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On closure operators of Jonsson sets

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The work is related to the study of the model-theoretic properties of Jonsson theories, which, generally speaking, are not complete. In the article, on the Boolean of Jonsson subsets of the semantic model of some fixed Jonsson theory, the concept of the Jonsson closure operator Jcl was introduced, defining the J-pregeometry on these subsets, and some results were obtained describing this closure operator.

Keywords: Jonsson theory, semantic model, Jonsson set, closure operator, J-pregeometry.

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Introduction

This work is related to the study of definable subsets of the semantic model of a fairly wide class of fixed Jonsson theories. The following reasons cause the appeal to definable subsets of the semantic model. The problem of describing the heredity of Jonsson theories is well known. To date, a complete description of this important model-theoretic property is unknown.

The Jonsson theory is called hereditary if the property of being a Jonsson theory is preserved for admissible enrichments of the signature under consideration. Moreover, with admissible enrichment, only those hereditary Jonsson theories are of interest that, in these enrichments, preserve the definability of the type for the stability obtained with such enrichment. If we take into account the fact that the classical version of stability is associated with enrichments only of constants, then the problem of describing heredity is naturally present for complete theories. In particular, if we consider the center of Jonsson theory of fields of a fixed characteristic, then in the case of characteristic 0, the enrichment of this center does not preserve Jonssoness when the enrichment is a one-place predicate interpreted in an algebraically closed field of characteristic 0 as an elementary subfield, i.e. an elementary submodel of an algebraically closed field of characteristic 0 is of sufficiently large power. At the same time, the center of Jonsson theory is always a complete theory by definition. Besides the fact that this is an example of the importance of this problem not only for Jonsson theories but also for complete theories, we also note that in this example, the dimension of the model of the center differs from the dimension of the model of any of the Abelian groups that define a given field over the field itself. This fact is due to the fact that the concept of dimension, which determines the maximum number of independent elements of the model of a given center, is differently connected in these examples with the relation of non-forking of the corresponding types of algebras considered: fields and Abelian groups. The main tool that distinguished these dimensions was once identified by S. Shelah when studying the classification of complete theories, and it was called forking. In the work [1], the basics of forking for Jonsson theories were defined when studying the model-theoretic properties of the semantic model of a fixed class of the Jonsson spectrum of a fixed class of models of the consideration language.

The next interesting and important issue discussed in this article is describing definable subsets of a semantic model using a closure operator that defines some pregeometry on the Boolean of the

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semantic model under consideration. Note that the idea of describing the classification of theories regarding various types of geometries specified by pregeometry was once proposed by B. Zilber [2]. Mustafin T.G. in work [3], within the framework of the study of complete theories, the concept of a pure pair and a semantic triple was proposed using a specific closure operator defined on subsets of a sufficiently large model of a stable complete theory. Theorems were obtained describing the properties of the closure operator defined on special subsets of the monster model of the considered complete stable theory. In contrast, the description of the properties of this operator was closely related to one of the elements of the semantic triple, namely, of the group of automorphisms of this monster model. Note that this approach to describing closure operators differs from the description in the works of B. Zilber.

Yeshkeyev A.R. obtained results on implementing the approach of Mustafin T.G. already within the framework of studying, generally speaking, the incomplete Jonsson stable fixed Jonsson theory [4]. It should also be noted that when determining the syntactic and semantic similarities of Jonsson theories, Yeshkeyev A.R. redefined the concepts of a pure pair and a semantic triple for Jonsson theories and their semantic models [4].

In the work [5], the Jonsson spectrum of a fixed class of models of an arbitrary signature was defined. Note that within the framework of the study of the corresponding invariants of Abelian groups and special types of rings [6, 7], a description of such an important model-theoretic concept as cosemanticness of models of Jonsson theories was obtained. It turned out that the concept of cosemanticness generalizes and clarifies such an important concept as the elementary equivalence of two models. In addition, the concept of cosemanticness of models is related to the syntactic concept of Jonsson theory in that two models are cosemantic with each other if their Jonsson spectra are equal.

Thus, we note the importance of this article in connection with the following circumstance that connects the relevance and novelty of this problem, regarding the fact that any Jonsson theory is a special case of such an important and fruitful concept as the Jonsson spectrum of a fixed class of models of a given signature.

When studying definable subsets of a complete theory, as a rule, one specifies some axioms satisfied by these formulaic subsets of fixed models of this complete theory. In our case, we will do the same for the special case of definable subsets of the semantic model of a fixed Jonsson theory. Namely, these axioms will predetermine the possibility of determining the Morley rank function in its "Jonsson" interpretation, i.e. in conditions where only existentially closed extensions under corresponding monomorphisms, which are not necessarily elementary, are considered.

At the same time, we note that all these arguments will have a positive development for the corresponding types of homomorphisms, i.e. we can transfer the results of this article to positive Jonsson theories. The concept of a positive Jonsson theory and the properties of morphisms of such theories were considered in [8, 9].

Those facts that are not indicated in this article but may be helpful for a deeper understanding of the results of this article can be obtained from the following sources [10–18].

1 Basic concepts and results concerning Jonsson theories

Let us present the necessary definitions and results concerning Jonsson theories.

Definition 1. [19] A theory T is called Jonsson if it has an infinite model, is inductive, and satisfies the joint embedding property (JEP) and amalgamation property (AP).

Note that Jonsson theory, by its definition, is, generally speaking, not complete, i.e. the class of its models can contain both infinite and finite models and, in addition, the definition of JEP and AP considers isomorphic embeddings rather than elementary monomorphisms. There are many examples from classical algebra that satisfy Jonsson theories. These include groups, Abelian groups, rings, fields of fixed characteristic, Boolean algebras, linear orders, vector spaces, modules over a fixed ring and others.

Definition 2. [20] Let $\kappa \geq \omega$. A model M of a theory T is called κ-universal for the theory T if for each model $A \in Mod(T)$ such that $|A| < \kappa$, there is an isomorphism $f\colon A \to M$.

Definition 3. [20] Let $\kappa \geq \omega$. A model M of a theory T is called κ-homogeneous for T if for any two models A, $A_1 \in Mod(T)$, which are submodels of M such that $|A| < \kappa$, $|A_1| < \kappa$, and the isomorphism $f : A \to A_1$, for every extension B of model A that is a submodel of M and model T of cardinality strictly less than κ, there is an extension B_1 of model A_1, which is a submodel of M and an isomorphism $g : B \to B_1$ extending f.

A *homogeneous-universal model* for T is a κ-homogeneous-universal model for T of cardinality $\kappa \geq \omega$.

Definition 4. [20] The semantic model C_T of Jonsson theory T is called the ω^+-homogeneous-universal model of the theory T.

Definition 5. [4] The semantic completion (center) of Jonsson theory T is the elementary theory T^* of the semantic model C_T of the theory T, i.e. $T^* = \mathrm{Th}(C_T)$.

In the case when universally homogeneous models in the Jonsson sense are saturated, a special class of Jonson theories is distinguished, the elements of which are called perfect Jonsson theories.

Definition 6. [4] A Johnson theory T is said to be perfect if every semantic model of the theory T is a saturated model of T^*.

The remarkable property of the existence of a model companion for such theories determines the feature of perfect Jonsson theories.

Theorem 1. [4] Jonsson theory T is perfect if and only if T^* is a model companion of theory T.

An important characteristic of any theory is stability. For complete theories, the concept of stability was introduced by S. Shelah in 1969. In the work [4] Yeshkeyev A.R., the concept of stability in the Jonsson sense was defined. Let us recall the definition of this concept.

Let T be a Jonsson theory. Let $S^J(X)$ denote the set of all existential complete n-types over X that are consistent with T for every finite n.

Definition 7. [4] We say that a Jonsson theory T is a J-λ-stable if for any T-existentially closed model A, for any subset X of the set A since $|X| \leq \lambda$ it follows that $|S^J(X)| \leq \lambda$. The Jonsson theory T is called a J-stable if it is a J-λ-stable for some λ.

In the article [6], a result was obtained showing that stability in the above sense is in good agreement with the classical concept of stability.

Theorem 2. [6] Let T be a perfect Jonsson theory complete for $\exists$-sentences, $\lambda \geq \omega$. Then the following conditions are equivalent:

1 T is a J-λ-stable;

2 T^* is a λ-stable, where T^* is the center of Jonsson theory T.

In the work [1], within the framework of the study of Jonsson theories, the concept of J-pregeometry was introduced.

Let T be some fixed Jonsson theory, $X \subseteq C_T$, $\mathcal{P}(X)$ be the Boolean of the set X and the map $cl\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is some closure operator on the set $\mathcal{P}(X)$. The pair (X, cl) is a *J-pregeometry* if the following conditions are satisfied:

1) if $A \subseteq X$, then $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$;

2) if $A \subseteq B \subseteq X$, then $cl(A) \subseteq cl(B)$;

3) (exchange) $A \subseteq X$, $a, b \in X$ and $a \in cl(A \cup \{b\}) \setminus cl(A)$, then $b \in cl(A \cup \{a\})$;

4) (finite character) If $A \subseteq X$ and $a \in cl(A)$, then there is a finite $A_0 \subseteq A$, such that $a \in cl(A_0)$.

Further, we will assume that the operator cl, which defines the J-pregeometry on a subset of the semantic model of some fixed Jonsson theory, will be the algebraic closure operator, which is equal to the definable closure operator, i.e. $cl = acl = dcl$.

Definition 8. [21] A set X is called Jonsson in the theory T if it satisfies the following properties:

1) X is a definable subset of C_T;

2) $cl(X)$ is the carrier of some existentially closed submodel C_T.

Next, we write down additional axioms to preserve the Morley rank of the above formulaic Jonsson subsets and denote this system of axioms by (*). Due to the result on the equivalence of the Jonsson stability of the Jonsson theory and the corresponding stability of its center for a perfect $\exists$-complete theory (Theorem 2), these axioms not only clarify the boundaries of conservation of the model-theoretic properties of the Morley rank of the considered formulas of a fixed Jonsson theory in a given context but are also correctly related to the closure operator, which specifies the pregeometry on the Boolean of Jonsson subsets of the semantic model of the theory under consideration. Moreover, we note that the semantic model itself is an element of this Boolean due to the fact that the formula $x = x$ is an existential formula. The main results of this article also use the correctness of this axiomatics and the closedness with respect to such subsets.

Let T be some Jonsson theory, let $\mathbb{J}$ denote the set of all Jonsson subsets of the semantic model C_T and let $|\mathbb{J}| = |I|$, where I is index set. It's clear that $\mathbb{J} \subseteq \mathcal{P}(C_T)$. Let us introduce the concept of a family of Jonsson subsets of the semantic model C_T. Let us denote by $\mathbb{J}^n$ the set of all definable Jonsson subsets of the semantic model C_T, the length of whose defining formulas is equal to n.

Let us present a system of axioms, which we denote by (*).

Let $Jset(C_T)$ be the smallest family of Jonsson subsets in $\bigcup\limits_{n\geq 1} C_T^n$ with the following properties:

1) For each $i \in I$, from the fact that $A_i \in \mathbb{J}$ it follows that $A_i \in Jset(C_T)$.

2) The set $Jset(C_T)$ is closed under finite Boolean combinations, i.e. from the fact that $A, B \in \mathbb{J}^n$ it follows that $A, B \in Jset(C_T)$, $A \bigcup B \in Jset(C_T)$, $A \bigcap B \in Jset(C_T)$ and $C_T^n \setminus A \in Jset(C_T)$.

3) The set $Jset(C_T)$ is closed under the Cartesian product, i.e. from the fact that $A, B \in Jset(C_T)$ it follows that $A \times B \in Jset(C_T)$.

4) The set $Jset(C_T)$ is closed under the projection, i.e. if $A \subseteq C_T^{n+m}$, $A \in Jset(C_T)$, $\pi_n(A)$ is the projection of the Jonsson set A onto C_T^n, then $\pi_n(A) \in Jset(C_T)$.

5) The set $Jset(C_T)$ is closed under specialization, i.e. if $A \in Jset(C_T)$, $A \subseteq C_T^{n+k}$ and $\bar{m} \in C_T^n$, then $A(\bar{m}) = \{\bar{b} \in C_T^k \colon (\bar{m}, \bar{b}) \in A\} \in Jset(C_T)$.

6) The set $Jset(C_T)$ is closed under permutation of coordinates, i.e. if $A \in Jset(C_T)$, $A \subseteq C_T^n$, and σ is a permutation of the set $\{1, ..., n\}$, then $\sigma(A) = \{(a_{\sigma(1)}, ..., a_{\sigma(n)}) \colon (a_1, ..., a_n) \in A\} \in Jset(C_T)$.

In work [4], within the framework of the study of Jonsson subsets of the semantic model of given Jonsson theory, the concept of forking was axiomatically introduced, and the equivalence of forking according to Shelah and the axiomatically given forking for existential types over subsets of the semantic model of some Jonsson theory was proven.

Let $\mathcal{X}$ be the class of all Jonsson subsets of the $\exists$-saturated semantic model C_T of some Jonsson theory T, $\mathcal{R}$ be the class of all existential types (not necessarily complete). Let $JNF \subseteq \mathcal{R} \times \mathcal{X}$ be some binary relation. Let us write down in the form of axioms some conditions imposed on JNF (Jonsson nonforking).

Axiom 1. If $(p, A) \in JNF$ and $f : A \to B$ are isomorphic embeddings, then $(f(p), f(A)) \in JNF$.

Axiom 2. If $(p, A) \in JNF$ and $q \subseteq p$, then $(q, A) \in JNF$.

Axiom 3. If $A \subseteq B \subseteq C$ and $p \in S^J(C)$, then $(p, A) \in JNF$ if and only if $(p, B) \in JNF$ and $(p{\upharpoonright}B, A) \in JNF$.

Axiom 4. If $A \subseteq B$, $dom(p) \subseteq B$ and $(p, A) \in JNF$, then there exists $q \in S^J(B)$ such that $p \subseteq q$ and $(q, A) \in JNF$.

Axiom 5. There is a cardinal μ such that if $A \subseteq B \subseteq C$, $p \in S^J(B)$ and $(p, A) \in JNF$, then $\left|\{q \in S^J(C) : p \subseteq q, (q, A) \in JNF\}\right| < \mu$.

Axiom 6. There is a cardinal $\varkappa$ such that for any $p \in \mathcal{R}$ and for each $A \in \mathcal{X}$, if $(p, A) \in JNF$, then there exists $A_1 \subseteq A$, such that $|A_1| < \varkappa$ and $(p, A_1) \in JNF$.

Axiom 7. If $p \in S^J(A)$, then $(p, A) \in JNF$.

Theorem 3. [4] Let T be a perfect Jonsson theory, complete for $\exists$-sentences. Then the following conditions are equivalent:

1) the relation JNF satisfies axioms 1–7 with respect to the theory T;

2) the theory T^* is stable and for any $p \in \mathcal{R}$, $A \in \mathcal{X}$ the pair $(p, A) \in JNF \Leftrightarrow p$ is not forks over A (in the classical sense of S. Shelah [22]).

Next entry $p \not\downarrow^J A$ will mean that $(p, A) \in JNF$. If $tp(\overline{a}, A \cup \overline{b}) \not\downarrow^J A$, then we will write $\overline{a} \underset{A}{\not\downarrow}^J \overline{b}$.

2 Jonsson theories with closure operator

In the work [3] Mustafin T.G., some properties of complete theories admitting a closure operator were considered. In this article, we will consider some properties of the closure operator within the framework of the study of Jonsson theories concerning those additional considerations as the above *Axiomatics (*)* concerning the model-theoretic properties of preserving the Morley rank and the correctness of the definition of Jonsson subsets satisfying given axiomatics and satisfying the properties closure operator defining the pregeometry on the Boolean of the semantic model under consideration.

Recall that a complete theory T admits a closure operator J if on the monster model $\mathcal{C}$ of the theory T one can define a closure operator J so that $J(g(X)) = g(J(X))$ for all $X \in \mathcal{P}(\mathcal{C})$ and $g \in Aut(\mathcal{C})$ [3].

We need the following technical lemma from [3] to prove Theorem 4.

Lemma 1. [3] Let J be some closure operator admitted by the full theory of T, then the following conditions are equivalent:

1) if $M < \mathcal{C}$, then $M = \cup\{J(m) \colon m \in M\}$;

2) $|J(a)| < |\mathcal{C}|$ for any $a \in \mathcal{C}$;

3) $J(a) \subseteq acl(a)$ for any $a \in \mathcal{C}$.

The following definition belongs to A.R. Yeshkeyev. It defines the closure operator for a generally speaking incomplete theory, and it can be used in a broad sense for application in specific algebras, the theory of which is Jonsson.

Let T be a Jonsson theory whose semantic model C_T satisfies Axiomatics (*).

Definition 9. We will say that a Jonsson theory T with a closure operator Jcl if $Jcl(g(X)) = g(Jcl(X))$ for all $X \in \mathcal{P}(C_T)$ and $g \in Aut(C_T)$.

Let T be some Jonsson theory with the closure operator Jcl, X be a Jonsson set, and $Jcl(X) = M \in E_T$, where E_T is the class of all existentially closed models of the theory T. If $a, b \in C_T \backslash M$ then $b \in C_M(a)$ means that there exist $n < \omega$ and the sequence $\langle b_0, ..., b_n\rangle$ elements from $C_T \backslash M$ such that $b_0 = a$, $b_n = b$, $b_i \in cl(b_{i+1})$ or $b_{i+1} \in cl(b_i)$ for all $i < n$. In this case, the sequence $\langle b_0, ..., b_n\rangle$ will be called a Jcl-path outside M between a and b of length n.

Let us consider some conditions imposed on the Jcl operator.

Axiom 1. If $M \in E_T$, then $M = \overline{M}$, where $\overline{M} \leftrightharpoons \cup\{Jcl(m) \mid m \in M\}$.

Axiom 2. If $M \in E_T$ and $M = \overline{M}$, $\bar{a}, \bar{b}$ are tuples of elements from $C_T \setminus M$, $C_M(\bar{a}) \cap C_M(\bar{b}) = \varnothing$, then $\bar{a} \underset{M}{\not\downarrow}^J \bar{b}$.

Further, instead of the closure operator Jcl, we mean the algebraic closure operator acl, which is also the definable closure operator dcl.

It is well-known that ω-stable complete theories are characterized by the fact that any type (respectively, any formula) of a given theory has some Morley rank, i.e., ranked according to Morley. First, J-ω-stability does not coincide with ω-stability in the general case, and in the context of the following Theorems 4 and 5, the condition ω-stability is not assumed even on the center of the Jonson theory under consideration. However, at the same time, earlier in this article, we defined the Axiomatics (*), which is consistent with the Morley rank of definable Jonsson subsets. Therefore, in Theorems 4 and 5, Axiomatics (*) is assumed under the assumption that the closure operator under consideration is related to Morley-ranked subsets of the semantic model under consideration. We also note that due to the perfectness of the theory, the semantic model is saturated in power ω^+, which is enough for the rank of formulaic subsets relative to the center of the perfect Jonsson theory to exist.

In connection with the above definitions, we have the following results:

Theorem 4. Let T be a perfect Jonsson J-λ-stable theory, complete for $\exists$-sentences with the closure operator Jcl, whose semantic model C_T satisfies Axiomatics (*). If Jcl satisfies Axioms 1, 2 and $M \in E_T$, then for all $a \in C_T \backslash M$ $C_M(a) = Jcl(C_M(a))$.

Proof. Since by condition the Jonsson theory T is $\exists$-complete, then all $\exists$-types are complete types, i.e., they are all true in C_T. However, C_T is an existentially closed model, so all $\exists$-types are true in M. Due to the fact that $acl = dcl = Jcl$, then Lemma 1 is true for T^*, and therefore for any existentially closed model of the theory T, since T is a perfect Jonsson theory. The inclusion of $C_M(a) \subset cl(C_M(a))$ follows from Lemma 1.

We prove the reverse inclusion by induction on the length of Jcl-paths. Let $\langle b, a\rangle$ be a Jcl-path outside M (of length I), i.e. either $b \in Jcl(a)$ or $a \in Jcl(b)$. In any case, $a \underset{M}{\downarrow} b$ in T^* theory. By Theorem 2, the theory T^* is λ-stable in the classical sense. Then, by Theorem 3 we have $a \underset{M}{\downarrow}^{J} b$ in the Jonsonian sense of J-forking. Hence, by virtue of the definition of the theory with the closure operator Jcl and Axiom 2, we obtain $b \in C_M(a)$. Let now $\langle b_0, ..., b_n\rangle$ Jcl-path outside M between b and a of length n. By induction $b_1 \in C_M(b_n)$ and $b_0 \in C_M(b_1)$. This means $b_0 \in C_M(b_n)$, i.e. $b \in C_M(a)$.

Theorem 5. If T is a perfect Jonsson J-λ-stable theory, complete for $\exists$-sentences with the closure operator Jcl whose semantic model C_T satisfies axiomatics (*), the operator Jcl satisfies the Axioms 1, 2, $M, N \in E_T$, $M \prec_{\exists_1} N$, $a \in N \backslash M$, then:

1) $M \prec_{\exists_1} M \cup (N \cap C_M(a)) \preceq_{\exists_1} N$;

2) $M \preceq_{\exists_1} N \backslash (N \cap C_M(a)) \prec_{\exists_1} N$.

Proof. 1) Let $K \leftrightharpoons M \cup (N \cap C_M(a))$. Let us assume that K is not an elementary submodel of N with respect to $\exists$-formulas, i.e. K is not an existentially closed submodel of N. Then there must exist an element $b \in N$, an existential formula $\theta(x, \bar{y}, \bar{z})$ and tuples $\bar{a} \in N \cap C_M(a)$ and $\bar{m} \in M$ such that $N \models \theta(b, \bar{m}, \bar{a})$, but $N \models \neg\theta(c, \bar{m}, \bar{a})$ for all $c \in K$. Hence $b \notin C_M(a)$ and $b \underset{M}{\downarrow} \bar{a}$ in T^* theory. By Theorem 2, the theory T^* is λ-stable in the classical sense. Then, by Theorem 3 we have $a \underset{M}{\downarrow}^{J} b$ in the Jonsson sense of J-forking. Since $\overline{a} \subseteq C_M(a)$, then $C_M(\overline{a}) = C_M(a)$.

Therefore $C_M(b) \cap C_M(\bar{a}) = \oslash$. By Axiom 1 $M = \overline{M}$, and by Axiom 2 $b \underset{M}{\not\downarrow}^{J} \overline{a}$ in the Jonsson sense of J-forking. We have a contradiction.

2) Let $a_i \in N \backslash M, i < \lambda$ such that $N \backslash (N \cap C_M(a)) = M \cup \bigcup\limits_{i<\lambda} (N \cap C_M(a_i))$. Applying point 1) by induction we obtain that $M \preceq_{\exists_1} N \backslash (N \cap C_M(a)) \prec_{\exists_1} N$.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Йонсондық жиындардың тұйықталу операторлары туралы

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Жұмыс жалпы айтқанда толық емес болып табылатын йонсондық теориялардың модельді-теоретикалық қасиеттерін зерттеумен байланысты. Мақала авторлары кейбір бекітілген йонсондық теорияның семантикалық моделінің йонсондық ішкі жиындарының булеанында осы жиындардағы J-алғашқы геометрияны анықтайтын Jcl йонсондық тұйықталу операторы ұғымын енгізді және тұйықталу операторын сипаттайтын негізгі нәтижелер алынды.

Кілт сөздер: йонсондық теория, семантикалық модель, йонсондық жиын, тұйықталу операторы, J-алғашқы геометрия.

Об операторах замыкания йонсоновских множеств

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Данная работа связана с изучением теоретико-модельных свойств йонсоновских теорий, которые, вообще говоря, не являются полными. Авторами статьи на булеане йонсоновских подмножеств семантической модели некоторой фиксированной йонсоновской теории было введено понятие йонсоновского оператора замыкания Jcl, задающего J-предгеометрию на этих подмножествах, и получены некоторые результаты, описывающие указанный выше оператор замыкания.

Ключевые слова: йонсоновская теория, семантическая модель, йонсоновское множество, оператор замыкания, J-предгеометрия.

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