ANNIVERSARIES

Dulat Syzdykbekovich Dzhumabaev

Life and scientific activity (dedicated to the 70th birthday anniversary)

Professor Dulat Syzdykbekovich Dzhumabaev, Doctor of Physical and Mathematical Sciences, was a prominent scientist, a well-known specialist in the field of the qualitative theory of differential and integro-differential equations, the theory of nonlinear operator equations, numerical and approximate methods for solving boundary value problems.

Dzhumabaev D.S. was born in Kantagi, Turkistan district, South Kazakhstan region, on April 11, 1954. From 1961 to 1971, he attended secondary school in Turkistan. In 1971, he entered Faculty of Mechanics and Mathematics of Kazakh State University named after S.M. Kirov (now Al-Farabi Kazakh National University). After graduating with honors from the Department of Mathematics in 1976, he continued to pursue postgraduate studies at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR. His scientific activity began under the guidance of Academician Orymbek Akhmetbekovich Zhautykov, an outstanding scientist and mathematician, who made a huge contribution to the creation and development of the mathematical science in Kazakhstan. After successful completion of postgraduate studies in 1979, Dzhumabaev D.S. joined the Laboratory of Ordinary Differential Equations headed by Academician Zhautykov O.A. He went from being a junior researcher to becoming the head of the Laboratory of Differential Equations, one of the leading divisions of the Institute of Mathematics. He chaired the laboratory from 1996 to 2012.

Dzhumabaev D.S. was a successful scientist and versatile specialist in the field of mathematics and its applications. He devoted his talent and hard work to the study of nonlinear operator equations, to the creation and development of qualitative methods in the theory of boundary value problems for differential equations.

The main research areas and the results obtained by Professor Dzhumabaev can be divided into several groups. The most significant and important scientific results are presented below in chronological order.

1 Boundary value problems for ordinary differential equations with a parameter in a Banach space

During postgraduate studies, his research was focused on nonlinear boundary value problems with parameter for ordinary differential equations of the following form:

$$\frac{dx}{dt} = f(t, x, \lambda), \qquad x(0) = x^0, \tag{1}$$

$$x(T) = x^1, \tag{2}$$

where $f : [0,T]\times B\times B \to B$ is a continuous function satisfying the existence conditions for the Cauchy problem (1) on $[0,T]$ for all values of a parameter λ from some set $G \subset B$; here B is a Banach space.

The problem is to find a pair $(\lambda^*, x^*(t))$, where $\lambda^* \in G$ and $x^*(t)$ is a solution to Cauchy problem (1) with $\lambda = \lambda^*$, satisfying the boundary condition (2).

Let the right-hand part of the differential equation be defined on the set

$$D^0 = \{(t,x,\lambda) : 0 \le t \le T, ||x - x^{(0)}(t)|| \le R(t)\rho, ||\lambda - \lambda^0|| \le \rho\}.$$

Here $\lambda^0 \in G$, $x^{(0)}(t)$ is a solution to Cauchy problem (1) with $\lambda = \lambda^0$, $R(t)$ is a positive function continuously differentiable on $[0,T]$, and ρ is a nonnegative number.

Let $M(f)$ denote a set of triples $(\lambda^0 \in G, R(t) > 0, \rho \ge 0)$ for which the Lipschitz condition

$$||f(t,x,\lambda) - f(t,\tilde{x},\tilde{\lambda})|| \le \alpha(t)\cdot(||x-\tilde{x}|| + ||\lambda - \tilde{\lambda}||)$$

is satisfied on the set D^0, and the inequality

$$(a_1) \qquad \exp\Big\{\int_0^t \alpha(\tau)d\tau\Big\} - 1 \le R(t)$$

holds ($\alpha(t) \in C([0,T])$).

The set $M(f)$ is non-empty if so is the set G.

For a triple $(\lambda^0, R(t), \rho)$, a solution of problem (1), (2) is sought in the set $\alpha^0 = \alpha^0_\lambda \times \alpha^0_{x(t)}$, where $\alpha^0_\lambda = \{\lambda : ||\lambda - \lambda^0|| \le \rho\}$ and $\alpha^0_{x(t)} = \{x(t) : ||x(t) - x^{(0)}(t)|| \le R(t)\rho\}$.

Theorem 1. Problem (1), (2) is solvable if and only if, given some $(\lambda^0, R(t), \rho) \in M(f)$, for any two pairs $(\lambda, x(t))$ and $(\tilde{\lambda}, x(t))$ from the set α^0, there exist an invertible operator $A \in L(B,B)$ and a number $\theta > 0$ satisfying the inequality

$$(a_2) \qquad \Big|\Big|\lambda - \tilde{\lambda} - A\Big[\int_0^T \{f(t,x(t),\lambda) - f(t,x(t),\tilde{\lambda})\}dt\Big]\Big|\Big| \le (1-\theta)||\lambda - \tilde{\lambda}||,$$

and the following inequality is true

$$(a_3) \qquad \frac{1}{\theta}\Big|\Big|A\Big[\int_0^T f(t,x^{(0)}(t),\lambda^0)dt - (x^1 - x^0)\Big]\Big|\Big| \le \rho(1-q),$$

where $q = \frac{||A||}{\theta}\cdot\Big[\exp\Big\{\int_0^T \alpha(t)dt\Big\} - 1 - \int_0^T \alpha(t)dt\Big] < 1$. Here $L(B,B)$ is a space of linear bounded operators mapping B into B.

Under the conditions of Theorem 1, problem (1), (2) is uniquely solvable on the domain α^0.

For the linear boundary value problem

$$\frac{dx}{dt} = Q_1(t)x + Q_2(t)\lambda + f(t), \qquad x(0) = x^0, \qquad x(T) = x^1,$$

the conditions of Theorem 1 are reduced to the bounded invertibility of the operator $\bar{Q} = \int_0^T Q_2(t)dt$.

The inequality (a_3) guarantees the existence and uniqueness of a solution to problem (1), (2) on the domain α^0.

The proposed approach was applied to semi-explicit differential equations with nonlinear boundary conditions:

$$\frac{dx}{dt} = f\Big(t, x, \frac{dx}{dt}, \lambda\Big), \qquad x(0) = x^0, \tag{3}$$

$$\Phi[x(T), \dot{x}(T), \lambda] = 0. \tag{4}$$

Here $f : [0, T] \times B \times B \times B \to B$ is a continuous function satisfying the conditions for the existence of a solution to the Cauchy problem (3) on $[0, T]$ for all $\lambda \in G$; $G \subset B$, $\Phi : B \times B \times B \to B$.

Analogously, the right-hand side of the differential equation is considered on the set $\tilde{D}^0 = \{(t, x, y, \lambda) : 0 \le t \le T, ||x - x^{(0)}(t)|| \le R(t)\rho, ||y - \dot{x}^{(0)}(t)|| \le \dot{R}(t)\rho, ||\lambda - \lambda^0|| \le \rho\}$, where $\lambda^0 \in G$, $x^{(0)}(t)$ is a solution to the Cauchy problem (3) with $\lambda = \lambda^0$, $R(t)$ is a positive function continuously differentiable on $[0, T]$, and ρ is a nonnegative number. Let $\tilde{M}(f)$ denote the set of triples $(\lambda^0 \in G, R(t) > 0, \rho \ge 0)$ for which the following inequalities are satisfied:

$$||f(t, x, y, \lambda) - f(t, \tilde{x}, \tilde{y}, \tilde{\lambda})|| \le \alpha_1(t) \cdot (||x - \tilde{x}|| + ||\lambda - \tilde{\lambda}||) + \alpha_2(t) \cdot ||y - \tilde{y}||,$$

$$\alpha_2(t) < 1 \quad \big(\alpha_i(t) \in C([0, T]), i = 1, 2\big); \quad c(t) \exp\Big\{\int_0^t c(\tau)d\tau\Big\} \le \dot{R}(t) \quad \Big(c(t) = \frac{\alpha_1(t)}{1 - \alpha_2(t)}\Big).$$

For a triple $(\lambda^0, R(t), \rho)$, the following sets are introduced:

$$\tilde{\alpha}^0_{x(t)} = \{x(t) : ||x(t) - x^{(0)}(t)|| \le R(t)\rho, ||\dot{x}(t) - \dot{x}^{(0)}(t)|| \le \dot{R}(t)\rho\},$$

$$\tilde{D}^0(T) = \{(u, v, \lambda) : ||u - x^{(0)}(T)|| \le R(T)\rho, ||v - \dot{x}^{(0)}(T)|| \le \dot{R}(T)\rho, ||\lambda - \lambda^0|| \le \rho\}.$$

Let the boundary function in (4) satisfy the Lipschitz condition $||\Phi(u, v, \lambda) - \Phi(\tilde{u}, \tilde{v}, \tilde{\lambda})|| \le \Phi_u||u - \tilde{u}|| + \Phi_v||v - \tilde{v}|| + \Phi_\lambda||\lambda - \tilde{\lambda}||$ on the set $\tilde{D}^0(T)$.

Theorem 2. Problem (3), (4) is solvable if and only if, given some $(\lambda^0, R(t), \rho) \in \tilde{M}(f)$, for any two pairs $(\lambda, x(t))$ and $(\tilde{\lambda}, x(t))$ from the set $\tilde{\alpha}^0 = \alpha^0_\lambda \times \tilde{\alpha}^0_{x(t)}$, there exist an invertible operator $A \in L(B, B)$ and a number $\theta > 0$ satisfying the inequality $||\lambda - \tilde{\lambda} - A\{\tilde{K}_1[\lambda, x(t)] - \tilde{K}_1[\tilde{\lambda}, x(t)]\}|| \le (1 - \theta)||\lambda - \tilde{\lambda}||$, and the following inequality is true:

$$\frac{1}{\theta}||A\tilde{K}_1[\lambda^0, x^{(0)}(t)]|| \le \rho(1 - q),$$

where $q = \frac{||A||}{\theta} \cdot \Big[\Phi_u \cdot \Big\{\exp\Big\{\int_0^T c(t)dt - 1 - \int_0^T \alpha_1(t)dt\Big\} + \Phi_v \cdot \Big\{c(T)\exp\Big\{\int_0^T c(t)dt - \alpha_1(T)\Big\}\Big] < 1$, $\tilde{K}_1[\lambda, x(t)] = \Phi\Big[x^0 + \int_0^T f(t, x(t), \dot{x}(t), \lambda), f(T, x(T), \dot{x}(T), \lambda), \lambda\Big]$.

Conditions for the continuous dependence of a solution on the initial data and a criterion for the existence of an isolated solution to problem (3), (4) were established.

Dzhumabaev D.S. justified a new version of the shooting method for nonlinear two-point boundary value problems of the following form

$$\frac{dz}{dt} = f(t, z), \tag{5}$$

$$g[z(0), z(T)] = 0, \tag{6}$$

where $f : [0, T] \times B \to B$ is continuous in t and z, $g : B \times B \to B$.

Let λ denote the value of $z(t)$ at the point $t = 0$. By the substitution $x(t) = z(t) - \lambda$, problem (5), (6) is reduced to the following boundary value problem with parameter

$$\frac{dx}{dt} = f(t, x + \lambda), \qquad x(0) = 0, \tag{7}$$

$$g[\lambda, \lambda + x(T)] = 0. \tag{8}$$

Assume that in the closed regions $D^0 = \{(t,x,\lambda) : 0 \le t \le T, ||x - x^{(0)}(t)|| \le R(t)\rho, ||\lambda - \lambda^0|| \le \rho\}$ and $D_1^0 = \{(\lambda,u) : ||\lambda - \lambda^0|| \le \rho, ||u - \lambda^0 - x^{(0)}(T)|| \le [1 + R(T)]\rho\}$ (here $x^{(0)}(t)$ is a solution to Cauchy problem (7) for $\lambda = \lambda^0$, $R(t) > 0$ for $t \in [0,T]$, and $\rho > 0$), the following inequalities hold:

$$||f(t,x+\lambda) - f(t,\tilde{x}+\tilde{\lambda})|| \le \alpha(t)(||x-\tilde{x}|| + ||\lambda - \tilde{\lambda}||),$$

$$||g(\lambda,u) - g(\tilde{\lambda},\tilde{u})|| \le g_\lambda||\lambda-\tilde{\lambda}|| + g_u||u-\tilde{u}||,$$

and $\exp\Big\{\int\limits_0^t \alpha(\tau)d\tau\Big\} - 1 \le R(t)$.

Theorem 3. If for any two pairs $(\lambda, x(t))$ and $(\tilde{\lambda}, x(t))$ from the domain $\alpha^0 = \alpha_\lambda^0 \times \alpha_{x(t)}^0$ and for some $N \ge 0$, there exist an invertible operator $A \in L(B,B)$ and a number $\theta > 0$ satisfying the inequality $||\lambda - \tilde{\lambda} - A\{K_N^{(1)}[\lambda, x(t)] - K_N^{(1)}[\tilde{\lambda}, x(t)]\}|| \le (1-\theta)||\lambda - \tilde{\lambda}||$, and the following inequality holds

$$\frac{1}{\theta}||A\{K_N^{(1)}[\lambda^0, x^{(0)}(t)]\}|| \le \rho(1 - q_N^{(1)}),$$

where $q_N^{(1)} = g_u \cdot \frac{||A||}{\theta} \cdot \Big[\exp\Big\{\int\limits_0^T \alpha(t)dt\Big\} - 1 - \int\limits_0^T \alpha(t)dt - ... - \frac{1}{N!}\Big(\int\limits_0^T \alpha(t)dt\Big)^N\Big] < 1$, then the boundary value problem (7), (8) has a unique solution in α^0.

Here $K_N^{(1)}[\lambda, x(t)] = g\Big[\lambda, \lambda + \int\limits_0^T f(t, \lambda + ... + \int\limits_0^{\tau_{N-3}} f(\tau_{N-2}, \lambda + x(\tau_{N-2}))d\tau_{N-2})...)dt\Big]$, $N = 0,1,2,...$.

For different values of N, various sufficient conditions for the unique solvability to problem (7), (8) can be derived from Theorem 3. The problem of choosing an initial approximation and other replacement versions in problems with parameter were also considered.

Dzhumabaev D.S. also studied nonlinear infinite systems of equations

$$Q_j(\lambda_1, \lambda_2, ..., \lambda_i, ...) = b_j, \qquad j = 1,2,..., \tag{9}$$

where $\lambda = (\lambda_1, \lambda_2, ...)$ and $b = (b_1, b_2, ...)$ are elements of l_p $(1 \le p \le \infty)$. It is supposed that in the domain $D' = \{\lambda : ||\lambda - \lambda^0|| < \rho\} \subset l_p$, for all i $(i = 1,2,...)$, functions $Q_i(\lambda_1, \lambda_2, ...)$ have continuous partial derivatives with respect to all arguments and

1) $\sum\limits_{j=1}^{\infty} \sup\limits_{\lambda \in D'} \Big|\frac{\partial Q_i(\lambda)}{\partial \lambda_j}\Big| \le k_1 < \infty$; 2) $\sum\limits_{k=1}^{\infty} \sup\limits_{\lambda \in D'} \Big|\frac{\partial Q_k(\lambda)}{\partial \lambda_i}\Big| \le k_2 < \infty$.

Then there exist numbers θ_1 and θ_2 satisfying the inequalities

3) $\Big|\frac{\partial Q_i(\lambda)}{\partial \lambda_i}\Big| \ge \sum\limits_{j \ne i} \Big|\frac{\partial Q_i(\lambda)}{\partial \lambda_j}\Big| + \theta_1$; 4) $\Big|\frac{\partial Q_i(\lambda)}{\partial \lambda_i}\Big| \ge \sum\limits_{k \ne i} \sup\limits_{\lambda \in D'} \Big|\frac{\partial Q_k(\lambda)}{\partial \lambda_i}\Big| + \theta_2$, for all $\lambda \in D'$ and $i = 1,2,\dots$

The following definition extends the concept of complete regularity to the case of nonlinear infinite systems in l_p.

Definition 1. An operator $Q = (Q_1, Q_2, ...)$ is called completely regular in the domain D', if it satisfies conditions 1)-4) wherein the numbers θ_1 and θ_2 are such that 5) $\frac{p-1}{p}\theta_1 + \frac{1}{p}\theta_2 = \theta > 0$.

Lemma 1. If Q is a completely regular operator in the domain D' and $\frac{1}{\theta}||Q(\lambda^0) - b|| < \rho$, then the infinite system of nonlinear equations (9) has a unique solution in D'.

Using Lemma 1, the results obtained for problems (1)-(2), (3)-(4), and (5)-(6) were concretized for infinite systems of differential equations. Effective conditions were established for the unique solvability of nonlinear boundary value problems for infinite systems of differential equations in the space l_p.

The findings described in this Section were published in [1–5] and formed the basis of his candidate thesis. In 1980, Dzhumabaev D.S. defended his dissertation “Boundary value problems with a parameter for ordinary differential equations in a Banach space" and earned a degree of Candidate of Physical and Mathematical Sciences in the specialty 01.01.02 – Differential Equations.

The methods and results of [1–5] were applied to nonlinear differential equations of various classes [6–12]. Dzhumabaev's research was then focused on various problems for nonlinear operator equations [13–17].

2 A linearizer and iterative processes for unbounded non-smooth operators

Consider the nonlinear operator equation

$$A(x) = 0, \tag{10}$$

where $x \in B_1$, $A(x) \in B_2$, and each B_i is a Banach space with norm $||\cdot||_i$, $i = 1,2$. Let $D(A)$ and $R(A)$ denote the domain and range of A, respectively.

For a point $x^0 \in D(A)$, the following sets are constructed: $S(x^0, r) = \{x \in B_1 : ||x - x^0||_1 \le r\}$, $U^0 = \{x \in D(A) : ||A(x)||_2 \le ||A(x^0)||_2 = u^0\}$, and $\Omega = S(x^0, r) \cap U^0$. Assume that the operator A is closed on Ω. As is known, iterative methods, that allow one to find a solution under some sufficient conditions for its existence, rely on certain linearizations of the nonlinear operator. Linearization of an unbounded operator naturally leads to unbounded linear operators. This motivated Dzhumabaev D.S. to introduce the concept of a linearizer of an operator A at a point $\hat{x} \in D(A)$ that generalizes the Frechet derivative for unbounded non-smooth operators.

Definition 2. A linear operator $C : B_1 \to B_2$ is called a linearizer of an operator A at a point $\hat{x} \in D(A)$, if $D(A) \subseteq D(C)$ and there exist numbers $\epsilon \ge 0$ and $\delta > 0$ such that

$$||A(x) - A(\hat{x}) - C(x - \hat{x})||_2 \le \epsilon ||x - \hat{x}||_1$$

for all $x \in D(A)$ satisfying $||x - \hat{x}||_1 < \delta$.

If $C \in L(B_1, B_2)$ is the Frechet derivative of A at a point $\hat{x} \in D(A)$, then it is also a linearizer. However, the definition of a linearizer, unlike that of the Frechet derivative, does not require: a) the boundedness of the operator C and 2) the dependence of ϵ on δ ($\epsilon(\delta) \to 0$ as $\delta \to 0$ for the Frechet derivative).

While the Frechet derivative of an operator A is uniquely determined, there can be infinitely many linearizers of this operator.

Distinctive advantages of linearizers make it possible to extend the domain of application of iterative methods to solving nonlinear operator equations. Dzhumabaev D.S. proposed a method for proving the convergence of iterative processes that takes into account the specificities of unbounded operator equations.

Theorem 4. Suppose that at each point $x \in \Omega$ the operator A has a linearizer C_x with constants ϵ_x and δ_x such that: 1) C_x is a one-to-one mapping of $D(C)$ onto $R(C)$, and $||C_x^{-1}|| \le \gamma_x \le \bar{\gamma}$; 2) $\epsilon_x \cdot \delta_x \le \Theta < 1$; and 3) $\frac{\gamma_x}{\delta_x} \cdot ||A(x)||_2 \le K$. If $\frac{\bar{\gamma}}{1-\Theta} \cdot ||A(x)||_2 < r$, then (10) has a solution $x^* \in \Omega$, to which the iteration process

$$x^{(n+1)} = x^{(n)} - \frac{1}{\alpha} C^{-1}_{x^{(n)}} \{A(x^{(n)})\}$$

converges, here $\alpha = \max\{1, K\}$, $n = 0, 1, 2, \ldots$

In the case when for a given $\delta > 0$ there exists $\epsilon(\delta)$ independent of x, the following assertion is true.

Theorem 5. Suppose that at each point $x \in \Omega$ and for each $\delta \in (0, h)$ the operator A has a linearizer C_x with constants δ and $\epsilon(\delta) \ge 0$ satisfying the following conditions: 1) C_x^{-1} exists on $R(C)$, and $||C_x^{-1}|| \le \gamma$, 2) $\lim\limits_{\delta \to 0} \epsilon(\delta) = 0$.

Then (10) has a solution $x^* \in \Omega$, if the following inequality holds: 3) $\gamma \cdot ||A(x)||_2 < r$.

Theorem 5 generalizes the local theorem of Hadamard to unbounded operator equations. This made it possible to extend the well-known Newton-Kantorovich method to unbounded nonsmooth operator equations and apply it to nonlinear boundary value problems for differential equations.

Consider the closed operator equation

$$A(x) \equiv Cx + F(x) = 0, \tag{11}$$

where $C : X \to Y$ is a closed linear operator, $F : X \to Y$ is a continuous operator, and X and Y are Banach spaces with respective norms $||\cdot||_1$ and $||\cdot||_2$.

Assume that F has a Frechet derivative in some domain containing the ball $\bar{S}(x^0, r) = \{x \in X : ||x - x^0||_1 \leq r\}$, $x^0 \in D(C)$, and $R(C + F'(x)) = Y$ for $x \in S(x^0, r)$. Then in $D(A) = D(C) \cap \bar{S}(x^0, r)$ the operator A has the linearizer $C_1(x) = C + F'(x)$, and $D(C_1) = D(C) \cap X = D(C)$.

Theorem 6. Assume that the following conditions hold:
(1) For all $x \in D(A)$, the linearizer $C_1(x)$ has a bounded inverse, and $||C_1^{-1}(x)||_{L(Y,X)} \leq \gamma$;
(2) $||F'(x) - F'(y)||_{L(X,Y)} \leq L \cdot ||x - y||_1$, $x, y \in \bar{S}(x^0, r)$;
(3) $\frac{m}{L\gamma} + \gamma \frac{b_m}{b_0} ||A(x^0)||_2 \sum\limits_{s=0}^{\infty} (b_m)^{2^s - 1} < r$, where $b_0 = \frac{L}{2}\gamma^2 u_0$, $u_0 = ||A(x^0)||_2$, $\beta_k = 1 - \frac{1}{4b_{k-1}}$, $b_k = \beta_k \cdot b_{k-1}$, $k = 1, 2, ..., m$, where m is a nonnegative number such that $b_m < 1$ and $b_{m-1} \geq 1$.

Then the damped Newton-Kantorovich method

$$x^{(k+1)} = x^{(k)} - \frac{1}{\alpha_k}[C + F'(x^k)]^{-1}[Cx^k + F(x^k)], \quad k = 0, 1, 2, ...,$$

where $\alpha_k = 2b_k$ for $k = 0, ..., m - 1$ and $\alpha_k = 1$ for $k = m, m + 1, ...$, converges to a solution of (11).

Theorem 7. Assume that the following conditions hold:
(1) For all $x \in D(A)$, the linearizer $C_1(x)$ has a bounded inverse, and $||C_1^{-1}(x)||_{L(Y,X)} \leq \gamma$;
(2) The Frechet derivative $F'(x)$ is uniformly continuous in $\bar{S}(x^0, r)$;
(3) $\gamma \cdot ||A(x^0)||_2 < r$.

Then there exist numbers $\alpha_n \geq 1$, $n = 0, 1, ...$, such that the iteration process

$$x^{(m+s+1)} = x^{(m+s)} - [C + F'(x^{m+s})]^{-1}[Cx^{m+s} + F(x^{m+s})], \quad s = 0, 1, 2, ...,$$

converges to an isolated solution $x^* \in D(A)$ of (11). Furthermore, starting with some k^0, we can take α_n $(n \geq k^0)$ equal to 1, and the convergence rate becomes superlinear.

These results were published in "News of the Academy of Sciences of Kazakh SSR. Series Physical and Mathematical", 1984 [13, 14], and, at the request of the American Mathematical Society, were translated and published in "American Mathematical Society Translations", 1989 [16, 17], as well as in "Mathematical Notes" [15]. Various aspects of applications of these results were considered in [18–20].

3 The parametrization method for solving boundary value problems

Dzhumabaev D.S. developed the parametrization method for investigation and solving boundary value problems. The method was originally offered in [21, 22] for solving two-point boundary value problems for a linear differential equation of the following form

$$\frac{dx}{dt} = A(t)x + f(t), \qquad x \in \mathbb{R}^n, \tag{12}$$

$$Bx(0) + Cx(T) = d, \tag{13}$$

where $A(t)$ and $f(t)$ are continuous in $(0, T]$, B and C are $n \times n$ matrices, $d \in \mathbb{R}^n$.

Consider a partition dividing the interval $[0,T)$ into N equal parts with step size $h>0$: $[0,T)=\bigcup_{r=1}^{N}[(r-1)h,rh)$. Let $x_r(t)$ denote the restriction of the function $x(t)$ to the r-th subinterval, i.e. $x_r(t)$, $r=\overline{1,N}$, is a vector function of dimension n defined on $[(r-1)h,rh)$ and coinciding there with $x(t)$. Problem (12), (13) is thus transformed into an equivalent multipoint boundary-value problem

$$\frac{dx_r}{dt}=A(t)x_r+f(t), \qquad t\in[(r-1)h,rh), \quad r=1,2,...,N, \tag{14}$$

$$Bx_1(0)+C\lim_{t\to T-0}x_N(t)=d, \tag{15}$$

$$\lim_{t\to sh-0}x_s(t)=x_{s+1}(sh), \qquad s=1,2,...,N-1. \tag{16}$$

Here (16) are the matching conditions for the solution at the interior points of the partition.

Obviously, if $x(t)$ is a solution of problem (12), (13), then the set of restrictions $(x_r(t))$, $r=1,2,...,N$, is a solution of the multi-point problem (14)–(16). Conversely, if a set of vector functions $(x_r(t))$, $r=1,2,....N$, is a solution of problem (14)–(16), then the function $x(t)$ obtained by piecing together $x_r(t)$ is a solution of the original boundary value problem.

On each subinterval $[(r-1)h,rh)$, the substitution $u_r(t)=x_r(t)-\lambda_r$ is made, where λ_r denotes the value of $x_r(t)$ at the point $t=(r-1)h$. Problem (14)–(16) is then reduced to the boundary value problem with parameter

$$\frac{du_r}{dt}=A(t)u_r+A(t)\lambda_r+f(t), \quad t\in[(r-1)h,rh), \quad u_r[(r-1)h]=0, \quad r=1,2,...,N, \tag{17}$$

$$B\lambda_1+C\lambda_N+C\lim_{t\to T-0}u_N(t)=d, \tag{18}$$

$$\lambda_s+\lim_{t\to sh-0}u_s(t)=\lambda_{s+1}, \qquad s=1,2,...,N-1. \tag{19}$$

An advantage of problem (17)–(19) is that it involves the initial conditions $u_r[(r-l)h]=0$, so that one can determine $u_r(t)$ from the integral equations

$$u_r(t)=\int\limits_{(r-1)h}^{t}[A(\tau)u_r+A(\tau)\lambda_r]d\tau+\int\limits_{(r-1)h}^{t}f(\tau)d\tau. \tag{20}$$

In (20), replacing $u_r(\tau)$ by the right-hand side of (20) and repeating the process ν ($\nu=1,2,...$) times, one obtains a representation of $u_r(t)$ by a sum of iterated integrals. Letting $t\to rh-0$ and substituting $\lim_{t\to rh-0}u_r(t)$, $r=1,2,\ldots,N$, into (18) and (19) results in a system of nN algebraic equations in the parameters λ_{ri}, $r=1,2,...,N$, $i=1,2,...,n$:

$$Q_\nu(h)\lambda=-F_\nu(h)-G_\nu(u,h), \qquad \lambda\in\mathbb{R}^{Nn}. \tag{21}$$

The basic idea behind the method is to reduce the problem in question to an equivalent problem with a parameter (17)–(19) whose solution is determined as the limit of a sequence of systems of pairs consisting of the parameter λ and the function u. The parameter is found from the system of linear equations (21) determined by the matrices of the differential equation (12) and boundary conditions (13). The functions u_r are solutions of Cauchy problems (17) on the partition subintervals $[(r-1)h,rh)$, $r=1,2,...,N$, for the found values of the parameter. The introduction of parameters made it possible to obtain conditions for the convergence of proposed algorithms and, at the same time, for the existence of a solution, in terms of the input data. This makes the parametrization method different from the shooting method and its modifications, where shooting parameters are found from some equations constructed via general solutions of differential equations, and convergence conditions are also given in terms of general solutions.

Theorem 8. Suppose that for some $h > 0 \quad (Nh = T)$ and $\nu \quad (\nu = 1, 2, \ldots)$ the matrix $Q_\nu(h) : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is invertible and the following inequalities hold:

(a) $||[Q_\nu(h)]^{-1}|| \le \gamma_\nu(h)$;

(b) $q_\nu(h) = \gamma_\nu(h) \max(1, h||C||)[e^{\alpha h} - 1 - \alpha h - \ldots - \frac{(\alpha h)^\nu}{\nu!}] < 1$, where $\alpha = \max\limits_{t \in [0,T]} ||A(t)||$.

Then the boundary-value problem (12), (13) has a unique solution $x^*(t)$, and the estimate

$$||x^*(t) - x^{(k)}(t)|| \le \gamma_\nu(h) \max(1, h||C||) \frac{(\alpha h)^\nu}{\nu!} e^{\alpha h} \frac{[q_\nu(h)]^\nu}{1 - q_\nu(h)} M(h), \qquad t \in [0, T],$$

holds true, where

$$M(h) = \gamma_\nu(h)[e^{\alpha h} - 1] \max\Big\{1 + h||C|| \sum_{j=0}^{\nu-1} \frac{(\alpha h)^j}{j!}, \sum_{j=0}^{\nu-1} \frac{(\alpha h)^j}{j!}\Big\} \max(||d||, \max_{t \in [0,T]} ||f(t)||)h$$

$$+ e^{\alpha h} \max_{t \in [0,T]} ||f(t)||h,$$

and $x^{(k)}(t)$ is a piecewise-continuously differentiable function on $[0, T]$, for which $\lambda_r^{(k)} + u_r^{(k)}(t)$ is the restriction to $[(r - l)h, rh)$, $r = 1, 2, ..., N$.

It was shown that the conditions of Theorem 8 are also necessary and sufficient for the unique solvability of problem (12), (13).

The parametrization method was then applied to the study of singular problems for which the problem of approximation by regular two-point boundary value problems was solved [23–27]. Necessary and sufficient conditions were obtained for the well-posed solvability of the problem of finding a solution to the system of differential equations (12), that is bounded on the whole axis $\mathbb{R}$. For systems whose matrices and right-hand sides are constant in the limit, approximating regular two-point boundary value problems were constructed. The connection between the well-posed solvability of the original singular problem and that of an approximating problem was established. In the general case, Lyapunov transformations possessing certain properties were used to construct regular two-point boundary value problems as approximations to the problem of determining a solution bounded on the entire real line. The concept of a solution "in the limit as $t \to \infty$" was introduced and the behavior of solutions of linear ordinary differential equations as $t \to \infty$ was investigated. Necessary and sufficient conditions were derived under which a singular boundary value problem with conditions assigned at infinity is uniquely solvable, and an appropriate approximating problem was constructed. These results were developed to the system of differential equations on the real axis:

$$\frac{dx}{dt} = f(t, x), \qquad x \in \mathbb{R}^n. \tag{22}$$

In [28, 29] the results of Section 2 were also extended to system (22) with the nonlinear boundary condition

$$g[x(0), x(T)] = 0.$$

Results of Sections 2 and 3 were included in the doctoral dissertation.

The doctoral dissertation by Dzhumabaev D.S. titled "Singular boundary value problems for ordinary differential equations and their approximation" is a fundamental scientific work that underwent comprehensive approbation in leading scientific centers, such as the Computing Center of the Russian Academy of Sciences (A.A. Abramov, N.B. Konyukhova), the Institute of Applied Mathematics of the Russian Academy of Sciences (K.I. Babenko), Lomonosov Moscow State University (V.M. Millionshchikov, V.A. Kondratiev, N.Kh. Rozov), Institute of Mathematics NAS of Ukraine (Y.A.

Mitropol'skii, A.M. Samoilenko, V.L. Makarov, V.L. Kulik), Voronezh State University (V.I. Perov), I. Vekua Institute of Applied Mathematics of Tbilisi State University (I.T. Kiguradze), Kiev State University named after T. Shevchenko (N.I. Perestyuk). Doctoral dissertation was defended at the Specialized Council of the Institute of Mathematics of the NAS of Ukraine in 1994.

The parametrization method was extended to various linear and nonlinear boundary value problems for ordinary differential equations on a finite interval and on the whole real line; necessary and sufficient solvability conditions for those problems were obtained in [28–49].

4 Nonlocal problems for systems of second-order hyperbolic equations

The results obtained in Sections 2 and 3 provided a basis for solving nonlocal boundary value problems for systems of second-order hyperbolic equations [50–70].

In the domain $\Omega = [0,T] \times [0,\omega]$, consider the following nonlocal boundary value problem for the system of hyperbolic equations with two independent variables:

$$\frac{\partial^2 u}{\partial t \partial x} = A(t,x)\frac{\partial u}{\partial x} + B(t,x)\frac{\partial u}{\partial t} + C(t,x)u + f(t,x), \tag{23}$$

$$P_2(x)\frac{\partial u(t,x)}{\partial x}\Big|_{t=0} + P_1(x)\frac{\partial u(t,x)}{\partial t}\Big|_{t=0} + P_0(x)u(t,x)|_{t=0} +$$
$$+ S_2(x)\frac{\partial u(t,x)}{\partial x}\Big|_{t=T} + S_1(x)\frac{\partial u(t,x)}{\partial t}\Big|_{t=T} + S_0(x)u(t,x)|_{t=T} = \varphi(x), \quad x \in [0,\omega], \tag{24}$$

$$u(t,0) = \psi(t), \qquad t \in [0,T], \tag{25}$$

where $u(t,x) = col(u_1(t,x),...,u_n(t,x))$ is an unknown function, the $n \times n$ matrices $A(t,x)$, $B(t,x)$, $C(t,x)$, $P_i(x)$, $S_i(x)$, $i = \overline{0,2}$, and the n-vector functions $f(t,x)$, $\varphi(x)$ are continuous on Ω and $[0,\omega]$, respectively; the n-vector function $\psi(t)$ is continuously differentiable on $[0,T]$.

Sufficient coefficient conditions for the existence and uniqueness of a classical solution of problem (23)–(25) were established by a modification of the parametrization method [50, 53, 55, 60, 61]. A relationship with the following family of boundary value problems for ordinary differential equations was established:

$$\frac{\partial v}{\partial t} = A(t,x)v + F(t,x), \qquad x \in [0,\omega], \tag{26}$$

$$P_2(x)v(0,x) + S_2(x)v(T,x) = \Phi(x), \tag{27}$$

here n-vector functions $F(t,x)$ and $\Phi(x)$ are continuous on Ω and $[0,\omega]$, respectively.

For fixed $x \in [0,\omega]$ problem (26), (27) is a linear boundary value problem for the system of ordinary differential equations. Suppose the variable x is changed on $[0,\omega]$; then we obtain a family of boundary value problems for ordinary differential equations.

Sufficient and necessary conditions for the well-posedness of nonlocal boundary value problem for the system of hyperbolic equations (25)–(27) were obtained in [59, 64, 66, 67].

Let $C([0,\omega], R^n)$ be a space of continuous on $[0,\omega]$ vector functions $\varphi(x)$ with the norm $||\varphi||_{0,1} = \max\limits_{x\in[0,\omega]} ||\varphi(x)||$;
$C^1([0,T], R^n)$ be a space of continuously differentiable on $[0,T]$ vector functions $\psi(t)$ with the norm $||\psi||_{1,0} = \max\Big(\max\limits_{t\in[0,T]} ||\psi(t)||, \max\limits_{t\in[0,T]} ||\dot{\psi}(t)||\Big)$;
$C^{1,1}(\Omega, R^n)$ be a space of functions $u(t,x) \in C(\Omega, R^n)$ with continuous on Ω partial derivatives $\frac{\partial u(t,x)}{\partial x}$, $\frac{\partial u(t,x)}{\partial t}$, $\frac{\partial^2 u(t,x)}{\partial t \partial x}$ with the norm $||u||_{1,1} = \max\Big(||u||_0, \Big|\Big|\frac{\partial u}{\partial x}\Big|\Big|_0, \Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_0, \Big|\Big|\frac{\partial^2 u}{\partial t \partial x}\Big|\Big|_0\Big)$.

Lemma 2. If problem (26), (27) has a solution for arbitrary $F(t,x) \in C(\Omega, R^n)$ and $\Phi(x) \in C([0,\omega], R^n)$, then this solution is unique.

Definition 3. Problem (26), (27) is called well-posed if for arbitrary $F(t,x) \in C(\Omega, R^n)$ and $\Phi(x) \in C([0,\omega], R^n)$ it has a unique solution $v(t,x) \in C(\Omega, R^n)$ and for it the estimate holds

$$\max_{t\in[0,T]} ||v(t,x)|| \le K \max\Big(\max_{t\in[0,T]} ||F(t,x)||, ||\Phi(x)||\Big), \tag{28}$$

where the constant K is independent of $F(t,x)$ and $\Phi(x)$, and $x \in [0,\omega]$.

Lemma 3. If $v(t,x)$ is a solution to problem (26), (27), and the estimate holds

$$||v||_0 \le K \max\Big(||F||_0, ||\Phi||_{0,1}\Big),$$

where K is a constant independent of the functions $F(t,x)$ and $\Phi(x)$, then for every $x \in [0,\omega]$ the inequality (28) is valid.

Denote by $\Omega_\eta = [0,T] \times [0,\eta]$ and $||u||_{0,\eta} = \max\limits_{(t,x)\in\Omega_\eta} ||u(t,x)||$.

Definition 4. Boundary value problem (23)–(25) is called well-posed if for arbitrary $f(t,x) \in C(\Omega, R^n)$ and $\psi(t) \in C^1([0,T], R^n)$ and $\varphi(x) \in C([0,\omega], R^n)$ it has a unique classical solution $u(t,x)$ and this solution satisfies the following estimate

$$\max\Big(||u||_{0,\eta}, \Big|\Big|\frac{\partial u}{\partial x}\Big|\Big|_{0,\eta}, \Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{0,\eta}\Big) \le \tilde{K} \max\Big(||f||_{0,\eta}, ||\psi||_{1,0}, \max_{x\in[0,\eta]} ||\varphi(x)||\Big),$$

where constant $\tilde{K}$ is independent of $f(t,x)$ and $\psi(t)$ and $\varphi(x)$ and $\eta \in [0,\omega]$.

Theorem 9. The boundary value problem (23)–(25) is well-posed if and only if so is problem (26), (27).

From Theorem 9 it follows that the well-posedness of problem (23)–(25) are equivalent to the well-posedness of problem (26), (27).

These results were extended to a nonlocal problem with an integral condition for system (25) (see [71]).

The problem of finding bounded solutions of system (23) and the families of systems (26) was solved in [54, 56–58, 61–63, 65, 72].

The parametrization method was further developed to nonlinear nonlocal problems for a system of hyperbolic equations [68–70, 73].

5 Boundary value problems for loaded and integro-differential equations

On the basis of the parametrization method, constructive algorithms were developed for finding solutions to various boundary value problems for integro-differential and loaded equations [72, 74–82].

In the interval $[0,T]$, consider the following linear two-point boundary value problem for an integro-differential equation:

$$\frac{dx}{dt} = A(t)x + \int_0^T K(t,s)x(s)ds + f(t), \qquad x \in \mathbb{R}^n, \tag{29}$$

$$Bx(0) + Cx(T) = d, \qquad d \in \mathbb{R}^n, \tag{30}$$

where $A(t)$ and $K(t,s)$ are continuous matrices on $[0,T]$ and $[0,T] \times [0,T]$, respectively; $f(t)$ is continuous on $[0,T]$.

It is well known that the basic techniques for analysis and solving boundary value problems for integro-differential equations are the Nekrasov method and the Green's function method. Nekrasov's method applies to problem (29), (30), if we assume the unique solvability of the second-kind Fredholm integral equation

$$x(t) = \int_0^T M(t,s)x(s)ds + F(t), \qquad t \in [0,T],$$

with the kernel $M(t,s) = \int_0^t X(t)X^{-1}(\tau)K(\tau,s)d\tau$, where $X(t)$ is the fundamental matrix of the differential part of equation (29) and $F(t) \in C([0,T],\mathbb{R}^n)$. The Green's function method applies to problem (29), (30) under assumption that the boundary value problem for the differential part of (29) is uniquely solvable; i.e., this method assumes the unique solvability of problem (29), (30) with $K(t,s) = 0$.

However, the assumptions of neither Nekrasov's method nor Green's function method are necessary conditions for the solvability of problem (29), (30).

In [83], a coefficient criterion for the well-posedness of problem (29), (30) was established in terms of approximating boundary value problems for the loaded differential equation

$$\frac{dx}{dt} = A(t)x + \sum_{i=1}^{m} K_i(t)x(\theta_i) + f(t), \qquad x \in \mathbb{R}^n,$$

subject to condition (30), by the parametrization method.

In [84], Dulat Dzhumabaev proposed a method for solving the problem (29), (30) that is based on the parametrization method and properties of a fundamental matrix of the differential part of (29). The interval $[0,T]$ is divided into N equal parts with step size $h > 0$: $[0,T) = \bigcup_{r=1}^{N}[(r-1)h, rh)$. Let $x_r(t)$ be the restriction of $x(t)$ to the rth subinterval $[(r-1)h, rh)$. The values of the solution at the left-endpoints of the subintervals are assumed as additional parameters $\lambda_r = x_r[(r-1)h]$. By the substitution $u_r(t) = x_r(t) - \lambda_r$ at every rth subinterval, the problem (29), (30) is reduced to the multi-point boundary value problem for a system of integro-differential equations with parameters

$$\frac{du_r}{dt} = A(t)u_r + A(t)\lambda_r + \sum_{j=1}^{N} \int_{(j-1)h}^{jh} K(t,s)[u_j(s)+\lambda_j]ds + f(t), \qquad t \in [(r-1)h, rh), \tag{31}$$

$$u_r[(r-1)h] = 0, \qquad r = 1,2,...,N, \tag{32}$$

$$B\lambda_1 + C\lambda_N + C\lim_{t\to T-0} u_N(t) = d, \tag{33}$$

$$\lambda_p + \lim_{t\to ph-0} u_p(t) - \lambda_{p+1} = 0, \qquad p = 1,2,...,N-1. \tag{34}$$

The introduction of additional parameters resulted in the emergence of the initial data (32) for the unknown functions $u_r(t)$, $r = 1,2,...,N$. For fixed parameter values $\lambda \in \mathbb{R}^{nN}$, the system of functions $u[t] = (u_1(t), u_2(t), ..., u_N(t))$ is determined from problem (31), (32), which is a special Cauchy problem for the system of integro-differential equations. Problem (31), (32) is equivalent to the system of integral equations

$$u_r(t) = X(t)\int_{(r-1)h}^{t} X^{-1}(\tau)A(\tau)d\tau\lambda_r + X(t)\int_{(r-1)h}^{t} X^{-1}(\tau)\sum_{j=1}^{N}\int_{(j-1)h}^{jh} K(\tau,s)[u_j(s)+\lambda_j]dsd\tau +$$

$$+X(t)\int\limits_{(r-1)h}^{t} X^{-1}(\tau)f(\tau)d\tau, \qquad t\in[(r-1)h,rh), \qquad r=1,2,...,N. \tag{35}$$

By solving (35), one can find the representations of $u_r(t)$ in terms of $\lambda \in \mathbb{R}^{nN}$ and $f(t)$. Substituting them into (33) and (34) yields a system of equations for finding the unknown parameters. Thus, when applying the parametrization method to problem (29), (30), one has to solve an auxiliary problem, namely, the special Cauchy problem (31), (32), or the equivalent system of integral equations (35). However, unlike the auxiliary problem of Nekrasov's method, the special Cauchy problem is uniquely solvable for any sufficiently small partition step size $h > 0$. Let a number $h_0 > 0$ satisfy the inequality

$$\sigma(h_0) = \beta T h_0 e^{\alpha h_0} < 1,$$

where $\beta = \max\limits_{(t,s)\in[0,T]\times[0,T]} ||K(t,s)||$ and $\alpha = \max\limits_{t\in[0,T]} ||A(t)||$. It was shown that, for any $h \in (0, h_0] : Nh = T$, system (35) is uniquely solvable. This property of the auxiliary problem of the parametrization method made it possible to establish solvability criteria for the boundary value problem considered.

Necessary and sufficient conditions for the solvability, including the unique solvability, of problem (29), (30) were obtained in terms of a matrix $Q_{*,*}(h)$ constructed via the fundamental matrix of the differential part of system (29), the matrices of boundary conditions (30), and the resolvent of an auxiliary Fredholm integral equation of the second kind.

In [85], a family of algorithms was proposed for solving problem (29), (30). The numerical parameters of the family are the partition step $h > 0 : Nh = T$, the number $\nu \in \mathbb{R}^n$ of iterated integrals used in the algorithm, and a nonnegative integer m specifying how many terms of the resolvent of the corresponding Fredholm integral equation of the second kind are used in the algorithm. The basic condition for the feasibility and convergence of the algorithm is that the matrix $Q_\nu^m(h)$ is invertible for chosen numerical parameters. The unknown parameters are found at the first stage of each step in the algorithm by using the invertibility of this matrix. The special Cauchy problem (31), (32) with the found parameter values is solved at the second stage of the algorithm. Necessary and sufficient conditions for the well-posedness of problem (29), (30) were established in terms of the input data without using the fundamental matrix or the resolvent.

In [86], the method and results of [84] were generalized to the case of an arbitrary partition. Let Δ_N denote a partition of $[0,T]$ into N parts: $t_0 = 0 < t_1 < \ldots < t_N = T$; the case of no partitioning is denoted by Δ_1. Each partition Δ_N is associated with a homogeneous Fredholm integral equation of the second kind. The partition Δ_N is called regular if the corresponding equation has only the trivial solution. The regularity of Δ_N leads to a unique solvability of the special Cauchy problem mentioned above. The solvability criteria for linear two-point boundary value problem for Equation (29) obtained in [86] are applicable for arbitrary regular partition Δ_N. The algorithms of the parameterization method for solving linear boundary value problems for Fredholm integro-differential equations were offered in [70].

These results were extended to boundary value problems for impulsive integro-differential equations in [87].

6 New general solutions to linear Fredholm integro-differential equations and their applications in solving boundary value problems

It is known that Volterra integro-differential equations are solvable for any right-hand side and have classical general solutions. However, there exist linear loaded differential equations and Fredholm integro-differential equations that do not admit classical general solutions. The question arises as to whether it is possible to construct such general solutions that exist for all differential and integro-differential equations and would allow solving boundary value problems for these equations.

Dzhumabaev D.S. proposed a novel approach to the concept of the general solution for a linear ordinary Fredholm integro-differential equation based on the parametrization method in [88]. The domain interval is partitioned and the values of the solution at the left endpoints of the subintervals are considered as additional parameters. By introducing new unknown functions on the partition subintervals, a special Cauchy problem for a system of integro-differential equations with parameters is obtained. Using the solution of this problem, a new general solution of the linear Fredholm integro-differential equation was constructed.

Suppose Δ_N is a partition $t_0 = 0 < t_1 < \ldots < t_N = T$. Let $x(t)$ be a function, piecewise continuous on $[0, T]$ with the possible points of discontinuity: $t = t_p$, $p = 1, 2, ..., N-1$. Let $x_r(t)$ be the restriction of $x(t)$ to the rth subinterval $[t_{r-1}, t_r)$, i.e. $x_r(t) = x(t)$, $t \in [t_{r-1}, t_r)$, $r = 1, 2, \ldots, N$. For definiteness, assume that $x_r(t_{r-1}) = \lim\limits_{t \to t_{r-1}+0} x_r(t)$, $r = 1, 2, ..., N$. If $x(t)$ is piecewise continuously differentiable on $(0, T)$ and satisfies the Fredholm integro-differential equation (29) for each $t \in (0, T) \backslash \{t_p, p = 1, 2, ..., N-1\}$, then the system of its restrictions $x[t] = (x_1(t), ..., x_N(t))$ satisfies the following system of integro-differential equations:

$$\frac{dx_r}{dt} = A(t)x_r + \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} K(t,\tau)x_j(\tau)d\tau + f(t), \qquad t \in [t_{r-1}, t_r), \quad r = 1, 2, ..., N. \tag{36}$$

Let $C([0, T], \Delta_N, \mathbb{R}^{nN})$ denote the space of function systems $x[t] = (x_1(t), x_2(t), ..., x_N(t))$, where $x_r : [t_{r-1}, t_r) \to \mathbb{R}^n$ is continuous and has the finite left-sided limit $\lim\limits_{t \to t_r - 0} x_r(t)$ for any $r = 1, 2, ..., N$, with the norm $x[\Delta]_2 = \max\limits_{r=1,2,...,N} \sup\limits_{t \in [t_{r-1}, t_r)} ||x_r(t)||$.

A function system $x[t] = (x_1(t), x_2(t), ..., x_N(t)) \in C([0, T], \Delta_N, \mathbb{R}^{nN})$ is called a solution to the system of integro-differential equations (35) if the functions $x_r(t)$, $r = 1, 2, ..., N$, are continuously differentiable on (t_{r-1}, t_r) and satisfy equations (36).

Suppose that the function system $x^*[t] = (x_1^*(t), x_2^*(t), ..., x_N^*(t))$ is a solution to (36). Then the function $x^*(t)$, defined as $x^*(t) = x_r^*(t)$ for $t \in [t_{r-1}, t_r)$, $r = 1, 2, ..., N$, and $x^*(T) = \lim\limits_{t \to T-0} x_N^*(t)$, is piecewise continuously differentiable and consistent with Eq. (29) for $t \in (0, T) \backslash \{t_p, p = 1, 2, ..., N-1\}$. The introduction of the parameters $\lambda_r = x_r(t_{r-1})$, $r = 1, 2, ..., N$, and substituting new unknown functions $u_r(t) = x_r(t) - \lambda_r$ on each subinterval $[t_{r-1}, t_r)$, yields the system of integro-differential equations with parameters

$$\frac{du_r}{dt} = A(t)u_r + A(t)\lambda_r + \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} K(t,\tau)[u_j(\tau) + \lambda_j]d\tau + f(t), \quad t \in [t_{r-1}, t_r), \; r = 1, ..., N, \tag{37}$$

subject to the initial conditions

$$u_r(t_{r-1}) = 0, \qquad r = 1, 2, ..., N. \tag{38}$$

Problem (37), (38) is called a special Cauchy problem for the system of integro-differential equations with parameters. Without the interval's partition, problem (37), (38) is the Cauchy problem with the initial condition at $t = 0$ for the Fredholm integro-differential equation with parameter.

A solution to the special Cauchy problem (37), (38) with fixed values of parameters $\lambda_r^* \in \mathbb{R}^n$, $r = 1, ..., N$, is a function system $u[t, \lambda^*] = (u_1(t, \lambda^*), u_2(t, \lambda^*), ..., u_N(t, \lambda^*)) \in C([0, T], \Delta_N, \mathbb{R}^{nN})$, which satisfies the system of integro-differential equations (37) with $\lambda = \lambda^*$ and initial conditions (38).

Let $X_r(t)$ be a fundamental matrix of the differential equation $\frac{dx}{dt} = A(t)x$ on the interval $[t_{r-1}, t_r]$. Then problem (37), (38) is equivalent to the system of integral equations

$$u_r(t) = X_r(t)\int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau_1)A(\tau_1)d\tau_1\lambda_r + X_r(t)\int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau_1)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j} K(\tau_1,\tau)[u_j(\tau)+\lambda_j]d\tau d\tau_1+$$

$$+X_r(t)\int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau_1)f(\tau_1)d\tau_1, \qquad t\in[t_{r-1},t_r), \quad r=1,2,...,N.$$

Take an arbitrary partition Δ_N and consider the corresponding homogeneous Fredholm integral equation of the second kind

$$y(t) = \int\limits_0^T M(\Delta_N,t,\tau)y(\tau)d\tau, \qquad t\in[0,T], \tag{39}$$

where $M(\Delta_N,t,\tau) = \int\limits_{\tau}^{t_1} K(t,\tau_1)X_1(\tau_1)d\tau_1 X_1^{-1}(\tau)$, $t\in[0,T]$, $\tau\in[0,t_1]$,

$M(\Delta_N,t,\tau) = \int\limits_{\tau}^{t_j} K(t,\tau_1)X_j(\tau_1)d\tau_1 X_j^{-1}(\tau)$, $t\in[0,T]$, $\tau\in(t_{j-1},t_j]$, $j=2,...,N$.

Definition 5. A partition Δ_N is called regular for Equation (29) if the integral equation (39) has only the trivial solution.

Let $\sigma([0,T])$ denote the set of regular partitions of the interval $[0,T]$. The set $\sigma([0,T])$ is not empty.

Definition 6. The special Cauchy problem (37), (38) is called uniquely solvable if it has a unique solution for any pair $(f(t),\lambda)$ with $f(t)\in C([0,T],\mathbb{R}^n)$ and $\lambda\in\mathbb{R}^{nN}$.

Definition 7. Suppose that $\Delta_N\in\sigma([0,T])$, $\lambda=(\lambda_1,\lambda_2,...,\lambda_N)\in\mathbb{R}^{nN}$, and the function system $u[t,\lambda]=(u_1(t,\lambda),u_2(t,\lambda),...,u_N(t,\lambda))$ is a solution to the special Cauchy problem for the system of integro-differential equations with parameters (37), (38). Then the function $x(\Delta_N,t,\lambda)$ defined by the equalities $x(\Delta_N,t,\lambda)=\lambda_r+u_r(t,\lambda)$, $t\in[t_{r-1},t_r)$, $r=1,2,...,N$, and $x(\Delta_N,T,\lambda)=\lambda_N+\lim\limits_{t\to T-0}u_N(t,\lambda)$ is called the Δ_N general solution to the integro-differential equation (29).

Theorem 10. For any $\Delta_N\in\sigma([0,T])$, there exists a unique Δ_N general solution to the linear Fredholm integro-differential equation (29).

In contrast to the classical general solution, the Δ_N general solution exists for all linear nonhomogeneous Fredholm integro-differential equations and contains N arbitrary parameters $\lambda_r\in\mathbb{R}^n$.

The concept of new general solution, introduced by Dzhumabaev, made it possible to derive the solvability criteria for the linear Fredholm integro-differential equations and boundary value problems for this equation. The proposed method consists of the construction of Δ_N general solutions and solving linear algebraic equations with respect to parameters of those solutions. The Cauchy problems for ordinary differential equations and problems of evaluation of the definite integrals on the subintervals are used as auxiliary problems. Depending on the choice of methods for solving auxiliary problems, either numerical or approximate methods were obtained in order to solve the linear boundary value problems for Fredholm integro-differential equations [89–92].

The new general solution made it possible to propose new numerical and approximate methods for solving boundary value problems with and without parameter for nonlinear ordinary differential equations [93–98]. These methods are based on the construction and solving a system of algebraic equations for arbitrary vectors of the new general solution. The coefficients and the right-hand sides of this system are determined using solutions of the Cauchy problems for ordinary differential equations on the subintervals. Using the new general solution, solvability criteria were established for boundary value problems for nonlinear ordinary differential equations.

The results and methods were extended to linear nonlocal boundary value problems for systems of loaded hyperbolic equations and Fredholm hyperbolic integro-differential equations [99, 100].

The new approach to the general solution became the basis of methods for research and solving nonlinear boundary value problems for loaded differential and integro-differential equations [101–111]. The methods are based on the construction and solving systems of nonlinear algebraic equations for arbitrary vectors of new general solutions. To solve nonlocal boundary value problems for nonlinear partial differential and integro-differential equations, a modification of Euler's broken lines method was developed.

These results were further extended to multi-point problems, periodic problems with impulse effects, and control problems for various classes of differential, loaded differential, integro-differential, and partial differential equations [112–114].

Conclusion

Dzhumabaev D.S. was a highly qualified expert in the theory of differential, integral and nonlinear operator equations, computer and mathematical modeling of applied problems. He has published over 300 papers in scientific journals, including authoritative periodicals like *Journal of Mathematical Analysis and Applications, Journal of Computational and Applied Mathematics, Mathematical Methods in Applied Sciences, Mathematical Notes, Computational Mathematics and Mathematical Physics, Differential Equations, Ukrainian Mathematical Journal, Journal of Integral Equations and Applications, Journal of Mathematical Sciences, Eurasian Mathematical Journal*, etc. The list of his major publications is given below.

The research findings were presented and discussed at many international symposia and conferences. His scientific results were widely recognized in Kazakhstan and at the international level by experts in the field of differential equations and computational mathematics. The scientific direction formed by Dzhumabaev D.S. has been further developed by his students, who successfully work at the Institute of Mathematics and Mathematical Modeling and leading universities in Kazakhstan.

In 1998, Dzhumabaev D.S. was awarded the title of professor (specialty 01.01.00 – Mathematics). Under his supervision, two doctoral, twenty candidate dissertations, and one PhD thesis were defended. He supervised five PhD students. In 2004-2005, Dzhumabaev D.S. was the chair of the Expert Commission on Mathematics and Computer Science of the Committee on Supervision and Certification in Education and Science of the Ministry Education and science of the Republic of Kazakhstan.

Professor Dzhumabaev made a great contribution to academic community. He led a scientific seminar on the qualitative theory of differential equations at the Institute of Mathematics and Mathematical Modeling. He was a scientific expert of the State Expertise of the Ministry of Education and Science of the Republic of Kazakhstan. For many years, Dzhumabaev D.S. was a member of Dissertation Councils at the Institute of Mathematics, Al-Farabi Kazakh National University, Abai Kazakh National Pedagogical University, K. Zhubanov Aktobe Regional State University.

In 2014, at the invitation of the university authorities, Professor Dzhumabaev began to deliver lectures at the International University of Information Technology. He taught such courses as “Mathematical Analysis”, “Methods of solving linear and nonlinear boundary value problems for ordinary differential equations”, “Problems for integro-differential equations of processes with consequences”, “Boundary value problems, their applications and methods for solving”. It should be noted that his scientific results of recent years were obtained under the influence of teaching at the

International University of Information Technology. While giving lectures and conducting practical classes, he realized with great clarity the importance of developing numerical methods for solving applied problems. Having set himself the goal of bringing to the final numerical implementation the theoretical results and algorithms of the parameterization method, he made a breakthrough in the field of mathematical and computer modeling. Under scientific supervision of Professor Dzhumabaev, master students and undergraduates of the International University of Information Technology carried out research in the area of numerical methods for solving boundary value problems for differential and integro-differential equations.

Professor Dzhumabaev chaired the Mathematics Section of Academic Council of the Institute of Mathematics and Mathematical Modeling. He was a member of the editorial board of the scientific journals *News of NAS RK. Series: Physics and Mathematics, Kazakh Mathematical Journal, Bulletin of Karaganda State University. Series: Mathematics.*

Dzhumabaev D.S. was awarded the lapel badge "For Contribution to the Development of Science and Technology" and the Certificate of Merit of the Ministry of Education and Science of the Republic of Kazakhstan (2014).

Since 2018, Dzhumabaev D.S. headed the Department of Mathematical Physics and Mathematical Modeling at the Institute of Mathematics and Mathematical Modeling. In 2019, his research team, together with mathematicians from Ukraine, Uzbekistan, Azerbaijan, Germany, and the Czech Republic, received funding from the European Union's Horizon 2020 research and innovation programme under EU grant agreement 873071-H2020-MSCA-PISE-2019 (Marie Sklodowska-Curie Research and Innovation Staff Exchange), project titled "Spectral Optimization: From Mathematics to Physics and Advanced Technology" (SOMPATY).

The first publication in the framework of this project is devoted to the application of the parameterization method to multipoint problems for Fredholm integro-differential equations and was published in *Kazakh Mathematical Journal* (2020, Vol. 20, No. 1).

At the end of 2019, having applied for the competition from the International University of Information Technology, Professor Dzhumabaev became the owner of the grant "The Best University Teacher 2019" of the Ministry of Education and Science of the Republic of Kazakhstan.

A prominent scientist, an outstanding teacher, and a talented organizer, Dulat Syzdykbekovich Dzhumabaev passed away on February 20, 2020. He will be lovingly remembered by his wife Klara Kabdygalymovna, daughters Dana and Damira, son Anuar, and fours grandchildren. His memory will live in the hearts of his friends, colleagues, as well as generations of grateful and adoring students. His research, scientific ideas and plans will be continued and implemented by his students.

The major publications by Dzhumabaev D.S.

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