https://doi.org/10.31489/2024M1/101-111 *Research article*

Some properties of the one-dimensional potentials

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The main aim of this paper is to study the properties of the one-dimensional potentials. In this paper, we have studied the connection between the one-dimensional potentials and the self-adjoint part of the operator L_K^{-1}, which L_K^{-1} is the solution to the one-dimensional Cauchy problem. Moreover, a new method is used that allows us to reduce the spectral problem for the Helmholtz potential to the equivalent problem.

Keywords: one-dimensional Helmholtz potential, spectral problem, Fredholm operator.

2020 Mathematics Subject Classification: 34B05.

Introduction

One-dimensional potentials are important in the field of mathematical physics, offering insight into the behavior and characteristics of physical systems in a simplified form. The study of one-dimensional potentials also involves the analysis of eigenvalues and eigenfunctions. These concepts provide valuable information about the energy levels and corresponding wavefunctions associated with the potential.

In the study of elliptic equations, the Laplace and Helmholtz equations hold significant importance due to their wide-ranging applications and deep implications. The solutions to these equations take the form of Newton and Helmholtz potentials, respectively, which offer fundamental insights into the behavior and properties of these equations.

The Newton potential's properties have important applications in various fields, including physics and engineering. Similarly, the Helmholtz potential finds extensive utilization in electromagnetic radiation, seismology, and acoustics due to its inherent connection with the wave equation. In recent years, new methods have been discovered for investigating the potentials of the elliptic equations in multidimensional cases [1–4].

Let Ω be a bounded simply-connected domain in $\mathbb{R}^n$. In multidimensional case, a Newton potential is defined as follows:

$$u(x) = \int\limits_{\Omega} \varepsilon_n(x-\xi) f(\xi) d\xi, \tag{1}$$

where

$$\varepsilon_n(x-\xi) = \begin{cases} -\frac{1}{2}\ln|x-\xi|, & n=2, \\ \frac{1}{(n-2)\sigma_n}|x-\xi|^{2-n}, & n \geq 3, \end{cases}$$

σ_n is the surface domain of the sphere in $\mathbb{R}^n$, and $\varepsilon_n(x-\xi)$ is a fundamental solution of the Laplace equation, such that

$$\Delta\varepsilon_n(x-\xi) = \delta(x-\xi),$$

here $\delta(x)$ is the Dirac delta function.

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This work is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP14871460).

Received: 31 August 2023; *Accepted:* 11 October 2023.

In [1], the general form of the boundary condition for the Newton potential (1) was discovered by T.Sh. Kalmenov and D. Suragan used their method

$$N[u] = -\frac{u(x)}{2} + \int\limits_{\partial\Omega} \left(\varepsilon_n(x-\xi)\frac{\partial u(\xi)}{\partial n_\xi} - \frac{\partial \varepsilon_n(x-\xi)}{\partial n_\xi} u(\xi) \right) d\xi = 0, \quad x \in \partial\Omega, \tag{2}$$

where $\frac{\partial}{\partial n_\xi}$ is a normal derivative. In the aforementioned study, the eigenvalues and eigenfunctions of the volume potential were discovered for both the 2-disk and the 3-ball. In [2], the eigenfunctions of two-dimensional Newton potential was studied.

In the work [3], for the n-dimensional Helmholtz equation, after transforming the entire space into a finite domain in $\mathbb{R}^n$, the Sommerfeld radiation condition is transferred into a general boundary value condition with the same form as boundary value condition (2). Additional and comprehensive references on this study can be found in [5–7].

In the present paper, we will study the connections between potentials in the Cauchy problem and investigates the eigenvalue problem of the one-dimensional Helmholtz potential as a Fredholm operator, employing a novel methodology. Furthermore, we will analyze the relationship between one-dimensional Newton potential and the solution of the Cauchy problem.

1 *Main functional relations*

It is well known that the one-dimensional Newton potential is defined as follows

$$u(x) = \int\limits_a^b \frac{|x-\xi|}{2} f(\xi) d\xi, \quad x \in (a,b) \subset \mathbb{R},$$

and satisfies the following Poisson equation

$$Lu = \frac{d^2}{dx^2} u(x) = f(x).$$

Let $f \in L_2(a,b)$, then we can find the solution of the one-dimensional Cauchy problem in the following form

$$u_K(x) = L_K^{-1} f := \int\limits_a^x (x-\xi) f(\xi) d\xi.$$

An adjoint operator to L_K^{-1} is

$$\left(L_K^{-1}\right)^* f := \int\limits_x^b (\xi - x) f(\xi) d\xi. \tag{3}$$

By using the Cartesian theorem for operators, we can rewrite operator (3) as follows

$$L_K^{-1} f = \mathfrak{Re}\left(L_K^{-1}\right) f + i \cdot \mathfrak{Im}\left(L_K^{-1}\right) f = \frac{L_K^{-1} + (L_K^{-1})^*}{2} f + i \frac{L_K^{-1} - (L_K^{-1})^*}{2i} f.$$

The operators $\mathfrak{Re}\left(L_K^{-1}\right)$ and $\mathfrak{Im}\left(L_K^{-1}\right)$ are respectively the real part and image part of the operator L_K^{-1}, and they are self-adjoint operators.

It is easily seen that the real part of the operator L_K^{-1} coincides with the one-dimensional Newton potential such that

$$\frac{L_K^{-1} + (L_K^{-1})^*}{2} f = \int\limits_a^b \frac{|x-\xi|}{2} f(\xi) d\xi.$$

Note that we have related the one-dimensional Newton potential to the classical Cauchy problem.

The proof above gives more, namely, we can generalize this fact to high-order differential equations.

Let us consider the following self-adjoint linear differential equation

$$Lu := \frac{d^{(2m)}}{dx^{(2m)}}u(x) + a_1\frac{d^{(2m-2)}}{dx^{(2m-2)}}u(x) + ... + a_m u(x) = f(x), \quad m \in \mathbb{N}, \quad x \in (a,b), \tag{4}$$

where coefficients $a_1, ..., a_m \in \mathbb{R}$ are constants.

By the Malgrange-Ehrenpreis theorem [8] we know that equation (4) has a fundamental solution. Now, we will construct the fundamental solution of the operator (4).

Lemma 1. Let $z(x)$ is a solution of the following homogeneous equation

$$Lz = 0,$$

which satisfies the conditions

$$\begin{aligned} &z(a) = z'(a) = ... = z^{(2m-2)}(a) = 0; \\ &z^{(2m-1)}(a) = 1. \end{aligned} \tag{5}$$

Then the function

$$\varepsilon(x) = \frac{1}{2}\mathrm{sgn}(x) \cdot z(x) \tag{6}$$

is a fundamental solution of the operator L.

Proof. By using the weak derivatives and properties of distributions, and taking into account conditions (5), we have

$$\begin{aligned} \varepsilon'(x) &= \delta(x) \cdot z(x) + \frac{1}{2}\mathrm{sgn}(x) \cdot z'(x) = \frac{1}{2}\mathrm{sgn}(x) \cdot z'(x), \\ \varepsilon''(x) &= \delta(x) \cdot z'(x) + \frac{1}{2}\mathrm{sgn}(x) \cdot z''(x) = \frac{1}{2}\mathrm{sgn}(x) \cdot z''(x), \\ &\cdots \\ \varepsilon^{(2m-2)}(x) &= \frac{1}{2}\mathrm{sgn}(x) \cdot z^{(2m-2)}(x), \\ \varepsilon^{(2m)}(x) &= \delta(x) \cdot z^{(2m-1)}(x) + \frac{1}{2}\mathrm{sgn}(x) \cdot z^{(2m)}(x) \\ &= \delta(x) + \frac{1}{2}\mathrm{sgn}(x) \cdot z^{(2m)}(x). \end{aligned} \tag{7}$$

We see at once that

$$\begin{aligned} L\varepsilon &= \frac{d^{(2m)}}{dx^{(2m)}}\varepsilon(x) + a_1\frac{d^{(2m-2)}}{dx^{(2m-2)}}\varepsilon(x) + ... + a_m\varepsilon(x) \\ &= \frac{1}{2}\mathrm{sgn}(x) \cdot Lz + \delta(x) \\ &= \delta(x), \end{aligned}$$

which is clear from (7).

Using fundamental solution (6) we can write the potential of equation (4) as follows

$$u(x) = L^{-1}f = \varepsilon * f = \frac{1}{2}\int_a^b \operatorname{sgn}(x-\xi)\cdot z(x-\xi)f(\xi)d\xi. \tag{8}$$

On the other hand, the solution of the Cauchy problem with zero-conditions of equation (4) is

$$u_K(x) = L_K^{-1}f = \int_a^x z(x-\xi)f(\xi)d\xi,$$

and

$$\left(L_K^{-1}\right)^* f = \int_x^b z(\xi - x)f(\xi)d\xi.$$

By the Cartesian theorem, the real part of the operator L_K^{-1} is

$$\begin{aligned}\mathfrak{Re}\left(L_K^{-1}\right)f =& \frac{L_K^{-1} + \left(L_K^{-1}\right)^*}{2}f = \\ =& \frac{1}{2}\left[\int_a^x z(x-\xi)f(\xi)d\xi + \int_x^b z(\xi-x)f(\xi)d\xi\right].\end{aligned} \tag{9}$$

Obviously $z(x)$ is an odd function, i.e. $z(-x) = -z(x)$, therefore we can rewrite (9) as

$$\begin{aligned}\mathfrak{Re}\left(L_K^{-1}\right)f =& \frac{1}{2}\left[\int_a^x z(x-\xi)f(\xi)d\xi + \int_x^b z(\xi-x)f(\xi)d\xi\right] = \\ =& \frac{1}{2}\left[\int_a^x \operatorname{sgn}(x-\xi)\cdot z(x-\xi)f(\xi)d\xi + \int_x^b \operatorname{sgn}(x-\xi)\cdot z(x-\xi)f(\xi)d\xi\right] = \\ =& \frac{1}{2}\int_a^b \operatorname{sgn}(x-\xi)\cdot z(x-\xi)f(\xi)d\xi,\end{aligned}$$

which proves the following theorem.

Theorem 1. Potential (8) and the real part of the solution of the Cauchy problem for the equation (4) are equal:

$$\mathfrak{Re}\left(L_K^{-1}\right)f = \frac{L_K^{-1} + \left(L_K^{-1}\right)^*}{2}f = L^{-1}f = \frac{1}{2}\int_a^b \operatorname{sgn}(x-\xi)\cdot z(x-\xi)f(\xi)d\xi.$$

Example. If $a_1 = a_2 = \ldots = a_m = 0$ in (4), which we may assume, then we have the following polyharmonic equation

$$Lu = \Delta^m u(x) = \frac{d^{2m}}{dx^{2m}}u(x) = f(x), \quad x \in (a,b) \subset \mathbb{R}. \tag{10}$$

By using (6) and (8) the polyharmonic Newton potential is given by

$$u(x) = L^{-1}f = \frac{1}{2}\int\limits_a^b \frac{1}{(2m-1)!}|x-\xi|^{2m-1}f(\xi)d\xi.$$

We find the solution of the Cauchy problem for equation (10) as

$$u_K(x) = L_K^{-1}f = \frac{1}{(2m-1)!}\int\limits_a^x (x-\xi)^{2m-1}f(\xi)d\xi,$$

and

$$\left(L_K^{-1}\right)^* f = \frac{1}{(2m-1)!}\int\limits_x^b (\xi-x)^{2m-1}f(\xi)d\xi,$$

where $\left(L_K^{-1}\right)^*$ is an adjoint operator to the operator L_K^{-1}. By direct calculation we obtain

$$\begin{aligned}\mathfrak{Re}\left(L_K^{-1}\right)f =& \frac{L_K^{-1} + \left(L_K^{-1}\right)^*}{2}f = \\ =& \frac{1}{2}\frac{1}{(2m-1)!}\left[\int\limits_a^x (x-\xi)^{2m-1}f(\xi)d\xi + \int\limits_x^b (\xi-x)^{2m-1}f(\xi)d\xi\right] = \\ =& \frac{1}{2}\int\limits_a^b \frac{1}{(2m-1)!}|x-\xi|^{2m-1}f(\xi)d\xi = L^{-1}f,\end{aligned}$$

and $\mathfrak{Re}\left(L_K^{-1}\right)f = L^{-1}f$ as claimed.

2 Spectral problem for Helmholtz potential

Let us consider a one-dimensional Helmholtz equation in $(a, b) \subset \mathbb{R}$

$$Lu = -\frac{d^2}{dx^2}u(x) - k^2u(x) = f(x), \quad x \in (a, b). \tag{11}$$

It is easy to check that a particular solution to the Helmholtz equation is defined as a one-dimensional Helmholtz potential [9]

$$u(x) = -\frac{1}{2}\int\limits_a^b \frac{\sin(k|x-\xi|)}{k}f(\xi)d\xi, \tag{12}$$

here $\varepsilon_1(x-\xi) := -\frac{1}{2}\frac{\sin(k|x-\xi|)}{k}$ is a fundamental solution of the Helmholtz equation, i.e.

$$-\frac{d^2}{dx^2}\left(\varepsilon_1(x-\xi)\right) - k^2\varepsilon_1(x-\xi) = \delta(x-\xi).$$

In [10] there are the considered boundary conditions of operator (12) with this fundamental solution and with $\frac{e^{ik|x|}}{2ik}$. In this work, we will study the integral operator with $\varepsilon_1(x-\xi) := -\frac{1}{2}\frac{\sin(k|x-\xi|)}{k}$.

Lemma 2. Let $f \in C[a,b]$. Then there is a unique solution to equation (11), defined by the Helmholtz potential (12), and satisfies the following boundary conditions:

$$\begin{aligned} N_1[u] =& \frac{1}{k}\cos(bk)u'(b) + \frac{1}{k}\cos(ak)u'(a) + \sin(bk)u(b) + \sin(ak)u(a) = 0; \\ N_2[u] =& \frac{1}{k}\sin(bk)u'(b) + \frac{1}{k}\sin(ak)u'(a) - \cos(bk)u(b) - \cos(ak)u(a) = 0. \end{aligned} \tag{13}$$

Proof. Replacing $f(\xi)$ by $-\frac{d^2}{d\xi^2}u(\xi) - k^2u(\xi)$ in (12) we can rewrite (12) as

$$\begin{aligned} u(x) =& -\frac{1}{2}\int_a^b \frac{\sin(k|x-\xi|)}{k} f(\xi)d\xi = \\ =& \frac{1}{2}\int_a^b \frac{\sin(k|x-\xi|)}{k}\left(\frac{d^2}{d\xi^2}u(\xi) + k^2u(\xi)\right)d\xi, \end{aligned}$$

hence using integration by parts, we have

$$\begin{aligned} u(x) =& \int_a^b \frac{\sin(k|x-\xi|)}{2k}\left(\frac{d^2}{d\xi^2}u(\xi)\right)d\xi + k^2\int_a^b \frac{\sin(k|x-\xi|)}{2\lambda}u(\xi)d\xi = \\ =& \frac{\sin(k|x-\xi|)}{2k}\frac{d}{d\xi}u(\xi)\Big|_a^b - u(\xi)\frac{d}{d\xi}\left(\frac{\sin(k|x-\xi|)}{2k}\right)\Big|_a^b + \int_a^b \frac{d^2}{d\xi^2}\left(\frac{\sin(k|x-\xi|)}{2k}\right)u(\xi)d\xi + \\ &+k^2\int_a^b \frac{\sin(k|x-\xi|)}{2k}u(\xi)d\xi, \end{aligned}$$

since

$$\frac{d}{d\xi}\left(\frac{\sin(k|x-\xi|)}{2k}\right) = -\frac{\cos(k|x-\xi|)}{2}\cdot \mathrm{sgn}(x-\xi),$$

we obtain

$$\begin{aligned} u(x) =& \frac{\sin(k|x-\xi|)}{2k}\frac{d}{d\xi}u(\xi)\Big|_a^b + u(\xi)\frac{\cos(k|x-\xi|)}{2}\mathrm{sgn}(x-\xi)\Big|_a^b + \int_a^b \delta(x-\xi)u(\xi)d\xi = \\ =& \frac{\sin(k|x-\xi|)}{2k}\frac{d}{d\xi}u(\xi)\Big|_a^b + u(\xi)\frac{\cos(k|x-\xi|)}{2}\mathrm{sgn}(x-\xi)\Big|_a^b + u(x), \end{aligned}$$

we see that the same terms in the equality cancel out, and it follows that

$$\begin{aligned} &\frac{\sin(k|x-\xi|)}{2k}\frac{d}{d\xi}u(\xi)\Big|_a^b + u(\xi)\frac{\cos(k|x-\xi|)}{2}\mathrm{sgn}(x-\xi)\Big|_a^b = \\ =& \frac{\sin(k|x-b|)}{2k}\frac{d}{d\xi}u(b) - \frac{\sin(k|x-a|)}{2k}\frac{d}{d\xi}u(a) + \\ &+u(b)\frac{\cos(k|x-b|)}{2}\mathrm{sgn}(x-b) - u(a)\frac{\cos(k|x-a|)}{2}\mathrm{sgn}(x-a) = 0. \end{aligned}$$

Since $a < x < b$, we have

$$-\frac{\sin(k(x-b))}{2k}\frac{d}{d\xi}u(b) - \frac{\sin(k(x-a))}{2k}\frac{d}{d\xi}u(a) - \frac{\cos(k(x-b))}{2}u(b) - \frac{\cos(k(x-a))}{2}u(a) =$$
$$=\frac{1}{2}\sin(kx)\left(-\frac{1}{k}\cos(kb)u'(b) - \frac{1}{k}\cos(ka)u'(a) - \sin(kb)u(b) - \sin(ka)u(a)\right) +$$
$$+\frac{1}{2}\cos(kx)\left(\frac{1}{k}\sin(kb)u'(b) + \frac{1}{k}\sin(ka)u'(a) - \cos(kb)u(b) - \cos(ka)u(a)\right) = 0.$$

Since $\sin(kx)$ and $\cos(kx)$ are linearly independent, then we obtain the general conditions for the one-dimensional Newton potential as follows:

$$N_1[u] = \frac{1}{k}\cos(kb)u'(b) + \frac{1}{k}\cos(ka)u'(a) + \sin(kb)u(b) + \sin(ka)u(a) = 0;$$
$$N_2[u] = \frac{1}{k}\sin(kb)u'(b) + \frac{1}{k}\sin(ka)u'(a) - \cos(kb)u(b) - \cos(ka)u(a) = 0.$$

On the other hand, the general solution to equation (11) is given by

$$u(x) = -\int_a^x \frac{\sin(k(x-\xi))}{k} f(\xi)d\xi + c_1 e^{ikx} + c_2 e^{-ikx}, \tag{14}$$

boundary conditions (13) determine c_1 and c_2:

$$c_1 = \frac{1}{4}\int_a^b \frac{e^{-ik\xi}}{ik} f(\xi)d\xi, \quad c_2 = -\frac{1}{4}\int_a^b \frac{e^{ik\xi}}{ik} f(\xi)d\xi. \tag{15}$$

Substituting (15) into (14) we can assert that

$$u(x) = -\int_a^x \frac{\sin(k(x-\xi))}{k} f(\xi)d\xi + \frac{1}{4}\int_a^b \frac{e^{ik(x-\xi)}}{ik} f(\xi)d\xi - \frac{1}{4}\int_a^b \frac{e^{-ik(x-\xi)}}{ik} f(\xi)d\xi =$$
$$= -\int_a^x \frac{\sin(k(x-\xi))}{k} f(\xi)d\xi + \frac{1}{2}\int_a^b \frac{\sin(k(x-\xi))}{k} f(\xi)d\xi =$$
$$= -\frac{1}{2}\int_a^b \frac{\sin(k|x-\xi|)}{k} f(\xi)d\xi.$$

The proof is completed.

Now, we will consider the spectral problem for potential (12) as Fredholm integral operator

$$-\frac{1}{2}\int_0^1 \frac{\sin(k|x-\xi|)}{k} u(\xi)d\xi = \frac{u(x)}{\lambda}, \quad x \in (0,1). \tag{16}$$

Applying lemma 2 we conclude that operator (11) with conditions (13) is inverse operator to potential (12). Therefore, eigenvalue problem (16) and the following problem is equivalent

$$-\frac{d^2}{dx^2}u(x) - k^2 u(x) = \lambda u(x) \tag{17}$$

that satisfies the conditions [10]

$$N_1[u] = \frac{1}{k}\cos(k)u'(1) + \frac{1}{k}u'(0) + \sin(k)u(1) = 0,$$
$$N_2[u] = \frac{1}{k}\sin(k)u'(1) - \cos(k)u(1) - u(0) = 0. \tag{18}$$

Since operator (12) is a self-adjoint operator, then it always has real eigenvalues. It follows the problem (17)-(18) also has the real eigenvalue.

A solution to equation (17) is

$$u(x) = C_1 e^{ix\sqrt{\lambda+k^2}} + C_2 e^{-ix\sqrt{\lambda+k^2}}.$$

The conditions (18) implies that

$$\begin{bmatrix} \frac{1}{k}i\sqrt{\lambda+k^2}\cos(k)e^{i\sqrt{\lambda+k^2}} + \frac{1}{k}i\sqrt{\lambda+k^2} + \sin(k)e^{i\sqrt{\lambda+k^2}} \\ \frac{1}{k}i\sqrt{\lambda+k^2}\sin(k)e^{i\sqrt{\lambda+k^2}} - \cos(k)e^{i\sqrt{k^2+\lambda}} - 1 \end{bmatrix}$$
$$\begin{matrix} -\frac{1}{k}i\sqrt{\lambda+k^2}\cos(k)e^{-i\sqrt{\lambda+k^2}} - \frac{1}{k}i\sqrt{\lambda+k^2} + \sin(k)e^{-i\sqrt{\lambda+k^2}} \\ -\frac{1}{k}i\sqrt{\lambda+k^2}\sin(k)e^{-i\sqrt{\lambda+k^2}} - \cos(k)e^{-i\sqrt{\lambda+k^2}} - 1 \end{matrix} \Bigg] \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0.$$

To have non-trivial solutions of this system, the determinant of the left-hand side matrix must be zero, then

$$-4i\frac{1}{k}\sqrt{\lambda+k^2} - \left(\frac{1}{k^2}(\lambda+k^2)+1\right)\sin(k)\left(e^{i\sqrt{\lambda+k^2}} - e^{-i\sqrt{\lambda+k^2}}\right) -$$
$$-\frac{2}{k}i\sqrt{\lambda+k^2}\cos(k)\left(e^{i\sqrt{\lambda+k^2}} + e^{-i\sqrt{\lambda+k^2}}\right) = 0, \tag{19}$$

equation (19) is a transcendental equation for eigenvalues of problem (17)-(18).

1) If $\lambda + k^2 > 0$, by Euler's formula, we get

$$2\sqrt{\lambda+k^2} + \frac{1}{k}(\lambda+k^2)\sin(k)\sin\left(\sqrt{\lambda+k^2}\right) +$$
$$+2\sqrt{\lambda+k^2}\cos(k)\cos\left(\sqrt{\lambda+k^2}\right) + k\sin(k)\sin\left(\sqrt{\lambda+k^2}\right) = 0. \tag{20}$$

From (20) let $\sqrt{\lambda+k^2} = \pi n + \alpha_n$, here $\alpha_n \to 0$ as $n \to \infty$, then we obtain

$$\frac{1}{2}\left(\frac{k}{\pi n + \alpha_n} + \frac{\pi n + \alpha_n}{k}\right)(-1)^n \sin\alpha_n = -\frac{1 + (-1)^n \cos k \cos\alpha_n}{\sin k},$$

it follows that as $n \to \infty$

$$n \cdot \sin\alpha_n = -2k\frac{1 + (-1)^n \cos k}{(-1)^n \pi \sin k},$$

thus we have α_n as

$$\alpha_n = -\frac{1 + (-1)^n \cos k}{(-1)^n \pi \sin k} \cdot \frac{2k}{n}.$$

Then we can obtain asymptotic behavior of the eigenvalue λ as $n \to \infty$

$$\lambda_n = (\pi n + \alpha_n)^2 - k^2 = \pi^2 n^2 - k^2 - \frac{4k}{(-1)^n}\left(\frac{1 + (-1)^n \cos(k)}{\sin(k)}\right) + O\left(\frac{1}{n^2}\right). \tag{21}$$

2) Now we turn to the case $\lambda + k^2 < 0$. Denote by $\mu = i\sqrt{\lambda + k^2} = \sqrt{-\lambda - k^2}$, of course $\mu > 0$. In this case, we can rewrite (19) as

$$-4\mu + \sin(k)\left(\frac{1}{k}\mu^2 - k\right)\left(e^{\mu} - e^{-\mu}\right) - 2\mu\cos(k)\left(e^{\mu} + e^{-\mu}\right) = 0. \quad (22)$$

It is easily seen that equation (22) has only one root

$$\lambda = -\mu^2 - k^2.$$

However, this equation does not have a simple analytical solution, and graphical methods may be needed to approximate the root.

Thus, we described the eigenvalues of the Helmholtz potential.

Theorem 2. The eigenvalues of problem (16) for $\lambda + k^2 > 0$ are the roots of transcendental equation (20) and asymptotic behavior as $n \to \infty$ has the form (21), for $\lambda + k^2 < 0$ has only one eigenvalue as

$$\lambda = -\mu - k^2,$$

where μ is a root of equation (22).

Acknowledgments

This work is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP14871460).

Author Contributions

T.Sh. Kalmenov served as the principal investigator of the research grant and supervised the research process. A. Kadirbek and A. Kydyrbaikyzy contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Бірөлшемді потенциалдардың кейбір қасиеттері

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Мақаланың негізгі мақсаты бірөлшемді потенциалдардың қасиеттерін зерттеу. Авторлар бірөлшемді потенциалдар мен бірөлшемді Коши есебінің шешімі болатын L_K^{-1} операторының өзіне-өзі түйіндес бөлігінің арасындағы байланысты зерттеген. Сонымен қатар, жаңа әдіс арқылы Гельмгольц потенциалының спектральды мәселесі эквикалентті есепке келтірілді.

Кілт сөздер: бірөлшемді Гельмгольц потенциалы, спектральды мәселе, Фредгольм операторы.

Некоторые свойства одномерных потенциалов

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Основной целью данной работы является изучение свойств одномерных потенциалов. В статье мы исследовали связь одномерных потенциалов с самосопряженной частью оператора L_K^{-1}, который является решением одномерной задачи Коши. Более того, использован новый метод, позволяющий свести спектральную задачу для потенциала Гельмгольца к эквивалентной задаче.

Ключевые слова: одномерный потенциал Гельмгольца, спектральная проблема, оператор Фредгольма.

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