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The cosemanticness of Kaiser hulls of fixed classes of models

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In this article, within the framework of the study of Jonsson theories, the model-theoretic properties of cosemanticness classes belonging to the factor set of the Jonsson spectrum of an existentially closed models' subclass of some Jonsson theory in a fixed language were studied. Various results have been obtained. In particular, the properties of the cosemanticness of models and classes of models are considered; some results concerning the Jonsson equivalence in generalization for classes of existentially closed models are obtained; a criterion for the cosemanticness of J-classes in connection with their Kaiser hulls has been found.

Keywords: Jonsson theory, cosemanticness, cosemantic Jonsson theories, Jonsson spectrum, cosemanticness classes, Kaiser hull, Jonsson equivalence, J-class, cosemantic models, cosemantic classes.

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Introduction

It is well known that in modern Model Theory, the issue of studying incomplete theories occupies a special place due to the small number of suitable methods and techniques. Anyway, this is a very difficult task, so, as a rule, model theorists use various limiting conditions to obtain results concerning incomplete theories. One of the relevant directions in this sense is studying Jonsson theories. The relevance is determined by various reasons, and the main one is that the Jonsson theories are of great applied importance in algebra due to the presence of many classical examples linking these two mathematical areas.

Traditionally, the Karaganda School of Model Theory uses the definition of the Jonsson theory given in the Russian-language edition of [1]. In recent years, the apparatus for studying Jonsson theories has been significantly expanded, which is demonstrated by the number and variety of approaches in the works [2–9].

At the same time, one of our essential areas of research in this area is not only to obtain results describing the properties of the Jonsson theories, but also to generalize these results. In 2018, Yeshkeyev A.R. introduced the concept of the Jonsson spectrum of a fixed class of models, which is a special set of Jonsson theories. When considering the Jonsson spectrum we also use the notion of the cosemanticness relation proposed by Mustafin T.G. Cosemanticness is a specific equivalence relation that generalizes and refines the elementary equivalence in terms of researching Jonsson theories. It is well known that equivalence relation is a classical instrument for studying and constructing the classification of theories in Model Theory. In this matter, in [10, 11], Yeshkeyev A.R. and Ulbrikht O.I. obtained some considerable results on abelian groups and R-modules concerning cosemanticness and other related concepts, such as consemanticness classes.

Thus, studying the properties of the cosemanticness classes of the Jonsson spectrum is of great importance not only for the development of the apparatus for the study of Jonsson theories. Firstly, this area is of interest from the point of view of research in Model Theory. In addition, it was found in [12–16] that the Jonsson theories and their cosemanticness classes in the Jonsson spectrum of fixed classes of

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structures have interesting structural properties that are important for Universal Algebra. In [17], the lattices of existential formulas of a fixed Jonsson theory are considered in terms of syntactic similarity. In this article, we present some basic results that demonstrate the relationship of the cosemanticness classes of the Jonsson spectrum with fixed classes of structures and the specific properties of Kaiser hulls of these classes. These results are the forerunner of the study of Jonsson theories from the point of view of lattice algebra and other related fields.

This paper consists of two sections. In Section 1, we give some basic information on Jonsson theories. In Section 2, we present our results obtained for cosemanticness classes of Jonsson spectrum, so-called J-classes of structures and their Kaiser hulls.

1 Preliminary information on Jonsson theories

In this section we describe the apparatus of the study of Jonsson theories. Let us start with some basic definitions.

Definition 1. [1] A theory T has the joint embedding property (JEP), if, for any models A and B of T, there exists a model M of T and isomorphic embeddings $f : A \to M$, $g : B \to M$.

Definition 2. [1] A theory T has the amalgamation property (AP), if for any models A, B_1, B_2 of T and isomorphic embeddings $f_1 : A \to B_1$, $f_2 : A \to B_2$ there are $M \models T$ and isomorphic embeddings $g_1 : B_1 \to M$, $g_2 : B_2 \to M$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Originally, the properties of amalgamation and joint embedding are algebraic notions. However they play a crucial role in studying various classes of structures in Model Theory, especially for incomplete theories.

There are syntactic criteria of AP and JEP. We give two classical theorems of them.

Theorem 1 (Robinson). [18] For the first order theory T of the language L (of arbitrary cardinality) the following conditions are equivalent:

1) T has JEP;

2) For all universal sentences α, β of L, if $T \vdash \alpha \vee \beta$ then $T \vdash \alpha$ or $T \vdash \beta$.

It is well known that the given theorem is equivalent to the following statement:

Theorem 2. Let T be a theory of the first-order language L. Then T has JEP iff whenever $T \cup \{\varphi\}$ and $T \cup \{\psi\}$ are consistent sets, where φ and ψ are arbitrary existential sentences of L, $T \cup \{\varphi \wedge \psi\}$ is also consistent.

Theorem 3 (Bryars). [18] The following are equivalent:

1) T has the amalgamation property;

2) for all $\alpha_1(\overline{x}), \alpha_2(\overline{x}) \in \forall_1$ with $T \vdash \alpha_1 \vee \alpha_2$ there are $\beta_1(\overline{x}), \beta_2(\overline{x}) \in \exists_1$ such that $T \vdash \beta_i \to \alpha_i$ $(i = 1, 2)$ and $T \vdash \beta_1 \vee \beta_2$.

The following theorem plays an important role in studying Jonsson Model Theory.

Theorem 4 (Hodges). [19; 363] Let T be a theory of the first-order language L, and let T have JEP. Suppose that A and B are existentially closed models of T. Then every $\forall\exists$-sentence true in A is true in B as well.

Now we recall the main definition of our study.

We are working within the framework of the following definition of Jonsson theory published in the Russian edition of [1].

Definition 3. [1; 80] A theory T is called Jonsson if the following conditions hold for T:

1. T has at least one infinite model;
2. T is an inductive theory;
3. T has the amalgam property (AP);
4. T has the joint embedding property (JEP).

There are a lot of algebraic examples of Jonsson theories. Classical examples include
1) group theory;
2) the theory of abelian groups;
3) the theory of Boolean algebras;
4) the theory of linear orders;
5) field theory of characteristic p, where p is zero or a prime number;
6) the theory of ordered fields;
7) the theory of modules.

In [20], it is proved that the theory of differentially closed fields of the fixed characteristic is a Jonsson theory as well.

It is important to note that, by Theorem 4, we can see that, for any Jonsson theory T, all existentially closed models of T are elementary equivalent by $\forall\exists$-sentences.

Further we give the notions and statements that are of great importance in research in Jonsson theories. Definitions 4, 5 and Theorem 5 were introduced by Mustafin T.G.

Definition 4. [21; 155] Let T be a Jonsson theory. A model C_T of power $2^{|T|}$ is called a semantic model of the theory T if C_T is a $|T|^+$-homogeneous $|T|^+$-universal model of the theory T.

Theorem 5. [21; 155] An inductive theory T is Jonsson iff it has a $|T|^+$-homogeneous $|T|^+$-universal model.

Definition 5. [21; 161] The elementary theory of the semantic model of the Jonsson theory T is called the center of this theory. The center is denoted by T^*, i.e. $Th(C) = T^*$.

Now we move to the central notion of this work.

Let L be a first-order language of a signature σ and let K be a class of L-structures. We consider a specific sets of theories for K that is called a Jonsson spectrum of K. The Jonsson spectrum can be described as follows.

Definition 6. [11] A set $JSp(K)$ of Jonsson theories of L, where

$$JSp(K) = \{T \mid T \text{ is a Jonsson theory and } K \subseteq Mod(T)\},$$

is said to be a Jonsson spectrum of K.

Jonsson spectra are well-described in [12, 22–24].

In terms of studying Jonsson theories, the notion of cosemanticness relation plays an important role. Let T_1 and T_2 be Jonsson theories, T_1^* and T_2^* be their centres, respectively.

Definition 7. [21; 40] T_1 and T_2 are said to be cosemantic Jonsson theories (denoted by $T_1 \bowtie T_2$), if $T_1^* = T_2^*$.

It is well known that the cosemanticness between two Jonsson theories is an equivalence relation. This means that, when introducing the relation of cosemanticness on the Jonsson spectrum $JSp(K)$, we get a partition of $JSp(K)$ into cosemanticness classes. The obtained factor set is denoted by $JSp(K)_{/\bowtie}$. This technique allows to obtain many significant generalizations when considering the cosemanticness classes instead of single theories. As it is mentioned before, applying this technique is the main idea of this paper, which will be revealed in Section 2.

2 The properties of Kaiser hulls for J-classes

In this section, we present the results of generalization of some well-known theorems published in different papers of the first author of this article. All of these theorems one can also find in [21].

Let T be a Jonsson theory in L, $K \subseteq E_T$. We consider the Jonsson spectrum of the given class K. Let us introduce the cosemanticness relation on $JSp(K)$. As it is well known, this relation is an equivalence relation, and therefore divides the spectrum into cosemanticness classes. Thus, we get a factor set $JSp(K)_{/\bowtie}$. Next, we will work with some fixed cosemanticness class $[T]$. It is clear that all the Jonsson theories in this class have the same semantic model, which we denote by $C_{[T]}$. In this section, we will work with this fixed class K, unless otherwise specified in the terms of the theorems or definitions.

Here we introduce the following notation. Let $T \in [T], [T] \in JSp(K)_{/\bowtie}$, A be an L-structure. Then $A \models [T]$ means that $A \models T$ for any $T \in [T]$. Similarly, this notation is generalized for the class of models as well, i.e. $K' \models [T]$ means that $A \models T$ for any $A \in K'$ and any theory $T \in [T]$.

Note that we only work within the framework of the fixed language L of signature σ.

Now we give the definitions of some necessary notions, which are actually generalizations of some well-known concepts from [21].

Definition 8. The class K' of existentially closed models of the signature σ is called a J-class, if the set of sentences $Th_{\forall\exists}(K')$ is a Jonsson theory.

Definition 9. The theory $Th_{\forall\exists}(K)$ that is a set of all $\forall\exists$-sentences of L true for each model of K, is said to be a Kaiser hull of K. We denote it by $T^0(K)$.

Note that the theory $T^0(K)$ is Jonsson, if it admits the amalgamation property, because it has infinite models, is inductive and, due to $\exists$-completeness, admits JEP. Moreover, in case of AP, $T^0(K)$ is a maximal Jonsson theory of K, and all theories T' such that $T \subseteq T' \subseteq T^0(K)$, where T is some Jonsson theory of K under consideration, are Jonsson.

Lemma 1. Let $[T] \in JSp(K)_{/\bowtie}$ consist only of $\exists$-complete theories, and let in $JSp(K)_{/\bowtie}$ there be such a class $[T']$, which consists of extensions of theories of the class $[T]$ in the same language. Then if $p(\bar{x}) \cup T$ is consistent for each theory $T \in [T]$, then $p(\bar{x}) \cup T'$ is also consistent for each theory $T' \in [T']$, where $p(\bar{x})$ is the set of $\exists$-formulas.

Proof. Let us consider an arbitrary theory $T \in [T]$. According to the condition of the Lemma, there is $T' \in [T']$ such that $T \subseteq T'$. It is obvious that if T is an $\exists$-complete theory, so is T'. Let $T \cup p(\bar{x})$ be a consistent set of formulas, for any $T \in [T]$, and $T' \cup p(\bar{x})$ be inconsistent, for any $T' \in [T']$. It means that there is $\exists\bar{y}\varphi(\bar{x}, \bar{y}) \in p(\bar{x})$, where $\varphi(\bar{x}, \bar{y})$ is a quantifier-free formula such that $T' \vdash \neg\exists\bar{x}\exists\bar{y}\varphi(\bar{x}, \bar{y})$. Consequently, $T' \vdash \forall\bar{x}\forall\bar{y}\neg\varphi(\bar{x}, \bar{y})$, and $T \vdash \forall\bar{x}\forall\bar{y}\neg\varphi(\bar{x}, \bar{y})$ as well, due to its $\forall$-completeness. The latter means that $T \cup p(\bar{x})$ is inconsistent, so we obtain a contradictory. Thus, for any theory $T \in [T]$ and any theory $T' \in [T']$, if $T \cup p(\bar{x})$ is consistent, $T' \cup p(\bar{x})$ is also consistent.

The following statement is one of the important properties of J-class.

Proposition 1. Let $[T'] \in JSp(K)_{/\bowtie}$ consist only of $\exists$-complete theories. Then any class $K' \subseteq K$ of infinite models is a J-class.

Proof. It is clear that K' is never empty, as soon as, by the conditions stated before, K consists of existentially closed models of some Jonsson theory T, which are infinite. We need to show that K' is a J-class, i.e., according to Definition 8, $Th_{\forall\exists}(K')$ is a Jonsson theory. Let us check it through Definition 3:

1) K' contains infinite models by the condition of the Proposition;

2) It is obvious that $Th_{\forall\exists}(K')$ is a set of $\forall\exists$-sentences, so this theory is inductive;

3) $Th_{\forall\exists}(K')$ is always an existentially complete theory, so it is easy to see that, by Theorem 1, it has JEP;

4) Let $Th_{\forall\exists}(K') \vdash \alpha_1 \vee \alpha_2$ for any L-formulas $\alpha_1(\overline{x}), \alpha_2(\overline{x}) \in \forall_1$. By JEP, it means that $Th_{\forall\exists}(K') \vdash \alpha_1$ or $Th_{\forall\exists}(K') \vdash \alpha_2$. Since every $T' \in [T']$ is an inductive theory, $T' \subseteq Th_{\forall\exists}(K')$, for all $T' \in [T']$. And due to the fact that each T' in this cosemanticness class is $\exists$-complete (and therefore $\forall$-complete), $T' \vdash \alpha_1$, if $Th_{\forall\exists}(K') \vdash \alpha_1$, and $T' \vdash \alpha_2$, if $Th_{\forall\exists}(K') \vdash \alpha_2$. Every theory in $[T']$ admits AP, so if $T' \vdash \alpha_1$ then $T' \vdash \alpha_1 \vee \alpha_1$ and, by Theorem 3, there are $\beta_1(\overline{x}), \beta_2(\overline{x}) \in \exists_1$ such that $T' \vdash \beta_i \to \alpha_i$ $(i = 1, 2)$ and

$T' \vdash \beta_1 \vee \beta_2$. The same is if $T' \vdash \alpha_2$. Therefore, $Th_{\forall\exists}(K') \vdash \beta_i \to \alpha_i$ $(i = 1, 2)$ and $Th_{\forall\exists}(K') \vdash \beta_1 \vee \beta_2$, which means that $Th_{\forall\exists}(K')$ admits AP.

To prove some further theorems, we need the following lemma.

Lemma 2. Let T_1 and T_2 be L-theories and let $T' = T_1 \vee T_2 = \{\varphi \vee \psi \mid \varphi \in T_1,\ \psi \in T_2\}$. Then $Mod(T') = Mod(T_1) \cup Mod(T_2)$.

Proof. Firstly, the inclusion $Mod(T_1) \cup Mod(T_2) \subseteq Mod(T')$ is true, as soon as all sentences of T' are deducible both in T_1 and T_2. Now we show the inclusion $Mod(T') \subseteq Mod(T_1) \cup Mod(T_2)$. Suppose that it is false; then there is a model $M \in Mod(T')$ such that $M \notin Mod(T_1) \cup Mod(T_2)$. It means that $M \notin Mod(T_1)$ and $M \notin Mod(T_2)$, which is equivalent to the fact that there are $\varphi \in T_1$ and $\psi \in T_2$ such that $M \nvDash \varphi$ and $M \nvDash \psi$. But according to the condition of the Lemma, for any model $M \in Mod(T')$, $M \vDash \varphi \vee \psi$ for all $\varphi \in T_1$ and $\psi \in T_2$ that is a contradiction. Hence $Mod(T') \subseteq Mod(T_1) \cup Mod(T_2)$ and $Mod(T') = Mod(T_1) \cup Mod(T_2)$.

Now we demonstrate the result that concerns to the lattices of Jonsson theories in terms of cosemanticness classes in the Jonsson spectrum.

Proposition 2. Let $K' \subseteq K$, $[T] \in JSp(K)_{/\bowtie}$, and let $C_{[T]}$ be a semantic model of $[T]$. Then $T' \in [T]$, where

$$T' = T^0(K') \vee T^0(C_{[T]}) = \{\varphi \vee \psi \mid \varphi \in T^0(K'),\ \psi \in T^0(C_{[T]})\}.$$

Proof. Firstly, we note that, according to Lemma 2, $Mod(T') = Mod(T^0(K')) \cup Mod(T^0(C_{[T]}))$. In addition, for any theory $T \in [T]$, $T \subseteq T'$, which means that T' is a Jonsson theory cosemantic to any $T \in [T]$. It remains to show that $T' \in JSp(K)$. Since $K \subseteq E_T$, $K' \subseteq K$, then $K' \equiv_{\forall\exists} K$, which means that $T^0(K) = T^0(K')$. Hence $T^0(K') \in JSp(K)$.

Now let us consider some specific relations between structures.

Definition 10. [21; 174] L-structures A and B are called Jonsson equivalent, if for any Jonsson theory T the following holds:

$$A \models T \Leftrightarrow B \models T.$$

Definition 11. [11] Structures A and B are called cosemantic, if $JSp(A) = JSp(B)$.

The following theorem also presents the result of structural approach in studying Jonsson theories and their cosemanticness classes.

Theorem 6. Let T be an arbitrary inductive L-theory such that $A \models T$ for any $A \in K$, where K is a class of infinite L-structures, and let the cosemanticness class $[T'] \in JSp(K)_{/\bowtie}$ consist only of $\exists$-complete theories. Then $[T''] \in JSp(K)_{/\bowtie}$, where

$$[T''] = \{T'' \mid T'' = T \cup T' \text{ for each } T' \in [T]\}.$$

Proof. Firstly, we should note that all theories of $[T'']$ are consistent, as soon as, for any $A \in K$, $A \models T'$ for each $T' \in [T']$, and $A \models T$, hence $A \models T''$ for any $T'' \in [T'']$. Let us consider an arbitrary $T' \in [T']$. It remains to show that $T'' = T' \cup T$ is a Jonsson theory. We do it by Definition 3.

1) All models in K are infinite, consequently T'' has infinite models;

2) Obviously, T'' is an inductive theory;

3) $T \subseteq T''$ and T is $\exists$-complete theory, hence T'' is $\exists$-complete as well. It means that T'' has JEP;

4) Here we use Theorem 3 again. Let $T'' \vdash \alpha_1 \vee \alpha_2$ for some L-formulas $\alpha_1(\overline{x}), \alpha_2(\overline{x}) \in \forall_1$. T'' has JEP, it means that $T'' \vdash \alpha_1$ or $T'' \vdash \alpha_2$. Since $T' \subseteq T''$ and T' is $\exists$-complete, $T' \vdash \alpha_1$, if $T'' \vdash \alpha_1$, and $T' \vdash \alpha_2$, if $T'' \vdash \alpha_2$. T' admits AP, so if $T' \vdash \alpha_1$ then $T' \vdash \alpha_1 \vee \alpha_1$ and, by Theorem 3, there are $\beta_1(\overline{x}), \beta_2(\overline{x}) \in \exists_1$ such that $T' \vdash \beta_i \to \alpha_i$ $(i = 1, 2)$ and $T' \vdash \beta_1 \vee \beta_2$. The same is if $T' \vdash \alpha_2$. Therefore, $T'' \vdash \beta_i \to \alpha_i$ $(i = 1, 2)$ and $T'' \vdash \beta_1 \vee \beta_2$, which means that T'' admits AP.

In [21], it was introduced the definition of cosemantic models. Now we give its analogue for classes of L-structures.

Definition 12. Let K_1 and K_2 be some classes of L-structures. Then K_1 and K_2 are said to be cosemantic ($K_1 \bowtie K_2$) if $JSp(K_1) = JSp(K_2)$.

Now we move to the main result of this paper. Theorem 7 is a criterion that connects the cosemanticness of J-classes with their Kaiser hulls.

Theorem 7. Let K_1, K_2 be J-classes. Then the following conditions are equivalent:
1) $K_1 \bowtie K_2$;
2) $T^0(K_1) = T^0(K_2)$.

Proof. Since K_1 and K_2 are J-classes, $T^0(K_1)$ and $T^0(K_2)$ are Jonsson theories. Let us prove (1) $\rightarrow$ (2). If $K_1 \bowtie K_2$, then $JSp(K_1) = JSp(K_2)$, which means that $T^0(K_1) \in JSp(K_2)$ and $T^0(K_2) \in JSp(K_1)$. But it follows that $T^0(K_1) \subseteq T^0(K_2)$ and $T^0(K_2) \subseteq T^0(K_1)$. Then $T^0(K_1) = T^0(K_2)$. The implication (2) $\rightarrow$ (1) is trivial due to inductiveness of Kaiser hulls and Jonsson theories.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Бекітілген модельдер класының Кайзер қабықшасының косемантикалылығы

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Мақалада йонсондық теорияны зерттеу барысында бекітілген тілде кейбір йонсондық теорияның экзистенциалды тұйық модельдерінің ішкі класының косемантикалылық класының модельді-теориялық қасиеттері зерттелді. Әртүрлі нәтижелер алынды. Сондай-ақ, модельдер мен модельдердің кластарының косемантикалылық қасиеттері қарастырылды; экзистенциалды тұйық модельдер класының жағдайында жалпылаудағы йонсондық эквиваленттікке қатысты кейбір нәтижелер алынды; J-класының Кайзер қабықшасына байланысты косемантикалылық критерийі табылды.

Кілт сөздер: йонсондық теория, косемантикалылық, косемантты йонсондық теориялар, йонсондық спектр, косемантикалылық кластары, Кайзер қабықшасы, йонсондық эквиваленттілік, J-класс, модельдердің косемантикалылығы, кластардың косемантикалылығы.

Косемантичность оболочек Кайзера фиксированных классов моделей

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В статье в рамках изучения йонсоновских теорий были рассмотрены теоретико-модельные свойства классов косемантичности, принадлежащих фактор-множеству йонсоновского спектра подкласса экзистенциально замкнутых моделей некоторой йонсоновской теории на фиксированном языке. Получены различные результаты. В частности, изучены свойства косемантичности моделей и классов моделей; получены некоторые результаты, касающиеся йонсоновской эквивалентности в обобщении на случаи классов экзистенциально замкнутых моделей; найден критерий косемантичности J-классов в связи с их оболочками Кайзера.

Ключевые слова: йонсоновская теория, косемантичные йонсоновские теории, йонсоновский спектр, классы косемантичности, оболочка Кайзера, йонсоновская эквивалентность, J-класс, косемантичность моделей, косемантичность классов.

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