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Factorization of abstract operators into two second degree operators and its applications to integro-differential equations

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Boundary value problem $\mathbf{B}_1 x = f$ with an abstract linear operator B_1, corresponding to an Fredholm integro-differential equation with ordinary or partial differential operator is researched. An exact solution to $\mathbf{B}_1 x = f$ in the case when a bijective operator $\mathbf{B}_1$ has a factorization of the form $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$ where $\mathbf{B}$ and $\mathbf{B}_0$ are two linear more simple than $\mathbf{B}_1$ second degree abstract operators, received. Conditions for factorization of the operator $\mathbf{B}_1$ and a criterion for bijectivity of $\mathbf{B}_1$ are found.

Keywords: correct operator, bijective operator, factorization (decomposition) of linear operators, Fredholm integro-differential equations, boundary value problems, exact solutions.

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Introduction

Integro-differential equations (IDEs) are used to model many problems in science, engineering, economics, medicine, control theory, micro-inhomogeneous media and viscoelasticity [1–9]. Very important tools in solving of Boundary Value Problems (BVPs) with IDEs are the Parametrization Method [10] and the Factorization (Decomposition) method, but the applicability of the last method is confined to certain kinds of integro-differential operators, corresponding to BVPs and cannot be universal for all problems. There are several types of decomposition methods for solving BVPs with IDEs. The most popular is Adomyan decomposition method and its modifications, where the Adomyan polynomials are used, and approximate solutions of given BVPs are obtained [11–19]. Other types of decomposition method were considered in [4], [5], [20]. Factorization of tensor integro-differential wave equations of the acoustics of dispersive viscoelastic anisotropic media is performed for the one-dimensional case in [4]. The integro-differential one-dimensional tensor wave equations of the electrodynamics of dispersive anisotropic media are factorized in [5]. The initial first order integro-differential operator with arbitrary nonpositive parameters was decomposed on three factors in [20] and further the sufficient conditions for the existence of a solution are obtained on half line.

We propose in this article another factorization method on two factors which successfully was applied in the articles by the authors [6], [7], [21]–[26] and by another author in [27]. Here we generalize the results of these papers and study a more complicated boundary value problem with an abstract operator equation

$$\mathbf{B}_1 u = \mathcal{A}^2 u - V\Phi(A_0 u) - Y\Phi(A_0^2 u) - S\Psi(\mathcal{A}A_0 u)-$$
$$-G\Psi(\mathcal{A}^2 u) = f, \quad D(\mathbf{B}_1) = D(\mathcal{A}^2)$$

on a Banach space X, where $\mathcal{A}, A_0$ are abstract linear differential operators, the functional vectors Φ, Ψ are defined on X_m and vectors $V, Y, S, G \in X_m$. We obtained the conditions on the vectors

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V, Y, S, G under which the operator $\mathbf{B}_1$ can be factorized in a product of two second degree operators, i.e. $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$ with

$$\mathbf{B}_0 u = A_0^2 u - S_0 \Phi(A_0 u) - G_0 \Phi(A_0^2 u) = f, \quad D(\mathbf{B}_0) = D(A_0^2),$$

$$\mathbf{B} u = A^2 u - S\Psi(Au) - G\Psi(A^2 u) = f, \qquad D(\mathbf{B}) = D(A^2)$$

and then found the exact solution in closed form of the given problem, using the exact solutions of the above two simple problems. Using of the obtained formula for the exact solution of the equation $\mathbf{B}_1 u = f$ makes it possible to easily obtain exact solutions of a class of Fredholm IDEs with ordinary or partial differential operators. The decomposition method, applied to abstract operator equation

$$Tu = Au - Ku - G\Psi(Au) = f, \quad D(T) = D(A) = \{u \in X_n : \Phi(u) = \mathbf{0}\}$$

on a Banach space X for solving boundary value problems for n-th order linear Volterra-Fredholm integro-differential equations of convolution type, was used in [6], [7], where were constructed the closed-form solutions to the two-phase integral model of Euler-Bernoulli nanobeams in bending under transverse distributed load and various types of boundary conditions. In [21] the operator B_1 corresponding to the abstract operator equation

$$B_1 u = AA_0 u - S\langle A_0 u, \Phi^t \rangle_{H^m} - G\langle AA_0 u, F^t \rangle_{H^m} = f$$

on a Hilbert space H was factorized in two operators, i.e. $B_1 = B_G B_{G_0}$, where

$$B_{G_0} u = A_0 u - G_0 \langle A_0 u, \Phi^t \rangle_{H^m} = f, \qquad D(B_{G_0}) = D(A_0),$$

$$B_G u = Au - G\langle Au, F^t \rangle_{H^m} = f, \qquad D(B_G) = D(A).$$

Further, using the exact solutions of these two simple equations, the exact solution of $B_1 u = f$ was obtained. An exact solution to the abstract operator equation

$$B_1 u = \mathcal{A}u - S\Phi(A_0 u) - G\Psi(\mathcal{A}u) = f, \quad D(B_1) = D(\mathcal{A})$$

on a Banach space was found in [22] by factorization of B_1 in two simple operators, and then the corresponding theory was applied for solving of Hyperbolic integro-differential equations with integral boundary conditions. The exact solution to the abstract operator equation

$$B_1 u = A^2 u - S\Phi(Au) - G\Psi(A^2 u) = f, \quad D(B_1) = D(A^2)$$

on a Banach space was obtained in [23]. The operator B_1 corresponding to the abstract operator equations

$$B_1 u = A^2 u - SF(Au) - SF(A^2 u) = f,$$
$$D(B_1) = \{u \in D(A^2) : \Phi(u) = N\Psi(Au),\, \Phi(Au) = DF(Au) + N\Psi(A^2 u)\}, \quad \text{and}$$
$$B_1 u = A^2 u - SF(Au) - SF(A^2 u) = f,$$
$$D(B_1) = \{u \in D(A^2) : \Phi(u) = N\Psi(u),\, \Phi(Au) = DF(Au) + N\Psi(Au)\},$$

where D, N are matrices, S, G are vectors, by decomposition method for squared operators is factorized in $B_1 = B^2$ and then the exact solution of $B_1 u = f$ in closed form is easily obtained in [24], [25], respectively. The exact solution to the abstract operator equation

$$B_1 u = \mathcal{A}u - S\Phi(u) - G\Psi(A_0 u) = f, \quad D(B_1) = D(AA_0)$$

on a Banach space by factorization of B_1 in two simple operators B, B_0, was investigated in [26]. The exact solution in closed analytical form to the abstract operator equation

$$B_1 u = \mathcal{A}u - S_0 F(Au) - G_0 \Phi(Au) = f, \quad D(B_1) = D(\mathcal{A})$$

was obtained by decomposition method in [27], and then was applied for solving some ordinary integro-differential and partial integro-differential equations. Our decomposition method is simple to use and can be easily incorporated into any Computer Algebra (CAS). The paper is organized as follows. In Section 1 we give an introduction, terminology and notation. In Section 2 we develop the theory for the solution of the problem $\mathbf{B}_1 x = f$ when $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$ with $\mathbf{B}$ and $\mathbf{B}_0$ being two linear second degree abstract operators and give an example of boundary problem with integro-differential equation demonstrating the power and usefulness of the methods presented.

Preliminaries

Throughout this paper by X we denote the complex Banach space and by X^* the adjoint space of X, i.e. the set of all complex-valued linear and bounded functionals f on X. We denote by $f(x)$ the value of f on x. We write $D(A)$ and $R(A)$ for the domain and the range of the operator $A : X \to Y$, respectively. An operator $A : X \to Y$ is said to be *injective or uniquelly solvable* if for all $u_1, u_2 \in D(A)$ such that $Au_1 = Au_2$, follows that $u_1 = u_2$. Remind that a linear operator A is injective if and only if $\ker A = \{0\}$. An operator $A : X \to Y$ is called *surjective* or *everywhere solvable* if $R(A) = Y$. The operator $A : X \to Y$ is said to be *bijective* if A is both injective and surjective. An operator A and the corresponding problem $Au = f$ are called *correct* if A is bijective and its inverse A^{-1} is bounded on Y. Lastly, if for operator $B_1 : X \to X$ there exist two operators B and B_0 such that $B_1 = BB_0$ then we say that BB_0 is *a decomposition (factorization)* of B_1. If $g_i \in X$ and $\psi_i \in X^*, i = 1, \dots, m$, $x \in X$, then we denote by $G = (g_1, \dots, g_m)$, $\Psi =$col$(\psi_1, \dots, \psi_m)$ and $\Psi(x) =$col$(\psi_1(x), \dots, \psi_m(x))$ and write $G \in X_m$, $\Psi \in X_m^*$. If $G = (g_1, ..., g_m)$, $g_1, ..., g_m \in D(A)$, then we write $G \in [D(A)]_m$. We will denote by $\Psi(G)$ the $m \times m$ matrix whose i,j-th entry $\psi_i(g_j)$ is the value of functional ψ_i on element g_j. Note that $\Psi(GC) = \Psi(G)C$, where C is a $m \times k$ constant matrix. We will also denote by I_m the identity $m \times m$ matrix.

We will use the following Theorem, that have been shown in [20] and is recalled here but with a different notation tailored to the needs of the present article.

Theorem 1. Let X be a complex Banach space, the vectors $G_0 = (g_{10}, ..., g_{m0})$, $G = (g_1, ..., g_m)$, $S = (s_1, ..., s_m) \in X_m$, the components of the vectors $\Psi = col(\psi_1, ..., \psi_m)$ and $\Phi = col(\phi_1, ..., \phi_m)$ belong to X^* and the operators $B_0, B, B_1 : X \to X$ defined by

$$B_0 u = A_0 u - G_0 \Phi(A_0 u) = f, \qquad D(B_0) = D(A_0),$$

$$Bu = Au - G\Psi(Au) = f, \qquad D(B) = D(A),$$

$$B_1 u = AA_0 u - S\Phi(A_0 u) - G\Psi(AA_0 u) = f, \quad D(B_1) = D(AA_0), \tag{1}$$

where $A_0, A : X \to X$ are linear correct operators and $G_0 \in [D(A)]_m$. Then the following statements are fulfilled:

(i) If

$$S \in [R(B)]_m \quad \text{and} \quad S = BG_0 = AG_0 - G\Psi(AG_0), \tag{2}$$

then the operator B_1 can be factorized in $B_1 = BB_0$.

(ii) If the components of the vector Φ are linearly independent elements of X^* and the operator B_1 can be factorized in $B_1 = BB_0$, then (2) is fulfilled.

(iii) If there exists a vector $G_0 \in [D(A)]_m$, satisfying the equation $AG_0 - G\Psi(AG_0) = S$, then B_1 is bijective if and only if the operators B_0 and B are bijective, which means that

$$\det V = \det[I_m - \Phi(G_0)] \neq 0 \quad \text{and} \quad \det L = \det[I_m - \Psi(G)] \neq 0.$$

In this case, the unique solution to the boundary value problem (1) for any $f \in X$, is given by

$$u = B_1^{-1} f = A_0^{-1} v + A_0^{-1} G_0 V^{-1} \Phi(v), \quad \text{where} \quad v = A^{-1} f + A^{-1} G L^{-1} \Psi(f). \tag{3}$$

Lemma 2. Let X be a complex Banach space. Then a linear operator $A : X \to X$ is bijective if and only if A^2 is bijective.

Proof. Let A be bijective and $u \in \ker A^2$. Then $A^2 u = 0$. Applying twise the operator A^{-1} to this equation we obtain $u = 0$ which proves that $\ker A^2 = \{0\}$. Consider the equation $A^2 u = f, \quad f \in X$. Applying twise the operator A^{-1} to this equation, we obtain $u = A^{-1}\left(A^{-1} f\right) = A^{-2} f$ for every $f \in X$, which proves that $R(A^2) = X$. Thus A^2 is a bijective operator.

Conversely, let A^2 be bijective. Then $\ker A^2 = \{0\}$ and $R(A^2) = X$, and from well-known relations

$$\ker A \subset \ker A^2, \quad R(A^2) \subset R(A),$$

for a linear operator $A : X \to X$, follows that $\ker A = \{0\}$ and $R(A) = X$. Hence A is a bijective operator.

Bellow we prove the main theorem.

Theorem 3. Let X be a complex Banach space, $A_0, A_0^2, A, A^2 : X \overset{on}{\to} X$ linear operators and the vectors $V, Y, G, S \in X_m$, $\Phi, \Psi \in [X^*]_m$, $S_0, G_0 \in [D(A^2)]_m$. Then for the operators $\mathbf{B}_0, \mathbf{B}, \mathbf{B}_1 : X \to X$, defined by

$$\mathbf{B}_0 u = A_0^2 u - S_0 \Phi(A_0 u) - G_0 \Phi(A_0^2 u) = f, \quad D(\mathbf{B}_0) = D(A_0^2), \tag{4}$$

$$\mathbf{B} u = A^2 u - S\Psi(Au) - G\Psi(A^2 u) = f, \qquad D(\mathbf{B}) = D(A^2), \tag{5}$$

$$\mathbf{B}_1 u = A^2 A_0^2 u - V\Phi(A_0 u) - Y\Phi(A_0^2 u) - S\Psi(A A_0^2 u) - G\Psi(A^2 A_0^2 u) = f,$$
$$D(\mathbf{B}_1) = D(A^2 A_0^2) = \{u \in D(A_0^2) : A_0^2 u \in D(A^2)\}, \tag{6}$$

hold true the next statements:

(i) If the vectors $G_0 = (g_{10}, ..., g_{m0})$ and $S_0 = (s_{10}, ..., s_{m0})$ belong to $[D(A^2)]_m$ and satisfy the system of equations

$$V = \mathbf{B} S_0 = A^2 S_0 - S\Psi(AS_0) - G\Psi(A^2 S_0), \tag{7}$$

$$Y = \mathbf{B} G_0 = A^2 G_0 - S\Psi(AG_0) - G\Psi(A^2 G_0), \tag{8}$$

then the operator $\mathbf{B}_1$ can be factorized in $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$.

(ii) If $G_0 = (g_{10}, ..., g_{m0})$, $S_0 = (s_{10}, ..., s_{m0}) \in [D(A^2)]_m$ and the operator $\mathbf{B}_1$ is factorized in $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$, where A, A_0 are bijective, and if the functional vectors

$$\hat{\mathbf{\Phi}}(f) = \left(\Phi * A_0^{-1} A^{-2}\right)(f) = \Phi(A_0^{-1} A^{-2} f), \quad \check{\mathbf{\Phi}}(f) = \left(\Phi * A^{-2}\right)(f) = \Phi(A^{-2} f)$$

are linearly independent on X, then (7), (8) hold true.

(iii) The operators $\mathbf{B}_0, \mathbf{B}$ are bijective if and only if, respectively,

$$\det \mathbf{L}_0 = \det \begin{pmatrix} I_m - \Phi(A^{-1} S_0) & -\Phi(A^{-1} G_0) \\ -\Phi(S_0) & I_m - \Phi(G_0) \end{pmatrix} \neq 0, \tag{9}$$

$$\det \mathbf{L} = \det \begin{pmatrix} I_m - \Psi(A^{-1} S) & -\Psi(A^{-1} G) \\ -\Psi(S) & I_m - \Psi(G) \end{pmatrix} \neq 0, \tag{10}$$

and in this case the unique solutions of (4), (5) for any $f \in X$ are given by

$$u = \mathbf{B}_0^{-1} f = A_0^{-2} f + (A_0^{-2} S_0, A_0^{-2} G_0)\mathbf{L}_0^{-1} \begin{pmatrix} \Phi(A_0^{-1} f) \\ \Phi(f) \end{pmatrix}, \tag{11}$$

$$u = \mathbf{B}^{-1} f = A^{-2} f + (A^{-2} S, A^{-2} G)\mathbf{L}^{-1} \begin{pmatrix} \Psi(A^{-1} f) \\ \Psi(f) \end{pmatrix}, \tag{12}$$

respectively.

(iv) If V, Y are defined by (7), (8) and A, A_0 are bijective operators, then $\mathbf{B}_1$ is bijective if and only if (9) and (10) are fulfilled, and the unique solution of (6) in this case for every $f \in X$ is given by

$$u = A_0^{-2} v + (A_0^{-2} S_0, A_0^{-2} G_0)\mathbf{L}_0^{-1} \begin{pmatrix} \Phi(A_0^{-1} v) \\ \Phi(v) \end{pmatrix}, \text{ where} \tag{13}$$

$$v = A^{-2} f + (A^{-2} S, A^{-2} G)\mathbf{L}^{-1} \begin{pmatrix} \Psi(A^{-1} f) \\ \Psi(f) \end{pmatrix}. \tag{14}$$

Proof (i). Taking into account that G_0, $S_0 \in [D(A^2)]_m$ we obtain

$$D(\mathbf{B}\mathbf{B}_0) = \{u \in D(\mathbf{B}_0) : \mathbf{B}_0 u \in D(\mathbf{B})\} =$$
$$= \{u \in D(A_0^2) : A_0^2 u - S_0 \Phi(A_0 u) - G_0 \Phi(A_0^2 u) \in D(A^2)\} =$$
$$= \{u \in D(A_0^2) : A_0^2 u \in D(A^2)\} = D(A^2 A_0^2) = D(\mathbf{B}_1).$$

We put $y = \mathbf{B}_0 u$. Then for each $u \in D(A^2 A_0^2)$ since (5) and (4) we have

$$\mathbf{B}\mathbf{B}_0 u = \mathbf{B} y = A^2 y - S\Psi(Ay) - G\Psi(A^2 y) =$$
$$= A^2 \mathbf{B}_0 u - S\Psi(A\mathbf{B}_0 u) - G\Psi(A^2 \mathbf{B}_0 u) =$$
$$= A^2 [A_0^2 u - S_0 \Phi(A_0 u) - G_0 \Phi(A_0^2 u)] -$$
$$- S\Psi\left(A[A_0^2 u - S_0 \Phi(A_0 u) - G_0 \Phi(A_0^2 u)]\right) -$$
$$- G\Psi\left(A^2 [A_0^2 u - S_0 \Phi(A_0 u) - G_0 \Phi(A_0^2 u)]\right) =$$
$$= A^2 A_0^2 u - A^2 S_0 \Phi(A_0 u) - A^2 G_0 \Phi(A_0^2 u) -$$
$$- S\Psi\left(A A_0^2 u - A S_0 \Phi(A_0 u) - A G_0 \Phi(A_0^2 u)\right) -$$
$$- G\Psi\left(A^2 A_0^2 u - A^2 S_0 \Phi(A_0 u) - A^2 G_0 \Phi(A_0^2 u)\right) =$$
$$= A^2 A_0^2 u - A^2 S_0 \Phi(A_0 u) - A^2 G_0 \Phi(A_0^2 u) -$$
$$- S\Psi(A A_0^2 u) + S\Psi(A S_0)\Phi(A_0 u) +$$
$$+ S\Psi(A G_0)\Phi(A_0^2 u) - G\Psi(A^2 A_0^2 u) +$$
$$+ G\Psi(A^2 S_0)\Phi(A_0 u) + G\Psi(A^2 G_0)\Phi(A_0^2 u).$$

So we obtain

$$\mathbf{B}\mathbf{B}_0 u = A^2 A_0^2 u - [A^2 S_0 - S\Psi(A S_0) - G\Psi(A^2 S_0)]\Phi(A_0 u) -$$
$$- [A^2 G_0 - S\Psi(A G_0) - G\Psi(A^2 G_0)]\Phi(A_0^2 u) -$$
$$- S\Psi(A A_0^2 u) - G\Psi(A^2 A_0^2 u), \quad \text{or}$$

$$\mathbf{B}\mathbf{B}_0 u = A^2 A_0^2 u - \mathbf{B} S_0 \Phi(A_0 u) - \mathbf{B} G_0 \Phi(A_0^2 u) - S\Psi(A A_0^2 u) - G\Psi(A^2 A_0^2 u), \tag{15}$$

where the relations

$$\mathbf{B}S_0 = A^2 S_0 - S\Psi(AS_0) - G\Psi(A^2 S_0),$$

$$\mathbf{B}G_0 = A^2 G_0 - S\Psi(AG_0) - G\Psi(A^2 G_0)$$

follow by substituting $u = S_0$ and $u = G_0$ in (5). By comparing (6) with (15), it is easy to verify that $\mathbf{B}\mathbf{B}_0 u = \mathbf{B}_1 u$ for each $u \in D(A^2 A_0^2)$ if (7), (8) hold true.

(ii) Let now $\mathbf{B}\mathbf{B}_0 u = \mathbf{B}_1 u$ for each $u \in D(A^2 A_0^2)$. Then by subtraction for each $u \in D(A^2 A_0^2)$, we get $\mathbf{B}\mathbf{B}_0 u - \mathbf{B}_1 u = 0,$ which implies

$$(\mathbf{B}S_0 - V)\Phi(A_0 u) + (\mathbf{B}G_0 - Y)\Phi(A_0^2 u) = 0,$$

or, since the operators A, A_0 are bijective and, by Lemma 2, the operators A^2, A_0^2 are bijective too, we have

$$(\mathbf{B}S_0 - V)\Phi(A_0^{-1} A^{-2} A^2 A_0^2 u) + (\mathbf{B}G_0 - Y)\Phi(A^{-2} A^2 A_0^2 u) = 0,$$

or denoting $f = A^2 A_0^2 u$, for each $f \in X$ we get

$$(\mathbf{B}S_0 - V)\Phi(A_0^{-1} A^{-2} f) + (\mathbf{B}G_0 - Y)\Phi(A^{-2} f) = 0,$$

which is

$$(\mathbf{B}S_0 - V)\hat{\boldsymbol{\Phi}}(f) + (\mathbf{B}G_0 - Y)\check{\boldsymbol{\Phi}}(f) = 0, \quad \forall f \in X.$$

The last equation, because of the vectors $\hat{\boldsymbol{\Phi}}, \check{\boldsymbol{\Phi}}$ are linear independent on X, gives $V = \mathbf{B}S_0$, $Y = \mathbf{B}G_0$.

(iii)-(iv) Let the operator $\mathbf{B}_1$ and the vectors V, Y be defined by (6), (7) and (8), respectively. Equation (6) can also be written in matrix notation as

$$\mathbf{B}_1 u = A^2 A_0^2 u - \big(\mathbf{B}S_0, \mathbf{B}G_0\big) \begin{pmatrix} \Phi(A_0 u) \\ \Phi(A_0^2 u) \end{pmatrix} - (S, G) \begin{pmatrix} \Psi(A A_0^2 u) \\ \Psi(A^2 A_0^2 u) \end{pmatrix} = f,$$

or

$$\mathbf{B}_1 u = A^2 A_0^2 u - \big(\mathbf{B}S_0, \mathbf{B}G_0\big) \begin{pmatrix} \Phi(A_0^{-1} A_0^2 u) \\ \Phi(A_0^2 u) \end{pmatrix} - (S, G) \begin{pmatrix} \Psi(A^{-1} A^2 A_0^2 u) \\ \Psi(A^2 A_0^2 u) \end{pmatrix} = f,$$

or

$$\mathbf{B}_1 u = \mathcal{A}\mathcal{A}_0 u - \widetilde{S}\widetilde{\Phi}(\mathcal{A}_0 u) - \widetilde{G}\widetilde{\Psi}(\mathcal{A}\mathcal{A}_0 u) = f, \quad D(\mathbf{B}_1) = D(\mathcal{A}\mathcal{A}_0), \tag{16}$$

where

$$\mathcal{A} = A^2,\ \mathcal{A}_0 = A_0^2, \quad \widetilde{S} = \mathbf{B}\widetilde{G}_0, \quad \widetilde{G} = (S, G),\ \widetilde{G}_0 = (S_0, G_0), \tag{17}$$

$$\widetilde{\Phi} = col\big(\Phi * A_0^{-1}, \Phi\big), \quad \widetilde{\Psi} = col\big(\Psi * A^{-1}, \Psi\big) \tag{18}$$

and $(\Phi * A_0^{-1})(v) = \Phi(A_0^{-1} v)$, $(\Psi * A^{-1})(v) = \Psi(A^{-1} v)$. Remind that by Lemma 2, the operators $\mathcal{A} = A^2$ and $\mathcal{A}_0 = A_0^2$ are bijective, because of A and A_0 are bijective. Notice that the components of the vectors $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are bounded on X, since the operators A_0^{-1}, A^{-1} are bounded, the components of the vectors Φ and Ψ belong to X^* and for any $f \in X$ the elements $A_0^{-1} A^{-2} f$, $A^{-2} f \in X$. It is easy to verify that Equations (4) and (5) can be equivalently represented in matrix form:

$$\mathbf{B}_0 u = \mathcal{A}_0 u - \widetilde{G}_0 \widetilde{\Phi}(\mathcal{A}_0 u) = f, \quad D(\mathbf{B}_0) = D(\mathcal{A}_0),$$

$$\mathbf{B} u = \mathcal{A} u - \widetilde{G}\widetilde{\Psi}(\mathcal{A} u) = f, \qquad D(\mathbf{B}) = D(\mathcal{A}).$$

Now by Theorem 1, where instead of $B, B_0, B_1, S, G, \Phi, \Psi, A, A_0, L, V$ and m we have $\mathbf{B}, \mathbf{B}_0, \mathbf{B}_1, \widetilde{S}, \widetilde{G}, \widetilde{\Phi}, \widetilde{\Psi}, \mathcal{A}, \mathcal{A}_0, \mathbf{L}, \mathbf{V}$ and $2m$, respectively, we conclude that the operator $\mathbf{B}_1$ can be factorized in $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$ if

$$\mathcal{A}\widetilde{G}_0 - \widetilde{G}\widetilde{\Psi}(\mathcal{A}\widetilde{G}_0) = \widetilde{G}. \tag{19}$$

It is easy to verify that Equation (19) is equivalent to System (7), (8). Also by Theorem 1, the operator $\mathbf{B}_1$ is bijective if and only if

$$\det \mathbf{V} = \det[I_{2m} - \widetilde{\Phi}(\widetilde{G}_0)] \neq 0 \quad \text{and} \quad \det \mathbf{L} = \det[I_{2m} - \widetilde{\Psi}(\widetilde{G})] \neq 0,$$

respectively. The last inequalities, since

$$\widetilde{\Phi}(\widetilde{G}_0) = \begin{pmatrix} (\Phi * A_0^{-1})(S_0) & (\Phi * A_0^{-1})(G_0) \\ \Phi(S_0) & \Phi(G_0) \end{pmatrix} = \begin{pmatrix} \Phi(A_0^{-1}S_0) & \Phi(A_0^{-1}G_0) \\ \Phi(S_0) & \Phi(G_0) \end{pmatrix},$$

$$\widetilde{\Psi}(\widetilde{G}) = \begin{pmatrix} (\Psi * A^{-1})(S) & (\Psi * A^{-1})(G) \\ \Psi(S) & \Psi(G) \end{pmatrix} = \begin{pmatrix} \Psi(A^{-1}S) & \Psi(A^{-1}G) \\ \Psi(S) & \Psi(G) \end{pmatrix},$$

give (9) and (10). Let $\mathbf{B}_1 u = \mathbf{B}\mathbf{B}_0 u = f$, $f \in X$. By Theorem 1 using (3), since $\mathbf{B}, \mathbf{B}_0$ are bijective operators, we obtain the unique solution of (16) or (6)

$$u = \mathbf{B}_0^{-1} v = \mathcal{A}_0^{-1} v + \mathcal{A}_0^{-1} \widetilde{G}_0 \mathbf{L}_0^{-1} \widetilde{\Phi}(v), \text{ where}$$

$$v = \mathbf{B}^{-1} f = \mathcal{A}^{-1} f + \mathcal{A}^{-1} \widetilde{G} \mathbf{L}^{-1} \widetilde{\Psi}(f),$$

which since (17), (18) gives

$$u = \mathbf{B}_0^{-1} v = A_0^{-2} v + (A_0^{-2} S_0, A_0^{-2} G_0) \mathbf{L}_0^{-1} \begin{pmatrix} \Phi(A_0^{-1} v) \\ \Phi(v) \end{pmatrix}, \text{ where} \tag{20}$$

$$v = \mathbf{B}^{-1} f = A^{-2} f + (A^{-2} S, A^{-2} G) \mathbf{L}^{-1} \begin{pmatrix} \Psi(A^{-1} f) \\ \Psi(f) \end{pmatrix}. \tag{21}$$

So we proved (13), (14). From (20), (21) we immediately obtain (11), (12). The theorem is proved.

The next theorem follows from Theorem 3 and is useful in applications and gives the decomposition $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$, where $\mathbf{B}, \mathbf{B}_0$ beforehand we do not know. Also this theorem gives a criterion for the bijectivity of $\mathbf{B}_1$ and the solution of $\mathbf{B}_1 u = f$ in an elegant way.

Theorem 4. Let the space X and the vectors V, Y, S, G, Φ, Ψ be defined as in Theorem 3 and the operator $B_1 : X \to X$ by

$$\mathbf{B}_1 u = \mathcal{A}^2 u - V\Phi(A_0 u) - Y\Phi(A_0^2 u) - S\Psi(\mathcal{A}A_0 u) - G\Psi(\mathcal{A}^2 u) = f, \; D(\mathbf{B}_1) = D(\mathcal{A}^2), \tag{22}$$

where $A_0 : X \to X$ is a bijective n_1-order differential operator and $\mathcal{A} : X \to X$ is a n-order differential operator, $n_1 < n$. Suppose that there exists a bijective linear differential $n - n_1$ order operator $A : X \to X$ such that

$$\mathcal{A} = AA_0, \quad D(\mathbf{B}_1) = D(A^2 A_0^2) \tag{23}$$

and

$$\det \mathbf{L} = \det \begin{pmatrix} I_m - \Psi(A^{-1}S) & -\Psi(A^{-1}G) \\ -\Psi(S) & I_m - \Psi(G) \end{pmatrix} \neq 0. \tag{24}$$

Then the operator $\mathbf{B}_1$ is factorized into $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$, where $\mathbf{B}_0$, $\mathbf{B}$ are defined by (4), (5), respectively, and

$$S_0 = A^{-2} V + (A^{-2} S, A^{-2} G) \mathbf{L}^{-1} \begin{pmatrix} \Psi(A^{-1} V) \\ \Psi(V) \end{pmatrix}, \tag{25}$$

$$G_0 = A^{-2} Y + (A^{-2} S, A^{-2} G) \mathbf{L}^{-1} \begin{pmatrix} \Psi(A^{-1} Y) \\ \Psi(Y) \end{pmatrix}. \tag{26}$$

Furtermore the operator $\mathbf{B}_1$ is bijective if (9) is fulfilled, and in this case a unique solution to the boundary value problem (22), (23) for any $f \in X$ is given by (13), (14).

Proof. Substituting $\mathcal{A} = AA_0$ into (22) we obtain the operator $\mathbf{B}_1$ in the form (6). Construct the operators $\mathbf{B}_0$ and $\mathbf{B}$ by using (4) and (5), respectively, where for $\mathbf{B}$ we take the elements G, S, Ψ and A from (22) and (23), and for $\mathbf{B}_0$ the elements A_0, Φ and S_0, G_0 from (22) and (25), (26).

Note that the operator $\mathbf{B}$, by Theorem 3 (iii), since (24) and bijectivity of A, is bijective, and that taking into account (12) the system of equations (25), (26) can be represented as $S_0 = \mathbf{B}^{-1}V$ and $G_0 = \mathbf{B}^{-1}Y$. The last system, because of bijectivity of $\mathbf{B}$, is equivalent to the system $V = \mathbf{B}S_0$ and $Y = \mathbf{B}G_0$, which is the system (7), (8). Then by Theorem 3 (i), the operator $\mathbf{B}_1$ can be factorized in $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$. Furthermore by Theorem 3 (iv), since (24) and bijectivity of A_0, the operator $\mathbf{B}_1$ is bijective if (9) holds. The unique solution to (22), (23), by Theorem 3 (iv), is given by (13), (14). The theorem is proved.

A reader can prove easily by Lemma 2 the next proposition.

Proposition 5. Let the operators $A_0, A : C[0,1] \to C[0,1]$ be defined by

$$A_0u(t) = u'(t) = f, \quad D(A_0) = \{u(t) \in C^1[0,1] : \, u(0) = 0\}, \tag{27}$$

$$Au(t) = u'(t) = f, \quad D(A) = \{u(t) \in C^1[0,1] : \, u(1) = 0\}. \tag{28}$$

Then:

(i) The operators A_0, A are bijective and the unique solution of the problem (27) and (28) is given by

$$u(t) = A_0^{-1}f(t) = \int_0^t f(x)dx, \tag{29}$$

$$u(t) = A^{-1}f(t) = \int_0^t f(x)dx - \int_0^1 f(x)dx, \tag{30}$$

respectively.

(ii) The operators $A_0^2, A^2 : C[0,1] \to C[0,1]$ are defined by

$$A_0^2u(t) = u''(t) = f, \quad D(A_0^2) = \{u(t) \in C^2[0,1] : \, u(0) = 0, \, u'(0) = 0\}, \tag{31}$$

$$A^2u(t) = u''(t) = f, \quad D(A^2) = \{u(t) \in C^2[0,1] : \, u(1) = 0, \, u'(1) = 0\}, \tag{32}$$

and bijective. The unique solution of the problem (31), (32) is given by

$$u(t) = A_0^{-2}f(t) = \int_0^t (t-x)f(x)dx, \tag{33}$$

$$u(t) = A^{-2}f(t) = \int_0^t (t-x)f(x)dx - \int_0^1 (t-x)f(x)dx, \tag{34}$$

respectively.

Example 6. Let the operator $\mathbf{B}_1 : C[0,1] \to C[0,1]$ be defined by

$$\mathbf{B}_1u = u^{(4)}(t) - (5-2t)\int_0^1 x^2\, u'(x)dx - (6t-3)\int_0^1 x^2\, u''(x)dx -$$

$$-12\int_0^1 x\, u'''(x)dx - (2t+1)\int_0^1 x\, u^{(4)}(x)dx = 2 - 3t, \tag{35}$$

$$D(\mathbf{B}_1) = \{u(x) \in C^4[0,1] : u(0) = u'(0) = u''(1) = u'''(1) = 0\}. \tag{36}$$

Then:

(i) $\mathbf{B}_1$ can be factorized as a product of two operators and is bijective.

(ii) The unique solution of Problem (35)-(36) is given by

$$u(t) = -\frac{t^2(12271t^3 - 46530t^2 + 63410t - 33760)}{531448}. \tag{37}$$

Proof (i). If we compare equation (35) with equation (22), it is natural to take $A_0 u = u'(x)$, $A_0^2 u = u''(x)$, $\mathcal{A}A_0 u = u'''(x)$, $\mathcal{A}^2 u = u^{(4)}$, $n_1 = 1$, $V = 5 - 2t$, $Y = 6t - 3$, $S = 12$, $G = 2t + 1$, f=2-3t, $\Phi(A_0 u) = \int_0^1 x^2\, u'(x)dx$, $\Phi(A_0^2 u) = \int_0^1 x^2\, u''(x)dx$, $\Psi(\mathcal{A}A_0 u) = \int_0^1 x\, u'''(x)dx$, $\Psi(\mathcal{A}^2 u) = \int_0^1 x\, u^{(4)}(x)dx$.

Then

$$\Phi(v) = \int_0^1 x^2\, v(x)dx, \quad \Psi(v) = \int_0^1 x\, v(x)dx. \tag{38}$$

It is evident that $\Phi, \Psi \in X^*$. We chooce the operator A to satisfy (23), namely $\mathcal{A}u = AA_0 u$, $D(\mathbf{B}_1) = D(A^2 A_0^2)$. From $\mathcal{A}u(x) = AA_0 u(x)$, $\mathcal{A}A_0 u = u'''(x)$ and $A_0 u(x) = u'(x)$ we get $\mathcal{A}A_0 u(x) = AA_0^2 u(x) = Au''(x) = u'''(x)$. Denote $v(x) = u''(x)$, then $Av(x) = v'(x)$. Let $D(A_0) = \{u(x) \in C^1[0,1] : u(0) = 0\}$, $D(A) = \{v(x) \in C^1[0,1] : v(1) = 0\}$. So we proved that the operators A_0, A are defined as in (27), (28). Then the operators A_0^2, A^2 are defined as in (31), (32), respectively. Further we find

$$D(A^2 A_0^2) = \{u(t) \in D(A_0^2) : A_0^2 u(t) \in D(A^2)\} =$$

$$= \{u(t) \in C^2[0,1] : u(0) = u'(0) = 0,\ u''(t) \in C^2[0,1],\ u''(1) = u'''(1) = 0\} =$$

$$= \{u(t) \in C^4[0,1] : u(0) = u'(0) = 0,\ u''(1) = u'''(1) = 0\} = D(\mathbf{B}_1).$$

This proves that the conditions (23) are satisfied and so we can apply Theorem 4. Using (30) and (38) by simple calculations we find

$A^{-1}S = \int_0^t S dx - \int_0^1 S dx = 12t - 12$,

$A^{-1}G = \int_0^t G dx - \int_0^1 G dx = \int_0^t (2x+1)dx - \int_0^1 (2x+1)dx = t^2 + t - 2$,

$\Psi(A^{-1}S) = \int_0^1 x(12x - 12)dx = -2$, $\quad \Psi(A^{-1}G) = \int_0^1 x(x^2 + x - 2)dx = -5/12$,

$\Psi(S) = \int_0^1 12x dx = 6$, $\quad \Psi(G) = \int_0^1 x(2x+1)dx = 7/6$.

By (10), we obtain $\mathbf{L} = \begin{pmatrix} 3 & 5/12 \\ -6 & -1/6 \end{pmatrix}$. Then $\det \mathbf{L} \neq 0$ and $\mathbf{L}^{-1} = \begin{pmatrix} -1/12 & -5/24 \\ 3 & 3/2 \end{pmatrix}$. By Theorem 4, the operator $\mathbf{B}_1$ is factorized in $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$, where $\mathbf{B}_0$, $\mathbf{B}$ and S_0, G_0 are defined by (4), (5) and (25), (26), respectively. Using (34) for $S = 12$, $G = 2x + 1$, we obtain

$A^{-2}S = \int_0^t (t-x)S dx - \int_0^1 (t-x)S dx = 6(t-1)^2$,

$A^{-2}G = t^3/3 + t^2/2 - 2t + 7/6$.

By (30), (34) for $V = 5 - 2x$, $Y = 6x - 3$ we get

$A^{-1}V = \int_0^t V(x)dx - \int_0^1 V(x)dx = 5t - t^2 - 4$,

$A^{-1}Y = \int_0^t Y(x)dx - \int_0^1 Y(x)dx = 3t^2 - 3t$,

$A^{-2}V = \int_0^t (t-x)V(x)dx - \int_0^1 (t-x)V(x)dx = -t^3/3 + 5t^2/2 - 4t + 11/6$,

$A^{-2}Y = t^3 - 3t^2/2 + 1/2$, and further by (38) we have

$\Psi(A^{-1}V) = \int_0^1 t(5t - t^2 - 4)dt = -7/12$, $\quad \Psi(A^{-1}Y) = -1/4$,

$\Psi(V) = \int_0^1 t(5 - 2t)dt = 11/6$, $\quad \Psi(Y) = 1/2$.

Applying (25), (26) and the above calculations we get

$S_0 = S_0(t) = (t-1)^2$, $\quad G_0 = G_0(t) = t(t-1)^2$.

By (29) we find $A_0^{-1}S_0 = \int_0^t S_0(x)dx = t(t^2 - 3t + 3)/3$,

$A_0^{-1}G_0 = \int_0^t G_0(x)dx = t^2(3t^2 - 8t + 6)/12$. Then

$\Phi(A_0^{-1}S_0) = \int_0^1 t^2 t(t^2 - 3t + 3)/3 dt = 19/180$,

$\Phi(A_0^{-1}G_0) = \int_0^1 t^2 t^2(3t^2 - 8t + 6)/12 dt = 31/1260$,

$\Phi(S_0) = \int_0^1 t^2(t-1)^2 dt = 1/30,$
$\Phi(G_0) = \int_0^1 t^3(t-1)^2 dt = 1/60.$
Using (9) we obtain $\mathbf{L}_0 = \begin{pmatrix} 161/180 & -31/1260 \\ -1/30 & 59/60 \end{pmatrix}$. It is easy to verify that

$$\det \mathbf{L}_0 \neq 0, \qquad \mathbf{L}_0^{-1} = \frac{1}{66431} \begin{pmatrix} 74340 & 1860 \\ 2520 & 67620 \end{pmatrix}.$$

Then, by the Theorem 4, the operator $\mathbf{B}_1$ is bijective.

(ii) Now we find the solution of (35)-(36). Using (29), (33), (30), (34), (38) we find

$$A_0^{-2} S_0 = \frac{t^2}{12}(t^2 - 4t + 6), \quad A_0^{-2} G_0 = \frac{t^3}{60}(3t^2 - 10t + 10),$$

and for $f = 2 - 3t$

$$A^{-1} f = \frac{1}{2}(4t - 3t^2 - 1), \quad A^{-2} f = \frac{1}{2}(-t^3 + 2t^2 - t),$$

$$\Psi(f) = \int_0^1 (2 - 3t) t dt = 0, \quad \Psi(A^{-1} f) = 1/24.$$

Using (14) from the above we get

$$v = v(t) = -\frac{11t^3 - 25t^2 + 17t - 3}{24}.$$

Then by (29), (33) and (38) we have

$$A_0^{-1} v(t) = \int_0^t v(x) dx = -\frac{t(33t^3 - 100t^2 + 102t - 36)}{288},$$

$$A_0^{-2} v(t) = \int_0^t (t - x) v(x) dx = -\frac{t^2(33t^3 - 125t^2 + 170t - 90)}{1440},$$

$$A_0^{-2} S_0 = \int_0^t (t - x) S_0(x) dx = \int_0^t (t - x)(x - 1)^2 dx = \frac{t^2(t^2 - 4t + 6)}{12},$$

$$A_0^{-2} G_0 = \int_0^t (t - x) G_0(x) dx = \int_0^t (t - x) x (x - 1)^2 dx = \frac{t^3(3t^2 - 10t + 10)}{60},$$

$$\Phi(v) = \int_0^1 x^2 v(x) dx = -\frac{1}{288},$$

$$\Phi(A_0^{-1} v) = -\int_0^1 x^2 A_0^{-1} v(x) dx = -\int_0^1 x^2 \left[\frac{x(33x^3 - 100x^2 + 102x - 36)}{288}\right] dx = \frac{29}{15120}.$$

Substituting these values into (13) we obtain (37).

Author Contributions

I.N. Parasidis collected and analyzed data, and led manuscript preparation. E. Providas assisted in data collection and analysis.

Conflict of Interest

The authors declare no conflict of interest.

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Абстрактілі операторларды екінші дәрежелі екі операторға факторизациялау және оны интегралдық дифференциалдық теңдеулерге қолдану

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Мақалада дербес туындылы дифференциалдық оператормен немесе Фредгольм интегралдық-дифференциалдық теңдеуіне сәйкес келетін қарапайым дифференциалдық операторы бар B_1 абстрактылы сызықтық операторымен $\mathbf{B}_1 x = f$ шекаралық есебі зерттелген. Биективті оператор $\mathbf{B}_1$ түріндегі факторизацияны өткізген жағдайда $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$, $\mathbf{B}_1 x = f$ есебінің дәл аналитикалық шешімі алынды, мұндағы $\mathbf{B}$, $\mathbf{B}_0$ $\mathbf{B}_1$ қарағанда қарапайым, екінші дәрежелі екі сызықтық абстрактылы оператор. $\mathbf{B}_1$ операторының факторизациялау шарттары және биективтіліктің критерийі табылды.

Кілт сөздер: корректілі оператор, биективті оператор, сызықтық операторларды факторизациялау (жіктеу), Фредгольм интегралдық-дифференциалдық теңдеулері, шекаралық есептер, дәл шешімдер.

Факторизация абстрактных операторов на два оператора второй степени и ее приложения к интегро-дифференциальным уравнениям

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Исследована краевая задача $\mathbf{B}_1 x = f$ с абстрактным линейным оператором B_1, соответствующая интегро-дифференциальному уравнению Фредгольма с обыкновенным дифференциальным оператором или дифференциальным оператором в частных производных. Получено точное аналитическое решение задачи $\mathbf{B}_1 x = f$ в случае, когда биективный оператор $\mathbf{B}_1$ допускает факторизацию вида $\mathbf{B}_1 = \mathbf{B}\mathbf{B}_0$, где $\mathbf{B}$, $\mathbf{B}_0$ — два линейных абстрактных оператора второй степени, более простых, чем $\mathbf{B}_1$. Найдены условия факторизации и критерий биективности оператора $\mathbf{B}_1$.

Ключевые слова: корректный оператор, биективный оператор, факторизация (разложение) линейных операторов, интегро-дифференциальные уравнения Фредгольма, краевые задачи, точные решения.

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