, χ^{0}, χ^{1} and χ^{r}, $r=k,k\pm1,k+2$ are $(2\Gamma+1)\times 1$ column vectors defined by

$$
\Delta=\begin{bmatrix}
1+\frac{28}{h} & -\frac{28}{h} & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0\\
a & b & a & \cdot & 0 & 0 & 0 & \cdot & a^{*} & b^{*} & a^{*}\\
0 & a & b & \cdot & 0 & 0 & 0 & \cdot & b^{*} & a^{*} & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & \cdot & a & 0 & a_{*} & \cdot & 0 & 0 & 0\\
0 & 0 & 0 & \cdot & b & w_{1} & b^{*} & \cdot & 0 & 0 & 0\\
0 & 0 & 0 & \cdot & w_{1} & w_{2} & w_{1} & \cdot & 0 & 0 & 0\\
0 & 0 & 0 & \cdot & b^{*} & w_{1} & b & \cdot & 0 & 0 & 0\\
0 & 0 & 0 & \cdot & a^{*} & 0 & a & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & a^{*} & b^{*} & \cdot & 0 & 0 & 0 & \cdot & b & a & 0\\
a^{*} & b^{*} & a^{*} & \cdot & 0 & 0 & 0 & \cdot & b & a & \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & -\frac{28}{h} & 1+\frac{28}{h}
\end{bmatrix},
$$

$$
\Theta=\begin{bmatrix}
0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0\\
l & c & l & \cdot & 0 & 0 & 0 & \cdot & l^{*} & c^{*} & l^{*}\\
0 & l & c & \cdot & 0 & 0 & 0 & \cdot & c^{*} & l^{*} & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & \cdot & l & 0 & l^{*} & \cdot & 0 & 0 & 0\\
0 & 0 & 0 & \cdot & c & l+l^{*} & c^{*} & \cdot & 0 & 0 & 0\\
0 & 0 & 0 & \cdot & l+l^{*} & c+c^{*} & l+l^{*} & \cdot & 0 & 0 & 0\\
0 & 0 & 0 & \cdot & c^{*} & l^{*}+l & c & \cdot & 0 & 0 & 0\\
0 & 0 & 0 & \cdot & l^{*} & 0 & l & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & l^{*} & c^{*} & \cdot & 0 & 0 & 0 & \cdot & c & l & 0\\
l^{*} & c^{*} & l^{*} & \cdot & 0 & 0 & 0 & \cdot & l & c & l\\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0
\end{bmatrix},
$$

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & d & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & d & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & d & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & d & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}, \ \Omega = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & e & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & e & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & e & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & e & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix},$$

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & w & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & w & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & w & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & w & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}, \ S = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & s & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & s & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & s & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & s & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & z & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & z & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & z & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & z & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}, \ F = \begin{bmatrix} 1+\frac{28}{h} & -\frac{28}{h} & 0 & \cdot & 0 & 0 & 0 \\ f^* & f & f^* & \cdot & 0 & 0 & 0 \\ 0 & f^* & f & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & f & f^* & 0 \\ 0 & 0 & 0 & \cdot & f^* & f & f^* \\ 0 & 0 & 0 & \cdot & 0 & -\frac{28}{h} & 1+\frac{28}{h} \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ p^* & p & p^* & \cdot & 0 & 0 & 0 \\ 0 & p^* & p & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & p & p^* & 0 \\ 0 & 0 & 0 & \cdot & p^* & p & p^* \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} 1+\frac{28}{h} & -\frac{28}{h} & 0 & \cdot & 0 & 0 & 0 \\ v* & v & v* & \cdot & 0 & 0 & 0 \\ 0 & v* & v & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & v & v* & 0 \\ 0 & 0 & 0 & \cdot & v* & v & v* \\ 0 & 0 & 0 & \cdot & 0 & -\frac{28}{h} & 1+\frac{28}{h} \end{bmatrix}, \ J = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ j^* & j & j^* & \cdot & 0 & 0 & 0 \\ 0 & j^* & j & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & j & j^* & 0 \\ 0 & 0 & 0 & \cdot & j^* & j & j^* \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ h^* & e^* & h^* & \cdot & 0 & 0 & 0 \\ 0 & h^* & e^* & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & e^* & h^* & 0 \\ 0 & 0 & 0 & \cdot & h^* & e^* & h^* \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}, \ X = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ x^* & s^* & x^* & \cdot & 0 & 0 & 0 \\ 0 & x^* & s^* & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & s^* & x^* & 0 \\ 0 & 0 & 0 & \cdot & x^* & s^* & x^* \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ q^* & q & q^* & \cdot & 0 & 0 & 0 \\ 0 & q^* & q & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & q & q^* & 0 \\ 0 & 0 & 0 & \cdot & q^* & q & q^* \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}, \ \varphi^{(k-\mathrm{M})} = \begin{bmatrix} \varphi^k_{-M} \\ \varphi^k_{-M+1} \\ \vdots \\ \varphi^k_{M-1} \\ \varphi^k_M \end{bmatrix}, \ \chi^r = \begin{bmatrix} \chi^r_0 \\ \chi^r_{-M+1} \\ \vdots \\ \chi^r_{M-1} \\ \chi^r_M \end{bmatrix},$$

$r = k, k \pm 1, k+2,$
where

$$\varphi^k_n = -(0.1)\left(-\frac{u^{k-\mathrm{M}}_{n+1} - 2u^{k-\mathrm{M}}_n + u^{k-\mathrm{M}}_{n-1}}{h^2} + 8u^{k-\mathrm{M}}_n\right) - 35e^{-2\zeta_k}\cos 2y_n + 1.2e^{-2(\zeta_{k-\mathrm{M}})}\cos 2y_n,$$

$$\zeta_k = k\eta, \ m\mathrm{M} + 1 \le k \le (m+1)\mathrm{M} - 2, \ m = 0, 1, ..., \ -\Gamma + 1 \le n \le \Gamma - 1.$$

Where, $a = -\frac{1}{\eta h^2}, \ a^* = -\frac{1}{8\eta h^2}, \ b = \frac{1}{\eta^3} + \frac{2}{\eta h^2} + \frac{8}{\eta}, \ b^* = \frac{2}{8\eta h^2} + \frac{1}{\eta}, \ c = -\frac{3}{\eta^3} - \frac{2}{\eta h^2} - \frac{8}{\eta}, \ c^* = -b^*, \ l = -a, \ l^* = -a^*, \ d = \frac{3}{\eta^3}, \ e = -\frac{1}{\eta^3}, w = -\frac{1}{\eta}, \ s = -\frac{2}{\eta^2} \ z = \frac{1}{\eta^2}, \ f = \frac{2\eta}{h^2} + \frac{1}{\eta} + 8\eta, \ f^* = -\frac{\eta}{h^2}, \ p = \frac{2}{\eta^2} + \frac{4}{h^2} + 16, \ p^* = -\frac{2}{h^2}, \ v = \frac{1}{2}p, \ v^* = \frac{1}{2}p^*, \ j = f + \frac{1}{\eta}, \ j^* = f^*, \ h^* = f^*, \ e^* = f - 2, \ s^* = p^* - 8, \ x^* = -v*, \ q = -\frac{1}{,} \ w_1 = a + a^*, \ w_2 = b + b^*, \eta^2 - \frac{2}{h^2} - 4, \ q^* = x^*, \ L = F^{-1}, \ Y = V^{-1}.$

3 Numerical analysis

Provided in Table below are the solutions obtained numerically for various values of M and Γ, with u^k_n representing the solution of this DS at $u(\zeta_k, y_n)$ numerically. The table consist of values for $\mathrm{M} = \Gamma = 30, 60, 120$ in $\zeta \in [0,1], \ \zeta \in [1,2], \ \zeta \in [2,3]$ respectively and the errors are calculated by

$$mE^{\mathrm{M}}_M = \max_{m\mathrm{M}+1 \le k \le (m+1)\mathrm{M}, \ -\Gamma \le n \le \Gamma} |u(\zeta_k, y_n) - u^k_n|.$$

T a b l e

Errors of DS (9)

(M, Γ)	$\mathrm{M} = \Gamma = 30$	$\mathrm{M} = \Gamma = 60$	$\mathrm{M} = \Gamma = 120$
$\zeta \in [0,1]$	0.1933	0.1012	0.0516
$\zeta \in [1,2]$	0.2350	0.1169	0.0583
$\zeta \in [2,3]$	0.1692	0.0780	0.0340

If M and Γ are doubled as shown in the above table, the values of the errors decrease by a factor of approximately $\frac{1}{2}$ for DS (9).

4 Conclusion

In this paper, the first order of accuracy DS for the numerical solution of the third order delay PDE with IRC is considered. Numerical results are given for illustration.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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*Author Information**

Allaberen Ashyralyev — Doctor of physical and mathematical sciences, Professor, Department of Mathematics, Bahcesehir University, 34353, Istanbul, Turkey; e-mail: *aallaberen@gmail.com*; https://orcid.org/0000-0002-4153-6624

Suleiman Ibrahim (*corresponding author*) — Doctor of mathematical sciences, Assistant Professor, Lecturer, Near East University Faculty of Arts and Sciences, Department of Mathematics, Near East University, Nicosia, TRNC, Mersin 10, Turkey; e-mail: *suleiman@neu.edu.tr*; https://orcid.org/0009-0009-8913-7217

Evren Hincal — Doctor of mathematical sciences, Professor, Head of the Department of Mathematics, Near East University Faculty of Arts and Sciences, Department of Mathematics, Near East University, Nicosia, TRNC, Mersin 10, Turkey; e-mail: *evren.hincal@neu.edu.tr*; https://orcid.org/0000-0001-6175-1455

*The author's name is presented in the order: First, Middle and Last Names.