MATHEMATICS

https://doi.org/10.31489/2024M3/5-12 *Research article*

The boundary value problem for an ordinary linear half-order differential equation

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This study is devoted to the study of the solution of a boundary value problem for an ordinary linear differential equation of half order with constant coefficients. Using of the fundamental solution of the main part of the considered equation, we obtained the principal relations, from which we obtain the necessary conditions for the Fredholm property of the original problem. Further, using the Mittag-Leffler function, a general solution of the homogeneous equation is obtained. Finally, the problem under consideration is reduced to an integral Fredholm equation of the second kind with a non-singular kernel, i.e., the Fredholm property of the stated problem is proved.

Keywords: half-order equations, boundary value problem, fundamental solution, basic relation, integral equations, Fredholm property, Mittag-Leffler functions, general solution of a homogeneous half-order equation.

2020 Mathematics Subject Classification: 34A08.

Introduction

Most investigations in different fields of science and engineering are modeled with the help of differential equations (or systems of equations) with fractional derivatives. The concept of fractional calculus has gained considerable popularity and importance during the past half decades. The concept of the fractional calculus takes beginning from outstanding learned as Marquis de L'Hopital, G.W. Leibniz, Fourier, Laplace, Liouville, Riemann, Letnikov etc, as gained considerable popularity and importance during the past half decades, in [1–5].

The study of solving boundary value problems is closely related to the Green's function. The construction of the Green's function is not an easy task, since it is related to the considered equations and the boundary condition [6–8].

Problems of the Cauchy type for an ordinary linear differential equation of fractional order, in particular half-order, are studied in $[1,2,4,5]$, where these problems are reduced to Volterra integral equations of the second kind. Constructing of a fundamental solution is much easier than constructing

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Received: 17 December 2023; *Accepted:* 15 May 2024.

the Green's function, since it is associated only with the equation under consideration. In [9] and [10] solutions of some classes of Cauchy problems containing fractional differential operators are established.

In [11–13] for some class of the fractional order equations of fundamental solutions constructed. This article is devoted to the study of a boundary value problem for an ordinary linear differential equation of half order. We used the fundamental solution proposed in [6], where a fundamental solution was constructed for a wide class of differential equations.

With the help of the fundamental solution of the main part of the considered equation, the main relations is obtained, from which the necessary conditions for Fredholm property are proved. Further, with the help of the Mittag-Leffler function the general solution of the homogeneous equation is obtained.

Let us consider the following problem

$$D^{\frac{1}{2}}_{x_1-}y(x) - ay(x) = f(x), \quad 0 < x_0 < x < x_1, \tag{1}$$

$$y(x_1) + \alpha y(x_0) = 0, \tag{2}$$

where a, x_0, x_1 and α are given constants, $f(x)$ is a known continuous function defined on $[x_0, x_1]$ and $y(x)$ is an unknown function that is required to define, and

$$D^{\frac{1}{2}}_{x_1-}y(x) = -\frac{d}{dx}\int_x^{x_1}\frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(t)dt, \quad x < x_1$$

is left half order derivative of the function $y(x)$ [1],

$$\Big(-\frac{1}{2}\Big)! = \Gamma\Big(\frac{1}{2}\Big) = \int_0^\infty e^{-t}t^{-\frac{1}{2}}dt.$$

Here, Γ is Euler's gamma function.

In order to construct solution of the considered problem we use of the fundamental solution of the conjugate equation corresponding to Eq. (1)

$$D^{\frac{1}{2}}_{x_0}y(x) - ay(x) = f(x), \quad 0 < x_0 < x < x_1, \tag{3}$$

where

$$D^{\frac{1}{2}}_{x_0}y(x) = \frac{d}{dx}\int_{x_0}^{x}\frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(t)dt, \quad x > x_0.$$

Let $f(x), g(x) \in C[x_0, x_1]$ and $D^{\frac{1}{2}}_{x_0}g(x)$, $D^{\frac{1}{2}}_{x_1-}f(x)$ exist on $[x_0, x_1]$, where $C[x_0, x_1]$ is a class of continuously functions in $[x_0, x_1]$. According to [13] the following equality holds

$$\int_{x_0}^{x_1}\Big(D^{\frac{1}{2}}_{x_0}f(x)\Big)g(x)dx = \int_{x_0}^{x_1}f(x)\Big(D^{\frac{1}{2}}_{x_1-}g(x)\Big)dx. \tag{4}$$

Easy to see that $\frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!}$ is a fundamental solution for the main part of Eq. (3). Indeed [1],

$$D^{\frac{1}{2}}_{x_0}\frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!} = \frac{x^{-1}}{(-1)!} = \delta(x).$$

Here $\delta(x)$ is Dirac's function.

Then multiplying Eq. (1) by

$$\frac{(t-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(t),$$

and integrating over x on the open interval (x_0, x_1), we have

$$\int_{x_0}^{x_1}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right)D_{x_1-}^{\frac{1}{2}}y(x)dx-a\int_{x_0}^{x_1}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right)y(x)dx=$$

$$=\int_{x_0}^{x_1}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right)f(x)dx. \tag{5}$$

Here,

$$y_h(x)=\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!} \tag{6}$$

is a partial solution and is the homogeneous solution corresponding to Eq. (3). Indeed,

$$D_{x_0}^{\frac{1}{2}}y(x)=\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-2}{2}}}{\left(\frac{k-2}{2}\right)!}=\frac{x^{-1}}{(-1)!}+a\frac{x^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+a^2\frac{x^0}{(0)!}+a^3\frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)!}+\cdots=$$

$$=\delta(x)+a\left[\frac{x^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+a\frac{x^0}{(0)!}+a^2\frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)!}+\cdots\right]=ay(x).$$

Then the general solution of Eq. (3) has the form

$$y_h(x)=C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!},$$

where C is an arbitrary constant.

Taking into account Eq. (4), we get from Eq. (5)

$$\int_{x_0}^{x_1}y(x)dxD_{x_0}^{\frac{1}{2}}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right]-a\int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right]y(x)dx=$$

$$=\int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right]f(x)dx$$

or

$$\int_{x_0}^{x_1}D_{x_0}^{\frac{1}{2}}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}\right)y(x)dx+\int_{x_0}^{x_1}D_{x_0}^{\frac{1}{2}}\left[Cy_h(x)\right]y(x)dx-$$

$$-a\int_{x_0}^{x_1}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}\right)y(x)dx-a\int_{x_0}^{x_1}Cy_h(x)y(x)dx=\int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}+Cy_h(x)\right]f(x)dx.$$

According to

$$D_{x_0}^{\frac{1}{2}}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}\right)=\frac{(x-\xi)^{-1}}{(-1)!}=\delta(x-\xi)$$

from the last relation, we have

$$\int_{x_0}^{x_1} y(x)\delta(x-\xi)dx + \int_{x_0}^{x_1} C\left[D_{x_0}^{\frac{1}{2}}\big(y_h(x)\big) - ay_h(x)\right]y(x)dx -$$

$$-a\int_{x_0}^{x_1}\left(\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}\right)y(x)dx = \int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + Cy_h(x)\right]f(x)dx.$$

From here we get the following main relation

$$a\int_{x_0}^{x_1}\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}y(x)dx + \int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + Cy_h(x)\right]f(x)dx =$$

$$= \begin{cases} y(\xi), & \xi \in (x_0, x_1), \\ \frac{1}{2}y(x_0), & \xi = x_0, \\ \frac{1}{2}y(x_1), & \xi = x_1, \end{cases}$$

or

$$a\int_{x_0}^{x_1}\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}y(x)dx + \int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!}\right]f(x)dx =$$

$$= \begin{cases} y(\xi), & \xi \in (x_0, x_1), \\ \frac{1}{2}y(x_0), & \xi = x_0, \\ \frac{1}{2}y(x_1), & \xi = x_1. \end{cases} \tag{7}$$

Thus, based on the fundamental solution for the main part of Eq. (3), for the fractional order equation, we obtained the main relation (7), which consists of two parts. The first part, where $\xi \in (x_0, x_1)$ gives any solution of Eq. (3), and the second part, where $\xi = x_0$, or $\xi = x_1$ gives us the necessary conditions. With this, for each solution of the inhomogeneous Eq. (1), the boundary values are obtained in the main relation (7).

Thus, for $x \in (x_0, x_1)$ for the general solution of Eq. (3) we have the following representation

$$y(\xi) = a\int_{x_0}^{x_1}\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}y(x)dx + \int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!}\right]f(x)dx \tag{8}$$

and for boundary points $\xi = x_0$, and $\xi = x_1$ we get relations

$$\begin{cases} \frac{1}{2}y(x_0) = \int_{x_0}^{x_1}\frac{(x-x_0)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}y(x)dx + \int_{x_0}^{x_1}\left[\frac{(x-x_0)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!}\right]f(x)dx, \\ \frac{1}{2}y(x_1) = \int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}y(x)dx + \int_{x_0}^{x_1}\left[\frac{(x-x_1)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!}\right]f(x)dx. \end{cases} \tag{9}$$

Putting (9) in boundary condition (2), we can define the arbitrary constant C

$$2a\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}y(x)dx + 2\int_{x_0}^{x_1}\left[\frac{(x-x_1)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} + C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{\left(\frac{k-1}{2}\right)!}\right]f(x)dx +$$

$$+2\alpha\left[a\int_{x_0}^{x_1}\frac{(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+\int_{x_0}^{x_1}\left[\frac{(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}+C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}\right]f(x)dx\right]=0$$

or

$$2a\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+2\alpha a\int_{x_0}^{x_1}\frac{(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+$$

$$+2\int_{x_0}^{x_1}\left[\frac{(x-x_1)^{-\frac{1}{2}}}{(-\frac{1}{2})!}\right]f(x)dx+2\alpha\int_{x_0}^{x_1}\left[\frac{(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}\right]f(x)dx+$$

$$+2\int_{x_0}^{x_1}\left[C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}\right]f(x)dx+2\alpha\int_{x_0}^{x_1}\left[C\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}\right]f(x)dx=0.$$

Grouping similar terms, we have

$$a\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}f(x)dx+$$

$$+C(1+\alpha)\int_{x_0}^{x_1}\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}f(x)dx=0,$$

or

$$C(1+\alpha)\int_{x_0}^{x_1}\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}f(x)dx=-a\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx-$$

$$-\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}f(x)dx. \tag{10}$$

If

$$\Delta=\int_{x_0}^{x_1}\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}f(x)dx\neq 0, \tag{11}$$

then from Eq. (10) we obtain

$$C=-\frac{1}{\Delta(1+\alpha)}\left[a\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+\right.$$

$$\left.+\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}f(x)dx\right]. \tag{12}$$

Finally, substituting Eq. (12) in Eq. (8), we have

$$y(\xi)=a\int_{x_0}^{x_1}\frac{(x-\xi)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+\int_{x_0}^{x_1}\left[\frac{(x-\xi)^{-\frac{1}{2}}}{(-\frac{1}{2})!}\right]f(x)dx-$$

$$-\int_{x_0}^{x_1}\sum_{k=0}^{\infty}a^k\frac{x^{\frac{k-1}{2}}}{(\frac{k-1}{2})!}f(x)dx\left\{\frac{1}{\Delta(1+\alpha)}\left[a\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}y(x)dx+\right.\right.$$

$$\left.\left.+\int_{x_0}^{x_1}\frac{(x-x_1)^{-\frac{1}{2}}+\alpha(x-x_0)^{-\frac{1}{2}}}{(-\frac{1}{2})!}f(x)dx\right]\right\}$$

or

$$y(\xi) = a\int_{x_0}^{x_1} \frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} y(x)dx + \int_{x_0}^{x_1} \left[\frac{(x-\xi)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}\right] f(x)dx -$$

$$-\frac{a}{1+\alpha}\int_{x_0}^{x_1} \frac{(x-x_1)^{-\frac{1}{2}} + \alpha(x-x_0)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} y(x)dx - \frac{a}{1+\alpha}\int_{x_0}^{x_1} \frac{(x-x_1)^{-\frac{1}{2}} + \alpha(x-x_0)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} f(x)dx.$$

Thus, the solution of problem (3), (2), we reduce to the following integral equation

$$y(\xi) = a\int_{x_0}^{x_1} \frac{(x-\xi)^{-\frac{1}{2}} - \frac{1}{1+\alpha}\left((x-x_1)^{-\frac{1}{2}} + \alpha(x-x_0)^{-\frac{1}{2}}\right)}{\left(-\frac{1}{2}\right)!} y(x)dx +$$

$$+\int_{x_0}^{x_1} \frac{(x-\xi)^{-\frac{1}{2}} - \frac{1}{1+\alpha}\left((x-x_1)^{-\frac{1}{2}} + \alpha(x-x_0)^{-\frac{1}{2}}\right)}{\left(-\frac{1}{2}\right)!} f(x)dx.$$

Let us denote by $K(x,\xi)$ kernel in the last integral

$$K(x,\xi) = \frac{(x-\xi)^{-\frac{1}{2}} - \frac{1}{1+\alpha}\left((x-x_1)^{-\frac{1}{2}} + \alpha(x-x_0)^{-\frac{1}{2}}\right)}{\left(-\frac{1}{2}\right)!}$$

then the solution of problem (3), (2) is reduced to the second type integral equation of the Fredholm with regular kernel as

$$y(\xi) = a\int_{x_0}^{x_1} K(x,\xi)y(x)dx + \int_{x_0}^{x_1} K(x,\xi)f(x)dx, \tag{13}$$

and so the following theorem is true.

Theorem 1. Let a and α be given positive constants and $f(x)$ by $x \in (x_0, x_1)$ known a continuous function. If series (6) is convergent and take place (11), then the boundary value problem (3), (2) has the Fredholm property.

The actual solution of problem (1), (2) can be obtained from Eq. (13) either by the method of successive approximations [14], or by replacing the integral entering in (13) with any approximate integration formula, for example method of trapeze, Simpson, etc.

Conclusion

For the first time, using the fundamental solution of the main part of the conjugate corresponding to the main equation, we obtained the main relations from which the necessary conditions for the Fredholm property of the original problem are obtained.

With this, for each solution of the inhomogeneous Eq. (3) the boundary values are obtained mainly from relation (9).

Author Contributions

N. Aliyev drew attention to the issue. B. Sinsoysal researched relevant literature and assisted in the research process. M. Rasulov and B. Sinsoysal together did all the theoretical work and implemented the process of writing the article. All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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