Geometric approach to define a railway plan model

A. Artykbaev*, M.M. Toshmatova

Tashkent State Transport University, Tashkent, Uzbekistan
(E-mail: aartykbaev@mail.ru, toshmatova_mm@mail.ru)

The construction of new railway lines is based on the railway plan. There are various ways to draw up a railway plan. The basis of all railway plans is a scheme of geometric point locations, the projection of the center of gravity of the carriage is on a horizontal plane and consists of a single flat line. The railway plan consists of linear and curved parts connecting straight sections. However, the curves determining the position of the rails of the railway track in the curved part will be spatial. To extinguish the centrifugal force arising in the curved part of the road, an external rail rises. In this case, the elevated curve representing the outer rail becomes spatial. Therefore, in the work, it is proposed to draw up a plan of a railway track as two curves, one of which is flat, and the other depicts an external spatial rail. In this case, the distance between the ends of the rectilinear parts and the angle between the rectilinear parts are selected as the main parameters. In the work, for the simplest case, when both linear parts belong to the same horizontal plane, it is proved that the curved part is a spatial curve. The curvature of the required curve was determined and a dynamic system was constructed, the solution of which would be a curve that satisfied the technical conditions presented for the railway route. This dynamic system is proposed as a mathematical model of the railway route. In the rectilinear parts, the railway plan is straight on a horizontal plane. The curve of the road should be spatial.

Keywords: railway plan, route, curvature, torsion, osculating plane, radius, osculating circle, angle of rotation, car motion profile, vector equation, Frenier's formula.

2024 Mathematics Subject Classification: 53A35, 53Z30.

Introduction

Railroad plan design has been the main issue of practical experience for many years. The railway plan consists of linear parts and curved parts connecting them [1]. The curved part of the plan has a curvature that is the inverse of the radius of the touching circle. Existing methods consider the curved part of the plan to be a flat curve. For technical reasons, the curved part of the plan must be spatial [2]. The work determines the steepness of the curve, which is the route plan, and provides a mathematical model with which one can determine the equation of the route.

There are methods for designing and reconstructing existing roads using laser technology [3]. Reconstruction of existing roads using modern technologies is expensive [4].

1 Elements of the railroad plan

The railway plan is the projection of the track axis onto a horizontal plane. The railway in plan is a combination of alternating linear and curved sections [5].

In straight sections, the main parameter is its direction, the technical name of which is azimuth. In this case, the railway plan is linear on a horizontal plane. But even on a horizontal plane, due to the terrain, the presence of settlements and other obstacles, there is a need to change the azimuth. The new azimuth defines a new straight section on the horizontal plane.

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*Corresponding author. *E-mail: aartykbaev@mail.ru*

The pairing of linear sections with each other is carried out using curves. The presence of curves in the railway plan is due to the need to deviate the route from the linear direction for the purpose of bypassing [6].

The angle α between linear sections is called the angle of rotation. When a railway plan is considered only the horizontal plane $\alpha \in (0, \pi)$. In general, α can take any value. If the linear sections belong to horizontal planes of different levels, the connecting curve has the form of a spiral, then the angle of rotation can take on arbitrary values. A section of a railway plan, on a horizontal plane, with a rotation angle α can be represented in the form of two rays that make up this rotation angle (Fig. 1).
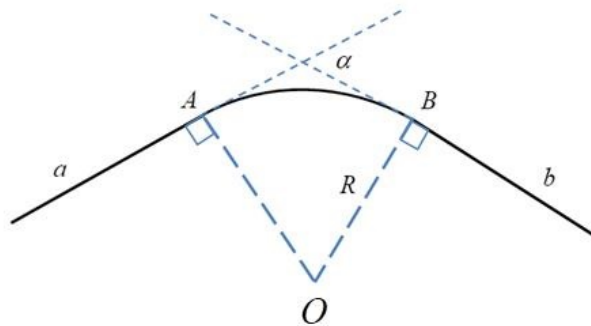

Figure 1. Curved section of the railway plan

The simplest, cost-effective solution for choosing a curved section is a circle of radius R and another connecting points A and B of the straight sections a and b of the turn (Fig. 1). The position of the railway significantly depends on the value of the radius of the circle R and it is called the radius of the curve [7].

In the railway there are special tables that determine the radius of the curve R, taking into account the technical and economic requirements for a given road [8]. In addition, the curved part of the road is considered clothoidal [9, 10]. This table is compiled taking into account the need to limit the speed of trains, removal of the designed line, increased wear of the rails, increased costs for the ongoing maintenance of the upper structure of the track and repair of rolling stock and other factors associated with the operation of the road.

Note that the curve of a railway plan section on a horizontal plane is considered as a flat curve on this plane.

2 Properties of the curve describing the railway plan

To study the curve of the railway plan, it is important to consider the point describing the curve in the profile section by the movement of the wagon.

When the wagon moves in a linear direction on a horizontal plane, the plan of the railway will be a straight line, which is obtained by moving point M in azimuth (Fig. 2).
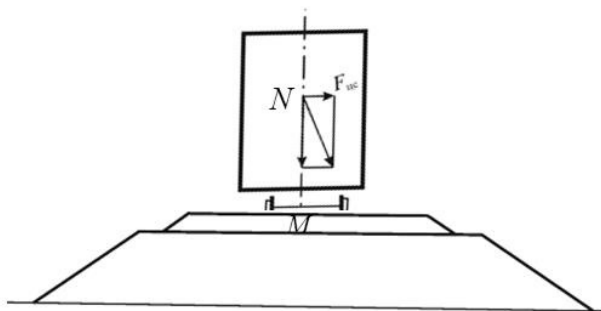
Figure 2. The profile of the movement of the wagon in a straight section

Here N is the center of gravity of the wagon and point M is its projection on the horizontal plane π_0. The plan of a railway track is understood as the geometric location of points M on a horizontal plane. These geometric locations of points generally determine the curve that defines the railroad plan. As stated above, this curve consists of linear and curved parts connecting straight parts.

To clarify the geometry of the curve formed by the point M, we consider the mechanics of the movement of the wagon when turning, that is, when moving from one straight section to the next, when these straight sections both belong to the same horizontal plane.

Theorem 1. With curved sections of the railway plan, the trace of point M will be a spatial curve.

Proof. We will prove the theorem for the simplest case of the railway plan, when both linear sections a and b of the railway plan belong to the same horizontal plane α.

The proof of the theorem in complex conditions, that is, at least in the case of a and b lying on different horizontal planes, becomes obvious, since the connection of points A and B at different levels ensures the spatiality of the curve of the railway plan.

To prove the theorem, let's consider the profile of the movement of the wagon in a curve, which looks like the one shown in Figure 3. To avoid the influence of centrifugal force when the part is curved, an elevation of the outer rail h is arranged in relation to the inner one. The value of h depends on the radius of R and the speed of the train in this section.
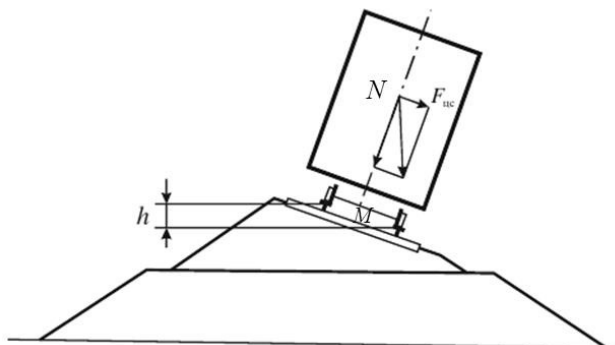

Figure 3. The profile of the movement of the wagon in a curve section

If we establish a Cartesian coordinate system with the origin at point A and the direction of the x-axis along the direction of the segment AB, the y-axis is perpendicular to the x-axis and the z-axis is along the normal of the horizontal plane, then the point M' has three coordinates (x_0, y_0, z_0). Moreover, the value of z_0 depends on the value of h and will be different from zero, if $h \neq 0$. In the curved part of the railway plan there is $h \neq 0$, therefore the curve that is the trace of point M is spatial. The theorem is proved.

To study the movement of point M of the railway plan, the horizontal plane π_0 is taken as the plane $z = 0$. We select the y-axis perpendicular to the x-axis with a positive direction to the corresponding direction of the linear part starting from point B.
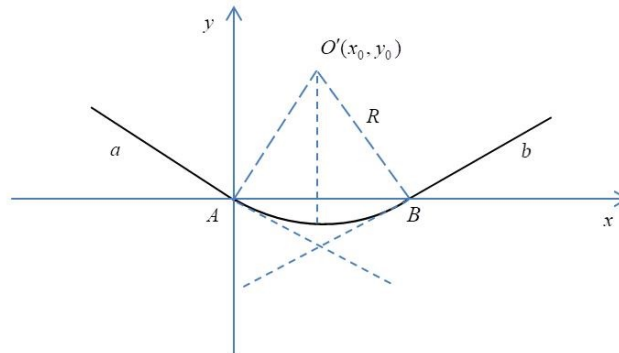

Figure 4. The inner path of the curve section

Let's assume that points A and B are connected to an arc of a circle with radius R and center at point $O'(x_0, y_0)$.

With the current selection of the coordinate system, the points A and B have the following coordinates $A(0,0)$ and $B(2x_0, 0)$. The equation of a circle with center at point $O'(x_0, y_0)$ and radius R has the form

$$(x - x_0)^2 + (y - y_0)^2 = x_0^2 + y_0^2, \tag{1}$$

since

$$R^2 = x_0^2 + y_0^2.$$

The same equation can be written in parametric form:

$$\begin{cases} x = x_0 + \left(x_0^2 + y_0^2\right)^{\frac{1}{2}} \cos t, \\ y = y_0 + \left(x_0^2 + y_0^2\right)^{\frac{1}{2}} \sin t, \end{cases}$$

where the parameter $t = \dfrac{S}{\left(x_0^2 + y_0^2\right)^{\frac{1}{2}}}$ is proportional to the length of the circular arc.

When the train moves along a curved part of the road, in order to extinguish the centrifugal force that appears when turning, the outer country of the rail is raised to a certain height H. The value of H depends on the radius of the curve R and on the speed of the train.

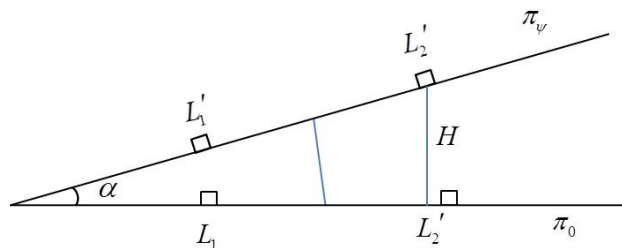The position of the base of the wagon when turning is shown schematically in Figure 5.

Figure 5. Deviation from the horizontal plane

The horizontal plane π_0 takes on another position π_ψ, where ψ is the angle between these planes. But the size of the angle ψ depends on the value of H. In this case, point M of the railway plan goes into point M' on the plane π_ψ. Since the point M' differs from the M belonging to the horizontal plane, it is spatial. Therefore, the curve describing the point M will also be spatial.

The technical and economic requirement for railways prefers not to change the internal part of the track rail. Therefore, it is advisable to select plane π_0 so that the inner part of the road rail remains on a horizontal plane. Then the road traffic pattern takes the following form.
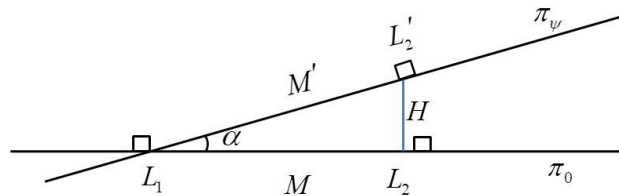

Figure 6. Deviation from the horizontal plane while maintaining the internal path on the horizontal plane

At the same time, the spatiality of the curve formed by the point M' remains.

In Figures 5 and 6, L_1 and L_2 indicate the track rails. It is obvious in Figure 6 that the inner part of the track rail is left unchanged and the outer part, that is, the point L_2 goes into the point L'_2.

Note that changes in point L_2 linearly depend on point M and on the track width, which is constant. The position of the point L'_2 is completely determined by the position of the point M'. Therefore, we can reason only with respect to the point M'.

Let us assume that, relative to the section AB, the radius R is selected and the speed V is the passage of the train. Then the lifting height of the outer rail H_{AB} can be accurately determined, so that the inner side of the track rail remains on a horizontal plane.

Then the plane π_0 to which the circular arc belongs can be accurately determined by the equation. This plane will be the plane passing through the points $(x_0, y_0, 0)$, $(0, 0, H)$ and $(2x_0, 0, H)$ of the equation of this plane.

$$z = H\left(\frac{x}{x_0} + \frac{y}{y_0} - 2\right). \tag{2}$$

The curve M' described by the point M is a spatial curve relative to the railway plan described by the point M, but belongs to the plane (2). The equation of this curve can be written in metric form:

$$\begin{cases} x = x_0 + \left(x_0^2 + y_0^2\right)^{\frac{1}{2}} \cos t, \\ y = y_0 + \left(x_0^2 + y_0^2\right)^{\frac{1}{2}} \sin t, \\ z = \frac{\left(x_0^2 + y_0^2\right)^{\frac{1}{2}}}{x_0} \cos t + \frac{\left(x_0^2 + y_0^2\right)^{\frac{1}{2}}}{y_0} \sin t - 2. \end{cases}$$

But curve γ will not be a continuous continuation of the path; it lies on a different plane relative to the horizontal plane. A connection should be established using an additional curve connecting the curve γ with the linear part a and b. For this purpose, we divide $\widehat{AB}$ into three parts $\widehat{AC}, \widehat{CD}$ and $\widehat{DB}$ with the condition that the length of the arc $\widehat{AC}$ and $\widehat{DB}$ in duration is greater than the length of the two wagons. This is a general requirement for a curved part to ensure smooth movement of the train along that part. Let's assume that points C' and D' are the images of points C and D on the curve γ. Then we take part of the curve γ with ends at points C' and D' as a part of the route.

It is required to construct a part of the railway plan connecting points A and C', also points D' and B, so that when crossing the curve $AC'D'B$, the smooth movement of the train is ensured.

We denote by γ_1 and γ_2 the curved parts of the curve connecting points A and B consisting of the arc $\widehat{AC'}$ and $\widehat{D'B}$, respectively. For convenience, curves γ_1 and γ_2 can be considered symmetrical with respect to the bisector x_0 of the angle formed by the straight part a and b of the railway plan (Fig. 7).
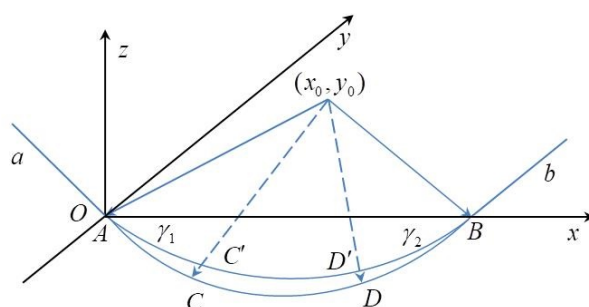

Figure 7. The curve of the outer rail

The curve equation of part γ_1 can be thought of as a spatial curve connecting points $A(0,0,0)$ and $C'(x_1, y_1, z_1) \in \gamma$ with curvature $k = \dfrac{1}{\left(x_0^2 + y_0^2\right)^{\frac{1}{2}}}$ and torsion σ.

But torsion σ can be considered a linearly increasing function of the length of the road in the form $\sigma = m \cdot s + e$. Moreover, given R and V, the values of m and e can be determined. Thus, curvature k and torsion σ are functions of R radius of curvature and speed of movement. Then, using the Frenier formula [11] for the curve γ_1 , one can determine the equation of the curve. Having obtained the equation of the curve, we can calculate the size of the railway track with the necessary accuracy, which ensures the safe movement of the train along this track.

3 Dynamic system for determining the route schedule

The railway with the curved part is called the railway route. It has been proven that the route is a spatial curve.

In the previous section it was shown that the radius of the curved part of the railway plan completely determines the curvature of the curved part of the route [12].

Torsion of the curved part of the road is defined as a change in the angle of the contacting plane of the curve representing the road route.

If the curve is given by the vector equation

$$\vec{r}(s) = \vec{x}(s)i + \vec{y}(s)j + \vec{z}(s)k,$$

where s is mastiff curve length, $\{i, j, k\}$ is basis vectors and $x(s), y(s), z(s) \in C^2$.

Then the touching plane at point (x_0, y_0, z_0) is determined by the formula:

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ x'(s_0) & y'(s_0) & z'(s_0) \\ x''(s_0) & y''(s_0) & z''(s_0) \end{vmatrix} = 0.$$

Consider at two points $M(x_0, y_0, z_0)$ and $N(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$ the osculating plane of the curve γ. Let us determine the angle $\Delta \psi$ between these planes. Speed change of angle ψ between osculating planes

$$\lim_{\Delta s \to 0} \frac{\Delta \varphi}{\Delta s} = \varphi'_s = \sigma$$

called torsion of a curve into points $M(x_0, y_0, z_0)$.

The torsion of plane curves is zero everywhere.

If we have a vector equation of a curve given by formula (1), then the curvature and torsion of the curve are calculated by the formulas [13, 14]:

$$k = \left| r''(s) \right|$$

and

$$\sigma = \frac{\left| (r'r''r''') \right|}{k^2}.$$

It should be noted that the curvature depends on the coordinates of the center of the osculating circle (x_0, y_0). The radius of the osculating circle is calculated using the formula $R = \sqrt{x_0^2 + y_0^2}$. The torsion of the curve is determined depending on the radius R and the rise h- the outer part of the track rail $\sigma = f(R, h)$.

But the parameters R, h can be selected depending on the requirements for the road, which are determined by technical and economic conditions. Therefore, Frenier's formula

$$\begin{cases} \dot{\tau} = k\nu, \\ \dot{\nu} = -k\tau - \sigma\beta \\ \dot{\beta} = \sigma\nu \end{cases} \tag{3}$$

is a dynamic system of differential equations overestimated from parameters R and V. The path equation is a solution to the dynamic system (3). Setting parameters R, h is completely determined by the solution of the system. Therefore, system (3) can be taken as a mathematical model of the railway route.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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*Author Information**

Abdullaaziz Artykbaev (*corresponding author*)— Doctor of physical and mathematical sciences, Professor, Head of the Department of Higher Mathematics, Tashkent State Transport University, Tashkent, Uzbekistan; e-mail: *aartykbaev@mail.ru*; https://orcid.org/0000-0001-6228-8749

Mokhiniso Murodulla kizi Toshmatova — PhD student, Tashkent State Transport University, Tashkent, Uzbekistan; e-mail: *toshmatova_mm@mail.ru*; https://orcid.org/ 0009-0006-2781-9325

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*The author's name is presented in the order: First, Middle and Last Names.