Existence of Hilfer fractional neutral stochastic differential systems with infinite delay

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The goal of this study is to propose the existence of mild solutions to delay fractional neutral stochastic differential systems with almost sectorial operators involving the Hilfer fractional (HF) derivative in Hilbert space, which generalized the famous Riemann-Liouville fractional derivative. The main techniques rely on the basic principles and concepts from fractional calculus, semigroup theory, almost sectorial operators, stochastic analysis, and the Mönch fixed point theorem via the measure of noncompactness (MNC). Particularly, the existence result of the equation is obtained under some weakly compactness conditions. An example is given at the end of this article to show the applications of the obtained abstract results.

Keywords: Hilfer fractional evolution system, Neutral system, Measure of noncompactness, Fixed point theorem.

2020 Mathematics Subject Classification: 47H10, 47H08, 34K30, 34K50.

Introduction

Applications for fractional calculus extend from engineering and natural phenomena to financial views and physical accomplishments, and the subject is always growing. Fields like viscoelasticity, electrical engineering circuits, the vibration of seismic movements, biological systems, etc. usually contain an increasing number of fractional frameworks. Numerous good monographs provide the essential scientific methods for the attractiveness of this research topic. It should be possible to compare frameworks with practical systems of fractional power to the framework of ordinary integer order. Regarding fractional order, the derivative of the framework sum in the practical system might be correct. Numerous models in scattering, sensor fusion, automation, and so forth might all be used using this system. Learners can examine the literature [1–3], as well as research articles [4–8] that deal with the concept of fractional evolution systems to gain a thorough understanding of the concepts as well as the specifics of how it is implemented.

Due to the prevalence of neutral differential equations in many applications of applied mathematics, only neutral systems have received substantial attention in recent decades. In most cases, neutral systems with or without delay serve as an optimal configuration of numerous partial neutral systems that emerge in problems related to heat stream in components, viscoelasticity, acoustic waves, and various natural processes. One may mention [9–11] for a very helpful discussion on neutral systems involved in differential equations. Instead of deterministic models, stochastic ones should be studied since both natural and manufactured systems are prone to noise or uncontrolled fluctuations. Differential equations with stochastic components contain unpredictability in their theoretical depiction of a specific event. For a general overview of stochastic differential equations (SDE) and its applications [12–15].
The R-L and Caputo fractional derivatives were among the additional fractional order derivatives that Hilfer [16] started. The significance and consequences of the Hilfer fractional derivative (HFD) have also been found through conceptual forecasts of experiments in hard materials, pharmaceutical industries, set architecture design, architecture, and other fields. Gu and Trujillo [17] recently showed that the HFD evolution problem has an integral solution using a fixed point approach and a MNC strategy. In order to identify the derivative’s order, he constructed the greatest current variable that the HFD evolution problem has an integral solution using a fixed point approach and a MNC. The research articles [20–22] to improve the fractional existence for fractional calculus by utilizing almost sectorial operators and a fixed point method.

The research articles [24–27] to improve the fractional existence for fractional calculus by utilizing almost sectorial operators. Investigators in the study by [20–22] employed almost sectorial operators to get their results using Schauder’s fixed point theorem. Researchers have subsequently constructed nonlocal fractional differential equations with or without delay using non-dense fields, semigroups, cosine families, many fixed point strategy, and the MNC. To the best of our knowledge, the existence and consequences of the Hilfer fractional derivative (HFD) have also been found through conceptual forecasts of experiments in hard materials, pharmaceutical industries, set architecture design, architecture, and other fields. The R-L and Caputo fractional derivatives were among the additional fractional order derivatives that Hilfer [16] started. The significance and consequences of the Hilfer fractional derivative (HFD) have also been found through conceptual forecasts of experiments in hard materials, pharmaceutical industries, set architecture design, architecture, and other fields. Gu and Trujillo [17] recently showed that the HFD evolution problem has an integral solution using a fixed point approach and a MNC strategy. In order to identify the derivative’s order, he constructed the greatest current variable that the HFD evolution problem has an integral solution using a fixed point approach and a MNC.

Now let’s break up our content into the following sections. In Section 1 we outline a few crucial fundamental concepts, theorems, and lemma that are used throughout the whole work are introduced here. The notations (Y, ||·||) and (F, ||·||) signify two real distinct Hilbert spaces. Suppose (Δ, F, P) is a full probability area connected with full family of right continuous growing sub σ-algebra {Fρ : ρ ∈ R} fulfills Fρ ⊂ F. Consider W = {W(ρ)}ρ≥0 is a Q-Wiener strategy identified on (Δ, F, P) with the correlation operator Q such that Tr(Q) < ∞. We assume there exists a full orthonormal system e_k, k ≥ 1 in U, a limited series of non-negative real integers χ_k such that Qe_k = χ_ke_k, k = 1, 2, ··· and {µ_k} of independent Brownian movements such that

\[ (W(ρ), e)_U = \sum_{k=1}^{∞} \sqrt{χ_k} (e, e)µ_k(ρ), \quad e ∈ U, ρ ≥ 0. \]

Assuming that the area of all Q-Hilbert-Schmidt operators φ : \( Q^1 F \rightarrow Y \) with the inner product \( ||φ||_Q^2 = (φ, φ) = Tr(φQφ) \) is signified by the symbols \( L^2_0 = L^2(Q^1 F, Y) \). Let us consider the
resolvent operator of \( \hat{A} \), \( 0 < \rho(\hat{A}) \), where \( S(\cdot) \) is uniformly bounded, that is, \( \|S(\rho)\| \leq M \), \( M \geq 1 \), and \( \rho \geq 0 \). Thus, given \( \delta \in (0, 1] \), the fractional power operator \( \hat{A}^\delta \) on its range \( D(\hat{A}^\delta) \) may be obtained. Furthermore, \( D(\hat{A}^\delta) \) is dense in \( \mathcal{Y} \).

The succeeding substantial characteristic of \( \hat{A}^\delta \) will be discussed.

**Theorem 1.** [1]

1. If \( 0 < \delta \leq 1 \), then \( \mathcal{Y}_{\delta} = D(\hat{A}^\delta) \) is a Banach space with \( \|z\|_\delta = \|\hat{A}^\delta z\|_\delta \), \( z \in \mathcal{Y}_{\delta} \).

2. Assume \( 0 < \gamma < \delta \leq 1 \), embedding \( D(\hat{A}^\delta) \rightarrow D(\hat{A}^\gamma) \) and the implementation are compact whenever \( \hat{A} \) is compact.

3. For all \( 0 < \delta \leq 1 \), there exists \( C_\delta > 0 \) such that

\[
\|\hat{A}^\delta S(\rho)\| \leq \frac{C_\delta}{\rho^\delta}, \quad 0 < \rho \leq d.
\]

Consider \( \mathcal{C} : \mathcal{D} \rightarrow \mathcal{Y} \) is the family of all continuous functions, where \( \mathcal{D} = [0, d] \) and \( \mathcal{D}' = (0, d] \) with \( d > 0 \). Choose

\[
\mathcal{Y} = \{ z \in \mathcal{C} : \lim_{\rho \rightarrow 0} \rho^{1-\zeta+\eta \zeta-\eta \gamma} z(\rho) \text{ exists and finite} \},
\]

which is the Banach space and its \( \| \cdot \|_\mathcal{Y} \), specified as

\[
\|z\|_\mathcal{Y} = \sup_{\rho \in \mathcal{D}'} \rho^{1-\zeta+\eta \zeta-\eta \gamma} \|z(\rho)\|.
\]

Fix \( \mathcal{B}_p(\mathcal{D}) = \{ u \in \mathcal{C} \text{ such that } \|u\| \leq P \} \). Let \( z(\rho) = \rho^{-1+\zeta+\eta \zeta-\eta \gamma} y(\rho) \), \( \rho \in (0, d) \) then, \( z \in \mathcal{Y} \) if and only if \( y \in \mathcal{C} \) and \( \|z\|_\mathcal{Y} = \|y\|_\mathcal{C} \). We produce \( \mathcal{H} \) with \( \|\mathcal{H}\|_{L^p(\mathcal{D}, \mathbb{R}^+)} \), where \( \mathcal{H} \in L^p(\mathcal{D}, \mathbb{R}^+) \) for some \( p \) along with \( 1 \leq p \leq \infty \). Also \( L^p(\mathcal{D}, \mathcal{Y}) \) represent the Banach space of functions \( \mathcal{H} : \mathcal{D} \times \mathcal{B} \rightarrow \mathcal{Y} \) which are the Bochner integrable normed by \( \|\mathcal{H}\|_{L^p(\mathcal{D}, \mathcal{Y})} \).

**Definition 1.** [16] For the function \( \mathcal{H} : [d, +\infty) \rightarrow \mathbb{R} \), the HFD of order \( 0 < \eta < 1 \) and type \( \zeta \in [0, 1] \), represented by

\[
D_{d+}^{\eta \zeta} \mathcal{F}(\rho) = [I_{d+}^{(1-q)}D(I_{d+}^{(1-q)}(1-\zeta)\mathcal{F}))(\rho).
\]

The abstract phase space \( \mathcal{B}_r \) is now specified. Assume that \( w : (\infty, 0) \rightarrow (0, +\infty) \) is continuous along \( l = \int_{-\infty}^0 w(\rho) \, d\rho < +\infty \). Now, for all \( n > 0 \), we obtain

\[
\mathcal{B} = \{ \varepsilon : [-n, 0] \rightarrow \mathcal{Y} \text{ such that } \varepsilon(\rho) \text{ is bounded and measurable} \},
\]

and set the space \( \mathcal{B} \) with the norm

\[
\|\varepsilon\|_{[-n, 0]} = \sup_{\tau \in [-n, 0]} \|\varepsilon(\tau)\|, \text{ for all } \varepsilon \in \mathcal{B}.
\]

We now specify,

\[
\mathcal{B}_r = \{ \varepsilon : (-\infty, 0] \rightarrow \mathcal{Y} \text{ such that for all } n > 0, \varepsilon_{[-n, 0]} \in \mathcal{B}, \text{ and } \int_{-\infty}^0 w(\tau)\varepsilon_{[-n, 0]}(\tau) d\tau < +\infty \}.
\]

Suppose \( \mathcal{B}_r \) is endowed with

\[
\|\varepsilon\|_{\mathcal{B}_r} = \int_{-\infty}^0 w(\tau)\varepsilon_{[\tau, 0]} d\tau, \text{ for all } \varepsilon \in \mathcal{B}_r,
\]
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Lemma 4. [21]
1. $K_\eta(\rho)$ and $S_{\eta, \zeta}(\rho)$ are strongly continuous, for $\rho > 0$.
2. $K_\eta(\rho)$ and $S_{\eta, \zeta}(\rho)$ are bounded linear operators on $\mathbb{Y}$, for all fixed $\rho \in S_{\bar{\pi}}$, we obtain
\[
\|K_\eta(\rho)z\| \leq \kappa_\rho^{-1+\eta} \|z\|, \quad \|Q_\eta(\rho)z\| \leq \kappa_\rho^{-\eta+\eta} \|z\|,
\]
\[
\|S_{\eta, \zeta}(\rho)z\| \leq \frac{\Gamma(\varnothing)}{\Gamma(1-\eta)+\eta} \kappa_\rho^{-1+\zeta-\eta} \|z\|.
\]

Proposition 1. [19] Let $\eta \in (0, 1)$, $q \in (0, 1]$ and for all $z \in D(\hat{A})$, then there exists a $\kappa_\eta > 0$ such that
\[
\|\hat{A}^q Q_\eta(\rho)z\| \leq \frac{\eta \kappa_\eta \Gamma(2-q)}{\rho^m \Gamma(1+\eta(1-q))} \|z\|, \quad 0 < \rho < d.
\]

The Hausdorff MNC will now be briefly discussed.

Definition 5. [29] For a bounded set $X$ in a Banach space $\mathbb{Y}$, the Hausdorff MNC $\mu$ is represented as
\[
\mu(X) = \inf\{\epsilon > 0 : X \text{ can be linked by a finite number of balls with radii } \epsilon\}.
\]

Theorem 3. [8] If $\{v_k\}_{k=1}^\infty$ is a sequence of Bochner integrable functions from $\mathbb{D} \to \mathbb{Y}$ with the measurement $\|v_k(\rho)\| \leq \mu(\rho)$, for all $\rho \in \mathbb{Y}$ and for all $k \geq 1$, where $\mu \in L^1(\mathbb{D}, \mathbb{R})$, then the function $\omega(\rho) = \mu(\{v(\rho) : k \geq 1\})$ is in $L^1(\mathbb{D}, \mathbb{R})$ and fulfills
\[
\mu\left(\left\{\int_0^\rho v_k(t)dt : k \geq 1\right\}\right) \leq \int_0^\rho \omega(t)dt.
\]

Lemma 5. [8] Let $X \subset \mathbb{Y}$ be a bounded set, then there exists a countable set $X_0 \subset X$ such that $\mu(X) \leq 2\mu(X_0)$.

Definition 6. [29] If $E^+$ is the positive cone of an order Banach space $(E, \leq)$. Let $\overline{U}$ be the function represented on the family of all bounded subset of the Banach space $\mathbb{Y}$ with values in $E^+$ is known as MNC on $\mathbb{Y}$ if and only if $U(\text{conv}(\nu)) = U(\nu)$ for all bounded subset $\nu \subset \mathbb{Y}$, where $\text{conv}(\nu)$ denoted the closed convex hull of $\nu$.

Lemma 6. [30] Let $G$ be a closed convex subset of a Banach space $\mathbb{Y}$ and $0 \in G$. Suppose $F : G \to \mathbb{Y}$ continuous map which fulfills Mönch’s requirements, i.e., if $G_1 \subset G$ is countable and, $G_1 \subset \overline{\text{co}}\{0\} \cup F(G_1)$ \implies $G_1$ is compact. Then $F$ has a fixed point in $G$.

2 Existence

We require the succeeding hypotheses:

$(H_1)$ Let $\hat{A}$ be the almost sectorial operator of the analytic semigroup $T(\rho)$, $\rho > 0$ in $\mathbb{Y}$ such that $\|T(\rho)\| \leq K_1$ where $K_1 \geq 0$ be the constant.

$(H_2)$ The function $F : \mathbb{X} \times B_1 \to \mathbb{Y}$ fulfills:

(a) Caratheodory circumstances: $F(\cdot, z)$ is strongly measurable for all $z \in B_1$ and $F(\rho, \cdot)$ is continuous for a.e. $\rho \in \mathbb{D}$, $F(\cdot, \cdot) : [0, S] \to \mathbb{Y}$ is strongly measurable;

(b) There exists a constant $0 < \eta_1 < \eta$ and $m_1 \in L^1(\mathbb{D}, \mathbb{R}^+)$ and non-decreasing continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|F(\rho, z)\| \leq m_1(\rho)f(\rho^{1-\zeta+\kappa-\eta} \|z\|)$, $z \in \mathbb{Y}$, $\rho \in \mathbb{D}$ where $f$ fulfills \limsup_{k \to \infty} \frac{\psi(k)}{k} = 0;
(c) There exists a constant $0 < \eta_2 < \eta$ and $c_1 \in L^\infty(\mathcal{D}, \mathbb{R}^+)$ such that, for any bounded subset $M \subset \mathcal{Y}$, $\mu(\mathcal{F}(\rho, M)) \leq c_1(\rho) \mu(M)$ for a.e. $\rho \in \mathcal{D}$.

$(H_3)$ The function $\mathcal{H} : \mathcal{D} \times B_t \rightarrow L^1_0(\mathcal{J}, \mathcal{Y})$ fulfills:

(a) Caratheodory circumstances: $\mathcal{H}(\cdot, z)$ is strongly measurable for all $z \in B_t$ and $\mathcal{H}(\rho, \cdot)$ is continuous for a.e. $\rho \in \mathcal{D}$, $\mathcal{H}(\cdot, \cdot) : [0, T] \rightarrow L^1_0(\mathcal{J}, \mathcal{Y})$ is strongly measurable;

(b) There exists a constant $0 < \eta_3 < \eta$ and $m_2 \in L^\infty(\mathcal{D}, \mathbb{R}^+)$ and non-decreasing continuous function $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\|\mathcal{H}(\rho, z)\| \leq m_2(\rho) h(\rho^{1-\zeta+\eta \kappa - \eta \theta} \|z\|)$, $z \in \mathcal{Y}$, $\rho \in \mathcal{D}$ where $h$ fulfills $\liminf_{k \to \infty} \frac{\sigma(k)}{\sigma} = 0$;

(c) There exists a constant $0 < \eta_4 < \eta$ and $c_2 \in L^\infty(\mathcal{D}, \mathbb{R}^+)$ such that, for any bounded subset $M \subset \mathcal{Y}$, $\mu(\mathcal{H}(\rho, M)) \leq c_2(\rho) \mu(M)$ for a.e. $\rho \in \mathcal{D}$.

$(H_4)$ The function $\mathcal{S} : C(\mathcal{D}, \mathcal{Y}) \rightarrow \mathcal{Y}$ is continuous, compact operator and there exists $L_1 > 0$ as the value such that $\|\mathcal{S}(z_1) - \mathcal{S}(z_2)\| \leq L_1 \|z_1 - z_2\|$.

$(H_5)$ The function $\mathcal{S} : \mathcal{D} \times B_t \rightarrow \mathcal{Y}$ is continuous and there exists $q > 0$, $0 < q < 1$ such that $\mathcal{S} \in D(\mathcal{A}^{q})$ for all $z \in \mathcal{Y}$, $\rho \in \mathcal{D}$, $\mathcal{A}^q \mathcal{S}(\cdot, z)$ is strongly measurable, then there exists $M_w > 0$, $M'_w > 0$ such that $\gamma_1, \gamma_2 \in \mathcal{Y}$ and $\mathcal{A}^q \mathcal{S}(\rho, z(\rho))$ satisfies the following:

$$\|\mathcal{A}^q \mathcal{S}(\rho, \gamma_1(\rho)) - \mathcal{A}^q \mathcal{S}(\rho, \gamma_2(\rho))\| \leq M_w \rho^{1-\zeta+\eta \kappa - \eta \theta}\|\gamma_1(\rho) - \gamma_2(\rho)\|_{B_t},$$

$$\|\mathcal{A}^q \mathcal{S}(\rho, z(\rho))\| \leq M'_w (1 + \rho^{1-\zeta+\eta \kappa - \eta \theta}\|z\|_{B_t}).$$

Take $\|\mathcal{A}^-\eta\| = M_0$.

**Theorem 4.** Suppose $(H_1)-(H_3)$ holds, then the HF neutral stochastic system (1)-(2) has a unique solution on $\mathcal{D}$ presented, $\alpha(0) \in D(\mathcal{A}^q)$ with $\theta > 1 + \theta$.

**Proof.** Consider the operator $\Psi : B'_t \rightarrow B'_t$, defined

$$\Psi(z(\rho)) = \left\{ \begin{array}{ll}
\Psi_1(\rho), & (\rho, (-\infty, 0],
S_{\eta, \kappa}(\rho) \left[ \alpha(0) - \mathcal{S}(\rho, z(\rho)) + \mathcal{S}(\rho, z(\rho)) \right] + \mathcal{S}(\rho, z(\rho)) \\
+ \int_0^\rho (\rho - \nu)^{\eta - 1} \mathcal{Q}_\eta(\rho - \nu) \mathcal{S}(\nu, z(\nu)) d\nu \\
+ \int_0^\rho (\rho - \nu)^{\eta - 1} \mathcal{Q}_{\eta}(\rho - \nu) \mathcal{S}(\nu, z(\nu)) d\nu \\
+ \int_0^\rho (\rho - \nu)^{\eta - 1} \mathcal{Q}_\eta(\rho - \nu) \mathcal{S}(\nu, z(\nu)) d\nu \end{array} \right.$$ for $\Psi_1 \in B'_t$, we specify $\mathcal{S}$ as

$$\widehat{\Psi}(\rho) = \left\{ \begin{array}{ll}
\Psi_1(\rho), & \rho \in (-\infty, 0],
S_{\eta, \kappa}(\rho) \alpha(0), & \rho \in \mathcal{D},
\end{array} \right.$$ then $\widehat{\Psi} \in B'_t$. Let $z(\rho) = \rho^{1-\zeta+\eta \kappa - \eta \theta} \nu(\rho, \widehat{\Psi}(\rho)), \eta < \rho \leq d$. It is trivial to establish that $u$ fulfills by the Definition 4 if and only if $v$ satisfies $v_0$ and

$$v(\rho) = -S_{\eta, \kappa}(\rho) \left[ \mathcal{S}(\rho^{1-\zeta+\eta \kappa - \eta \theta} \nu_\rho + \widehat{\Psi}(\rho)) + \mathcal{S}(\mathcal{H}(\rho, z(\rho))) + \mathcal{S}(\rho, (\rho^{1-\zeta+\eta \kappa - \eta \theta} \nu_\rho + \widehat{\Psi}(\rho))) \right]$$

$$+ \int_0^\rho (\rho - \nu)^{\eta - 1} \mathcal{Q}_\eta(\rho - \nu) \mathcal{S}(\nu, \nu^{1-\zeta+\eta \kappa - \eta \theta} \nu_\nu + \nu_\nu) d\nu$$

$$+ \int_0^\rho (\rho - \nu)^{\eta - 1} \mathcal{Q}_{\eta}(\rho - \nu) \mathcal{S}(\nu, \nu^{1-\zeta+\eta \kappa - \eta \theta} \nu_\nu + \nu_\nu) d\nu$$

$$+ \int_0^\rho (\rho - \nu)^{\eta - 1} \mathcal{Q}_\eta(\rho - \nu) \mathcal{S}(\nu, \nu^{1-\zeta+\eta \kappa - \eta \theta} \nu_\nu + \nu_\nu) d\nu \nu dW(\nu).$$
Let $B''_r = \{ v \in B'_r : v_0 \in B_r \}$. For any $v \in B''_r$,
$$\|v\|_d = \|v_0\|_{B_r} + \sup\{\|v(t)\| : 0 \leq t \leq d\}$$
$$= \sup\{\|v(t)\| : 0 \leq t \leq d\}.$$ 
Hence, $(B''_r, \| \cdot \|)$ is a Banach space.

For $P > 0$, take $\mathcal{B}_P = \{ v \in B''_r : \|v\|_d \leq P \}$, then $\mathcal{B}_P \subset B''_r$ is uniformly bounded, and for $v \in \mathcal{B}_P$, by Lemma 1,
$$\|v + \tilde{\Psi} \|_{B_r} \leq \|v\|_{B_r} + \|\tilde{\Psi}\|_{B_r}$$
$$\leq l \left( P + \frac{\Gamma(\vartheta)}{\Gamma(\zeta(1 - \eta) + \eta \vartheta)} \kappa_P \rho^{1 + \zeta - \eta \vartheta} \right) + \|\Psi\|_{B_r},$$
$$= P'.$$

Consider an operator $\mathcal{U} : B''_r \to B''_r$, specified by
\[
\mathcal{U}v(\rho) = \left\{ \begin{array}{cl}
0, & \rho \in (-\infty, 0], \\
-S\eta,\zeta(\rho)\left[ \mathcal{N}(\rho^{1 - \zeta + \kappa - \eta \vartheta}[v + \tilde{\Psi}] + \mathcal{C}(0, \alpha(0)) \right) \\
+\mathcal{C}(\rho, \rho^{1 - \zeta + \kappa - \eta \vartheta}[v + \tilde{\Psi}]) \\
+\int_0^\rho (\rho - \xi)^{-1} \mathcal{A}Q_\eta(\rho - \xi)\mathcal{C}(\xi, \xi^{1 - \zeta + \kappa - \eta \vartheta}[v + \tilde{\Psi}])d\xi \\
+\int_0^\rho (\rho - \xi)^{-1} \mathcal{A}Q_\eta(\rho - \xi)\mathcal{F}(\xi, \xi^{1 - \zeta + \kappa - \eta \vartheta}[v + \tilde{\Psi}])d\xi \\
+\int_0^\rho (\rho - \xi)^{-1} \mathcal{A}Q_\eta(\rho - \xi)\mathcal{H}(\xi, \xi^{1 - \zeta + \kappa - \eta \vartheta}[v + \tilde{\Psi}])d\xi \\
\end{array} \right\}, \quad \rho \in \mathcal{D}.
\]

Then to prove $\mathcal{U}$ has a fixed point.

**Step 1:** To prove there exists a positive value $P$ such that $\mathcal{U}(\mathcal{B}_P(\mathcal{D})) \subseteq \mathcal{B}_P(\mathcal{D})$. Suppose the claim is incorrect i.e., for all $P > 0$, there exists $v^P \in \mathcal{B}_P(\mathcal{D})$, but $\mathcal{U}(v^P)$ not in $\mathcal{B}_P(\mathcal{D})$, that is,
$$E\|v^P\|^2 \leq P < E\sup_{\rho \in [0,d]} \rho^{1 - \zeta + \kappa - \eta \vartheta}(\mathcal{U}v^P(\rho))$$
$$\leq E\sup_{\rho \in [0,d]} \rho^{1 - \zeta + \kappa - \eta \vartheta} \left\{ -S\eta,\zeta(\rho)\left[ \mathcal{N}(\rho^{1 - \zeta + \kappa - \eta \vartheta}[v + \tilde{\Psi}] + \mathcal{C}(0, \alpha(0)) \right) \\
+\mathcal{C}(\rho, \rho^{1 - \zeta + \kappa - \eta \vartheta}[v + \tilde{\Psi}]) \\
+\int_0^\rho (\rho - \xi)^{-1} \mathcal{A}Q_\eta(\rho - \xi)\mathcal{C}(\xi, \xi^{1 - \zeta + \kappa - \eta \vartheta}[v + \tilde{\Psi}])d\xi \\
+\int_0^\rho (\rho - \xi)^{-1} \mathcal{A}Q_\eta(\rho - \xi)\mathcal{F}(\xi, \xi^{1 - \zeta + \kappa - \eta \vartheta}[v + \tilde{\Psi}])d\xi \\
+\int_0^\rho (\rho - \xi)^{-1} \mathcal{A}Q_\eta(\rho - \xi)\mathcal{H}(\xi, \xi^{1 - \zeta + \kappa - \eta \vartheta}[v + \tilde{\Psi}])d\xi \\
\right\}^2$$
$$\leq 5d^2(1 - \zeta + \kappa - \eta \vartheta) \left[ E\|\mathcal{N}(\rho^{1 - \zeta + \kappa - \eta \vartheta}[v^P + \tilde{\Psi}] + \mathcal{C}(0, \alpha(0)) \right) \\
+ E\|\mathcal{C}(\rho, \rho^{1 - \zeta + \kappa - \eta \vartheta}[v^P + \tilde{\Psi}]) \right\|^2$$
$$+ E\left\| \int_0^\rho (\rho - \xi)^{-1} \mathcal{A}Q_\eta(\rho - \xi)\mathcal{C}(\xi, \xi^{1 - \zeta + \kappa - \eta \vartheta}[v^P + \tilde{\Psi}])d\xi \right\|^2$$
$$+ E\left\| \int_0^\rho (\rho - \xi)^{-1} \mathcal{A}Q_\eta(\rho - \xi)\mathcal{F}(\xi, \xi^{1 - \zeta + \kappa - \eta \vartheta}[v^P + \tilde{\Psi}])d\xi \right\|^2$$
$$+ E\left\| \int_0^\rho (\rho - \xi)^{-1} \mathcal{A}Q_\eta(\rho - \xi)\mathcal{H}(\xi, \xi^{1 - \zeta + \kappa - \eta \vartheta}[v^P + \tilde{\Psi}])d\xi \right\|^2$$
$$\leq \frac{\kappa_P \rho^{1 + \zeta - \eta \vartheta}}{l^2 \Gamma(\zeta(1 - \eta) + \eta \vartheta)}.$$
Then, using the assumption

\[ E \left\| \int_0^\rho (\rho - \iota)^{\eta - 1}Q_H(\rho - \iota)H(\iota, \rho)^{1-\zeta+\eta}\{v_\iota^P + \hat{\Psi}_i\}dW(i) \right\|^2 \]

\leq 5d^2(1-\zeta+\eta-\theta)^2 \left[ \|S_H(\rho, \iota, \rho)^{1-\zeta+\eta}(v_\iota^P + \hat{\Psi}_i)\|^2 + \|N(0)\|^2 + M_w^2\|\alpha\|^2 \right]

+ \int_0^\rho (\rho - \iota)^{2(\eta-1)}\left\| A_{\iota, \rho}^2\right\|^2 E\left\| A_{\iota, \rho}^2(\iota, \rho)^{1-\zeta+\eta}(v_\iota^P + \hat{\Psi}_i)\right\|^2 dt

+ \int_0^\rho (\rho - \iota)^{2(\eta-1)}\|Q_H(\rho - \iota)\|^2 \kappa_2^2\left(f^2(P')d\right)\]

\leq 5d^2(1-\zeta+\eta-\theta)^2 \left[ \left( \frac{\Gamma(\theta)}{\Gamma(1-\eta + \eta\theta)} \right)^2 \kappa_2^2d^2(1-\zeta+\eta\theta) \right]

+ \int_0^\rho (\rho - \iota)^{2(\eta-1)}\left\| A_{\iota, \rho}^2\right\|^2 E\left\| A_{\iota, \rho}^2(\iota, \rho)^{1-\zeta+\eta}(v_\iota^P + \hat{\Psi}_i)\right\|^2 dt

+ Tr(Q)\int_0^\rho (\rho - \iota)^{2(\eta-1)}\|Q_H(\rho - \iota)\|^2 \kappa_2^2\left(f^2(P')d\right)

\leq 5d^2(1-\zeta+\eta-\theta)^2 M^{**},

where

\[ M^{**} = \left[ \left( \frac{\Gamma(\theta)}{\Gamma(1-\eta + \eta\theta)} \right)^2 \kappa_2^2d^2(1-\zeta+\eta\theta) \right]

+ \left( \frac{\Gamma(\theta)}{\Gamma(1-\eta + \eta\theta)} \right)^2 \kappa_2^2d^2(1-\zeta+\eta\theta) \right]

+ \left( \frac{\Gamma(\theta)}{\Gamma(1-\eta + \eta\theta)} \right)^2 \kappa_2^2d^2(1-\zeta+\eta\theta) \right]

+ \left( \frac{\Gamma(\theta)}{\Gamma(1-\eta + \eta\theta)} \right)^2 \kappa_2^2d^2(1-\zeta+\eta\theta) \right]

By dividing the aforementioned inequality by \( P \) and using the limit as \( P \to \infty \), we arrive at the contradiction, which is 1 ≤ 0. Consequently, \( \overline{U}(B_P(\mathcal{D})) \subseteq B_P(\mathcal{D}) \).

Step 2: The operator \( \overline{U} \) is continuous on \( B_P(\mathcal{D}) \). For \( \overline{U} : B_P(\mathcal{D}) \to B_P(\mathcal{D}) \) and for all \( v_k, v \in B_P(\mathcal{D}), k = 0, 1, 2, \ldots \) such that \( \lim_{k \to \infty} v_k = v \), then we get \( \lim_{k \to \infty} v_k(\rho) = v(\rho) \).

By \( (H_2) \),

\[ F_k(\iota) = F(\iota, \iota) \rho^1-\zeta+\eta-\theta\{v_\iota^k + \hat{\Psi}_i\} \to F(\rho, v(\rho)) \rho^1-\zeta+\eta-\theta\{v(\rho) + \hat{\Psi}_i\} \]

\[ = F(\rho, z(\rho)) \text{ as } k \to \infty. \]

Take

\[ F_k(\iota) = F(\iota, \iota) \rho^1-\zeta+\eta-\theta\{v_\iota^k + \hat{\Psi}_i\} \text{ and } F(\iota) = F(\iota, \iota) \rho^1-\zeta+\eta-\theta\{v(\iota) + \hat{\Psi}_i\}. \]

Then, using the assumption \( (H_2) \) and Lebesgue’s dominated convergence theorem (LDCT), we can obtain

\[ \int_0^\rho (\rho - \iota)^{2(\eta-1)}\|Q_H(\rho - \iota)\|^2 E\|F_k(\iota) - F(\iota)\|^2 dt \to 0 \text{ as } k \to \infty, \rho \in \mathcal{D}. \]
By \((H_3)\),
\[
\mathcal{H}(\rho, z_k(\rho)) = \mathcal{H}(\rho, \rho^{1-\zeta+\kappa-\eta}\eta^0[v^k(\rho) + \hat{\Psi}(\rho)]) \\
= \mathcal{H}(\rho, z(\rho)) \text{ as } k \to \infty.
\]
Take
\[
H_k(\iota) = \mathcal{H}(\iota, \iota^{1-\zeta+\kappa-\eta}\eta^0[v^k_\iota + \hat{\Psi}_\iota]) \text{ and } H(\iota) = \mathcal{H}(\iota, \iota^{1-\zeta+\kappa-\eta}\eta^0[v_i + \hat{\Psi}_i]).
\]
Then, using the hypotheses \((H_3)\) and LDCT, we can obtain
\[
\int_0^\rho (\rho - \iota)^{2(\eta - 1)}\|Q_\iota(\rho - \iota)\|^2 E\|H_k(\iota) - H(\iota)\|^2 dW(\iota) \to 0 \text{ as } k \to \infty, \rho \in \mathcal{D}. \tag{4}
\]
Take \(N_k(\rho) = \mathcal{N}(\rho^{1-\zeta+\kappa-\eta}\eta^0[v^k_\rho + \hat{\Psi}_\rho])\) and \(N(\rho) = \mathcal{N}(\rho^{1-\zeta+\kappa-\eta}\eta^0[v_\rho + \hat{\Psi}_\rho])\), from \((H_4)\), we get
\[
E\|N_k(\rho) - N(\rho)\|^2 \to 0 \text{ as } k \to \infty. \tag{5}
\]
Then, \(\mathcal{C}_k(\rho, z_k(\rho)) = \mathcal{C}(\rho, \rho^{1-\zeta+\kappa-\eta}\eta^0[v^k_\rho + \hat{\Psi}_\rho])\) and \(\mathcal{C}(\rho, z(\rho)) = \mathcal{C}(\rho, \rho^{1-\zeta+\kappa-\eta}\eta^0[v_\rho + \hat{\Psi}_\rho])\). From hypotheses \((H_5)\), we obtain
\[
E\|\mathcal{C}(\rho, z_k(\rho)) - \mathcal{C}(\rho, z(\rho))\|^2 \to 0 \text{ as } k \to \infty. \tag{6}
\]
Now,
\[
E\|\mathcal{O}v^k - \mathcal{O}v\|^2 \leq 4\left(\frac{\Gamma(\eta)}{\Gamma(1-\eta) + \eta^0}\right)^2 \kappa_p^2 d^{2(1+\zeta+\kappa+\eta^0)}E\|N_k(\rho) - N(\rho)\|^2
\]
\[
+ 4E\|\mathcal{C}_k(\rho, z_k(\rho)) - \mathcal{C}(\rho, z_\rho)\|^2 + 4\kappa^2 \left(\frac{d\eta^0}{\eta^0}\right)^2 E\|F_k(\iota) - F(\iota)\|^2
\]
\[
+ 4Tr(Q)\kappa_p^2 \left(\frac{d\eta^0}{\eta^0}\right)^2 E\|H_k(\iota) - H(\iota)\|^2.
\]
Using (3), (4), (5) and (6), we obtain
\[
E\|\mathcal{O}v^k - \mathcal{O}v\|^2 \to 0 \text{ as } k \to \infty.
\]
As a result, \(\mathcal{O}\) is continuous on \(\mathcal{B}_F\).

**Step 3:** To prove \(\mathcal{O}\) is equicontinuous.

For \(z \in \mathcal{B}_F(\mathcal{D})\), and \(0 \leq \rho_1 < \rho_2 \leq d\), we obtain
\[
E\|\mathcal{O}z(\rho_2) - \mathcal{O}z(\rho_1)\|^2
\]
\[
= E\|\rho_2^{1-\zeta+\kappa-\eta}\eta^0 - S_{\eta^0}(\rho_2)\mathcal{N}(\rho_2^{1-\zeta+\kappa-\eta}\eta^0[v_\rho_2 + \hat{\Psi}_\rho_2]) + \mathcal{C}(0, \alpha(0))
\]
\[
+ \mathcal{C}(\rho_2, (\rho_2^{1-\zeta+\kappa-\eta}\eta^0[v_\rho_2 + \hat{\Psi}_\rho_2]))
\]
\[
+ \int_0^{\rho_2} (\rho_2 - \iota)^{\eta - 1} \hat{A}Q_\eta(\rho_2 - \iota)\mathcal{C}(\iota, \iota^{1-\zeta+\kappa-\eta}\eta^0[v_i + \hat{\Psi}_i]) d\iota
\]
\[
+ \int_0^{\rho_2} (\rho_2 - \iota)^{\eta - 1} Q_\eta(\rho_2 - \iota)F(\iota, \iota^{1-\zeta+\kappa-\eta}\eta^0[v_i + \hat{\Psi}_i]) d\iota.
\]
Existence of Hilfer fractional ...
\[-\rho_1^{1-\zeta+\eta\zeta-\eta^\theta} \circ \left( \rho_1 \left( \rho_1^{1-\zeta+\eta\zeta-\eta^\theta} [v_{p_1} + \hat{\Psi}_{p_1}] \right) \right) \leq 2 \]
\[+ 15E \left\| \rho_2^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_2 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_2 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[-\rho_1^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_1 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_1 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[+ 15E \left\| \rho_1^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_1 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_1 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[+ 15E \left\| \rho_2^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_2 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_2 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[+ 15E \left\| \rho_2^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_2 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_2 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[-\rho_1^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_1 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_1 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[+ 15E \left\| \rho_1^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_1 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_1 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[-\rho_1^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_1 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_1 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[+ 15E \left\| \rho_1^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_1 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_1 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[-\rho_1^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_1 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_1 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[-\rho_1^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_1 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_1 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[-\rho_1^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_1 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_1 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[-\rho_1^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_1 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_1 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[-\rho_1^{1-\zeta+\eta\zeta-\eta^\theta} \int_0^{p_1} (\rho_1 - \iota) \eta^{-1} \hat{A} Q_{\eta}(\rho_1 - \iota) \circ (\iota, \iota^{1-\zeta+\eta\zeta-\eta^\theta} [v_\iota + \hat{\Psi}_\iota]) d\iota \right\|^2 \]
\[\leq 12 \sum_{i=1}^{12} I_i. \]

\[I_1 = 10E \left\| \rho_2^{1-\zeta+\eta\zeta-\eta^\theta} S_{\eta\zeta}(\rho_2) \left[ \mathcal{N} \left( \rho_2^{1-\zeta+\eta\zeta-\eta^\theta} [v_{p_2} + \hat{\Psi}_{p_2}] \right) \right] + \mathbb{D}(0, \alpha(0)) \right\|^2 \]
\[-\rho_2^{1-\zeta+\eta\zeta-\eta^\theta} S_{\eta\zeta}(\rho_2) \left[ \mathcal{N} \left( \rho_1^{1-\zeta+\eta\zeta-\eta^\theta} [v_{p_1} + \hat{\Psi}_{p_1}] \right) \right] + \mathbb{D}(0, \alpha(0)) \right\|^2 \]
\[\leq 10E \left\| \rho_2^{1-\zeta+\eta\zeta-\eta^\theta} S_{\eta\zeta}(\rho_2) \left( \mathcal{N} \left( \rho_2^{1-\zeta+\eta\zeta-\eta^\theta} [v_{p_2} + \hat{\Psi}_{p_2}] \right) - \mathcal{N} \left( \rho_1^{1-\zeta+\eta\zeta-\eta^\theta} [v_{p_1} + \hat{\Psi}_{p_1}] \right) \right) \right\|^2 . \]
From hypotheses \((H_4)\) and \((5)\), we obtain \(I_1\) tends to zero as \(\rho_2 \to \rho_1\).

\[
I_2 = 10\left\|\rho_2^{1-\zeta+\kappa-\eta^\theta}S_{\eta,\zeta}(\rho_2)\right\|_2^2 \left[\|\mathcal{N}(\rho_1^{1-\zeta+\kappa-\eta^\theta}[v_{\rho_1} + \hat{\Psi}_{\rho_1}]) + \mathcal{O}(0, \alpha(0))\| + \mathcal{O}(0, \alpha(0))\right]^2 \\
- \rho_1^{1-\zeta+\kappa-\eta^\theta}S_{\eta,\zeta}(\rho_1)\left[\|\mathcal{N}(\rho_1^{1-\zeta+\kappa-\eta^\theta}[v_{\rho_1} + \hat{\Psi}_{\rho_1}]) + \mathcal{O}(0, \alpha(0))\| + \mathcal{O}(0, \alpha(0))\right]^2 \\
\leq 10\left\|\rho_2^{1-\zeta+\kappa-\eta^\theta}S_{\eta,\zeta}(\rho_2) - \rho_1^{1-\zeta+\kappa-\eta^\theta}S_{\eta,\zeta}(\rho_1)\right\|^2 \\
\leq 10\left\|\rho_1^{1-\zeta+\kappa-\eta^\theta}[v_{\rho_1} + \hat{\Psi}_{\rho_1}] + \mathcal{O}(0, \alpha(0))\right\|^2.
\]

By the strong continuity of \(S_{\eta,\zeta}(\rho)\) and \((H_4)\), we get \(I_2 \to 0\) as \(\rho_2 \to \rho_1\).

\[
I_3 = 5\left\|\rho_2^{1-\zeta+\kappa-\eta^\theta}F(\rho_2, \rho_2^{1-\zeta+\kappa-\eta^\theta}[v_{\rho_2} + \hat{\Psi}_{\rho_2}])\right\|^2 \\
- \rho_1^{1-\zeta+\kappa-\eta^\theta}F(\rho_1, \rho_1^{1-\zeta+\kappa-\eta^\theta}[v_{\rho_1} + \hat{\Psi}_{\rho_1}])\left\|^2 \\
\leq 5M_0^2 M_\omega^2 (1 + P^2)\left\|\rho_2^{1-\zeta+\kappa-\eta^\theta} - \rho_1^{1-\zeta+\kappa-\eta^\theta}\right\|^2.
\]

From hypotheses \((H_5)\), we obtain \(I_3 \to 0\) as \(\rho_2 \to \rho_1\).

\[
I_4 = 15\left\|\rho_2^{1-\zeta+\kappa-\eta^\theta}\int_0^{\rho_1} (\rho_2 - \iota)^{\eta-1} \hat{\mathcal{Q}}(\rho_2 - \iota)\hat{\mathcal{O}}(\iota, \iota^{1-\zeta+\kappa-\eta^\theta}[v_\iota + \hat{\Psi}_\iota])\|_2^2 \\
- \rho_1^{1-\zeta+\kappa-\eta^\theta}\int_0^{\rho_1} (\rho_1 - \iota)^{\eta-1} \hat{\mathcal{Q}}(\rho_1 - \iota)\hat{\mathcal{O}}(\iota, \iota^{1-\zeta+\kappa-\eta^\theta}[v_\iota + \hat{\Psi}_\iota])\|_2^2 \\
\leq 15\left\|\rho_1^{1-\zeta+\kappa-\eta^\theta}\int_0^{\rho_1} (\rho_2 - \iota)^{\eta-1} - \rho_1^{1-\zeta+\kappa-\eta^\theta}(\rho_1 - \iota)^{\eta-1}\right\|^2 \\
\times \left\|\hat{\mathcal{A}}^{-q}\mathcal{Q} (\rho_2 - \iota)\right\|^2 E \left\|\hat{\mathcal{A}}\hat{\mathcal{O}}(\iota, \iota^{1-\zeta+\kappa-\eta^\theta}[v_\iota + \hat{\Psi}_\iota])\|_2^2 \\
\leq 15\left(\frac{M^2_\omega K_1}{\eta q} \frac{\eta \Gamma(1 + q)}{q \Gamma(1 + \eta q)}\right)^2 (1 + P^2) (1 - \rho_1^{2(1-\zeta+\kappa-\eta^\theta)}P^2) \\
\times \left\|\int_0^{\rho_1} (\rho_2 - \iota)^{\eta(q-1)} (\rho_2 - \iota)^{\eta-1} - \rho_1^{1-\zeta+\kappa-\eta^\theta}(\rho_1 - \iota)^{\eta-1}\right\|^2 \|d\iota\|.
\]

Implies \(I_4 \to 0\) as \(\rho_2 \to \rho_1\).

\[
I_5 = 15\left\|\rho_1^{1-\zeta+\kappa-\eta^\theta}\int_0^{\rho_1} (\rho_1 - \iota)^{\eta-1} \hat{\mathcal{Q}}(\rho_1 - \iota)\hat{\mathcal{O}}(\iota, \iota^{1-\zeta+\kappa-\eta^\theta}[v_\iota + \hat{\Psi}_\iota])\|_2^2 \\
- \rho_1^{1-\zeta+\kappa-\eta^\theta}\int_0^{\rho_1} (\rho_1 - \iota)^{\eta-1} \hat{\mathcal{Q}}(\rho_1 - \iota)\hat{\mathcal{O}}(\iota, \iota^{1-\zeta+\kappa-\eta^\theta}[v_\iota + \hat{\Psi}_\iota])\|_2^2 \\
\leq 15M_0^2 M_\omega^2 (1 + P^2) \left\|\rho_1^{1-\zeta+\kappa-\eta^\theta}\int_0^{\rho_1} (\rho_1 - \iota)^{\eta-1} \left[\mathcal{Q}_\eta (\rho_1 - \iota) - \mathcal{Q}_\eta (\rho_1 - \iota)\right]\|_2^2.
\]
Since $Q_{\eta}(\rho)$ is uniformly continuous in operator norm topology, we obtain $I_5 \to 0$ as $\rho_2 \to \rho_1$.

\[
I_6 = 15E \left\| \rho_2^{1-\zeta+\eta\zeta-\eta^d} \int_{\rho_1}^{\rho_2} (\rho_2 - \iota)^{y-1} \tilde{A}Q_{\eta}(\rho_2 - \iota) \mathcal{C}(\iota, \iota^{1-\zeta+\eta\zeta-\eta^d}[\nu_t + \tilde{\Psi}_t]) d\iota \right\|^2 \\
\leq 15 \left( \frac{M_{\zeta_1,1-\zeta+\eta\zeta-\eta^d}}{q(1 + \eta^d)} \right)^2 (1 + \rho_2^{2(1-\zeta+\eta\zeta-\eta^d)}P^2) \rho_2^{2(1-\zeta+\eta\zeta-\eta^d)} \int_{\rho_1}^{\rho_2} (\rho_2 - \iota)^{2y(1-\eta^d)} d\iota.
\]

Integrating and $\rho_2 \to \rho_1 \implies I_6 = 0$.

\[
I_7 = 15E \left\| \rho_2^{1-\zeta+\eta\zeta-\eta^d} \int_{0}^{\rho_1} (\rho_2 - \iota)^{y-1} Q_{\eta}(\rho_2 - \iota) \mathcal{F}(\iota, \iota^{1-\zeta+\eta\zeta-\eta^d}[\nu_t + \tilde{\Psi}_t]) d\iota \right\|^2 \\
- \rho_1^{1-\zeta+\eta\zeta-\eta^d} \int_{0}^{\rho_1} (\rho_1 - \iota)^{y-1} Q_{\eta}(\rho_1 - \iota) \mathcal{F}(\iota, \iota^{1-\zeta+\eta\zeta-\eta^d}[\nu_t + \tilde{\Psi}_t]) d\iota \right\|^2 \\
\leq 15E \left\| \int_{0}^{\rho_1} \left( \rho_2^{1-\zeta+\eta\zeta-\eta^d} (\rho_2 - \iota)^{y-1} - \rho_1^{1-\zeta+\eta\zeta-\eta^d} (\rho_1 - \iota)^{y-1} \right) \right\|^2 \\
\times \left( \rho_2 - \iota)^{2y(\theta-1)} m_2^2(d) f^2(P^d) d\iota.
\]

Implies $I_7 \to 0$ as $\rho_2 \to \rho_1$.

\[
I_8 = 15E \left\| \rho_1^{1-\zeta+\eta\zeta-\eta^d} \int_{0}^{\rho_1} (\rho_1 - \iota)^{y-1} Q_{\eta}(\rho_2 - \iota) \mathcal{F}(\iota, \iota^{1-\zeta+\eta\zeta-\eta^d}[\nu_t + \tilde{\Psi}_t]) d\iota \right\|^2 \\
- \rho_1^{1-\zeta+\eta\zeta-\eta^d} \int_{0}^{\rho_1} (\rho_1 - \iota)^{y-1} Q_{\eta}(\rho_1 - \iota) \mathcal{F}(\iota, \iota^{1-\zeta+\eta\zeta-\eta^d}[\nu_t + \tilde{\Psi}_t]) d\iota \right\|^2 \\
\leq 15 \rho_1^{2(1-\zeta+\eta\zeta-\eta^d)} \int_{0}^{\rho_1} (\rho_1 - \iota)^{2(\eta-1)} \left\| Q_{\eta}(\rho_2 - \iota) - Q_{\eta}(\rho_1 - \iota) \right\| m_2^2(d) f^2(P^d) d\iota.
\]

Since $Q_{\eta}(\rho)$ is uniformly continuous in operator norm topology, we obtain $I_8 \to 0$ as $\rho_2 \to \rho_1$.

\[
I_9 = 15E \left\| \rho_2^{1-\zeta+\eta\zeta-\eta^d} \int_{\rho_1}^{\rho_2} (\rho_2 - \iota)^{y-1} Q_{\eta}(\rho_2 - \iota) \mathcal{F}(\iota, \iota^{1-\zeta+\eta\zeta-\eta^d}[\nu_t + \tilde{\Psi}_t]) d\iota \right\|^2 \\
\leq 15 \rho_2^{2(1-\zeta+\eta\zeta-\eta^d)} \int_{\rho_1}^{\rho_2} (\rho_2 - \iota)^{2(\eta-1)} m_2^2(d) f^2(P^d) d\iota.
\]

Integrating and $\rho_2 \to \rho_1 \implies I_9 = 0$.

\[
I_{10} = 15E \left\| \rho_2^{1-\zeta+\eta\zeta-\eta^d} \int_{0}^{\rho_1} (\rho_2 - \iota)^{y-1} Q_{\eta}(\rho_2 - \iota) \mathcal{H}(\iota, \iota^{1-\zeta+\eta\zeta-\eta^d}[\nu_t + \tilde{\Psi}_t]) dW(\iota) \\
- \rho_1^{1-\zeta+\eta\zeta-\eta^d} \int_{0}^{\rho_1} (\rho_1 - \iota)^{y-1} Q_{\eta}(\rho_1 - \iota) \mathcal{H}(\iota, \iota^{1-\zeta+\eta\zeta-\eta^d}[\nu_t + \tilde{\Psi}_t]) dW(\iota) \right\|^2 \\
\leq 15E \left\| \int_{0}^{\rho_1} \left( \rho_2^{1-\zeta+\eta\zeta-\eta^d} (\rho_2 - \iota)^{y-1} - \rho_1^{1-\zeta+\eta\zeta-\eta^d} (\rho_1 - \iota)^{y-1} \right) \right\|^2.
\]
Since $\mathcal{Q}$ is equicontinuous on $\mathcal{D}$, we get

$$\rho_1 \leq \frac{2}{\eta-1}m_2^2(d)h^2(P')dt.\] $$

Implies $I_{10} \to 0$ as $\rho_2 \to \rho_1$.

$$I_{11} = 15E\left\| \rho_1 \mathcal{Q}_n(\rho_2 - \rho) \right\|^2 \int_0^{\rho_1} (\rho_2 - \rho)^{\eta-1} \mathcal{Q}_n(\rho_2 - \rho) \mathcal{H}(\epsilon, \epsilon^{1-\zeta+\eta-\eta^d}[v_t + \hat{\Psi}_t])dW(\epsilon)$$

$$- \rho_1 \mathcal{Q}_n(\rho_2 - \rho) \int_0^{\rho_1} (\rho_2 - \rho)^{\eta-1} \mathcal{Q}_n(\rho_2 - \rho) \mathcal{H}(\epsilon, \epsilon^{1-\zeta+\eta-\eta^d}[v_t + \hat{\Psi}_t])dW(\epsilon)$$

$$\leq 15Tr(Q)\rho_1^{2(1-\zeta+\eta-\eta^d)} \int_0^{\rho_1} (\rho_2 - \rho)^{2(\eta-1)}dW(\epsilon)$$

Since $\mathcal{Q}_n(\rho)$ is uniformly continuous in operator norm topology, we get $I_{11} \to 0$ as $\rho_2 \to \rho_1$.

$$I_{12} = 15E\left\| \rho_2 \mathcal{Q}_n(\rho_2 - \rho) \right\|^2 \int_0^{\rho_2} (\rho_2 - \rho)^{\eta-1} \mathcal{Q}_n(\rho_2 - \rho) \mathcal{H}(\epsilon, \epsilon^{1-\zeta+\eta-\eta^d}[v_t + \hat{\Psi}_t])dW(\epsilon)$$

$$\leq 15Tr(Q)\rho_2^{2(1-\zeta+\eta-\eta^d)} \int_0^{\rho_2} (\rho_2 - \rho)^{2(\eta-1)}dW(\epsilon).$$

Integrating and $\rho_2 \to \rho_1 \implies I_{12} = 0$.

Hence, $\mathcal{U}$ is equicontinuous on $\mathcal{D}$.

Step 4: The Mönch statement is true.

Let $\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3 + \mathcal{U}_4 + \mathcal{U}_5$, where

$$\mathcal{U}_1(v(\rho)) = -S_{\eta, \zeta}(\rho)\left[ \mathcal{Q}(\rho^{1-\zeta+\eta-\eta^d}[v_\rho + \hat{\Psi}_\rho]) + \mathcal{O}(0, \alpha(0)) \right],$$

$$\mathcal{U}_2(v(\rho)) = \mathcal{O}(\rho^{1-\zeta+\eta-\eta^d}[v_\rho + \hat{\Psi}_\rho]),$$

$$\mathcal{U}_3(v(\rho)) = \int_0^\rho (\rho - \epsilon)^{\eta-1} \mathcal{A}\mathcal{Q}_n(\rho - \epsilon) \mathcal{H}(\epsilon, \epsilon^{1-\zeta+\eta-\eta^d}[v_t + \hat{\Psi}_t])d\epsilon,$$

$$\mathcal{U}_4(v(\rho)) = \int_0^\rho (\rho - \epsilon)^{\eta-1} \mathcal{Q}_n(\rho - \epsilon) \mathcal{F}(\epsilon, \epsilon^{1-\zeta+\eta-\eta^d}[v_t + \hat{\Psi}_t])d\epsilon,$$

$$\mathcal{U}_5(v(\rho)) = \int_0^\rho (\rho - \epsilon)^{\eta-1} \mathcal{Q}_n(\rho - \epsilon) \mathcal{H}(\epsilon, \epsilon^{1-\zeta+\eta-\eta^d}[v_t + \hat{\Psi}_t])d\epsilon.$$

Suppose $G_1 \subseteq \mathcal{B}_P$ is countable and $G_1 \subseteq \mathcal{D}(\{0\} \cup F(G_1))$. We demonstrate that $\mu(G_1) = 0$, where $\mu$ is the Hausdorff MNC. Without loss of generality, suppose $G_1 = \{v^k\}_{k=1}^{\infty}$. Since $\mathcal{U}(G_1)$ is equicontinuous on $\mathcal{D}$ as well.

Utilising Lemma (see [29]), and the hypotheses $(H_2)(c)$, $(H_3)(c)$, and $(H_4)$, we have

$$\mu\left(\{\mathcal{U}_1 v^k(\rho)\}_{k=1}^{\infty}\right) \leq \mu\left\{ -S_{\eta, \zeta}(\rho)\left[ \mathcal{Q}(\rho^{1-\zeta+\eta-\eta^d}[v_\rho + \hat{\Psi}_\rho]) + \mathcal{O}(0, \alpha(0)) \right] \right\}_{k=1}^{\infty},$$

since $\mathcal{N}$ is compact, then $S_{\eta, \zeta}(\rho)$ is relatively compact, we get $\mathcal{U}_1 v(\rho)$ becomes zero. Next consider,

$$\mu\left(\{\mathcal{U}_2 v^k(\rho)\}_{k=1}^{\infty}\right) \leq \mu\left\{ \mathcal{O}(\rho^{1-\zeta+\eta-\eta^d}[v_\rho + \hat{\Psi}_\rho]) \right\}_{k=1}^{\infty},$$

$$\mu\left(\{\mathcal{U}_3 v^k(\rho)\}_{k=1}^{\infty}\right) \leq \mu\left\{ \int_0^\rho (\rho - \epsilon)^{\eta-1} \mathcal{A}\mathcal{Q}_n(\rho - \epsilon) \mathcal{H}(\epsilon, \epsilon^{1-\zeta+\eta-\eta^d}[v_t + \hat{\Psi}_t])d\epsilon \right\}_{k=1}^{\infty}.$$
From hypotheses \((H_5)\) and properties of function \(\mathcal{C}\) and \(\hat{A}Q\), we obtain, that terms are relatively compact. So \(\mathcal{U}v(\rho)\) and \(\mathcal{U}v(\rho)\) become zero.

\[
\mu\left(\{\mathcal{U}v^{k}(\rho)\}_{k=1}^{\infty}\right) \leq \mu\left\{ \int_{0}^{\rho} (\rho - \iota)^{n-1}Q_{\eta}(\rho - \iota)F(\iota, \iota^{1-\zeta+\eta \kappa - \eta \delta}[v^{k} + \hat{\Psi}_{1}]) d\iota \right\}_{k=1}^{\infty} \\
\leq 2 \int_{0}^{\rho} (\rho - \iota)^{n-1}Q_{\eta}(\rho - \iota)e_{1}(\iota) \sup_{-\infty < \theta \leq 0} \mu\left(\{v^{k}_{\rho}(\theta)\}_{k=1}^{\infty}\right) d\iota \\
\leq 2 \left( \frac{d^{\eta \delta}}{\eta \delta} \right) \|e_{1}\|_{L_{T}^{2}(\mathcal{D}, \mathbb{R}^{+})} \sup_{-\infty < \theta \leq 0} \mu\left(\{v^{k}_{\rho}(\theta)\}_{k=1}^{\infty}\right),
\]

\[
\mu\left(\{\mathcal{U}v^{k}(\rho)\}_{k=1}^{\infty}\right) \leq \mu\left\{ \int_{0}^{\rho} (\rho - \iota)^{n-1}Q_{\eta}(\rho - \iota)H(\iota, \iota^{1-\zeta+\eta \kappa - \eta \delta}[v^{k} + \hat{\Psi}_{1}]) dW(\iota) \right\}_{k=1}^{\infty} \\
\leq 2Tr(Q) \int_{0}^{\rho} (\rho - \iota)^{n-1}Q_{\eta}(\rho - \iota)e_{2}(\iota) \sup_{-\infty < \theta \leq 0} \mu\left(\{v^{k}_{\rho}(\theta)\}_{k=1}^{\infty}\right) d\iota \\
\leq 2Tr(Q) \left( \frac{d^{\eta \delta}}{\eta \delta} \right) \|e_{2}\|_{L_{T}^{2}(\mathcal{D}, \mathbb{R}^{+})} \sup_{-\infty < \theta \leq 0} \mu\left(\{v^{k}_{\rho}(\theta)\}_{k=1}^{\infty}\right).
\]

Thus, we have

\[
\mu\left(\{\mathcal{U}v^{k}(\rho)\}_{k=1}^{\infty}\right) \leq \mu\left(\{\mathcal{U}v^{k}(\rho)\}_{k=1}^{\infty}\right) + \mu\left(\{\mathcal{U}v^{k}(\rho)\}_{k=1}^{\infty}\right) + \mu\left(\{\mathcal{U}v^{k}(\rho)\}_{k=1}^{\infty}\right) + \mu\left(\{\mathcal{U}v^{k}(\rho)\}_{k=1}^{\infty}\right) \\
\leq 2 \left( \frac{d^{\eta \delta}}{\eta \delta} \right) \|e_{1}\|_{L_{T}^{2}(\mathcal{D}, \mathbb{R}^{+})} \sup_{-\infty < \theta \leq 0} \mu\left(\{v^{k}_{\rho}(\theta)\}_{k=1}^{\infty}\right) \\
+ 2Tr(Q) \left( \frac{d^{\eta \delta}}{\eta \delta} \right) \|e_{2}\|_{L_{T}^{2}(\mathcal{D}, \mathbb{R}^{+})} \sup_{-\infty < \theta \leq 0} \mu\left(\{v^{k}_{\rho}(\theta)\}_{k=1}^{\infty}\right) \\
\leq 2 \left( \frac{d^{\eta \delta}}{\eta \delta} \right) \|\|e_{1}\|_{L_{T}^{2}(\mathcal{D}, \mathbb{R}^{+})} + Tr(Q)\|e_{2}\|_{L_{T}^{2}(\mathcal{D}, \mathbb{R}^{+})}\right) \mu\left(\{v^{k}(\rho)\}_{k=1}^{\infty}\right) \\
\leq M^{*} \mu\left(\{v^{k}(\rho)\}_{k=1}^{\infty}\right),
\]

where \(M^{*} = 2 \left( \frac{d^{\eta \delta}}{\eta \delta} \right) \|\|e_{1}\|_{L_{T}^{2}(\mathcal{D}, \mathbb{R}^{+})} + Tr(Q)\|e_{2}\|_{L_{T}^{2}(\mathcal{D}, \mathbb{R}^{+})}\right)\).

Since \(G_{1}\) and \(\mathcal{U}(G_{1})\) are equicontinuous on \(\mathcal{D}\), it follows from Lemma (see [29]) that the constraint implies that \(\mu(\mathcal{U}G_{1}) \leq M^{*} \mu(G_{1})\).

Therefore, given the requirements of the Mönch’s, we get

\[
\mu(G_{1}) \leq \mu(\{G_{1}\} \cup \mathcal{U}(G_{1})) = \mu(\mathcal{U}G_{1}) \leq M^{*} \mu G_{1}.
\]

Given \(M^{*} < 1\), we have \(\mu(G_{1}) = 0\). Therefore, \(G_{1}\) is relatively compact. As a result, \(\mathcal{U}\) has a fixed point \(v\) in \(G_{1}\) from Lemma 6.

Hence, completed the proof.
3 Example

Consider the HF neutral stochastic differential systems with infinite delay of the form

\[
\begin{aligned}
D_{0+}^{\frac{2}{3}+\zeta} & \left[ z(\rho, \tau) + \int_0^\rho \rho(\beta, \tau) z(\rho, \tau) d\beta \right] = z_{\tau\tau}(\rho, \tau) + \gamma \left( \rho, \int_0^\rho \chi_1(\tau - \rho) z(\rho, \tau) d\tau \right) \\
& + \chi \left( \rho, \int_0^\rho \chi_2(\tau - \rho) z(\rho, \tau) dW(\tau) \right), \\
\end{aligned}
\]

(7)

where \( D_{0+}^{\frac{2}{3}+\zeta} \) denoted the HFD of order \( \eta = 2/3 \), type \( \zeta \) and \( \chi \), \( \chi_1 \), \( \rho \) and \( \mathcal{N} \) are the necessary functions. Consider, \( W(\rho) \) is the one-dimensional Brownian movements in \( \mathcal{Y} \) on the filtered probability space \( (\Delta, \mathcal{F}, P) \) and with \( ||.||_{\mathcal{Y}} \) to write the system (7) in the abstract form of (1)-(2). Let \( \mathcal{Y} = L^2[0, \pi] \), to transform this structure into an abstract structure, and \( \hat{A} : D(\hat{A}) \subset \mathcal{Y} \rightarrow \mathcal{Y} \) is classified as \( Ax = x' \) with

\[
D(\hat{A}) = \{ x \in \mathcal{Y} : x, x' \text{ are absolutely continuous, } x'' \in \mathcal{Y}, x(0) = x(\pi) = 0 \}
\]

and

\[
\hat{A}x = \sum_{k=1}^\infty k^2 \langle x, g_k \rangle g_k, \quad g \in D(\hat{A}),
\]

where the orthogonal set of eigen vectors of \( \hat{A} \) is \( g_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx), \quad k \in \mathbb{N} \).

Here, \( \hat{A} \) is the almost sectorial operator of the analytic semigroup \( \{ T(\rho), \quad \rho \geq 0 \} \) in \( \mathcal{Y} \), \( T(\rho) \) is noncompact semigroup on \( \mathcal{Y} \) with \( \zeta(T(\rho)B) \leq \zeta(B) \), where \( \zeta \) denoted the Hausdorff measure of noncompactness and there exists a constant \( K_1 \geq 1 \), satisfy \( \sup_{\rho \in \mathcal{D}} ||T(\rho)|| \leq K_1 \).

Specify, \( \mathcal{F} : \mathcal{D} \times B_\varepsilon \rightarrow \mathcal{Y}, \quad \mathcal{H} : \mathcal{D} \times B_\varepsilon \rightarrow L^0_{\mathcal{Y}}(\mathcal{J}, \mathcal{Y}), \quad \mathcal{O} : \mathcal{D} \times B_\varepsilon \rightarrow \mathcal{Y} \) and \( \mathcal{N} : B_\varepsilon \rightarrow \mathcal{Y} \) are the suitable functions, which fulfils the assumptions \( (H_1) - (H_5) \),

\[
\begin{aligned}
\mathcal{F}(\rho, z_\rho)(\tau) &= \gamma \left( \rho, \int_0^\rho \chi_1(\tau - \rho) z(\rho, \tau) d\tau \right), \\
\mathcal{H}(\rho, z_\rho)(\tau) &= \chi \left( \rho, \int_0^\rho \chi_2(\tau - \rho) z(\rho, \tau) dW(\tau) \right), \\
\mathcal{O}(\rho, z_\rho)(\tau) &= \int_0^\pi \rho(\beta, \tau) z(\rho, \tau) d\beta, \\
\mathcal{N}(z_\rho)(\tau) &= \int_0^\pi \mathcal{N}(\beta, \tau) z(\rho, \tau) d\beta.
\end{aligned}
\]

We also establish some acceptable requirements for the above-mentioned functions in order to validate all of the Theorem 4’s hypotheses, and we confirm that the HF stochastic system (1)-(2) has a mild solution.

Conclusion

The existence of a mild solution to HF neutral stochastic differential systems was the main emphasis of this research. Almost sectorial operators, fractional calculus, MNC, and the fixed point approach are used to establish the key conclusions. We offered an example to further illustrate the idea. In the following years, we’ll use the fixed point approach to examine the exact controllability of HF stochastic differential systems with delay.
Acknowledgments

The authors thank the referees very much for their valuable advice on this paper.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

Existence of Hilfer fractional ...
Шексіз кешігі бар бөлшек-нейтралды стохастикалық Хильфер дифференциалдық жүйелерінің болуы

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Зерттеудің мақсаты ойылған Риман-Лиувилл бөлшек туындысының жалпылайтын Гильберт кеңістігінде
гі Хильфер бөлшек туындысы қатысатына әрекет ететін дерлік секторлық операторлары бар кешігетін бөлшек-
нейтралды стохастикалық дифференциалдық жүйелер үшін жұмысқа шешімдірдің болуын ұсыňу.

Негізгі арнайы бөлшек септетудің, жартылай группа теориясының, дерлік секторлық операторлардың,
стояхастікалық талдаудың және компакттылық емес жоғары аралықтың қосылған теоремасын негіздеу.
Ал, тәндедің бар болуына қатыстығы алғашқы атап айтқанда, теңдеудің жоғары аралықтың қосылған теоремасы.

Кілт сөздер: Хильфердің бөлшек эволюциялық жүйесі, нейтралды жүйе, компакттылық емес жоғары,
қосылған теоремасы.

Существование дробно-нейтральных стохастических
дифференциальных систем Хильфера с бесконечным
запаздыванием

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Цель данного исследования — предложить существование мягких решений для запаздывающих дробно-
нейтральных стохастических дифференциальных систем с почти секториальными операторами, вклю-
чая дробную производную Хильфера в гильбертовом пространстве, которая обобщает знаме-
нитую дробную производную Римана-Лиувилля. Основные методы построены на базовых принципах
и концепциях дробного исчисления, теории полугрупп, почти секториальных операторах, стохасти-
ческом анализе и теореме Мёнха о неподвижной точке через меру некompактности. В частности,
результат существования уравнения получен при некоторых условиях слабой компактности. В конце
статьи приведен пример, демонстрирующий применение полученных абстрактных результатов.

Ключевые слова: дробная эволюционная система Хильфера, нейтральная система, мера некompакт-
ности, теорема о неподвижной точке.
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