https://doi.org/10.31489/2024M1/174-193 *Research article*

Existence of Hilfer fractional neutral stochastic differential systems with infinite delay

S. Sivasankar[1], R. Udhayakumar[1,*], V. Muthukumaran[2], G. Gokul[1], S. Al-Omari[3]

[1] *School of Advanced Sciences, Vellore Institute of Technology, Vellore, India;*
[2] *College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, India;*
[3] *Faculty of Science, Al-Balqa Applied University, Amman, Jordan*
(E-mail: sivajisivasankar@gmail.com, udhayaram.v@gmail.com, muthu.v2404@gmail.com, gokulg2110@gmail.com, shridehalomari@bau.edu.jo)

The goal of this study is to propose the existence of mild solutions to delay fractional neutral stochastic differential systems with almost sectorial operators involving the Hilfer fractional (HF) derivative in Hilbert space, which generalized the famous Riemann-Liouville fractional derivative. The main techniques rely on the basic principles and concepts from fractional calculus, semigroup theory, almost sectorial operators, stochastic analysis, and the Mönch fixed point theorem via the measure of noncompactness (MNC). Particularly, the existence result of the equation is obtained under some weakly compactness conditions. An example is given at the end of this article to show the applications of the obtained abstract results.

Keywords: Hilfer fractional evolution system, Neutral system, Measure of noncompactness, Fixed point theorem.

2020 Mathematics Subject Classification: 47H10, 47H08, 34K30, 34K50.

Introduction

Applications for fractional calculus extend from engineering and natural phenomena to financial views and physical accomplishments, and the subject is always growing. Fields like viscoelasticity, electrical engineering circuits, the vibration of seismic movements, biological systems, etc. usually contain an increasing number of fractional frameworks. Numerous good monographs provide the essential scientific methods for the attractiveness of this research topic. It should be possible to compare frameworks with practical systems of fractional power to the framework of ordinary integer order. Regarding fractional order, the derivative of the framework sum in the practical system might be correct. Numerous models in scattering, sensor fusion, automation, and so forth might all be used using this system. Learners can examine the literature [1–3], as well as research articles [4–8] that deal with the concept of fractional evolution systems to gain a thorough understanding of the concepts as well as the specifics of how it is implemented.

Due to the prevalence of neutral differential equations in many applications of applied mathematics, only neutral systems have received substantial attention in recent decades. In most cases, neutral systems with or without delay serve as an optimal configuration of numerous partial neutral systems that emerge in problems related to heat stream in components, viscoelasticity, acoustic waves, and various natural processes. One may mention [9–11] for a very helpful discussion on neutral systems involved in differential equations. Instead of deterministic models, stochastic ones should be studied since both natural and manufactured systems are prone to noise or uncontrolled fluctuations. Differential equations with stochastic components contain unpredictability in their theoretical depiction of a specific event. For a general overview of stochastic differential equations (SDE) and its applications [12–15].

*Corresponding author. *E-mail: udhayaram.v@gmail.com*
Received: 27 July 2023; *Accepted:* 06 November 2023.

The R-L and Caputo fractional derivatives were among the additional fractional order derivatives that Hilfer [16] started. The significance and consequences of the Hilfer fractional derivative (HFD) have also been found through conceptual forecasts of experiments in hard materials, pharmaceutical industries, set architecture design, architecture, and other fields. Gu and Trujillo [17] recently showed that the HFD evolution problem has an integral solution using a fixed point approach and a MNC strategy. In order to identify the derivative's order, he constructed the greatest current variable $\zeta \in [0,1]$ and a fractional variable η so that $\zeta = 0$ generates the R-L derivative and $\zeta = 1$ generates the Caputo derivative. Numerous papers have been written about Hilfer fractional calculus [18, 19]. According to [20–23], researchers discovered a mild solution for HF differential systems employing almost sectorial operators and a fixed point method.

The research articles [24–27] to improve the fractional existence for fractional calculus by utilizing almost sectorial operators. Investigators in the study by [20–22] employed almost sectorial operators to get their results using Schauder's fixed point theorem. Researchers have subsequently constructed nonlocal fractional differential equations with or without delay using non-dense fields, semigroups, cosine families, many fixed point strategy, and the MNC. To the best of our knowledge, the existence of HF neutral stochastic differential systems using the measure of noncompactness mentioned in this study is an exposed area of research that appears to give an extra incentive for completing this research.

The following subject will be looked at in this article: HF stochastic differential systems contain almost sectorial operators with nonlocal condition

$$D_{0^+}^{\eta,\zeta}\big[z(\rho) - \eth(\rho, z_\rho))\big] = \widehat{A}z(\rho) + \mathcal{F}\big(\rho, z_\rho\big) + \mathcal{H}\big(\rho, z_\rho\big)\frac{dW(\rho)}{d\rho}, \qquad \rho \in \mathfrak{D}' = (0, d], \tag{1}$$

$$I_{0^+}^{(1-\eta)(1-\zeta)}z(0) + \aleph(z_\rho) = \alpha \in L^2(\Delta, B_{\mathrm{r}}), \quad \rho \in (-\infty, 0], \tag{2}$$

where $\widehat{A}$ denote the almost sectorial operator, which generate an analytic semigroup $\{T(\rho), \rho \geq 0\}$ on $\mathbb{Y}$. Consider $z(\cdot)$ is the value in a Hilbert space $\mathbb{Y}$ with $\|\cdot\|$ and $D_{0^+}^{\eta,\zeta}$ represents the HFD of order $\eta,\ 0 < \eta < 1$ and type $\zeta,\ 0 \leq \zeta \leq 1$. The histories $z_\rho : (-\infty, 0] \to B_{\mathrm{r}},\ z_\rho(a) = z(\rho + a),\ a \leq 0$ connected with the abstract phase space B_{r}. Fix $\mathfrak{D} = [0, d]$, and let $\mathcal{F} : \mathfrak{D} \times B_{\mathrm{r}} \to \mathbb{Y}$, $\mathcal{H} : \mathfrak{D} \times B_{\mathrm{r}} \to L_2^0(\mathcal{J}, \mathbb{Y})$ and $\eth : \mathfrak{D} \times B_{\mathrm{r}} \to \mathbb{Y}$ are the $\mathbb{Y}$-valued function and non-local term $\aleph : B_{\mathrm{r}} \to \mathbb{Y}$.

Now let's break up our content into the following sections. In Section 1 we outline a few crucial ideas and details from our study that are referenced throughout the body of this article. We discussed the existence of a mild solution to the problem in Section 2. We provide an illustration of our main notion in Section 3. Then, a few conclusions are offered.

1 *Preliminaries*

The fundamental concepts, theorems, and lemma that are used throughout the whole work are introduced here.

The notations $(\mathbb{Y}, \|\cdot\|)$ and $(\mathcal{J}, \|\cdot\|)$ signify two real distinct Hilbert spaces. Suppose $(\Delta, \mathscr{F}, P)$ is a full probability area connected with full family of right continuous growing sub σ-algebra $\{\mathscr{F}_\rho : \rho \in \mathfrak{D}\}$ fulfills $\mathscr{F}_\rho \subset \mathscr{F}$. Consider $W = \{W(\rho)\}_{\rho \geq 0}$ is a Q-Wiener strategy identified on $(\Delta, \mathscr{F}, P)$ with the correlation operator Q such that $Tr(Q) < \infty$. We assume there exists a full orthonormal system e_k, $k \geq 1$ in U, a limited series of non-negative real integers χ_k such that $Qe_k = \chi_k e_k,\ k = 1, 2, \cdots$ and $\{\mu_k\}$ of independent Brownian movements such that

$$\big(W(\rho), e\big)_U = \sum_{k=1}^{\infty} \sqrt{\chi_k}(e_k, e)\mu_k(\rho), \quad e \in U \ \rho \geq 0.$$

Assuming that the area of all Q-Hilbert-Schmidt operators $\varphi : Q^{\frac{1}{2}}\mathcal{J} \to \mathbb{Y}$ with the inner product $\|\varphi\|_Q^2 = \langle \varphi, \varphi \rangle = Tr(\varphi Q \varphi)$ is signified by the symbols $L_2^0 = L_2\big(Q^{\frac{1}{2}}\mathcal{J}, \mathbb{Y}\big)$. Let us consider the resolvent operator of $\widehat{A}$, $0 \in \rho(\widehat{A})$, where $\mathsf{S}(\cdot)$ is uniformly bounded, that is, $\|\mathsf{S}(\rho)\| \leq M,\ M \geq 1$, and $\rho \geq 0$. Thus, given $\delta \in (0,1]$, the fractional power operator $\widehat{A}^{\delta}$ on its range $D(\widehat{A}^{\delta})$ may be obtained. Furthermore, $D(\widehat{A}^{\delta})$ is dense in $\mathbb{Y}$.

The succeeding substantial characteristic of $\widehat{A}^{\delta}$ will be discussed.

Theorem 1. [1]

1 If $0 < \delta \leq 1$, then $\mathbb{Y}_{\delta} = D(\widehat{A}^{\delta})$ is a Banach space with $\|z\|_{\delta} = \|\widehat{A}^{\delta} z\|,\ z \in \mathbb{Y}_{\chi}$.

2 Assume $0 < \gamma < \delta \leq 1$, embedding $D(\widehat{A}^{\delta}) \to D(\widehat{A}^{\gamma})$ and the implementation are compact whenever $\widehat{A}$ is compact.

3 For all $0 < \delta \leq 1$, there exists $C_{\delta} > 0$ such that

$$\|\widehat{A}^{\delta}\mathsf{S}(\rho)\| \leq \frac{C_{\delta}}{\rho^{\delta}},\ 0 < \rho \leq d.$$

Consider $\mathsf{C} : \mathfrak{D} \to \mathbb{Y}$ is the family of all continuous functions, where $\mathfrak{D} = [0,d]$ and $\mathfrak{D}' = (0,d]$ with $d > 0$. Choose

$$Y = \big\{z \in \mathsf{C} : \lim_{\rho \to 0} \rho^{1-\zeta+\eta\zeta-\eta\vartheta} z(\rho) \text{ exists and finite}\big\},$$

which is the Banach space and its $\|\cdot\|_{Y}$, specified as

$$\|z\|_{Y} = \sup_{\rho \in \mathfrak{D}'} \big\{\rho^{1-\zeta+\eta\zeta-\eta\vartheta}\|z(\rho)\|\big\}.$$

Fix $\mathscr{B}_{P}(\mathfrak{D}) = \{u \in \mathsf{C} \text{ such that } \|u\| \leq P\}$. Let $z(\rho) = \rho^{-1+\zeta-\eta\zeta+\eta\vartheta} y(\rho),\ \rho \in (0,d]$ then, $z \in Y$ if and only if $y \in \mathsf{C}$ and $\|z\|_{Y} = \|y\|$. We produce $\mathcal{H}$ with $\|\mathcal{H}\|_{L^{p}(\mathfrak{D},\mathbb{R}^{+})}$, where $\mathcal{H} \in L^{p}(\mathfrak{D},\mathbb{R}^{+})$ for some p along with $1 \leq p \leq \infty$. Also $L^{p}(\mathfrak{D},\mathbb{Y})$ represent the Banach space of functions $\mathcal{H} : \mathfrak{D} \times B_{\mathrm{r}} \to \mathbb{Y}$ which are the Bochner integrable normed by $\|\mathcal{H}\|_{L^{p}(\mathfrak{D},\mathbb{Y})}$.

Definition 1. [16] For the function $\mathcal{H} : [d,+\infty) \to \mathbb{R}$, the HFD of order $0 < \eta < 1$ and type $\zeta \in [0,1]$, presented by

$$D^{\eta,\zeta}_{d^{+}}\mathcal{F}(\rho) = [I^{(1-\eta)\zeta}_{d^{+}} D(I^{(1-\eta)(1-\zeta)}_{d^{+}}\mathcal{F})](\rho).$$

The abstract phase space B_{r} is now specified. Assume that $w : (-\infty,0] \to (0,+\infty)$ is continuous along $l = \int_{-\infty}^{0} w(\rho)d\rho < +\infty$. Now, for all $n > 0$, we obtain

$$B = \big\{\varepsilon : [-n,0] \to \mathbb{Y} \text{ such that } \varepsilon(\rho) \text{ is bounded and measurable}\big\},$$

and set the space B with the norm

$$\|\varepsilon\|_{[-n,0]} = \sup_{\tau \in [-n,0]} \|\varepsilon(\tau)\|,\ \text{for all } \varepsilon \in \mathscr{B}.$$

We now specify,

$$B_{\mathrm{r}} = \bigg\{\varepsilon : (-\infty,0] \to \mathbb{Y} \text{ such that for all } n > 0,\ \varepsilon|_{[-n,0]} \in \mathscr{B} \\ \text{and } \int_{-\infty}^{0} w(\tau)\|\varepsilon\|_{[\tau,0]}d\tau < +\infty\bigg\}.$$

Suppose B_{r} is endowed with

$$\|\varepsilon\|_{B_{\mathrm{r}}} = \int_{-\infty}^{0} w(\tau)\|\varepsilon\|_{[\tau,0]}d\tau,\ \text{for all } \varepsilon \in B_{\mathrm{r}},$$

therefore $(B_{\mathrm{r}}, \|\cdot\|)$ is a Banach space.

Now, we specify the space

$$B'_{\mathrm{r}} = \big\{z : (-\infty, d] \to \mathbb{Y} \text{ such that } z|_{\mathfrak{D}} \in \mathsf{C},\ \alpha \in B_{\mathrm{r}}\big\}.$$

Let us consider the seminorm $\|\cdot\|_d$ in B'_{r} defined as

$$\|z\|_d = \|\alpha\|_{B_{\mathrm{r}}} + \sup\{\|z(\tau)\| : \tau \in [0, d]\},\ z \in B'_{\mathrm{r}}.$$

Lemma 1. If $z \in B'_{\mathrm{r}}$, then for all $\rho \in \mathfrak{D},\ z_\rho \in B_{\mathrm{r}}$. Furthermore,

$$l|z(\rho)| \le \|z_\rho\|_{B_{\mathrm{r}}} \le \|\alpha\|_{B_{\mathrm{r}}} + l \sup_{r\in[0,\rho]} |z(r)|,$$

where $l = \int_{-\infty}^{0} w(\rho)d\rho < \infty$.

Definition 2. [25] We explain the family of closed linear operators Θ_ω^ϑ, for $0 < \vartheta < 1,\ 0 < \omega < \frac{\pi}{2}$, the sector $S_\omega = \{\theta \in \mathbb{C}\backslash\{0\} \text{ with } |arg\ \theta| \le \omega\}$ and $\widehat{A} : D(\widehat{A}) \subset \mathbb{Y} \to \mathbb{Y}$ that fulfills
(i) $\sigma(\widehat{A}) \subseteq S_\omega$;
(ii) $\big\|(\theta - \widehat{A})^{-1}\big\| \le \mathcal{K}_\varepsilon |\iota|^{-\vartheta}$, for all $\omega < \varepsilon < \pi$ and there exists $\mathcal{K}_\varepsilon$ as a constant,
afterward $\widehat{A} \in \Theta_\omega^{-\vartheta}$ is specified like almost sectorial operator on $\mathbb{Y}$.

Theorem 2. [3] $\mathsf{S}_\eta(\rho)$ and $\mathsf{Q}_\eta(\rho)$ are continuous in the uniform operator topology, for $\rho > 0$, for all $d > 0$, the continuity is uniform on $[d, \infty)$.

Lemma 2. [28] Suppose $\{T_\eta(\rho)\}_{\rho>0}$ is a compact operator, then $\{\mathsf{S}_{\eta,\zeta}(\rho)\}_{\rho>0}$ and $\{\mathsf{Q}_\eta(\rho)\}_{\rho>0}$ are also compact linear operators.

Lemma 3. [17] System (1)-(2) is unique to an integral equation offered by

$$\begin{aligned} z(\rho) =& \frac{\alpha(0) - \aleph(z_\rho) - \eth(0, \alpha(0))}{\Gamma(\zeta(1-\eta)+\eta)} \rho^{-(1-\eta)(\zeta-1)} + \eth(\rho, z_\rho) \\ &+ \frac{1}{\Gamma(\eta)} \int_0^\rho (\rho - \iota)^{\eta-1} \big[\widehat{A} z_\iota d\iota + \mathcal{F}\big(\iota, z_\iota\big) d\iota + \mathcal{H}\big(\iota, z_\iota\big) dW(\iota)\big]. \end{aligned}$$

Definition 3. [17] Let $z(\rho)$ be the solution of the integral equation offered by Lemma 3 then $z(\rho)$ fulfills

$$\begin{aligned} z(\rho) = & \mathsf{S}_{\eta,\zeta}(\rho)\big[\alpha(0) - \aleph(z_\rho) - \eth(0, \alpha(0))\big] + \eth(\rho, z_\rho) + \int_0^\rho \mathsf{K}_\eta(\rho - \iota)\mathcal{F}\big(\iota, z_\iota\big)d\iota \\ &+ \int_0^\rho \mathsf{K}_\eta(\rho - \iota)\mathcal{H}\big(\iota, z_\iota\big)dW(\iota),\ \rho \in \mathfrak{D}, \end{aligned}$$

where $\mathsf{S}_{\eta,\zeta}(\rho) = I_0^{\zeta(1-\eta)} \mathsf{K}_\eta(\rho),\ \mathsf{K}_\eta(\rho) = \rho^{\eta-1}\mathsf{Q}_\eta(\rho)$ and $\mathsf{Q}_\eta(\rho) = \int_0^\infty \eta\epsilon \mathfrak{W}_\eta(\epsilon) T(\rho^\eta \epsilon) d\epsilon$.

Definition 4. [7] A stochastic process $z : (-\infty, d] \to \mathbb{Y}$ is said to be a mild solution of the system (1)-(2) if $I_{0^+}^{(1-\eta)(1-\zeta)} z(0) + \aleph(z_\rho) = \alpha \in L^2(\eta, B_{\mathrm{r}}),\ \rho \in (-\infty, 0]$ and the preceding integral equation that fulfills

$$\begin{aligned} z(\rho) = & \mathsf{S}_{\eta,\zeta}(\rho)\big[\alpha(0) - \aleph(z_\rho) - \eth(0, \alpha(0))\big] + \eth(\rho, z_\rho) + \int_0^\rho (\rho - \iota)^{\eta-1} \widehat{A}\mathsf{Q}_\eta(\rho - \iota)\eth(\iota, z_\iota)d\iota \\ &+ \int_0^\rho (\rho - \iota)^{(\eta-1)}\mathsf{Q}_\eta(\rho - \iota)\mathcal{F}(\iota, z_\iota)d\iota + \int_0^\rho (\rho - \iota)^{(\eta-1)}\mathsf{Q}_\eta(\rho - \iota)\mathcal{H}(\iota, z_\iota)dW(\iota). \end{aligned}$$

Lemma 4. [21]
1 $\mathsf{K}_\eta(\rho)$ and $\mathsf{S}_{\eta,\zeta}(\rho)$ are strongly continuous, for $\rho > 0$.
2 $\mathsf{K}_\eta(\rho)$ and $\mathsf{S}_{\eta,\zeta}(\rho)$ are bounded linear operators on $\mathbb{Y}$, for all fixed $\rho \in S_{\frac{\pi}{2}-\omega}$, we obtain

$$\big\|\mathsf{K}_\eta(\rho)z\big\| \leq \kappa_p \rho^{-1+\eta\vartheta}\|z\|, \qquad \big\|\mathsf{Q}_\eta(\rho)z\big\| \leq \kappa_p \rho^{-\eta+\eta\vartheta}\|z\|,$$
$$\big\|\mathsf{S}_{\eta,\zeta}(\rho)z\big\| \leq \frac{\Gamma(\vartheta)}{\Gamma(\zeta(1-\eta)+\eta\vartheta)}\kappa_p \rho^{-1+\zeta-\eta\zeta+\eta\vartheta}\|z\|.$$

Proposition 1. [19] Let $\eta \in (0,1),\ q \in (0,1]$ and for all $z \in D(\widehat{A})$, then there exists a $\kappa_q > 0$ such that

$$\|\widehat{A}^q \mathcal{Q}_\eta(\rho)z\| \leq \frac{\eta\kappa_q \Gamma(2-q)}{\rho^{\eta q}\Gamma(1+\eta(1-q))}\|z\|,\ 0 < \rho < d.$$

The Hausdorff MNC will now be briefly discussed.

Definition 5. [29] For a bounded set X in a Banach space $\mathbb{Y}$, the Hausdorff MNC μ is represented as

$$\mu(X) = \inf\{\epsilon > 0 : X \text{ can be linked by a finite number of balls with radii } \epsilon\}.$$

Theorem 3. [8] If $\{v_k\}_{k=1}^{\infty}$ is a sequence of Bochner integrable functions from $\mathfrak{D} \to \mathbb{Y}$ with the measurement $\|v_k(\rho)\| \leq \mu(\rho)$, for all $\rho \in \mathcal{V}$ and for all $k \geq 1$, where $\mu \in L^1(\mathfrak{D}, \mathbb{R})$, then the function $\omega(\rho) = \mu\big(\{v(\rho) : k \geq 1\}\big)$ is in $L^1(\mathfrak{D}, \mathbb{R})$ and fulfills

$$\mu\bigg(\bigg\{\int_0^\rho v_k(\iota)d\iota :\ k \geq 1\bigg\}\bigg) \leq 2\int_0^\rho \omega(\iota)d\iota.$$

Lemma 5. [8] Let $X \subset \mathbb{Y}$ be a bounded set, then there exists a countable set $X_0 \subset X$ such that $\mu(X) \leq 2\mu(X_0)$.

Definition 6. [29] If E^+ is the positive cone of an order Banach space $(E, \leq)$. Let $\mho$ be the function represented on the family of all bounded subset of the Banach space $\mathbb{Y}$ with values in E^+ is known as MNC on $\mathbb{Y}$ if and only if $\mho(conv(\iota)) = \mho(\iota)$ for all bounded subset $\iota \subset \mathbb{Y}$, where $conv(\iota)$ denoted the closed convex hull of ι.

Lemma 6. [30] Let G be a closed convex subset of a Banach space $\mathbb{Y}$ and $0 \in G$. Suppose $F : G \to \mathbb{Y}$ continuous map which fulfils Mönch's requirements, i.e., if $G_1 \subset G$ is countable and, $G_1 \subset \overline{co}\big(\{0\} \cup F(G_1)\big) \implies \overline{G_1}$ is compact. Then F has a fixed point in G.

2 *Existence*

We require the succeeding hypotheses:

(H_1) Let $\widehat{A}$ be the almost sectorial operator of the analytic semigroup $T(\rho),\ \rho > 0$ in $\mathbb{Y}$ such that $\|T(\rho)\| \leq \mathsf{K}_1$ where $\mathsf{K}_1 \geq 0$ be the constant.

(H_2) The function $\mathcal{F} : \mathfrak{D} \times B_{\mathrm{r}} \to \mathbb{Y}$ fulfills:
- (a) Caratheodory circumstances: $\mathcal{F}(\cdot, z)$ is strongly measurable for all $z \in B_{\mathrm{r}}$ and $\mathcal{F}(\rho, \cdot)$ is continuous for a.e. $\rho \in \mathfrak{D},\ \mathcal{F}(\cdot,\cdot) : [0, S] \to \mathbb{Y}$ is strongly measurable;
- (b) There exists a constant $0 < \eta_1 < \eta$ and $m_1 \in L^{\frac{1}{\eta_1}}(\mathfrak{D}, \mathbb{R}^+)$ and non-decreasing continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\big\|\mathcal{F}(\rho, z)\big\| \leq m_1(\rho) f(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}\|z\|),\ z \in \mathbb{Y},\ \rho \in \mathfrak{D}$ where f fulfills $\liminf_{k\to\infty} \frac{\psi(k)}{k} = 0$;

(c) There exists a constant $0<\eta_2<\eta$ and $e_1 \in L^{\frac{1}{\eta_2}}(\mathfrak{D},\mathbb{R}^+)$ such that, for any bounded subset $M\subset\mathbb{Y},\ \mu(\mathcal{F}(\rho,M))\leq e_1(\rho)\mu(M)$ for a.e. $\rho\in\mathfrak{D}$.

(H_3) The function $\mathcal{H}:\mathfrak{D}\times B_{\mathrm{r}}\to L_2^0(\mathcal{J},\mathbb{Y})$ fulfills:

(a) Caratheodory circumstances: $\mathcal{H}(\cdot,z)$ is strongly measurable for all $z\in B_{\mathrm{r}}$ and $\mathcal{H}(\rho,\cdot)$ is continuous for a.e. $\rho\in\mathfrak{D},\ \mathcal{H}(\cdot,\cdot):[0,S]\to L_2^0(\mathcal{J},\mathbb{Y})$ is strongly measurable;

(b) There exists a constant $0<\eta_3<\eta$ and $m_2\in L^{\frac{1}{\eta_3}}(\mathfrak{D},\mathbb{R}^+)$ and non-decreasing continuous function $\hbar:\mathbb{R}^+\to\mathbb{R}^+$ such that $\big\|\mathcal{H}(\rho,z)\big\|\leq m_2(\rho)\hbar(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}\|z\|),\ z\in\mathbb{Y},\ \rho\in\mathfrak{D}$ where $\hbar$ fulfills $\liminf_{k\to\infty}\frac{\sigma(k)}{\sigma}=0$;

(c) There exists a constant $0<\eta_4<\eta$ and $e_2\in L^{\frac{1}{\eta_4}}(\mathfrak{D},\mathbb{R}^+)$ such that, for any bounded subset $M\subset\mathbb{Y},\ \mu(\mathcal{H}(\rho,M))\leq e_2(\rho)\mu(M)$ for a.e. $\rho\in\mathfrak{D}$.

(H_4) The function $\aleph:C(\mathfrak{D},\mathbb{Y})\to\mathbb{Y}$ is continuous, compact operator and there exists $L_1>0$ as the value such that $\|\aleph(z_1)-\aleph(z_2)\|\leq L_1\|z_1-z_2\|$.

(H_5) The function $\eth:\mathfrak{D}\times B_{\mathrm{r}}\to\mathbb{Y}$ is continuous and there exists $q>0,\ 0<q<1$ such that $\eth\in D(\widehat{A}^q)$ for all $z\in\mathbb{Y},\ \rho\in\mathfrak{D},\ \widehat{A}^q\eth(\cdot,z)$ is strongly measurable, then there exists $M_w>0,\ M_w'>0$ such that $\gamma_1,\gamma_2\in\mathbb{Y}$ and $\widehat{A}^q\eth(\rho,\cdot)$ satisfies the following:

$$\big\|\widehat{A}^q\eth(\rho,\gamma_1(\rho))-\widehat{A}^q\eth(\rho,\gamma_2(\rho))\big\|\leq M_w\rho^{1-\zeta+\eta\zeta-\eta\vartheta}\|\gamma_1(\rho)-\gamma_2(\rho)\|_{B_{\mathrm{r}}},$$
$$\big\|\widehat{A}^q\eth(\rho,z(\rho))\big\|\leq M_w'\big(1+\rho^{1-\zeta+\eta\zeta-\eta\vartheta}\|z\|_{B_{\mathrm{r}}}\big).$$

Take $\|\widehat{A}^{-q}\|=M_0$.

Theorem 4. Suppose $(H_1)-(H_5)$ holds, then the HF neutral stochastic system (1)-(2) has a unique solution on $\mathfrak{D}$ presented, $\alpha(0)\in D(\widehat{A}^\theta)$ with $\theta>1+\vartheta$.

Proof. Consider the operator $\Psi:B_{\mathrm{r}}'\to B_{\mathrm{r}}'$, defined

$$\Psi(z(\rho))=\begin{cases}\Psi_1(\rho),\ (-\infty,0],\\ \mathsf{S}_{\eta,\zeta}(\rho)\big[\alpha(0)-\aleph(z_\rho)-\eth(0,\alpha(0))\big]+\eth(\rho,z_\rho)\\ +\int_0^\rho(\rho-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho-\iota)\eth(\iota,z_\iota)d\iota\\ +\int_0^\rho(\rho-\iota)^{(\eta-1)}\mathsf{Q}_\eta(\rho-\iota)\mathcal{F}(\iota,z_\iota)d\iota\\ +\int_0^\rho(\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{H}\big(\iota,z_\iota\big)dW(\iota),\ \rho\in\mathfrak{D}.\end{cases}$$

For $\Psi_1\in B_{\mathrm{r}}$, we specify $\widehat{\Psi}$ as

$$\widehat{\Psi}(\rho)=\begin{cases}\Psi_1(\rho), & \rho\in(-\infty,0],\\ \mathsf{S}_{\eta,\zeta}(\rho)\alpha(0), & \rho\in\mathfrak{D},\end{cases}$$

then $\widehat{\Psi}\in B_{\mathrm{r}}'$. Let $z(\rho)=\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v(\rho)+\widehat{\Psi}(\rho)],\ \infty<\rho\leq d$. It is trivial to establish that $\mathfrak{u}$ fulfills by the Definition 4 if and only if v satisfies v_0 and

$$\begin{aligned}v(\rho)=&-\mathsf{S}_{\eta,\zeta}(\rho)\big[\aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho+\widehat{\Psi}_\rho]\big)+\eth(0,\alpha(0))\big]+\eth\big(\rho,\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho+\widehat{\Psi}_\rho]\big)\big)\\ &+\int_0^\rho(\rho-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\ &+\int_0^\rho(\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\ &+\int_0^\rho(\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota).\end{aligned}$$

Let $B''_{\mathrm{r}} = \{v \in B'_{\mathrm{r}} : v_0 \in B_{\mathrm{r}}\}$. For any $v \in B'_{\mathrm{r}}$,

$$\begin{aligned}\|v\|_d =&\|v_0\|_{B_{\mathrm{r}}} + \sup\{\|v(\iota)\| : 0 \le \iota \le d\}\\ =&\sup\{\|v(\iota)\| : 0 \le \iota \le d\}.\end{aligned}$$

Hence, $(B''_{\mathrm{r}}, \|\cdot\|)$ is a Banach space.

For $P > 0$, take $\mathscr{B}_P = \{v \in B''_{\mathrm{r}} : \|v\|_d \le P\}$, then $\mathscr{B}_P \subset B''_{\mathrm{r}}$ is uniformly bounded, and for $v \in \mathscr{B}_P$, by Lemma 1,

$$\begin{aligned}\|v_\rho + \widehat{\Psi}_\rho\|_{B_{\mathrm{r}}} &\le \|v_\rho\|_{B_{\mathrm{r}}} + \|\widehat{\Psi}\|_{B_{\mathrm{r}}}\\ &\le l\Big(P + \frac{\Gamma(\vartheta)}{\Gamma(\zeta(1-\eta)+\eta\vartheta)}\kappa_P \rho^{-1+\zeta-\eta\zeta+\eta\vartheta}\Big) + \|\Psi_1\|_{B_{\mathrm{r}}}\\ &= P'.\end{aligned}$$

Consider an operator $\mho : B''_{\mathrm{r}} \to B''_{\mathrm{r}}$, specified by

$$\mho v(\rho) = \begin{cases} 0, \ \rho \in (-\infty, 0], \\ -\mathsf{S}_{\eta,\zeta}(\rho)\big[\aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big) + \eth(0,\alpha(0))\big] \\ \quad +\eth\big(\rho, \big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big)\big) \\ \quad + \int_0^\rho (\rho-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho-\iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\ \quad + \int_0^\rho (\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\ \quad + \int_0^\rho (\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota), \qquad \rho \in \mathfrak{D}. \end{cases}$$

Then to prove $\mho$ has a fixed point.

Step 1: To prove there exists a positive value P such that $\mho(\mathscr{B}_P(\mathfrak{D})) \subseteq \mathscr{B}_P(\mathfrak{D})$. Suppose the claim is incorrect i.e., for all $P > 0$, there exists $v^P \in \mathscr{B}_P(\mathfrak{D})$, but $\mho(v^P)$ not in $\mathscr{B}_P(\mathfrak{D})$, that is,

$$\begin{aligned}E\big\|v^P\big\|^2 \le P < E&\Big\|\sup_{\rho\in[0,d]} \rho^{1-\zeta+\eta\zeta-\eta\vartheta}\big(\mho v^P(\rho)\big)\Big\|^2\\ \le E&\Big\|\sup_{\rho\in[0,d]} \rho^{1-\zeta+\eta\zeta-\eta\vartheta}\Big\{ - \mathsf{S}_{\eta,\zeta}(\rho)\big[\aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big) + \eth(0,\alpha(0))\big]\\ &+ \eth\big(\rho, \big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big)\big)\\ &+ \int_0^\rho (\rho-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho-\iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\\ &+ \int_0^\rho (\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\\ &+ \int_0^\rho (\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota)\Big\}\Big\|^2\\ \le 5d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}&\Big[E\Big\|\mathsf{S}_{\eta,\zeta}(\rho)\big[\aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v^P_\rho + \widehat{\Psi}_\rho]\big) + \eth(0,\alpha(0))\big]\Big\|^2\\ &+ E\Big\|\eth\big(\rho, \rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v^P_\rho + \widehat{\Psi}_\rho]\big)\Big\|^2\\ &+ E\Big\|\int_0^\rho (\rho-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho-\iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v^P_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\ &+ E\Big\|\int_0^\rho (\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v^P_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2\end{aligned}$$

$$
\begin{aligned}
&+E\Big\| \int_0^\rho (\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota^P+\widehat{\Psi}_\iota]\big)dW(\iota)\Big\|^2\Bigg]\\
&\le 5d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\Bigg[\|\mathsf{S}_{\eta,\zeta}(\rho)\|^2\big[L_1^2\|v_\rho^P+\widehat{\Psi}_\rho\|^2+\|\aleph(0)\|^2+M_w'^2\|\alpha\|^2\big]\\
&+M_0^2M_w'^2(1+d^{2(1-\zeta+\eta\zeta-\eta\vartheta)})P'\\
&+\int_0^\rho(\rho-\iota)^{2(\eta-1)}\|\widehat{A}^{1-q}\mathsf{Q}_\eta(\rho-\iota)\|^2E\|\widehat{A}^q\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota^P+\widehat{\Psi}_\iota]\big)\|^2d\iota\\
&+\int_0^\rho(\rho-\iota)^{2(\eta-1)}\|\mathsf{Q}_\eta(\rho-\iota)\|^2m_1^2(d)f^2(P')d\iota\\
&+Tr(Q)\int_0^\rho(\rho-\iota)^{2(\eta-1)}\|\mathsf{Q}_\eta(\rho-\iota)\|^2m_2^2(d)\hbar^2(P')d\iota\Bigg]\\
&\le 5d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\Bigg[\left(\frac{\Gamma(\vartheta)}{\Gamma(\zeta(1-\eta)+\eta\vartheta)}\right)^2\kappa_p^2d^{2(-1+\zeta-\eta\zeta+\eta\vartheta)}\\
&\qquad\qquad\big[L_1^2P'^2+\|\aleph(0)\|^2+M_w'^2\|\alpha\|^2\big]\\
&+M_0^2M_w'^2\big(1+d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\big)P'^2\\
&+\left(\frac{M_w'\kappa_{1-q}\Gamma(1+q)}{q\Gamma(1+\eta q)}\right)^2\big(1+d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}P'\big)d^{2\eta q}\\
&+\left(\frac{d^{\eta\vartheta}}{\eta\vartheta}\right)^2\kappa_p^2m_1^2(d)f^2(P')+Tr(Q)\left(\frac{d^{\eta\vartheta}}{\eta\vartheta}\right)^2\kappa_p^2m_2^2(d)\hbar^2(P')\Bigg]\\
&\le 5d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}M^{**},
\end{aligned}
$$

where

$$
\begin{aligned}
M^{**}=&\Bigg[\left(\frac{\Gamma(\vartheta)}{\Gamma(\zeta(1-\eta)+\eta\vartheta)}\right)^2\kappa_p^2d^{2(-1+\zeta-\eta\zeta+\eta\vartheta)}\big[L_1^2P'^2+\|\aleph(0)\|^2+M_w'^2\|\alpha\|^2\big]\\
&+M_0^2M_w'^2\big(1+d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\big)P'^2+\left(\frac{M_w'\kappa_{1-q}\Gamma(1+q)}{q\Gamma(1+\eta q)}\right)^2\big(1+d^{2(1-\zeta+\eta\zeta-\eta\vartheta)}P'\big)d^{2\eta q}\\
&+\left(\frac{d^{\eta\vartheta}}{\eta\vartheta}\right)^2\kappa_p^2m_1^2(d)f^2(P')+Tr(Q)\left(\frac{d^{\eta\vartheta}}{\eta\vartheta}\right)^2\kappa_p^2m_2^2(d)\hbar^2(P')\Bigg].
\end{aligned}
$$

By dividing the aforementioned inequality by P and using the limit as $P\to\infty$, we arrive at the contradiction, which is $1\le 0$. Consequently, $\mho(\mathscr{B}_P(\mathfrak{D}))\subseteq\mathscr{B}_P(\mathfrak{D})$.

Step 2: The operator $\mho$ is continuous on $\mathscr{B}_P(\mathfrak{D})$. For $\mho:\mathscr{B}_P(\mathfrak{D})\to\mathscr{B}_P(\mathfrak{D})$ and for all v^k, $v\in\mathscr{B}_P(\mathfrak{D})$, $k=0,1,2,\cdots$ such that $\lim_{k\to\infty}v^k=v$, then we get $\lim_{k\to\infty}v^k(\rho)=v(\rho)$ and $\lim_{k\to\infty}\rho^{1-\zeta+\eta\zeta-\eta\vartheta}v^k(\rho)=\rho^{1-\zeta+\eta\zeta-\eta\vartheta}v(\rho)$.

By (H_2),

$$
\begin{aligned}
\mathcal{F}(\rho,z_k(\rho))&=\mathcal{F}\big(\rho,\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v^k(\rho)+\widehat{\Psi}(\rho)]\big)\to\mathcal{F}\big(\rho,\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v(\rho)+\widehat{\Psi}(\rho)]\big)\\
&=\mathcal{F}(\rho,z(\rho))\text{ as }k\to\infty.
\end{aligned}
$$

Take

$$F_k(\iota)=\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota^k+\widehat{\Psi}_\iota]\big)\text{ and }F(\iota)=\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big).$$

Then, using the assumption (H_2) and Lebesgue's dominated convergence theorem (LDCT), we can obtain

$$\int_0^\rho(\rho-\iota)^{2(\eta-1)}\|\mathsf{Q}_\eta(\rho-\iota)\|^2E\|F_k(\iota)-F(\iota)\|^2d\iota\to 0\text{ as }k\to\infty,\ \rho\in\mathfrak{D}.\tag{3}$$

By (H_3),

$$\begin{aligned}\mathcal{H}(\rho, z_k(\rho)) &= \mathcal{H}\big(\rho, \rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v^k(\rho) + \widehat{\Psi}(\rho)]\big) \to \mathcal{H}\big(\rho, \rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v(\rho) + \widehat{\Psi}(\rho)]\big) \\ &= \mathcal{H}(\rho, z(\rho)) \text{ as } k \to \infty.\end{aligned}$$

Take

$$H_k(\iota) = \mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota^k + \widehat{\Psi}_\iota]\big) \text{ and } H(\iota) = \mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big).$$

Then, using the hypotheses (H_3) and LDCT, we can obtain

$$\int_0^\rho (\rho-\iota)^{2(\eta-1)} \|\mathsf{Q}_\eta(\rho-\iota)\|^2 E\|H_k(\iota) - H(\iota)\|^2 dW(\iota) \to 0 \text{ as } k \to \infty, \ \rho \in \mathfrak{D}. \tag{4}$$

Take $\mathcal{N}_k(\rho) = \aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho^k + \widehat{\Psi}_\rho]\big)$ and $\mathcal{N}(\rho) = \aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big)$, from (H_4), we get

$$E\|\mathcal{N}_k(\rho) - \mathcal{N}(\rho)\|^2 \to 0 \text{ as } k \to \infty. \tag{5}$$

Then, $\eth_k(\rho, z_k(\rho)) = \eth\big(\rho, \rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho^k + \widehat{\Psi}_\rho]\big)$ and $\eth(\rho, z(\rho)) = \eth\big(\rho, \rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho + \widehat{\Psi}_\rho]\big)$. From hypotheses (H_5), we obtain

$$E\|\eth(\rho, z_k(\rho)) - \eth(\rho, z(\rho))\|^2 \to 0 \text{ as } k \to \infty. \tag{6}$$

Now,

$$\begin{aligned}E\big\|\mho v^k - \mho v\big\|_d^2 &\le 4\bigg(\frac{\Gamma(\vartheta)}{\Gamma(\zeta(1-\eta)+\eta\vartheta)}\bigg)^2 \kappa_p^2 d^{2(-1+\zeta-\eta\zeta+\eta\vartheta)} E\big\|\mathcal{N}_k(\rho) - \mathcal{N}(\rho)\big\|^2 \\ &\quad + 4E\|\eth_k(\rho, z_k(\rho)) - \eth(\rho, z_\rho)\|^2 + 4\kappa_p^2\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)^2 E\big\|F_k(\iota) - F(\iota)\big\|^2 \\ &\quad + 4Tr(Q)\kappa_p^2\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)^2 E\big\|H_k(\iota) - H(\iota)\big\|^2.\end{aligned}$$

Using (3), (4), (5) and (6), we obtain

$$E\big\|\mho v^k - \mho v\big\|_d^2 \to 0 \text{ as } k \to \infty.$$

As a result, $\mho$ is continuous on $\mathscr{B}_P$.

Step 3: To prove $\mho$ is equicontinuous.

For $z \in \mathscr{B}_P(\mathfrak{D})$, and $0 \le \rho_1 < \rho_2 \le d$, we obtain

$$\begin{aligned}&E\bigg\|\mho z(\rho_2) - \mho z(\rho_1)\bigg\|^2 \\ &\quad = E\bigg\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\bigg(-\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2} + \widehat{\Psi}_{\rho_2}]\big) + \eth(0, \alpha(0))\big] \\ &\quad + \eth\big(\rho_2, \big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2} + \widehat{\Psi}_{\rho_2}]\big)\big) \\ &\quad + \int_0^{\rho_2} (\rho_2-\iota)^{\eta-1} \widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\ &\quad + \int_0^{\rho_2} (\rho_2-\iota)^{\eta-1} \mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{\rho_2}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\bigg)\\
&-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\bigg(-\mathsf{S}_{\eta,\zeta}(\rho_1)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\backepsilon(0,\alpha(0))\big]\\
&+\backepsilon\big(\rho_1,\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)\big)\\
&+\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_1-\iota)\backepsilon\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\
&+\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\
&+\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\bigg)\bigg\|^2\\
\le{}&5E\bigg\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\bigg(-\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2}+\widehat{\Psi}_{\rho_2}]\big)+\backepsilon(0,\alpha(0))\big]\bigg)\\
&-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\bigg(-\mathsf{S}_{\eta,\zeta}(\rho_1)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\backepsilon(0,\alpha(0))\big]\bigg)\bigg\|^2\\
&+5E\bigg\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\backepsilon\big(\rho_2,\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2}+\widehat{\Psi}_{\rho_2}]\big)\big)\\
&-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\backepsilon\big(\rho_1,\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)\big)\bigg\|^2\\
&+5E\bigg\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_2}(\rho_2-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\backepsilon\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\
&-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_1-\iota)\backepsilon\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\bigg\|^2\\
&+5E\bigg\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_2}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\
&-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\bigg\|^2\\
&+5E\bigg\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_2}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\\
&-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\bigg\|^2\\
\le{}&10E\bigg\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2}+\widehat{\Psi}_{\rho_2}]\big)+\backepsilon(0,\alpha(0))\big]\\
&-\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\backepsilon(0,\alpha(0))\big]\bigg\|^2\\
&+10E\bigg\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\backepsilon(0,\alpha(0))\big]\\
&-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_1)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\backepsilon(0,\alpha(0))\big]\bigg\|^2\\
&+5E\bigg\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\backepsilon\big(\rho_2,\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2}+\widehat{\Psi}_{\rho_2}]\big)\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta} \eth\big(\rho_1, \big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1} + \widehat{\Psi}_{\rho_1}]\big)\big)\Big\|^2 \\
&+ 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\rho_1} (\rho_2 - \iota)^{\eta-1} \widehat{A}\mathsf{Q}_\eta(\rho_2 - \iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\rho_1} (\rho_1 - \iota)^{\eta-1} \widehat{A}\mathsf{Q}_\eta(\rho_2 - \iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&+ 15E\Big\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\rho_1} (\rho_1 - \iota)^{\eta-1} \widehat{A}\mathsf{Q}_\eta(\rho_2 - \iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\rho_1} (\rho_1 - \iota)^{\eta-1} \widehat{A}\mathsf{Q}_\eta(\rho_1 - \iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&+ 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\rho_1}^{\rho_2} (\rho_2 - \iota)^{\eta-1} \widehat{A}\mathsf{Q}_\eta(\rho_2 - \iota)\eth\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&+ 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\rho_1} (\rho_2 - \iota)^{\eta-1} \mathsf{Q}_\eta(\rho_2 - \iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\rho_1} (\rho_1 - \iota)^{\eta-1} \mathsf{Q}_\eta(\rho_2 - \iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&+ 15E\Big\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\rho_1} (\rho_1 - \iota)^{\eta-1} \mathsf{Q}_\eta(\rho_2 - \iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\rho_1} (\rho_1 - \iota)^{\eta-1} \mathsf{Q}_\eta(\rho_1 - \iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&+ 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\rho_1}^{\rho_2} (\rho_2 - \iota)^{\eta-1} \mathsf{Q}_\eta(\rho_2 - \iota)\mathcal{F}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&+ 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\rho_1} (\rho_2 - \iota)^{\eta-1} \mathsf{Q}_\eta(\rho_2 - \iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota) \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\rho_1} (\rho_1 - \iota)^{\eta-1} \mathsf{Q}_\eta(\rho_2 - \iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota)\Big\|^2 \\
&+ 15E\Big\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\rho_1} (\rho_1 - \iota)^{\eta-1} \mathsf{Q}_\eta(\rho_2 - \iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota) \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\rho_1} (\rho_1 - \iota)^{\eta-1} \mathsf{Q}_\eta(\rho_1 - \iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota)\Big\|^2 \\
&+ 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\rho_1}^{\rho_2} (\rho_2 - \iota)^{\eta-1} \mathsf{Q}_\eta(\rho_2 - \iota)\mathcal{H}\big(\iota, \iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota + \widehat{\Psi}_\iota]\big)dW(\iota)\Big\|^2 \\
\leq{}& \sum_{i=1}^{12} I_i.
\end{aligned}
$$

$$
\begin{aligned}
I_1 ={}& 10E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2} + \widehat{\Psi}_{\rho_2}]\big) + \eth(0, \alpha(0))\big] \\
&- \rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1} + \widehat{\Psi}_{\rho_1}]\big) + \eth(0, \alpha(0))\big]\Big\|^2 \\
\leq{}& 10E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\Big(\aleph\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2} + \widehat{\Psi}_{\rho_2}]\big) - \aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1} + \widehat{\Psi}_{\rho_1}]\big)\Big)\Big\|^2.
\end{aligned}
$$

From hypotheses (H_4) and (5), we obtain I_1 tends to zero as $\rho_2 \to \rho_1$.

$$
\begin{aligned}
I_2 = 10E\Big\|&\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\eth(0,\alpha(0))\big] \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_1)\big[\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\eth(0,\alpha(0))\big]\Big\|^2 \\
\leq 10\Big\|&\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_2)-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathsf{S}_{\eta,\zeta}(\rho_1)\Big\|^2 \\
&E\Big\|\aleph\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)+\eth(0,\alpha(0))\Big\|^2.
\end{aligned}
$$

By the strong continuity of $\mathsf{S}_{\eta,\zeta}(\rho)$ and (H_4), we get $I_2 \to 0$ as $\rho_2 \to \rho_1$.

$$
\begin{aligned}
I_3 = 5E\Big\|&\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\eth\big(\rho_2,\big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_2}+\widehat{\Psi}_{\rho_2}]\big)\big) \\
&- \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\eth\big(\rho_1,\big(\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\rho_1}+\widehat{\Psi}_{\rho_1}]\big)\big)\Big\|^2 \\
\leq 5M_0^2M_w'^2&(1+P'^2)\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\Big\|^2.
\end{aligned}
$$

From hypotheses (H_5), we obtain $I_3 \to 0$ as $\rho_2 \to \rho_1$.

$$
\begin{aligned}
I_4 &= 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_2-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota \\
&\quad - \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&\leq 15E\Big\|\int_0^{\rho_1}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big) \\
&\qquad \widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&\leq 15\Big\|\int_0^{\rho_1}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big)\Big\|^2 \\
&\qquad \times\Big\|\widehat{A}^{1-q}\mathsf{Q}_\eta(\rho_2-\iota)\Big\|^2E\Big\|\widehat{A}^q\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)\Big\|^2d\iota \\
&\leq 15\Big(\frac{M_w'\kappa_{1-q}\eta\Gamma(1+q)}{q\Gamma(1+\eta q)}\Big)^2\big(1+\rho_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}P'^2\big) \\
&\quad \times\Big\|\int_0^{\rho_1}(\rho_2-\iota)^{\eta(q-1)}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big)d\iota\Big\|^2.
\end{aligned}
$$

Implies $I_4 \to 0$ as $\rho_2 \to \rho_1$.

$$
\begin{aligned}
I_5 &= 15E\Big\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota \\
&\quad - \rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_1-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2 \\
&\leq 15M_0^2M_w'^2(1+\rho_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}P'^2)\Big\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\big[\mathsf{Q}_\eta(\rho_2-\iota)-\mathsf{Q}_\eta(\rho_1-\iota)\big]d\iota\Big\|^2.
\end{aligned}
$$

Since $\mathsf{Q}_\eta(\rho)$ is uniformly continuous in operator norm topology, we obtain $I_5 \to 0$ as $\rho_2 \to \rho_1$.

$$\begin{aligned}
I_6 &= 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho_2-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\
&\le 15\Big(\frac{M_w'\kappa_{1-q}\eta\Gamma(1+q)}{q\Gamma(1+\eta q)}\Big)^2\big(1+\rho_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}P'^2\big)\rho_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{2\eta(1-q)}d\iota.
\end{aligned}$$

Integrating and $\rho_2 \to \rho_1 \Longrightarrow I_6 = 0$.

$$\begin{aligned}
I_7 &= 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\
&\quad-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\
&\le 15E\Big\|\int_0^{\rho_1}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big)\\
&\qquad\qquad\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\
&\le 15\kappa_p^2\Big\|\int_0^{\rho_1}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big)\Big\|^2\\
&\qquad\qquad(\rho_2-\iota)^{2\eta(\vartheta-1)}m_1^2(d)f^2(P')d\iota.
\end{aligned}$$

Implies $I_7 \to 0$ as $\rho_2 \to \rho_1$.

$$\begin{aligned}
I_8 &= 15E\Big\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\\
&\quad-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\
&\le 15\rho_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_0^{\rho_1}(\rho_1-\iota)^{2(\eta-1)}\big\|\mathsf{Q}_\eta(\rho_2-\iota)-\mathsf{Q}_\eta(\rho_1-\iota)\big\|m_1^2(d)f^2(P')d\iota.
\end{aligned}$$

Since $\mathsf{Q}_\eta(\rho)$ is uniformly continuous in operator norm topology, we obtain $I_8 \to 0$ as $\rho_2 \to \rho_1$.

$$\begin{aligned}
I_9 &= 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota\Big\|^2\\
&\le 15\kappa_p^2\rho_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{2(\eta\vartheta-1)}m_1^2(d)f^2(P')d\iota.
\end{aligned}$$

Integrating and $\rho_2 \to \rho_1 \Longrightarrow I_9 = 0$.

$$\begin{aligned}
I_{10} &= 15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\\
&\quad-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\Big\|^2\\
&\le 15E\Big\|\int_0^{\rho_1}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big)
\end{aligned}$$

$$\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\Big\|^2$$
$$\leq 15Tr(Q)\kappa_p^2\Big\|\int_0^{\rho_1}\Big(\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_2-\iota)^{\eta-1}-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\rho_1-\iota)^{\eta-1}\Big)\Big\|^2 (\rho_2-\iota)^{2\eta(\vartheta-1)}m_2^2(d)\hbar^2(P')d\iota.$$

Implies $I_{10}\to 0$ as $\rho_2\to\rho_1$.

$$I_{11}=15E\Big\|\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)$$
$$-\rho_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\rho_1}(\rho_1-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_1-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\Big\|^2$$
$$\leq 15Tr(Q)\rho_1^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_0^{\rho_1}(\rho_1-\iota)^{2(\eta-1)}\big\|\mathsf{Q}_\eta(\rho_2-\iota)-\mathsf{Q}_\eta(\rho_1-\iota)\big\|^2m_2^2(d)\hbar^2(P')d\iota.$$

Since $\mathsf{Q}_\eta(\rho)$ is uniformly continuous in operator norm topology, we get $I_{11}\to 0$ as $\rho_2\to\rho_1$.

$$I_{12}=15E\Big\|\rho_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho_2-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota)\Big\|^2$$
$$\leq 15Tr(Q)\kappa_p^2\rho_2^{2(1-\zeta+\eta\zeta-\eta\vartheta)}\int_{\rho_1}^{\rho_2}(\rho_2-\iota)^{2(\eta\vartheta-1)}m_2^2(d)\hbar^2(P')d\iota.$$

Integrating and $\rho_2\to\rho_1\implies I_{12}=0$.

Hence, $\mho$ is equicontinuous on $\mathfrak{D}$.

Step 4: The Mönch statement is true.

Let $\mho=\mho_1+\mho_2+\mho_3+\mho_4+\mho_5$, where

$$\begin{aligned}
\mho_1v(\rho)&=-\mathsf{S}_{\eta,\zeta}(\rho)\big[\aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho+\widehat{\Psi}_\rho]\big)+\eth(0,\alpha(0))\big],\\
\mho_2v(\rho)&=\eth\big(\rho,\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho+\widehat{\Psi}_\rho]\big)\big),\\
\mho_3v(\rho)&=\int_0^\rho(\rho-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota,\\
\mho_4v(\rho)&=\int_0^\rho(\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)d\iota,\\
\mho_5v(\rho)&=\int_0^\rho(\rho-\iota)^{\eta-1}\mathsf{Q}_\eta(\rho-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota+\widehat{\Psi}_\iota]\big)dW(\iota).
\end{aligned}$$

Suppose $G_1\subseteq\mathfrak{B}_P$ is countable and $G_1\subset\overline{co}\big(\{0\}\cup F(G_1)\big)$. We demonstrate that $\mu(G_1)=0$, where μ is the Hausdorff MNC. Without loss of generality, suppose $G_1=\{v^k\}_{k=1}^\infty$. Since $\mho(G_1)$ is equicontinuous on $\mathfrak{D}$ as well.

Utilising Lemma (see [29]), and the hypotheses $(H_2)(c)$, $(H_3)(c)$, and (H_4), we have

$$\mu\big(\{\mho_1v^k(\rho)\}_{k=1}^\infty\big)\leq\mu\big\{-\mathsf{S}_{\eta,\zeta}(\rho)\big[\aleph\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho^k+\widehat{\Psi}_\rho]\big)+\eth(0,\alpha(0))\big]\big\}_{k=1}^\infty,$$

since $\aleph$ is compact, then $\mathsf{S}_{\eta,\zeta}(\rho)$ is relatively compact, we get $\mho_1v(\rho)$ becomes zero. Next consider,

$$\mu\big(\{\mho_2v^k(\rho)\}_{k=1}^\infty\big)\leq\mu\big\{\eth\big(\rho,\big(\rho^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\rho^k+\widehat{\Psi}_\rho]\big)\big)\big\}_{k=1}^\infty,$$
$$\mu\big(\{\mho_3v^k(\rho)\}_{k=1}^\infty\big)\leq\mu\Big\{\int_0^\rho(\rho-\iota)^{\eta-1}\widehat{A}\mathsf{Q}_\eta(\rho-\iota)\eth\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_\iota^k+\widehat{\Psi}_\iota]\big)d\iota\Big\}_{k=1}^\infty.$$

From hypotheses (H_5) and properties of function $\eth$ and $\widehat{A}\mathsf{Q}$, we obtain, that terms are relatively compact. So $\mho_2 v(\rho)$ and $\mho_3 v(\rho)$ become zero.

$$\begin{aligned}\mu\big(\{\mho_4 v^k(\rho)\}_{k=1}^{\infty}\big) &\le \mu\bigg\{\int_0^{\rho}(\rho-\iota)^{\eta-1}\mathsf{Q}_{\eta}(\rho-\iota)\mathcal{F}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\iota}^k+\widehat{\Psi}_{\iota}]\big)d\iota\bigg\}_{k=1}^{\infty}\\ &\le 2\int_0^{\rho}(\rho-\iota)^{\eta-1}\mathsf{Q}_{\eta}(\rho-\iota)e_1(\iota)\sup_{-\infty<\theta\le 0}\mu\big(\{v_{\rho}^k(\theta)\}_{k=1}^{\infty}\big)d\iota\\ &\le 2\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)\|e_1\|_{L^{\frac{1}{\eta_2}}(\mathfrak{D},\mathbb{R}^+)}\sup_{-\infty<\theta\le 0}\mu\big(\{v_{\rho}^k(\theta)\}_{k=1}^{\infty}\big),\end{aligned}$$

$$\begin{aligned}\mu\big(\{\mho_5 v^k(\rho)\}_{k=1}^{\infty}\big) &\le \mu\bigg\{\int_0^{\rho}(\rho-\iota)^{\eta-1}\mathsf{Q}_{\eta}(\rho-\iota)\mathcal{H}\big(\iota,\iota^{1-\zeta+\eta\zeta-\eta\vartheta}[v_{\iota}^k+\widehat{\Psi}_{\iota}]\big)dW(\iota)\bigg\}_{k=1}^{\infty}\\ &\le 2Tr(Q)\int_0^{\rho}(\rho-\iota)^{\eta-1}\mathsf{Q}_{\eta}(\rho-\iota)e_2(\iota)\sup_{-\infty<\theta\le 0}\mu\big(\{v_{\rho}^k(\theta)\}_{k=1}^{\infty}\big)d\iota\\ &\le 2Tr(Q)\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)\|e_2\|_{L^{\frac{1}{\eta_4}}(\mathfrak{D},\mathbb{R}^+)}\sup_{-\infty<\theta\le 0}\mu\big(\{v_{\rho}^k(\theta)\}_{k=1}^{\infty}\big).\end{aligned}$$

Thus, we have

$$\begin{aligned}\mu\big(\{\mho v^k(\rho)\}_{k=1}^{\infty}\big) &\le \mu\big(\{\mho_1 v^k(\rho)\}_{k=1}^{\infty}\big)+\mu\big(\{\mho_2 v^k(\rho)\}_{k=1}^{\infty}\big)+\mu\big(\{\mho_3 v^k(\rho)\}_{k=1}^{\infty}\big)\\ &\quad+\mu\big(\{\mho_4 v^k(\rho)\}_{k=1}^{\infty}\big)+\mu\big(\{\mho_5 v^k(\rho)\}_{k=1}^{\infty}\big)\\ &\le 2\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)\|e_1\|_{L^{\frac{1}{\eta_2}}(\mathfrak{D},\mathbb{R}^+)}\sup_{-\infty<\theta\le 0}\mu\big(\{v_{\rho}^k(\theta)\}_{k=1}^{\infty}\big)\\ &\quad+2Tr(Q)\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)\|e_2\|_{L^{\frac{1}{\eta_4}}(\mathfrak{D},\mathbb{R}^+)}\sup_{-\infty<\theta\le 0}\mu\big(\{v_{\rho}^k(\theta)\}_{k=1}^{\infty}\big)\\ &\le 2\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)\big[\|e_1\|_{L^{\frac{1}{\eta_2}}(\mathfrak{D},\mathbb{R}^+)}+Tr(Q)\|e_2\|_{L^{\frac{1}{\eta_4}}(\mathfrak{D},\mathbb{R}^+)}\big]\mu\big(\{v^k(\rho)\}_{k=1}^{\infty}\big)\\ &\le M^*\mu\big(\{v^k(\rho)\}_{k=1}^{\infty}\big),\end{aligned}$$

where $M^*=2\bigg(\frac{d^{\eta\vartheta}}{\eta\vartheta}\bigg)\big[\|e_1\|_{L^{\frac{1}{\eta_2}}(\mathfrak{D},\mathbb{R}^+)}+Tr(Q)\|e_2\|_{L^{\frac{1}{\eta_4}}(\mathfrak{D},\mathbb{R}^+)}\big]$.

Since G_1 and $\mho(G_1)$ are equicontinuous on $\mathfrak{D}$, it follows from Lemma (see [29]) that the constraint implies that $\mu(\mho G_1)\le M^*\mu(G_1)$.

Therefore, given the requirements of the Mönch's, we get

$$\mu(G_1)\le\mu\big(\overline{co}\{0\}\cup\mho(G_1)\big)=\mu(\mho G_1)\le M^*\mu G_1.$$

Given $M^*<1$, we have $\mu(G_1)=0$. Therefore, G_1 is relatively compact. As a result, $\mho$ has a fixed point v in G_1 from Lemma 6.

Hence, completed the proof.

3 Example

Consider the HF neutral stochastic differential systems with infinite delay of the form

$$\begin{cases} D_{0^+}^{\frac{2}{3},\zeta}\big[z(\rho,\tau)+\int_0^\pi \rho(\beta,\tau)z(\rho,\tau)d\beta\big] = z_{\tau\tau}(\rho,\tau)+\gamma\Big(\rho,\int_\infty^\rho \chi_1(\iota-\rho)z(\rho,\tau)d\iota\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad +\chi\Big(\rho,\int_\infty^\rho \chi_2(\iota-\rho)z(\rho,\tau)dW(\iota)\Big), \\ z(\rho,0)=z(\rho,\pi)=0,\ \rho\in\mathfrak{D}, \\ I_{0^+}^{(1-\frac{2}{3})(1-\zeta)}z(0,\tau)+\int_0^\pi \mathcal{N}(\beta,\tau)z(\rho,\tau)d\beta = z(0,\tau),\ \tau\in[0,\pi],\ \rho\in(-\infty,0), \end{cases} \tag{7}$$

where $D_{0^+}^{\frac{2}{3},\zeta}$ denoted the HFD of order $\eta=2/3$, type ζ and $\chi,\ \chi_1,\ \rho$ and $\mathcal{N}$ are the necessary functions. Consider, $W(\rho)$ is the one-dimensional Brownian movements in $\mathbb{Y}$ represented on the filtered probability space $(\Delta,\mathscr{F},P)$ and with $\|\cdot\|_{\mathbb{Y}}$ to write the system (7) in the abstract form of (1)-(2). Let $\mathbb{Y}=L^2[0,\pi]$, to transform this structure into an abstract structure, and $\widehat{A}: D(\widehat{A})\subset\mathbb{Y}\to\mathbb{Y}$ is classified as $\widehat{A}x=x'$ with

$$D(\widehat{A})=\big\{x\in\mathbb{Y}: x,x' \text{ are absolutely continuous},\ x''\in\mathbb{Y},\ x(0)=x(\pi)=0\big\}$$

and

$$\widehat{A}x=\sum_{k=1}^{\infty}k^2\langle x,\varrho_k\rangle\varrho_k,\ \varrho\in D(\widehat{A}),$$

where the orthogonal set of eigen vectors of $\widehat{A}$ is $\varrho_k(x)=\sqrt{\frac{2}{\pi}}\sin(kx),\ k\in\mathbb{N}$.

Here, $\widehat{A}$ is the almost sectorial operator of the analytic semigroup $\{T(\rho),\ \rho\geq 0\}$ in $\mathbb{Y}$, $T(\rho)$ is noncompact semigroup on $\mathbb{Y}$ with $\zeta(T(\rho)B)\leq\zeta(B)$, where ζ denoted the Hausdorff measure of noncompactness and there exists a constant $\mathcal{K}_1\geq 1$, satisfy $\sup_{\rho\in\mathfrak{D}}\|T(\rho)\|\leq\mathcal{K}_1$.

Specify, $\mathcal{F}:\mathfrak{D}\times B_{\mathrm{r}}\to\mathbb{Y}$, $\mathcal{H}:\mathfrak{D}\times B_{\mathrm{r}}\to L_2^0(\mathcal{J},\mathbb{Y})$, $\eth:\mathfrak{D}\times B_{\mathrm{r}}\to\mathbb{Y}$ and $\aleph: B_{\mathrm{r}}\to\mathbb{Y}$ are the suitable functions, which fulfils the assumptions $(H_1)-(H_5)$,

$$\begin{aligned} \mathcal{F}\big(\rho,z_\rho\big)(\tau), &= \gamma\Big(\rho,\int_\infty^\rho \chi_1(\iota-\rho)z(\rho,\tau)d\iota\Big),\\ \mathcal{H}\big(\rho,z_\rho\big)(\tau), &= \chi\Big(\rho,\int_\infty^\rho \chi_2(\iota-\rho)z(\rho,\tau)dW(\iota)\Big),\\ \eth(\rho,z_\rho)(\tau) &= \int_0^\pi \rho(\beta,\tau)z(\rho,\tau)d\beta,\\ \aleph(z_\rho)(\tau) &= \int_0^\pi \mathcal{N}(\beta,\tau)z(\rho,\tau)d\beta. \end{aligned}$$

We also establish some acceptable requirements for the above-mentioned functions in order to validate all of the Theorem 4's hypotheses, and we confirm that the HF stochastic system (1)-(2) has a mild solution.

Conclusion

The existence of a mild solution to HF neutral stochastic differential systems was the main emphasis of this research. Almost sectorial operators, fractional calculus, MNC, and the fixed point approach are used to establish the key conclusions. We offered an example to further illustrate the idea. In the following years, we'll use the fixed point approach to examine the exact controllability of HF stochastic differential systems with delay.

Acknowledgments

The authors thank the referees very much for their valuable advice on this paper.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Pazy, A. (1983). Semigroups of Linear Operators and Applications to Partial Differential Equations. *Applied Mathematical Sciences, Vol. 44*. Springer, New York. https://doi.org/10.1007/978-1-4612-5561-1

2 Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press, San Diego.

3 Zhou, Y. (2014). *Basic Theory of Fractional Differential Equations*. World Scientific, Singapore. https://doi.org/10.1142/9069

4 Agarwal, R.P., Lakshmikanthan, V. & Nieto, J.J. (2010). On the concept of solution for fractional differential equations with uncertainty. *Nonlinear Analysis, Theory Methods and Applications 72*(6), 2859–2862. https://doi.org/10.1016/j.na.2009.11.029

5 Ahmad, B., Alsaedi, A., Ntouyas, S.K., & Tariboon, J. (2017). *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*. Springer International Publishing AG. https://doi.org/10.1007/978-3-319-52141-1

6 Diethelm, K. (2010). The analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type. *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

7 Wang, J., & Zhou, Y. (2011). Existence and Controllability results for fractional semilinear differential inclusions. *Nonlinear Analysis: Real World Applications, 12*, 3642–3653. https://doi.org/10.1016/j.nonrwa.2011.06.021

8 Wang, J.R., Fan, Z., & Zhou, Y. (2012). Nonlocal controllability of semilinear dynamic systems with fractional derivative in Banach spaces. *Journal of Optimization Theory and Applications, 154*(1), 292–302. https://doi.org/10.1007/s10957-012-9999-3

9 Guo, Y., Shu, X.B., Li, Y., & Xu, F. (2019). The existence and Hyers-Ulam stability of solution for an impulsive Riemann-Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$. *Boundary Value Problems, 59*, 1–18. https://doi.org/10.1186/s13661-019-1172-6.

10 Li, F., Xiao, T.J., & Xu, H.K. (2012). On nonlinear neutral fractional integro-differential inclusions with infinite delay. *Journal of Applied Mathematics, 2012*, 916543, 1–19. https://doi.org/10.1155/2012/916543

11 Ma, X., Shu, X.B., & Mao, J. (2020). Existence of almost periodic solutions for fractional impulsive neutral stochastic differential equations with infinite delay. *Stochastics and Dynamics, 20*(1), 1–31. https://doi.org/10.1142/S0219493720500033

12 Boudaoui, A., & Slama, A. (2016). Approximate controllability of nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delay. *Nonlinear Dynamics and Systems Theory, 16*(1), 35–48.

13 Evans, L.C. (2013). *An Introduction to Stochastic Differential Equations.* Berkeley, CA: University of California, Berkeley.

14 Mao, X. (1997). *Stochastic Differential Equations and Applications.* Horwood, Chichester, UK.

15 Sivasankar, S., & Udhayakumar, R. (2022). A note on approximate controllability of second-order neutral stochastic delay integro-differential evolution inclusions with impulses. *Mathematical Methods in the Applied Sciences, 45*(11), 6650–6676. https://doi.org/10.1002/mma.8198

16 Hilfer, R. (2000). *Application of fractional calculus in physics.* World Scientific, Singapore.

17 Gu, H., & Trujillo, J.J. (2015). Existence of integral solution for evolution equation with Hilfer fractional derivative. *Applied Mathematics and Computation, 257*, 344–354. http://dx.doi.org/10.1016/j.amc.2014.10.083

18 Sivasankar, S., & Udhayakumar, R. (2022). Hilfer Fractional Neutral Stochastic Volterra Integro-Differential Inclusions via Almost Sectorial Operators. *Mathematics, 10*(12), 2074. https://doi.org/10.3390/math10122074.

19 Yang, M., & Wang, Q. (2017). Existence of mild solutions for a class of Hilfer fractional evolution equations with nonlocal conditions. *Fractional Calculus and Applied Analysis, 20*(3), 679–705. https://doi.org/10.1515/fca-2017-0036

20 Bedi, P., Kumar, A., Abdeljawad, T., Khan, Z.A., & Khan, A. (2020). Existence and approximate controllability of Hilfer fractional evolution equations with almost sectorial operators. *Advances in Differential Equations, 615*, 1–15.

21 Jaiswal, A., & Bahuguna, D. (2020). Hilfer fractional differential equations with almost sectorial operators. *Differential Equations and Dynamical Systems, 31*, 301–317. https://doi.org/10.1007/s12591-020-00514-y.

22 Karthikeyan, K., Debbouche, A., & Torres, D.F.M. (2021). Analysis of Hilfer fractional integro-differential equations with almost sectorial operators. *Fractal and Fractional, 5*(1), 1–14. https://doi.org/10.3390/fractalfract5010022

23 Varun Bose, C.S., & Udhayakumar, R. (2022). A note on the existence of Hilfer fractional differential inclusions with almost sectorial operators. *Mathematical Methods in the Applied Sciences, 45*(5), 2530–2541. https://doi.org/10.1002/mma.7938

24 Li, F. (2013). Mild solutions for abstract differential equations with almost sectorial operators and infinite delay. *Advances in Differential Equations, 2013*(327), 1–11.

25 Periago, F., & Straub, B. (2002). A functional calculus for almost sectorial operators and applications to abstract evolution equations. *Journal of Evolution Equations, 2*, 41–68. https://doi.org/10.1007/s00028-002-8079-9

26 Wang, R.N., De-Han Chen, & Xiao, T.J. (2012). Abstract fractional Cauchy problems with almost sectorial operators. *Journal of Differential Equations, 252*(1), 202–235. https://doi.org/10.1016/j.jde.2011.08.048

27 Zhang, L., & Zhou, Y. (2014). Fractional Cauchy problems with almost sectorial operators. *Applied Mathematics and Computation, 257*, 145–157. https://doi.org/10.1016/j.amc.2014.07.024

28 Zhou, M., Li, C., & Zhou, Y. (2022). Existence of mild solutions for Hilfer fractional differential evolution equations with almost sectorial operators. *Axioms, 11*, 144. https://doi.org/10.3390/axioms11040144.

29 Ji, S., Li, G., & Wang, M. (2011). Controllability of impulsive differential systems with nonlocal conditions. *Applied Mathematics and Computation, 217*, 6981–6989. https://doi.org/10.1016/j.amc.2011.01.107

30 Mönch, H. (1980). Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. *Nonlinear Analysis, 4*(5), 985–999.

Шексіз кешігуі бар бөлшек-нейтралды стохастикалық Хильфер дифференциалдық жүйелерінің болуы

С. Сивасанкар[1], Р. Удхаякумар[1], В. Мутхумаран[2], Г. Гокул[1], Ш. Аль-Омари[3]

[1] *Жетілдірілген ғылымдар мектебі, Веллуру технологиялық институты, Веллуру, Үндістан;*
[2] *Инженерлік және технологиялар колледжі, SRM Ғылым және технология институты, Каттанкулатур, Үндістан;*
[3] *Ғылым факультеті, Әл-Балқа қолданбалы университеті, Амман, Иордания*

Зерттеудің мақсаты әйгілі Риман-Лиувилл бөлшек туындысын жалпылайтын Гильберт кеңістігіндегі Хильфер бөлшек туындысы қатысатын дерлік секторлық операторлары бар кешігетін бөлшек-нейтралды стохастикалық дифференциалдық жүйелер үшін жұмсақ шешімдердің болуын ұсыну. Негізгі әдістер бөлшек есептеудің, жартылай группа теориясының, дерлік секторлық операторлардың, стохастикалық талдаудың және компактылы емес өлшемі арқылы Мёнхтің қозғалмайтын нүкте теоремасының негізгі қағидалары мен тұжырымдамаларына негізделген. Атап айтқанда, теңдеудің бар болуының нәтижесі әлсіз компактылықтың белгілі бір жағдайында алынды. Мақаланың соңында алынған абстрактылы нәтижелердің қолдану аясын көрсететін мысал бар.

Кілт сөздер: Хильфердің бөлшек эволюциялық жүйесі, нейтралды жүйе, компактылы емес өлшем, қозғалмайтын нүкте теоремасы.

Существование дробно-нейтральных стохастических дифференциальных систем Хильфера с бесконечным запаздыванием

С. Сивасанкар[1], Р. Удхаякумар[1], В. Мутхумаран[2], Г. Гокул[1], Ш. Аль-Омари[3]

[1] *Школа передовых наук, Технологический институт Веллуру, Веллуру, Индия;*
[2] *Инженерно-технологический колледж; Институт науки и технологий SRM, Каттанкулатур, Индия;*
[3] *Прикладной университет Аль-Балка, Амман, Иордания*

Цель данного исследования — предложить существование мягких решений для запаздывающих дробно-нейтральных стохастических дифференциальных систем с почти секториальными операторами, включающими дробную производную Хильфера в гильбертовом пространстве, которая обобщает знаменитую дробную производную Римана-Лиувилля. Основные методы построены на базовых принципах и концепциях дробного исчисления, теории полугрупп, почти секториальных операторах, стохастическом анализе и теореме Мёнха о неподвижной точке через меру некомпактности. В частности, результат существования уравнения получен при некоторых условиях слабой компактности. В конце статьи приведен пример, демонстрирующий применение полученных абстрактных результатов.

Ключевые слова: дробная эволюционная система Хильфера, нейтральная система, мера некомпактности, теорема о неподвижной точке.

*Author Information**

Sivajiganesan Sivasankar — Master of science, Research Scholar, Department of Mathematics, Vellore Institute of Technology, Vellore- 632014, Tamilnadu, India; e-mail: *sivajisivasankar@gmail.com*; https://orcid.org/0000-0003-2422-9670.

Ramalingam Udhayakumar (*corresponding author*) — Doctor of philosophy, Assistant Professor Senior Grade1, Department of Mathematics, Vellore Institute of Technology, Vellore- 632014, Tamilnadu, India; e-mail: *udhayaram.v@gmail.com*; https://orcid.org/0000-0002-7020-3466.

Venkatesan Muthukumaran — Doctor of Philosophy, Assistant professor, Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur-603203, India; e-mail: *muthu.v2404@gmail.com*; https://orcid.org/0000-0002-3393-5596.

Gunasekaran Gokul — Master of science, Research Scholar, Department of Mathematics, Vellore Institute of Technology, Vellore- 632014, Tamilnadu, India; e-mail: *gokulg2110@gmail.com*.

Shrideh Al-Omari — Doctor of philosophy, Professor, Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Amman 11134, Jordan; e-mail: *shridehalomari@bau.edu.jo*; https://orcid.org/0000-0001-8955-5552.

*The author's name is presented in the order: First, Middle and Last Names.