https://doi.org/10.31489/2024M3/77-92 *Research article*

Controllability and Optimal Fast Operation of Nonlinear Systems

S. Aisagaliev, G. Korpebay*

Al-Farabi Kazakh National University, Almaty, Kazakhstan
(E-mail: Serikbai.Aisagaliev@kaznu.kz, korpebay.guldana1@gmail.com)

A new method for solving the problem of controllability and optimal transient behavior of nonlinear systems subject to boundary conditions and constraints on control values was proposed. Unlike existing methods, this new approach is based on constructing a general solution of the integral equation for a linear controlled system, followed by transforming the original problem into a special initial optimal control problem. We propose a new method for studying the global asymptotic stability of dynamical systems with a cylindrical phase space with a countable equilibrium position based on a non-singular transformation of the equation of motion and estimation of improper integrals along the solution of the system. Conditions for global asymptotic stability were obtained without involving any periodic Lyapunov function, as well as the frequency theorem. The effectiveness of the proposed method is shown with an example.

Keywords: optimal performance, integrity constraints, functional gradient, integral equation.

2020 Mathematics Subject Classification: 49K15, 49K20, 49K21.

Introduction

The first work on controllability of linear systems without constraints on control values is the paper by R.E. Kalman [1]. In [1], minimal norm control is constructed for systems with constant coefficients, and a rank criterion for controllability is established. Controllability of linear systems based on l-problem methods is explored in [2]. Various issues such as minimal control vector dimension, controllability of nonlinear systems with small parameters, and consequences of controllability for linear systems are discussed in [3]. Positional control of linear systems based on Lyapunov functions is examined in [4]. Geometric interpretations of controllability of linear systems are studied in [5], and the relationship between controllability and stabilization of dynamic systems is investigated in [6].

The problem of optimal transient performance was first studied by L.S. Pontryagin and his students [7]. Optimal fast operation under phase coordinate constraints is detailed in [8], and solutions under uncertainty conditions are considered in [9]. Applications of the maximum principle to various specific problems are presented in [10].

It is noteworthy that the problem of optimal fast operation is closely related to controllability. The aforementioned works explore specific cases of the general problems of controllability and fast operation without phase or integral constraints and without boundary condition restrictions. Current and unresolved issues in controllability and optimal fast operation include obtaining necessary and sufficient conditions for the solvability of general controllability and fast operation problems and developing constructive methods for solving general problems of controllability and fast operation of ordinary differential equations.

This paper proposes a new method for investigating controllability and optimal transient behavior of ordinary differential equations based on the study of solvability and the construction of a general solution of a Fredholm integral equation of the first kind with a fixed parameter.

*Corresponding author. *E-mail: korpebay.guldana1@gmail.com*

This research is supported by Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08856324).

Received: 27 February 2024; *Accepted:* 20 May 2024.

The solvability and construction of solutions of Fredholm integral equations of the first kind are among the complex and unresolved problems in mathematics [11]. Known results on the solvability of integral equations apply when the operator kernel is symmetric [12].

Results on solvability and construction of solutions of Fredholm integral equations of the first kind and their applications to the qualitative theory of differential equations are presented in [12, 13]. Specific results on applying the study of Fredholm integral equations of the first kind to solving problems of controllability and optimal control are found in [13]. A general theory of boundary value problems for dynamic systems is provided in [12], and research on the dynamics of processes described by integro-differential equations is detailed in [9].

The theory of controllability for nonlinear systems described by ordinary differential equations remains a relatively underexplored area in the mathematical theory of control. It is shown that the problem of controllability of ordinary differential equations, by constructing a general solution of a Fredholm integral equation of the first kind with a fixed parameter, can be reduced to an initial optimal control problem. Solutions to the problem of optimal fast operation can be derived from solving the general controllability problem.

1 Problem Statement

Consider a controlled process described by ordinary differential equations:

$$\dot{x} = A(t)x + B(t)\ f(x, u, t), \quad t \in \mathrm{I} = [t_0,\ t_1], \tag{1}$$

with boundary conditions

$$x(t_0) = x_0 \in R^n, \quad x(t_1) = x_1 \in R^n, \tag{2}$$

subject to control constraints

$$u(t) \in \Lambda(t) = \{u(t) \in L_2(I, R^{m_1}) | u(t) \in V(t) \subset R^{m_1} \ \text{ almost everywhere } \ t \in I\}. \tag{3}$$

Here, $A(t),\ B(t)$ are matrices with piecewise continuous elements of sizes $n \times n$ and $n \times m$, respectively. The vector function $f(x,\ u,\ t)$ is continuous in all variables $(x,\ u,\ t) \in R^n \times R^{m_1} \times I$, satisfying conditions

$$|f(x,\ u,\ t) - f(y,\ u,\ t)| \le l(t)|x - y|, \ \ \forall (x, u, t),\ (y,\ u,\ t) \in R^n \times R^{m_1} \times I, \tag{4}$$

$$|f(x,\ u,\ t)| \le c_0(|x| + |u|^2) + c_1(t), \ \ t \in I, \tag{5}$$

$$l(t) > 0, \ \ l(t) \in L_1(I,\ R^1), \ \ c_0 = const > 0, \ \ c_1(t) \ge 0, \ \ c_1(t) \in L_2\,(I,\ R^1). \tag{6}$$

From (4)–(6) it follows that differential equation (1) with initial condition $x(t_0) = x_0$, for any fixed control $u(t) \in L_2(I, R^{m_1})$, has a unique solution. Assume $\Lambda(t),\ t \in I$ is a given bounded convex closed set in $L_2(I,\ R^{m_1})$. In particular, if $A(t) \equiv 0,\ B(t) = I_n$, where I_n, is the $n \times n$, identity matrix, then equation (1) takes the form $\dot{x} = f(x,\ u,\ t)$.

Definition 1. The system (1)–(3) is called controllable, if there exists a control $u(t) \in \Lambda(t)$, that transforms the solution of differential equation (1) from initial state $x_0 = x(t_0)$ at time t_0 to state $x_1 = x(t_1)$ at time t_1.

Along with system (1)–(3), consider the linear controllable system

$$\dot{y} = A(t)y + B(t)\ w(t), \quad t \in I = [t_0, t_1], \tag{7}$$

$$y(t_0) = x_0 \in R^n, \ y(t_1) = x_1 \in R^n, \tag{8}$$

$$w(t) \in L_2(I,\ R^m). \tag{9}$$

The following problems are solved:

Problem 1. Find all control sets $U(t) \subset L_2(I,\ R^m)$, where each element $U(t)$ function $w(t) \in U(t)$ transforms the solution of differential equation (7) under conditions (8), (9) from initial point $x_0 = y(t_0)$ to point $x_1 = y(t_1)$.

Problem 2. Find control $u(t) \in \Lambda(t)$, that transforms the trajectory of system (1)–(3) from initial state $x_0 = x(t_0)$ at time t_0, to state $x_1 = x(t_1)$ at time t_1.

Problem 3. (Optimal Quick Action). Find control $u(t) \in \Lambda(t) \subset L_2(I,\ R^m)$ that moves the trajectory of system (1)–(3) from poin $x_0 = x(t_0)$ to point $x_1 = x(t_1)$ in the shortest time, where t_0 is fixed and t_1 is not fixed.

The problem of optimal quick action is formulated as

$$J(x,\ u,\ t_1) = \int_{t_0}^{t_1} 1 \cdot dt = t_1 - t_0\ \to \inf$$

subject to conditions (1)–(3).

2 Linear Controllable System

Consider solving Problem 1.

The solution of differential equation (7) takes the form

$$y(t) = \Phi(t, t_0)\ x_0 + \int_{t_0}^{t} \Phi(t, \tau) B(\tau) w(\tau) d\tau, \quad t \in I, \tag{10}$$

where $\Phi(t,\ \tau) = \theta(t)\theta^{-1}(\tau)$, $\theta(t)$ is the fundamental matrix of solutions of the linear homogeneous equation $\dot{\xi} = A(t)\xi$. Note that the matrix $\theta(t),\ t \in I$ of order $n \times n$ is a solution of the matrix equation $\dot{\theta}(t) = A(t)\ \theta(t),\ \theta(t_0) = I_n$, where I_n is the identity matrix of order $n \times n$. From (10) at $t \in t_1$, considering $y(t_1) = x_0$, we obtain

$$y(t_1) = x_1 = \Phi(t_1, t_0)\, x_0 + \int_{t_0}^{t_1} \Phi\,(t_1, t)\, B(t)\, w(t) dt.$$

Then

$$\int_{t_0}^{t_1} \Phi\,(t_1, t)\, B(t)\, w(t) dt = x_1 - \Phi(t_1, t_0)\, x_0.$$

Here, considering $\Phi(t_1, t) = \Phi(t_1, t_0)\ \Phi(t_0, t),\ \ \Phi^{-1}(t_1, t_0) = \Phi(t_0, t_1)$, we have

$$\int_{t_0}^{t_1} \Phi(t_0, t)\, B(t)\, w(t)\, dt = \Phi\,(t_0, t_1) x_1 - x_0. \tag{11}$$

Let

$$K(t) = \Phi(t_0, t) B(t),\ \ a = \Phi(t_0, t_1) x_1 - x_0,\ \ t \in I,\ \ a \in R^n. \tag{12}$$

From (11) it follows that the control $w(t) \in L_2(I,\ R^m)$ drives the trajectory of system (7)–(9) from any point x_0 to any point x_1, when $u(t)$ satisfies the integral equation (11). The following theorem establishes the necessary and sufficient condition for the solvability of integral equation (11) for any vector $a \in R^n$ from (12).

Theorem 1. The integral equation (11) has solutions for any vector $a \in R^n$ if and only if the matrix

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)\ B(t)\ B^*(t)\ \Phi^*(t_0, t)\, dt = \int_{t_0}^{t_1} K(t)\ K^*(t) dt, \tag{13}$$

of order $n \times n$ is positive definite, where (*) denotes transposition.

The proof of Theorem 1 can be found in reference [1]. The following two theorems present new results in the theory of controllability of linear systems.

Theorem 2. Suppose the matrix $W(t_0, t_1)$ defined by formula (13) is positive definite. Then the general solution of the integral equation (11) for any $a \in R^n$ is given by

$$w(t) = v(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)\ z(t_1,\ v) \in L_2(I, R^m), \tag{14}$$

where $v(t) \in L_2(I,\ R^m)$ is any function. The function $z(t) = z(t,\ v),\ \ t \in I$ is the solution of the differential equation

$$\dot{z} = A(t)z + B(t)\ v(t),\ \ z(t_0) = 0,\ \ v(t) \in L_2(I, R^m), \tag{15}$$

where

$$\lambda_1(t,\ x_0,\ x_1) = B^*(t)\ \Phi^*(t_0, t)\ W^{-1}(t_0, t_1)\ a,\ \ N_1(t) = -B^*(t)\ \Phi^*(t_0, t)\ W^{-1}(t_0, t_1)\ \Phi(t_0, t_1),\ \ t \in I. \tag{16}$$

Proof. Introduce the following sets

$$W = \{w\,(t) \in L_2(I, R^m)|\int_{t_0}^{t_1} K(t)\ w(t)\,dt = a\}, \tag{17}$$

$$U = \{w(t) \in L_2(I, R^m)|w(t) = v(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)\ z(t_1, v),\ \ v(t) \in L_2(I, R^m) - \text{any function}\}. \tag{18}$$

The set W contains all solutions of the integral equation (11), when $W(t_0, t_1) > 0$. The theorem asserts that a function $w(t) \in L_2(I,\ R^m)$ belongs to W if and only if it belongs to U. To prove $W = U$, it suffices to show $U \subset W$ and $W \subset U$.

Show $U \subset W$. Indeed, if $w(t) \in U$, then from (18) the equality

$$\int_{t_0}^{t_1} K(t)\,w(t)dt = \int_{t_0}^{t_1} K(t)\,[v(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)\ z(t_1,\ v)]dt = \int_{t_0}^{t_1} K(t)\ v(t)dt +$$

$$+ \int_{t_0}^{t_1} K(t)\ \lambda_1(t,\ x_0,\ x_1)dt + \int_{t_0}^{t_1} K(t)\ N_1(t)dt\ z(t_1,\,v) =$$

$$= \int_{t_0}^{t_1} K(t)\ v(t)d + \int_{t_0}^{t_1} K(t)\,B^*(t)\Phi^*(t_0, t)dt\ W^{-1}(t_0, t_1)\,a +$$

$$+ \int_{t_0}^{t_1} K(t)\,[-B^*(t)\ \Phi^*(t_0, t)]dt\ W^{-1}(t_0, t_1)\ \Phi(t_0, t_1)\ z(t_1,\ v).$$

Hence, considering that the solution of differential equation (15) has the form

$$z(t) = \Phi(t, t_0)\ z(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\ B(\tau)\ v(\tau)d\tau = \int_{t_0}^{t} \Phi(t, \tau)\ B(\tau)\ v(\tau)d\tau,$$

$$z(t_1) = \int_{t_0}^{t_1} \Phi(t_1, t)\ B(\tau)\ v(\tau)dt = \Phi(t_1, t_0)\int_{t_0}^{t} \Phi(t_0, t)\ B(t)\ v(t)dt,$$

we get $(K(t) = \Phi(t_0, t)\ B(t))$

$$\int_{t_0}^{t_1} K(t)\ w(t)dt = \int_{t_0}^{t_1} \Phi(t_0, t)\,B(t)\ v(t)\,dt + \int_{t_0}^{t} \Phi(t_0, t)\ B(t)\ B^*(t)\ \Phi^*(t_0, t)dt\ W^{-1}(t_0, t_1)\ a -$$

$$-\int_{t_0}^{t}\Phi(t_0,t)\;B(t)\;B^*(t)\Phi^*(t_0,t)dt\;W^{-1}(t_0,t_1)\;\Phi(t_0,t_1)\;\Phi(t_1,t_0)\;\int_{t_0}^{t_1}\Phi(t_0,t)\;B(t)\;v(t)\,dt=$$

$$=\int_{t_0}^{t_1}\Phi(t_0,t)\;B(t)\;v(t)\,dt+a-\int_{t_0}^{t_1}\Phi(t_0,t)\;B(t)\;v(t)\,dt=a.$$

Therefore, $w(t)\in W,\;\;U\subset W$.

Show that $W\subset U$. Suppose $w_*(\tau)\in W$. Then from (17) it follows that

$$\int_{t_0}^{t_1}K(t)\;w_*(t)dt=a.$$

Note that in relation (14), the function $v(t)\in L_2(I,\;R^m)$ is arbitrary. In particular, we can choose $v(t)=w_*(\tau),\;\;t\in I$. Now, the function $w(t)\in U$ can be expressed as.

$$w(t)=v(t)+\lambda_1(t,\;x_0,\;x_1)+N_1(t)\;z(t_1,\,v)=w_*(t)+B^*(t)\,\Phi^*(t_0,t)\,W^{-1}(t_0,t_1)a-$$

$$-B^*(t)\;\Phi^*(t_0,t)\;W^{-1}(t_0,t_1)\;\Phi(t_0,t_1)\;\Phi(t_1,t_0)\;\int_{t_0}^{t}\Phi(t_0,t)\,B(t)\;w_*(t)dt=w_*(t)+$$

$$+B^*(t)\,\Phi^*(t_0,t)\;W^{-1}(t_0,t_1)\,a-B^*(t)\;\Phi^*(t_0,t)\;W^{-1}(t_0,t_1)a\;=w_*(t)\in U.$$

Therefore, $w_*(\tau)=w(\tau)\in U$. Hence, $W\subset U$. From $U\subset W$ and $W\subset U$, it follows that $U=W$. The theorem is proved.

From (14)–(18), it follows that all control sets, each element of which transforms the trajectory of the system (7)–(9) from point x_0 to point x_1, are determined by formula (18).

Key properties of solutions to integral equation (11):

1. Function $w(t)\in U$ can be represented as $w(t)=w_1(t)+w_2(t)$, where $w_1(t)=K^*(t)W^{-1}(t_0,t_1)a$ is a particular solution of integral equation (11), and

$$w_2(t)=v(t)-K^*(t)W^{-1}(t_0,t_1)\int_{t_0}^{t_1}K(\eta)\,v(\eta)\,d\eta,\;\;t\in I$$

is a solution of the homogeneous integral equation.

$$\int_{t_0}^{t_1}K(t)\;w_2(t)dt=0.$$

Indeed,

$$\int_{t_0}^{t_1}K(t)\;w_1(t)dt=\int_{t_0}^{t_1}K(t)\;K^*(t)\;W^{-1}(t_0,t_1)\,a=a,$$

$$\int_{t_0}^{t_1}K(t)\;w_2(t)dt=\int_{t_0}^{t_1}K(t)\;v(t)dt-\int_{t_0}^{t_1}K(t)\;K^*(t)W^{-1}(t_0,t_1)\;\int_{t_0}^{t_1}K(\eta)\;v(\eta)d\eta=0.$$

2. Functions $w_1(t)\in L_2(I,\;R^m)$, $w_2(t)\in L_2(I,\;R^m)$ are orthogonal in L_2, $w_1\perp w_2$. Indeed,

$$\langle w_1,\;w_2\rangle_{L_2}=\int_{t_0}^{t_1}w_1^*(t)\;w_2(t)dt=a^*\,W^{-1}(t_0,t_1)\int_{t_0}^{t_1}K(t)\;v(t)dt-$$

$$-a^*\,W^{-1}(t_0,t_1)\int_{t_0}^{t_1}K(t)K^*(t)W^{-1}(t_0,t_1)\;\int_{t_0}^{t_1}K(\eta)\;v(\eta)d\eta=0.$$

3. Function $w_1(t)=K^*(t)W^{-1}(t_0,t_1)a,\;\;t\in I$, is a solution of integral equation (11) with minimal norm in $L_2(I,\;R^m)$. Indeed, $\|w\|^2\geq\|w_1\|^2+\|w_2\|^2$, due to $w_1\perp w_2$. Hence, $\|w\|^2\geq\|w_1\|^2$. If the

function $v(t) \equiv 0,\ t \in I$, then the function $w_2(t) \equiv 0,\ t \in I$. Therefore $\|w\| = \|w_1\|,\ w(t) = w_1(t)$, $t \in I$.

4. The set of solutions of integral equation (11) is convex. Since $w(t) \in U$, U is a convex set.

Theorem 3. Let the matrix $W(t_0, t_1) > 0$. Then the solution of the differential equation (7) corresponding to the control $w(t) \in U$ is determined by the formula

$$y(t) = z(t_1,\ v) + \lambda_2(t,\ x_0,\ x_1) + N_2(t)z(t_1,\ v),\ \ t \in I,\ \ \forall v,\ \ v(t) \in L_2(I, R^m), \tag{19}$$

where

$$\begin{aligned} &\lambda_2(t,\ x_0,\ x_1) = \Phi(t,t_0)\,W(t,t_1)\,W^{-1}(t_0,t_1)x_0 + \Phi(t,t_0)W(t_0,t)\,W^{-1}(t_0,t_1)\Phi(t_0,t_1)x_1,\\ &N_2(t) = -\Phi(t,t_0)\,W(t_0,t)\,W^{-1}(t_0,t_1)\Phi(t_0,t_1),\ \ t \in I,\\ &W(t_0,t) = \textstyle\int_{t_0}^{t} K(\tau)\,K^*(\tau)d\tau,\ \ W(t,t_1) = \int_{t}^{t_1} K(\tau)\,K^*(\tau)d\tau,\ \ t \in I. \end{aligned} \tag{20}$$

Proof. Suppose the control is determined by formula (14). Then the function.

$$y(t) = \Phi(t,t_0)\ x_0 + \int_{t_0}^{t} \Phi(t,\tau)\ B(\tau)[v(\tau) + \lambda_1(\tau,\ x_0,\ x_1) + N_1(\tau)\ z(t_1,\ v)]d\tau =$$

$$= \int_{t_0}^{t} \Phi(t,\tau)\ B(\tau)\ v(\tau)\ d\tau + \Phi(t,t_0)\ x_0 + \int_{t_0}^{t} \Phi(t,\tau)\ B(\tau)\ B^*(\tau)\ \Phi^*(t_0,\tau)\,d\tau\ \ W^{-1}(t_0,t_1),$$

$$[\Phi(t_1,t_0)x_1 - x_0] - \int_{t_0}^{t} \Phi(t,\tau)\ B(\tau)\ B^*(\tau)\ \Phi^*(t_0,\tau)\,d\tau\ \ W^{-1}(t_0,t_1)\ \Phi(t_1,t_0)\ z(t_1,\ v).$$

Thus, considering that

$$W(t_0,t) = \int_{t_0}^{t} K(\tau)\,K^*(\tau)d\tau = \int_{t_0}^{t} \Phi(t_0,\tau)\,B(\tau)\ B^*(\tau)\ \Phi^*(t_0,\tau)d\tau,\quad W(t,t_1) = W(t_0,t_1) - W(t_0,t),$$

we obtain

$$y(t) = z(t,\ v) + [\Phi(t,t_0) - \Phi(t,t_0)\ W(t_0,t)W^{-1}(t_0,t_1)]\ x_0 + \Phi(t,t_0)\ W(t_0,t)\ W^{-1}(t_0,t_1)\ \Phi(t_1,t_0)\ x_1 -$$

$$-\Phi(t,t_0)\ W(t_0,t)\ W^{-1}(t_0,t_1)\ \Phi(t_1,t_0)\ z(t_1,\ v) = z(t,\ v) + \Phi(t,t_0)\ W(t,t_1)\ W^{-1}(t_0,t_1)\ x_0 +$$

$$+\Phi(t,t_0)\ W(t_0,t)\ W^{-1}(t_0,t_1)\ \Phi(t_0,t_1)\,x_1 - \Phi(t,t_0)\,W(t_0,t)W^{-1}(t_0,t_1)\ \Phi(t_0,t_1)z(t_1,\ v) =$$

$$= z(t,\,v) + \lambda_2(t,\ x_0,\ x_1) + N_2(t)z(t_1,\,v),$$

where $\lambda_2(t,\ x_0,\ x_1),\ N_2(t),\ t \in I,$ are from (20). The theorem is proved.

3 Controllability of Nonlinear Systems

Consider the solution to problem 2.

Comparing systems (1)–(3) and (7)–(9), it is easy to see that they coincide when replacing the function $w(t)$ with $f(x,\,u,\,t)$. This leads to considering the following optimization problem: minimize the functional

$$J(v,u) = \int_{t_0}^{t_1} |v(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)z(t_1,v) - f(y(t),u(t),t)|^2 dt \to \inf, \tag{21}$$

subject to the constraints

$$\dot{z} = A(t)z + B(t)v(t),\quad z(t_0) = 0,\ \ t \in I = [t_0,t_1], \tag{22}$$

$$v(t) \in L_2(I, R^m),\ u(t) \in \Lambda(t) \subset L_2(I, R^{m_1}), \tag{23}$$

where the function $y(t),\ t \in I$, is determined by formula (19).

Theorem 4. Suppose the matrix $W(t_0, t_1) > 0$. Then the system (1)–(3) is controllable if and only if the value $J(v_*, u_*) = 0$, where the pair $(v_*(t), u_*(t)) \in L_2(I, R^m) \times \Lambda(t)$ is the optimal control in problem (21)–(23).

Proof. Necessity. Suppose the system (1)–(3) is controllable. We will show that $J(v_*, u_*) = 0$. From the controllability of the system (1)–(3), it follows that there exists a solution to the differential equation (1) the function $x(t) = x(t;\ t_0, x_0, u_*),\ t \in I$, such that $x(t_0) = x_0$, $x(t_1) = x_1$ for $u_* = u_*(t),\ t \in I$. Then $f(x(t;\, t_0,\, x_0,\, u_*),\, u_*(t), t) = w_*(t) \in L_2(I, R^m)$, and the system (1)–(3) can be written as $(x(t) = x(t;\ t_0, x_0, u_*))$.

$$\dot{x}(t; t_0, x_0, u_*) = A(t)x(t; t_0, x_0, u_*) + B(t)\ w_*(t),\ \ t \in I = [t_0, t_1],$$

$$x(t_0; t_0, x_0, u_*) = x_0,\ \ x(t_1; t_0, x_0, u_*) = x_1,\ \ u_*(t) \in L_2(I, R^m).$$

Let $y(t) = x(t;\, t_0, x_0, u_*),\ t \in I$. The function $y(t),\ t \in I$ satisfies $\dot{y} = A(t)y + B(t)w_*(t),\ y(t_0) = x_0$, $y(t_1) = x_1$. Therefore, the function $w_*(t) \in L_2(I, R^m)$ translates the trajectory $y(t),\ t \in I$ from the point x_0 the point x_1. According to Theorem 1, $w_*(t) \in U$, where $w_*(t) = v_*(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)z(t_1, v_*),\ \ t \in I$. Thus,

$$J(v_*, u_*) = \int_{t_0}^{t_1} |v_*(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)z(t_1, v_*) - f(y(t),\ u_*(t), t)|^2 dt = 0.$$

Necessity is proved.

Sufficiency. Let the functional value $J(v_*, u_*) = 0$, for the pair $(v_*(t),\ u_*(t)) \in L_2(I, R^m) \times \Lambda(t)$. We will demonstrate that the system (1)–(3) is controllable. Note that $J(v, u) \geq 0$. Hence, $J(v_*, u_*) = 0$ if and only if

$$v_*(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)\ z(t_1, v_*) = f(y(t,\ v_*),\ u_*(t), t),\ \ t \in I,$$

where we denote

$$w_*(t) = v_*(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)\ z(t_1, v_*) = f(y(t,\ v_*),\ u_*(t), t),\ \ t \in I,$$

with $y(t_0,\ v_*) = x_0,\ \ y(t_1,\ v_*) = x_1$. Now the system (7)–(9) can be written as

$$\dot{y}(t, v_*) = A(t)y(t, v_*) + B(t)\ w_*(t),\ \ y(t_0) = x_0,\ \ y(t_1) = x_1,\ \ w_*(t) \in L_2(I, R^m).$$

From this, it follows that $y(t,\ v_*) = x(t;\ t_0, x_0, u_*),\ x(t_0) = x_0,\ x(t_1) = x_1$. Therefore, system (1)–(3) is controllable. Sufficiency is proven. The theorem is proved.

Below are solutions to the optimization problem (21)–(23). It should be noted that: 1) in the optimization problem (21)–(23), unlike the original boundary value problem (1)–(3), boundary conditions are absent; 2) the optimization problem (21)–(23) is an initial problem of optimal control and can be solved using known methods of successive approximations.

Let us introduce the following notations:

$$F_0(q_0, t) = |v + T_1(t)\, x_0 + T_2(t)\, x_1 + N_1(t)\ z(t_1, v) - f(y, u.t)|^2\,, \tag{24}$$

where

$$\lambda_1(t,\ x_0,\ x_1) = T_1(t)\, x_0 + T_2(t)\, x_1,$$

$$T_1(t) = -B^*(t)\Phi^*(t_0, t)\, W^{-1}(t_0, t_1),\ T_2(t) = B^*(t)\Phi^*(t_0, t)\, W^{-1}(t_0, t_1)\ \Phi^*(t_0, t_1),$$

$$\lambda_2(t,\ x_0,\ x_1) = C_1(t)\,x_0 + C_2(t)\,x_1,$$

$$C_1(t) = \Phi(t,t_0)\,W(t,t_1),\ W^{-1}(t_0,t_1),\ C_2(t) = \Phi(t,t_0)\,W(t_0,t),\ W^{-1}(t_0,t_1)\ \Phi(t_0,t_1),$$

$$y(t) = z(t,v)\ + C_1(t)\,x_0 + C_2(t)\,x_1 + N_2(t)\,z(t_1,v),\ t \in I,$$

$$q = (v,u,z,\ z(t_1)) \in R^m \times R^{m_1} \times R^n \times R^n.$$

Lemma 1. Suppose matrix $W\,(t_0,t_1) > 0$, the function $f(y,\ u,\ t)$ is defined and continuous with respect to $(y,u,t) \in R^n \times R^{m_1} \times I$ together with partial derivatives with respect to $(y,u) \in R^n \times R^{m_1}$. Then the partial derivatives are

$$\frac{\partial\ F_0(q,t)}{\partial v} = 2[v + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1) - f(z + C_1(t) + C_2(t) + N_2(t)z(t_1),\ u,\ t)], \quad (25)$$

$$\frac{\partial\ F_0(q,t)}{\partial u} = -2f_u(y,u,t)[v + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1) - f(y,\ u,\ t)], \quad (26)$$

$$\frac{\partial\ F_0(q,t)}{\partial z} = -2f_x(y,u,t)[v + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1) - f(y,\ u,\ t)], \quad (27)$$

$$\frac{\partial\ F_0(q,t)}{\partial z(t_1)} = 2[N_1^*(t) + N_2^*(t)f_x(y,\ u,\ t)][v + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1) - f(y,\ u,\ t)]. \quad (28)$$

Relations (25)–(28) are derived directly from (24) by differentiation.

Lemma 2. Suppose the conditions of Lemma 1 hold and the inequality

$$\langle F_{0q}(q_1,t) - F_{0q}(q_2,t),\ q_1 - q_2\rangle \geq 0,\ \ \forall q_1,\ q_2 \in R^{m+m_1+2n}, \quad (29)$$

is satisfied, where

$$F_{0q}(q,t) = \frac{\partial\,F_0(q,t)}{\partial q} = (\frac{\partial\,F_0}{\partial v},\ \frac{\partial\,F_0}{\partial u},\ \frac{\partial\,F_0}{\partial z},\frac{\partial\,F_0}{\partial z(t_1)}),\ \ t \in I.$$

Then the functional (21) under conditions (22), (23) is convex.

Proof. Inequality (29) is a necessary and sufficient condition for the convexity of the function $F_0(q,t)$ with respect to q. Therefore,

$$F_0(\alpha q_1 + (1-\alpha)q_2)\ \leq \alpha F_0(q_1,t) + (1-\alpha)F_0(q_2,t),\ \ t \in I,$$

$$\forall q_1,\ q_2 \in R^N,\ \ N = m_1 + m + 2n,\ \ \forall \alpha, \alpha \in [0,1].$$

Since for any $v_1(t),\ v_2(t) \in L_2(I,R^m)$, the value $z(t,\alpha\,v_1 + (1-\alpha)\,v_2) = \alpha\,z(t,\ v_1) + (1-\alpha)\,z\,(t,\ v_2)$, $\forall \alpha, \alpha \in [0,1],\ \ t \in I$, then

$$J(\alpha\,v_1 + (1-\alpha)\,v_2,\ \alpha\,u_1 + (1-\alpha)\,u_2) = \int_{t_0}^{t_1} F_0(\alpha\,v_1 + (1-\alpha)\,v_2,\ \alpha\,u_1 + (1-\alpha)\,u_2),$$

$$z(t,\alpha\,v_1 + (1-\alpha)\,v_2),\ z(t_1,\ \alpha v_1 + (1-\alpha)\,v_2))dt \leq \alpha \int_{t_0}^{t_1} F_0(q_1,t)dt + (1-\alpha)\int_{t_0}^{t_1} F_0(q_2,t)dt =$$

$$= \alpha\,J\,(v_1,u_1) + (1-\alpha)J\,(v_2,u_2),\ \ \forall\,v_1,\ v_2 \in L_2(I,R^m),\ \ \forall\,u_1,\ u_2 \in L_2(I,R^{m_1}).$$

Thus, the lemma statement follows. Lemma is proved.

Definition 2. The partial derivatives (25)–(28) are said to satisfy the Lipschitz condition if

$$\begin{vmatrix} \frac{\partial\,F_0(q+\Delta q,t)}{\partial v} - \frac{\partial\,F_0(q,t)}{\partial v} \\ \frac{\partial\,F_0(q+\Delta q,t)}{\partial z} - \frac{\partial\,F_0(q,t)}{\partial z} \end{vmatrix} \begin{matrix} \leq L_1\,|\Delta q|\,, \\ \leq L_3\,|\Delta q|\,, \end{matrix} \qquad \begin{vmatrix} \frac{\partial\,F_0(q+\Delta q,t)}{\partial u} - \frac{\partial\,F_0(q,t)}{\partial u} \\ \frac{\partial\,F_0(q+\Delta q,t)}{\partial z(t_1)} - \frac{\partial\,F_0(q,t)}{\partial z(t_1)} \end{vmatrix} \begin{matrix} \leq L_2\,|\Delta q|\,, \\ \leq L_4\,|\Delta q|\,, \end{matrix} \quad (30)$$

where $L_i = const > 0, \ \ i = \overline{1,4}, \Delta q = (\Delta v, \ \Delta u, \ \Delta z, \ \Delta z(t_1))$.

Theorem 5. Suppose the conditions of Lemma 1 and inequalities (30). Then the functional (21) under conditions (22), (23) is continuously differentiable in the Frechet sense, and the gradient

$$J'(v,u) = (J'_v(v,u), \ (J'_u(v,u)) \in L_2(I, R^m) \times L_2(I, R^{m_1})$$

at any point $(v,u) \in L_2(I, R^m) \times L_2(I, R^{m_1})$ is defined by

$$J'_v(v,u) = \frac{\partial F_0(q(t),t)}{\partial v} - B^*(t)\psi(t), \quad J'_u(v,u) = \frac{\partial F_0(q(t),t)}{\partial u}, \tag{31}$$

where $q(t) = (v(t), \ u(t), \ z(t,v), \ z(t_1,v))$, the function $z(t) = z(t,v), \ t \in I$ is a solution of differential equation (22), and $\psi(t), \ t \in I$ is a solution of equation

$$\dot{\varphi} = \frac{\partial F_0(q(t),t)}{\partial z} - A^*(t)\psi, \quad \psi(t_1) = -\int_{t_0}^{t_1} \frac{\partial F_0(q(t),t)}{\partial z(t_1)} dt. \tag{32}$$

Moreover, the gradients $J'(v,u)$ satisfy the Lipschitz condition

$$\begin{aligned} &\|J'(v_1,u_1) - J'(v_2,u_2)\| \le l_1(\|v_1 - v_2\|^2 + \|u_1 - u_2\|^2)^{1/2}, \\ &\forall (v_1,v_2) \in L_2(I,R^m), \quad \forall (u_1,u_2) \in L_2(I,R^{m_1}). \end{aligned} \tag{33}$$

Proof. Note that for any $v(t)$, $v(t) + h(t) \in L_2(I, R^m)$, $\Delta z(t) = z(t, v+h) - z(t,v)$ satisfies the differential equation

$$\Delta \dot{z}(t) = A(t)\Delta z(t) + B(t)h(t), \ \Delta z(t_0) = 0, \ \ t \in I,$$

where

$$\Delta z(t) = \int_{t_0}^{t} \Phi(t,\tau) \ B(\tau) \ h(\tau)d\tau, \quad |\Delta z(t)| \le \int_{t_0}^{t_1} \|\Phi(t,\tau)\| \ \|l(\tau)\| \ |h(\tau)|d\tau \le c_1 \|h\|_{L_2}.$$

The increment of the functional

$$\Delta J = J(v+h, \ u + \Delta u) - J(v,u) = \int_{t_0}^{t_1} [h^*(\tau) F_{ov}(q(t),t) + \Delta u^*(t) F_{ou}(q(t),t) +$$

$$+ z^*(t) F_{oz}(q(t),t) + \Delta z^*(t_1) F_{oz(t_1)}(q(t),t)]dt + R,$$

where $|R| \le c_2(\|h\|^2 + \|\Delta u\|^2)$, due to estimate (30),

$$F_{0v}(q,t) = \frac{\partial F_0(q,t)}{\partial v}, \ \ F_{0u}(q,t) = \frac{\partial F_0(q,t)}{\partial u},$$

$$F_{0z}(q,t) = \frac{\partial F_0(q,t)}{\partial z}, \ \ F_{0z(t_1)}(q,t) = \frac{\partial F_0(q,t)}{\partial z(t_1)}.$$

The term

$$\Delta z^*(t_1) \int_{t_0}^{t_1} F_{oz(t_1)}(q(t),t) = -\int_{t_0}^{t_1} \Delta z^*(t) \ \psi(t)dt - \int_{t_0}^{t_1} \Delta z^*(t) \ \dot{\psi}(t)dt =$$

$$= -\int_{t_0}^{t_1} h^*(t) \ B^*(t) \ \psi(t) - \int_{t_0}^{t_1} \Delta z^*(t) F_{oz}(q(t),t)dt.$$

Thus, the increment of the functional

$$\Delta J = \int_{t_0}^{t_1} \{h^*(t) \ [F_{ov}(q(t),t) - B^*(t) \ \psi(t)] + \Delta u^*(t) \ F_{ou}(q(t),t)\}dt + R.$$

From here, the first statement (31) of the theorem follows. Let's show that estimate (33), where $\psi(t),\ t \in I$ is a solution of differential equation (32).

Let $\xi(t) = (v(t),\ u(t)),\ t \in I$. Then,

$$J'(\xi_1) - J'(\xi_2) = (F_{ov}(q(t) + \Delta q(t), t) - F_{ov}(q(t), t) - B^*(t)\ \Delta\psi(t),$$

$$F_{ou}(\, q(t) + \Delta q(t), t) - F_{ou}(q(t), t)),\quad \xi_1 = (v_1,\ u_1),\ \ \xi_2 = (v_2,\ u_2).$$

Therefore,

$$|J'(\xi_1) - J'(\xi_2)| = |F_{ov}(q(t) + \Delta q(t), t) - F_{ov}(q(t), t)| + B^*_{\max}\ |\Delta\psi(t)| +$$

$$+ |F_{ou}(\, q(t) + \Delta q(t), t) - F_{ou}(q(t), t)| \leq (L_1 + L_2)\ |\Delta q(t), t)| + B^*_{\max}\ |\Delta\psi(t)|,$$

where $B^*_{\max} = \sup\limits_{t_0 \leq t \leq t_1} \|B^*(t)\|$. Norm

$$\|J'(\xi_1) - J'(\xi_2)\|^2 = \int_{t_0}^{t_1} |J'(\xi_1) - J'(\xi_2)|^2 dt \leq 2\ (L_1 + L_2)\ \int_{t_0}^{t_1} |\Delta q(t)|^2 dt + 2(B^*_{\max})^2 \int_{t_0}^{t_1} |\Delta\psi(t)|^2 dt \leq$$

$$\leq 2\, c_3^2\, (L_1 + L_2)\ \|\Delta\xi\|^2 + 2\, (B^*_{\max})^2 \int_{t_0}^{t_1} |\Delta\psi(t)|^2 dt,$$

where $\|\Delta q\| \leq c_3\, \|\Delta\xi\|^2,\ \|\Delta\xi\|^2 = (\|h\|^2 + \|\Delta u\|^2),\ \ \Delta\xi = (h,\ \Delta u)$. It can be shown that $|\Delta\psi(t)| \leq (L_4\, c_3\ \sqrt{t_1 - t_0} + L_3\, c_3\ \sqrt{t_1 - t_0})\ e^{A^*_{\max}(t_1 - t_0)} \|\Delta\xi\|,\ \ t \in I$, where $A^*_{\max} = \sup\limits_{t_0 \leq t \leq t_1} \|A^*(t)\|$. Then

$\|J'(\xi_1) - J'(\xi_2)\|^2 \leq l_1^2\ \|\Delta\xi\|^2$, where

$$l_1 = [2\, c_3^2\, (L_1 + L_2)^2 + 2\, (B^*_{\max})^2\, (t_1 - t_0)^2 (L_3 + L_4)^2\ c_3^2\, e^{A^*_{\max}(t_1 - t_0)}]^{1/2}.$$

Hence, estimate (33) is proven. Theorem is proved.

Theorem 6. Suppose the conditions of Theorem 5 are satisfied, and the sequences $\{v_n\} \subset L_2(I, R^m)$, $\{u_n\} \subset \Lambda(t) \subset L_2(I, R^{m_1})$ are defined by relations

$$\begin{array}{l} v_{n+1} = v_n - \alpha_n\, J'_v(v_n,\, u_n),\ \ u_{n+1} = P_\Lambda[u_n - \alpha_n\, J'_u(v_n,\, u_n)],\quad n = 0,\, 1,\, 2,\, \ldots\, . \\ 0 < \varepsilon_0 \leq \alpha_n \leq \frac{2}{l_1 + 2\varepsilon_1},\ \ \varepsilon_1 > 0,\ \ n = 0,\, 1,\, 2,\, \ldots\, , \end{array} \tag{34}$$

where $P_\Lambda[\cdot]$ is the projection of a point onto the set Λ. Then:

1) The numerical sequence $\{J(v_n,\, u_n)\}$ strictly decreases;

2) $\|v_n - v_{n+1}\| \to 0, \quad \|u_n - u_{n+1}\| \to 0$ as $n \to 0$.

If, in addition, inequality (29), is satisfied, the set $M(v_0, u_0) = \{(v, u) \in L_2(I, R^m) \times \Lambda(t) | J(v, u) \leq J(v_0, u_0)\}$ is bounded, then

3) The sequences $\{v_n\}$, $\{u_n\}$ are minimizing sequences,

$$\lim_{n \to \infty} (v_n, u_n) = J_* = \inf J(v, u),\ \ (v, u) \in X \in L_2(I, R^m) \times \Lambda(t);$$

4) The sequences $\{v_n\}$, $\{u_n\}$, weakly converge to the set U_*, where

$$U_* = \{(v_*, u_*) \in X | J(v_*, u_*) = J_* = \inf J(v, u) = \min J\,(v, u),\ \ (v, u) \in X\};$$

5) The rate of convergence estimate is valid:

$$0 \leq J(v_n, u_n) - J_* \leq \frac{m_0}{n},\ \ n = 1,\, 2,\, \ldots\, ,\ \ m_0 = const > 0;$$

6) The controllability problem (1), (2), (4) has a solution if and only if $J(v_*,u_*)=J_*=0$, in which case $x_*(t)=z(t,\,v_*)+\lambda_2(t,\,x_0,\,x_1)+N_2(t)\;z(t_1,\,v_*),\;\;t\in I$;

7) If $J(v_*,u_*)>0$, это $x_*(t),\;\;t\in I$ is the best approximate solution to the controllability problem (1), (2), (4).

Proof. From the property of projection onto sets (34), we have

$$\langle v_{n+1}-v_n+\alpha_n\,J'_v\,(v_n,u_n),\;v-v_{n+1}\rangle_{L_2}=0,\;\;\forall v,\;\;v\in L_2(I,R^m) \tag{35}$$

$$\langle u_{n+1}-u_n+\alpha_n\,J'_u\,(v_n,u_n),\;u-u_{n+1}\rangle_{L_2}\geq 0,\;\;\forall u,\,u\in\Lambda. \tag{36}$$

Let $\theta=(v,\,u),\;\;\theta_n=(v_n,u_n),\;\;J'(v_n,u_n)=(J'_v(v_n,u_n),\;\;J'_u(v_n,u_n))$. Then (35), (36) can be written as

$$\langle J'(\theta_n),\;\theta-\theta_{n+1}\rangle_{L_2}\geq\frac{1}{\alpha_n}\langle\theta_n-\theta_{n-1},\;\theta-\theta_{n-1}\rangle,\;\;\forall\theta,\;\theta\in X. \tag{37}$$

From the inclusion $J(v,u)\in C^{1,1}(X)$ the inequality

$$J(\theta^1)-J(\theta^2)\geq\langle J'(\theta^1),\;\theta^1-\theta^2\rangle_H-\frac{l_1}{2}\|\theta^1-\theta^2\|^2,\;\;\forall\theta^1,\;\theta^2\in X.$$

Therefore, specifically for $\theta^1=\theta_n,\;\;\theta^2=\theta_{n+1}$, we obtain

$$J(\theta_n)-J(\theta_{n-1})\geq\langle J'(\theta_n),\;\theta_n-\theta_{n+1}\rangle-\frac{l_1}{2}\;\|\theta_n-\theta_{n-1}\|^2. \tag{38}$$

From (37), (38), (34), we have

$$J(\theta_n)-J(\theta_{n-1})\geq(\frac{1}{\alpha_n}-\frac{l_1}{2})\;\|\theta_n-\theta_{n-1}\|^2\geq\varepsilon_1\;\|\theta_n-\theta_{n-1}\|^2,\;\;n=0,\,1,\,2,\,... \tag{39}$$

From here, statements 1) and 2) of the theorem follow.

If inequality (29), is satisfied, then the functional (21) under conditions (22), (23) is convex, the set $M(v_0,u_0)$ is bounded, closed, and convex in H. Therefore, the set $M(v_0,u_0)$ is weakly precompact. The functional $J(v,u)$ is weakly lower semicontinuous on the set $M(v_0,u_0)$ and achieves its infimum, $U_*\neq\oslash,\;\;\oslash$ empty set.

Let's show that the sequence $\{\xi_n\}=\{v_n,\,u_n\}$ is minimizing. Indeed, from the convexity of $J(\xi)\in C^{1,1}(M(v_0,u_0))$, it follows that

$$J(\xi_n)-J(\xi_*)\leq\langle J'(\xi_n),\;\xi_n-\xi_*\rangle_H\leq\|J'(\xi_n)\|\;\|\xi_n-\xi_*\|\leq\|J'(\xi_n)\|\;D, \tag{40}$$

where $\xi_*=(v_*,\;\;u_*)\in U_*\subset M(v_0,u_0),\;\;D$ is diameter of $M(v_0,u_0)$.

From (40), it follows that the sequence $\{\xi_n\}\subset M(\xi_0)$ is minimizing, and $\xi_n\overset{weak}{\to}\xi_*$ weakly as $n\to\infty$, where $\xi_n\overset{weak}{\to}\xi_*$ as $n\to\infty$ means a special convergence of the sequence $\{\xi_n\}$ to an element ξ_*. Thus, statements 3) and 4) are proven.

Let $a_n=J(\xi_n)-J(\xi_*)$. Then from (39), (40) we have

$$a_n-a_{n-1}\geq\frac{1}{2l_1}\|J'(\xi_n)\|^2,\;\;a_n\leq D\;\|J'(\xi_n)\|. \tag{41}$$

From (41) the rate of convergence estimate 5) follows. The theorem is proven.

Optimal Performance. Let t_0 be fixed, t_1 be unfixed. It is necessary to find the smallest value $t_1=t_*$, for which the system (1), (2), (4) is controllable. It is necessary to find a pair $(t_*,\;u_*(t))$, where $u_*(t)\in\Lambda(t)\subset L_2(I,R^{m_1})$.

I. Setting $t_1 > t_*,$. Using the algorithm outlined above, we find the control $u_{*t_1}(t)$, where $t_0,\ t_1$ are known quantities.

Next, we choose $t_{11} = \frac{t_1}{2}$. We find a pair $(v_{**},\ u_{**}) \in X,\ \ t \in [t_0,\ t_{11}]$. If $J(v_{**},\ u_{**}) = 0$, for this pair, then we choose $t_{12} = \frac{t_1}{4},\ \ t_{12} < t_{11}$ and solve optimization problem (41).

In case where $J(v_{**},\ u_{**}) > 0$, optimization problem (41) is solved for $\frac{3t_1}{4}$ and so on. As a result, the value t_* is determined with the given accuracy $\varepsilon = t_{1n} - t_*$.

II. Sequential Approximation Method. Consider the following optimization problem: minimize the functional

$$J(v,u,t_1) = \int_{t_0}^{t_1} |v(t) + \lambda_1(t,\ x_0,\ x_1) + N_1(t)z(t_1,v) - f(y(t),\ u(t),t)|^2 dt = \int_{t_0}^{t_1} F_0(q(t),\ t_1,\ t)dt \to \inf$$

subject to conditions (42), (43), $t_1 > t_0$. Find Frechet derivatives, $J'_v(v,u,t_1),\ \ J'_u(v,u,t_1)$,

$$J'_{t_1}(v,u,t_1) = F_0(q(t_1),\ t_1,\ t_1) + \int_{t_0}^{t_1} \frac{\partial F_0(q(t),t_1,\ t)}{\partial t_1} dt.$$

Next, we construct sequences $\{v_n\},\ \ \{u_n\},\ \ \{t_{1n}\}$, where

$$t_{1n+1} = t_{1n} - \alpha_n J'_{t_1}(v_n,\ \ u_n,\ \ t_{1n}),\ \ n = 0,\ 1,\ 2,\ ...$$

4 Solution of the Model Problem

As an example, consider the Duffing equation with control [12].

$$\ddot{x} + x + 2x^3 = u(t),\ \ t \in I = [0,\ t_1].$$

This equation can be represented as

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 - 2x_1^3 + u(t),\ \ t \in [0,\ t_1] = I, \tag{42}$$

where

$$x_1(0) = 1,\ \ x_2(0) = 0,\ x_1(t_1) = 0,\ \ x_2(t_1) = 0, \tag{43}$$

$$u(t) \in \Lambda = \{u(t) \in L_2(I, R^1)| - 2 \le u(t) \le +2 \text{ almost everywhere } t \in I\}. \tag{44}$$

The system (42)–(44) is a mathematical model describing the motion of a rigid spring under the influence of external force $u(t) \in \Lambda$. Consider the problem of optimal performance. For (42)–(44), the linear controllable system takes the form

$$\dot{y}_1 = y_2,\ \dot{y}_2 = w(t),\ \ t \in [0,\ t_1] = I,\ u(t) \in \Lambda,$$

$$y_1(0) = 1,\ \ y_2(0),\ y_1(t_1) = 0,\ \ y_2(t_1) = 0.$$

For this example,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\quad y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x_0,\quad y(t_1) = x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Matrices

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\quad e^{-At} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix},\quad \theta(t) = e^{At},\ \ \Phi(t,\ \tau) = e^{A(t-\tau)}.$$

Calculate the following vectors and matrices:

$$a = \Phi(\tau, t_1)x_1 - x_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad W(0,t_1) = \int_0^{t_1} e^{-At} BB^* e^{-A^*\tau} d\tau = \begin{pmatrix} \frac{t_1^3}{3} & -\frac{t_1^2}{2} \\ -\frac{t_1^2}{2} & t_1 \end{pmatrix} > 0,$$

$$W^{-1}(0,t_1) = \begin{pmatrix} \frac{12}{t_1^3} & \frac{6}{t_1^2} \\ \frac{6}{t_1^2} & \frac{4}{t_1} \end{pmatrix}, \quad \lambda_1(t, x_0, x_1) = T_1(t)x_0 + T_2(t)x_1 = \frac{12}{t_1^3} - \frac{6}{t_1^2},$$

$$N_1(t) = \left(\frac{12}{t_1^3} - \frac{6}{t_1^2}, \ -\frac{6t}{t_1^2} - \frac{2}{t_1}\right), \quad \lambda_2(t, x_0, x_1) = \begin{pmatrix} \frac{t_1^3 + 2t^3 - 3t_1 t^2}{t_1^3} \\ \frac{6t^2 - 6t\, t_1}{t_1^3} \end{pmatrix},$$

$$N_2(t) = \begin{pmatrix} \frac{2t^3 - 3t^2 t_1}{t_1^3} & \frac{-t^3 + t_1 t^2}{t_1^2} \\ \frac{6t^3 - 6t\, t_1}{t_1^3} & \frac{-3t^2 + 2t\, t_1}{t_1^2} \end{pmatrix}.$$

Then

$$\begin{aligned} &w(t) = v(t) + \left(\tfrac{12t}{t_1^3} - \tfrac{6}{t_1^2}\right) + \left(\tfrac{12t}{t_1^3} - \tfrac{6}{t_1^2}\right) \ z_1(t_1, v) \left(-\tfrac{6t}{t_1^2} + \tfrac{2}{t_1^3}\right) \ z_2(t_1, v), \\ &y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \ y_1(t) = z_1(t) + \tfrac{t_1^3 + 2t^3 - 3t_1 t^2}{t_1^3} + \left(\tfrac{2t^3 - 3t^2 t_1}{t_1^3}\right) \ z_1(t_1, v) + \tfrac{-t^3 + t_1 t^2}{t_1^2} \ z_2(t_1, v), \\ &y_2(t) = z_2(t) + \tfrac{6t^2 - 6t\, t_1}{t_1^3} + \left(\tfrac{6t^2 - 3t^2 t_1}{t_1^3}\right) \ z_1(t_1, v) + \left(\tfrac{-3t^2 + 2t\, t_1}{t_1^2}\right) \ z_2(t_1, v). \end{aligned} \tag{45}$$

The optimal control problem (1) (21)–(23) for this example takes the form

$$J(v,u) = \int_{t_0}^{t_1} |v(t) + \lambda_1(t, \ x_0, \ x_1) + N_1(t) z(t_1, v) - (-y_1 - 2y_1^3 + u(t))|^2 dt \to \inf \tag{46}$$

subject to conditions

$$\dot z_1 = z_2, \ \dot z_2 = v(t), \ z_1(0) = 0, \ z_2(0) = 0, \ v(t) \in L_2(I, R^1), \ u \in \Lambda, \tag{47}$$

where $f(y, \ u, \ t) = -y_1 - 2y_1^3 + u(t), \ F_0 = |w(t) - (-y_1 - 2y_1^3 + u)|^2$.

Partial derivatives:

$$\frac{\partial F_0}{\partial v} = 2[v(t) - (-y_1 - 2y_1^3 + u(t)], \quad \frac{\partial F_0}{\partial u} = -2[w(t) - (-y_1 - 2y_1^3 + u(t)],$$

$$\frac{\partial F_0}{\partial z_1} = -2(-1 - 6y_1^2)[w(t) - (-y_1 - 2y_1^3 + u], \quad \frac{\partial F_0}{\partial z_2} = 0,$$

$$\frac{\partial F_0}{\partial z_1(t_1)} = 2[N_1^*(t) + N_2^*(t) f_x(y,u,t)[w(t) - (-y_1 - y_1^3 + u],$$

where $f_x(y,u,t) = \begin{pmatrix} -1 - 3y_1^2 \\ 0 \end{pmatrix}$, $w(t), \ y_1(t), \ y_2(t), \ t \in I$ are determined by formula (45).

The Frechet derivative of the functional (46) under condition (47) is $J'(v,u) = (J'_v(v,u), \ J'_u(v,u))$, where $J'_v(v,u) = \frac{\partial F_0}{\partial v} - B^*\psi(t), \ J'_u(v,u) = \frac{\partial F_0}{\partial u}$. The function $\psi(t), \ t \in I = [0, t_1]$ solving a differential equation

$$\dot\psi = \frac{\partial F_0}{\partial z} - A^*\psi, \ \ \psi(t_1) = -\int_0^{t_1} \frac{\partial F_0}{\partial z(t_1)} dt.$$

Sequences $\{v_n\}$, $\{u_n\}$ are determined by the formulas:

$$v_{n+1} = v_n - \alpha_n J'_v(v_n, u_n), \ \ u_{n+1} = P_\Lambda[u_n - \alpha_n J'_u(v_n, u_n)], \quad n = 0, \, 1, \, 2, \, ...$$

The solution of the optimization problem (44), (45) for $t_1 = 4$ is:

$$v_*(t) = \begin{cases} -1, & 0 \le t < \frac{5}{4}, \\ +1, & \frac{5}{4} \le t < \frac{13}{4} \\ -1, & \frac{13}{4} \le t < 4, \end{cases}, \qquad u_*(t) = \begin{cases} -\frac{t^2}{2} + 2(1 - \frac{t^2}{2})^3, & \tau \le t < \frac{5}{4}, \\ \frac{t^2}{6} + \frac{5t}{2} + \frac{57}{16} + 2(\frac{t^2}{2} - \frac{5t}{2} + \frac{41}{16})^3, & 0 \le t < \frac{13}{4}, \\ (\frac{t^2}{2} + 4t - 9) + 2(-\frac{t^2}{2} + 4t - 8)^3, & \frac{13}{4} \le t < 4. \end{cases}$$

$$-2 \le u_*(t) \le +2, \quad t \in I = [0,\ 4],$$

$$x_{1*}(t) = \begin{cases} 1 - \frac{t^2}{2}, & 0 \le t \le \frac{5}{4}, \\ \frac{t^2}{2} - \frac{5t}{2} + \frac{41}{16}, & \frac{5}{4} \le t \le \frac{13}{4}, \\ -\frac{t^2}{2} + 4t - 8, & \frac{13}{4} \le t \le 4, \end{cases} \qquad x_{2*}(t) = \begin{cases} -t, & 0 \le t \le \frac{5}{4}, \\ t - \frac{5t}{2}, & \frac{5}{4} \le t \le \frac{13}{4}, \\ -t + 4, & \frac{13}{4} \le t \le 4. \end{cases}$$

The solution to the optimal performance problem for $t_{1*} = 2$ is:

$$v_*(t) = \begin{cases} -1, & 0 \le t < 1, \\ 1, & 1 \le t < 2, \end{cases} \qquad u_*(t) = \begin{cases} -\frac{t^6}{4} + \frac{3t^4}{2} - \frac{7t^2}{2} + 2, & 0 \le t < 1, \\ \frac{t^6}{4} - 3t^5 + 15t^4 - 40t^3 + \frac{121t^2}{2} - 50t + 19, & 1 \le t < 2, \end{cases}$$

$$-2 \le u_*(t) \le 2, \quad t \in I = [0,\ 2].$$

$$x_{1*}(t) = \begin{cases} 1 - \frac{t^2}{2}, & 0 \le t \le 1, \\ \frac{t^2}{2} - 2t + 2, & 1 \le t \le 2, \end{cases} \qquad x_{2*}(t) = \begin{cases} -t, & 0 \le t \le 1, \\ t - 2, & 1 \le t \le 2. \end{cases}$$

5 Conclusion

A new method for solving the controllability problem of nonlinear systems described by ordinary differential equations has been developed. The scientific novelty of the obtained results lies in the following:

– all sets of controls for linear systems have been found, each element of which transforms the system trajectory from any initial state to any desired final state (Theorem 2);

– a general solution to the linear controllable system corresponding to the control from the selected set of all controls has been constructed (Theorem 3);

– necessary and sufficient conditions for the controllability of nonlinear systems have been derived (Theorem 4);

– the controllability problem has been reduced to solving the initial optimal control problem for nonlinear control systems (Lemmas 1, 2);

– the gradient of the functional has been found, minimizing sequences have been constructed, and their convergence has been studied (Theorems 5, 6);

– an algorithm for solving the problem of optimal speed was formulated;

– theoretical research results have been demonstrated using an example by solving the nonlinear Duffing equation control problem.

This completes the summary and conclusions of the paper regarding the methods and results obtained for solving the optimal speed control problem for nonlinear systems.

Acknowledgments

This research is supported by Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08856324).

Author Contributions

S.A. Aisagaliev collected and analyzed data, and led manuscript preparation. G.T. Korpebay assisted in data collection and analysis. S.A. Aisagaliev, G.T. Korpebay served as the principal investigator of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Aisagaliev, S.A. (2022). *Kachestvennaia teoriia integro-differentsialnykh uravnenii [Qualitative theory of integro-differential equations].* Almaty: Faraby University [in Russian].

2 Aisagaliev, S.A. (2021). *Teoriia kraevykh zadach dinamicheskikh sistem [Theory of boundary value problems of dynamical systems].* Almaty: Faraby University [in Russian].

3 Kalimoldayev, M.N., Abdildayeva, A.A., Mamyrbayev, O.Z., & Akhmetzhanov, M. (2016). Information system based on the mathematical model of the EPS. *Open Engineering, 6*(1), 464–469. https://doi.org/10.1515/eng-2016-0053

4 Dzyuba, A., & Torskyy, A. (2022). Algorithm of the successive approximation method for optimal control problems with phase restrictions for mechanics tasks. *Mathematical Modeling and Computing, 9*(3), 734–749 https://doi.org/10.23939/mmc2022.03.734

5 Ananyevsky, I.M., Anokhin, N.V., & Ovseevich, A.I. (2010). Sintez ogranichennogo upravleniia lineinymi dinamicheskimi sistemami s pomoshchiu obshchei funktsii Liapunova [Synthesis of limited control of linear dynamic systems using the general Lyapunov function]. *Doklady Akademii nauk — Reports of the Academy of Sciences, 434*(3), 319–323 [in Russian].

6 Semenov, Yu.M. (2012). On the Complete Controllability of Linear Nonautonomous Systems. *Differential Equations, 48*(9), 1245–1257.

7 Aisagaliev, S.A., & Zhunussova, Z.K. (2017). Constructive method for solvability of fredholm equation of the first kind. *Electronic Journal of Qualitative Theory of Differential Equations, 45*, 1–11. https://doi.org/10.14232/ejqtde.2017.1.45

8 Benner, P., Chuiko, S., & Zuyev, A. (2023). A periodic boundary value problem with switchings under nonlinear perturbations. *Boundary Value Problems, 1*, 50. https://doi.org/10.1186/s13661-023-01734-1

9 Aisagaliev, S.A. (2018). On Periodic Solutions of Autonomous Systems. *Journal of Mathematical Sciences, 229*(4), 335–353. https://doi.org/10.1007/s10958-018-3681-8

10 Dong, X.X., Liu, Z.B., & Li, S.C. (2019). Similar constructing method for solving nonhomogeneous mixed boundary value problem of n-interval composite ode and its application. *Mathematical Methods in the Applied Sciences, 42*(5), 1702–1723 https://doi.org/10.1002/mma.5468

11 Aisagaliev, S.A., & Korpebay, G.T. (2023). Controllability and optimal speed-in-action of linear systems with boundary conditions. *Bulletin of the Karaganda University. Mathematics Series,* 2(110), 21–34. https://doi.org/10.1134/S0012266112060079

12 Kalimoldayev, M., Jenaliyev, M., Abdildayeva, A., Zhukabayeva, T., & Akhmetzhanov, M. (2020). Optimal control of power systems. *News of the National Academy of Sciences of the Republic of Kazakhstan-Series Physico-Mathematical, 5*(333), 86–94. https://doi.org/10.32014/2020.2518-1726.86

13 Kvitko, A.N. (2021). On a Method for Solving a Local Boundary Value Problem for a Nonlinear Stationary Controlled System in the Class of Differentiable Controls. *Computational Mathematics and Mathematical Physics, 61*(4), 527–541. https://dx.doi.org/10.1134/S0965542521040072

*Author Information**

Serikbay Aisagaliev — Doctor of technical sciences, Professor Department of Mathematics, al-Farabi Kazakh National University, Almaty, 050040, Kazakhstan; e-mail: *Serikbai.Aisagaliev@kaznu.kz*; https://orcid.org/0000-0002-6507-2916

Guldana Korpebay (*corresponding author*) — PhD student, Department of Mathematics, al-Farabi Kazakh National University, Almaty, 050040, Kazakhstan; e-mail: *korpebay.guldana1@gmail.com*; https://orcid.org/0000-0002-5023-9402

*The author's name is presented in the order: First, Middle and Last Names.