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Recurrence free decomposition formulas for the Lauricella special functions

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Expansion formulas associated with the multidimensional Lauricella hypergeometric functions are well-established and extensively utilized. However, the recurrence relations inherit in these formulas add extra complexities to their use. A thorough analysis of the characteristics of these expansion formulas shows that they can be simplified and converted into a more convenient form. This paper presents new recurrence free decomposition formulas, which are employed to solve boundary value problems.

Keywords: Appell Functions, Lauricella Functions, Recurrence Decomposition Formula, Recurrence Free Decomposition Formula.

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Introduction

The theory of multidimensional hypergeometric functions has gained significant interest because of its capability to solve numerous applied problems involving partial differential equations (for details, see [1]; also the references quoted in to [2] and [3]). As shown in work [4], for instance, hypergeometric special functions with many arguments can be widely used to estimate the energy absorbed by the non ferromagnetic conducting sphere located inside an internal magnetic field. In addition to their using in solving partial deferential equations, hypergeometric series of several variables are utilized into different quantum physical problems and also in quantum chemical applications [2,5]. Inter alia, the second order degenerate differential equations in partial derivatives of elliptic-parabolic types, which are particularly widespread in studying gas dynamics problems may be solved by means of diverse multidimensional Gaussian series. Interesting examples consist of the studying problem of the adiabatic plane parallel to the liquid or gas flow without any vortex. Also the problem of the flow of supersonic current from a container with smooth walls and several other technical issues of gas-liquid flow may arise in various applications [6, 7].

It is very essential to highlight that Riemann's and Green's special functions, as well as the fundamental solutions with singularity of the second order degenerate differential equations with partial derivatives may be also expressed by multidimensional Gaussian series. When we research problems with boundary values for similar differential equations in partial derivatives, we need to expand hypergeometric special functions of several variables into more simpler types of special functions, like Gauss or Appell functions.

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The known operator method of Burchnall and Chaundy [8] has been very widely used by scientists to receive formulas for expanding of hypergeometric functions of two independent arguments, the known operator method of Burchnall and Chaundy [8] has been very widely used by scientists to receive formulas for expanding of hypergeometric Gaussian series of two variables, expressing through the use of simple Gauss' hypergeometric series of one variable.

Based on the fundamental work of Burchnall and Chaundy [8], Hasanov and Srivastava [9, 10] introduced formulas which extend the capabilities Burchnall-Chaundy operator, this leads to another expansion formulas for various hypergeometric series of three variables. They also established recurrent formulas for higher-dimensional hypergeometric functions. Nonetheless, the recurrence introduces potential complications when applying these decomposition formulas.

In this study, we develop novel decomposition formulas for all four multiple Lauricella's hypergeometric functions, providing they are independent of recurrence.

1 *The expansions of Appell's two-variable functions*

The decomposition of a hypergeometric series with many arguments into several simpler components is one of the main problems of the special functions theory. Such a decomposition is valuable because it enables the simplification of complex calculations, reduces the dimensionality of the problem, and facilitates the development of new identities and relationships between special functions.

In 1940, Burchnall and Chaundy [8] introduce the operators

$$\nabla(h) = \frac{\Gamma(h)\,\Gamma(\delta_1+\delta_2+h)}{\Gamma(\delta_1+h)\,\Gamma(\delta_2+h)}, \quad \Delta(h) = \frac{\Gamma(\delta_1+h)\,\Gamma(\delta_2+h)}{\Gamma(h)\,\Gamma(\delta_1+\delta_2+h)}, \tag{1}$$

where $\delta_1 = x\dfrac{\partial}{\partial x}$ and $\delta_2 = y\dfrac{\partial}{\partial y}$, through which they penned

$$F_2\left(a,b,b';c,c';x,y\right) = \nabla(a)F(a,b;c;x)F(a,b';c';y), \tag{2}$$

$$F_3\left(a,a',b,b';c;x,y\right) = \Delta(c)F(a,b;c;x)F(a',b';c;y),$$

$$F_1\left(a,b,b';c;x,y\right) = \nabla(a)\Delta(c)F(a,b;c;x)F(a,b';c;y),$$

$$F_4\left(a,b;c,c';x,y\right) = \nabla(a)\nabla(b)F(a,b;c;x)F(a,b;c';y),$$

thus decomposing Appell's functions using operators Δ and ∇; they also obtained transformations of Appell's functions including

$$F_1\left(a,b,b';c;x,y\right) = \nabla(a)F_3\left(a,a,b,b';c;x,y\right),$$

$$F_1\left(a,b,b';c;x,y\right) = \Delta(c)F_2\left(a,b,b';c,c;x,y\right),$$

$$F_4\left(a,b;c,c';x,y\right) = \nabla(b)F_2\left(a,b,b;c,c';x,y\right),$$

and some others.

These symbolic representations are utilized to derive numerous expansions of Appell's functions either as products of ordinary hypergeometric functions or conversely. For instance, employing Gauss' formula [11; 73],

$$F(a,b;c;x) \equiv F\left[\begin{matrix} a,b; \\ c; \end{matrix}\; x\right] = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}, \ c \neq 0,-1,-2,...$$

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \ c \neq 0,-1,-2,..., \ Re(c-a-b) > 0 \tag{3}$$

we symbolically express

$$\nabla(h) = \sum_{r=0}^{\infty} \frac{(-\delta)_r(-\delta')_r}{(h)_r r!}.$$

Now, by virtue of Poole's formula [12; 26]

$$(-\delta)_r f(r) = (-1)^r x^r \frac{d^r f(r)}{dx^r},$$

we obtain

$$(-\delta)_r F(a,b;c;x) = (-1)^r \frac{(a)_r (b)_r}{(c)_r} x^r F(a+r, b+r; c+r; x)$$

and therefore (2) indicates the decomposition formula [8]

$$\begin{gathered} F_2\left(a,b,b';c,c';x,y\right) = \sum_{r=0}^{\infty} \frac{(a)_r\,(b)_r\,(b')_r}{r!\,(c)_r\,(c')_r} \times \\ \times\, x^r y^r F\left(a+r, b+r; c+r; x\right) F\left(a+r, b'+r; c'+r; y\right). \end{gathered} \tag{4}$$

Through the inversion of (2) in the following form

$$F(a,b;c;x)F(a,b';c';y) = \Delta(a) F_2\left(a,b,b';c,c';x,y\right)$$

and an associated expansion of $\Delta(a)$, which is related to (4),

$$\begin{gathered} F\left(a,b;c;x\right) F\left(a,b';c';y\right) = \sum_{r=0}^{\infty} (-1)^r \frac{(a)_r\,(b)_r\,(b')_r}{r!\,(c)_r\,(c')_r} \times \\ \times\, x^r y^r F_2\left(a+r, b+r, b'+r; c+r, c'+r; x, y\right) \end{gathered}$$

is obtained.

These expansions can be established through coefficient comparison of corresponding powers of x and y.

Applying their way, Burchnall and Chaundy enacted 15 couples of expansions that binds Appell's two-variables functions to one-variables ordinary hypergeometric functions, along with many additional expansion formulas involving hypergeometric series of many variables and confluent hypergeometric series of Humbert.

The introduced method is applicable to functions with two arguments, relies on symbolic operators that are mutually inverse, as detailed in subsequent literature [8].

2 Decomposition formulas for multiple Lauricella hypergeometric functions

To extend the operators $\nabla(h)$ and $\Delta(h)$ introduced in (1), Hasanov and Srivastava [9,10] proposed new operators

$$\tilde{\nabla}_{x_1;x_2,\ldots,x_n}(h) = \frac{\Gamma(h)\,\Gamma(\delta_1 + \ldots + \delta_n + h)}{\Gamma(\delta_1 + h)\,\Gamma(\delta_2 + \ldots + \delta_n + h)},$$

$$\tilde{\Delta}_{x_1;x_2,\ldots,x_n}(h) = \frac{\Gamma(\delta_1 + h)\,\Gamma(\delta_2 + \ldots + \delta_n + h)}{\Gamma(h)\,\Gamma(\delta_1 + \ldots + \delta_n + h)},$$

where $\delta_k = x_k \dfrac{\partial}{\partial x_k}$ $(k = \overline{1,n})$, through which they successfully derived decomposition formulas for the entire class of multiple Gauss series.

Based on the ideas presented in [8], Hasanov and Srivastava [9] demonstrated that the recurrence formulas [10] hold for all $n \in N \backslash \{1\}$.

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = \sum_{|\mathbf{k}'|=0}^{\infty} \frac{(a)_{|\mathbf{k}'|}(b_1)_{|\mathbf{k}'|}}{(c_1)_{|\mathbf{k}'|}} x_1^{|\mathbf{k}'|} \prod_{j=2}^{n} \frac{(b_j)_{k_j}}{k_j!\,(c_j)_{k_j}} x_j^{k_j} \times$$
$$\times F\left(a+|\mathbf{k}'|, b_1+|\mathbf{k}'|; c_1+|\mathbf{k}'|; x_1\right) F_A^{(n-1)}\left(a+|\mathbf{k}'|, \mathbf{b}'+\mathbf{k}'; \mathbf{c}'+\mathbf{k}'; \mathbf{x}'\right), \quad (5)$$

$$F_B^{(n)}(\mathbf{a},\mathbf{b};c;\mathbf{x}) = \sum_{|\mathbf{k}'|=0}^{\infty} \frac{(-1)^{|\mathbf{k}'|}(a_1)_{|\mathbf{k}'|}(b_1)_{|\mathbf{k}'|}}{(c-1+|\mathbf{k}'|)_{|\mathbf{k}'|}(c)_{2|\mathbf{k}'|}} x_1^{|\mathbf{k}'|} \prod_{j=2}^{n} \frac{(a_j)_{k_j}(b_j)_{k_j}}{k_j!} x_j^{k_j} \times$$
$$\times F\left(a_1+|\mathbf{k}'|, b_1+|\mathbf{k}'|; c+2|\mathbf{k}'|; x_1\right) F_B^{(n-1)}\left(\mathbf{a}'+\mathbf{k}', \mathbf{b}'+\mathbf{k}'; c+2|\mathbf{k}'|; \mathbf{x}'\right), \quad (6)$$

$$F_C^{(n)}(a,b;\mathbf{c};\mathbf{x}) = \sum_{|\mathbf{k}'|+|\mathbf{l}'|=0}^{\infty} \frac{\left[(a)_{|\mathbf{k}'|}\right]^2 (b)_{2|\mathbf{k}'|+|\mathbf{l}'|}}{(a)_{|\mathbf{k}'|}(c_1)_{|\mathbf{k}'|+|\mathbf{l}'|}} x_1^{|\mathbf{k}'|+|\mathbf{l}'|} \prod_{j=2}^{n} \frac{x_j^{k_j+l_j}}{k_j! l_j!\,(c_j)_{k_j+l_j}} \times$$
$$\times F\left(a+|\mathbf{k}'|+|\mathbf{l}'|, b+2|\mathbf{k}'|+|\mathbf{l}'|; c_1+|\mathbf{k}'|+|\mathbf{l}'|; x_1\right),$$
$$F_C^{(n-1)}\left(a+|\mathbf{k}'|+|\mathbf{l}'|, b+2|\mathbf{k}'|+|\mathbf{l}'|; \mathbf{c}'+\mathbf{k}'; \mathbf{x}'\right), \quad (7)$$

$$F_D^{(n)}(a,\mathbf{b};c;\mathbf{x}) = \sum_{|\mathbf{k}'|+|\mathbf{l}'|=0}^{\infty} \frac{(-1)^{|\mathbf{k}'|}(a)_{2|\mathbf{k}'|+|\mathbf{l}'|}(b_1)_{|\mathbf{k}'|+|\mathbf{l}'|}(c)_{2|\mathbf{k}'|}}{(c-1+|\mathbf{k}'|)_{|\mathbf{k}'|}\left[(c)_{2|\mathbf{k}'|+|\mathbf{l}'|}\right]^2} x_1^{|\mathbf{k}'|+|\mathbf{l}'|} \times$$
$$\times \prod_{j=2}^{n} \frac{(b_j)_{k_j+l_j}}{k_j! l_j!} x_j^{k_j+l_j} F\left(a+2|\mathbf{k}'|+|\mathbf{l}'|, b_1+|\mathbf{k}'|+|\mathbf{l}'|; c+2|\mathbf{k}'|+|\mathbf{l}'|; x_1\right),$$
$$F_D^{(n-1)}\left(a+2|\mathbf{k}'|+|\mathbf{l}'|, \mathbf{b}'+\mathbf{k}'+\mathbf{l}'; c+2|\mathbf{k}'|+|\mathbf{l}'|; \mathbf{x}'\right), \quad (8)$$

where

$|\mathbf{k}'| := k_2 + ... + k_n,\ k_2 \geq 0, ..., k_n \geq 0;\ |\mathbf{l}'| := l_2 + ... + l_n,\ l_2 \geq 0, ..., l_n \geq 0;\ \mathbf{x}' := (x_2, ..., x_n);$ $\mathbf{a}' + \mathbf{a}' := (a_2 + k_2, ..., a_n + k_n)$ and so on.

Certain properties of the Lauricella $F_A^{(n)}$ function have been studied previously, differentiation formulas, limit formulas, new integral representations and several decomposition formulas have been derived [13]. Nevertheless, the recurrence that presents in formulas (5)–(8) may introduce additional complexities when applying these expansions. Further investigation into the properties of Lauricella functions has shown that these recurrence formulas can be simplified into more manageable forms.

3 New recurrence free decomposition formulas for the Lauricella hypergeometric functions

Until the presentation the main results, let's determine some necessary notations

$$A(k) = A(k,n) = \sum_{i=2}^{k+1}\sum_{j=i}^{n} m_{i,j},\ A(0) = 0;\ B(k) \equiv B(k,n) = \sum_{i=2}^{k} m_{i,k} + \sum_{i=k+1}^{n} m_{k+1,i},$$

$$|\mathbf{m}_n| := \sum_{i=2}^{n}\sum_{j=i}^{n} m_{i,j},\quad M_n! := \prod_{i=2}^{n}\prod_{j=i}^{n} m_{i,j}!,$$

$$C(k) \equiv C(k,n) = \sum_{i=2}^{k+1}\sum_{j=i}^{n} p_{i,j},\ C(0) = 0;\ D(k) \equiv D(k,n) = \sum_{i=2}^{k} m_{i,k} + \sum_{i=k+1}^{n} p_{k+1,i},$$

$$|\mathbf{p}_n| := \sum_{i=2}^{n}\sum_{j=i}^{n} p_{i,j}, \quad P_n! := \prod_{i=2}^{n}\prod_{j=i}^{n} m_{i,j}!,$$

where $k, n \in \mathrm{N}$, $k \leq n$; $m_{i,j} \in \mathrm{N} \cap \{0\}$ $(2 \leq i \leq j \leq n)$; if we interpret the $\sum\limits_{i=2}^{s}$ as zero when $s = 1$, for instance, our notations $A(0) = B(1) = C(0) = D(1) = 0$ are adopted.

Theorem 1. The following expansion formulas hold at $n \in \mathrm{N}$

$$F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n)}}{M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B(k)}}{(c_k)_{B(k)}} \times \\ \times \prod_{k=1}^{n} x_k^{B(k)} F\left[\begin{array}{l} a + A(k), b_k + B(k); \\ c_k + B(k); \end{array} x_k\right], \tag{9}$$

$$F_B^{(n)}(\mathbf{a}, \mathbf{b}; c; \mathbf{x}) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(-1)^{A(n)}}{(c)_{2A(n)}\, M_n!} \times \\ \times \prod_{k=1}^{n} \frac{(a_k)_{B(k)}\,(b_k)_{B(k)}}{(c - 1 + A(k) - A(k-1))_{A(k)-A(k-1)}} \times \\ \times \prod_{k=1}^{n} x_k^{B(k)} F\left[\begin{array}{l} a_k + B(k), b_k + B(k); \\ c + 2A(k); \end{array} x_k\right], \tag{10}$$

$$F_C^{(n)}(a, b; \mathbf{c}; \mathbf{x}) = \sum_{|\mathbf{m}_n|+|\mathbf{p}_n|=0}^{\infty} \frac{\left[(a)_{A(n)+C(n)}\right]^2 (b)_{2A(n)+C(n)}}{M_n! P_n!} \times \\ \times \prod_{k=1}^{n} \frac{x_k^{B(k)+D(k)}}{(c_k)_{B(k)+D(k)}\,(a + A(k-1) + C(k-1))_{A(k)-A(k-1)}} \times \\ \times \prod_{k=1}^{n} F\left[\begin{array}{l} a + A(k) + C(k), b + 2A(k) + C(k); \\ c_k + B(k) + D(k); \end{array} x_k\right], \tag{11}$$

$$F_D^{(n)}(a, \mathbf{b}; c; \mathbf{x}) = \sum_{|\mathbf{m}_n|+|\mathbf{p}_n|=0}^{\infty} \frac{(-1)^{A(n)}(a)_{2A(n)+C(n)}}{M_n! P_n! \left[(c)_{2A(n)+C(n)}\right]^2} \times \\ \times \prod_{k=1}^{n} \frac{(c + 2A(k-1) + C(k-1))_{2A(k)-2A(k-1)}\,(b_k)_{B(k)+D(k)}}{(c + A(k) + A(k-1) + C(k-1))_{A(k)-A(k-1)}} \times \\ \times \prod_{k=1}^{n} x_k^{B(k)+D(k)} F\left[\begin{array}{l} a + 2A(k) + C(k), b_k + B(k) + D(k); \\ c_k + 2A(k) + C(k); \end{array} x_k\right]. \tag{12}$$

Proof. Equality (9) is proved with the help of the mathematical induction method. Three new equalities (10)–(12) are also proved by mathematical induction.

Corollary 1. Let $a, b_1, \ldots, b_n$ be real numbers with a, c_k, $c_k - b_k \neq 0, -1, -2, \ldots$ and $a > |\mathbf{b}|$. Then the ensuing limit formulas valid at $n \in \mathrm{N}$

$$\lim_{\mathbf{x}\to 0}\left\{\mathbf{x}^{-\mathbf{b}} F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; 1 - \frac{1}{\mathbf{x}}\right)\right\} = \frac{\Gamma(a - |\mathbf{b}|)}{\Gamma(a)} \prod_{k=1}^{n} \frac{\Gamma(c_k)}{\Gamma(c_k - b_k)}; \tag{13}$$

$$\lim_{\mathbf{x}\to 0}\left\{\mathbf{x}^{-\mathbf{b}}F_B^{(n)}\left(\mathbf{a};\mathbf{b};c;1-\frac{1}{\mathbf{x}}\right)\right\}=\frac{\Gamma(c)}{\Gamma(c-|\mathbf{b}|)}\prod_{k=1}^{n}\frac{\Gamma(a_k-b_k)}{\Gamma(a_k)}, \tag{14}$$

where

$$\mathbf{x}^{-\mathbf{b}}:=x_1^{-b_1}...x_n^{-b_n};\ \frac{1}{\mathbf{x}}:=\left(\frac{1}{x_1},...,\frac{1}{x_n}\right).$$

Proof. Due to the above decomposition formula (9) we get next formula

$$\begin{gathered}F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};1-\frac{1}{\mathbf{x}}\right)=\sum_{|\mathbf{m}_n|=0}^{\infty}\frac{(a)_{A(n,n)}}{M_n!}\prod_{k=1}^{n}\frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}}\times\\ \times\prod_{k=1}^{n}\left(1-\frac{1}{x_k}\right)^{B(k,n)}F\left[\begin{matrix}a+A(k,n),b_k+B(k,n);\\ c_k+B(k,n);\end{matrix}\ 1-\frac{1}{x_k}\right].\end{gathered} \tag{15}$$

Now applying the well-known Boltz's formula

$$F(a,b;c;z)=(1-z)^{-b}F\left(c-a,b;c;\frac{z}{z-1}\right)$$

for each hypergeometric function within sum (15), we obtain

$$\begin{gathered}F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};1-\frac{1}{\mathbf{x}}\right)=\mathbf{x}^{\mathbf{b}}\sum_{|\mathbf{m}_n|=0}^{\infty}\frac{(a)_{A(n,n)}}{M_n!}\prod_{k=1}^{n}\frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}}(x_k-1)^{B(k,n)}\times\\ \times\prod_{k=1}^{n}F\left[\begin{matrix}c_k-a+B(k,n)-A(k,n),b_k+B(k,n);\\ c_k+B(k,n);\end{matrix}\ 1-x_k\right].\end{gathered}$$

Utilizing the property parity of the sum

$$\sum_{k=1}^{n}B(k)=2\sum_{k=2}^{n}\sum_{i=2}^{k}m_{i,k}=2\sum_{k=1}^{n-1}\sum_{i=k+1}^{n}m_{k+1,i},$$

we calculate the limit

$$\begin{gathered}\lim_{\mathbf{x}\to 0}\left\{\mathbf{x}^{-\mathbf{b}}F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};1-\frac{1}{\mathbf{x}}\right)\right\}=\sum_{|\mathbf{m}_n|=0}^{\infty}\frac{(a)_{A(n,n)}}{M_n!}\prod_{k=1}^{n}\frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}}\times\\ \times\prod_{k=1}^{n}F\left[\begin{matrix}c_k-a+B(k,n)-A(k,n),b_k+B(k,n);\\ c_k+B(k,n);\end{matrix}\ 1\right]\end{gathered}$$

and utilizing identity (3) to transform the hypergeometric Gauss series in the final summation, by virtue of the previously received equality [14]

$$\begin{gathered}\sum_{|\mathbf{m}_n|=0}^{\infty}\frac{(a)_{A(n,n)}}{M_n!}\prod_{k=1}^{n}\frac{(b_k)_{B(k,n)}(a-b_k)_{A(k,n)-B(k,n)}}{(a)_{A(k,n)}}=\\ =\frac{\Gamma(a-|\mathbf{b}|)}{\Gamma(a)}\prod_{k=1}^{n}\frac{\Gamma(a)}{\Gamma(a-b_k)},\end{gathered}$$

we obtain equality (13). Equality (14) is proved analogously to the proof of (13).

4 Applications of the recurrence free decomposition formulas

Two dimensional case. In case $n = 2$, the formula (9) was known since 1940 in the work [8] (see the expansion (4)) and it was effectively used in studying problems with boundary values for the differential equation of elliptic type with two singular coefficients

$$u_{xx} + u_{yy} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y = 0, \;\; [2\alpha, 2\beta \in (0,1)]$$

in the works [15, 16].

Three dimensional case. A following decomposition formula

$$\begin{aligned} &F_A^{(3)}\left(a, b_1, b_2, b_3; c_1, c_2, c_3; x_1, x_2, x_3\right) = \\ &= \sum_{i,j,k=0}^{\infty} \frac{(a)_{i+j+k}(b_1)_{j+k}(b_2)_{i+k}(b_3)_{i+j}}{i!j!k!(c_1)_{j+k}(c_2)_{i+k}(c_3)_{i+j}} \times \\ &\times x_1^{j+k} F\left(a + j + k, b_1 + j + k; c_1 + j + k; x_1\right) \times \\ &\times x_2^{i+k} F\left(a + i + j + k, b_2 + i + k; c_2 + i + k; x_2\right) \times \\ &\times x_3^{i+j} F\left(a + i + j + k, b_3 + i + j; c_3 + i + j; x_3\right) \end{aligned}$$

is used in solving various problems with boundary values for the three dimensional differential equation of elliptic type with the three singular coefficients

$$u_{xx} + u_{yy} + u_{yy} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y + \frac{2\gamma}{z}u_z = 0, \;\; 0 < 2\alpha, 2\beta, 2\gamma < 1$$

in the works [17–19].

Four dimensional case. Sixteen fundamental solutions were constructed for degenerate elliptic type equation with four variables [20]

$$y^m z^k t^l u_{xx} + x^n z^k t^l u_{yy} + x^n y^m t^l u_{zz} + x^n y^m z^k u_{tt} = 0, \quad m, n, k, l \equiv const > 0, \tag{16}$$

by means of following recurrence free expansion formula for the hypergeometric Lauricella's series of four independent variables

$$\begin{gathered} F_A^{(4)}\left(a; b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; x_1, x_2, x_3, x_4\right) = \\ = \sum_{\substack{m_2, m_3, m_4,\\ i,j,k=0}}^{\infty} \frac{(a)_{m_2+m_3+m_4+i+j+k}(b_1)_{m_2+m_3+m_4}(b_2)_{m_2+i+j}(b_3)_{m_3+i+k}(b_4)_{m_4+j+k}}{(c_1)_{m_2+m_3+m_4}(c_2)_{m_2+i+j}(c_3)_{m_3+i+k}(c_4)_{m_4+j+k} m_2! m_3! m_4! i! j! k!} \times \\ \times x_1^{m_2+m_3+m_4} x_2^{m_2+i+j} x_3^{m_3+i+k} x_4^{m_4+j+k} \\ \times F\left(a + m_2 + m_3 + m_4, b_1 + m_2 + m_3 + m_4; c + m_2 + m_3 + m_4; x_1\right) \\ \times F\left(a + m_2 + m_3 + m_4 + i + j, b_2 + m_2 + i + j; c_2 + m_2 + i + j; x_2\right) \\ \times F\left(a + m_2 + m_3 + m_4 + i + j + k, b_3 + m_3 + i + k; c_3 + m_3 + i + k; x_3\right) \\ \times F\left(a + m_2 + m_3 + m_4 + i + j + k, b_4 + m_4 + j + k; c_4 + m_4 + i + k; x_4\right). \end{gathered}$$

Using the obtained fundamental solutions, several boundary value problems were solved in both finite and infinite domains. For equation (16) in an infinite domain, Neumann, Dirichlet, and several mixed boundary value problems were solved [21, 22]. In a finite domain the Holmgren's problem analogue was solved [23].

Multidimensional case. It is known that all fundamental solutions of the elliptic type differential equation of many variables with singular coefficients

$$L_{\alpha,\lambda}^{(m,n)}(u) \equiv \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} + \sum_{j=1}^{n} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} = 0,\ 0 < 2\alpha_j < 1, j = \overline{1, n} \tag{17}$$

in the domain $R_m^{n+} := \{(x_1, ..., x_m) : x_1 > 0, ..., x_n > 0\}$ $(m \geq 2,\ 1 \leq n \leq m)$ are expressed by the Lauricella hypergeometric function $F_A^{(n)}$ in the forms

$$\begin{gathered} q_k(x;\xi) = \gamma_k r^{-2\beta_k} \prod_{i=1}^{n} x_i^{2\alpha_i} \prod_{i=1}^{k} (x_i \xi_i)^{1-2\alpha_i} \times \\ \times F_A^{(n)} \left[\begin{matrix} \beta_k, 1-\alpha_1, ..., 1-\alpha_k, \alpha_{k+1}, ..., \alpha_n; \\ 2-2\alpha_1, ..., 2-2\alpha_k, 2\alpha_{k+1}, ..., 2\alpha_n; \end{matrix} \sigma \right],\quad k = \overline{0,\, n}, \end{gathered} \tag{18}$$

where

$$\beta_k = \frac{m-2}{2} + k - \sum_{i=1}^{k} \alpha_i + \sum_{i=k+1}^{n} \alpha_i,\quad k = \overline{0,\, n};$$

$$\gamma_k = 2^{2\beta_k - m} \frac{\Gamma(\beta_k)}{\pi^{m/2}} \prod_{i=1}^{k} \frac{\Gamma(1-\alpha_i)}{\Gamma(2-2\alpha_i)} \prod_{i=k+1}^{n} \frac{\Gamma(\alpha_i)}{\Gamma(2\alpha_i)}, k = \overline{0,\, n};$$

$$\xi = (\xi_1, ..., \xi_m) : \xi_1 > 0, ..., \xi_n > 0;\ \ \sigma = (\sigma_1, ..., \sigma_n),\quad \sigma_j = 1 - \frac{r_j^2}{r^2},$$

$$r^2 = \sum_{i=1}^{m} (x_i - \xi_i)^2,\ r_j^2 = (x_j + \xi_j)^2 + \sum_{i=1, i\neq j}^{m} (x_i - \xi_i)^2,\ \ j = \overline{1, n}.$$

The singularity of fundamental solutions. By means of the expansion formula (9), it can be shown that the received fundamental solutions (18) have their singularity at $r = 0$. Indeed, it is easy to rewrite a fundamental solution $q_k(x;\xi)$ in the form

$$q_k(x;\xi) = \frac{1}{r^{m-2}}\, \tilde{q}_k(x;\xi),\ \ m > 2,$$

where

$$\tilde{q}_k(x;\xi) = \gamma_k \mathbf{X}^{-|\mathbf{b}|} \prod_{i=1}^{k} \frac{x_i \xi_i^{1-2\alpha_i}}{r_i^{2-2\alpha_i}} \prod_{i=k+1}^{n} \left(\frac{x_i}{r_i}\right)^{2\alpha_i} F_A^{(n)} \left(\beta_k, \mathbf{b}; \mathbf{c}; 1 - \frac{1}{\mathbf{X}}\right),$$

$$\mathbf{X} := \left(\frac{r^2}{r_1^2}, ..., \frac{r^2}{r_n^2}\right), \mathbf{X}^{-|\mathbf{b}|} := \prod_{i=1}^{k} \left(\frac{r}{r_i}\right)^{2-2\alpha_i} \prod_{i=k+1}^{n} \left(\frac{r}{r_i}\right)^{2\alpha_i},$$

$$|\mathbf{b}| := k - \sum_{i=1}^{k} \alpha_i + \sum_{i=k+1}^{n} \alpha_i,\ \ \mathbf{b} := (1-\alpha_1, ..., 1-\alpha_k, \alpha_{k+1}, ..., \alpha_n),$$

$$\mathbf{c} := (2-2\alpha_1, ..., 2-2\alpha_k, 2\alpha_{k+1}, ..., 2\alpha_n),\ \ k = \overline{0,\, n}.$$

Now using limit relation (13), we see that the function $\tilde{q}_k(x;\xi)$ is a limited expression at $x \to \xi$:

$$\lim_{r\to 0} \tilde{q}_k(x;\xi) = \frac{1}{4\pi^{m/2}} \Gamma\left(\frac{m-2}{2}\right).$$

So, the constructed fundamental solutions of the differential equation (17) have a singularity of the order $m-2$ when $r \to 0$.

Introduce the following notation:

$$S_p = \{x : x_1 > 0, ..., x_{p-1} > 0,\ x_p = 0,\\ x_{p+1} > 0, ..., x_n > 0,\ -\infty < x_{n+1} < +\infty, ..., -\infty < x_m < +\infty\},$$

$$X_p^2 := 1 + x_1^2 + ... + x_{p-1}^2 + x_{p+1}^2 + ... + x_m^2,\ p = \overline{1,n}.$$

Dirichlet-Neumann problem $\left(D^k N^{n-k}\right)^\infty$ *in unbounded domains.* Find a regular solution $u_k(x)$ of equation (17) from the function class $C\left(\overline{\Omega}\right) \cap C^2(\Omega)$, satisfying conditions

$$u_k(x)|_{x_p=0} = \tau_p(\tilde{x}_p),\ \ p = \overline{1,k}, \tag{19}$$

$$\left.\left(x_p^{2\alpha_p} \frac{\partial u_k(x)}{\partial x_p}\right)\right|_{x_p=0} = \nu_p(\tilde{x}_p),\ \ p = \overline{k+1,n}, \tag{20}$$

and

$$\lim_{R\to\infty} u_k(x) = 0,\ m > 2,\ k = \overline{0,n} \tag{21}$$

(if $m = 2$, then the boundedless of the desired solution at infinity is required as well), where $\tau_p(\tilde{x}_p)$ and $\nu_p(\tilde{x}_p)$ are defined functions in the following form:

$$\tau_p(\tilde{x}_p) = \frac{\tilde{\tau}_p(\tilde{x}_p)}{X_p^{\varepsilon_p}},\quad \tilde{\tau}_p(\tilde{x}_p) \in C\left(\overline{S_p}\right),\quad \varepsilon_p > 0,\ p = \overline{1,k},$$

and

$$\nu_p(\tilde{x}_p) = \frac{\tilde{\nu}_p(\tilde{x}_p)}{X_p^{1-2\alpha_p+\varepsilon_p}},\quad \tilde{\nu}_p(\tilde{x}_p) \in C\left(\overline{S}_p\right),\quad \varepsilon_p > 0,\ p = \overline{k+1,n}.$$

The functions $\tau_p(\tilde{x})$ $(p = \overline{1,k})$ satisfy the coordination conditions on the initial k lateral faces S_p of the domain and at the origin:

$$\tau_1|_{x_2=0} = \tau_2|_{x_1=0},\ \tau_2|_{x_3=0} = \tau_3|_{x_2=0},\ ....,\ \tau_{k-1}|_{x_k=0} = \tau_k|_{x_{k-1}=0};\\ \tau_1(0,0,...,0) = \tau_2(0,0,...,0) = ... = \tau_k(0,0,...,0).$$

The vector $\tilde{x}_p$ occurring in the problem setting is obtained from a vector x by excluding its pth component:

$$\tilde{x}_p := (x_1, ..., x_{p-1}, x_{p+1}, ..., x_m),\ p = \overline{1,n}.$$

The problem's unique solution $\left(D^k N^{n-k}\right)^\infty$ is represented in the next form

$$u_k(\xi) = \sum_{p=1}^{k} \int_{S_p} \tau_p(\tilde{x}_p)\, \tilde{x}_p^{(2\alpha)} \left.\left(x_p^{2\alpha_p} \frac{\partial q_k(x,\xi)}{\partial x_p}\right)\right|_{x_p=0} dS_p -\\ - \sum_{p=k+1}^{n} \int_{S_p} \nu_p(\tilde{x}_p)\, \tilde{x}_p^{(2\alpha)}\, q_k(x,\xi)|_{x_p=0} dS_p. \tag{22}$$

In (22), we use the notation

$$\int_{S_p} ...dS_p := \underbrace{\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty}}_{m-n} \underbrace{\int_{0}^{+\infty} ... \int_{0}^{+\infty}}_{n-1} ...dx_1...dx_{p-1}dx_{p+1}...dx_n dx_{n+1}...dx_m.$$

By direct calculation, we establish that the function $u_k(\xi)$, defined in (22), is a solution to the equation (17). Using the decomposition formula (9) and limit relation (13) we can prove that the function $u_k(\xi)$ satisfies the conditions (19)–(21) of the problem $\left(D^k N^{n-k}\right)^\infty$ (for details, see [24]).

Other applications of the expansion formula (9) for the multiple Lauricella special function $F_A^{(n)}$ are found in [25].

We do not yet know any applications of the decomposition formulas (10), (11) and (12) for the well-known Lauricella's hypergeometric series $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$, respectively.

Conclusion

In this paper, recurrence free decomposition formulas for the four Lauricella functions were presented. The obtained formulas were proved using the mathematical induction. These expansions can be demonstrated by comparing the coefficients of equal powers of the variables $x_1, ..., x_n$ on both sides. Formulas (21) and (22) indicate a reciprocity property of the hypergeometric Lauricella functions F_A and F_B, as these functions exhibit reciprocal values in the limit. Do the F_C and F_D functions have similar properties? One of these decomposition formulas for the Lauricella's series F_A is often used in studying problems with boundary values for partial differential equations of various types.

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Author Contributions

T.G. Ergashev collected and analyzed data, led manuscript preparation and supervised the research process. A.R. Ryskan analyzed data, led manuscript preparation, served as the principal investigator of the research grant. N.N. Yuldashev collected and analyzed data.

All authors participated in the revision of the manuscript and approved the final submission.

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Srivastava, H.M., & Karlsson, P.W. (1985). *Multiple Gaussian Hypergeometric Series.* Halsted Press New York Chichester Brisbane Toronto.

2 Opps, S.B., Saad, N., & Srivastava, H.M. (2005). Some reduction and transformation formulas for the Appell hypergeometric functions F_2. *Journal of Mathematical Analysis and Applications, 302*, 180–195. https://doi.org/10.1016/j.jmaa.2004.07.052

3 Padmanabham, P.A., & Srivastava, H.M. (2000). Summation formulas associated with the Lauricella function $F_A^{(r)}$. *Applied Mathematics Letters, 13*(1), 65–70. https://doi.org/10.1016/S0893-9659(99)00146-9

4 Lohöfer, G. (1989). Theory of an electromagnetically deviated metal sphere, I: Absorbed power. *SIAM Journal on Applied Mathematics, 49*, 567–581. https://doi.org/10.1137/0149032

5 Niukkanen, A.W. (1983). Generalized hypergeometric series ${}^{N}F(x_1, ..., x_N)$ arising in physical and quantum chemical applications. *Journal of Physics A: Mathematical and General, 16*, 1813–1825. https://doi.org/10.1088/0305-4470/16/9/007

6 Frankl, F.I. (1973). *Izbrannye trudy po gazovoi dinamike [Selected works on gas dynamics]*. Moscow: Nauka [in Russian].

7 Bers, L. (1958). *Mathematical Aspects of Subsonic and Transonic Gas Dynamics.* Dover Publications Inc. New York.

8 Burchnall, J.L., & Chaundy, T.W. (1940). Expansions of Appell's double hypergeometric functions. *The Quarterly Journal of Mathematics, Oxford, Ser., 11*, 249–270. https://doi.org/10.1093/qmath/os-11.1.249

9 Hasanov, A., & Srivastava, H.M. (2006). Decomposition formulas associated with the Lauricella function $F_A^{(r)}$ and other multiple hypergeometric functions. *Applied Mathematic Letters, 19*(2), 113–121. https://doi.org/10.1016/j.aml.2005.03.009

10 Hasanov, A., & Srivastava, H.M. (2007). Decomposition Formulas Associated with the Lauricella Multivariable Hypergeometric Functions. *Computers and Mathematics with Applications, 53*(7), 1119–1128. https://doi.org/10.1016/j.camwa.2006.07.007

11 Erdelyi, A., Magnus, W., Oberhettinger, F., & Tricomi, F.G. (1953). *Higher Transcendental Functions, Vol. I.* McGraw-Hill Book Company, New York, Toronto, London.

12 Poole, E.G. (1936). *Introduction to the Theory of Linear Differential Equations.* Oxford University Press Oxford.

13 Ryskan, A., & Ergashev T. (2023). On Some Formulas for the Lauricella Function. *Mathematics, 11*(24), https://doi.org/10.3390/math11244978

14 Hasanov, A., & Ergashev, T.G. (2021). New decomposition formulas associated with the Lauricella multivariable hypergeometric functions. *Montes Taurus Journal of Pure and Applied Mathematics, 3*(3), 317–326. Article ID: MTJPAM-D-20-00049.

15 Karimov, E.T., & Nieto, J.J. (2011). The Dirichlet problem for a 3D elliptic equation with two singular coefficients. *Computers and Mathematics with Applications, 62*, 214–224. https://doi.org/10.1016/j.camwa.2011.04.068

16 Srivastava, H.M., Hasanov, A., & Choi J. (2015). Double-layer potentials for a generalized biaxially symmetric Helmholtz equation. *Sohag Journal of Mathematics, 2*(1), 1–10.

17 Karimov, E.T. (2010). On a boundary problem with Neumann's condition for 3D singular elliptic equations. *Applied Mathematics Letters, 23*(5), 517–522. https://doi.org/10.1016/j.aml.2010.01.002

18 Karimov, E.T. (2010). A boundary-value problem for 3-D elliptic equation with singular coefficients. *Progress in analysis and its applications*, 619–625. https://doi.org/10.1142/9789814313179_0081

19 Nieto, J.J. (2012). On an analogue of the Holmdreen's problem for 3D singular elliptic equation. *Asian-European Journal of Mathematics 5*(2), 1–18. https://doi.org/10.1142/S1793557112500210

20 Hasanov, A., Berdyshev, A.S., & Ryskan, A. Fundamental solutions for a class of four-dimensional degenerate elliptic equation. *Complex Variables and Elliptic Equations, 65*, 632–647. https://doi.org/10.1080/17476933.2019.1606803

21 Berdyshev, A.S., & Ryskan, A. (2020). The Neumann and Dirichlet problems for one four-dimensional degenerate elliptic equation. *Lobachevskii Journal of Mathematics, 41*(6), 1051–1066. https://doi.org/10.1134/S1995080220060062

22 Berdyshev, A.S., & Ryskan, A. (2021). Boundary value problem for the four-dimensional Gellerstedt equation. *Bulletin of the Karaganda University. Mathematics Series, 4*(104), 35–48.

https://doi.org/10.31489/2021M4/35-48

23 Baishemirov, Z., Berdyshev, & Ryskan A. (2022). A Solution of a Boundary Value Problem with Mixed Conditions for a Four-Dimensional Degenerate Elliptic Equation. *Mathematics, 10*(7), 1094. https://doi.org/10.3390/math10071094

24 Ergashev, T.G., & Tulakova, Z.R. (2022). A Problem with Mixed Boundary Conditions for a Singular Elliptic in an Infinite Domain. *Russian Mathematics, 66*(7), 51–63. https://doi.org/10.26907/0021-3446-2022-7-58-72

25 Ergashev, T.G., & Tulakova, Z.R. (2022). The Neumann problem for a multidimensional elliptic equation with several singular coefficients in an infinite domain. *Lobachevskii Journal of Mathematics, 43*(1), 199–206. https://doi.org/10.1134/S1995080222040102

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