MATHEMATICS

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Structural properties of the sets of positively curved Riemannian metrics on generalized Wallach spaces

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In the present paper sets related to invariant Riemannian metrics of positive sectional and (or) Ricci curvature on generalized Wallach spaces are considered. The problem arises in studying of the evolution of such metrics under the influence of the normalized Ricci flow. For invariant Riemannian metrics of the Wallach spaces which admit positive sectional curvature and belong to a given invariant surface of the normalized Ricci flow equation we establish that they form a set bounded by three connected and pairwise disjoint regular space curves such that each of them approaches two others asymptotically at infinity. Analogously, for all generalized Wallach spaces with coincided parameters the set of Riemannian metrics which belong to the invariant surface of the normalized Ricci flow and admit positive Ricci curvature is bounded by three space curves each consisting of exactly two connected components as regular curves. Mutual intersections and asymptotical behaviors of these components are studied as well. We also establish that curves corresponding to Kähler metrics of spaces under consideration form separatrices of saddles of a three-dimensional system of nonlinear autonomous ordinary differential equations obtained from the normalized Ricci flow equation.

Keywords: generalized Wallach space, Riemannian metric, Kähler metric, normalized Ricci flow, sectional curvature, Ricci curvature, dynamical system, singular point.

2020 Mathematics Subject Classification: 53C30, 53E20, 37C10.

Introduction

The paper is devoted to the study of structural properties of two important sets responsible for positivity of the sectional and the Ricci curvatures of invariant Riemannian metrics on the Wallach spaces and generalized Wallach spaces. The Wallach spaces

$$W_6 := \mathrm{SU}(3)/T_{\max}, \quad W_{12} := \mathrm{Sp}(3)\,/\,\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1), \quad W_{24} := F_4/\,\mathrm{Spin}(8) \tag{1}$$

are well-known and admit invariant Riemannian metrics of positive sectional curvature as it was shown in [1]. As for generalized Wallach space, firstly, recall its definition and basic properties (see [2, 3]). Let G/H be a homogeneous almost effective compact space with a (compact) semisimple connected Lie group G and its closed subgroup H. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras of G

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and H. Then $[\cdot\,,\cdot]$ is a corresponding Lie bracket of $\mathfrak{g}$ whereas $B(\cdot\,,\cdot)$ is the Killing form of $\mathfrak{g}$. Note that $\langle\cdot\,,\cdot\rangle = -B(\cdot\,,\cdot)$ is a bi-invariant inner product on $\mathfrak{g}$. In this way invariant Riemannian metrics on G/H can be identified with $\mathrm{Ad}(H)$-invariant inner products on the orthogonal complement $\mathfrak{p}$ of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to $\langle\cdot\,,\cdot\rangle$. Compact homogeneous spaces G/H whose isotropy representation admits a decomposition into a direct sum $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ of three $\mathrm{Ad}(H)$-invariant irreducible modules $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ satisfying $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for each $i \in \{1, 2, 3\}$ were called *generalized Wallach spaces* in the terminology of [3]. The main characteristic of these spaces is that every generalized Wallach space can be described by a triple of real parameters $a_i := A/d_i \in (0, 1/2]$, $i = 1, 2, 3$, where $d_i = \dim(\mathfrak{p}_i)$ and A is some important positive constant (see [2] for details). It should be also noted that not every triple $(a_1, a_2, a_3) \in (0, 1/2] \times (0, 1/2] \times (0, 1/2]$ corresponds to some generalized Wallach spaces. An interesting fact is the fact that the Wallach spaces (1) are partial cases $a_1 = a_2 = a_3 = a$ of generalized Wallach spaces with $a = 1/6$, $a = 1/8$ and $a = 1/9$ respectively (see [4]).

As noted above for a fixed bi-invariant inner product $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{g}$ of the Lie group G, any G-invariant Riemannian metric $\mathbf{g}$ on G/H can be determined by an $\mathrm{Ad}(H)$-invariant inner product

$$(\cdot,\cdot) = x_1\langle\cdot,\cdot\rangle|_{\mathfrak{p}_1} + x_2\langle\cdot,\cdot\rangle|_{\mathfrak{p}_2} + x_3\langle\cdot,\cdot\rangle|_{\mathfrak{p}_3}\,, \tag{2}$$

where x_1, x_2, x_3 are positive real numbers (a detailed exposition can be found in [2,3,5] and references therein). In [2] the explicit expressions $\mathrm{Ric}_{\mathbf{g}} = \mathbf{r_1}\ \mathrm{Id}|_{\mathfrak{p}_1} + \mathbf{r_2}\ \mathrm{Id}|_{\mathfrak{p}_2} + \mathbf{r_3}\ \mathrm{Id}|_{\mathfrak{p}_3}$ and $S_{\mathbf{g}} = d_1\mathbf{r_1} + d_2\mathbf{r_2} + d_3\mathbf{r_3}$ were derived for the Ricci tensor $\mathrm{Ric}_{\mathbf{g}}$ and the scalar curvature $S_{\mathbf{g}}$ of the metric (2) on generalized Wallach spaces, where

$$\mathbf{r_i} := \frac{1}{2x_i} + \frac{1}{2a_i}\left(\frac{x_i}{x_jx_k} - \frac{x_k}{x_ix_j} - \frac{x_j}{x_ix_k}\right) \tag{3}$$

are the principal Ricci curvatures, $\{i, j, k\} = \{1, 2, 3\}$.

Knowing $\mathrm{Ric}_{\mathbf{g}}$ and $S_{\mathbf{g}}$ allowed us to initiate in [6,7] the study of the normalized Ricci flow equation

$$\frac{\partial}{\partial t}\mathbf{g}(t) = -2\,\mathrm{Ric}_{\mathbf{g}} + 2\mathbf{g}(t)\frac{S_{\mathbf{g}}}{n} \tag{4}$$

introduced by R. Hamilton in [8] on generalized Wallach spaces. Since then studies related to this topic were continued in [9–14] concerning classifications of singular (equilibria) points of (4) being Einstein metrics and their bifurcations. The authors of [15–17] studied an interesting and quite complicated surface of bifurcations of (4) defined by a symmetric polynomial equation in three variables a_1, a_2, a_3 of degree 12. In the sequel authors of [4,18] considered the evolution of positively curved Riemannian metrics under the influence of (4) on an interesting class of generalized Wallach spaces with coincided parameters $a_1 = a_2 = a_3 := a \in (0, 1/2)$ generalizing some results of [19,20]. In this case (4) can be reduced to the following system of three autonomous ordinary differential equations (see [4]):

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3) := \frac{x_i}{x_j} + \frac{x_i}{x_k} + 2a\left(\frac{x_j}{x_k} + \frac{x_k}{x_j} - 2\,\frac{x_i^2}{x_jx_k}\right) - 2 \tag{5}$$

with $\{i, j, k\} = \{1, 2, 3\}$.

In [4] it was proved that (4) deforms all generic metrics with positive sectional curvature into metrics with mixed sectional curvature on each Wallach space in (1) (Theorem 1 in [4]) and all generic metrics with positive Ricci curvature will be deformed into metrics with mixed Ricci curvature for W_{12} and W_{24} (see Theorem 2 in [4]), where given metric is said to be generic if $x_i \neq x_j \neq x_k \neq x_i$ for $i, j, k \in \{1, 2, 3\}$. According to Theorems 3 and 4 in [4] and Theorem 3 in [18] positiveness of the Ricci curvature will be preserved for all generic metrics at $a \in (1/6, 1/2)$ and for a special kind of metrics satisfying $x_k < x_i + x_j$ at $a = 1/6$ (the equalities $x_k = x_i + x_j$ correspond to Kähler metrics), whereas all positively curved metrics will be deformed into metrics with mixed Ricci curvature if $a \in (0, 1/6)$. In [4,18] we used the description $S := \big\{(x_1,x_2,x_3) \in (0,+\infty)^3 \mid \gamma_1 > 0,\, \gamma_2 > 0,\, \gamma_3 > 0\big\} \setminus \big\{(r,r,r) \in \mathbb{R}^3 \mid r > 0\big\}$ of the set of Riemannian metrics with positive sectional curvature on the Wallach spaces (1) given in [21], where

$$\gamma_i := (x_j - x_k)^2 + 2x_i(x_j + x_k) - 3x_i^2, \quad \{i,j,k\} = \{1,2,3\}. \tag{6}$$

Analogously, $R := \big\{(x_1,x_2,x_3) \in (0,+\infty)^3 \mid \lambda_1 > 0,\, \lambda_2 > 0,\, \lambda_3 > 0\big\} \setminus \big\{(r,r,r) \in \mathbb{R}^3 \mid r > 0\big\}$ is the set of all Riemannian metrics of positive Ricci curvature on every generalized Wallach spaces with $a_1 = a_2 = a_3 := a \in (0, 1/2)$, where

$$\lambda_i := x_j x_k + a\left(x_i^2 - x_j^2 - x_k^2\right), \quad \{i,j,k\} = \{1,2,3\} \tag{7}$$

in accordance with (3).

The present paper is devoted to detailed proof of our observations in [4,18] concerning structural properties of surfaces and curves obtained from (6) and (7). For each $i = 1,2,3$ introduce the surfaces (cones) $\Gamma_i := \{(x_1,x_2,x_3) \in (0,+\infty)^3 \mid \gamma_i = 0\}$ and $\Lambda_i := \{(x_1,x_2,x_3) \in (0,+\infty)^3 \mid \lambda_i = 0\}$.

Denote by Σ the surface defined by the equation $V = 1$, where $V := x_1x_2x_3$. Introduce also space curves $s_i := \Sigma \cap \Gamma_i$, $r_i := \Sigma \cap \Lambda_i$. The main result of this paper is contained in the following two theorems.

Theorem 1. The following assertions hold for all indices with $\{i,j,k\} = \{1,2,3\}$:

1 For each Wallach space in (1) the set of invariant Riemannian metrics (2) which belong to the invariant surface Σ of the differential system (5) and admit positive sectional curvature is bounded by the pairwise disjoint regular space curves s_1, s_2 and s_3 in Σ such that each s_k is connected and can be parameterized as

$$x_k = t^{-1}\alpha^{-2}, \quad x_i = t\alpha, \quad x_j = \alpha,$$

where

$$\alpha = \alpha(t) := \begin{cases} \sqrt[3]{\left(-t - 1 + 2\sqrt{t^2 - t + 1}\right) t^{-1}(t-1)^{-2}}, & \text{if} \quad t > 0,\ t \neq 1, \\ \sqrt[3]{6}/2, & \text{if} \quad t = 1, \end{cases}$$

and $\alpha(t) > 0$ for all $t > 0$.

2 Every invariant curve I_k of the differential system (5) given by the equations $x_i = x_j = p$, $x_k = p^{-2}$, $p > 0$, intersects the only border curve s_k at the unique point with coordinates $x_i = x_j = p_0$, $x_k = p_0^{-2}$ approaching at infinity the other two curves s_i and s_j as close as we like, where $p_0 = \sqrt[3]{6}/2$.

The results of Theorem 1 are illustrated in the left panel of Figure 1, where the curves s_1, s_2 and s_3 are depicted respectively in red, teal and blue colors, the invariant curves I_1, I_2, I_3 are all yellow colored.

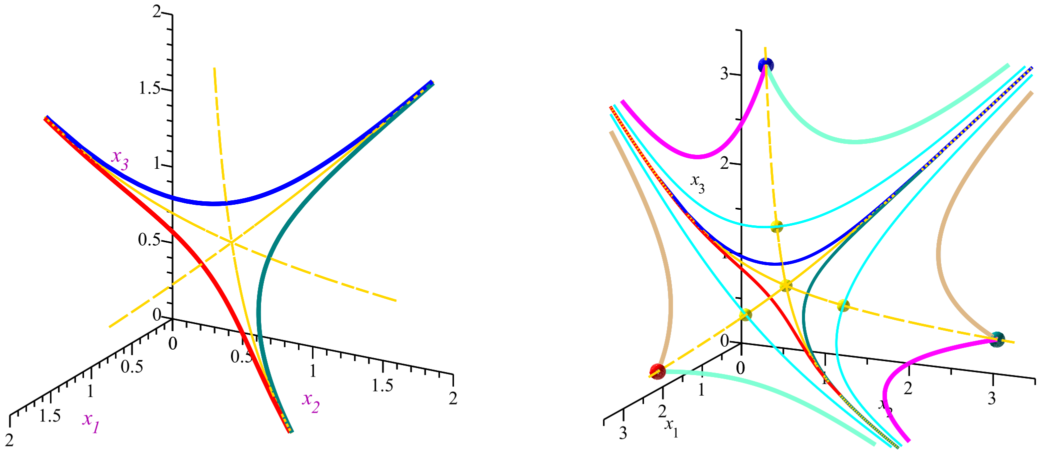

Figure 1. The curves s_1, s_2, s_3 (the left panel); the curves r_1, r_2, r_3, l_1, l_2, l_3 and singular points $\mathbf{o}_0, \mathbf{o}_1, \mathbf{o}_2, \mathbf{o}_3$ corresponding to $a = 1/6$ (the right panel)

Theorem 2. The following assertions hold for all indices with $\{i, j, k\} = \{1, 2, 3\}$:

1 For every generalized Wallach space with $a_1 = a_2 = a_3 = a \in (0, 1/2)$ the set of invariant Riemannian metrics (2) which belong to the invariant surface Σ of the differential system (5) and admit positive Ricci curvature is bounded by the space curves r_1, r_2 and r_3 in Σ such that each r_k consists of two regular connected components r_{ki} and r_{kj} parameterized by equations

$$x_k = t^{-1}\beta^{-2}, \quad x_i = t\beta, \quad x_j = \beta \tag{8}$$

and

$$x_k = t^{-1}\beta^{-2}, \quad x_j = t\beta, \quad x_i = \beta \tag{9}$$

respectively, where

$$\beta = \beta(t) := \sqrt[6]{(t^4 - a^{-1}t^3 + t^2)^{-1}} > 0,$$

$t \in (0, a]$.

2 Every pair of the curves r_i and r_j admits a unique common point P_{ij} with coordinates $x_i = x_j = a^{\frac{1}{3}}$, $x_k = a^{-\frac{2}{3}}$ which belong to the components r_{ij} and r_{ji}; In addition, every invariant curve I_k of the system (5) meets the components r_{ij} and r_{ji} of r_i and r_j exactly at the point P_{ij} approaching their another components r_{ik} and r_{jk} at infinity as close as we like.

3 For every $a \in (0, 1/2)$ all singular (equilibria) points of the differential system (5) belong to the set $\Sigma \cap R$.

4 Kähler metrics $x_k = x_i + x_j$ of generalized Wallach spaces with $a = 1/6$ form separatrices l_k of saddles of (5) in Σ which can be defined by parametric equations

$$x_k = t^{-1}\phi^{-2}, \quad x_i = t\phi, \quad x_j = \phi, \tag{10}$$

where $\phi = \phi(t) := \sqrt[3]{(t^2 + t)^{-1}}, \quad t > 0.$

The results of Theorem 2 are illustrated in the right panel of Figure 1 for the case $a = 1/6$, where the curves r_1, r_2 and r_3 are depicted respectively in magenta, aquamarine and burlywood colors, the curves l_1, l_2 and l_3 are depicted by cyan colored curves and yellow colored points correspond to singular points of (5).

It should be noted that we will consider only Riemannian metrics satisfying the unit volume condition $V := x_1x_2x_3 = 1$ (see [4,6]). In general, surfaces $V = c$, where $c > 0$, play the significant role for study (5) on generalized Wallach spaces. It is known that any set determined by the equation $V = c$ is invariant under (5), moreover $V = c$ is its first integral. Surfaces $V = c$ will also be unstable (or stable) manifolds of (5) and contain leading directions of motions of its trajectories (see [22]). Since the right hand sides of (5) are all homogeneous, namely $f_i(cx_1, cx_2, cx_3) = f_i(x_1, x_2, x_3)$ for any c, we can pass to a new differential system of the same form as the original one, but with $\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3 = 1$. Actually this is reachable by replacings $x_i(t) = \widetilde{x}_i(\tau)\sqrt[3]{c}$ and $t = \tau\sqrt[3]{c}$. Therefore without loss of generality we assume that the invariant surface is given by $V \equiv 1$.

1 *Proofs of Theorems 1 and 2*

Observe that the expressions for γ_i and λ_i in (6) and (7) are symmetric under the permutations $i \to j \to k \to i$. Therefore it suffices to consider representatives only at fixed (i, j, k), where $\{i, j, k\} = \{1, 2, 3\}$.

1.1 *Proof of Theorem 1*

Proof. (1) *The curves s_1, s_2, s_3 are pairwise disjoint and form the boundary of the set $\Sigma \cap S$.* For each Wallach space in (1) the set S of Riemannian metrics (2) admitting positive sectional curvature is bounded by the pairwise disjoint cones Γ_1, Γ_2 and Γ_3 (these cones are depicted in the left panel of Figure 2 in red, teal and blue colors respectively). Although this fact was proved in [22] we repeat here the sketch of reasonings for convenience of the readers. Indeed the equation $\gamma_k = 0$ defines two connected components $x_k = 3^{-1}\left(x_i + x_j - 2\sqrt{x_i^2 - x_ix_j + x_j^2}\right)$ and $x_k = \Phi_k(x_i, x_j) := 3^{-1}\left(x_i + x_j + 2\sqrt{x_i^2 - x_ix_j + x_j^2}\right)$ of the cone Γ_k. Since the first of them gives $x_k < 0$ for all $x_i, x_j > 0$ then $\gamma_k > 0$ is equivalent to $0 < x_k < \Phi_k(x_i, x_j)$ meaning that S is bounded by the plane $x_k = 0$ and the positive part Γ_k of the cone $\gamma_k = 0$. By symmetry we have the same for Γ_i and Γ_j. Thus $\partial(S) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and hence $\partial(\Sigma \cap S) = s_1 \cup s_2 \cup s_3$.

Consider now the pair (i, j). The equations $\gamma_i = 0$ and $\gamma_j = 0$ defining the surfaces Γ_i and Γ_j can admit only the following two family of common solutions $x_i = x_j$, $x_k = 0$ and $x_i = x_k$, $x_j = 0$. But we need in positive solutions only. Hence $\Gamma_i \cap \Gamma_j = \emptyset$ for all positive x_1, x_2, x_3. By symmetry the same assertions hold for the pairs (i, k) and (j, k).

Parameterizations of the curves s_1, s_2 and s_3. Due to symmetry fix any unordered triple (i, j, k). The parametric representation $x_k = t^{-1}\alpha^{-2}$, $x_i = t\alpha$, $x_j = \alpha$ of the curve s_k can be obtained putting $x_k = x_i^{-1}x_j^{-1}$ in $\gamma_k = 0$. Then we have the following polynomial equation of degree 6 in two variables x_i and x_j: $x_i^2x_j^2(x_i - x_j)^2 + 2x_ix_j(x_i + x_j) - 3 = 0$.

Substituting $x_i = tx_j$, $x_j = \sqrt[3]{u}$ into the obtained equation and solving it with respect to u we find its two different roots $u_1 := \left(-t - 1 + 2\sqrt{t^2 - t + 1}\right)t^{-1}(t-1)^{-2}$, $u_2 := \left(-t - 1 - 2\sqrt{t^2 - t + 1}\right)t^{-1}(t-1)^{-2}$, where $t > 0$, $t \neq 1$, but the second of them, taken with the minus sign, gives only negative values of x_i and x_j.

Denote $\widetilde{\alpha}(t) = \sqrt[3]{u_1(t)} > 0$. Note that $\lim_{t\to 0+}\widetilde{\alpha}(t) = +\infty$ and $\lim_{t\to +\infty}\widetilde{\alpha}(t) = 0$. This predicts the behavior of the curve s_k for values $t \to 0+$ and $t \to +\infty$ of the parameter t: $\lim_{t\to+\infty} x_j(t) = 0$, $\lim_{t\to+\infty} x_i(t) = \lim_{t\to+\infty} x_k(t) = +\infty$ and $\lim_{t\to 0+} x_i(t) = 0$, $\lim_{t\to 0+} x_j(t) = \lim_{t\to 0+} x_k(t) = +\infty$.

Connectedness of the curves s_1, s_2 *and* s_3. Note also that $\lim_{t\to 1+}\widetilde{\alpha}(t)=\lim_{t\to 1-}\widetilde{\alpha}(t)=p_0:=\sqrt[3]{6}/2$. Hence assigning $\alpha(1):=p_0$ and

$$\alpha(t):=\begin{cases}\widetilde{\alpha}(t), & \text{if } \ t>0,\ t\neq 1,\\ p_0, & \text{if } \ t=1\end{cases}$$

we define a continuous function $\alpha\colon G\to G$ on $G:=(0,+\infty)$. Therefore in the standard topology of $\mathbb{R}^3$ the set (curve) $s_k=F(G)\subset G^3$ must be connected as a continuous image of the connected set G under a function $F\colon G\to G^3$ with continuous coordinate components $x_i,x_j,x_k\colon G\to G$ such that $x_i(t)=t\alpha(t)$, $x_j(t)=\alpha(t)$ and $x_k(t)=t^{-1}\alpha(t)^{-2}$.

Smoothness of the curves s_1, s_2 *and* s_3 can be proved using their parametric equations. But we prefer another way. Due to symmetry it suffices to prove smoothness of the curve $s_i=\Sigma\cap\Gamma_i$. Since Σ and Γ_i are smooth (regular) surfaces it remains to show that their intersection is transversal, in other words their gradient vectors $\nabla V=(x_2x_3,x_1x_3,x_1x_2)=\left(x_1^{-1},x_2^{-1},x_3^{-1}\right)$ and $\nabla\gamma_i=(\gamma_{i1},\gamma_{i2},\gamma_{i3})$ are linearly independent along s_i, where

$$\gamma_{ij}:=\frac{\partial\gamma_i}{\partial x_j}=\begin{cases}x_i+x_j-x_k, & \text{if } \ j\neq i,\\ -3x_i+x_j+x_k, & \text{if } \ j=i,\end{cases}$$

for $i,j\in\{1,2,3\}$. Due to symmetry fix any i and suppose by contrary that $\nabla\gamma_i=c\,\nabla V$ for some real $c\neq 0$. This means that the equalities $\gamma_{ij}=cx_j^{-1}$ hold for $j\in 1,2,3$. Then for $j\neq i$ and $k\neq i$ we obtain equalities $(x_i+x_j-x_k)\,x_j=(x_i+x_k-x_j)\,x_k=c$ equivalent to $(x_j-x_k)(x_i+x_j+x_k)=0$ which is impossible for $x_i\neq x_j\neq x_k\neq x_i$. Actually we proved the more strong fact that the normal vectors ∇V and $\nabla\gamma_i$ are linearly independent not only along s_i, but everywhere where the surfaces Σ and Γ_i are defined excepting points (x_1,x_2,x_3) with non positive or coincided components.

(2) *Intersections of* s_1,s_2,s_3 *with* I_1,I_2,I_3. Due to symmetry it suffices to take the invariant curve I_k of the system (5) defined as $x_i=x_j=p$, $x_k=p^{-2}$, $p>0$. Consider the curve s_k. The question is whether I_k will cross the curve s_k or not. It suffices to answer this question for I_k and the surface Γ_k because existing of a point Z in $(0,+\infty)^3$ such that $Z\in I_k\cap\Gamma_k$ implies $Z\in I_k\subset\Sigma$ and hence $Z\in\Sigma\cap\Gamma_k=s_k$. Thus substituting $x_i=x_j=p$, $x_k=p^{-2}$ into the equation $\gamma_k=0$ of Γ_k, we obtain the equation $\gamma_k=(4p^3-3)p^{-4}=0$ which can admit the single root $p=p_0=\sqrt[3]{6}/2$ providing the unique common point $x_i=x_j=p_0$, $x_k=p_0^{-2}$ of I_k with s_k.

Consider now any curve s_i such that $i\neq k$. Then we obtain an incompatible system of equations $x_i=x_j=p$, $x_k=p^{-2}$ and $\gamma_i=0$ because of $\gamma_i=p^{-4}\neq 0$. Moreover, s_i asymptotically tends to I_k as $p\to+\infty$ according to $\lim_{p\to+\infty}\gamma_i=\lim_{p\to+\infty}p^{-4}=0$. The same result holds for s_j by symmetry in the equation of I_k. Theorem 1 is proved.

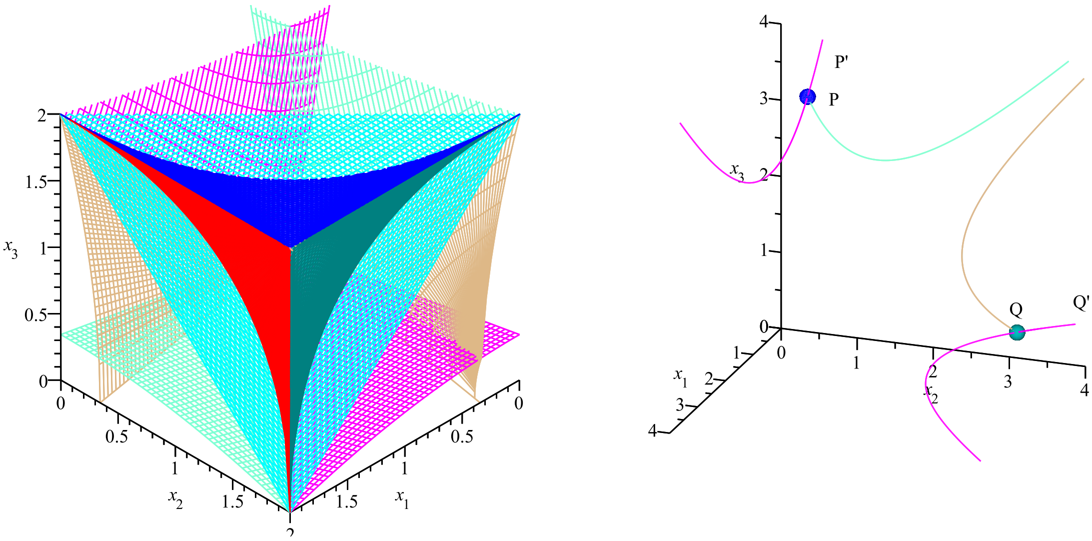

Figure 2. The cones $\Gamma_1, \Gamma_2, \Gamma_3$, $\Lambda_1, \Lambda_2, \Lambda_3$ and the planes $x_k = x_i + x_j$ for $\{i, j, k\} = \{1, 2, 3\}$ (the left panel); Crossing r_2 and r_3 by r_1 (the right panel)

1.2 Proof of Theorem 2

To prove Theorem 2 we need the following Lemma containing auxiliary results.

Lemma 1. For every generalized Wallach space with $a \in (0, 1/2)$ the set R is bounded by the conic surfaces Λ_1, Λ_2 and Λ_3. Each pair Λ_i and Λ_j has intersections along two different straight lines $x_i = x_j = u$, $x_k = 0$ and $x_i = x_j = av$, $x_k = v$, where $u, v > 0$.

The cones Λ_1, Λ_2 and Λ_3 are depicted in the left panel of Figure 2 in magenta, aquamarine and burlywood colors respectively.

Proof of Lemma 1. Consider the surface Λ_k. Since $D := x_i^2 - a^{-1}x_i x_j + x_j^2$ is symmetric with respect to x_i and x_j it can be considered as a quadratic polynomial in x_j without loss of generality. Then $D \le 0$ if $mx_i \le x_j \le Mx_i$ and $D > 0$ if $0 < x_j < mx_i$ or $x_j > Mx_i$, where

$$m = m(a) := \left(1 - \sqrt{1 - 4a^2}\right)(2a)^{-1}, \qquad M = M(a) := \left(1 + \sqrt{1 - 4a^2}\right)(2a)^{-1}. \tag{11}$$

It is easy to see that $0 < m(a) < M(a)$ for all $a \in (0, 1/2)$.

Depending on the sign of D the inequality $\lambda_k > 0$ admits the positive solution $x_k > \sqrt{D}$ if $D > 0$ and any $x_k > 0$ can satisfy $\lambda_k > 0$ if $D \le 0$. This means that besides the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ the set R is bounded by two disjoint connected components Λ_{kj} and Λ_{ki} of the surface $\Lambda_k = \Lambda_{ki} \cup \Lambda_{kj}$ defined by the same equation $x_k = \Psi(x_i, x_j) := \sqrt{x_i^2 - a^{-1}x_i x_j + x_j^2}$ but on the different domains $\left\{(x_i, x_j) \in \mathbb{R}^2 \;\middle|\; x_i > 0,\, 0 < x_j < mx_i\right\}$ and $\left\{(x_i, x_j) \in \mathbb{R}^2 \;\middle|\; x_i > 0,\, x_j > Mx_i\right\}$ respectively.

Due to symmetry the same properties hold for the surfaces Λ_i and Λ_j as well. Thus $\partial(R) = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$.

By the same reason it suffices to analyze only $\Lambda_i \cap \Lambda_j$. Assume that some triple (x_1, x_2, x_3) satisfies both of $\lambda_i = 0$ and $\lambda_j = 0$. Then $\lambda_i - \lambda_j = 0$ and $\lambda_i + \lambda_j = 0$ imply the system of equations $(x_i - x_j)(x_k - 2a(x_i + x_j)) = 0$ and $x_k(x_i + x_j - 2ax_k) = 0$. In what follows that the system of the equations $\lambda_i = 0$ and $\lambda_j = 0$ can admit only the following two different families of one-parametric solutions $x_i = x_j = u$, $x_k = 0$ and $x_i = x_j = av$, $x_k = v$ with parameters $u, v > 0$. Lemma 1 is proved.

Proof of Theorem 2. (1) Clearly $\partial(\Sigma \cap R) = r_1 \cup r_2 \cup r_3$ directly follows from $\partial(R) = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ proved in Lemma 1. Intersecting both of the connected components Λ_{ki} and Λ_{kj} of the cone Λ_k the surface Σ forms components $r_{ki} = \Sigma \cap \Lambda_{ki}$ and $r_{kj} = \Sigma \cap \Lambda_{kj}$ of the curve r_k such that $r_k = r_{ki} \cup r_{kj}$ and $r_{ki} \cap r_{kj} = \emptyset$.

Smoothness of the components of r_1*,* r_2 *and* r_3. Consider the curve r_k. We claim that the gradient vectors $\nabla V = \left(x_1^{-1}, x_2^{-1}, x_3^{-1}\right)$ and $\nabla\lambda_k = (\lambda_{k1}, \lambda_{k2}, \lambda_{k3})$ of the surfaces Σ and Λ_k are linearly independent for all positive x_1, x_2, x_3 such that $x_1 \neq x_2 \neq x_3 \neq x_1$, where

$$\lambda_{kj} := \frac{\partial\lambda_k}{\partial x_j} = \begin{cases} x_i - 2ax_j, & \text{if } \ j \neq k, \\ 2ax_k, & \text{if } \ j = k, \end{cases}$$

for $k, j \in \{1, 2, 3\}$. Indeed supposing $\nabla\lambda_k = c\nabla V$, where c is a nonzero real number, we obtain immediately an unreachable equality $(x_j - x_i)(x_j + x_i) = 0$. In what follows that each component r_{k1} and r_{k2} of the curve r_k is a smooth curve as a transversal intersection of two smooth surfaces.

Connectedness of the components of r_1, r_2, r_3. The variable x_k can be eliminated from the system of equations $x_i x_j x_k = 1$ and $\lambda_k = 0$ to obtain the equation

$$a x_i^2 x_j^2 \left(x_i^2 + x_j^2\right) - x_i^3 x_j^3 - a = 0$$

of the projection of the curve r_k onto the coordinate plane (x_i, x_j). By the same way as in Theorem 1 substituting $x_i = tx_j$, $x_j = \sqrt[3]{u}$ into the last equation and solving it with respect to u we obtain the parametric equation

$$x_k = t^{-1}\beta^{-2}, \quad x_i = t\beta, \quad x_j = \beta$$

of the curve r_k, where $\beta = \beta(t) := \left(t^4 - a^{-1}t^3 + t^2\right)^{-\frac{1}{6}} > 0$. It is easy to see that the numbers $m = m(a)$ and $M = M(a)$, $0 < m < M$, given in (11) are different real roots of the polynomial $t^2 - a^{-1}t + 1$ for all $a \in (0, 1/2)$. Therefore β can be rewritten in the form

$$\beta = \beta(t) := \left(t^2(t - m)(t - M)\right)^{-\frac{1}{6}}.$$

In what follows that the function $\beta(t)$ is well defined, continuous and positive valued for $t \in (0, m) \cup (M, +\infty)$. We conclude now that the components r_{ki} and r_{kj} of r_k are respectively continuous images of the connected sets $(0, m)$ and $(M, +\infty)$ under a vector-function with coordinates $x_i(t)$, $x_j(t)$ and $x_k(t)$. Therefore r_{ki} and r_{kj} are connected too.

Note that the components r_{ki} and r_{kj} are symmetric under the permutation $i \to j \to i$. Therefore we can parameterize them on the same interval but using different formulas (8) and (9) respectively. For simplicity we choose the interval $(0, m)$.

Intersections of r_1*,* r_2 *and* r_3. Consider the pair r_1 and r_2. By Lemma 1 the only common line of the surfaces Λ_1 and Λ_2 which consists of points with nonzero coordinates is the straight line $x_1 = x_2 = av$, $x_3 = v$, $v > 0$. This line intersects the surface Σ at a unique point, denote it P_{12}. Indeed substituting $x_1 = x_2 = av$, $x_3 = v$ into $x_1x_2x_3 = 1$ we get the unique value $v = v_0 := a^{-2/3}$. This yields coordinates $\left(a^{1/3}, a^{1/3}, a^{-2/3}\right)$ of P_{12}. Note that P_{12} (the point P in the right panel of Figure 2) is also the only intersection point of the curves r_1 and r_2 (their components r_{12} and r_{21}).

Now a value of t at which P_{12} is located in r_1 can be found from the parametric representation $x_1(t) = t^{-1}\beta(t)^{-2}$, $x_2(t) = t\beta(t)$, $x_3(t) = \beta(t)$ of r_{12}. The condition $x_1 = x_2$ implies an equation $t^{-1}\beta^{-2} = t\beta$ admitting the single root $t_0 = a$ for all $a \in (0, 1/2)$. Therefore the curve r_1 passes through P_{12} at $t = t_0$ only. It should be noted that the curves r_1 and r_2 leave extra pieces after crossing each other. In principle, we can preserve them, but it is advisable to remove them for greater clarity of pictures. Basing on the values of the limits $\lim_{t\to 0+} x_2(t) = 0$, $\lim_{t\to 0+} x_1(t) = \lim_{t\to 0+} x_3(t) = +\infty$

and $\lim_{t\to m-} x_1(t) = 0$, $\lim_{t\to m-} x_2(t) = \lim_{t\to m-} x_3(t) = +\infty$ we conclude that the tail PP' corresponds to values $t \in (a, m)$. Therefore the original interval of parametrization $(0, m)$ can be reduced to the interval $(0, a]$ shown in the text of Theorem 2.

By symmetry the analysis of the pairs $r_1 \cap r_3$ and $r_2 \cap r_3$ (points in teal and red color in the right panel of Figure 1) will be the same using permutations of the indices $\{i, j, k\} = \{1, 2, 3\}$. For example, the equations $x_1(t) = t^{-1}\beta(t)^{-2}$, $x_2(t) = \beta(t)$ and $x_3(t) = t\beta(t)$ define another connected component r_{13} of the curve r_1 (which intersects r_3) on the same interval $(0, a]$. Then coordinates $\left(a^{1/3}, a^{-2/3}, a^{1/3}\right)$ of the point P_{13} (in fact $\{P_{13}\} = r_{13} \cap r_{31}$) can be obtained at the same boundary value $t = a$ (the point Q in the right panel of Figure 2). Analogously at $t \in (a, m)$ we get the tail QQ' of r_{13}.

(2) *Intersections of* r_1, r_2, r_3 *with* I_1, I_2, I_3. Without loss of generality consider the invariant curve I_k. As in Theorem 1 it suffices to consider the surfaces Λ_i instead of the corresponding curves r_i. The curve I_k crosses both of the curves r_i and r_j (the components r_{ij} and r_{ji}) exactly at their common point P_{ij} because substituting $x_i = x_j = p$, $x_k = p^{-2}$ into $\lambda_i = 0$ and $\lambda_j = 0$ yields the equation

$$\lambda_i = \lambda_j = (p^3 - a)p^{-4} = 0$$

which admit a single root $p = a^{1/3}$ corresponding to P_{ij}. Therefore $I_k \cap r_{ij} \cap r_{ji} = \{P_{ij}\}$.

At the same time I_k approximates both of r_i and r_j (their components r_{ik} and r_{jk}) at infinity. Indeed

$$\lim_{p\to+\infty} \lambda_i = \lim_{p\to+\infty} \lambda_j = \lim_{p\to+\infty} (p^3 - a)p^{-4} = 0.$$

For the curve r_k we have $\lambda_k = (1 - 2a)p^2 + p^{-4} > 0$ under the same substitutions. Therefore I_k never cross r_k, moreover, $\lim_{p\to+\infty} \lambda_k = +\infty$.

(3) *Every singular point of* (5) *belongs to* $\Sigma \cap R$. As it follows from [6] the system of algebraic equations $f_i(x_1, x_2, x_3) = 0$ has the following four families of one-parametric solutions for every $a \in (0, 1/2) \setminus \{1/4\}$:

$$x_1 = x_2 = x_3 = \tau, \qquad x_i = \tau\kappa,\ x_j = x_k = \tau, \quad \tau > 0, \quad \{i, j, k\} = \{1, 2, 3\}, \tag{12}$$

where $\kappa := (1 - 2a)(2a)^{-1}$. At $a = 1/4$ these families merge to the unique family $x_1 = x_2 = x_3 = \tau$.

Substituting $x_i = \tau\kappa$ and $x_j = x_k = \tau$ into the expressions (7) for λ_1, λ_2 and λ_3 we obtain

$$\lambda_1 = \lambda_2 = \lambda_3 = (1 - 2a)(1 + 2a)(4a)^{-1}\, \tau^2 > 0,$$

because $a \in (0, 1/2)$. Obviously,

$$\lambda_1 = \lambda_2 = \lambda_3 = (1 - a)\, \tau^2 > 0$$

at $x_1 = x_2 = x_3 = \tau$. Therefore the straight lines (12) lye in the set R for all $a \in (0, 1/2)$ according to the definition of R. These lines cross the invariant surface Σ at the points (see also [22])

$$\mathbf{o}_0 := (1, 1, 1), \quad \mathbf{o}_1 := (q\kappa, q, q)\,, \quad \mathbf{o}_2 := (q, q\kappa, q)\,, \quad \mathbf{o}_3 := (q, q, q\kappa)\,,$$

being the singular points of the system (5) on Σ, where $q := \sqrt[3]{\kappa^{-1}}$ (obviously, the unique singular point $(1, 1, 1)$ will be obtained if $a = 1/4$). Thus we conclude that $\mathbf{o}_i \in \Sigma \cap R$ for every $a \in (0, 1/2)$ and $i \in \{0, 1, 2, 3\}$.

(4) *Invariancy of the curves* l_1, l_2, l_3. According to [6] the curves I_1, I_2 and I_3 are separatrices of the unique saddle point $\mathbf{o}_0$ (which has the *linear zero type*) of the system (5) if $a = 1/4$. For $a \in (0, 1/2) \setminus \{1/4\}$ the points $\mathbf{o}_1, \mathbf{o}_2, \mathbf{o}_3$ are all *hyperbolic type* saddles and $\mathbf{o}_0$ is a stable (respectively

unstable) *hyperbolic* node if $1/4 < a < 1/2$ (respectively if $0 < a < 1/4$). Additionally, each invariant curve I_k is one of two separatrices of the saddle $\mathbf{o}_k$ (see [22]), where $k = 1, 2, 3$. At $a = 1/6$ we have an opportunity to find analytically the second separatrice of each $\mathbf{o}_k$ different from I_k. Indeed it is easy to see that coordinates of $\mathbf{o}_k$ satisfy the system of equations

$$\begin{cases} x_k = x_i + x_j, \quad \{i, j, k\} = \{1, 2, 3\}, \\ x_i x_j x_k = 1, \end{cases} \tag{13}$$

where the equalities $x_k = x_i + x_j$ describe the set of Kähler metrics on a given generalized Wallach space G/H with $a_1 = a_2 = a_3 = a = 1/6$ (see also [4]). Therefore each saddle $\mathbf{o}_k$ belongs to the intersection l_k of the invariant surface Σ with the plane $x_k = x_i + x_j$ (the curves l_1, l_2, l_3 are depicted in the right panel of Figure 1 in cyan color for all indices $\{i, j, k\} = \{1, 2, 3\}$). Parametric equations (10) of the curves l_k can be obtained repeating similar procedures as in the case of the curves s_k and r_k.

It is easy to show that $l_i \cap l_j = \emptyset$ for $i \neq j$. Moreover, we claim that each of l_1, l_2, l_3 is also an invariant curve of the differential system (5). To show it consider the case $k = 3$ due to symmetry. Substitute the parametric representation $x_1 = \phi$, $x_2 = t\phi$, $x_3 = t^{-1}\phi^{-2}$ of the curve l_3 into f_1, f_2 and f_3 in (5), where

$$\phi = \phi(t) := \sqrt[3]{(t^2 + t)^{-1}}, \quad t > 0.$$

For $x_1 = \phi$, $x_2 = t\phi$ and $x_3 = t^{-1}\phi^{-2}$ the functions f_1, f_2, f_3 take the following forms

$$f_1 = -\frac{2}{9}\,\frac{(2t + 1)(t - 1)}{t(t + 1)}, \qquad f_2 = \frac{2}{9}\,\frac{(t + 2)(t - 1)}{t + 1}, \qquad f_3 = \frac{2}{9}\,\frac{(t - 1)^2}{t}.$$

The value $t = 1$ providing $f_1 = f_2 = f_3 = 0$ gives a stationary trajectory, namely it is the singular point $\mathbf{o}_3 = (q, q, q\kappa)$ itself. Thus assume $t \neq 1$. The identities

$$\begin{aligned}
\frac{dx_2}{dx_1} &= \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} \equiv \frac{(t\phi)'}{\phi'} = -\frac{(t + 2)t}{2t + 1} = \frac{f_2}{f_1}, \\
\frac{dx_3}{dx_1} &= \frac{\left(t^{-1}\phi^{-2}\right)'}{\phi'} = -\frac{t^2 - 1}{2t + 1} = \frac{f_3}{f_1}, \\
\frac{dx_3}{dx_2} &= \frac{\left(t^{-1}\phi^{-2}\right)'}{(t\phi)'} = \frac{t^2 - 1}{t(t + 2)} = \frac{f_3}{f_2}
\end{aligned}$$

imply that l_3 is a trajectory of (5) for $t > 0$ and $t \neq 1$. Moreover, l_3 passes through the singular point $\mathbf{o}_3$. This means that l_3 is a separatrice of $\mathbf{o}_3$. Invariancy of the curves l_1 and l_2 respectively passing through $\mathbf{o}_1$ and $\mathbf{o}_2$ can be proved using the same idea. Theorem 2 is proved.

Remark 1. As it was noted in the proof of Theorem 2 the equations (8) define for $t \in (M, +\infty)$ the same curve as (9) for $t \in (0, m)$. In the case $t \in (M, +\infty)$ the tail removing procedure leads to the equation $t^{-1}\beta^{-2} = \beta$ equivalent to $at^2 - (a^2 + 1)t + a = 0$. Its first root $t = a$ corresponds to the point P_{ki} and the second root $t = 1/a$ gives the point P_{kj}. Obviously $0 < a < m < M < 1/a$ for all $a \in (0, 1/2)$. Therefore both components of each curve r_k can be parameterized by one formula, say (8), but using the different intervals $(0, a]$ and $[1/a, +\infty)$.

Remark 2. We proved that all singular points $\mathbf{o}_0, \mathbf{o}_1, \mathbf{o}_2$ and $\mathbf{o}_3$ of the normalized Ricci flow on generalized Wallach spaces with $a_1 = a_2 = a_3 = a$ belong to the set $\Sigma \cap R$ of metrics with positive Ricci curvature. Unfortunately a similar assertion does not hold for the set $\Sigma \cap S$. Lemma 3 in [22] shows that there exists a critical value $a = 3/14$ such that $\mathbf{o}_1, \mathbf{o}_2, \mathbf{o}_3 \in \Sigma S$ only if $a \in (3/14, 1/2)$ and the boundary cases $\mathbf{o}_i \in s_i$ $(i = 1, 2, 3)$ hold if $a = 3/14$. The only generalized Wallach spaces which admit metrics with positive sectional curvature are the Wallach spaces (1) which satisfy the condition $a \in (0, 3/14)$.

Remark 3. The case $a = 1/6$ is original, where Kähler metrics provide separatrices of saddles $\mathbf{o}_i$. For $a \neq 1/6$ it is a difficult problem to find similar separatrices analytically. Knowing all separatrices allows to predict the dynamics of the Ricci flow in more detail. To demonstrate the main idea consider an arbitrarily chosen singular point in the case $a = 1/6$. Without loss of generality take $\mathbf{o}_3 = \left(2^{-1/3}, 2^{-1/3}, 2^{2/3}\right)$ (the Kähler-Einstein metric) and observe that the curve l_3 defined by the equations $x_3 = x_1 + x_2$ and $x_1x_2x_3 = 1$ coincides with the unstable manifold W_3^u of $\mathbf{o}_3$ as it was shown in [22]. The stable manifold of $\mathbf{o}_3$ is $W_3^s := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_3 = p^{-2},\ x_1 = x_2 = p,\ 0 < p < 1\right\} \subset I_3$. It is clear now that controlling by W_i^s and W_i^u trajectories of (5) never can leave the domain bounded by the curves (13) because of no trajectory originated in that domain can intersect separatrices by the uniqueness of a solution of an initial value problem. This explains the fact proved in Theorem 4 in [4] that Riemannian metrics (2) on generalized Wallach spaces with $a = 1/6$ (on the Wallach space $\mathrm{SU}(3)/T_{\max}$ in particularly) preserve the positivity of their Ricci curvature for $x_k < x_i + x_j$ $(\{i, j, k\} = \{1, 2, 3\})$. In Figure 3 the separatrices l_1, l_2, l_3 and some trajectories of (5) are depicted for illustrations.

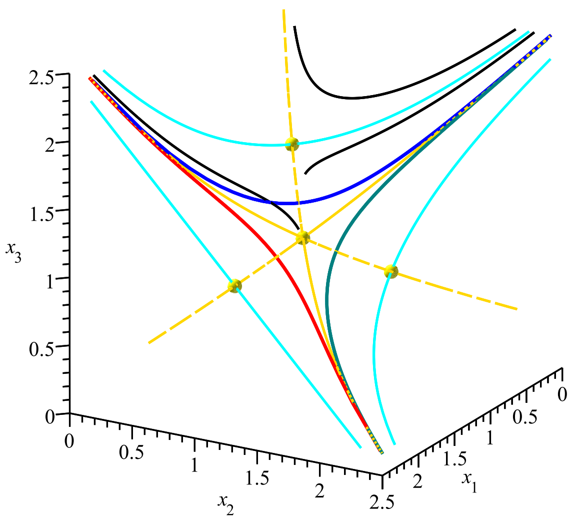

Figure 3. The case $a = 1/6$: the separatrices l_i (in cyan color), I_i (in yellow color) of the saddles $\mathbf{o}_i$ and some trajectories (in black color) of system (5), $i = 1, 2, 3$

2 *Additional remarks*

i) The well known fact that the positivity of the Ricci curvature follows from the positivity of the sectional curvature can be justified and illustrated via inclusion $S \subset R$, where S is depicted in Figure 2 as a set bounded by three cones in red, teal and blue colors, respectively R is bounded by six conic surfaces in magenta, aquamarine and burlywood colors.

To establish $S \subset R$ for all $a \in (0, 1/2)$ it suffices to show the inclusion $\partial(S) \subset R$. We will follow this opportunity since a direct attempt to establish $S \subset R$ leads to pairs of inequalities of the kind $\gamma_i > 0$ and $\lambda_i > 0$ whose analysis is much more complicated than to deal with the system consisting of one equation $\gamma_i = 0$ and one inequality $\lambda_i > 0$:

$$\begin{cases} (x_j - x_k)^2 + 2x_i(x_j + x_k) - 3x_i^2 = 0, \\ x_j x_k + a\left(x_i^2 - x_j^2 - x_k^2\right) > 0. \end{cases} \tag{14}$$

By symmetry fix any $i \in \{1, 2, 3\}$ and consider the component Γ_i of the boundary of S. Every point of the cone Γ_i belongs to some its generator line $x_i = \nu t,\ x_j = \mu t,\ x_k = t,\ t > 0$, where $\mu = 1 - \nu + 2\sqrt{\nu(\nu-1)} > 0,\ \nu > 1$ (see also [22]). Indeed generators satisfy the equation in (14) and the inequality in (14) takes the form $(X - Y)\,t^2 > 0$ with $X := \big(4a(\nu-1) + 2\big)\sqrt{\nu(\nu-1)}$ and $Y := 4a\nu^2 + (1-6a)\nu + 2a - 1$.

Obviously $X > 0$ for all $a \in (0, 1/2)$ and $\nu > 1$. Since $Y = 0$ has roots $\nu_1 = \frac{2a-1}{4a} < 0$ and $\nu_2 = 1$ the inequality $Y > 0$ holds as well at $\nu > 1$. Thus $X - Y > 0$ is equivalent to $X^2 - Y^2 = (\nu - 1)\,p(\nu) > 0$, where the quadratic polynomial $p(\nu) = 8a\nu^2 - (2a+3)(2a-1)\nu + (2a-1)^2$ admits two different negative roots $\nu_1 = \frac{2a-1}{16a}\left(2a + 3 + \sqrt{(2a-1)(2a-9)}\right)$ and $\nu_2 = \frac{2a-1}{16a}\left(2a + 3 - \sqrt{(2a-1)(2a-9)}\right)$ for every $a \in (0, 1/2)$. It follows then $p(\nu) > 0$ and hence $X^2 - Y^2 > 0$ at $\nu > 1$ independently on $a \in (0, 1/2)$. Therefore $\lambda_i > 0$ for any point of Γ_i which means that $\Gamma_i \subset R$. Since i was chosen arbitrarily we obtain $\partial(S) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \subset R$ and hence $S \subset R$ with the obvious consequence $\Sigma \cap S \subset \Sigma \cap R$.

ii) There are useful asymptotical representations for practical aims. For instance, at $t \to 0$ the expressions $x_1(t) = x_3(t) = t^{-1/3} + \mathcal{O}\big(t^{5/3}\big)$, $x_2(t) = t^{2/3} + \mathcal{O}\big(t^{8/3}\big)$ are valid for coordinates of points of the curve s_3 defined as a variety of solutions of the system

$$\begin{cases} (x_1 - x_2)^2 + 2x_3(x_1 + x_2) - 3x_3^2 = 0, \\ x_1 x_2 x_3 = 1. \end{cases} \tag{15}$$

For t tending to 0 the curve $r_1\ :\ \begin{cases} x_2 x_3 + a\left(x_1^2 - x_2^2 - x_3^2\right) = 0, \\ x_1 x_2 x_3 = 1 \end{cases}$ has a similar asymptotic $x_1(t) = t^{-1/3} + \mathcal{O}\big(t^{5/3}\big)$, $x_2(t) = t^{2/3} + \frac{t^{5/3}}{6a} + \mathcal{O}\big(t^{8/3}\big)$, $x_3(t) = t^{-1/3} + \frac{t^{2/3}}{6a} + \mathcal{O}\big(t^{5/3}\big)$ in accordance with the fact that s_3 and $r_{12} \subset r_1$ approach the same invariant curve I_2 at infinity.

iii) Often it is easier to deal with a planar analysis of the dynamics of the normalized Ricci flow. Choose the coordinate plane $x_3 = 0$ without loss of generality. Then the projection of the set $\Sigma \cap S$ of Riemannian metrics with positive sectional curvature onto the plane $x_3 = 0$ is bounded by the following plane curves s_1', s_2' and s_3' defined implicitly $3x_1^4x_2^2 - 2x_1^3x_2^3 - x_1^2x_2^4 - 2x_1^2x_2 + 2x_1x_2^2 - 1 = 0$, $3x_2^4x_1^2 - 2x_2^3x_1^3 - x_2^2x_1^4 - 2x_2^2x_1 + 2x_2x_1^2 - 1 = 0$ and $x_1^4x_2^2 - 2x_1^3x_2^3 + x_1^2x_2^4 + 2x_1^2x_2 + 2x_1x_2^2 - 3 = 0$.

For example the equation of s_3' can be obtained eliminating x_3 in the system (15).

Analogously, boundary curves of the projection of the set $\Sigma \cap R$ of Ricci positive metrics onto the plane $x_3 = 0$ have equations $ax_1^4x_2^2 - ax_1^2x_2^4 + x_1x_2^2 - a = 0$, $ax_2^4x_1^2 - ax_2^2x_1^4 + x_2x_1^2 - a = 0$ and $ax_1^4x_2^2 + ax_1^2x_2^4 - x_1^3x_2^3 - a = 0$.

Projections of the Kähler metrics $x_1 = x_2 + x_3$, $x_2 = x_1 + x_3$ and $x_3 = x_1 + x_2$ will be defined by $x_1x_2(x_1 - x_2) = 1$, $x_1x_2(x_2 - x_1) = 1$ and $x_1x_2(x_1 + x_2) = 1$ respectively.

We recommend to compare the pictures demonstrated in this paper with planar pictures depicted in the right panels of Figures 3, 6 and 7 obtained in [4] in the coordinate plane (x_1, x_2).

Author Contributions

The single author contributed to this work.

Conflict of Interest

The author declares no conflict of interest.

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