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On a non-local problem for a fractional differential equation of the Boussinesq type

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In recent years, the fractional partial differential equation of the Boussinesq type has attracted much attention from researchers due to its practical importance. In this paper, we study a non-local problem for the Boussinesq type equation $D_t^\alpha u(t) + AD_t^\alpha u(t) + \nu^2 Au(t) = 0$, $0 < t < T$, $1 < \alpha < 3/2$, where D_t^α is the Caputo fractional derivative, and A is an abstract operator. In the classical case, i.e., when $\alpha = 2$, this problem has been studied previously, and an interesting effect has been discovered: the existence and uniqueness of a solution depend significantly on the length of the time interval and the parameter ν. In this note, we show that in the case of a fractional equation, there is no such effect: a solution of the problem exists and is unique for any T and ν.

Keywords: fractional equation, Caputo derivative, forward and inverse problems, Fourier method.

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Introduction

Let H be a separable Hilbert space, and let $A : D(A) \subset H \to H$ be an arbitrary unbounded, positive self-adjoint operator, and we assume that A has a compact inverse A^{-1}, where $D(A)$ is the domain of A. Let λ_k and $\{v_k\}$ be the eigenvalues and corresponding eigenfunctions of A.

Let us introduce the Caputo fractional derivative D_t^α of order $\alpha \in (1, 2)$ of a vector-valued function $h(t) \in H$ (see, for example [1])

$$D_t^\alpha h(t) = \frac{1}{\Gamma(2-\alpha)} \int\limits_0^t \frac{h''(\xi)}{(t-\xi)^{\alpha-1}} d\xi, \quad t > 0,$$

provided the right-hand side exists. Here $\Gamma(\alpha)$ is Euler's gamma function.

Let $1 < \alpha < 3/2$. The object of study of this work is the following fractional differential equation

$$D_t^\alpha u(t) + AD_t^\alpha u(t) + \nu^2 Au(t) = 0, \quad 0 < t < T \tag{1}$$

with non-local conditions

$$u(0) = u(T), \tag{2}$$

and

$$\int\limits_0^T u(t)dt = \varphi, \tag{3}$$

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where $\varphi \in H$ is a given vector and $\nu > 0$ is a fixed number.

Note that since the abstract operator A is only required to have a complete orthonormal system of eigenfunctions, any elliptic operator can be considered as A. For example, if we take $L_2(\Omega)$, $\Omega \subset \mathbb{R}^N$, as the Hilbert space H, then we can take the Laplace operator $(-\Delta)$ with the Dirichlet condition as A.

The equation (1) has different names for different values of the parameter α. Thus, if $\alpha = 1$, it is called a differential equation of the Barenblatt-Zheltov-Kochina type (see [2]), and if $\alpha = 2$, it is called a differential equation of the Boussinesq type (see [3]). If $0 < \alpha < 1$, it is called a fractional differential equation of the Barenblatt-Zheltov-Kochina type, in the case $1 < \alpha < 2$, it is called a fractional differential equation of the Boussinesq type. Differential equations of the Boussinesq type were introduced by Joseph Boussinesq in 1872 (see [3], eq. 26). The Boussinesq equations are widely used in numerical modeling in coastal engineering for modeling waves in shallow water and harbors. Although wave modeling in such cases is well described by the Navier-Stokes equations, it is currently extremely difficult to solve three-dimensional equations in complex models. Therefore, approximate models, such as the Boussinesq equations can be used to reduce three-dimensional problems to two-dimensional states (see, e.g., [4]).

There is a number of works (see, for example, [2], [5]–[7]) in which specialists consider various initial-boundary value problems for differential and fractional differential equations of the Barenblatt-Zheltov-Kochina type. Since our study relates to the Boussinesq type differential equation, we present some results related specifically to these equations.

Due to the mathematical and physical importance, over the last couple of decades, existence and nonexistence of solutions of the Boussinesq type equations have been extensively studied by many mathematicians and physicists (see, for example [8]–[12] with fractional order, and literature therein). Nonlinear Boussinesq type equations arise in a number of mathematical models of physical processes, for example, in the modeling of surface waves in shallow waters or considering the possibility of energy exchange through the lateral surfaces of the wave guide in the physical study of nonlinear wave propagation in wave guide (see, for example, [13] and [14], and literature therein). In [13], the authors consider the Cauchy problem of the two-dimensional generalized Boussinesq type equation $u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u + \Delta f(u) = 0$. Under the assumption that $f(u)$ is a function with exponential growth at infinity and under some assumptions on the initial data, the authors prove the existence and, in some cases the nonexistence of a global weak solution.

Model equations of the Boussinesq type (the problem (1)–(3) with $\alpha = 2$, $\nu = 1$ and $A = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$, $x, y \in (0, l)$) and equations of mixed type and nonlinear equations containing equations of the Boussinesq type are systematically studied in a series of works [15]–[17]. In these works, the existence and uniqueness of the classical solution of initial-boundary value problems were proved and some inverse problems were studied. In the work [18], problems for the Boussinesq equation with a spectral parameter were investigated.

Let us cite two more works [19] and [20] that motivated the appearance of our research. In these works, the above non-local problem (1)–(3) was studied for a classical partial differential equation in which A is the Laplace operator with the Dirichlet condition. So in the fundamental work [19], Alimov and Khalmukhamedov studied the following non-local problem in the cylinder $\Omega \times (0, T)$:

$$\begin{cases} u_{tt} - \Delta u_{tt} - \nu^2 \Delta u = 0, \quad x \in \Omega, \quad 0 < t < T, \\ u(x,0) = u(x,T), \quad x \in \Omega, \\ \int\limits_0^T u(x,t)dt = \varphi(x), \end{cases} \tag{4}$$

where $\varphi(x)$ is a given function. The authors discovered an interesting effect: it turns out that the existence and uniqueness of the solution of this problem significantly depend on the length of the time interval and the parameter ν. If $\frac{\nu T}{2\pi} \in (0, 1)$, then the solution exists and is unique for all $\varphi \in D(A)$. The case $\frac{\nu T}{2\pi} \geq 1$ is more complicated: if $\frac{\nu T}{2\pi} > 1$, and this number is not a natural number, then for the existence of a solution, it is necessary that the function φ is orthogonal to some eigenfunctions of the Laplace operator, and in this case, the solution is not unique. If the number $\frac{\nu T}{2\pi}$ is a natural number, then only orthogonality is not enough; it is necessary that the function φ is smoother: $\varphi \in D(A^2)$.

Since the parameter ν in the equation is fixed, this result means that if the process under study lasts "not so long", then a solution to the problem exists for any measurements φ. However, if the process lasts "a little" longer, then the solution does not exist for all data φ.

In the recent work [20], problem (4) was studied with the kernel $tu(x, t)$ in the integral condition. Similar to the paper [19], conditions have been found for the time interval $(0, T]$, function φ and parameter ν, which guarantees the existence of a solution to the problem.

A natural question arises: will the effect found in [19] be preserved, if instead of the second time derivative in equation (4) we take the fractional derivative of order $\alpha \in (1, 3/2)$, in other words, instead of equation in (4), consider equation (1)? In this paper it will be shown that the above parameter $\frac{\nu T}{2\pi}$ does not play a significant role in solving the non-local problem (1)–(3) and the solution to this problem exists and is unique for any function $\varphi \in D(A)$, regardless of the value of the number $\frac{\nu T}{2\pi}$.

The article is organized as follows: Section 2 provides some information about the domain of definition of the operator A and proves the necessary estimates for the Mittag-Leffler functions. In Section 3, we will formulate the main result of the work and construct a formal solution to the problem (1)–(3). Section 4 is devoted to the proof of Theorem 1. In the "Conclusions" section discusses possible further developments of the obtained results.

1 Preliminaries

In this section, we provide some information about the operator A and present new bounds for the Mittag-Leffler function in the case $1 < \rho < 3/2$, based on the findings of the study conducted by [21].

The action of the abstract operator A under consideration on the element $h \in H$ can be written as

$$Ah = \sum_{k=1}^{\infty} \lambda_k h_k v_k,$$

where h_k is the Fourier coefficient of the element h: $h_k = (h, v_k)$. Obviously, the domain of this operator has the form

$$D(A) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^2 |h_k|^2 < \infty\}.$$

For elements h and g of $D(A)$ we introduce the norm and inner product as

$$||h||_1^2 = \sum_{k=1}^{\infty} \lambda_k^2 |h_k|^2 = ||Ah||^2,$$

$$(h, g)_1 = \sum_{k=1}^{\infty} \lambda_k^2 h_k \overline{g}_k,$$

respectively. Together with this norm $D(A)$ turns into a Hilbert space.

Let us denote by $C((a, b); H)$ the sets of continuous vector functions $u(t)$ on the interval $t \in (a, b)$, whose values lie in H, and by $AC^1((a, b); H)$ the sets of vector functions whose derivatives are absolutely continuous with respect to $t \in (a, b)$.

Recall, the Mittag-Leffler function $E_{\rho,\mu}(t)$ has the form (see e.g. [22], p. 56):

$$E_{\rho,\mu}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\rho n + \mu)},$$

where $\rho > 0$ and μ complex number.

Next, we establish some two-sided estimates for the Mittag-Leffler function $E_{\rho,\mu}(-t)$, $1 < \rho < 3/2$, $t \geq 0$, $\mu = 1, 2, 3, \rho$. The following simple method for obtaining these estimates was suggested to the authors by Professor A.V. Pskhu (see, [21]).

Let $\phi(\delta, \beta; z)$ stand for the Wright function, defined as

$$\phi(\delta, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\beta + \delta k)}, \ \delta > -1, \ \beta \in \mathbb{R}, \ z \in \mathbb{C}.$$

Let $0 < \xi < 1$. In the work of A.V. Pskhu [21] for functions $h(t)$ defined at $t \geq 0$, the following integral transform is introduced and studied:

$$P^{\xi,\eta}h(t) = t^{\eta-1} \int\limits_0^{\infty} h(s)\phi\Big(-\xi, \eta; -\frac{s}{t^{\xi}}\Big) ds.$$

Note that $P^{\xi,\eta}h(t)$ is some modification of the integral transform introduced by B. Stanković in 1955 (see [23]).

Let us present the following statement from [21].

Lemma 1. Let $\gamma > 0$. Then

$$P^{\xi,\eta}t^{\gamma-1} = t^{\xi\gamma+\eta-1} \frac{\Gamma(\gamma)}{\Gamma(\xi\gamma + \eta)}.$$

From Lemma 1, by the definition of the Mittag-Leffler function, we get

$$P^{\xi,\eta}[t^{\mu-1}E_{\rho,\mu}(\lambda t^{\rho})] = t^{\mu\xi+\eta-1}E_{\rho\xi,\mu\xi+\eta}(\lambda t^{\rho\xi}). \tag{5}$$

Lemma 2. (see [24], p. 372, 373) There is a function $f(\alpha)$ decreasing on the interval $(1, 3/2)$ such that for any $\alpha \in (1, 3/2)$ and $\beta > f(\alpha)$ function $E_{\alpha,\beta}(z)$ does not vanish, where $f(\alpha)$ satisfies the following inequality:

$$\alpha + h(\alpha) < f(\alpha) < \frac{4}{3}\alpha, \quad 1 < \alpha < 3/2,$$

where

$$h(\alpha) = \exp\left[-\pi(1 - 1/\alpha)\right].$$

Lemma 3. Let $\alpha \in (1, 2)$. Then the following estimate holds:

$$E_{\alpha,1}(-t^{\alpha}) \leq 1 \quad t > 0.$$

Proof. Let $\mu = 1, \rho = 2, \xi = \frac{\alpha}{2}, \eta = 1 - \frac{\alpha}{2}$ and $\lambda = -1$ in equality (5). Then, we have:

$$E_{\alpha,1}(-t^{\alpha}) = P^{\frac{\alpha}{2},1-\frac{\alpha}{2}}(E_{2,1}(-t^2)) = P^{\frac{\alpha}{2},1-\frac{\alpha}{2}}\cos t.$$

Using the inequality $|\cos t| \leq 1$ and Lemma 1, we get

$$|E_{\alpha,1}(-t^{\alpha})| \leq P^{\frac{\alpha}{2},1-\frac{\alpha}{2}}1 = \frac{\Gamma(1)}{\Gamma(1)} = 1.$$

Lemma 3 is proved.

Lemma 4. Let $\alpha \in (1, 3/2)$ and $0 < a < \infty$. Then there exists a number $\varepsilon_1 = \varepsilon_1(a) > 0$, depending on a such that the following estimate holds

$$0 < \varepsilon_1 < E_{\alpha,2}(-t^\alpha) \le 1, \quad 0 < t \le a.$$

Proof. Let $\mu = 1, \rho = 2, \xi = \frac{\alpha}{2}, \eta = 2 - \frac{\alpha}{2}$ and $\lambda = -1$. Then from (5) it follows the following equality

$$tE_{\alpha,2}(-t^\alpha) = P^{\frac{\alpha}{2},2-\frac{\alpha}{2}}(E_{2,1}(-t^2)) = P^{\frac{\alpha}{2},2-\frac{\alpha}{2}} \cos t.$$

Using the inequality $|\cos t| \le 1$ and Lemma 1, we get

$$|tE_{\alpha,2}(-t^\alpha)| \le P^{\frac{\alpha}{2},2-\frac{\alpha}{2}} 1 = t\frac{\Gamma(1)}{\Gamma(2)} = t.$$

Therefore, for $t > 0$, we have that

$$|E_{\alpha,2}(-t^\alpha)| \le 1.$$

Let $0 < a < \infty$. First we show that $E_{\alpha,2}(-t^\alpha) > 0$. Since $\beta = 2 > f(\alpha)$, then from Lemma 2 it follows $E_{\alpha,2}(-t^\alpha) \ne 0$, and therefore $E_{\alpha,2}(-t^\alpha)$ function keeps its sign for all $t \ge 0$. On the other hand, we know that $E_{\alpha,2}(0) = 1 > 0$ and therefore $E_{\alpha,2}(-t^\alpha) > 0$ for all $t \ge 0$. Further it is well known that $E_{\alpha,2}(-t^\alpha) \in C[0,\infty)$. Since function $E_{\alpha,2}(-t^\alpha)$, continuous in a closed domain $[0, a]$, reaches its minimum and this minimum is obviously positive, denoting it by $\varepsilon_1 = \varepsilon_1(a) > 0$, we obtain the statement of the lemma. Lemma 4 is proved.

Lemma 5. Let $\alpha \in (1, 3/2)$. Then the following estimate holds

$$0 < E_{\alpha,3}(-t^\alpha) \le \frac{1}{2}, \quad 0 < t \le b.$$

Proof. Let $\mu = 1, \rho = 2, \xi = \frac{\alpha}{2}, \eta = 3 - \frac{\alpha}{2}$ and $\lambda = -1$. Then from (5) it follows the following equality

$$t^2 E_{\alpha,3}(-t^\alpha) = P^{\frac{\alpha}{2},3-\frac{\alpha}{2}}(E_{2,1}(-t^2)) = P^{\frac{\alpha}{2},3-\frac{\alpha}{2}} \cos t.$$

Using the inequality $|\cos t| \le 1$ and Lemma 1, we get

$$|t^2 E_{\alpha,3}(-t^\alpha)| \le P^{\frac{\alpha}{2},3-\frac{\alpha}{2}} 1 = t^2\frac{\Gamma(1)}{\Gamma(3)} = \frac{t^2}{2}.$$

Therefore

$$|E_{\alpha,3}(-t^\alpha)| \le \frac{1}{2}, \quad t > 0.$$

Now we show that $E_{\alpha,3}(-t^\alpha) > 0$. Since $\beta = 3 > f(\alpha)$, then from Lemma 2 it follows $E_{\alpha,3}(-t^\alpha) \ne 0$, and therefore $E_{\alpha,3}(-t^\alpha)$ function keeps its sign for all $t \ge 0$. Also, we know that $E_{\alpha,3}(0) = \frac{1}{2} > 0$ and therefore $E_{\alpha,3}(-t^\alpha) > 0$ for all $t \ge 0$. Lemma 5 is proved.

Lemma 6. Let $\alpha \in (1, 3/2)$. Then there exists a number $C_0 > 0$, such that the following estimate holds:

$$(E_{\alpha,2}(-T^\alpha))^2 + E_{\alpha,3}(-T^\alpha)(1 - E_{\alpha,1}(-T^\alpha)) > C_0.$$

Proof. We have that

$$(E_{\alpha,2}(-T^\alpha))^2 + E_{\alpha,3}(-T^\alpha)(1 - E_{\alpha,1}(-T^\alpha)) \ge (E_{\alpha,2}(-T^\alpha))^2.$$

According to Lemma 4, there exists a positive number C_0, such that

$$(E_{\alpha,2}(-T^\alpha))^2 > C_0,$$

where $C_0 = \varepsilon_1^2$. Lemma 6 is proved.

Lemma 7. Let $\alpha \in (1,2)$. Then, the following estimate holds

$$|E_{\alpha,\alpha}(-t^\alpha)| \le \frac{1}{\Gamma(\alpha)}, \quad t \ge 0.$$

Proof. Let $\mu = 1, \rho = 2, \xi = \frac{\alpha}{2}, \eta = 0$ and $\lambda = -1$. Then from (5) it follows the equality

$$t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha) = P^{\frac{\alpha}{2},0}(E_{2,1}(-t^2)) = P^{\frac{\alpha}{2},0}\cos t.$$

Using the inequality $|\cos t| \le 1$ and Lemma 1, we get

$$|t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha)| \le P^{\frac{\alpha}{2},0}1 = t^{\alpha-1}\frac{\Gamma(1)}{\Gamma(\alpha)} = t^{\alpha-1}\frac{1}{\Gamma(\alpha)}.$$

Therefore, for $t > 0$, we have that

$$|E_{\alpha,\alpha}(-t^\alpha)| \le \frac{1}{\Gamma(\alpha)}.$$

Lemma 7 is proved.

2 Formulation of the main result and formal solution of the problem (1)–(3)

The solution of problem (1)–(3) will be understood in the sense of the following definition:

Definition 1. If a function $u(t) \in AC^1([0,T];H)$, $D_t^\alpha u(t)$, $Au(t)$, $AD_t^\alpha u(t) \in C((0,T);H)$ and satisfies all the conditions of problem (1)–(3), then it is called the solution of problem (1)–(3).

Note that here the absolute continuity of the derivative $u'(t)$ is necessary to avoid non-uniqueness of solutions due to singular functions.

Here is the main result of this paper.

Theorem 1. Let $\varphi \in D(A)$. Then, there is a unique solution of problem (1)–(3) and it has the form:

$$u(t) = \sum_{k=1}^{\infty}\left(\frac{\varphi_k E_{\alpha,2}(-\nu_k^2T^\alpha)E_{\alpha,1}(-\nu_k^2t^\alpha)}{T((E_{\alpha,2}(-\nu_k^2T^\alpha))^2 + E_{\alpha,3}(-\nu_k^2T^\alpha)(1 - E_{\alpha,1}(-\nu_k^2T^\alpha)))} + \right.$$
$$\left. + \frac{\varphi_k t(1 - E_{\alpha,1}(-\nu_k^2T^\alpha))E_{\alpha,2}(-\nu_k^2t^\alpha)}{T^2((E_{\alpha,2}(-\nu_k^2T^\alpha))^2 + E_{\alpha,3}(-\nu_k^2T^\alpha)(1 - E_{\alpha,1}(-\nu_k^2T^\alpha)))}\right)v_k, \tag{6}$$

where $\nu_k = \nu\sqrt{\frac{\lambda_k}{1+\lambda_k}}$ and $\varphi_k = (\varphi, v_k)$ are the Fourier coefficients of function φ.

In this section we will construct a formal solution of problem (1)–(3) and prove the uniqueness of the solution.

Let $u(t)$ be any solution of the non-local problem (1)–(3). Then since the system $\{v_k\}$ is complete in H, the solution has the form:

$$u(t) = \sum_{k=1}^{\infty} T_k(t)v_k. \tag{7}$$

If we multiply both sides of this equality scalarly by v_j, then from the orthonormality of the system of eigenfunctions $\{v_k\}$, we obtain the equalities $T_j(t) = (u(t), v_j)$.

Substituting (7) into equation (1), we get

$$D_t^\alpha T_k(t) + \lambda_k D_t^\alpha T_k(t) + \nu^2\lambda_k T_k(t) = 0.$$

Then, we have that

$$(1+\lambda_k)D_t^\alpha T_k(t)+\nu^2\lambda_k T_k(t)=0.$$

If we divide above equation to $1+\lambda_k$ and by ν_k we denote $\nu\sqrt{\frac{\lambda_k}{1+\lambda_k}}$, then we obtain:

$$D_t^\alpha T_k(t)+\nu_k^2 T_k(t)=0, \tag{8}$$

and using the conditions (2) and (3), we have:

$$T_k(0)=T_k(T), \tag{9}$$

and

$$\int\limits_0^T T_k(t)dt=\varphi_k. \tag{10}$$

The solution of the equation (8) has the form (see, for example [25], p. 231.)

$$T_k(t)=a_k E_{\alpha,1}(-\nu_k^2 t^\alpha)+b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha). \tag{11}$$

To find the unknown coefficients a_k and b_k, we use the non-local conditions (9) and (10).

Apply conditions (9) and (10) to (11), we get:

$$\begin{cases} a_k=a_k E_{\alpha,1}(-\nu_k^2 T^\alpha)+b_k T E_{\alpha,2}(-\nu_k^2 T^\alpha), \\ \int\limits_0^T (a_k E_{\alpha,1}(-\nu_k^2 t^\alpha)+b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha))dt=\varphi_k. \end{cases}$$

Solving this system of equations, we will have

$$a_k=\frac{\varphi_k E_{\alpha,2}(-\nu_k^2 T^\alpha)}{T((E_{\alpha,2}(-\nu_k^2 T^\alpha))^2+E_{\alpha,3}(-\nu_k^2 T^\alpha)(1-E_{\alpha,1}(-\nu_k^2 T^\alpha)))}, \tag{12}$$

$$b_k=\frac{\varphi_k(1-E_{\alpha,1}(-\nu_k^2 T^\alpha))}{T^2((E_{\alpha,2}(-\nu_k^2 T^\alpha))^2+E_{\alpha,3}(-\nu_k^2 T^\alpha)(1-E_{\alpha,1}(-\nu_k^2 T^\alpha)))}. \tag{13}$$

Using the equalities (7), (11), (12) and (13) we get the formal solution (6) for the problem (1)–(3). It remains to prove that the constructed formal solution satisfies all the requirements of Definition 1, i.e. is indeed a solution to problem (1)–(3). We will do this in the next section.

On the other hand, the uniqueness of the solution follows from the already established equalities (12) and (13). Indeed, let us show that the solution to the homogeneous problem (1)–(3) with function $\varphi=0$ is identically zero. From equalities (12) and (13) it follows that $a_k=b_k=0$, and then all coefficients $T_k(t)$ of series (7) are equal to zero. Due to the completeness of system $\{v_k\}$, it follows that $u(t)\equiv 0$.

3 Proof of Theorem 1

Let $S_j(t)$ be the partial sums of (6). Then

$$AS_j(t)=\sum_{k=1}^{j}\lambda_k(a_k E_{\alpha,1}(-\nu_k^2 t^\alpha)+b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha))v_k.$$

By Parseval equality, we obtain

$$||AS_j(t)||^2 = \sum_{k=1}^{j} \lambda_k^2 |a_k E_{\alpha,1}(-\nu_k^2 t^\alpha) + b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha)|^2 \le$$

$$\le C \sum_{k=1}^{j} \lambda_k^2 |a_k E_{\alpha,1}(-\nu_k^2 t^\alpha)|^2 + C \sum_{k=1}^{j} \lambda_k^2 |b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha)|^2.$$

Let us estimate the following two terms, separately

$$I_1 = |a_k E_{\alpha,1}(-\nu_k^2 t^\alpha)| = \left| \frac{\varphi_k E_{\alpha,2}(-\nu_k^2 T^\alpha) E_{\alpha,1}(-\nu_k^2 t^\alpha)}{T((E_{\alpha,2}(-\nu_k^2 T^\alpha))^2 + E_{\alpha,3}(-\nu_k^2 T^\alpha)(1 - E_{\alpha,1}(-\nu_k^2 T^\alpha)))} \right|$$

and

$$I_2 = |b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha)| = \left| \frac{\varphi_k t (1 - E_{\alpha,1}(-\nu_k^2 T^\alpha)) E_{\alpha,2}(-\nu_k^2 t^\alpha)}{T^2((E_{\alpha,2}(-\nu_k^2 T^\alpha))^2 + E_{\alpha,3}(-\nu_k^2 T^\alpha)(1 - E_{\alpha,1}(-\nu_k^2 T^\alpha)))} \right|.$$

To estimate I_1, we apply Lemma 3, Lemma 4 and Lemma 6. Then

$$I_1 \le \frac{|\varphi_k|}{T} \frac{1}{C_0} \le C T^{-1} |\varphi_k|. \tag{14}$$

Similarly

$$I_2 \le \frac{t|\varphi_k|}{T^2} \frac{1}{C_0} \le C T^{-2} t |\varphi_k|. \tag{15}$$

Using estimates (14) and (15), we obtain:

$$||AS_j(t)||^2 \le C^2 T^{-2} \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2 + C^2 T^{-4} t^2 \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2.$$

Therefore, if $\varphi \in D(A)$, then

$$C^2 T^{-2} \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2 + C^2 T^{-4} t^2 \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2 \le const.$$

Thus $Au(t) \in C([0,T]; D(A))$.

Now we will show that the termwise differentiated series (6) converges uniformly on $[0,T]$, which will mean that $u'(t) \in C([0,T], H)$. We have that

$$S_j'(t) = \sum_{k=1}^{j} (a_k t^{\alpha-1} E_{\alpha,\alpha}(-\nu_k^2 t^\alpha) + b_k E_{\alpha,1}(-\nu_k^2 t^\alpha)) v_k.$$

By Parseval equality, we obtain that

$$||S_j'(t)||^2 = \sum_{k=1}^{j} |a_k t^{\alpha-1} E_{\alpha,\alpha}(-\nu_k^2 t^\alpha) + b_k E_{\alpha,1}(-\nu_k^2 t^\alpha)|^2 \le$$

$$\le C \sum_{k=1}^{j} |a_k t^{\alpha-1} E_{\alpha,\alpha}(-\nu_k^2 t^\alpha)|^2 + C \sum_{k=1}^{j} |b_k E_{\alpha,1}(-\nu_k^2 t^\alpha)|^2.$$

Let us estimate the following two terms, separately

$$I_1 = |a_k t^{\alpha-1} E_{\alpha,\alpha}(-\nu_k^2 t^\alpha)| = \left| \frac{\varphi_k t^{\alpha-1} E_{\alpha,2}(-\nu_k^2 T^\alpha) E_{\alpha,\alpha}(-\nu_k^2 t^\alpha)}{T((E_{\alpha,2}(-\nu_k^2 T^\alpha))^2 + E_{\alpha,3}(-\nu_k^2 T^\alpha)(1 - E_{\alpha,1}(-\nu_k^2 T^\alpha)))} \right|,$$

and

$$I_2 = |b_k E_{\alpha,1}(-\nu_k^2 t^\alpha)| = \left| \frac{\varphi_k (1 - E_{\alpha,1}(-\nu_k^2 T^\alpha)) E_{\alpha,1}(-\nu_k^2 t^\alpha)}{T^2((E_{\alpha,2}(-\nu_k^2 T^\alpha))^2 + E_{\alpha,3}(-\nu_k^2 T^\alpha)(1 - E_{\alpha,1}(-\nu_k^2 T^\alpha)))} \right|.$$

To estimate I_1, we apply Lemmas 3–7. Then

$$I_1 \le t^{\alpha-1} \frac{|\varphi_k|}{T} \frac{1}{C_0 \Gamma(\alpha)} \le C t^{\alpha-1} T^{-1} |\varphi_k|. \tag{16}$$

Similarly

$$I_2 \le \frac{|\varphi_k|}{T^2} \frac{1}{C_0} \le C T^{-2} |\varphi_k|. \tag{17}$$

Apply estimates (16) and (17), we get

$$||S_j'(t)||^2 \le C^2 t^{2(\alpha-1)} T^{-2} \sum_{k=1}^{j} |\varphi_k|^2 + C^2 T^{-4} \sum_{k=1}^{j} |\varphi_k|^2.$$

Hence

$$||S_j'(t)||^2 \le C ||\varphi||, \ t \ge 0.$$

Further let us show that $u'(t)$ is absolutely continuous. For this, we take the first-order derivative with respect to t from the partial sums $S_j'(t)$:

$$S_j''(t) = \sum_{k=1}^{j} (a_k t^{\alpha-2} E_{\alpha,\alpha-1}(-\nu_k^2 t^\alpha) + b_k t^{\alpha-1} E_{\alpha,\alpha}(-\nu_k^2 t^\alpha)) v_k.$$

From this it is easy to see that $S_j''(t) \in L((0,T), H)$. Therefore, we get $u(t) \in AC^1([0,T]; H)$.

Now we show that the following sum $D_t^\alpha S_j(t)$ converge uniformly in $t \in (0,T)$. To do this, first consider the sums

$$(I + A)^{-1} A S_j(t) = \sum_{k=1}^{j} \frac{\lambda_k}{1 + \lambda_k} (a_k E_{\alpha,1}(-\nu_k^2 t^\alpha) + b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha)) v_k.$$

By Parseval equality, we get

$$||(I + A)^{-1} A S_j(t)||^2 = \sum_{k=1}^{j} \frac{\lambda_k^2}{(1 + \lambda_k)^2} |a_k E_{\alpha,1}(-\nu_k^2 t^\alpha) + b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha)|^2 \le$$

$$\le C \sum_{k=1}^{j} \frac{\lambda_k^2}{(1 + \lambda_k)^2} |a_k E_{\alpha,1}(-\nu_k^2 t^\alpha)|^2 + C \sum_{k=1}^{j} \frac{\lambda_k^2}{(1 + \lambda_k)^2} |b_k t E_{\alpha,2}(-\nu_k^2 t^\alpha)|^2.$$

By estimates (14), (15) and $\frac{\lambda_k}{1+\lambda_k} \le 1$ we have that

$$||(I + A)^{-1} A S_j(t)||^2 \le C^2 T^{-2} \sum_{k=1}^{j} |\varphi_k|^2 + C^2 T^{-4} t^2 \sum_{k=1}^{j} |\varphi_k|^2.$$

From this, since $\varphi \in H$, we have that

$$C^2T^{-2}\sum_{k=1}^{j}|\varphi_k|^2 + C^2T^{-4}t^2\sum_{k=1}^{j}|\varphi_k|^2 \le const.$$

Therefore $(I+A)^{-1}Au(t) \in C((0,T);H)$. Now applying the obvious equality
$D_t^\alpha u(t) = -\nu^2(I+A)^{-1}Au(t)$, which follows from the commutativity of the corresponding operators, we obtain $D_t^\alpha u(t) \in C((0,T),H)$.

It remains to prove the continuity of $AD_t^\alpha u(t)$. From equality $AD_t^\alpha u(t) = -D_t^\alpha u(t) - \nu^2 Au(t)$ and continuity of $D_t^\alpha u(t)$ and $Au(t)$, it follows $AD_t^\alpha u(t) \in C((0,T),D(A))$. Theorem 1 is proved.

4 Conclusions

The work is devoted to the study of the correctness of a certain non-local problem (1)–(3) for equations of Busineski type. Namely, the question of the existence and uniqueness of a solution to the corresponding non-local problem is analyzed. In recent years, a number of works have appeared where initial boundary value problems for various types of equations of Busineski type have been studied. The motivation for this was primarily the numerous applications of such problems in the modeling of various processes in physics and mechanics.

Recently, a fundamental work [19] (see also [20]) appeared, where the correctness of a similar non-local problem was studied in the case when $\alpha = 2$. Here the authors discovered an interesting phenomenon: the correctness of the problem significantly depends on the duration of the process T and the parameter ν. It turned out that the most optimal case is when the process does not last that long, i.e. $\frac{\nu T}{2\pi} \in (0,1)$: here the problem is correct for any $\varphi \in D(A)$. If the process lasts longer, i.e. $\frac{\nu T}{2\pi} \ge 1$, then additional conditions will appear on the function φ and these conditions depend on whether the number $\frac{\nu T}{2\pi}$ is a natural number or not.

The question naturally arises: does this phenomenon persist in the case when, instead of the second derivative with respect to time, we take a derivative in the sense of Caputo D_t^α of order $1 < \alpha < 3/2$. In this paper it is shown that there is no such effect and the corresponding non-local problem has a unique solution for any $\varphi \in D(A)$.

In the future, it would be interesting to consider other fractional derivatives instead of Caputo derivatives, to see if the corresponding effect would take place. Also interesting is the study of inverse problems to determine the right-hand side of the equation for such non-local problems.

These tasks are the subject of further research.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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