https://doi.org/10.31489/2024M3/13-25 *Research article*

On stability of nonlinear difference equations and some of their applications

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The issues of stability in solving nonlinear difference equations were considered. Based on a generalized difference analog of the well-known Bihari lemma, stability conditions for a trivial solution based on initial data were obtained, and an a priori estimate of stability under permanent disturbances was determined. The results were used to study the stability of solving explicit and implicit difference schemes approximating nonlinear parabolic equations.

Keywords: nonlinear difference equations, difference schemes, a priori estimates, stability.

2020 Mathematics Subject Classification: 65M06, 65M50.

Introduction

Many applied problems are reduced to the solution of nonlinear difference equations. One of the main issues in the theory of difference equations is the study of the stability of their solution. Therefore, it is of particular interest to study the stability of solutions of linear and nonlinear difference equations. The concept of stability of solutions to difference equations was first formulated by O. Perron [1] as an analog of the stability of differential equations. Then, numerous works appeared devoted to the study of the stability of difference equations. Currently, methods for studying the stability of linear difference equations with constant coefficients are quite well-developed (we do not consider equations with periodic coefficients). However, the study of the stability of linear difference equations with variable coefficients and nonlinear difference equations were not sufficient, since there were no effective criteria for the stability of their solutions. It should be noted that many problems are reduced to the solution of difference equations with variable coefficients and nonlinear difference equations. For example, such problems are posed when numerically solving differential equations using finite difference or finite element methods [2–6].

The stability of systems of linear difference equations with constant and variable coefficients was studied in [7, 8]. O. Perron [7] formulated the concept of stability of solutions of a system of difference equations with constant coefficients by analogy with this concept for differential equations. In [8] P.I. Koval studied the stability of linear difference equations with variable coefficients. He considered the difference equation in vector-matrix form:

$$y_{n+1} = Ay_n + b_n, \; n = 1, 2, ..., \tag{1}$$

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Received: 23 December 2023; *Accepted:* 19 May 2024.

where $\{y_n\}$ is the sought-for sequence of vectors, $\{A_n\}$ is the given sequence of matrices, and $\{b_n\}$ is the given norm-bounded sequence of vectors. It was proven that the solution of system (1) is stable if matrix A_n of the corresponding homogeneous system

$$y_{n+1} = A_n y_n$$

satisfies condition $\|A_n\| \leq 1 + q_n$, where $\sum\limits_{n=n_0}^{\infty} q_n < \infty$, and asymptotically stable, if $\|A_n\| \leq a < 1$ $(n > n_0)$. Next, the so-called limiting matrix $A = \lim\limits_{n\to\infty} A_n$ is introduced and, on its basis, the stability and instability of difference equations of the form (1) are studied. In [8] P.I. Koval considered linear difference equations that could be reduced to almost triangular form using linear substitutions. The asymptotic behavior of linear difference equations with almost triangular matrices was also studied there.

M.A. Skalkina in [9] showed the connection between the stability of differential and difference equations. V.B. Demidovich in [10, 11] studied the stability of nonlinear difference equations based on the first Lyapunov method. At that point, the concept of characteristic numbers of a system of linear difference equations was introduced. The concepts of reducible and regular systems of linear difference equations were introduced. In particular, it was shown that every reducible system is regular. In addition, stability under the first approximation was studied. The main result of these studies is the theorem on the asymptotic stability of the system

$$y_{n+1} = S_n y_n + f_n(y_n),$$

where $f_n(y_n)$ is the nonlinear term, S_n is the transition operator.

Nonlinear difference equations, the right-hand sides of which are linear combinations of power functions of phase variables, were studied in [12]. In addition, similar studies for differential and difference equations were carried out in [13–15].

In this article, issues of stability of the solution of nonlinear difference equations are studied. Various stability criteria are obtained, based on which nonlinear two-layer difference schemes are studied. A theorem on the stability of a trivial solution with respect to initial data is proven. The difference analog of Behari's lemma is generalized and, on its basis, an a priori estimate of the stability under permanent disturbances of a nonlinear difference equation is obtained. Examples of application of the theorem to explicit and implicit difference schemes approximating nonlinear parabolic equations are considered. Examples are given that confirm the theoretical results obtained.

1 Statement of the problem

Let us consider the Cauchy problem

$$\dot{u}(t) + A(t)u(t) = f(t, u), \;\; u(0) = u_0, \;\; \dot{u} = dt/du, \tag{2}$$

where A is a slowly varying matrix.

Equation (1) is obtained by spatial discretization of a partial differential equation of parabolic type

$$\partial t/\partial u = Lu, \;\; u(0) = u_0, \tag{3}$$

where $Lu \in H$ is some general form of a nonlinear differential operator. Here, H is the Hilbert space with scalar product (u, ϑ) and norm $\|u\| = \sqrt{(u, u)}$. Such problems arise in the mathematical modeling of processes of chemical kinetics, combustion theory, biophysics, various kinds of biochemical reactions (reaction-diffusion), convection-diffusion, processes of population growth and migration, etc.

Any two-layer difference scheme [1] that approximates problem (2) or (3) can be written in the following form of the difference equation:

$$y_{n+1} = S_n y_n + \tau f_n(y_n), \;\; y(0) = y_0, \;\; n = 0, 1, ..., \tag{4}$$

where y is a grid function that approximates function u, $y_n = y(t_n)$, $t_n \in \overline{\omega}_\tau$, $\overline{\omega}_\tau = \{t_n = n\tau,$ $n = 0, 1, ...\}$, $\tau > 0$ is a uniform grid in time $t \in [0, T]$, S_n is a certain operator (transition operator), $f_n(y_n)$ is a nonlinear term.

Let us study the stability of the trivial solution of equation (4).

Along with (4), we consider the following homogeneous equation:

$$y_{n+1} = S_n y_n, \;\; y(0) = y_0, \;\; n = 0, 1, ... \tag{5}$$

The stability of solutions of the nonlinear non-homogeneous equation (4) is completely determined by the stability of the trivial solution of its homogeneous equation (5) [10].

We consider the difference equation (4), where the nonlinear disturbance $f_n(y_n)$ satisfies the following conditions:

$$\|f_n(y_n)\| = K_n \|y_n\|^r, \;\; r > 1, \;\; f_n(0) = 0, \;\; \sum_{m=0}^{n-1} K_m \le M_1 < \infty, \tag{6}$$

where M_1 is some positive constant. In this case, the trivial sequence $y_n = 0$ is a solution to equation (4).

2 Stability theorems

Lemma 1. (The discrete analogue of Bihari's lemma) [10]. Let

$$0 \le y_0 \le c \;\; (c > 0) \tag{7}$$

and

$$y_n \le c + \sum_{v=0}^{n-1} a_v \varphi(y_v), \;\; n = 1, 2, ...,$$

where c is a positive constant, the sequence $y_i \ge 0$, $a_i \ge 0$, $i = 0, 1, ...$, $\varphi(y)$ is a continuous monotonically increasing positive function for $y > 0$, and $\varphi(0) \ge 0$, and let the inequality $\sum\limits_{v=0}^{n-1} a_v < \varphi(\infty)$ be satisfied, where $\varphi(z) = \int\limits_c^z \frac{dz_1}{\varphi(z_1)}$. Then the following estimate is valid:

$$y_n \le \varphi^{-1}\left(\sum_{v=0}^{n-1} a_v\right), \;\; n = 1, 2, ...$$

Corollary 1. Let $\varphi(y) = y^r$ $(r > 0)$, i.e. inequalities (7) be satisfied and

$$y_n \le c + \sum_{v=0}^{n-1} a_v y_v^r, \;\; n = 1, 2, ... \;,$$

where the sequence $y_i \ge 0$, $a_i \ge 0$, $i = 0, 1, ...$ Then, based on Lemma 1, we have:

$$y_n \le c / \left[1 - (r-1)c^{r-1} \sum_{v=0}^{n-1} a_v\right]^{1/(r-1)},$$

only if

$$\sum_{v=0}^{n-1} a_v < 1/\left[(r-1)c^{r-1}\right].$$

Let us generalize Lemma 1.

Lemma 2. Let the following inequality hold

$$\begin{gathered} 0 \le y_0 \le c_0 \ \ (c_0 > 0), \\ y_n \le c_n + \sum_{v=0}^{n-1} a_v \varphi(y_v), \ \ n = 1, 2, ..., \end{gathered} \tag{8}$$

where $y_i \ge 0$, $c_i > 0$, $a_i \ge 0$, $i = 0, 1, ...,$ c_i is a non-decreasing sequence ($c_{i+1} \ge c_i$), $\varphi(y)$ is a homogeneous continuous monotonically increasing function ($\varphi(0) \ge 0$) of r-th order; and let the following inequality be satisfied:

$$c_n^{-1} \sum_{v=0}^{n-1} c_v^r a_v < \phi(\infty),$$

where

$$\phi(z) = \int_1^z \frac{dz_1}{\varphi(z_1)}.$$

Then the following estimate holds:

$$y_n \le c_n \phi^{-1}\left(c_n^{-1} \sum_{v=0}^{n-1} \widetilde{a}_v\right), \ \ n = 1, 2, ..., \tag{9}$$

where $\widetilde{a}_v = c_v^r a_v$.

Proof. We divide (8) by $c_n > 0$:

$$\frac{y_n}{c_n} \le 1 + \sum_{v=0}^{n-1} \frac{a_v}{c_n} \varphi(y_v), \quad \frac{y_0}{c_0} \le 1.$$

Since $c_n \ge c_v$, then from the last inequality considering homogeneity of $\varphi(y)$ it follows that

$$\frac{y_n}{c_n} \le 1 + \frac{1}{c_n} \sum_{v=0}^{n-1} c_v^r a_v \varphi\left(\frac{y_v}{c_v}\right), \quad \frac{y_0}{c_0} \le 1. \tag{10}$$

For inequality (10), we apply Lemma 1, which gives the following estimate:

$$\frac{y_n}{c_n} \le \phi^{-1}\left(\frac{1}{c_n} \sum_{v=0}^{n-1} c_v^r a_v\right),$$

where $\phi^{-1}(z)$ is the inverse function of $\phi(z)$. This gives us estimate (9).

Corollary 2. Let $\varphi(y) = y^r$ $(r > 1)$ and the following inequalities be satisfied

$$0 < y_0 < c_0 \ \ (c_0 > 0),$$

$$y_n \le c_n + \sum_{v=0}^{n-1} a_v y_v^r, \ \ r > 1, \ \ n = 1, 2, ... \ ,$$

where $y_i \geq 0,\ c_i > 0,\ a_i \geq 0,\ \ i = 0, 1, ...$ Therefore, if

$$\sum_{v=0}^{n-1} c_v^{r-1} a_v < \frac{1}{r-1},$$

then based on Lemma 2, we have that

$$y_n \leq c_n / \left[1 - (r-1) c_n^{-1} \sum_{v=0}^{n-1} c_v^r a_v \right]^{1/(r-1)}.$$

Thus, the following theorem holds.

Theorem 1. Let the following conditions be satisfied:

a) the trivial solution of equation (5) is uniformly stable, i.e. $\forall j > 0,\ j \leq n$, estimate $\|y_n\| \leq M_2 \|y_j\|$ holds; M_2 is a positive constant;

b) the nonlinear right-hand side of equation (4) satisfies conditions (6);

c) the initial disturbance y_0 is small.

Then the trivial solution of equation (4) is stable, i.e. the following estimate holds:

$$\|y_n\| \leq \widetilde{M_2} \|y_0\|,\ \ \forall n = 0, 1, ..., \tag{11}$$

where $\widetilde{M_2}$ is a positive constant.

Proof. The solution of equation (4) satisfies the following relationship:

$$y_n = T_{n,0} y_0 + \sum_{m=0}^{n-1} T_{n,m} f_m(y_m),$$

where $T_{n,m} = S_{n-1} S_n \cdots S_m$ is the resolving operator of equation (5) from layer m to layer n. Due to assumptions a) and b) for the solution (4), we have the following estimates

$$\|T_{n,m}\| \leq M_2,\ \ \|y_n\| \leq M_2 \|y_0\| + \sum_{m=0}^{n-1} M_2 K_m \|y_m\|^r.$$

Applying the discrete analogue of Bihari's lemma (Lemma 1) to this inequality, we obtain

$$\|y_n\| \leq \frac{M_2 \|y_0\|}{\varphi(\|y_0\|)}, \tag{12}$$

where

$$\varphi(\|y_0\|) = \left[1 - (r-1)(M_2 \|y_0\|)^{r-1} M_2 \sum_{m=0}^{n-1} K_m \right]^{1/(r-1)},\ \ \varphi(0) = 1.$$

Let us estimate the lower bound $\varphi(\|y_0\|)$. We assume that

$$(r-1) M_2^r \|y_0\|^{r-1} \sum_{m=0}^{n-1} K_m \leq \delta,\ \ 0 < \delta < 1, \tag{13}$$

i.e. y_0 is a small value. Then $\varphi(\|y_0\|) \geq (1-\delta)^{1/(r-1)}$. Inequality (13) is satisfied, for example, if $\sum\limits_{m=0}^{n-1} K_m \leq M_3,\ \forall n > 1$, and the initial data satisfies the following condition

$$\|y_0\| \leq \left(\delta / \left[(r-1) M_3 M_2^r\right]\right)^{1/(r-1)}, \tag{14}$$

where M_3 is a positive constant. From (12) and (14) for the solution (4), we obtain estimate (11), which means stability based on the initial data of difference equation (4), where $\widetilde{M_2} = M_2 / \left[(1-\delta)^{1/(r-1)}\right]$.

3 Stability under permanent disturbances

Let us study the stability of the trivial solution of the difference equation (4) under permanent disturbances g_n, i.e. consider the following difference equation:

$$y_{n+1} = S_n y_n + f_n(y_n) + g_n, \;\; y(0) = y_0, \;\; g_n(0) \neq 0, \;\; n = 0, 1, ... \tag{15}$$

The nonlinear disturbance $f_n(y_n)$ satisfies condition (6) and the permanent disturbance g_n is such that

$$\sum_{m=0}^{n=1} \|g_m\| \leq \delta_0, \;\; \forall m, \;\; \delta_0 > 0, \tag{16}$$

where δ_0 is quite small.

The following theorem holds.

Theorem 2. Let the conditions of Theorem 1 be satisfied. In addition, a permanent disturbance satisfies condition (16). Then the trivial solution of equation (15) is stable under permanent disturbances and the following estimate is valid for its solution

$$\|y_n\| \leq \widetilde{M_2} \left(\|y_0\| + \sum_{m=0}^{n=1} \|g_m\| \right). \tag{17}$$

Proof. The solution of equation (15) satisfies the following relationship

$$y_n = T_{n,0} y_0 + \sum_{m=0}^{n-1} T_{n,m} \left[f_m(y_m) + g_m \right].$$

Hence, considering conditions of the theorem, we have

$$\|y_n\| \leq M_2 \left(\|y_0\| + \sum_{m=0}^{n-1} \|g_m\| + \sum_{m=0}^{n-1} K_m \|y_m\|^r \right), \;\; r > 1. \tag{18}$$

Let $\|y_0\| + \sum\limits_{m=0}^{n=1} \|g_m\| = c_n$. Then, applying Lemma 2 to inequality (18), we obtain

$$\|y_n\| \leq M_2 \left(\|y_0\| + \sum_{m=0}^{n=1} \|g_m\| \right) / \left[1 - (r-1) \sum_{m=0}^{n=1} K_m M_2^r \left(\|y_0\| + \sum_{v=0}^{m=1} \|g_v\| \right)^{r-1} \right]^{1/(r-1)},$$

i.e.

$$\|y_n\| \leq \frac{M_2}{\widetilde{\varphi}(\|y_0\|)} \left(\|y_0\| + \sum_{m=0}^{n=1} \|g_m\| \right),$$

where $\widetilde{\varphi}(\|y_0\|) \geq (1-\delta)^{1/(r-1)}$, if

$$\delta = (r-1) \sum_{m=0}^{n=1} M_2^r K_m \left(\|y_0\| + \sum_{v=0}^{m=1} \|g_v\| \right)^{r-1} \leq \delta_1, \;\; 0 \leq \delta_1 < 1. \tag{19}$$

We rewrite inequality (19) in the following form

$$(r-1)M_2^r \sum_{m=0}^{n=1} K_m \left(\|y_0\| + \sum_{v=0}^{m=1} \|g_v\| \right)^{r-1} \leq \delta_1, \;\; 0 \leq \delta_1 < 1.$$

Then

$$\|y_0\| + \sum_{v=0}^{m=1} \|g_v\| \leq (\delta_1 / [(r-1)M_2^r M_3])^{1/(r-1)},$$

where $\sum_{v=0}^{m=1} K_m \leq M_3 < \infty$, for all m.

Consequently, estimate (17) of Theorem 2 holds.

4 Study of the stability of nonlinear difference schemes

Let us consider the Cauchy problem

$$\frac{\partial u}{\partial t} = f(u), \quad u(0) = u_0, \tag{20}$$

where u is a certain variable describing the state of the system, $f(u)$ is a nonlinear operator (functional). Similar problems include equations of the following form:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(u),$$

with linear derivative terms, but containing a nonlinear in u term. For example, the following semilinear equations [16, 17]:

– Zeldovich's equation, found in combustion theory, for which

$$q(u) = ku^v(1-u), \;\; v > 1, \, q(u) > 0, \;\; 0 < u < 1,$$

$$q(0) = q(1) = 0, q'(0) = 0, \;\; q'(1) < 0;$$

– Semenov's equation describing autocatalytic chain reactions:

$$q(u) = u(u-\alpha)(1-u), \, 0 < u < 1, \;\; 0 < \alpha < 1,$$

$$q(0) = q(\alpha) = q(1) = 0, q'(0) < 0, \;\; q'(\alpha) > 0, \;\; q'(1) < 0;$$

– Fisher's equation (or Kolmogorov-Petrovsky-Piskunov's equation) found in problems of mathematical biology, for which

$$q(u) = ku(1-u), \;\; q(u) > 0, \;\; 0 < u < 1,$$

$$q(0) = q(1) = 0, \;\; q'(0) > 0, \;\; q'(1) < 0,$$

$k > 0$ is the constant.

1°. Let us approximate (20) with an explicit difference scheme of the following form (Eulerian scheme)

$$y_t = f(y_n), \quad y(0) = y_0, \tag{21}$$

where $y_t = (y_{n+1} - y_n)/\tau$.

The error of scheme (21) $z = y - u$ $(y = z + u)$ satisfies the following equation:

$$z_t = f(y_n) - f(u_n) + g_n, \;\; z_0 = 0, \tag{22}$$

where $g_n = O(\tau)$ is the approximation error.

Using the Fréchet derivative for functional f, we obtain:

$$f(y) - f(u) = f(u+z) - f(u) = f'(u)z + O(z),$$

i.e. from (22) it follows that

$$z_t = f'(u_n)z_n + q(z_n) + g_n, \quad z_0 = 0, \tag{23}$$

where $q(z_n) = O\left(\|z_n\|^r\right), \quad r > 1$.

From (23) it follows that

$$z_{n+1} = S_n z_n + \tau q(z_n) + \tau g_n, \quad z_0 = 0, \tag{24}$$

where $S_n = 1 + \tau f'(u_n)$.

By Theorem 2, scheme (24) is stable, if the solution of its first approximation is uniformly stable

$$z_{n+1} = S_n z_n, \quad z_0 = 0. \tag{25}$$

The condition for uniform stability of solution (25) is $\|S_n\| \leq M$. If this condition is met, we obtain estimate

$$\|z_{n+1}\| \leq M_1 \|z_n\|$$

for all $n > 0$. Since $f'(u)$ is the bounded linear functional, estimate $\|1 + \tau f'(u_n)\| \leq M_1$ holds, and the remaining conditions of Theorem 2 are satisfied.

Now we prove the convergence of the scheme. Since (21) is uniformly stable according to the initial data (the first condition of Theorem 1), then from (24) it follows that

$$\|z_{n+1}\| \leq \|S_n\| \|z_n\| + \tau \|q(z_n)\| + \tau \|g_n\|.$$

Hence

$$\|z_{n+1}\| \leq M_1 \left(\|z_0\| + \tau \sum_{m=0}^{n-1} K_m \|z_n\| + \tau \sum_{m=0}^{n-1} \|g_m\| \right),$$

by Theorem 2, the following estimate holds:

$$\|z_{n+1}\| \leq \widetilde{M}_1 \left(\|z_0\| + \tau \sum_{m=0}^{n-1} \|g_m\| \right). \tag{26}$$

From $\|S_n\| \leq 1$ we obtain condition $|1 + \tau f'(u_n)| \leq 1$ or $-1 \leq 1 + \tau f'(u_n) \leq 1$, i.e.

a) inequality

$$1 + \tau f'(u_n) \leq 1$$

fulfilled for $f'(u_n) \leq 0$;

b) inequalities $-1 \leq 1 + \tau f'(u_n)$, $\tau \left| f'(u_n) \right| \leq 2$, $\tau \leq 2/\left| f'(u_n) \right|$ are the conditions for uniform stability of scheme (21). Thus, the following theorem is proven.

Theorem 3. Let conditions $f'(u_n) \leq 0$, $\tau \leq 2/\left| f'(u_n) \right|$ be satisfied. Then the solution of the explicit difference scheme (21) is stable with respect to the initial data and the right-hand side, and for its solution, there is an a priori estimate (26).

2°. Let us approximate problem (20) with the following implicit difference scheme

$$y_t = f(\hat{y}), \tag{27}$$

where

$$\hat{y} = y^{n+1}.$$

Then for the error we get problem ($z = y - u,\ \ y = z + u$):

$$z_t = f(\hat{y}) - f(\hat{u}) + g_n.$$

Using the Frechet derivative, we get

$$z_t = f'(u_{n+1})z_{n+1} + q(z_{n+1}) + g_n, \tag{28}$$

where $\|q(z_{n+1})\| = K_{n+1}\|z_{n+1}\|^r,\ r > 1\,(n = 0, 1, ...),\ f'(u_{n+1})$ is a bounded linear operator (functional). To study the convergence of scheme (28), we obtain the first approximation equation

$$z_{n+1} = S_n z_n, \quad S_n = \left(1 - \tau f'(u_{n+1})\right)^{-1}.$$

Let the solution to this equation be uniformly stable, i.e. $\|S_n\| \le 1$. Then we obtain the condition for uniform stability of solution $1/\left(1 - \tau f'(u_{n+1})\right) \le 1.$ This condition is always satisfied, if

$$f'(u_{n+1}) \le 0. \tag{29}$$

Therefore, taking (6) into account, the following estimate holds:

$$\|z_{n+1}\| \le M_1 \left(\|z_0\| + \tau \sum_{m=0}^{n} k_{m+1}\|z_{m+1}\|^r + \tau \sum_{m=0}^{n-1} \|g_m\| \right).$$

From here, we get

$$\|z_{n+1}\| \left(1 - M_1 k_{n+1}\tau\|z_{n+1}\|^{r-1}\right) \le M_1 \left(\|z_0\| + \tau \sum_{m=0}^{n-1} K_{m+1}\|z_{n+1}\|^r + \tau \sum_{m=0}^{n-1} \|g_m\| \right). \tag{30}$$

Let

$$1 - M_1 k_{n+1}\tau\|z_{n+1}\|^{r-1} \ge \delta, \quad 0 < \delta < 1.$$

Then, from (30), we have the following estimate:

$$\|z_{n+1}\| \le \frac{M_1}{\delta} \left(\|z_0\| + \tau \sum_{m=0}^{n-1} K_{m+1}\|z_{m+1}\|^r + \tau \sum_{m=0}^{n-1} \|g_m\| \right).$$

Based on Lemma 2, we obtain the following estimate:

$$\|z_{n+1}\| \le \widetilde{M_1} \left(\|z_0\| + \tau \sum_{m=0}^{n-1} \|g_m\| \right), \tag{31}$$

if

$$\|z_{n+1}\|^{r-1} \le \frac{1-\delta}{M_1 K_{n+1}} \ \text{ or } \ \|z_{n+1}\| \le \left(\frac{1-\delta}{M_1 K_{n+1}}\right)^{\frac{1}{r-1}}.$$

Thus, the following theorem is proven.

Theorem 4. Let condition (29) be satisfied. Then the solution of the implicit difference scheme (27) is stable with respect to the initial data and the right-hand side, and its solution has a priori estimate (31).

3°. Let us approximate problem (20) with the following one-parameter difference scheme

$$y_t = y_n + \tau f^2(y_n)/\left[(1+\alpha)f(y_n) - \alpha f(y_n + \alpha\tau f(y_n))\right]. \tag{32}$$

Here

$$F(y) = y + \tau f^2(y)/\left[(1+\alpha)f(y) - \alpha f(y + \alpha\tau f(y))\right].$$

From (32) for $\alpha = 0$, we obtain difference scheme (21), for $\alpha = -1$ and $\alpha = 1$, we obtain V.V. Bobkov's A-stable difference schemes.

Let us obtain the problem for the scheme error (32)

$$z_t = F(y_n) - F(u_n) + g_n, \quad z_0 = 0.$$

Using the Fréchet derivative for $F(y)$, we obtain

$$z_t = F'(u_n)z_n + q(z_n) + g_n, \quad z_0 = 0,$$

where

$$F'(u) = 1 + \tau\widetilde{f}'(u),$$

$$\begin{gathered}\widetilde{f}'(u) = \frac{f^2(u)}{\left[(1+\alpha)f(y) - \alpha f(y+\alpha\tau f(y))\right]^2}\left[f'(u) + \alpha f'(u) -\right. \\ \left. -\frac{2\alpha}{f(u)} f(u+\alpha\tau f(u))f'(u) + \alpha f'(u+\alpha\tau f(u)) + \alpha^2\tau f'(u) f'(u+\alpha\tau f(u)\right).\end{gathered} \tag{33}$$

Thus, we obtained the first approximation equations

$$z_{n+1} = (1 + \widetilde{f}'(u_n))z_n.$$

Let us check, under what terms the uniform stability condition $\widetilde{f}'(u) \leq 0$ is satisfied. From (33), it follows that $\widetilde{f}'(u) \leq 0$, if

$$\begin{gathered} f'(u) + \alpha f'(u) - \frac{2\alpha}{f(u)} f(u+\alpha\tau f(u))f'(u) + \\ +\alpha f'(u+\alpha\tau f(u)) + \alpha^2\tau f'(u) f'(u+\alpha\tau f(u))] \leq 0. \end{gathered} \tag{34}$$

Applying the Taylor formula for $f(u+\alpha\tau f(u))$ and $f'(u+\alpha\tau f(u))$, we obtain the condition for the fulfillment of inequality (34)

$$f'(u) - \alpha^2\tau f'^2(u) + \alpha^2\tau f(u)f''(u) + \frac{\alpha^3\tau^2}{2} f^2(u)f'''(u) + O(\tau^3) \leq 0. \tag{35}$$

This proves the following theorem.

Theorem 5. Let condition (35) be satisfied. Then the solution of the difference scheme (32) is stable with respect to the initial data and the right-hand side, and for its solution, there is a priori estimate (17).

Let us check condition (35) using a test example. Let $f(u) = -\lambda u,\ \ \lambda > 0$.

Then, substituting $f(u) = -\lambda u, f'(u) = -\lambda, f''(u) = 0$ into (35), we obtain inequality, $-\lambda - \tau\lambda^2 \leq 0$, from which it follows that $f'(u) \leq 0$. Let now $f(u) = k[A(1-u) - Bu^2]$, where $k > 0$, $A > 0$, $B > 0$. Hence $f'(u) = -k[A + 2Bu]$, $f''(u) = -2kB$, and the remaining derivatives are zero. Then to satisfy (35), we obtain the following condition:

$$-k[A + 2Bu + \tau k A^2 + 2AB\tau k u + 2B^2\tau k u^2 + 2\tau k AB] \leq 0,$$

or

$$A + 2Bu + \tau k A^2 + 2AB\tau k u + 2B^2\tau k u^2 + 2\tau k AB \geq 0,$$

which is valid for $0 \leq u \leq 1$.

The results of Theorem 5 are also valid for the multi-parameter explicit absolutely A-stable Bobkov difference scheme

$$\hat{y} = y + \tau(A+B)\frac{f^{k+1}(y, t+\alpha\tau)}{Af^k(y + a\tau f(y, t+\alpha\tau), t+\beta\tau) + Bf^k(y, t+\alpha\tau)},$$

where $y \approx u(t)$, $\hat{y} \approx u(t+\tau)$ are approximate solutions, $u(t)$ is the solution of equation $\dot{u} = f(t,u)$, A, B, α, β, a, k are some parameters that control the order of accuracy of the scheme.

Conclusion

Stability conditions for solutions of nonlinear difference equations are obtained. Based on the generalized discrete analogue of Bihari's lemma, an a priori estimate of the stability under permanent disturbances of a nonlinear difference equation is obtained. Theorems on the stability of the solution of nonlinear difference equations are proven. Examples of application of the stability theorem to explicit and implicit difference schemes that approximate nonlinear parabolic equations are considered. Based on the proposed methodology for studying the stability of difference equations, it is possible to study the stability of difference schemes for the above semi-linear equations of Zeldovich, Semenov and Fisher, as well as the stability of difference schemes for nonlinear equations of pseudo-parabolic type [18–21].

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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