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On the solvability of a boundary value problem for a two-dimensional system of Navier-Stokes equations in a cone

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Due to the fact that the Navier-Stokes equations are involved in the formulation of a large number of interesting problems that are important from an applied point of view, these equations have been the object of attention of mechanics, mathematicians and other scientists for several decades in a row. But despite this, many problems for the Navier-Stokes equation remain unexplored to this day. In this work, we are exploring the solvability of a boundary value problem for a two-dimensional Navier-Stokes system in a non-cylindrical degenerating domain, namely, in a cone with its vertex at the origin. Previously, we studied cases of the linearized Navier-Stokes system or non-degenerating cylindrical domains, so this work is a logical continuation of our previous research in this direction. To the above-mentioned degenerate domain we associate a family of non-degenerate truncated cones, which, in turn, are formed by a one-to-one transformation into cylindrical domains, where for the problem under consideration we established uniform a priori estimates with respect to changes in the index of the domains. Further, using a priori estimates and the Faedo-Galerkin method, we established the existence, uniqueness of solution in Sobolev classes, and its regularity as the smoothness of the given functions increases.

Keywords: Navier-Stokes system, degenerating domain, Galerkin method.

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Introduction

As mentioned above, the Navier-Stokes equations have been the object of research by many scientists due to their applied importance ([1–5], and others). A significant number of practical problems have not been solved to this day.

Boundary value problems for parabolic equations in domains with moving boundaries are often models for ecological and medical processes [6], thermal processes in electrical contacts [7], thermomechanics processes [8, 9], and so on.

Among the works in this direction, we would like to mention the works [10] and [11], where the solvability of boundary value problems for the Burgers equation (the so-called one-dimensional version of the Navier-Stokes system) in domains with moving boundaries was researched. The results of these works were continued in [12], where by using the Faedo-Galerkin method and a priori estimates, the existence, uniqueness of the regular solution of the researched boundary value problems in Sobolev spaces is established.

Previously, in [13–15], it was shown by the authors that homogeneous boundary value problems for the Burgers equation and the nonlinear heat equation in an angular domain that degenerates at the initial time, along with the trivial solution, have nontrivial solutions. For boundary value problems with different inhomogeneities along the boundary, both unique and non-unique solvability were established

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in the work [16]. Also, the following works [15, 17, 18] devoted to problems in degenerating domains may be of interest to the readers.

In this work, we research the solvability of a boundary value problem with Dirichlet conditions for a two-dimensional Navier-Stokes system in a cone with its vertex at the origin. In Section 1, we present the formulation of the main boundary value problem and a sequence of auxiliary boundary value problems in the truncated cones. Then, in Section 2, these problems are transformed into boundary value problems in cylindrical domains by a change of independent variables. In Section 3, using the previously obtained results from the work [19], we obtain unique solvability of each of the above sequence of problems. In Section 4, auxiliary lemmas and the theorem on uniform a priori estimates are given. Section 5 is devoted to the main result.

1 *Preliminary statement of the problem*

Let us consider the next cone $Q_{xt_1} = \{x, t_1 : \ |x| < t_1, \ 0 < t_1 < T_1 < \infty\}$, which has its vertex at the origin. Let Ω_{xt_1} be the section of the cone Q_{xt_1} for a given $t_1 \in (0, T_1)$.

In the cone Q_{xt_1}, which degenerates into a point at $t_1 \in (0, T_1)$, we will consider the following boundary value problem (BVP) for a system of Navier-Stokes equations with respect to a two-dimensional vector-function of the fluid velocity $u(x, t_1) = \{u_1(x, t_1), \ u_2(x, t_1)\}$ and the fluid pressure function $p(x, t_1)$:

$$\frac{\partial u}{\partial t_1} - \nu \Delta u + \sum_{i=1}^{2} u_i \frac{\partial u}{\partial x_i} = f - \nabla p, \tag{1}$$

$$\operatorname{div} u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \tag{2}$$

$$u = 0, \ \ \{x, \, t_1\} \in \Sigma_{xt_1} \text{ is the lateral surface of the cone.} \tag{3}$$

Remark 1. Since at the initial moment of time the considering domain degenerates into a point, in the formulation of problem (1)–(3) we do not set the initial condition.

To the problem (1)–(3) we will set a sequence of BVPs, each of which will be considered in the corresponding truncated cone.

Let $n \in \mathbb{N}^* \equiv \{n \in \mathbb{N} : n \geq n_1, 1/n_1 < T_1\}$, $x = \{x_1, x_2\}$, and consider the domain $Q^n_{xt_1} = \{x, t_1 : |x| < t_1, \ 1/n < t_1 < T_1 < \infty\}$ which is an inverted truncated cone and let Ω_{xt_1} be the section of the cone $Q^n_{xt_1}$ for a given $t_1 \in (1/n, T_1)$. As we can see, now the domain $Q^n_{xt_1}$ does not degenerate into a point at the initial moment of time $t_1 = 1/n$. For domains Q_{xt_1} and $Q^n_{xt_1}$, the following inclusions are also true: $Q^{n_1}_{xt_1} \subset Q^{n_1+1}_{xt_1} \subset ... \subset Q_{xt_1}$, moreover, $\lim\limits_{n\to\infty} Q^n_{xt_1} = Q_{xt_1}$.

Now in the non-degenerating domain $Q^n_{xt_1}$ (for each finite $n \in \mathbb{N}^*$) we consider the following BVP:

$$\frac{\partial u_n}{\partial t_1} - \nu \Delta u_n + \sum_{i=1}^{2} u_{in} \frac{\partial u_n}{\partial x_i} = f_n - \nabla p_n, \tag{4}$$

$$\operatorname{div} u_n = \frac{\partial u_{1n}}{\partial x_1} + \frac{\partial u_{2n}}{\partial x_2} = 0, \tag{5}$$

$$u_n = 0 \ \ \{x, \, t_1\} \in \Sigma^n_{xt_1} \text{ is the lateral surface of the cone } \ Q^n_{xt_1}. \tag{6}$$

$$u_n(x, 1/n) = 0, \ \ x \in \Omega_{x1/n} \text{ is the section of the cone at } t_1 = 1/n. \tag{7}$$

A BVP of form (4)–(7) (for each fixed finite $n \in \mathbb{N}^*$) was studied by us in [19], in which we established theorems on unique solvability in Sobolev spaces.

2 Transformation of the problem (4)–(7) and its meaningful statement

Now we transform BVP (4)–(7) so that it would be posed in a cylindrical domain. For this purpose we use the transformation of independent variables and pass from the variables $\{x, t_1\}$ to variables $\{y, t\}$. Then we obtain

$$x_i = \frac{1}{n-t} y_i,\ t_1 = \frac{t}{n-t},\ y_i = \frac{x_i}{t_1},\ t = n - \frac{1}{t_1},\ i = 1,2;$$

$Q^n_{yt} = \{y, t : |y| < 1,\ 0 < t < T\}$ is a cylindrical domain, and Ω is a section of the cylinder Q^n_{yt} for any fixed $t \in [0, T]$,

$$t_1 = 1/n \Leftrightarrow t = 0,\ t_1 = T_1 \Leftrightarrow t = T = n - \frac{1}{T_1}.$$

Since

$$\tilde{u}_{in}(y,t) \triangleq u_{in}\left(\frac{y}{n-t}, \frac{1}{n-t}\right),\ \ \tilde{p}_n(y,t) \triangleq p_n\left(\frac{y}{n-t}, \frac{1}{n-t}\right), \tag{8}$$

we obtain the next derivatives with respect to t_1 of function $u_{in}(x, t_1)$ (8)

$$\frac{\partial u_{in}}{\partial t_1} = \frac{\partial \tilde{u}_{in}(y,t)}{\partial t}(n-t)^2 - \sum_{k=1}^{2} \frac{\partial \tilde{u}_{in}(y,t)}{\partial y_k}(n-t) y_k.$$

As for the derivatives with respect to x_k of function $u_{in}(x, t_1)$ (8), we have:

$$\frac{\partial u_{in}}{\partial x_k} = \frac{\partial \tilde{u}_{in}}{\partial y_k}(n-t), \quad \frac{\partial^2 u_{in}}{\partial x_k^2} = \frac{\partial^2 \tilde{u}_{in}}{\partial y_k^2}(n-t)^2.$$

Using the above we write down the BVP (4)–(7) in the cylindrical domain Q^n_{yt}:

$$\frac{\partial \tilde{u}_n}{\partial t} - \nu \Delta \tilde{u}_n + \frac{1}{n-t} \sum_{i=1}^{2} (\tilde{u}_{in} - y_i) \frac{\partial \tilde{u}_n}{\partial y_i} = \frac{1}{(n-t)^2} \tilde{f}_n - \frac{1}{n-t} \nabla p_n, \tag{9}$$

$$\operatorname{div} \tilde{u}_n = 0,\ \ \{y,t\} \in Q^n_{yt}, \tag{10}$$

$$\tilde{u}_n(y,t) = 0,\ \ \{y,t\} \in \Sigma^n_{yt} = \{y, t : |y| = 1,\ 0 < t < T\}, \tag{11}$$

$$\tilde{u}_n(y,0) = 0,\ \ y \in \Omega = \{y : |y| < 1\}. \tag{12}$$

Now instead of BVP (9)–(12) we will consider a more general BVP:

$$\frac{\partial \tilde{u}_n}{\partial t} - \nu \Delta \tilde{u}_n + \alpha(t) \sum_{i=1}^{2} \tilde{u}_{in} \frac{\partial \tilde{u}_n}{\partial y_i} = \sum_{i=1}^{2} \gamma_i(y_i, t) \frac{\partial \tilde{u}_n}{\partial y_i} + \beta(t) \tilde{f}_n - \delta(t) \nabla \tilde{p}_n, \tag{13}$$

$$\operatorname{div} \tilde{u}_n = 0,\ \ \{y,t\} \in Q^n_{yt} = \{y, t : |y| < 1,\ 0 < t < T\}, \tag{14}$$

$$\tilde{u}_n(y,t) = 0,\ \ \{y,t\} \in \Sigma^n_{yt} = \{y, t : |y| = 1,\ 0 < t < T\}, \tag{15}$$

$$\tilde{u}_n(y,0) = 0,\ \ y \in \Omega = \{y : |y| < 1\}, \tag{16}$$

where the given functions $\alpha(t), \beta(t), \gamma_i(y_i, t),\ i = 1, 2$, and $\delta(t)$, satisfy the following conditions

$$\begin{aligned} &\alpha_1 \le \alpha(t),\, \alpha'(t) \le \alpha_2,\ |\beta(t)| \le \beta_1,\ |\beta'(t)| \le \beta_1,\ |\delta(t)| \le \delta_1, \quad \forall\, t \in [0,T], \\ &|\gamma_i(y_i,t)| \le \gamma_1, \quad \left|\frac{\partial \gamma_i(y_i,t)}{\partial t}\right| \le \gamma_1,\ i = 1,2, \qquad \forall\, \{y,t\} \in Q^n_{yt}, \end{aligned} \tag{17}$$

where $\alpha_1,\ \alpha_2,\ \gamma_1,\ \beta_1,\ \delta_1$, are given positive constants.

It is easy to see that for the coefficients of equations (9) conditions (17) are aslo met.

Let us give a definition of a weak solution to problem (13)–(16). For this purpose we use the following notation [3, 4, 20–22] (here and further the designation $U^2 = U \times U$ is accepted):

$$V = \{\varphi |\, \varphi \in (D(\Omega))^2,\ \mathrm{div}\varphi = 0\},$$

$$H = \text{the closure of } V \text{ in } (L_2(\Omega))^2,$$

$$V = \text{the closure of } V \text{ in } (H^1(\Omega))^2.$$

For $\tilde{f}, \tilde{g} \in H$ we set

$$(\tilde{f}, \tilde{g}) = \int\limits_{\Omega} \tilde{f}(y)\tilde{g}(y)d\,y,\ \ |\tilde{f}| = (\tilde{f}, \tilde{f})^{1/2},$$

and for $\tilde{u}, \tilde{v} \in V$ we set

$$((\tilde{u}, \tilde{v})) = \sum_{i,j=1}^{2} \int\limits_{\Omega} \frac{\partial \tilde{u}_j(y)}{\partial y_i} \frac{\partial \tilde{v}_j(y)}{\partial y_i} d\,y,\ \ \|\tilde{u}\| = ((\tilde{u}, \tilde{u}))^{1/2}.$$

Then, identifying H with its conjugate: $H = H'$, we obtain the following inclusions

$$V \subset H = H' \subset V',$$

and each of these spaces is dense in the subsequent with completely continuous embedding operators. We can understand conditions (15) as conditions of belonging the function $\tilde{u}(y,t)$ to space V for almost all t.

Now we assume that

$$a(\tilde{u}, \tilde{v}) = \sum_{i,j=1}^{2} \int\limits_{\Omega} \frac{\partial \tilde{u}_j}{\partial y_i} \frac{\partial \tilde{v}_j}{\partial y_i} d\,y,\ \ \tilde{u},\, \tilde{v} \in V,\ \forall\, t \in (0,T),$$

$$b(\tilde{u}, \tilde{v}, \tilde{w}) = \sum_{i,k=1}^{2} \int\limits_{\Omega} \tilde{u}_k \frac{\partial \tilde{v}_i}{\partial y_k} \tilde{w}_i\, d\,y,\ \forall\, t \in (0,T),$$

for a triple of such two-dimensional vectors $\tilde{u}, \tilde{v}, \tilde{w}$, for which the corresponding integrals converge.

Problem 1. Let

$$\tilde{f}_n \in L_2(0,T; (H^{-1}(\Omega))^2),$$

be given and functions $\alpha(t), \beta(t), \gamma_i(y_i, t),\ i = 1,2$, and $\delta(t)$ satisfy conditions (17).

It is required to find such $\tilde{u}_n$ and $\tilde{p}_n$, $\tilde{p}_n \in D'(Q^n_{yt})$, that

$$\tilde{u}_n \in L_2(0,T;V) \cap L_\infty(0,T;H),$$

$$\frac{\partial \tilde{u}_n}{\partial t} - \nu \Delta \tilde{u}_n + \alpha(t) \sum_{i=1}^{2} \tilde{u}_{in} \frac{\partial \tilde{u}_n}{\partial y_i} = \sum_{i=1}^{2} \gamma_i(y_i, t) \frac{\partial \tilde{u}_n}{\partial y_i} + \beta(t)\tilde{f}_n - \delta(t)\nabla \tilde{p}_n, \tag{18}$$

$$\tilde{u}_n(y,0) = 0. \tag{19}$$

Despite the apparent accuracy, in the formulation of Problem 1 we have one ambiguity: there is no information regarding the derivative $\frac{\partial \tilde{u}_n}{\partial t}(y,t)$ and $\tilde{p}_n(y,t)$, there is only the following relation

$$\frac{\partial \tilde{u}_n}{\partial t} + \delta(t)\nabla \tilde{p}_n = \nu\Delta\tilde{u}_n - \alpha(t)\sum_{i=1}^{2} \tilde{u}_{in}\frac{\partial \tilde{u}_n}{\partial y_i} + \sum_{i=1}^{2}\gamma_i(y_i,t)\frac{\partial \tilde{u}_n}{\partial y_i} + \beta(t)\tilde{f}_n \quad \text{on } Q^n_{yt},$$

therefore the meaning of condition (19) is not obvious.

If we take $\varphi(y) \in V$, then $(\nabla\tilde{p}_n, \varphi) = 0$ in $(D'(0,T))^2$, and (18) leads to equality

$$\left(\frac{\partial \tilde{u}_n}{\partial t}, \varphi\right) = -\nu a(\tilde{u}_n, \varphi) - \alpha(t) b(\tilde{u}_n, \tilde{u}_n, \varphi) + \sum_{i=1}^{2}\left(\gamma_i(y_i,t)\frac{\partial \tilde{u}_n}{\partial y_i}, \varphi\right) + \beta(t)(\tilde{f}_n, \varphi) \text{ for any } \varphi \in V. \quad (20)$$

Using the following equality

$$b(\tilde{u}_n, \tilde{u}_n, \varphi) = -b(\tilde{u}_n, \varphi, \tilde{u}_n),$$

we get that (20) is equivalent to

$$\left(\frac{\partial \tilde{u}_n}{\partial t}, \varphi\right) = -\nu a(\tilde{u}_n, \varphi) + \alpha(t) b(\tilde{u}_n, \varphi, \tilde{u}_n) + \sum_{i=1}^{2}\left(\gamma_i(y_i,t)\frac{\partial \tilde{u}_n}{\partial y_i}, \varphi\right) + \beta(t)(\tilde{f}_n, \varphi) \text{ for any } \varphi \in V. \quad (21)$$

Let

$$X = \text{the closure of } V \text{ in } (W_1^1(\Omega))^2,$$

we have

$$|b(\tilde{u}_n, \varphi, \tilde{u}_n)| \leq C_1 \|\tilde{u}_n\|^2_{(L_\infty(\Omega))^2} \sum_{i,j=1}^{2} \left\|\frac{\partial \varphi_j}{\partial y_i}\right\|_{L_1(\Omega)} \leq C\|\tilde{u}_n\|^2 \|\varphi\|_X,$$

since $V \subset (L_\infty(\Omega))^2$, and therefore

$$b(\tilde{u}_n, \varphi, \tilde{u}_n) = (g, \varphi), \quad \|g\|_{X'} \leq C\|\tilde{u}_n\|^2_{L_\infty(\Omega))^2},$$

whence it follows that $g \in L_1(0,T;X')$.

From (21) we obtain that

$$\frac{\partial \tilde{u}_n}{\partial t} \in L_2(0,T;V') + L_1(0,T;X'),$$

so that (19) makes sense (for example, in X').

Thus, we obtain a different formulation of Problem 1.

Problem 2. Let

$$\tilde{f}_n \in L_2(0,T;V') \quad (22)$$

be given and functions $\alpha(t), \beta(t), \gamma_i(y_i,t),\ i = 1,2,$ and $\delta(t)$, satisfy conditions (17).

It is required to find such $\tilde{u}_n$, that

$$\tilde{u}_n \in L_2(0,T;V) \cap L_\infty(0,T;H), \quad (23)$$

$$\left(\frac{\partial \tilde{u}_n}{\partial t}, \tilde{v}\right) + \nu a(\tilde{u}_n, \tilde{v}) + \alpha(t) b(\tilde{u}_n, \tilde{u}_n, \tilde{v}) - \sum_{i=1}^{2}\left(\gamma_i(y_i,t)\frac{\partial \tilde{u}_n}{\partial y_i}, \tilde{v}\right) = \beta(t)(\tilde{f}_n, \tilde{v}) \ \forall\, \tilde{v} \in V, \quad (24)$$

$$\tilde{u}_n(y,0) = 0, \ y \in \Omega. \quad (25)$$

Next we want to formulate Problem 2 in relation to the BVP (4)–(7). To do this, first, we need the following correspondence of function spaces in terms of independent variables $\{y,t\} \in Q^n_{yt}$ and $\{x,t_1\} \in Q^n_{xt_1}$:

$$\tilde{f}_n(y,t) \in L_2(0,T;V') \Leftrightarrow f_n(x,t_1) \in L_2(1/n,T_1;V'_{t_1}),$$

$$\tilde{u}_n(y,t) \in L_2(0,T;V) \cap L_\infty(0,T;H) \Leftrightarrow u_n(x,t_1) \in L_2(1/n,T_1;V_{t_1}) \cap L_\infty(1/n,T_1;H_{t_1}),$$

etc., where for almost all $t_1 \in [1/n,T_1]$,

$$V_{t_1} = \{\varphi|\, \varphi \in (D(\Omega_{xt_1}))^2,\ \mathrm{div}\varphi = 0\},$$

$$H_{t_1} = \text{the closure of } V_{t_1} \text{ in } (L_2(\Omega_{xt_1}))^2,$$

$$V_{t_1} = \text{the closure of } V_{t_1} \text{ in } (W^1_2(\Omega_{xt_1}))^2.$$

Problem 3. Let

$$f_n(x,t_1) \in L_2(1/n,T_1;V'_{t_1}) \tag{26}$$

be given. It is required to find such $u(x,t_1)$, that

$$u_n(x,t_1) \in L_2(1/n,T_1;V_{t_1}) \cap L_\infty(1/n,T_1;H_{t_1}), \tag{27}$$

$$\left(\frac{\partial u_n}{\partial t_1}, v\right) + \nu a(u_n,v) + b(u_n,u_n,v) = (f,v)\ \forall\, v \in V_{t_1}, \tag{28}$$

$$u_n(x,1/n) = 0,\ x \in \Omega_{x1/n}. \tag{29}$$

Finally, we formulate Problem 4 in relation to the original BVP (1)–(3), which is given in a degenerating cone.

Problem 4. Let

$$f(x,t_1) \in L_2(0,T_1;V'_{t_1}) \tag{30}$$

be given. It is required to find such $u(x,t_1)$, that

$$u(x,t_1) \in L_2(0,T_1;V_{t_1}) \cap L_\infty(0,T_1;H_{t_1}), \tag{31}$$

$$\left(\frac{\partial u}{\partial t_1}, v\right) + \nu a(u,v) + b(u,u,v) = (f,v)\ \forall\, v \in V_{t_1}. \tag{32}$$

Further, we will use the following Lemma ([3], Lemma I.6.1); [4], Lemma II.1.1):

Lemma 1. The trilinear form $\{\tilde{u},\tilde{v},\tilde{w}\} \to b(\tilde{u},\tilde{v},\tilde{w})$ is continuous on $V \times V \times V$, $\forall\, t \in (0,T)$, and the following estimate is valid

$$\left|\int\limits_\Omega \tilde{u}_i \frac{\partial \tilde{v}_k}{\partial y_i} \tilde{w}_k\, d\, y\right| \le \|\tilde{u}_i\|_{L_4(\Omega)} \left\|\frac{\partial \tilde{v}_k}{\partial y_i}\right\|_{L_2(\Omega)} \|\tilde{w}_k\|_{L_4(\Omega)},\ i,k = 1,2.$$

3 Solvability theorems for problems (22)–(25), (26)–(29) and (30)–(32)

According to the results of [19] we have:

(a) in the case of each of the domains represented by the cylinders Q^n_{yt}, $n \in \mathbb{N}^*$, Theorems 1–3 and Corollary 1 are valid;

(b) and in the case of each of the domains represented by truncated cones $Q^n_{xt_1}$, $n \in \mathbb{N}^*$, Theorems 4–6 and Corollary 2 are valid.

Theorem 1. Let for the functions $\alpha(t), \beta(t), \gamma_i(y_i, t)$ and $\delta(t)$ conditions (17) met. Then Problem 2 (22)–(25) has a unique (weak) solution

$$\tilde{u}_n(y,t) \in \widetilde{W}(0,T) = \{v|\, v \in L_2(0,T;V),\ \frac{\partial v}{\partial t} \in L_2(0,T;V')\}.$$

Theorem 2. Let the following be true along with the conditions of Theorem 1:

$$\frac{\partial \tilde{f}_n}{\partial t} \in L_2(0,T;V'),\ \tilde{f}_n(y,0) \in H.$$

Then for the solution $\tilde{u}_n(y,t)$ to Problem 2 (22)–(25) we have the following inclusion

$$\frac{\partial \tilde{u}_n}{\partial t} \in L_2(0,T;V) \cap L_\infty(0,T;H).$$

Theorem 3. Let the following be true along with the conditions of Theorem 2:

$$\tilde{f}_n \in L_\infty(0,T;H).$$

Then for the solution $\tilde{u}_n(y,t)$ to Problem 2 (22)–(25) we have the following inclusion

$$\tilde{u}_n \in L_\infty(0,T;(W_2^2(\Omega))^2).$$

Corollary 1. Let the following be true along with the conditions of Theorem 3:

$$\tilde{f}_n \in L_2(0,T;H).$$

Then for the solution $\tilde{u}_n(y,t)$ to Problem 2 (22)–(25) we have the following inclusion

$$\tilde{u}_n \in L_2(0,T;(W_2^2(\Omega))^2).$$

Theorem 4. Let $f_n(x,t_1) \in L_2(1/n,T_1;V'_{t_1})$. Then Problem 3 (26)–(29) has a unique (weak) solution

$$u_n(x,t_1) \in W(1/n,T_1) = \{v|\, v \in L_2(1/n,T_1;V_{t_1}),\ \frac{\partial v}{\partial t_1} \in L_2(1/n,T_1;V'_{t_1})\}.$$

Theorem 5. Let the following be true along with the conditions of Theorem 4:

$$\frac{\partial f_n}{\partial t_1} \in L_2(1/n,T_1;V'_{t_1}),\ f_n(x,1/n) \in H_{1/n}.$$

Then for the solution $u_n(x,t_1)$ to Problem 3 (26)–(29) we have the following inclusion

$$\frac{\partial u_n}{\partial t_1} \in L_2(1/n,T_1;V_{t_1}) \cap L_\infty(1/n,T_1;H_{t_1}).$$

Theorem 6. Let the following be true along with the conditions of Theorem 5

$$f_n \in L_\infty(1/n, T_1; H_{t_1}).$$

Then for the solution $u_n(x, t_1)$ to Problem 3 (26)–(29) we have the following inclusion

$$u_n \in L_\infty(1/n, T_1; (W_2^2(\Omega_{xt_1}^n))^2).$$

Corollary 2. Let the following be true along with the conditions of Theorem 6

$$f_n \in L_2(1/n, T_1; H_{t_1}).$$

Then for the solution $u_n(x, t_1)$ to Problem 3 (26)–(29) we have the following inclusion

$$u_n \in L_2(1/n, T_1; (W_2^2(\Omega_{xt_1}^n))^2).$$

Remark 2. Note that Problem 3 (26)–(29) corresponds to BVP (4)–(7).

Further, by using the results of Theorems 4–6 and Corollaries 2, for BVP 3 (26)–(29) we will show the validity of the following theorem.

Theorem 7. Let the conditions of Theorem 6 and Corollary 2 be satisfied. Then there exists a positive constant K independent of n, such that for the solution $u_n(x, t_1)$ to BVP 3 (26)–(29) we have the following estimate

$$\|u_n(x, t_1)\|^2_{(W_2^{2,1}(Q_{xt_1}^n))^2} + \|\nabla p_n(x, t_1)\|^2_{(L_2(Q_{xt_1}^n))^2} \leq K \cdot F_n \leq K \cdot F,$$

where

$$F_n = |f_n(x, 1/n)|^2 + \|f_n(x, t_1)\|^2_{W_2^1(1/n, T_1; V'_{t_1})} + \|f_n(x, t_1)\|^2_{(L_2(Q_{xt_1}^n))^2},$$

$$F = |f(x, 0)|^2 + \|f(x, t_1)\|^2_{W_2^1(0, T_1; V'_{t_1})} + \|f(x, t_1)\|^2_{(L_2(Q_{xt_1}))^2}.$$

$$Q_{xt_1}^{n_1} \subset Q_{xt_1}^{n_1+1} \subset ... \subset Q_{xt_1} \text{ and obviously } \lim_{n\to\infty} Q_{xt_1}^n = Q_{xt_1}.$$

The proof of this theorem will be given in the next section.

Now we can formulate the main result of the paper, which will be proved in Section 5 on the basis of the assertion of Theorem 7.

Theorem 8. Let the conditions of Theorem 7 be met. Then in the degenerating domain Q_{xt_1} the two-dimensional BVP 4 for the system of Navier-Stokes equations (30)–(32) has a unique solution $\{u(x, t_1), p(x, t_1)\}$ in space

$$(W_2^{2,1}(Q_{xt_1}))^2 \times L_2(0, T_1; W_2^1(\Omega_{xt_1})/X_{xt_1}),$$

where $W_2^1(\Omega_{xt_1})/X_{xt_1}$ and $\|\psi(x)\|_{W_2^1(\Omega_{xt_1})/X_{xt_1}} \equiv \inf_{k\in X_{xt_1}} \|\psi(x)+k\|_{W_2^1(\Omega_{xt_1})}$ are, respectively, a quotient space and a quotient norm in the subspace X_{xt_1} consisting of all possible constants $k = \text{const}$ defined on the set Ω_{xt_1}.

Remark 3. Problem 4 (30)–(32) corresponds to BVP (1)–(3).

4 Auxiliary lemmas. Proof of Theorem 7

To prove Theorem 7, we need to establish the following lemmas.

Lemma 2. Let the conditions of Theorem 4 be met. Then there exists a positive constant K_1 independent of n, such that for the solution $u_n(x,t_1)$ to BVP (4)–(7) we have the following estimate

$$\|u_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2} + \|\nabla u_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2} \le K_1 \|f_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2}, \tag{33}$$

where

$$\|u_n(x,t_1)\|^2_{L_2(Q^n_{xt_1})} \equiv \int\limits_{1/n}^{T_1} |u_n(x,t_1)|^2 \, dt_1,$$

$$|u_n(x,t_1)|^2 = \int\limits_{\Omega_{xt_1}} \left\{[u_{1n}(x,t_1)]^2 + [u_{2n}(x,t_1)]^2\right\} dx.$$

Proof. By multiplying equation (4) scalarly by the function $u_n(x,t_1)$ in space $L_2(\Omega_{xt_1})$, we obtain

$$\frac{1}{2}\frac{d}{dt}|u_n(x,t_1)|^2 + a(u_n(x,t_1),u_n(x,t_1)) = (f_n(x,t_1),u_n(x,t_1)),$$

since $b(u_n(x,t_1),u_n(x,t_1),u_n(x,t_1)) = 0$. From here, according to the Cauchy ε-inequality, using the Poincare inequality ([21], 6.30) and integrating the result from $1/n$ to T_1, we obtain the required inequality (33).

Lemma 3. Let the conditions of Theorem 5 be met. Then there exists a positive constant K_2 independent of n, such that at all $t_1 \in [1/n, T_1]$ for the solution $u_n(x,t_1)$ to BVP (4)–(7) we have the following estimate

$$\left|\frac{\partial u_n(x,t_1)}{\partial t_1}\right|^2 + \int\limits_{1/n}^{t_1} \left\|\frac{\partial u_n(x,t_1)}{\partial t_1}\right\|^2 dt_1 \le K_2\left[|f_n(x,1/n)|^2 + \|f_n(x,t_1)\|^2_{W^1_2(1/n,T_1;V'_{t_1})}\right]. \tag{34}$$

Proof. By multiplying equation (4) scalarly by the function $\frac{\partial u_n(x,t_1)}{\partial t_1}$ in space $L_2(\Omega_{xt_1})$, for $t_1 = 1/n$ we will obtain:

$$\left|\frac{\partial u_n(x,1/n)}{\partial t_1}\right|^2 = \left(f(x,1/n), \frac{\partial u_n(x,1/n)}{\partial t}\right) \le |f(x,1/n)| \left|\frac{\partial u_n(x,1/n)}{\partial t}\right|,$$

i.e., we get

$$\left|\frac{\partial u_n(x,1/n)}{\partial t_1}\right|^2 \le C_0 |f(x,1/n)|^2. \tag{35}$$

Now we differentiate equation (4) with respect to t_1, then by multiplying the equation scalarly by the function $\frac{\partial u_n(x,t_1)}{\partial t_1}$ in space $L_2(\Omega^n_{xt_1})$, and considering (by virtue of Lemma II.1.3 from [4]) the following equality

$$b\left(u_n, \frac{\partial u_n}{\partial t_1}, \frac{\partial u_n}{\partial t_1}\right) = 0,$$

we get

$$\frac{1}{2}\frac{d}{dt_1}\left|\frac{\partial u_n}{\partial t_1}\right|^2 + \nu\left\|\frac{\partial u_n}{\partial t_1}\right\|^2 + b\left(\frac{\partial u_n}{\partial t_1}, u_n, \frac{\partial u_n}{\partial t_1}\right) = \left(\frac{\partial f_n}{\partial t_1}, \frac{\partial u_n}{\partial t_1}\right). \tag{36}$$

We have

$$\left|b\left(\frac{\partial u_n}{\partial t_1}, u_n, \frac{\partial u_n}{\partial t_1}\right)\right| = \left|-b\left(\frac{\partial u_n}{\partial t_1}, \frac{\partial u_n}{\partial t_1}, u_n\right)\right| \le C_5 \left\|\frac{\partial u_n}{\partial t_1}\right\|_{\left(L_4(\Omega^n_{xt_1})\right)^2} \left\|\frac{\partial u_n}{\partial t_1}\right\| \|u_n\|_{\left(L_4(\Omega^n_{xt_1})\right)^2} \le$$

$$\le C_6 \left\|\frac{\partial u_n}{\partial t_1}\right\|^{3/2} \left|\frac{\partial u_n}{\partial t_1}\right|^{1/2} \|u_n\|_{\left(L_4(\Omega^n_{xt_1})\right)^2} \le \frac{\nu}{2}\left\|\frac{\partial u_n}{\partial t_1}\right\|^2 + C_7 \left|\frac{\partial u_n}{\partial t_1}\right|^2 \|u_n\|^4_{\left(L_4(\Omega^n_{xt_1})\right)^2}.$$

Here we have used Lemma I.6.2 from [3] and Young's inequality ($p^{-1}+q^{-1}=1$):

$$|AB| = \left|\left(a^{1/p}A\right)\left(a^{1/q}\frac{B}{a}\right)\right| \le \frac{a}{p}|A|^p + \frac{a}{qa^q}|B|^q,$$

where

$$A = \left\|\frac{\partial u_n}{\partial t_1}\right\|^{3/2},\ B = C_6\,|u_n|^{1/2}\left\|\frac{\partial u_n}{\partial t_1}\right\|_{(L_4(\Omega^n_{xt_1}))^2},\ a=\frac{2\nu}{3},\ p=\frac{4}{3},\ q=4,$$

$$\left(\frac{\partial f_n}{\partial t_1}, \frac{\partial u_n}{\partial t_1}\right) \le C_8\left|\frac{\partial f_n}{\partial t_1}\right|\left|\frac{\partial u_n}{\partial t_1}\right| \le \frac{C_8^2}{2}\left|\frac{\partial f_n}{\partial t_1}\right|^2 + \frac{1}{2}\left|\frac{\partial u_n}{\partial t_1}\right|^2.$$

By using these inequalities and relations (35)–(36), we get uniform in t_1 and n required estimate (34). The statement of Lemma 3 is proved.

Lemma 4. Let the conditions of Theorem 6 and Corollary 2 be met. Then there exists a positive constant K_3 independent of n, such that at all $t_1 \in [1/n, T_1]$ for the solution $u_n(x,t_1)$ to BVP (4)–(7) we have the following estimate

$$|\nabla u_n(x,t_1)|^2 + \int\limits_{1/n}^{t_1} |\Delta u_n(x,t_1)|^2\, dt_1 \le K_3 \cdot F_{1n},$$

where

$$F_{1n} = |f_n(x,1/n)|^2 + \|f_n(x,t_1)\|^2_{W_2^1(1/n,T_1;V'_{t_1})} + \|f_n(x,t_1)\|^2_{L_\infty(1/n,T_1;H_{t_1})}.$$

Proof. First, note that, by Lemma III.3.1 from [4] function $B\,u_n(x,t_1)$, defined by equality

$$< B\,u_n, v >= b(u_n,u_n,v)\ \forall\, v \in V_{t_1}\ \text{ almost everywhere on }\ [1/n, T_1],$$

belongs to space $L_1(1/n,T_1;V'_{t_1})$.

We write equation (24) in the form

$$\nu a(u_n(x,t_1), v(x)) = (g_n(x,t_1), v(x))\ \ \forall\, v \in V_{t_1}, \tag{37}$$

where

$$g_n(x,t_1) = -\frac{\partial u_n}{\partial t_1} - B\,u_n + f_n. \tag{38}$$

Since $u_n \in L_\infty(1/n,T_1;V_{t_1})$ and according to Lemmas I.6.1–I.6.2 from [3]

$$|b(u_n(x,t_1), u_n(x,t_1), v(x)| \le$$

$$\le C_0\|u_n(x,t_1)\|_{(L_4(\Omega^n_{xt_1}))^2}\|u_n(x,t_1)\|\|v\|_{(L_4(\Omega^n_{xt_1}))^2} \le C_1\|u_n(x,t_1)\|^2\|\|v\|_{(L_4(\Omega^n_{xt_1}))^2}, \tag{39}$$

then $B\,u_n \in L_\infty(1/n, T_1; (L_{4/3}(\Omega^n_{xt_1}))^2)$. From (38) and the inclusion

$$-\frac{\partial u_n}{\partial t_1} + f_n \in L_\infty(1/n, T_1; H_{t_1})$$

(here we have used the statement of Lemma 3) we have

$$g_n(x,t_1) \in L_\infty(1/n, T_1; (L_{4/3}(\Omega^n_{xt_1}))^2). \tag{40}$$

Further, applying the theorem from ([23], 309–311) and ([4], I.2.5) for the elliptic BVP (37), we get $u_n \in L_\infty(1/n, T_1; (W^2_{4/3}(\Omega^n_{xt_1}))^2)$, and the following estimates

$$\|u_n(x,t_1)\|^2_{L_\infty(1/n,T_1;(W^2_{4/3}(\Omega^n_{xt_1}))^2)} + \|p_n(x,t_1)\|^2_{L_\infty(1/n,T_1;W^1_{4/3}(\Omega^n_{xt_1})/X^n_{xt_1})} \leq$$

$$\leq K\,\|g_n(x,t_1)\|^2_{L_\infty(1/n,T_1;(L_{4/3}(\Omega^n_{xt_1}))^2)}\,, \tag{41}$$

where $W^1_{4/3}(\Omega^n_{xt_1})/X^n_{xt_1}$ is a quotient space in the subspace $X^n_{xt_1}$ consisting of all possible constants $k =$ const defined on the set $\Omega^n_{xt_1}$. But according to Sobolev embedding theorem $W^2_{4/3}(\Omega^n_{xt_1}) \subset L_\infty(\Omega^n_{xt_1})$, then $u_n \in (L_\infty(Q^n_{xt_1}))^2$.

Now we can improve the inclusion (40). We replace inequality (39) with the following

$$|b(u_n(x,t_1), u_n(x,t_1), v(x))| \leq C_2\|u_n\|_{(L_\infty(Q^n_{xt_1}))^2}\|u_n(x,t_1)\|\|v\|,$$

from which it follows that $B\,u_n \in L_\infty(1/n, T_1; H_{t_1})$. Thus, we obtain that $g_n \in L_\infty(1/n, T_1; H_{t_1})$.

Again, applying the theorem from ([23], 309–311) and ([4], I.2.5) for the elliptic BVP (37), we get that $u_n \in L_\infty(1/n, T_1; (W^2_2(\Omega^n_{xt_1}))^2) \subset L_2(1/n, T_1; (W^2_2(\Omega^n_{xt_1}))^2)$, and estimates for the case of Theorem 6:

$$\|u_n(x,t_1)\|^2_{L_\infty(1/n,T_1;(W^2_2(\Omega^n_{xt_1}))^2)} + \|p_n(x,t_1)\|^2_{L_\infty(1/n,T_1;W^1_2(\Omega^n_{xt_1})/X^n_{xt_1})} \leq$$

$$\leq K\,\|g_n(x,t_1)\|^2_{L_\infty(1/n,T_1;H_{t_1})}\,, \tag{42}$$

and for the case of Corollary 2:

$$\|u_n(x,t_1)\|^2_{L_2(1/n,T_1;(W^2_2(\Omega^n_{xt_1}))^2)} + \|p_n(x,t_1)\|^2_{L_2(1/n,T_1;W^1_2(\Omega^n_{xt_1})/X^n_{xt_1})} \leq$$

$$\leq K\,\|g_n(x,t_1)\|^2_{L_2(1/n,T_1;H_{t_1})}\,, \tag{43}$$

where $W^1_2(\Omega^n_{xt_1})/X^n_{xt_1}$ is a quotient space in the subspace $X^n_{xt_1}$ consisting of all possible constants $k =$ const defined on the set $\Omega^n_{xt_1}$. From here we also get that $\nabla u_n \in L_\infty(1/n, T_1; (W^1_2(\Omega^n_{xt_1}))^2) \subset L_2(1/n, T_1; (W^1_2(\Omega^n_{xt_1}))^2)$.

It remains to estimate the right-hand side in (41)–(43) with respect to the function $f_n(x,t_1)$. According to (38), it remains to estimate only the summand $B\,u_n$. We have

$$\|B\,u_n\|_{L_\infty(1/n,T_1;H_{t_1})} \leq C_3\|u_n(x,t_1)\|,$$

the right-hand side of which is estimated in Lemma 3. This completes the proof of Lemma 4.

Lemma 5. Let the conditions of Theorem 6 and Corollary 2 be satisfied. Then there exists a positive constant K_4 independent of n, such that for a solution to the boundary value problem (4)–(7) the following estimate takes place

$$\left\|\frac{\partial u_n(x,t_1)}{\partial t_1}\right\|^2_{(L_2(Q^n_{xt_1}))^2} + \|\Delta u_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2} + \|\nabla p_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2} \leq K_4 \cdot F_2, \tag{44}$$

where

$$F_2 = |f(x,0)|^2 + \|f(x,t_1)\|^2_{W^1_2(0,T_1;V'_{t_1})} + \|f(x,t_1)\|^2_{(L_2(Q_{xt_1}))^2}.$$

Proof. The proof of Lemma 5 directly follows from the statements of Lemmas 3 and 4.

Thus, the statement of Theorem 7 follows from Lemmas 2, 5 and inequalities

$$\|f_n\|^2_{(L_2(Q^n_{xt_1}))^2} \le \|f\|^2_{(L_2(Q_{xt_1}))^2}\,,$$

i.e., we obtain the required estimate (44):

$$\|u_n(x,t_1)\|^2_{(W_2^{2,1}(Q^n_{xt_1}))^2} + \|\nabla p_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2} \le K_4 \cdot F_{2n} \le K_4 \cdot F_2\,,$$

where

$$F_{2n} = |f_n(x,1/n)|^2 + \|f_n(x,t_1)\|^2_{W_2^1(1/n,T_1;V'_{t_1})} + \|f_n(x,t_1)\|^2_{(L_2(Q^n_{xt_1}))^2}\,.$$

5 *Proof of Theorem 8: the existence and uniqueness of a solution to boundary value problem (1)–(3)*

Let $\{u_n(x,t_1), p_n(x,t_1)\}$ be a solution to boundary value problem (4)–(7), which exists and is unique according to Theorems 4–6, Corollary 2 and Theorem 7. Denote by $\{\widetilde{u_n}(x,t_1), \widetilde{p_n}(x,t_1)\}$ the continuation of solutions $\{u_n(x,t_1), p_n(x,t_1)\}$ by zero to the entire cone Q_{xt_1}. Theorem 7 implies the following inequality

$$\|\widetilde{u_n}(x,t_1)\|^2_{(W_2^{2,1}(Q_{xt_1}))^2} + \|\nabla\widetilde{p_n}(x,t_1)\|^2_{(L_2(Q_{xt_1}))^2} \le K \cdot F,$$

that is uniform over n, where

$$F = |f(x,0)|^2 + \|f(x,t_1)\|^2_{W_2^1(0,T_1;V'_{t_1})} + \|f(x,t_1)\|^2_{(L_2(Q_{xt_1}))^2}.$$

It follows that from the bounded sequence $\{\widetilde{u_n}(x,t_1), \nabla\widetilde{p_n}(x,t_1)\}_{n=1}^{\infty}$ it is possible to extract a subsequence (to denote the index of which we keep the letter n), such that the following limit relations take place:

$$\frac{\partial\widetilde{u_n}(x,t_1)}{\partial t_1} \to \frac{\partial u(x,t_1)}{\partial t_1} \;\text{ weakly in }\; (L_2(Q_{xt_1}))^2,$$

$$\Delta\widetilde{u_n}(x,t_1) \to \Delta u(x,t_1) \;\text{ weakly in }\; (L_2(Q_{xt_1}))^2,$$

$$\widetilde{u_n}(x,t_1) \to u(x,t_1) \;\text{ strongly in }\; (L_2(Q_{xt_1}))^2,$$

$$\widetilde{u_{in}}(x,t_1)\frac{\partial\widetilde{u_{kn}}(x,t_1)}{\partial x_i}\widetilde{u_{kn}}(x,t_1) \to u_i(x,t_1)\frac{\partial u_k(x,t_1)}{\partial x_i}u_k(x,t_1) \;\text{ weakly in }\; L_2(Q_{xt_1}),\; i,k=1,2,$$

$$\nabla\widetilde{p_n}(x,t_1) \to \nabla p(x,t_1) \;\text{ weakly in }\; (L_2(Q_{xt_1}))^2.$$

Further, in a standard way, it is easy to show that

$$\{u(x,t_1), p(x,t_1)\} \in \Big\{(W_2^{2,1}(Q_{xt_1}))^2 \times L_2(0,T_1; W_2^1(\Omega_{xt_1})/X_{xt_1})\Big\}$$

is the solution of the boundary value problem (1)–(3), where $W_2^1(\Omega_{xt_1})/X_{xt_1}$ is a quotient space in the subspace X_{xt_1} consisting of all possible constants $k = \text{const}$ defined on the set Ω_{xt_1}.

We pass to the proof of uniqueness in problem (1)–(3). Let $\{\bar{u}(x,t_1), \bar{p}(x,t_1)\}$ and $\{u^*(x,t_1), p^*(x,t_1)\}$ be two solutions of the boundary value problem (1)–(3), and let

$$u(x,t_1) = \bar{u}(x,t_1) - u^*(x,t_1),\; p(x,t_1) = \bar{p}(x,t_1) - p^*(x,t_1),$$

which according to (1)–(3) satisfy the following equation:

$$\left(\frac{\partial u}{\partial t_1}, w\right) + \nu a(u,w) + b(u,\bar{u},w) + b(\bar{u},u,w) - b(u,u,w) = 0.$$

If we take as the test function $w = u$, then we will have the equality

$$\frac{1}{2}\frac{d}{d\,t_1}\|u\|^2_{H_{t_1}} + \nu\|\nabla u\|^2_{H_{t_1}} = b(u,u,\bar{u}), \tag{45}$$

since $b(u,u,\bar{u}) = -b(u,\bar{u},u),\ b(\bar{u},u,u) = 0,\ b(u,u,u) = 0.$

Further, proceeding in the same way as in the proof of Lemma 3, from (45) we obtain

$$\frac{d}{d\,t_1}\|u\|^2_{H_{t_1}} \le K\|u\|^2_{H_{t_1}},$$

where K is a positive constant, and by Gronwall's lemma it follows that $u \equiv 0$, and thus the property of uniqueness is proved.

This completes the proof of the main result of the work formulated in the following theorem.

Conclusion

The results of the work can be generalized to the case when the section of the cone for each fixed t_1 can change according to the rule $r = \sqrt{x_1^2 + x_2^2} \le \varphi(t_1),\ t_1 \in [0, T_1],\ \varphi(0) = 0$, under some natural requirements for the function $\varphi(t_1)$. For example, the function $\varphi(t_1)$ must satisfy the following two conditions: 1^0. in a sufficiently short period of time $(0, t_1^*)$ the function $\varphi(t_1)$ could have the representation $\varphi(t_1) = \mu\, t_1$, where μ is the given positive constant (in our work it was equal to one); 2^0. on the interval $[t_1^*, T_1]$ the function $\varphi(t_1)$ would be continuously differentiable and possess the property of monotonicity, providing a one-to-one transformation from independent variables $\{x, t_1\}$ to variables $\{y, t\}$.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Burgers J.M. The nonlinear diffusion equation / J.M. Burgers. — Dordrecht: Reidel, 1974. — 174 p.

2 Batchelor G.K. An intronduction to fluid dynamics / G.K. Batchelor. — Cambridge: CUP, 2000. — 615 p.

3 Lions J.-L. Quelques methodes de resolution des problemes aux limites non lineaires / J.-L. Lions. — Paris: Dunod, 1969. — 554 p.

4 Temam R. Navier-Stokes equations. Theory and numerical analysis / R. Temam. — Amsterdam: North Holland Publishing Company, 1979. — 530 p.

5 Vishik M.I. Mathematical problems of statistical hydromechanics / M.I. Vishik, A.B. Fursikov. — Dordrecht: Springer, 1988. — 576 p.

6 Mitropol'ski Yu.A. Free boundary problems for nonlinear evolution equations in metallurgy, medicine, and ecology / Yu.A. Mitropol'ski, A.A. Berezovskii, T.A. Plotnizkii // Ukrainian mathematical journal. — 1992. — 44. — No. 1. — P. 67–75. https://doi.org/10.1007/BF01062627

7 Ким Е.И. Математические модели тепловых процессов в электрических контактах / Е.И. Ким, В.Т. Омельченко, С.Н. Харин. — Алма-Ата: Наука, 1977. — 236 с.

8 Веригин Н.Н. Об одном классе гидромеханических задач для областей с подвижными границами / Н.Н. Веригин // Динамика жидкости со свободными границами. Динамика сплошной среды. — 1980. — 46. — С. 23–32.

9 Карташов Э.М. Проблема теплового удара в областях с движущейся границей на основе новых интегральных соотношений / Э.М. Карташов // Изв. АН РФ. Энергетика. — 1997. — № 4. – С. 122–137.

10 Benia Y. Existence of solutions to Burgers equations in domains that can be transformed into rectangles / Y. Benia, B.-K. Sadallah // Electronic Journal of Differential Equations. — 2016. — 157. — P. 1–13.

11 Benia Y. Existence of solutions to Burgers equations in a non-parabolic domain / Y. Benia, B.-K. Sadallah // Electronic Journal of Differential Equations. — 2018. — 20. — P. 1–13.

12 Sadallah B.-K. Etude d'un probleme 2m-parabolique dans des domaines plan non rectangulaires / B.-K. Sadallah // Bollettino dell'Unione Matematica Italiana. — 1983. — 6. — No. 2-B. — P. 51–112.

13 Amangaliyeva M.M. On the existence of non-trivial solution to homogeneous Burgers equation in a corner domain / M.M. Amangaliyeva, M.T. Jenaliyev, M.I. Ramazanov // Abstracts of Int. Conf. "Mathematics in the Modern World". — Novosibirsk, 2017. — P. 187.

14 Jenaliyev M. On linear and nonlinear heat equations in degenerating domains / M. Jenaliyev, M. Ramazanov, M. Yergaliyev // AIP Conference Proceedings. — 2017. — 1910. — 040001-1–040001-10. https://doi.org/10.1063/1.5013968

15 Jenaliyev M.T. On Solonnikov-Fasano problem for the Burgers equation / M.T. Jenaliyev, M.I. Ramazanov, A.A. Assetov // Bulletin of the Karaganda University. Mathematics series. — 2020. — No. 2(98). — P. 69–83. https://doi.org/10.31489/2020m2/69-83

16 Amangaliyeva A.A. On the solvability of nonhomogeneous boundary value problem for the Burgers equation in the angular domain and related integral equations / M.M. Amangaliyeva, M.T. Jenaliyev, M.T. Kosmakova, M.I. Ramazanov // Springer Proceedings in Mathematics and Statistics. — 2017. — 216. — P. 123–141.

17 Kosmakova M.I. On the solution to a two-dimensional boundary value problem of heat conduction in a degenerating domain / M.T. Kosmakova, V.G. Romanovski, D.M. Akhmanova, Zh.M. Tuleutaeva, A.Yu. Bartashevich // Bulletin of the Karaganda University. Mathematics series. — 2020. — No. 2(98). — P. 100–109. https://doi.org/10.31489/2020m2/100-109

18 Kosmakova M.T. A fractionally loaded boundary value problem two-dimensional in the spatial variable / M.T. Kosmakova, K.A. Izhanova, L.Z. Kasymova // Bulletin of the Karaganda University. Mathematics series. — 2023. — No. 2(110). — P. 72–83. https://doi.org/10.31489/2023m2/72-83

19 Jenaliyev M.T. On the solvability of a boundary value problem for a two-dimensional system of

Navier-Stokes equations in a truncated cone / M.T. Jenaliyev, M.G. Yergaliyev // Lobachevskii Journal of Mathematics. — 2023. — 44. — No. 8. — P. 3307–3320. https://doi.org/10.1134/S199508022308022X

20 Lions J.-L. Problemes aux limites non homogenes et applications, Vol. 1 / J.-L. Lions, E. Magenes. — Dunod: Paris, 1968. — 372 p.

21 Adams R.A. Sobolev spaces (Second Ed.) / R.A. Adams, J.J.F. Fournier. — New York: Academic Press, 2003. — 320 p.

22 Brezis H. Functional analysis, Sobolev spaces and partial differential equations / H. Brezis. — New York: Springer, 2011. — 600 p.

23 Cattabriga L. Su un problema al contorno relativo al sistema di equazioni di Stokes / L. Cattabriga // Rendiconti del Seminario Matematico della Universita di Padova. — 1961. — 31. — P. 308–340.

Конустағы екiөлшемдi Навье-Стокс теңдеулерiнiң жүйесi үшiн қойылған шекаралық есептiң шешiмдiлiгi туралы

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Навье-Стокс теңдеулерi қолданбалы тұрғыдан маңызды көптеген қызықты есептердiң қойылуында кездесетiндiктен, бұл теңдеулер бiрнеше ондаған жылдар бойы механиктердiң, математиктердiң және басқа да ғалымдардың назарында болды. Бiрақ бұған қарамастан Навье-Стокс теңдеуiне арналған көптеген есептер осы күнге дейiн әлi де зерттелмеген. Жұмыста цилиндрлiк емес өзгешеленетiн облыстағы, атап айтқанда төбесi координаталардың басында орналасқан конуста екiөлшемдi Навье-Стокс жүйесi үшiн шекаралық есептiң шешiмдiлiгi зерттелген. Бұған дейiн осы есептiң сызықты Навье-Стокс жүйесi үшiн қойылуы немесе өзгешеленетiн емес цилиндрлiк облыстардағы қойылуы зерттелген, сондықтан бұл жұмыс осы бағыттағы алдыңғы зерттеулердiң логикалық жалғасы болып табылады. Жоғарыда аталған өзгешеленетiн облысқа өзгешеленетiн емес кесiлген конустар жиыны сәйкестiкке қойылады. Бұл облыстар өз кезегiнде цилиндрлiк облыстарға өзара бiрмәндi түрлендiру арқылы келтiрiледi. Бұдан кейiн қарастырылып отырған есеп үшiн облыс индексiнiң өзгеруiне қатысты бiртектi априорлық бағалаулары айқындалып, әрi қарай, априорлық бағалаулар мен Фаедо-Галеркин әдiсiн қолдана отырып, Соболев кластарындағы шешiмнiң бар және жалғыз екендiгiн дәлелдеп, берiлген функциялардың тегiстiгi артқан сайын оның регулярлығын анықтаған.

Кiлт сөздер: Навье-Стокс жүйесi, өзгешеленетiн облыс, Галеркин әдiсi.

О разрешимости одной граничной задачи для двумерной системы уравнений Навье-Стокса в конусе

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В силу того, что уравнения Навье-Стокса участвуют в постановках большого количества интересных задач, важных с прикладной точки зрения, данные уравнения в течение нескольких десятилетий подряд были объектом внимания механиков, математиков и других ученых. Но, несмотря на это, множество задач для уравнения Навье-Стокса остаются неисследованными и по сей день. В этой работе мы исследуем разрешимость граничной задачи для двумерной системы Навье-Стокса в нецилиндрической вырождающейся области, а именно в конусе с вершиной в начале координат. Ранее мы изучали случаи линеаризованной системы Навье-Стокса, или невырождающихся цилиндрических областей, поэтому данная работа является логическим продолжением наших предыдущих исследований в этом направлении. Вышеупомянутой вырождающейся области мы сопоставляем семейство невырождающихся усеченных конусов, которые, в свою очередь, формируются путем взаимооднозначного преобразования в цилиндрические области, где для рассматриваемой задачи устанавливаются априорные оценки, однородные относительно изменения индекса областей. Далее, используя априорные оценки и метод Фаэдо-Галеркина, мы установили существование, единственность решения в классах Соболева и его регулярность по мере увеличения гладкости заданных функций.

Ключевые слова: система Навье-Стокса, вырождающаяся область, метод Галеркина.

References

1 Burgers, J.M. (1974). *The nonlinear diffusion equation.* Dordrecht: Reidel.

2 Batchelor, G.K. (2000). *An intronduction to fluid dynamics.* Cambridge: CUP.

3 Lions, J.-L. (1969). *Quelques methodes de resolution des problemes aux limites non lineaires.* Paris: Dunod.

4 Temam, R. (1979). *Navier-Stokes equations. Theory and numerical analysis.* Amsterdam: North Holland Publishing Company.

5 Vishik, M.I., & Fursikov, A.B. (1988). *Mathematical problems of statistical hydromechanics.* Dordrecht: Springer.

6 Mitropol'ski, Yu.A., Berezovskii, A.A., & Plotnizkii, T.A. (1992). Free boundary problems for nonlinear evolution equations in metallurgy, medicine, and ecology. *Ukrainian mathematical journal, 44*(1), 67–75.

7 Kim, E.I., Omelchenko, V.T., & Kharin, S.N. (1977). *Matematicheskie modeli teplovykh protsessov v elektricheskikh kontaktakh [Mathematical models of thermal processes in electrical contacts].* Alma-Ata: Nauka [in Russian].

8 Verigin, N.N. (1980). Ob odnom klasse gidromekhanicheskikh zadach dlia oblastei s podvizhnymi granitsami [On a class of hydromechanical problems for areas with moving boundaries]. *Dinamika zhidkosti so svobodnymi granitsami. Dinamika sploshnoi sredy — Dynamics of a liquid with free boundaries. Dynamics of a solid medium, 46*, 23–32 [in Russian].

9 Kartashov, E.M. (1997). Problema teplovogo udara v oblastiakh s dvizhushcheisia granitsei na osnove novykh integralnykh sootnoshenii [The problem of heat stroke in domains with a moving boundary based on new integral relations]. *Izvestiia Akademii nauk Rossiiskoi Federatsii. Energetika — Proceedings of the Academy of Sciences of Russian Federation. Energetics, 4*, 122–137 [in Russian].

10 Benia, Y., & Sadallah, B.-K. (2016). Existence of solutions to Burgers equations in domains that can be transformed into rectangles. *Electronic Journal of Differential Equations, 157*, 1–13.

11 Benia, Y., & Sadallah, B.-K. (2018). Existence of solutions to Burgers equations in a non-parabolic domain. *Electronic Journal of Differential Equations, 20*, 1–13.

12 Sadallah, B.-K. (1983). Etude d'un probleme 2m-parabolique dans des domaines plan non rectangulaires. *Bollettino dell'Unione Matematica Italiana, 6*(2-B), 51–112.

13 Amangaliyeva, M.M., Jenaliyev, M.T., & Ramazanov, M.I. (2017). On the existence of nontrivial solution to homogeneous Burgers equation in a corner domain. *Abstracts of Int. Conf. "Mathematics in the Modern World"*, 187.

14 Jenaliyev, M., Ramazanov, M., & Yergaliyev, M. (2017). On linear and nonlinear heat equations in degenerating domains. *AIP Conference Proceedings,* 1910.

15 Jenaliyev, M.T., Ramazanov, M.I., & Assetov, A.A. (2020). On Solonnikov-Fasano problem for the Burgers equation. *Bulletin of the Karaganda University. Mathematics series,* 2(98), 69–83. https://doi.org/10.31489/2020m2/69-83

16 Amangaliyeva, A.A., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2017). On the solvability of nonhomogeneous boundary value problem for the Burgers equation in the angular domain and related integral equations. *Springer Proceedings in Mathematics and Statistics, 216*, 123–141.

17 Kosmakova, M.I., Romanovski, V.G., Akhmanova, D.M., Tuleutaeva, Zh.M. & Bartashevich A.Yu. (2020). On the solution to a two-dimensional boundary value problem of heat conduction in a degenerating domain. *Bulletin of the Karaganda University. Mathematics series,* 2(98), 100–109. https://doi.org/10.31489/2020m2/100-109

18 Kosmakova, M.T., Izhanova, K.A., & Kasymova, L.Z. (2023). A fractionally loaded boundary value problem two-dimensional in the spatial variable. *Bulletin of the Karaganda University. Mathematics series,* 2(110), 72–83. https://doi.org/10.31489/2023m2/72-83

19 Jenaliyev, M.T., & Yergaliyev, M.G. (2023). On the solvability of a boundary value problem for a two-dimensional system of Navier-Stokes equations in a truncated cone. *Lobachevskii Journal of Mathematics, 44*(8), 3307–3320. https://doi.org/10.1134/S199508022308022X

20 Lions, J.-L., & Magenes, E. (1968). *Problemes aux limites non homogenes et applications.* Paris: Dunod.

21 Adams, R.A., & Fournier, J.J.F. (2003). *Sobolev spaces (Second Ed.).* New York: Academic Press.

22 Brezis, H. (2011). *Functional analysis, Sobolev spaces and partial differential equations.* New York: Springer.

23 Cattabriga, L. (1961). Su un problema al contorno relativo al sistema di equazioni di Stokes. *Rendiconti del Seminario Matematico della Universita di Padova, 31*, 308–340.

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