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Algebras of binary formulas for $\aleph_0$-categorical weakly circularly minimal theories: monotonic case

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This article concerns the notion of weak circular minimality being a variant of o-minimality for circularly ordered structures. Algebras of binary isolating formulas are studied for countably categorical weakly circularly minimal theories of convexity rank greater than 1 having both a 1-transitive non-primitive automorphism group and a non-trivial strictly monotonic function acting on the universe of a structure. On the basis of the study, the authors present a description of these algebras. It is shown that there exist both commutative and non-commutative algebras among these ones. A strict m-deterministicity of such algebras for some natural number m is also established.

Keywords: circularly ordered structure, binary formula, isolating formula, algebra of formulas, $\aleph_0$-categorical theory, weak circular minimality, convexity rank, automorphism group, transitivity, primitiveness, m-deterministicity.

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1 Preliminaries

Algebras of binary formulas are a tool for describing relationships between elements of the sets of realizations of a type at the binary level with respect to the superposition of binary definable sets. A *binary isolating formula* is a formula of the form $\varphi(x, y)$ such that for some parameter a the formula $\varphi(a, y)$ isolates a complete type in $S(\{a\})$. The concepts and notations related to these algebras can be found in the papers [1,2]. In recent years, algebras of binary formulas have been studied intensively and have been continued in the works [3–7].

Let L be a countable first-order language. Throughout we consider L-structures and assume that L contains a ternary relational symbol K, interpreted as a circular order in these structures (unless otherwise stated).

Let $\mathcal{M} = \langle M, \leq \rangle$ be a linearly ordered set. If we connect two endpoints of $\mathcal{M}$ (possibly, $-\infty$ and $+\infty$), then we obtain a circular order. More formally, the *circular order* is described by a ternary relation K satisfying the following conditions:

(co1) $\forall x \forall y \forall z (K(x, y, z) \to K(y, z, x))$;

(co2) $\forall x \forall y \forall z (K(x, y, z) \wedge K(y, x, z) \Leftrightarrow x = y \vee y = z \vee z = x)$;

(co3) $\forall x \forall y \forall z (K(x, y, z) \to \forall t [K(x, y, t) \vee K(t, y, z)])$;

(co4) $\forall x \forall y \forall z (K(x, y, z) \vee K(y, x, z))$.

Sometimes we will identify $\mathcal{M}$ and the universe M if a linear/circular order is fixed.

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The notion of *weak circular minimality* was studied initially in [8]. Let $A \subseteq M$, where $\mathcal{M}$ is a circularly ordered structure. The set A is called *convex* if for any $a, b \in A$ the following property is satisfied: for any $c \in M$ with $K(a, c, b)$, $c \in A$ holds, or for any $c \in M$ with $K(b, c, a)$, $c \in A$ holds. A *weakly circularly minimal structure* is a circularly ordered structure $\mathcal{M} = \langle M, K, \ldots \rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in $\mathcal{M}$. Recall [9] that such a structure $\mathcal{M}$ is called *circularly minimal* if any definable (with parameters) of M is a union of finitely many intervals and points in $\mathcal{M}$. Clearly, the weak circular minimality is a generalization of circular minimality. Notice also that any weakly o-minimal structure is weakly circular minimal. The converse, in general, fails. The study of weakly circularly minimal structures was continued in the papers [10–16].

Let $\mathcal{M}$ be an $\aleph_0$-categorical weakly circularly minimal structure, $G := \mathrm{Aut}(\mathcal{M})$. Following the standard group theory terminology, the group G is called *k-transitive* if for any pairwise distinct $a_1, a_2, \ldots, a_k \in M$ and pairwise distinct $b_1, b_2, \ldots, b_k \in M$ there exists $g \in G$ such that $g(a_1) = b_1$, $g(a_2) = b_2, \ldots, g(a_k) = b_k$. A *congruence* on $\mathcal{M}$ is an arbitrary G-invariant equivalence relation on $\mathcal{M}$. The group G is called *primitive* if G is 1-transitive and there are no non-trivial proper congruences on $\mathcal{M}$.

Let $\mathcal{M}$, $\mathcal{N}$ be circularly ordered structures. The 2-*reduct* of $\mathcal{M}$ is a circularly ordered structure with the same universe of $\mathcal{M}$ and consisting of predicates for each $\emptyset$-definable relation on $\mathcal{M}$ of arity ≤ 2 as well as of the ternary predicate K for the circular order, but does not have other predicates of arities more than two. We say that the structure $\mathcal{M}$ is *isomorphic* to $\mathcal{N}$ *up to binarity* or *binarily isomorphic to* $\mathcal{N}$ if the 2-reduct of $\mathcal{M}$ is isomorphic to the 2-reduct of $\mathcal{N}$.

Notation. (1) $K_0(x, y, z) := K(x, y, z) \wedge y \neq x \wedge y \neq z \wedge x \neq z$.

(2) $K(u_1, \ldots, u_n)$ denotes a formula saying that all subtuples of the tuple $\langle u_1, \ldots, u_n \rangle$ having the length 3 (in ascending order) satisfy K; similar notations are used for K_0.

(3) Let A, B, C be disjoint convex subsets of a circularly ordered structure $\mathcal{M}$. We write $K(A, B, C)$ if for any $a, b, c \in M$ with $a \in A$, $b \in B$, $c \in C$ we have $K(a, b, c)$. We extend naturally that notation using, for instance, the notation $K_0(A, d, B, C)$ if $d \notin A \cup B \cup C$ and $K_0(A, d, B) \wedge K_0(d, B, C)$ holds.

Let $f : M \to M$ be an $\emptyset$-definable function with $\mathrm{Dom}(f) = I \subseteq M$, where I is an open convex set. We say that f is *monotonic-to-right (left) on I* if it preserves (reverses) the relation K_0, i.e. for any $a, b, c \in I$ such that $K_0(a, b, c)$ we have $K_0(f(a), f(b), f(c))$ $(K_0(f(c), f(b), f(a)))$.

The following definition can be used in a circular ordered structure as well.

Definition 1. [17, 18] Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T, $A \subseteq M$. *The rank of convexity of the set A* ($RC(A)$) is defined as follows:

1) $RC(A) = -1$ if $A = \emptyset$.

2) $RC(A) = 0$ if A is finite and non-empty.

3) $RC(A) \geq 1$ if A is infinite.

4) $RC(A) \geq \alpha + 1$ if there exists a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A$, $i \in \omega$ such that:

- For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
- For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.

5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A)) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

The rank of convexity of an 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

In particular, a theory has convexity rank 1 if there are no definable (with parameters) equivalence relations with infinitely many infinite convex classes.

The following theorem characterizes $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structures of convexity rank greater than 1 having a non-trivial strictly monotonic function up to binarity:

Theorem 1. [11] (monotonic case) Let M be an $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater than 1 having a non-trivial strictly monotonic function so that $dcl(a) \neq \{a\}$ for some $a \in M$. Then M is isomorphic up to binarity to $M_{s,m,k} := \langle M, K, f^1, E_1^2, \ldots, E_s^2, E_{s+1}^2 \rangle$, where

- M is a circularly ordered structure, M is densely ordered, $s \geq 1$, $k \geq 2$, $m = 1$ or k divides m;
- E_{s+1} is an equivalence relation partitioning M into m infinite convex classes without endpoints, for every $1 \leq i \leq s$ the relation E_i is an equivalence relation partitioning every E_{i+1}–class into infinitely many infinite convex E_i–subclasses without endpoints so that the induced order on E_i–subclasses is dense without endpoints;
- f is a bijection on M so that $f^k(a) = a$ for any $a \in M$, for every $1 \leq i \leq s+1$ $f(E_i(M,a)) = E_i(M, f(a))$ and $\neg E_i(a, f(a))$, and either f is monotonic-to-right on M or f is monotonic-to-left on M (and in this case $k = m = 2$).

In [19] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories with a primitive automorphism group. In [20] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. Here we describe algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and having a non-trivial strictly monotonic function.

2 Results

Example 1. Consider the structure $M_{1,1,2} := \langle M, K^3, f^1, E^2 \rangle$ from Theorem 1, where f is monotonic-to-right on M, E is an equivalence relation partitioning M into infinitely many infinite convex classes. We assert that $Th(M_{1,1,2})$ has eight binary isolating formulas:

$$\theta_0(x,y) := x = y, \theta_1(x,y) := K_0(x,y,f(x)) \wedge E(x,y),$$

$$\theta_2(x,y) := K_0(x,y,f(x)) \wedge \neg E(x,y) \wedge \neg E(f(x),y),$$

$$\theta_3(x,y) := K_0(x,y,f(x)) \wedge E(f(x),y),$$

$$\theta_4(x,y) := f(x) = y, \theta_5(x,y) := K_0(f(x),y,x) \wedge E(f(x),y),$$

$$\theta_6(x,y) := K_0(f(x),y,x) \wedge \neg E(x,y) \wedge \neg E(f(x),y),$$

$$\theta_7(x,y) := K_0(f(x),y,x) \wedge E(x,y),$$

and

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \theta_3(a,M), \theta_4(a,M), \theta_5(a,M), \theta_6(a,M), \theta_7(a,M))$$

holds for any $a \in M$.

Define labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y) \text{ where } 0 \leq k \leq 7.$$

It is easy to check that for the algebra $\mathfrak{P}_{M_{1,1,2}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5	6	7
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3,4,5\}$	$\{5\}$	$\{5\}$	$\{6\}$	$\{7,0,1\}$
2	$\{2\}$	$\{2\}$	$\{2,3,4,5\}$	$\{6\}$	$\{6\}$	$\{6\}$	$\{6,7,0,1,2\}$	$\{2\}$
3	$\{3\}$	$\{3,4,5\}$	$\{6\}$	$\{7\}$	$\{7\}$	$\{7,0,1\}$	$\{2\}$	$\{3\}$
4	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
5	$\{5\}$	$\{5\}$	$\{6\}$	$\{7,0,1\}$	$\{1\}$	$\{1\}$	$\{2\}$	$\{3,4,5\}$
6	$\{6\}$	$\{6\}$	$\{6,7,0,1,2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2,3,4,5,6\}$	$\{6\}$
7	$\{7\}$	$\{7,0,1\}$	$\{2\}$	$\{3\}$	$\{3\}$	$\{3,4,5\}$	$\{6\}$	$\{7\}$

By the Cayley table the algebra $\mathfrak{P}_{M_{1,1,2}}$ is commutative.

Theorem 2. The algebra $\mathfrak{P}_{M_{s,1,k}}$ of binary isolating formulas having a monotonic-to-right function on M has $2k(s+1)$ labels and is commutative.

Proof of Theorem 2. Indeed, since $f^k(a)=a$, we have the following isolating formulas:

$$f^l(x)=y \ \text{ for every } \ 0\le l\le k-1.$$

Since for every $1\le i\le s$ the relation E_i is an equivalence relation partitioning every E_{i+1}–class into infinitely many infinite convex E_i–subclasses without endpoints so that the induced order on E_i–subclasses is dense without endpoints, we obtain the following binary isolating formulas:

$$K_0(f^l(x),y,f^{l+1}(x))\wedge E_1(f^l(x),y), \text{ where } 0\le l\le k-1,$$

$$K_0(f^l(x),y,f^{l+1}(x))\wedge\neg E_j(f^l(x),y)\wedge E_{j+1}(f^l(x),y), \text{ where } 0\le l\le k-1, 1\le j\le s-1,$$

$$K_0(f^l(x),y,f^{l+1}(x))\wedge\neg E_s(f^l(x),y)\wedge\neg E_s(f^{l+1}(x),y), \text{ where } 0\le l\le k-1,$$

$$K_0(f^l(x),y,f^{l+1}(x))\wedge\neg E_j(f^{l+1}(x),y)\wedge E_{j+1}(f^{l+1}(x),y), \text{ where } 0\le l\le k-1, 1\le j\le s-1,$$

$$K_0(f^l(x),y,f^{l+1}(x))\wedge E_1(f^{l+1}(x),y), \text{ where } 0\le l\le k-1.$$

Thus, we obtain $4k+2k(s-1)=2k(s+1)$ binary isolating formulas.

Now we establish commutativity of this algebra. Since for any binary isolating formula $\theta(x,y)$ the following holds:

$$\exists t[x=t\wedge\theta(t,y)]\equiv\theta(x,y) \text{ and } \exists t[\theta(x,t)\wedge t=y]\equiv\theta(x,y),$$

we obtain that $0\cdot l=l\cdot 0=\{l\}$ for any label l with the condition $0\le l\le 2k(s+1)-1$.

Obviously,

$$\text{both } \exists t[f^{l_1}(x)=t\wedge f^{l_2}(t)=y], \text{ and } \exists t[f^{l_2}(x)=t\wedge f^{l_1}(t)=y],$$

uniquely determine the formula $f^{l_1+l_2(\text{mod } k)}(x)=y$.

Further, since $K_0(a,f(a),f^2(a),\ldots,f^{k-1}(a))$ holds for any $a\in M$,

$$\exists t[f^{l_1}(x)=t\wedge E_i(f^{l_2}(t),y)\wedge K_0(f^{l_2}(t),y,f^{l_2+1}(t))],$$

where $1\le i\le s$, uniquely determines the formula

$$E_i(f^{l_1+l_2(\text{mod } k)}(x),y)\wedge K_0(f^{l_1+l_2(\text{mod } k)}(x),y,f^{l_1+l_2+1(\text{mod } k)}(x))$$

independently from behaviour of the function f. On the other hand, since f is monotonic-to-right on M,

$$\exists t[E_i(f^{l_2}(x),y)\wedge K_0(f^{l_2}(x),y,f^{l_2+1}(x))\wedge f^{l_1}(t)=y]$$

also uniquely determines the formula

$$E_i(f^{l_2+l_1(\text{mod } k)}(x),y)\wedge K_0(f^{l_2+l_1(\text{mod } k)}(x),y,f^{l_2+l_1+1(\text{mod } k)}(x)).$$

Further it is also easy to understand that the formulas

$$\exists t[K_0(f^{l_1}(x),t,f^{l_1+1}(x))\wedge E_1(f^{l_1}(x),t)\wedge K_0(f^{l_2}(t),y,f^{l_2+1}(t))\wedge E_1(f^{l_2}(t),y)] \text{ and}$$

$$\exists t[K_0(f^{l_2}(x),t,f^{l_2+1}(x))\wedge E_1(f^{l_2}(x),t)\wedge K_0(f^{l_1}(t),y,f^{l_1+1}(t))\wedge E_1(f^{l_1}(t),y)]$$

uniquely determine the formula

$$K_0(f^{l_1+l_2(\text{mod } k)}(x),y,f^{l_1+l_2+1(\text{mod } k)}(x))\wedge E_1(f^{l_1+l_2(\text{mod } k)}(x),y).$$

Now if we consider the formulas

$$\exists t[K_0(f^{l_1}(x),t,f^{l_1+1}(x))\wedge E_1(f^{l_1}(x),t)\wedge K_0(f^{l_2}(t),y,f^{l_2+1}(t))\wedge E_1(f^{l_2+1}(t),y)] \text{ and}$$

$$\exists t[K_0(f^{l_2}(x),t,f^{l_2+1}(x))\wedge E_1(f^{l_2+1}(x),t)\wedge K_0(f^{l_1}(t),y,f^{l_1+1}(t))\wedge E_1(f^{l_1}(t),y)],$$

there is no uniqueness, but both these formulas are compatible with the formulas

$$K_0(f^{l_1+l_2(\text{mod } k)}(x),y,f^{l_1+l_2+1(\text{mod } k)}(x))\wedge E_1(f^{l_1+l_2+1(\text{mod } k)}(x),y),$$

$$f^{l_1+l_2+1(\text{mod } k)}(x)=y,$$

$$K_0(f^{l_1+l_2+1(\text{mod } k)}(x),y,f^{l_1+l_2+2(\text{mod } k)}(x))\wedge E_1(f^{l_1+l_2+1(\text{mod } k)}(x),y).$$

Further, we consider the following formulas:

$$\exists t[K_0(f^{l_1}(x),t,f^{l_1+1}(x))\wedge E_1(f^{l_1}(x),t)\wedge K_0(f^{l_2}(t),y,f^{l_2+1}(t))$$
$$\wedge\neg E_j(f^{l_2}(t),y)\wedge E_{j+1}(f^{l_2}(t),y)] \qquad (*)$$
$$\text{and } \exists t[K_0(f^{l_2}(x),t,f^{l_2+1}(x))\wedge\neg E_j(f^{l_2}(x),t)\wedge E_{j+1}(f^{l_2}(x),t)$$
$$\wedge K_0(f^{l_1}(t),y,f^{l_1+1}(t))\wedge E_1(f^{l_1}(t),y)]. \qquad (**)$$

Since $E_1(f^{l_1}(x),t)$ implies $E_{j+1}(f^{l_1}(x),t)$, the formula

$$\exists t[E_1(f^{l_1}(x),t)\wedge\neg E_j(f^{l_2}(t),y)\wedge E_{j+1}(f^{l_2}(t),y)]$$

is compatible with the formula $E_{j+1}(f^{l_1+l_2(\text{mod } k)}(x),y)\wedge\neg E_j(f^{l_1+l_2(\text{mod } k)}(x),y)$. Consequently, the formulas $(*)$ and $(**)$ uniquely determine the formula

$$K_0(f^{l_1+l_2(\text{mod } k)}(x),y,f^{l_1+l_2+1(\text{mod } k)}(x))\wedge\neg E_j(f^{l_1+l_2(\text{mod } k)}(x),y)$$
$$\wedge E_{j+1}(f^{l_1+l_2(\text{mod } k)}(x),y).$$

Similarly we can show that

$$\exists t[K_0(f^{l_1}(x),t,f^{l_1+1}(x))\wedge E_1(f^{l_1}(x),t)\wedge K_0(f^{l_2}(t),y,f^{l_2+1}(t))$$
$$\wedge\neg E_s(f^{l_2}(t),y)\wedge\neg E_s(f^{l_2+1}(t),y)]$$
$$\text{and } \exists t[K_0(f^{l_2}(x),t,f^{l_2+1}(x))\wedge\neg E_s(f^{l_2}(x),t)\wedge\neg E_s(f^{l_2+1}(x),t)$$

$$\wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t), y)]$$

uniquely determine the formula

$$K_0(f^{l_1+l_2 (\text{mod } k)}(x), y, f^{l_1+l_2+1 (\text{mod } k)}(x)) \wedge \neg E_s(f^{l_1+l_2 (\text{mod } k)}(x), y)$$

$$\wedge \neg E_s(f^{l_1+l_2+1 (\text{mod } k)}(x), y).$$

Further, considering the following formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge \neg E_j(f^{l_1}(x), t) \wedge E_{j+1}(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$

$$\wedge \neg E_s(f^{l_2}(t), y) \wedge \neg E_s(f^{l_2+1}(t), y)]$$

$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_s(f^{l_2}(x), t) \wedge \neg E_s(f^{l_2+1}(x), t)$$

$$\wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge \neg E_j(f^{l_1}(t), y) \wedge E_{j+1}(f^{l_1}(t), y)],$$

we also obtain that they uniquely determine the formula

$$K_0(f^{l_1+l_2 (\text{mod } k)}(x), y, f^{l_1+l_2+1 (\text{mod } k)}(x)) \wedge \neg E_s(f^{l_1+l_2 (\text{mod } k)}(x), y)$$

$$\wedge \neg E_s(f^{l_1+l_2+1 (\text{mod } k)}(x), y).$$

Consider now the following formulas:

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge \neg E_{j_1}(f^{l_1}(x), t) \wedge E_{j_1+1}(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$

$$\wedge \neg E_{j_2}(f^{l_2}(t), y) \wedge E_{j_2+1}(f^{l_2}(t), y)]$$

$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_{j_2}(f^{l_2}(x), t) \wedge E_{j_2+1}(f^{l_2}(x), t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t))$$

$$\wedge \neg E_{j_1}(f^{l_1}(t), y) \wedge E_{j_1+1}(f^{l_1}(t), y)].$$

Let $j = \max\{j_1, j_2\}$. Then it is easy to establish that these formulas uniquely determine the formula

$$K_0(f^{l_1+l_2 (\text{mod } k)}(x), y, f^{l_1+l_2+1 (\text{mod } k)}(x)) \wedge \neg E_j(f^{l_1+l_2 (\text{mod } k)}(x), y)$$

$$\wedge E_{j+1}(f^{l_1+l_2 (\text{mod } k)}(x), y).$$

At last, consider the following formulas:

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge \neg E_s(f^{l_1}(x), t) \wedge \neg E_s(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$

$$\wedge \neg E_s(f^{l_2}(t), y) \wedge \neg E_s(f^{l_2}(t), y)]$$

$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_s(f^{l_2}(x), t) \wedge \neg E_s(f^{l_2}(x), t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t))$$

$$\wedge \neg E_s(f^{l_1}(t), y) \wedge \neg E_s(f^{l_1}(t), y)].$$

Here we loss the uniqueness: these formulas are compatible with the formula $K_0(f^{l_1+l_2}(x), y, f^{l_1+l_2+2}(x))$ which is in its turn compatible with the following $2s+3$ formulas:

$$f^{l_1+l_2+1}(x) = y,$$

$$K_0(f^{l_1+l_2}(x), y, f^{l_1+l_2+1}(x)) \wedge E_1(f^{l_1+l_2+1}(x), y),$$

$$K_0(f^{l_1+l_2}(x), y, f^{l_1+l_2+1}(x)) \wedge \neg E_1(f^{l_1+l_2+1}(x), y) \wedge E_2(f^{l_1+l_2+1}(x), y),$$

$$\ldots \quad \ldots \quad \ldots$$
$$K_0(f^{l_1+l_2}(x), y, f^{l_1+l_2+1}(x)) \wedge \neg E_{s-1}(f^{l_1+l_2+1}(x), y) \wedge E_s(f^{l_1+l_2+1}(x), y),$$
$$K_0(f^{l_1+l_2+1}(x), y, f^{l_1+l_2+2}(x)) \wedge \neg E_{s-1}(f^{l_1+l_2+1}(x), y) \wedge E_s(f^{l_1+l_2+1}(x), y),$$
$$\ldots \quad \ldots \quad \ldots$$
$$K_0(f^{l_1+l_2+1}(x), y, f^{l_1+l_2+2}(x)) \wedge \neg E_1(f^{l_1+l_2+1}(x), y) \wedge E_2(f^{l_1+l_2+1}(x), y),$$
$$K_0(f^{l_1+l_2+1}(x), y, f^{l_1+l_2+1+2}(x)) \wedge E_1(f^{l_1+l_2+1}(x), y).$$

Definition 2. [1] Let $p \in S_1(\emptyset)$ be non-algebraic. The algebra $\mathcal{P}_{\nu(p)}$ is said to be *deterministic* if $u_1 \cdot u_2$ is a singleton for any labels $u_1, u_2 \in \rho_{\nu(p)}$.

Generalizing the last definition, we say that the algebra $\mathcal{P}_{\nu(p)}$ is *m-deterministic* if the product $u_1 \cdot u_2$ consists of at most m elements for any labels $u_1, u_2 \in \rho_{\nu(p)}$. We also say that an m-deterministic algebra $\mathcal{P}_{\nu(p)}$ is *strictly m-deterministic* if it is not $(m-1)$-deterministic.

Corollary. The algebra $\mathfrak{P}_{M_{s,1,k}}$ of binary isolating formulas having a monotonic-to-right function on M is strictly $(2s+3)$-deterministic.

Example 2. Consider the structure $M_{1,2,2} := \langle M, K^3, f^1, E_1^2, E_2^2 \rangle$ from Theorem 1, where f is monotonic-to-right on M, E_1 is an equivalence relation partitioning M into infinitely many infinite convex classes, E_2 is an equivalence relation partitioning M into two infinite convex classes. We assert that $Th(M_{1,2,2})$ has ten binary isolating formulas:

$$\theta_0(x,y) := x = y, \theta_1(x,y) := K_0(x, y, f(x)) \wedge E_1(x,y),$$
$$\theta_2(x,y) := K_0(x, y, f(x)) \wedge \neg E_1(x,y) \wedge E_2(x,y),$$
$$\theta_3(x,y) := K_0(x, y, f(x)) \wedge \neg E_2(x,y) \wedge \neg E_1(f(x),y),$$
$$\theta_4(x,y) := K_0(x, y, f(x)) \wedge E_1(f(x),y),$$
$$\theta_5(x,y) := f(x) = y, \theta_6(x,y) := K_0(f(x), y, x) \wedge E_1(f(x),y),$$
$$\theta_7(x,y) := K_0(f(x), y, x) \wedge \neg E_1(f(x),y) \wedge E_2(f(x),y),$$
$$\theta_8(x,y) := K_0(f(x), y, x) \wedge \neg E_2(f(x),y) \wedge \neg E_1(x,y),$$
$$\theta_9(x,y) := K_0(f(x), y, x) \wedge E_1(x,y),$$

and
$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \theta_3(a,M), \theta_4(a,M), \theta_5(a,M), \theta_6(a,M), \theta_7(a,M))$$
$$\text{and } K_0(\theta_7(a,M), \theta_8(a,M), \theta_9(a,M), \theta_0(a,M))$$
holds for any $a \in M$.

Define labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y), \text{ where } 0 \leq k \leq 9.$$

It is easy to check that for the algebra $\mathfrak{P}_{M_{1,2,2}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5	6	7	8	9
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4,5,6\}$	$\{6\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9,0,1\}$
2	$\{2\}$	$\{2\}$	$\{2\}$	$\{3,4,5,6,7\}$	$\{7\}$	$\{7\}$	$\{7\}$	$\{7\}$	$\{8,9,0,1,2\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3,4,5,6,7\}$	$\{8\}$	$\{8\}$	$\{8\}$	$\{8\}$	$\{8,9,0,1,2\}$	$\{3\}$	$\{3\}$
4	$\{4\}$	$\{4,5,6\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{9\}$	$\{9,0,1\}$	$\{2\}$	$\{3\}$	$\{4\}$
5	$\{5\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
6	$\{6\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9,0,1\}$	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4,5,6\}$
7	$\{7\}$	$\{7\}$	$\{7\}$	$\{8,9,0,1,2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{3,4,5,6,7\}$	$\{7\}$
8	$\{8\}$	$\{8\}$	$\{8,9,0,1,2\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3,4,5,6,7\}$	$\{8\}$	$\{8\}$
9	$\{9\}$	$\{9,0,1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{4\}$	$\{4,5,6\}$	$\{7\}$	$\{8\}$	$\{9\}$

By the Cayley table the algebra $\mathfrak{P}_{M_{1,2,2}}$ is commutative.

Theorem 3. The algebra $\mathfrak{P}_{M_{s,m,k}}$ of binary isolating formulas having a monotonic-to-right function on M for $m \neq 1$ has $2k(s+1)+m$ labels, is commutative and strictly $(2s+3)$-deterministic.

Proof of Theorem 3. Similarly as in Theorem 2 we have the following binary isolating formulas:

$$f^l(x) = y \text{ for every } 0 \leq l \leq k-1,$$

$$K_0(f^l(x), y, f^{l+1}(x)) \wedge E_1(f^l(x), y), \text{ where } 0 \leq l \leq k-1,$$
$$K_0(f^l(x), y, f^{l+1}(x)) \wedge \neg E_j(f^l(x), y) \wedge E_{j+1}(f^l(x), y), \text{ where } 0 \leq l \leq k-1, 1 \leq j \leq s-1,$$
$$K_0(f^l(x), y, f^{l+1}(x)) \wedge \neg E_j(f^{l+1}(x), y) \wedge E_{j+1}(f^{l+1}(x), y), \text{ where } 0 \leq l \leq k-1, 1 \leq j \leq s-1,$$
$$K_0(f^l(x), y, f^{l+1}(x)) \wedge E_1(f^{l+1}(x), y), \text{ where } 0 \leq l \leq k-1.$$

Since in this structure there exists additionally the equivalence relation $E_{s+1}(x,y)$ partitioning M into m infinite convex classes, instead of the formulas

$$K_0(f^l(x), y, f^{l+1}(x)) \wedge \neg E_s(f^l(x), y) \wedge \neg E_s(f^{l+1}(x), y), \text{ where } 0 \leq l \leq k-1,$$

additionally the following binary isolating formulas appear:

$$K_0(f^l(x), y, f^{l+1}(x)) \wedge \neg E_s(f^l(x), y) \wedge E_{s+1}(f^l(x), y), \text{ where } 0 \leq l \leq k-1,$$

$$K_0(f^l(x), y, f^{l+1}(x)) \wedge \neg E_s(f^{l+1}(x), y) \wedge E_{s+1}(f^{l+1}(x), y), \text{ where } 0 \leq l \leq k-1.$$

Also, the formulas $\theta^{l,i}(x,y)$ containing the conjunctive term $K_0(f^l(x), y, f^{l+1}(x))$ and extracting the i-th E_{s+1}-class to the right of E_{s+1}-class containing $f^l(x)$ for some $1 \leq i \leq m/k-1$ (here also $0 \leq l \leq k-1$) will be binary isolating formulas. For example, the formula $\theta^{l,1}(x,y)$ has the following form:

$$\theta^{l,1}(x,y) := K_0(f^l(x), y, f^{l+1}(x)) \wedge \neg E_{s+1}(f^l(x), y) \wedge$$
$$\forall t[K_0(f^l(x), t, y) \wedge \neg E_{s+1}(t, y) \to E_{s+1}(f^l(x), t)].$$

Thus, we obtain $k+k+2k(s-1)+k+2k+k(m/k-1) = 2k(s+1)+m$ binary isolating formulas.

Take arbitrary labels l_1, l_2 and show that $l_1 \cdot l_2 = l_2 \cdot l_1$.

It is easy to establish that the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$
$$\wedge \neg E_s(f^{l_2}(t), y) \wedge E_{s+1}(f^{l_2}(t), y)]$$
$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_s(f^{l_2}(x), t) \wedge E_{s+1}(f^{l_2}(x), t)$$
$$\wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t), y)]$$

uniquely determine the formula

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge \neg E_s(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y)$$
$$\wedge E_{s+1}(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y).$$

Similarly, the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$
$$\wedge \neg E_s(f^{l_2+1}(t), y) \wedge E_{s+1}(f^{l_2+1}(t), y)]$$

$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_s(f^{l_2+1}(x), t) \wedge E_{s+1}(f^{l_2+1}(x), t)$$
$$\wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t), y)]$$

uniquely determine the formula

$$K_0(f^{l_1+l_2(\text{mod } k)}(x), y, f^{l_1+l_2+1(\text{mod } k)}(x)) \wedge \neg E_s(f^{l_1+l_2+1(\text{mod } k)}(x), y)$$
$$\wedge E_{s+1}(f^{l_1+l_2+1(\text{mod } k)}(x), y).$$

Further, considering the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x), t) \wedge \theta^{l_2,j}(t, y)]$$
$$\text{and } \exists t[\theta^{l_2,j}(x, t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t), y)],$$

we establish that they uniquely determine the formula $\theta^{l_1+l_2(\text{mod } k),j}(x, y)$ for arbitrary $1 \leq j \leq m/k - 1$.

Similarly, the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge \neg E_s(f^{l_1}(x), t) \wedge E_{s+1}(f^{l_1}(x), t) \wedge \theta^{l_2,j}(t, y)]$$
$$\text{and } \exists t[\theta^{l_2,j}(x, t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge \neg E_s(f^{l_1}(t), y) \wedge E_{s+1}(f^{l_1}(t), y)]$$

uniquely determine the formula $\theta^{l_1+l_2+1(\text{mod } k),j}(x, y)$ for every $1 \leq j \leq m/k - 1$.

On the other hand, the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge \neg E_s(f^{l_1+1}(x), t) \wedge E_{s+1}(f^{l_1+1}(x), t) \wedge \theta^{l_2,j}(t, y)]$$
$$\text{and } \exists t[\theta^{l_2,j}(x, t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge \neg E_s(f^{l_1+1}(t), y) \wedge E_{s+1}(f^{l_1+1}(t), y)]$$

uniquely determine the formula $\theta^{l_1+l_2(\text{mod } k),j}(x, y)$ for arbitrary $1 \leq j \leq m/k - 1$.

Also observe that the formulas

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge \neg E_j(f^{l_1}(x), t) \wedge E_{j+1}(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t))$$
$$\wedge \neg E_s(f^{l_2}(t), y) \wedge E_{s+1}(f^{l_2}(t), y)]$$
$$\text{and } \exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge \neg E_s(f^{l_2}(x), t) \wedge E_{s+1}(f^{l_2}(x), t)$$
$$\wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge \neg E_j(f^{l_1}(t), y) \wedge E_{j+1}(f^{l_1}(t), y)],$$

uniquely determine the formula

$$K_0(f^{l_1+l_2(\text{mod } k)}(x), y, f^{l_1+l_2+1(\text{mod } k)}(x)) \wedge \neg E_s(f^{l_1+l_2(\text{mod } k)}(x), y)$$
$$\wedge E_{s+1}(f^{l_1+l_2(\text{mod } k)}(x), y).$$

If we consider the following formulas:

$$\exists t[K_0(f^{l_1}(x), t, f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x), t) \wedge K_0(f^{l_2}(t), y, f^{l_2+1}(t)) \wedge E_1(f^{l_2+1}(t), y)] \text{ and}$$
$$\exists t[K_0(f^{l_2}(x), t, f^{l_2+1}(x)) \wedge E_1(f^{l_2+1}(x), t) \wedge K_0(f^{l_1}(t), y, f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t), y)],$$

there is no uniqueness, but both these formulas are compatible with the formulas

$$K_0(f^{l_1+l_2(\text{mod } k)}(x), y, f^{l_1+l_2+1(\text{mod } k)}(x)) \wedge E_1(f^{l_1+l_2+1(\text{mod } k)}(x), y),$$

$$f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x) = y,$$

$$K_0(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+2 (\mathrm{mod}\ k)}(x)) \wedge E_1(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y).$$

Consider now the formulas $\theta^{l_1,i}(x,y)$ and $\theta^{l_2,j}(x,y)$ for arbitrary $1 \leq i,j \leq m/k-1$. If $i+j$ $(\mathrm{mod}\ m/k) \neq 0$, it is easy to check that the formulas

$$\exists t[\theta^{l_1,i}(x,t) \wedge \theta^{l_2,j}(t,y)] \text{ and } \exists t[\theta^{l_2,j}(x,t) \wedge \theta^{l_1,i}(t,y)]$$

uniquely determine the formula $\theta^{l_1+l_2 (\mathrm{mod}\ k), i+j (\mathrm{mod}\ m/k)}(x,y)$.

If $i+j$ $(\mathrm{mod}\ m/k) = 0$, these formulas are compatible with the following $2s+3$ formulas:

$$f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x) = y,$$

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge E_1(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y),$$

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge \neg E_j(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y)$$

$$\wedge E_{j+1}(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y), \quad 1 \leq j \leq s,$$

$$K_0(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+2 (\mathrm{mod}\ k)}(x)) \wedge E_1(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y),$$

$$K_0(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+2 (\mathrm{mod}\ k)}(x)) \wedge \neg E_j(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y)$$

$$\wedge E_{j+1}(f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x), y), \quad 1 \leq j \leq s.$$

Example 3. Consider the structure $M_{1,2,2} := \langle M, K^3, f^1, E_1^2, E_2^2 \rangle$ from Theorem 1, where f is monotonic-to-left on M, E_1 is an equivalence relation partitioning M into infinitely many infinite convex classes, E_2 is an equivalence relation partitioning M into two infinite convex classes. We assert that $Th(M_{1,2,2})$ has ten binary isolating formulas:

$$\theta_0(x,y) := x = y, \theta_1(x,y) := K_0(x,y,f(x)) \wedge E_1(x,y),$$

$$\theta_2(x,y) := K_0(x,y,f(x)) \wedge \neg E_1(x,y) \wedge E_2(x,y),$$

$$\theta_3(x,y) := K_0(x,y,f(x)) \wedge \neg E_2(x,y) \wedge \neg E_1(f(x),y),$$

$$\theta_4(x,y) := K_0(x,y,f(x)) \wedge E_1(f(x),y),$$

$$\theta_5(x,y) := f(x) = y, \theta_6(x,y) := K_0(f(x),y,x) \wedge E_1(f(x),y),$$

$$\theta_7(x,y) := K_0(f(x),y,x) \wedge \neg E_1(f(x),y) \wedge E_2(f(x),y),$$

$$\theta_8(x,y) := K_0(f(x),y,x) \wedge \neg E_2(f(x),y) \wedge \neg E_1(x,y),$$

$$\theta_9(x,y) := K_0(f(x),y,x) \wedge E_1(x,y),$$

and both

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \theta_3(a,M), \theta_4(a,M), \theta_5(a,M), \theta_6(a,M), \theta_7(a,M))$$

$$\text{and } K_0(\theta_7(a,M), \theta_8(a,M), \theta_9(a,M), \theta_0(a,M))$$

hold for any $a \in M$.

Define labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y), \text{ where } 0 \leq k \leq 9.$$

It is easy to check that for the algebra $\mathfrak{P}_{M_{1,2,2}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5	6	7	8	9
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{4\}$	$\{4,5,6\}$	$\{7\}$	$\{8\}$	$\{9,0,1\}$
2	$\{2\}$	$\{2\}$	$\{2\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3,4,5,6,7\}$	$\{8,9,0,1,2\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3,4,5,6,7\}$	$\{8,9,0,1,2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{3\}$	$\{3\}$
4	$\{4\}$	$\{4,5,6\}$	$\{7\}$	$\{8\}$	$\{9,0,1\}$	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
5	$\{5\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
6	$\{6\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{9\}$	$\{9,0,1\}$	$\{2\}$	$\{3\}$	$\{4,5,6\}$
7	$\{7\}$	$\{7\}$	$\{7\}$	$\{8\}$	$\{8\}$	$\{8\}$	$\{8\}$	$\{8,9,0,1,2\}$	$\{3,4,5,6,7\}$	$\{7\}$
8	$\{8\}$	$\{8\}$	$\{8,9,0,1,2\}$	$\{3,4,5,6,7\}$	$\{7\}$	$\{7\}$	$\{7\}$	$\{7\}$	$\{8\}$	$\{8\}$
9	$\{9\}$	$\{9,0,1\}$	$\{2\}$	$\{3\}$	$\{4,5,6\}$	$\{6\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$

By the Cayley table the algebra $\mathfrak{P}_{M_{1,2,2}}$ is not commutative.

Theorem 4. The algebra $\mathfrak{P}_{M_{s,2,2}}$ of binary isolating formulas having a monotonic-to-left function on M has $4s+6$ labels, is strictly $(2s+3)$-deterministic and is not commutative.

Proof of Theorem 4. In this case we have the following binary isolating formulas:

$$x = y, \quad f(x) = y,$$

$$K_0(x, y, f(x)) \wedge E_1(x, y),$$

$$K_0(x, y, f(x)) \wedge \neg E_j(x, y) \wedge E_{j+1}(x, y), 1 \leq j \leq s,$$

$$K_0(x, y, f(x)) \wedge \neg E_j(f(x), y) \wedge E_{j+1}(f(x), y), 1 \leq j \leq s,$$

$$K_0(x, y, f(x)) \wedge E_1(f(x), y),$$

$$K_0(f(x), y, x) \wedge E_1(f(x), y),$$

$$K_0(f(x), y, x) \wedge \neg E_j(f(x), y) \wedge E_{j+1}(f(x), y), 1 \leq j \leq s,$$

$$K_0(f(x), y, x) \wedge \neg E_j(x, y) \wedge E_{j+1}(x, y), 1 \leq j \leq s,$$

$$K_0(f(x), y, x) \wedge E_1(x, y).$$

Thus, we obtain $4s+6$ binary isolating formulas.

The formula

$$\exists t[K_0(x, t, f(x)) \wedge E_1(x, t) \wedge K_0(t, y, f(t)) \wedge E_1(t, y)]$$

uniquely determines the formula $K_0(x, y, f(x)) \wedge E_1(x, y)$. Further, the formulas

$$\exists t[K_0(x, t, f(x)) \wedge E_1(x, t) \wedge K_0(t, y, f(t)) \wedge \neg E_j(t, y) \wedge E_{j+1}(t, y)]$$

$$\text{and } \exists t[K_0(x, t, f(x)) \wedge \neg E_j(x, t) \wedge E_{j+1}(x, t) \wedge K_0(t, y, f(t)) \wedge E_1(t, y)]$$

for every $1 \leq j \leq s$ uniquely determine the formula

$$K_0(x, y, f(x)) \wedge \neg E_j(x, y) \wedge E_{j+1}(x, y).$$

Consider now the formulas

$$\exists t[K_0(x, t, f(x)) \wedge E_1(x, t) \wedge K_0(t, y, f(t)) \wedge \neg E_j(f(t), y) \wedge E_{j+1}(f(t), y)]$$

$$\text{and } \exists t[K_0(x, t, f(x)) \wedge \neg E_j(f(x), t) \wedge E_{j+1}(f(x), t) \wedge K_0(t, y, f(t)) \wedge E_1(t, y)],$$

where $1 \leq j \leq s$. It is easy to establish that they uniquely determine the formula

$$K_0(x, y, f(x)) \wedge \neg E_j(f(x), y) \wedge E_{j+1}(f(x), y).$$

The formula

$$\exists t[K_0(x, t, f(x)) \wedge E_1(x, t) \wedge K_0(t, y, f(t)) \wedge E_1(f(t), y)]$$

uniquely determines the formula $K_0(x, y, f(x)) \wedge E_1(f(x), y)$. While the formula

$$\exists t[K_0(x, t, f(x)) \wedge E_1(f(x), t) \wedge K_0(t, y, f(t)) \wedge E_1(t, y)]$$

is compatible with the following three formulas:

$$K_0(x, y, f(x)) \wedge E_1(f(x), y), \quad f(x) = y, \quad K_0(f(x), y, x) \wedge E_1(f(x), y).$$

Thus, we established that the algebra $\mathfrak{P}_{M_{s,2,2}}$ is not commutative.

Further, considering the formulas

$$\exists t[K_0(x, t, f(x)) \wedge E_1(x, t) \wedge f(t) = y] \text{ and } \exists t[f(x) = t \wedge K_0(t, y, f(t)) \wedge E_1(t, y)],$$

we obtain that they uniquely determine the formulas

$$K_0(x, y, f(x)) \wedge E_1(f(x), y) \text{ and } K_0(f(x), y, x) \wedge E_1(f(x), y)$$

respectively, also confirming non-commutativity of the algebra $\mathfrak{P}_{M_{s,2,2}}$.

Similarly, the formulas

$$\exists t[K_0(x, t, f(x)) \wedge E_1(f(x), t) \wedge f(t) = y] \text{ and } \exists t[f(x) = t \wedge K_0(t, y, f(t)) \wedge E_1(f(t), y)],$$

uniquely determine the formulas

$$K_0(x, y, f(x)) \wedge E_1(x, y) \text{ and } K_0(f(x), y, x) \wedge E_1(x, y), \text{ respectively.}$$

The formula

$$\exists t[K_0(x, t, f(x)) \wedge E_1((x, t) \wedge K_0(f(t), y, t) \wedge E_1(f(t), y)]$$

is compatible with the following three formulas:

$$K_0(x, y, f(x)) \wedge E_1(f(x), y), \quad f(x) = y, \quad K_0(f(x), y, x) \wedge E_1(f(x), y).$$

While the formula

$$\exists t[K_0(f(x), t, x) \wedge E_1(f(x), t) \wedge K_0(t, y, f(t)) \wedge E_1(t, y)]$$

uniquely determines the formula $K_0(f(x), y, x) \wedge E_1(f(x), y)$.

Further, the formulas

$$\exists t[K_0(x, t, f(x)) \wedge E_1(x, t) \wedge K_0(f(t), y, t) \wedge \neg E_j(f(t), y) \wedge E_{j+1}(f(t), y)]$$

$$\text{and } \exists t[K_0(f(x), t, x) \wedge \neg E_j(f(x), t) \wedge E_{j+1}(f(x), t) \wedge K_0(t, y, f(t)) \wedge E_1(t, y)]$$

for every $1 \leq j \leq s$ uniquely determine the formula

$$K_0(f(x), y, x) \wedge \neg E_j(f(x), y) \wedge E_{j+1}(f(x), y).$$

Similarly, the formulas

$$\exists t[K_0(x, t, f(x)) \wedge E_1(x, t) \wedge K_0(f(t), y, t) \wedge \neg E_j(t, y) \wedge E_{j+1}(t, y)]$$

$$\text{and } \exists t[K_0(f(x), t, x) \wedge \neg E_j(x, t) \wedge E_{j+1}(x, t) \wedge K_0(t, y, f(t)) \wedge E_1(t, y)]$$

for every $1 \leq j \leq s$ uniquely determine the formula

$$K_0(f(x), y, x) \wedge \neg E_j(x, y) \wedge E_{j+1}(x, y).$$

Further, the formulas

$$\exists t[K_0(x,t,f(x)) \wedge E_1(x,t) \wedge K_0(f(t),y,t) \wedge E_1(t,y)]$$

$$\text{and } \exists t[K_0(f(x),t,x) \wedge E_1(x,t) \wedge K_0(t,y,f(t)) \wedge E_1(t,y)]$$

are compatible with the following three formulas:

$$K_0(f(x),y,x) \wedge E_1(x,y), \quad x=y, \quad K_0(x,y,f(x)) \wedge E_1(x,y).$$

The formula

$$\exists t[K_0(x,t,f(x)) \wedge \neg E_s(x,t) \wedge E_{s+1}(x,t) \wedge K_0(f(t),y,t) \wedge \neg E_s(f(t),y) \wedge E_{s+1}(f(t),y)]$$

is compatible with the following $2s+3$ formulas:

$$K_0(x,y,f(x)) \wedge \neg E_j(f(x),y) \wedge E_{j+1}(f(x),y), \quad 1 \le j \le s,$$

$$K_0(x,y,f(x)) \wedge E_1(f(x),y),$$

$$f(x)=y,$$

$$K_0(f(x),y,x) \wedge E_1(f(x),y),$$

$$K_0(f(x),y,x) \wedge \neg E_j(f(x),y) \wedge E_{j+1}(f(x),y), \quad 1 \le j \le s.$$

While the formula

$$\exists t[K_0(f(x),t,x) \wedge \neg E_s(f(x),t) \wedge E_{s+1}(f(x),t) \wedge K_0(t,y,f(t)) \wedge \neg E_s(t,y) \wedge E_{s+1}(t,y)]$$

uniquely determines the formula

$$K_0(f(x),y,x) \wedge \neg E_s(f(x),y) \wedge E_{s+1}(f(x),y).$$

On the other hand, the formulas

$$\exists t[K_0(x,t,f(x)) \wedge \neg E_s(x,t) \wedge E_{s+1}(x,t) \wedge K_0(f(t),y,t) \wedge \neg E_s(t,y) \wedge E_{s+1}(t,y)]$$

$$\text{and } \exists t[K_0(f(x),t,x) \wedge \neg E_s(x,t) \wedge E_{s+1}(x,t) \wedge K_0(t,y,f(t)) \wedge \neg E_s(t,y) \wedge E_{s+1}(t,y)]$$

are compatible with the same $2s+3$ formulas:

$$K_0(f(x),y,x) \wedge \neg E_j(x,y) \wedge E_{j+1}(x,y), \quad 1 \le j \le s,$$

$$K_0(f(x),y,x) \wedge E_1(x,y),$$

$$x=y,$$

$$K_0(x,y,f(x)) \wedge E_1(x,y),$$

$$K_0(x,y,f(x)) \wedge \neg E_j(x,y) \wedge E_{j+1}(x,y), \quad 1 \le j \le s.$$

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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$\aleph_0$-категориялық әлсіз циклдік минималды теориялар үшін бинарлық формулалар алгебралары: монотонды жағдай

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Мақалада о-минималдылық тұжырымдамасына қатысты циклдік реттелген құрылымдар үшін нұсқа болып табылатын әлсіз циклдік минималдылық түсінігі қарастырылған. 1-транзитивтілік автоморфизмдердің примитивтілік емес группасы және құрылымның негізгі жиынында әрекет ететін тривиальды емес қатаң монотонды функцияның екеуі де бар дөңестік рангісі бірден үлкен саналымды категориялық әлсіз циклдік минималды теориялары үшін бинарлы оқшаулау формулалар алгебралары зерттелген. Зерттеу нәтижесінде авторлар осы алгебралардың сипаттамасын ұсынған. Олардың арасында коммутативті және коммутативті емес алгебралар бар екені көрсетілген. Мұндай алгебралардың қатаң m-детерминаттылығы кейбір m натурал саны үшін де анықталған.

Кілт сөздер: циклдік реттелген құрылым, бинарлық формула, оқшаулау формуласы, формулалар алгебрасы, саналымды категориялық теория, әлсіз циклдік минималдылық, дөңестік рангісі, автоморфизм группасы, транзитивтілік, примитивтілік, m-детерминаттылық.

Алгебры бинарных формул для $\aleph_0$-категоричных слабо циклически минимальных теорий: монотонный случай

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В настоящей статье рассмотрено понятие слабой циклической минимальности, являющейся вариантом o-минимальности для циклически упорядоченных структур. Исследованы алгебры бинарных изолирующих формул для счетно категоричных слабо циклически минимальных теорий ранга выпуклости, большего единицы, имеющих как *1*-транзитивную непримитивную группу автоморфизмов, так и нетривиальную строго монотонную функцию, действующую на основном множестве структуры. В результате исследования авторы представляют описание этих алгебр. Показано, что среди них имеются как коммутативные, так и некоммутативные алгебры. Кроме того, установлена строгая m-детерминированность таких алгебр для некоторого натурального числа m.

Ключевые слова: циклически упорядоченная структура, бинарная формула, изолирующая формула, алгебра формул, счетно категоричная теория, слабая циклическая минимальность, ранг выпуклости, группа автоморфизмов, транзитивность, примитивность, m-детерминированность.

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