MATHEMATICS

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On estimates of M-term approximations of the Sobolev class in the Lorentz space

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In the paper spaces of periodic functions of several variables were considered, namely the Lorentz space $L_{2,\tau}(\mathrm{T}^m)$, the class of functions with bounded mixed fractional derivative $W_{2,\tau}^{\overline{r}}$, $1 \le \tau < \infty$, and the order of the best M-term approximation of a function $f \in L_{p,\tau}(\mathrm{T}^m)$ by trigonometric polynomials was studied. The article consists of an introduction, a main part, and a conclusion. In the introduction, basic concepts, definitions and necessary statements for the proof of the main results were considered. One can be found information about previous results on the mentioned topic. In the main part, exact-order estimates are established for the best M-term approximations of functions of the Sobolev class $W_{2,\tau_1}^{\overline{r}}$ in the norm of the space $L_{p,\tau_2}(\mathrm{T}^m)$ for various relations between the parameters p, τ_1, τ_2.

Keywords: Lorentz space, Sobolev class, mixed derivative, trigonometric polynomial, M-term approximation.

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Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ be the sets of natural, integer, and real numbers, respectively, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{R}^m$ is m-dimensional Euclidean space of points $\bar{x} = (x_1, \dots, x_m)$ with real coordinates; $\mathbb{T}^m = [0, 2\pi)^m$ and $\mathbb{I}^m = [0,1)^m$ are m-dimensional cubes.

We denote by $L_{p,\tau}(\mathbb{T}^m)$ the Lorentz space of all real-valued Lebesgue measurable functions f that have 2π-period in each variable and for which the quantity

$$\|f\|_{p,\tau} = \left\{ \frac{\tau}{p} \int_0^1 \left(f^*(t)\right)^\tau t^{\frac{\tau}{p}-1} dt \right\}^{\frac{1}{\tau}}, \ 1 < p < \infty, \ 1 \le \tau < \infty$$

is finite, where $f^*(t)$ is a non-increasing rearrangement of the function $|f(2\pi\overline{x})|$, $\overline{x} \in \mathbb{I}^m$ (see [1]).

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In case when $\tau = p$, the Lorentz space $L_{p,\tau}(\mathbb{T}^m)$ coincides with the Lebesgue space $L_p(\mathbb{T}^m)$ with the norm (see, for example, [2])

$$\|f\|_p = \left[\int_0^{2\pi} ... \int_0^{2\pi} |f(x_1, ..., x_m)|^p dx_1...dx_m\right]^{\frac{1}{p}},\ 1 \le p < \infty.$$

Let us begin by introducing some notation: $a_{\overline{n}}(f)$ are Fourier coefficients of the function $f \in L_1(\mathbb{T}^m)$ by the system $\{e^{i\langle \overline{n}, \overline{x}\rangle}\}_{\overline{n}\in\mathbb{Z}^m}$ and $\langle \overline{y}, \overline{x}\rangle = \sum\limits_{j=1}^{m} y_j x_j$;

$$\delta_{\overline{s}}(f, \overline{x}) = \sum_{\overline{n}\in\rho(\overline{s})} a_{\overline{n}}(f)\, e^{i\langle \overline{n}, \overline{x}\rangle},$$

where

$$\rho(\overline{s}) = \left\{\overline{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m \colon [2^{s_j - 1}] \le |k_j| < 2^{s_j}, j = 1, \ldots, m\right\},$$

and $[a]$ is an integer part of a, $\overline{s} = (s_1, ..., s_m)$, $s_j = 0, 1, 2, \ldots$

For a given vector $\overline{r} = (r_1, \ldots, r_m) > \overline{0} = (0, \ldots, 0)$ we set $\overline{\gamma} = \frac{\overline{r}}{r_1}$ and

$$Q_n^{(\overline{\gamma})} = \cup_{\langle \overline{s}, \overline{\gamma}\rangle < n} \rho(\overline{s}),$$

$S_n^{(\overline{\gamma})}(f, \overline{x}) = \sum_{\overline{k}\in Q_n^{(\overline{\gamma})}} a_{\overline{k}}(f) e^{i\langle \overline{k}, \overline{x}\rangle}$ is a partial sum of the Fourier series of the function f (see [2]).

Let us consider an one-dimensional Bernoulli kernel (see, for example, [2])

$$F_r(x) = 1 + 2\sum_{k=1}^{\infty} k^{-r} \cos(kx - r\pi/2),\ r > 0.$$

Next, for the vector $\overline{r} = (r_1, ..., r_m)$, $r_j > 0$, $j = 1, \ldots, m$, we set

$$F_{\overline{r}}(\overline{x}) = \prod_{j=1}^{m} F_{r_j}(x_j).$$

Let us consider a Sobolev functional class

$$W_{p,\tau}^{\overline{r}} = \{f :\ f = \varphi \star F_{\overline{r}},\ \|\varphi\|_{p,\tau} \leqslant 1\},$$

where $1 < p < \infty$, $1 \le \tau < \infty$,

$$(\varphi \star F_{\overline{r}})(\overline{x}) = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} \varphi(\overline{x} - \overline{u}) F_{\overline{r}})(\overline{u}) d\overline{u}.$$

In case when $\tau = p$, the class $W_{p,\tau}^{\overline{r}}$ has been considered in [3] and [4], so in this case, instead of $W_{p,p}^{\overline{r}}$ we write $W_p^{\overline{r}}$.

The value

$$e_M(f)_{p,\tau} = \inf_{\bar{k}^{(j)}, b_j} \left\| f - \sum_{j=1}^{M} b_j e^{i\langle \bar{k}^{(j)}, \bar{x}\rangle} \right\|_{p,\tau}$$

is called the best M-term trigonometric approximation of the function $f \in L_{p,\tau}(\mathbb{T}^m)$, $n \in \mathbb{N}$.

If $F \subset L_{p,\tau}(\mathbb{T}^m)$ is some functional class, then we set $e_M(F)_{p,\tau} = \sup_{f\in F} e_M(f)_{p,\tau}$. In case when $\tau = p$, instead of $e_M(F)_{p,\tau}$ we wrire $e_M(F)_p$.

The best M-term approximation of a function $f \in L_2[0,1]$ by polynomials in an orthonormal system has been first determined by S.B. Stechkin [5] and he has established a criterion for the absolute convergence of the Fourier series in this system. The advantage of the M-term approximation with respect to the one-dimensional trigonometric system over the linear approximation by M-order trigonometric polynomials has been shown by R.S. Ismagilov [6].

Exact order estimates of the best M-term approximation of the Bernoulli kernel have been established by V.E. Maiorov [7] and Yu. Makovoz [8], E.S. Belinsky [9,10]. In the one-dimensional case, the value $e_M(W_q^{\overline{r}})_p$ has been estimated by S. Belinsky [9]. At present, many important results on estimates of M-term approximations of functions from various Sobolev, Nikol'skii–Besov and Lizorkin–Triebel classes are known [11,12]. In the multidimensional case, for $1 < q \leq p < 2$ and $r_1 > \frac{1}{2}(\frac{1}{q} - \frac{1}{p})$, order-exact estimates of the best M-term approximation of functions of $W_q^{\overline{r}}$ in the norm of $L_p(\mathbb{T}^m)$ have been obtained by V.N. Temlyakov [3,4], and for $1 < q \leq p < 2$ and $r_1 \leq \frac{1}{2}(\frac{1}{q} - \frac{1}{p})$, E.S. Belinsky [10] has proved the following theorem:

Theorem. Let $1 < q \leq 2 < p < \infty$ and $r_1 = \ldots = r_\nu < r_{\nu+1} \leq \ldots r_m$. Then

$$e_M(W_q^{\overline{r}})_p \asymp M^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)(p-1)(r_1-p'(\frac{1}{q}-\frac{1}{p}))_+}$$

in case $\frac{1}{q} - \frac{1}{p} < r_1 < \frac{1}{q}$, where $q' = \frac{q}{q-1}$.

Note that a generalization of this theorem on the Lorentz space $L_{p,\tau}(\mathbb{T}^m)$ has been proved in [13–15].

Throughout the paper, $A_n \asymp B_n$ means that there are positive numbers C_1, C_2 independent of $n \in \mathbb{N}$ such that $C_1 A_n \leq B_n \leq C_2 A_n$ for $n \in \mathbb{N}$ and $\log M$, where $\log M$ is the logarithm with base 2 of the number $M > 1$.

By the constructive method, V.N. Temlyakov [16,17] has established estimates for M–term approximations of functions of the class $W_q^{\overline{r}}$ in the space $L_p(\mathbb{T}^m)$ for $1 < q \leq 2 < p < \infty$ and $(\frac{1}{q}-\frac{1}{p})p' < r_1 < \frac{1}{q}$, $p' = \frac{p}{p-1}$ and has raised the question of finding constructive evaluation method for $\frac{1}{q} - \frac{1}{p} < r_1 \leq (\frac{1}{q} - \frac{1}{p})p'$. Further application of the constructive method is given in [18,19].

In the first section, some auxiliary assertions are formulated that are necessary for proving main results. The main results of the article are formulated as a theorem and proved in the second section. In conclusion, we compare the proved Theorem 1 with previously known results.

1 *Auxiliary statements*

Theorem A. [20] Let $1 < q < \lambda < \infty$, $1 < \tau, \ \theta < \infty$. If a function $f \in L_{q,\tau}(\mathrm{T}^m)$, then

$$\|f\|_{q,\tau} \geq C\Big(\sum_{\overline{s}\in \mathrm{Z}_+^m}\prod_{l=1}^{m} 2^{s_l(1/\lambda-1/q)\tau}\|\delta_{\overline{s}}(f)\|_{\lambda,\theta}^{\tau}\Big)^{1/\tau}.$$

Theorem B. [20] Let $1 < p < q < \infty$, $1 < \tau_1, \tau_2 < \infty$. If a function $f \in L_{p,\tau_1}(\mathrm{T}^m)$ satisfies the condition

$$\sum_{\overline{s}\in \mathrm{Z}_+^m}\prod_{j=1}^{m} 2^{s_j\tau_2(1/p-1/q)}\|\delta_{\overline{s}}(f)\|_{p,\tau_1}^{\tau_2} < \infty,$$

then $f \in L_{q,\tau_2}(\mathrm{T}^m)$ and the inequality

$$\|f\|_{q,\tau_2} \leq C\left(\sum_{\overline{s}\in \mathrm{Z}_+^m}\prod_{j=1}^{m} 2^{s_j\tau_2(1/p-1/q)}\|\delta_{\overline{s}}(f)\|_{p,\tau_1}^{\tau_2}\right)^{1/\tau_2}$$

holds.

For a function $f \in L_1(\mathbb{T}^m)$ we set

$$f_{l,\bar{r}}(\bar{x}) = \sum_{l \le \langle \bar{s}, \bar{\gamma} \rangle < l+1} \delta_{\bar{s}}(f, \bar{x}), l \in \mathbb{Z}_+,$$

where $\bar{\gamma} = (\gamma_1, \ldots, \gamma_m)$, $\gamma_1 = \ldots = \gamma_\nu < \gamma_{\nu+1} \le \cdots \le \gamma_m$, $\gamma_j = \frac{r_j}{r_1}$, $r_j > 0$, $j = 1, \ldots, m$.

Let us consider the following class defined in [5, 6]

$$W_A^{a,b,\bar{r}} = \left\{ f \in L_1(\mathbb{T}^m) \colon \|f_{l,\bar{r}}\|_A \le 2^{-la} l^{(\nu-1)b} \right\},$$

where

$$\|f_{l,\bar{r}}\|_A = \sum_{l \le \langle \bar{s}, \bar{\gamma} \rangle < l+1} \sum_{\overline{n} \in \rho(\overline{s})} |a_{\overline{n}}(f)|.$$

The following lemma is a consequence of Lemma 6.1 in [16] (see also Lemma 2.1 in [17]), which we often use in proofs of main results.

Lemma 1. [15] Let $2 \le p < \infty$ and $1 < \tau < \infty$, $a > 0$. Then for $f \in W_A^{a,b,\bar{r}}$ there are constructive approximation methods of the greedy algorithm type of $G_M(f)$ with the property:

$$\|f - G_M(f)\|_{p,\tau} \le C(m) M^{-a-\frac{1}{2}} (\log M)^{(\nu-1)(a+b)}.$$

2 Main results

Theorem 1. Let $0 < r_1 = \ldots = r_\nu < r_{\nu+1} \le \ldots r_m$, $2 < p < \infty$, $1 < \max\{\tau_1, 2\} \le \tau_2 < \infty$, $\tau_2^{'} = \frac{\tau_2}{\tau_2 - 1}$.

a) If $\frac{1}{2} - \frac{1}{p} < r_1 < (\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2})\tau_2^{'}$, then

$$e_M(W_{2,\tau_1}^{\overline{r}})_{p,\tau_2} \le C M^{-\frac{p}{2}(r_1 + \frac{1}{p} - \frac{1}{2})} (\log_2 M)^{\frac{1}{2} - \frac{1}{\tau_1}}, \; M > 1.$$

b) If $\tau_2' \left(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2} \right) < r_1 < \frac{1}{2}$, then

$$e_M(W_{2,\tau_1}^{\overline{r}})_{p,\tau_2} \le C M^{-\frac{p}{2}(r_1 + \frac{1}{p} - \frac{1}{2})} (\log_2 M)^{\frac{1}{2} - \frac{1}{\tau_1}} (\log_2 M)^{(\nu-1)\frac{p}{\tau_2'}\left(r_1 - \tau_2'\left(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}\right)\right)}.$$

Proof. Let us introduce some notation

$$Q_{n,\bar{\gamma}} = \cup_{\langle \bar{s}, \bar{\gamma} \rangle \le n} \rho(\overline{s}), \; S_{Q_{n,\bar{\gamma}}}(f, \overline{x}) = \sum_{\langle \bar{s}, \bar{\gamma} \rangle \le n} \delta_{\bar{s}}(f, \bar{x}).$$

For a natural number M, there exists a number $n \in \mathbb{N}$ such that $M \asymp 2^n n^{\nu-1}$.

Let $\nu \ge 2$. We set

$$n_1 = \frac{p}{2} n - p\Big(\frac{1}{2} - \frac{1}{\tau_2}\Big)(\nu - 1) \log n,$$

$$n_2 = \frac{p}{2} n + \frac{p}{2}(\nu - 1) \log n.$$

Also, let us introduce

$$S_l = \Big(2^{l r_1 \tau_1} \sum_{l \le \langle \bar{s}, \bar{\gamma} \rangle < l+1} 2^{\langle \bar{s}, \bar{1} \rangle (\frac{1}{2} - \frac{1}{q})\tau_1} \|\delta_{\bar{s}}(\varphi)\|_2^{\tau_1} \Big)^{1/\tau_1}$$

and

$$m_l = \left[2^{-l\frac{\tau_2^{'}}{p}} S_l^{\tau_1} 2^{n\frac{\tau_2^{'}}{2}} n^{(\nu-1)\frac{\tau_2^{'}}{2}}\right] + 1,$$

where $\langle \bar{s}, \bar{1}\rangle = \sum\limits_{j=1}^{m} s_j$, $p^{'} = \frac{p}{p-1}$ and $[y]$ is an integer part of a number y.

By $G(l)$ is denoted the set of indices $\overline{s}$, $l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1$, with the largest $\|\delta_{\overline{s}}(\varphi)\|_2$, and $m_l = |G(l)|$ is the number of elements of $G(l)$.

Let us consider the functions

$$F_1(\overline{x}) = \sum_{n \leq l < n_1} f_l(\overline{x}),$$

$$F_2(\overline{x}) = \sum_{n_1 \leq l < n_2} \sum_{\bar{s} \notin G(l)} \delta_{\overline{s}}(f, \overline{x}),$$

$$F_3(\overline{x}) = \sum_{n_1 \leq l < n_2} \sum_{\bar{s} \in G(l)} \delta_{\overline{s}}(f, \overline{x}).$$

Let us estimate $\|F_1\|_A$. Applying Hölder's inequality for the sum and Parseval's equality, we have

$$\|F_1\|_A = \sum_{l=n}^{n_1-1} \sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1} \sum_{\overline{k} \in \rho(\overline{s})} |a_{\overline{k}}(f)| \leq 2^{-\frac{m}{2}} \sum_{l=n}^{n_1-1} \sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1} 2^{\langle \bar{s}, \bar{1}\rangle \frac{1}{2}} \|\delta_{\overline{s}}(f)\|_2 =$$

$$= 2^{-\frac{m}{2}} \sum_{l=n}^{n_1-1} \sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1} 2^{\langle \bar{s}, \bar{1}\rangle (\frac{1}{2} - \frac{1}{q})} \|\delta_{\overline{s}}(f)\|_2 2^{\langle \bar{s}, \bar{1}\rangle \frac{1}{q}}. \tag{1}$$

It is known that the Fourier coefficients of the convolution $f = \varphi \star F_{\overline{r}}$ are equal to $a_{\overline{k}}(\varphi) a_{\overline{k}}(F_{\overline{r}})$, $\overline{k} \in \mathbb{Z}^m$. Therefore, using Parseval's equality, it is easy to verify that

$$\|\delta_{\overline{s}}(f)\|_2 << 2^{-\langle \bar{s}, \bar{r}\rangle} \|\delta_{\overline{s}}(\varphi)\|_2, \ \overline{s} \in \mathbb{Z}_+^m. \tag{2}$$

Hense, from (1) and (2) we get

$$\|F_1\|_A \leq 2^{-\frac{m}{2}} \sum_{l=n}^{n_1-1} \sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1} 2^{\langle \bar{s}, \bar{1}\rangle \frac{1}{2}} \|\delta_{\overline{s}}(f)\|_2 \leq C \sum_{l=n}^{n_1-1} \sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1} 2^{\langle \bar{s}, \bar{1}\rangle \frac{1}{2}} 2^{-\langle \bar{s}, \bar{r}\rangle} \|\delta_{\overline{s}}(\varphi)\|_2. \tag{3}$$

If $2 < \tau_1 < \infty$, then according to the inequality of different metrics for trigonometric polynomials in the Lorentz space [20] we have

$$\|\delta_{\overline{s}}(\varphi)\|_2 \leq C \Big(\sum_{j=1}^{m} (s_j + 1)\Big)^{\frac{1}{2} - \frac{1}{\tau_1}} \|\delta_{\overline{s}}(\varphi)\|_{2,\tau_1}.$$

From Lemma 1.6 [21] for $p = 2$ and $2 < \tau_1 < \infty$ we get

$$\left(\sum_{\overline{s} \in \mathbb{Z}_+} \left(\sum_{j=1}^{m} (s_j + 1)\right)^{(\frac{1}{\tau_1} - \frac{1}{2})\tau_1} \|\delta_{\overline{s}}(\varphi)\|_2^{\tau_1}\right)^{\frac{1}{\tau_1}} \leq C \left(\sum_{\overline{s} \in \mathbb{Z}_+} \|\delta_{\overline{s}}(\varphi)\|_{2,\tau_1}^{\tau_1}\right)^{\frac{1}{\tau_1}} \leq C\|\varphi\|_{2,\tau_1}. \tag{4}$$

By virtue of inequality (4) and Hölder's inequality, we obtain

$$\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{\langle\bar{s},\bar{1}\rangle\frac{1}{2}}\|\delta_{\overline{s}}(f)\|_2 \leq \left(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}\|\delta_{\overline{s}}(\varphi)\|_2^{\tau_1}\right)^{\frac{1}{\tau_1}} \times$$
$$\times\left(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{-\langle\bar{s},\bar{\gamma}\rangle(r_1-\frac{1}{2})\tau_1^{'}}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_1^{'}}\right)^{\frac{1}{\tau_1^{'}}} \leq \tag{5}$$
$$\leq C\|\varphi\|_{2,\tau_1}\left(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{-\langle\bar{s},\bar{\gamma}\rangle(r_1-\frac{1}{2})\tau_1^{'}}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_1^{'}}\right)^{\frac{1}{\tau_1^{'}}} \leq$$
$$\leq C2^{-l\left(r_1-\frac{1}{2}\right)}l^{(\nu-1)\frac{1}{\tau_1^{'}}}l^{\frac{1}{2}-\frac{1}{\tau_1}}\|\varphi\|_{2,\tau_1},$$

where $\tau_1^{'}=\frac{\tau_1}{\tau_1-1}$, $1<\tau_1<\infty$.

(3) and (5) imply that

$$\|F_1\|_A \leq C\sum_{l=n}^{n_1-1}2^{\frac{l}{2}}l^{(\nu-1)\frac{1}{\tau_1^{'}}}l^{\frac{1}{2}-\frac{1}{\tau_1}}2^{-lr_1} \leq C2^{-n_1(r_1-\frac{1}{2})}n_1^{(\nu-1)\frac{1}{\tau_1^{'}}}n_1^{\frac{1}{2}-\frac{1}{\tau_1}} \tag{6}$$

for a function $f\in W^{\overline{r}}_{2,\tau_1}$ when $r_1<\frac{1}{2}$ and $2<\tau_1<\infty$.

By Lemma 1 for the function F_1 using a constructive method, one can find an M-term trigonometric polynomial $G_M(F_1)$ such that

$$\|F_1-G_M(F_1)\|_{p,\tau_2} \leq CM^{-\frac{1}{2}}2^{-n_1(r_1-\frac{1}{2})}n_1^{(\nu-1)\frac{1}{\tau_1^{'}}}n_1^{\frac{1}{2}-\frac{1}{\tau_1}}. \tag{7}$$

Therefore, according to inequality (6) and (7) and taking into account the definition of the number n_1 and the relation $M\asymp 2^n n^{\nu-1}$, we obtain

$$\|F_1-G_M(F_1)\|_{p,\tau_2} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}} \tag{8}$$

in case when $q=2<p<\infty$, $2<\tau_1<\infty$, $1<\tau_2<\infty$, $r_1<\frac{1}{2}$.

Let us estimate $\|F_3\|_A$. Applying Hölder's inequality for the sum and Parseval's equality, we obtain

$$\|F_3\|_A = \sum_{l=n_1}^{n_2-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)}\sum_{\overline{k}\in\rho(\overline{s})}|a_{\overline{k}}(f)| \leq 2^{-\frac{m}{2}}\sum_{l=n_1}^{n_2-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle\frac{1}{2}}\|\delta_{\overline{s}}(f)\|_2 \leq$$
$$\leq C\sum_{l=n_1}^{n_2-1}2^{\frac{l}{2}}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})}\|\delta_{\overline{s}}(f)\|_2. \tag{9}$$

Now, to the inner sum on the right side of inequality (9), applying Hölder's inequality for $\frac{1}{\tau_1}+\frac{1}{\tau_1^{'}}=1$, $1<\tau_1<\infty$, we have

$$\|F_3\|_A \leq C\sum_{l=n_1}^{n_2-1}2^{\frac{l}{2}}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}\|\delta_{\overline{s}}(f)\|_2^{\tau_1}\Big)^{\frac{1}{\tau_1}}|G(l)|^{\frac{1}{\tau_1^{'}}}.$$

Then, using (2) we get

$$\|F_3\|_A \leq C \sum_{l=n_1}^{n_2-1} 2^{\frac{l}{2}} (l+1)^{\frac{1}{2}-\frac{1}{\tau_1}} \times$$

$$\times \Big(\sum_{l \leq \langle \bar{s}, \bar{\gamma} \rangle < l+1, \bar{s} \in G(l)} \Big(\sum_{j=1}^{m} (s_j+1) \Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1} 2^{-\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(\varphi)\|_2^{\tau_1} \Big)^{\frac{1}{\tau_1}} |G(l)|^{\frac{1}{\tau_1'}} \leq$$

$$\leq C \sum_{l=n_1}^{n_2-1} 2^{l\left(\frac{1}{2}-r_1\right)} (l+1)^{\frac{1}{2}-\frac{1}{\tau_1}} \times \tag{10}$$

$$\times \Big(\sum_{l \leq \langle \bar{s}, \bar{\gamma} \rangle < l+1, \bar{s} \in G(l)} \Big(\sum_{j=1}^{m} (s_j+1) \Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1} \|\delta_{\bar{s}}(\varphi)\|_2^{\tau_1} \Big)^{\frac{1}{\tau_1}} |G(l)|^{\frac{1}{\tau_1'}}.$$

We set

$$\tilde{S}_l = \Big(\sum_{l \leq \langle \bar{s}, \bar{\gamma} \rangle < l+1} \Big(\sum_{j=1}^{m} (s_j+1) \Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1} \|\delta_{\bar{s}}(f)\|_2^{\tau_1} \Big)^{1/\tau_1}$$

and

$$m_l := |G(l)| := \Big[2^{-l\frac{\tau_2'}{p}} \tilde{S}_l^{\tau_1} 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}} \Big] + 1.$$

Then (10) implies that

$$\|F_3\|_A \leq C \sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2})} (l+1)^{\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l m_l^{\frac{1}{\tau_1'}} \leq$$

$$\leq C \sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2})} (l+1)^{\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l \Big\{ 2^{-l\frac{\tau_2'}{p}} \tilde{S}_l^{\tau_1} 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}} + 1 \Big\}^{\frac{1}{\tau_1'}} \leq \tag{11}$$

$$\leq C \Big\{ \Big(2^n n^{\nu-1} \Big)^{\frac{\tau_2'}{2\tau_1'}} \sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2}+\frac{\tau_2'}{p\tau_1'})} (l+1)^{\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}} + \sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2})} (l+1)^{\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l \Big\}.$$

Since $\tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}} = \tilde{S}_l^{\tau_1}$ and $-\frac{1}{2} + \frac{\tau_2'}{p\tau_1'} = \tau_2'(-\frac{1}{2} + \frac{1}{p} - \frac{1}{p\tau_1} + \frac{1}{2\tau_2})$, then by (4) we have

$$\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2}+\frac{\tau_2'}{p\tau_1'})} (l+1)^{\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}} = \sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} (l+1)^{\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l^{\tau_1} \leq$$

$$\leq C \sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} (l+1)^{\frac{1}{2}-\frac{1}{\tau_1}} \|\varphi\|_{2,\tau_1}^{\tau_1} \leq$$

$$\leq C \sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} l^{\frac{1}{2}-\frac{1}{\tau_1}}$$

for a function $f \in W_{2,\tau_1}^{\bar{r}}$ and $2 < \tau_1 < \infty$. Since $r_1 - \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < 0$, then, taking into account the definition of the number n_2, from here we obtain

$$\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2}+\frac{\tau_2'}{p\tau_1'})}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}} \leq C2^{-n_2(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}n_2^{\frac{1}{2}-\frac{1}{\tau_1}} \leq$$

$$\leq C2^{-n\frac{p}{2}(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}n^{-(\nu-1)\frac{p}{2}(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}n^{\frac{1}{2}-\frac{1}{\tau_1}} \qquad (12)$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$, $2 < \tau_1 < \infty$.

Next, due to inequality (4), taking into account that a function $f \in W^{\overline{r}}_{2,\tau_1}$ and $r_1 - \frac{1}{2} < 0$, we have

$$\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2})}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l \leq C\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2})}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\|\varphi\|_{2,\tau_1} \leq$$

$$\leq C\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2})}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}} \leq C2^{-n_2(r_1-\frac{1}{2})}(n_2+1)^{\frac{1}{2}-\frac{1}{\tau_1}} \leq \qquad (13)$$

$$\leq C2^{-n\frac{p}{2}(r_1-\frac{1}{2})}n^{-(\nu-1)\frac{p}{2}(r_1-\frac{1}{2})}n^{\frac{1}{2}-\frac{1}{\tau_1}}.$$

Now it follows from inequalities (11)–(13) that

$$\|F_3\|_A \leq C\Big\{\Big(2^n n^{\nu-1}\Big)^{\frac{\tau_2'}{2\tau_1'}} 2^{-n\frac{p}{2}(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}n^{-(\nu-1)\frac{p}{2}(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}n^{\frac{1}{2}-\frac{1}{\tau_1}}+$$

$$+(2^n n^{\nu-1})^{-\frac{p}{2}(r_1-\frac{1}{2})}n^{\frac{1}{2}-\frac{1}{\tau_1}}\Big\}$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$, $2 < \tau_1 < \infty$, $1 < \tau_2 < \infty$, $r_1 - \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < 0$.

Since $\frac{p}{2}(r_1 - \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2})) - \frac{\tau_2'}{2\tau_1'} = \frac{p}{2}(r_1 - \frac{1}{2})$, then it follows that

$$\|F_3\|_A \leq C(2^n n^{\nu-1})^{-\frac{p}{2}(r_1-\frac{1}{2})}n^{\frac{1}{2}-\frac{1}{\tau_1}}. \qquad (14)$$

Since $2 < p < \infty$, then by Lemma 1 for the function F_3, by a constructive method, there is an M-term trigonometric polynomial $G_M(F_3)$ such that

$$\|F_3 - G_M(F_3)\|_{p,\tau_2} \leq CM^{-\frac{1}{2}}(2^n n^{\nu-1})^{-\frac{p}{2}(r_1-\frac{1}{2})}n^{\frac{1}{2}-\frac{1}{\tau_1}}.$$

Hence, in accordance with (14), we have

$$\|F_3 - G_M(F_3)\|_{p,\tau_2} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}} \qquad (15)$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$ for $2 < p < \infty$, $2 < \tau_1 < \infty$, $1 < \tau_2 < \infty$ and $r_1 < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2})$.

Let us estimate $\|F_2\|_{p,\tau_2}$. So,

$$\|F_2\|_{p,\tau_2} \leq C\Big(\sum_{l=n_1}^{n_2-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_2-\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_2}.$$

Taking into account that

$$\|\delta_{\overline{s}}(f)\|_2 \leq m_l^{-\frac{1}{\tau_1}}2^{-lr_1}l^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l$$

for $\overline{s} \notin G(l)$ and substituting the values of the numbers m_l for $\tau_2 - \tau_1 \geq 0$, we have

$$\|F_2\|_{p,\tau_2} \leq C\Big(\sum_{l=n_1}^{n_2-1} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1, \bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2} \|\delta_{\overline{s}}(f)\|_2^{\tau_1} \Big(m_l^{-\frac{1}{\tau_1}} 2^{-lr_1} l^{\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l\Big)^{\tau_2-\tau_1}\Big)^{1/\tau_2} \leq$$

$$\leq C\Big(\sum_{l=n_1}^{n_2-1} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1, \bar{s}\notin G(l)} 2^{l(\frac{1}{2}-\frac{1}{p})\tau_2} 2^{-lr_1\tau_1} \|\delta_{\overline{s}}(\varphi)\|_2^{\tau_1} \Big(m_l^{-\frac{1}{\tau_1}} 2^{-lr_1} l^{\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l\Big)^{\tau_2-\tau_1}\Big)^{1/\tau_2} \leq$$

$$\leq C\Big(\sum_{l=n_1}^{n_2-1}\Big(\Big(2^{-l\frac{\tau_2'}{p}} \tilde{S}_l^{\tau_1} 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big)^{-\frac{1}{\tau_1}} 2^{-lr_1} \tilde{S}_l l^{\frac{1}{2}-\frac{1}{\tau_1}}\Big)^{\tau_2-\tau_1} \times \tag{16}$$

$$\times 2^{l(\frac{1}{2}-\frac{1}{p})\tau_2} 2^{-lr_1\tau_1} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1, \bar{s}\notin G(l)} \|\delta_{\overline{s}}(\varphi)\|_2^{\tau_1}\Big)^{1/\tau_2} =$$

$$= C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}} \Big(\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{\tau_2'}{p\tau_1})(\tau_2-\tau_1)} l^{(\frac{1}{2}-\frac{1}{\tau_1})(\tau_2-\tau_1)} l^{-(\frac{1}{\tau_1}-\frac{1}{2})\tau_1} \tilde{S}_l^{\tau_1}\Big)^{1/\tau_2}.$$

Using inequality (4), it is easy to verify that

$$\tilde{S}_l = \left(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \left(\sum_{j=1}^{m}(s_j+1)\right)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1} \|\delta_{\overline{s}}(\varphi)\|_2^{\tau_1}\right)^{1/\tau_1} \leq$$

$$\leq C\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\overline{s}}(\varphi)\Big\|_{2,\tau_1} \leq C\|\varphi\|_{2,\tau_1} \tag{17}$$

for a function $f \in W_{2,\tau_1}^{\overline{r}}$, $2 < \tau_1 \leq \tau_2 < \infty$.

Now it follows from inequalities (16) and (17) that

$$\|F_2\|_{p,\tau_2} \leq C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}} \times$$

$$\times\Big(\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{\tau_2'}{p\tau_1})(\tau_2-\tau_1)} l^{(\frac{1}{2}-\frac{1}{\tau_1})(\tau_2-\tau_1)} 2^{l(\frac{1}{2}-\frac{1}{p})\tau_2} l^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_1} 2^{-lr_1\tau_1}\Big)^{1/\tau_2} =$$

$$= C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}} \Big(\sum_{l=n_1}^{n_2-1} 2^{-l\tau_2(r_1-\frac{\tau_2'}{p\tau_1\tau_2}(\tau_2-\tau_1)-(\frac{1}{2}-\frac{1}{p}))} l^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_2}\Big)^{1/\tau_2}.$$

Since

$$r_1 - \frac{\tau_2'}{p\tau_1\tau_2}(\tau_2-\tau_1) - (\frac{1}{2}-\frac{1}{p}) = r_1 - \tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}),$$

then taking into account the definition of the number n_2, from here we get

$$\|F_2\|_{p,\tau_2} \leq C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}} 2^{-n_2(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} n_2^{\frac{1}{2}-\frac{1}{\tau_1}} \leq$$

$$\leq C 2^{-n\frac{p}{2}(r_1-\frac{1}{p}-\frac{1}{2})} n^{\frac{1}{2}-\frac{1}{\tau_1}} \tag{18}$$

for function $f \in W_{2,\tau_1}^{\overline{r}}$ when $2 < p < \infty$, $2 < \tau_1 \leq \tau_2 < \infty$, $r_1 < \tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})$.

Now it follows from inequalities (8), (15), and (18) that

$$\|f-(S_{Q_{n,\bar\gamma}}(f)+G_M(F_1)+G_M(F_3))\|_{p,\tau_2}\le$$
$$\le\|F_1-G_M(F_1)\|_{p,\tau_2}+\|F_3-G_M(F_3)\|_{p,\tau_2}+\|F_2\|_{p,\tau_2}+$$
$$+\Big\|\sum_{\langle\bar s,\bar\gamma\rangle\ge n_2}\delta_{\bar s}(f)\Big\|_{p,\tau_2}\le CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac12)}(\log M)^{\frac12-\frac{1}{\tau_1}}+\Big\|\sum_{\langle\bar s,\bar\gamma\rangle\ge n_2}\delta_{\bar s}(f)\Big\|_{p,\tau_2}$$

for a function $f\in W^{\bar r}_{2,\tau_1}$ when $2<p<\infty$, $2<\tau_1\le\tau_2<\infty$, $\frac12-\frac1p<r_1<\tau_2'(\frac12-\frac1p+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})$.

Further, taking into account that $2<\tau_1<\tau_2<\infty$ and $r_1+\frac1p-\frac12>0$, and successively applying Theorem B, Jensen's inequality, Theorem A, then Lemma 1.3 [21] and Theorem 1.1 [21], we obtain

$$\Big\|\sum_{\langle\bar s,\bar\gamma\rangle\ge n_2}\delta_{\bar s}(f)\Big\|_{p,\tau_2}=\Big\|\sum_{l=n_2}^{\infty}\sum_{l\leqslant\langle\bar s,\bar\gamma\rangle<l+1}\delta_{\bar s}(f)\Big\|_{p,\tau_2}\le$$
$$\le C\left(\sum_{l=n_2}^{\infty}\sum_{l\le\langle\bar s,\bar\gamma\rangle<l+1}\prod_{j=1}^{m}2^{s_j(\frac12-\frac1p)\tau_2}\|\delta_{\bar s}(f)\|^{\tau_2}_{2,\tau_1}\right)^{\frac{1}{\tau_2}}\le$$
$$\le C\left(\sum_{l=n_2}^{\infty}2^{l(\frac12-\frac1p)\tau_2}\left[\sum_{l\le\langle\bar s,\bar\gamma\rangle<l+1}\|\delta_{\bar s}(f)\|^{\tau_1}_{2,\tau_1}\right]^{\frac{\tau_2}{\tau_1}}\right)^{\frac{1}{\tau_2}}\le$$
$$\le C\Big(\sum_{l=n_2}^{\infty}2^{l(\frac12-\frac1p)\tau_2}\Big\|\sum_{l\le\langle\bar s,\bar\gamma\rangle<l+1}\delta_{\bar s}(f)\Big\|^{\tau_2}_{2,\tau_1}\Big)^{\frac{1}{\tau_2}}\le C\Big(\sum_{l=n_2}^{\infty}2^{-l(r_1+\frac1p-\frac12)p}\Big)^{\frac1p}\le$$
$$\le C2^{-n_2(r_1+\frac1p-\frac12)}\le CM^{-\frac{p}{2}(r_1+\frac1p-\frac12)},$$

that leads to

$$e_M(f)_{p,\tau_2}\le\|f-(S_{Q_{n,\bar\gamma}}(f)+G_M(F_1)+G_M(F_3))\|_{p,\tau_2}\le CM^{-\frac{p}{2}(r_1+\frac1p-\frac12)}(\log M)^{\frac12-\frac{1}{\tau_1}}$$

For a function $f\in W^{\bar r}_{2,\tau_1}$ when $2<p<\infty$, $2<\tau_1\le\tau_2<\infty$, $\frac12-\frac1p<r_1<\tau_2'(\frac12-\frac1p+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})$.

Assume that $\tau_2'(\frac12-\frac1p+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})<r_1<\frac12$. Then, taking into account the definition of the number n_1, we get

$$\sum_{l=n_1}^{n_2-1}2^{-l\left(r_1-\tau_2'\left(\frac12-\frac1p+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}(l+1)^{\frac12-\frac{1}{\tau_1}}\le C2^{-n_1\left(r_1-\tau_2'\left(\frac12-\frac1p+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}n_1^{\frac12-\frac{1}{\tau_1}}\le$$
$$\le C2^{-\frac{p}{2}n\left(r_1-\tau_2'\left(\frac12-\frac1p+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}n^{p(\frac12-\frac{1}{\tau_2})(\nu-1)\left(r_1-\tau_2'\left(\frac12-\frac1p+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}n^{\frac12-\frac{1}{\tau_1}}\qquad(19)$$

for a function $f\in W^{\bar r}_{2,\tau_1}$ when $\tau_2'(\frac12-\frac1p+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})<r_1<\frac12$.

(11), (13) and (19) imply that

$$\|F_3\|_A\le(2^n n^{\nu-1})^{-\frac{p}{2}(r_1-\frac12)}n^{\frac{p}{\tau_2'}(\nu-1)\left(r_1-\tau_2'\left(\frac12-\frac1p+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}n^{\frac12-\frac{1}{\tau_1}}$$

for a function $f\in W^{\bar r}_{2,\tau_1}$ when $\tau_2'(\frac12-\frac1p+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})<r_1<\frac12$.

Hence, by Lemma 1 we obtain

$$\|F_3-G_M(F_3)\|_{p,\tau_2}\le CM^{-\frac{p}{2}(r_1+\frac1p-\frac12)}(\log M)^{\frac{p}{\tau_2'}(\nu-1)\left(r_1-\tau_2'\left(\frac12-\frac1p+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}(\log M)^{\frac12-\frac{1}{\tau_1}}\qquad(20)$$

for a function $f \in W^{\bar{r}}_{2,\tau_1}$ when $\tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < r_1 < \frac{1}{2}$.

Let us estimate $\|F_2\|_{p,\tau_2}$ in case when $\tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < r_1 < \frac{1}{2}$. (16) and (17) imply that

$$\|F_2\|_{p,\tau_2} \leq C(2^n n^{\nu-1})^{-\frac{\tau_2^{'}}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}} \Big(\sum_{l=n_1}^{n_2-1} 2^{-l\tau_2(r_1 - \frac{\tau_2^{'}}{p\tau_1\tau_2}(\tau_2-\tau_1) - (\frac{1}{2}-\frac{1}{p}))} l^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_2}\Big)^{1/\tau_2} \leq$$

$$\leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{p}{\tau_2^{'}}(\nu-1)\left(r_1 - \tau_2^{'}\left(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}} \tag{21}$$

in case when $\tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < r_1 < \frac{1}{2}$.

Since $\tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < r_1 < \frac{1}{2}$, (8) implies that

$$\|F_1 - G_M(F_1)\|_{p,\tau_2} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}} \leq$$

$$\leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{p}{\tau_2^{'}}(\nu-1)\left(r_1 - \tau_2^{'}\left(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}} \tag{22}$$

(20)–(22) (see (18)) imply that

$$\|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M(F_1) + G_M(F_3))\|_{p,\tau_2} \leq$$

$$\leq \|F_1 - G_M(F_1)\|_{p,\tau_2} + \|F_3 - G_M(F_3)\|_{p,\tau_2} + \|F_2\|_{p,\tau_2} + \Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geq n_2}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2} \leq$$

$$\leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{p}{\tau_2^{'}}(\nu-1)\left(r_1 - \tau_2^{'}\left(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}} + \Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geq n_2}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2}.$$

Then, taking into account that $\tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < r_1 < \frac{1}{2}$ and following the same steps as in [20], we have

$$\Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geq n_2}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{p}{\tau_2^{'}}(\nu-1)\left(r_1 - \tau_2^{'}\left(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}}.$$

Hence,

$$e_M(f)_{p,\tau_2} \leqslant \|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M(F_1) + G_M(F_3))\|_{p,\tau_2} \leq$$

$$\leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{p}{\tau_2^{'}}(\nu-1)\left(r_1 - \tau_2^{'}\left(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}}$$

for a function $f \in W^{\bar{r}}_{2,\tau_1}$ when $2 < p < \infty$, $2 < \tau_1 \leq \tau_2 < \infty$, $\tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < r_1 < \frac{1}{2}$.

Let $1 < \tau_1 \leq 2$. Then by Lemma 1.5 [21] the inequality

$$\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\|\delta_{\bar{s}}(f)\|^2_{2,\tau_1}\Big)^{1/2} \leq C\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{2,\tau_1}. \tag{23}$$

Since $1 < \tau_1 \leq 2$, then (see [1; 217])

$$\|\delta_{\bar{s}}(f)\|_2 \leq C\|\delta_{\bar{s}}(f)\|_{2,\tau_1}. \tag{24}$$

It follows from inequalities (1), (23), and (24) that

$$\|F_1\|_A \leq C\sum_{l=n}^{n_1-1}2^{l/2}\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{2,\tau_1}.$$

Now, given that the function $f \in W^{\overline{r}}_{2,\tau_1}$ and the choice of the number n_1, we get

$$\|F_1\|_A \leq CM^{-\frac{p}{2}(r_1-\frac{1}{2})}(\log M)^{(\nu-1)\frac{p}{\tau_2'}(r_1-\frac{1}{2})}$$

for $r_1 < 1/2$. Further, arguing as in the proof of inequality (8), we obtain

$$\|F_1 - G_M(F_1)\|_{p,\tau_2} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{l}{2}-\frac{1}{\tau_1}} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})} \tag{25}$$

in case when $q = 2 < p < \infty$, $1 < \tau_1 \leq 2$, $1 < \tau_2 < \infty$, $r_1 < \frac{1}{2}$.

Let us estimate $\|F_3\|_A$. For this we set

$$\tilde{S}_l = \Big(2^{lr_1\tau_1} \sum_{l \leq \langle \overline{s}, \overline{\gamma} \rangle < l+1} \|\delta_{\overline{s}}(f)\|_2^2\Big)^{1/2}$$

and

$$\tilde{m}_l := |G(l)| := \Big[2^{-l\frac{\tau_2'}{p}}\tilde{S}_l^2 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big] + 1.$$

In inequality (9) it is proved that

$$\|F_3\|_A \leq 2^{-\frac{m}{2}} \sum_{l=n_1}^{n_2-1} \sum_{l \leq \langle \overline{s}, \overline{\gamma} \rangle < l+1, \overline{s} \in G(l)} 2^{\langle \overline{s}, \overline{1} \rangle \frac{1}{2}} \|\delta_{\overline{s}}(f)\|_2 \leq$$

$$\leq 2^{-\frac{m}{2}} \sum_{l=n_1}^{n_2-1} 2^{(l+1)/2} \sum_{l \leq \langle \overline{s}, \overline{\gamma} \rangle < l+1, \overline{s} \in G(l)} \|\delta_{\overline{s}}(f)\|_2. \tag{26}$$

Applying Hölder's inequality to the inner sum and substituting the value of the number $\tilde{m}_l := |G(l)|$ from (26), we obtain

$$\|F_3\|_A \leq 2^{-\frac{m}{2}} \sum_{l=n_1}^{n_2-1} 2^{(l+1)/2} \Big(\sum_{l \leq \langle \overline{s}, \overline{\gamma} \rangle < l+1, \overline{s} \in G(l)} \|\delta_{\overline{s}}(f)\|_2^2\Big)^{1/2} |G(l)|^{1/2} \times$$

$$\times 2^{-\frac{m-1}{2}} \Big\{ \sum_{l=n_1}^{n_2-1} 2^{l(\frac{1}{2}-r_1)} 2^{-l\frac{\tau_2'}{2p}} \tilde{S}_l^2 (2^n n^{(\nu-1)})^{\frac{\tau_2'}{4}} + \sum_{l=n_1}^{n_2-1} 2^{l(\frac{1}{2}-r_1)} \tilde{S}_l \Big\}. \tag{27}$$

Using inequalities (23) and (24) and taking into account the value of the numbers $\tilde{S}_l$, we obtain

$$\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2}+\frac{\tau_2'}{2p})} \tilde{S}_l^2 \leq \sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2}+\frac{\tau_2'}{2p})} \Big(2^{lr_1} \Big\| \sum_{l \leq \langle \overline{s}, \overline{\gamma} \rangle < l+1} \delta_{\overline{s}}(f) \Big\|_{2,\tau_1}\Big). \tag{28}$$

Since a function $f \in W^{\overline{r}}_{2,\tau_1}$ and

$$r_1 - \frac{1}{2} + \frac{\tau_2'}{2p} = r_1 - \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2}) \leq r_1 - \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < 0,$$

then from inequality (28) we have

$$\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2}+\frac{\tau_2'}{2p})} \tilde{S}_l^2 \leq C \sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{2p}-\frac{1}{2\tau_2}))} \leq C 2^{-n_2(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{2p}-\frac{1}{2\tau_2}))}. \tag{29}$$

Since the function $f \in W^{\overline{r}}_{2,\tau_1}$ and $r_1 - \frac{1}{2} < 0$, we can prove similarly that

$$\sum_{l=n_1}^{n_2-1} 2^{l(\frac{1}{2}-r_1)} \tilde{S}_l \leq C 2^{n_2(\frac{1}{2}-r_1)}. \tag{30}$$

Now it follows from inequalities (27), (29), and (30) that

$$\|F_3\|_A \leq C\Big\{ (2^n n^{(\nu-1)})^{\frac{\tau_2'}{4}} 2^{-n_2\left(r_1 - \tau_2'\left(\frac{1}{2}-\frac{1}{p}+\frac{1}{2p}-\frac{1}{2\tau_2}\right)\right)} + 2^{n_2(\frac{1}{2}-r_1)} \Big\} \leq$$
$$\leq C(2^n n^{\nu-1})^{-\frac{p}{2}(r_1-\frac{1}{2})}$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$ when $2 < p < \infty$, $1 < \tau_1 \leq 2$ and $1 < \tau_2 < \infty$, $r_1 < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2})$.

Therefore, according to Lemma 1, for the function F_3, by a constructive method, there is an M-term trigonometric polynomial $G_M(F_3)$ such that

$$\|F_3 - G_M(F_3)\|_{p,\tau_2} \leq CM^{-\frac{1}{2}}(2^n n^{\nu-1})^{-\frac{p}{2}(r_1-\frac{1}{2})} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})} \tag{31}$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$ for $2 < p < \infty$, $1 < \tau_1 \leq 2$, $1 < \tau_2 < \infty$, $r_1 < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2})$.

Let us estimate $\|F_2\|_{p,\tau_2}$. To do this, note that if $\overline{s} \notin G(l)$, then

$$\|\delta_{\overline{s}}(f)\|_2 \leq \tilde{m}_l^{-\frac{1}{2}} 2^{-lr_1} \tilde{S}_l \tag{32}$$

and

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(\sum_{l=n_1}^{n_2-1} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1, \bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2} \|\delta_{\overline{s}}(f)\|_2^{\tau_2}\Big)^{1/\tau_2} =$$
$$= C\Big(\sum_{l=n_1}^{n_2-1} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1, \bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2} \|\delta_{\overline{s}}(f)\|_2^{\tau_2-2} \|\delta_{\overline{s}}(f)\|_2^2\Big)^{1/\tau_2}.$$

Further, if $\tau_2 - 2 \geq 0$, then using inequality (32) and repeating the arguments of the proof (18), we obtain

$$\|F_2\|_{p,\tau_2} \leq C(2^n n^{\nu-1})^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})} \tag{33}$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$ when $q = 2 < p < \infty$, $1 < \tau_1 \leq 2 \leq \tau_2 < \infty$, $r_1 < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2})$.

Now inequalities (25), (31), (33) imply that

$$e_M(f)_{p,\tau_2} \leq \|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M^p(F_1) + G_M^p(F_3))\|_{p,\tau_2} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{q})}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}}$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$ when $2 < p < \infty$, $1 < \tau_1 \leq 2 \leq \tau_2 < \infty$, $r_1 < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2})$. The proof is complete.

Remark 1. In case when $\tau_1 = 2$, Theorem 1 complements Theorem 4 in [14].

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Author Contributions

G. Akishev and A.Kh. Myrzagaliyeva collected and analyzed data. Both authors participated in the revision of the manuscript and approved the final submission.

All authors contributed equally to this work.

Conflict of Interest

The authors declare no relevant financial or non-financial competing interests.

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Лоренц кеңiстiгiндегi Соболев класының M-мүшелiк жуықтауларын бағалау туралы

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Жұмыста бiрнеше айнымалы периодты функциялар кеңiстiктерi зерделенген, атап айтқанда Лоренц кеңiстiгi $L_{2,\tau}(\mathbb{T}^m)$, шектеулi аралас бөлшек туындысы бар функциялар класы $W^{\overline{r}}_{2,\tau}$, $1 \leq \tau < \infty$ және $f \in L_{p,\tau}(\mathbb{T}^m)$ функциясының тригонометриялық көпмүшелiктермен ең жақсы M-мүшелiк жуықтауларының ретi зерттелген. Мақала кiрiспеден, негiзгi бөлiмнен және қорытындыдан тұрады. Кiрiспеде негiзгi нәтижелердi дәлелдеу үшiн ұғымдар, анықтамалар және қажеттi тұжырымдар қарастырылған. Сонымен қатар, осы тақырып бойынша алдыңғы зерттеулер жайлы ақпаратты табуға болады. Негiзгi бөлiмде $W^{\overline{r}}_{2,\tau_1}$ Соболев класы функцияларының $L_{p,\tau_2}(\mathbb{T}^m)$ кеңiстiгiнiң нормасы бойынша p, τ_1, τ_2 параметрлерi арасындағы қатынастар үшiн ең жақсы M-мүшелiк жуықтауларының нақты реттiк бағалаулары анықталған.

Кiлт сөздер: Лоренц кеңiстiгi, Соболев класы, аралас туынды, тригонометриялық көпмүшелiктер, M-мүшелiк жуықтау.

Об оценках M-членных приближений класса Соболева в пространстве Лоренца

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В работе изучены пространства периодических функций нескольких переменных, а именно пространство Лоренца $L_{2,\tau}(\mathrm{T}^m)$, класс функций с ограниченной смешанной дробной производной $W^{\overline{r}}_{2,\tau}$, $1 \leq \tau < \infty$, и порядок наилучшего M-членного приближения функции $f \in L_{p,\tau}(\mathrm{T}^m)$ тригонометрическими полиномами. Статья состоит из введения, основной части и заключения. Во введении рассмотрены основные понятия, определения и необходимые утверждения для доказательства основных результатов. Также можно найти информацию о предыдущих результатах по этой теме. В основной части установлены точные по порядку оценки для наилучших M-членных приближений функций класса Соболева $W^{\overline{r}}_{2,\tau_1}$ по норме пространства $L_{p,\tau_2}(\mathrm{T}^m)$ для различных соотношений между параметрами p, τ_1, τ_2.

Ключевые слова: пространство Лоренца, класс Соболева, смешанная производная, тригонометрический полином, M-членное приближение.

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