https://doi.org/10.31489/2024M1/73-83 *Research article*

Embeddings of a Multi-Weighted Anisotropic Sobolev Type Space

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Parameters such as various integral and differential characteristics of functions, smoothness properties of regions and their boundaries, as well as many classes of weight functions cause complex relationships and embedding conditions for multi-weighted anisotropic Sobolev type spaces. The desire not to restrict these parameters leads to the development of new approaches based on the introduction of alternative definitions of spaces and norms in them or on special localization methods. This article examines the embeddings of multi-weighted anisotropic Sobolev type spaces with anisotropy in all the defining characteristics of the norm of space, including differential indices, summability indices, as well as weight coefficients. The applied localization method made it possible to obtain an embedding for the case of an arbitrary domain and weights of a general type, which is important in applications in differential operators' theory, numerical analysis.

Keywords: Anisotropic Sobolev Spaces, Multi-Weighted Spaces, Embedding Theorems, Localization Methods, Weighted Functions.

2020 Mathematics Subject Classification: 46E35.

Introduction

In the article embeddings of multi-weighted anisotropic Sobolev type spaces $W^{\bar{l}}_{\bar{p},p}(G,\bar{\rho},\upsilon)$ described by a finite norm

$$|u; W^{\bar{l}}_{\bar{p},p}(G,\bar{\rho},\upsilon)| = \sum_{i=1}^{n}\Big(\int\limits_G |D_i^{l_i}u|_i^p \rho_i dx\Big)^{1/p_i} + \Big(\int\limits_G |u|^p \upsilon dx\Big)^{1/p}$$

are investigated in the case when $\rho_1, ..., \rho_n$ and υ are connected by certain relations on average on parallelepipeds in G with an adjustable edge length.

The article extensively utilizes approaches developed in the works [1–5]. Nonetheless, they have enabled a slight expansion of the class of weights for which the considered embedding is valid. As before [5], the spaces are anisotropic in terms of derivative orders, integrability indices, and weight factors for these derivatives. Also, for the introduced class of weights, the localization method [1–4] allows not imposing conditions on the domains, in which the spaces are considered and embeddings are set. So, the conditions under which the embeddings (1) occur for a sufficiently broad class of weights, restricted by a special condition $\Pi_{(\delta,\,\varepsilon)}$ introduced in Definition 1 are studied. The localization method (Lemma 3) with the introduction of so-called "characteristic parallelepipeds" (3) allows considering the domain G, with no additional conditions imposed.

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This work is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP15473253).

Received: 21 September 2023; *Accepted:* 11 December 2023.

For domains with conditions, fundamental embedding results have been obtained for anisotropic Sobolev spaces [6, 7]; as for weighted cases, in [8–10] the weights are functions of the distance to the boundary, and inequalities in [8] are considered in cylindrical domains, in [9] — on a domain with cusp, and in [10] an open connected domain is considered with various conditions. In [11], along with other findings, an anisotropic Sobolev inequality is obtained for smooth bounded domains and the class of p-admissible weights. In the work [12], Sobolev spaces are considered in open domains with certain conditions of smoothness imposed on the functions introduced in them, anisotropy in terms of derivative orders, and integrable classes of weights; alternative descriptions of these spaces are presented, including norms and properties of the density of smooth functions within them, and the relationships between the weights and anisotropic properties of Sobolev spaces are described. In [13], alternative definitions of spaces are introduced, through which embeddings of weighted spaces are obtained. Here, we also present studies of anisotropic Sobolev type spaces [14, 15], and their embeddings [16–18].

1 Set up

Let us introduce the notation. Let R^n be an n-dimensional arithmetic space with a norm

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \quad x = (x_1, ..., x_n).$$

Denote by

$$\bar{l} = (l_1, \ldots, l_n), \quad \bar{a} = (a_1, \ldots, a_n), \quad \text{and } \bar{b} = (b_1, \ldots, b_n)$$

fixed vectors with coordinates $l_i, a_i > 0, b_i \geq 0,\ \ i = 1, \ldots, n$. Set for $\lambda > 0$:

$$\bar{a} \pm \lambda = (a_1 \pm \lambda, \ldots, a_n \pm \lambda), \lambda\bar{a} = (\lambda_1 a_1, \ldots, \lambda_n a_n),$$

$$\bar{b} \ : \ \bar{a} = (b_1/a_1, \ldots, b_n/a_n), \bar{b}\bar{a} = (b_1 a_1, \ldots, b_n a_n),$$

$$\lambda^{\bar{b}} = (\lambda^{b_1}, \ldots, \lambda^{b_n}), \bar{a}^{\lambda} = (a_1^{\lambda}, \ldots, a_n^{\lambda_n}), \bar{a}^{\bar{b}} = (a_1^{b_1}, \ldots, a_n^{b_n}),$$

$$|b| = \sum_{i=1}^{n} b_i, \ \prod \bar{b} = \prod_{i=1}^{n} b_i, \ \sum \bar{b} = \sum_{i=1}^{n} b_i.$$

For a multi-index $\alpha = (\alpha_1, ..., \alpha_n)$ $D^\alpha = \frac{\partial^{\alpha_1 + ... + \alpha_n}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$, for integers l_i $D_i^{l_i} u = \frac{\partial^{l_i}}{\partial x_i^{l_i}} u$.

Let $L(G;\, loc)$ be the space of locally-summable functions, $L_q^\alpha(G;\, \omega)$ is the Lebesgue weighted space of functions $u(\cdot)$, with a finite semi-norm

$$\left|u; L_q^\alpha(G;\omega)\right| = \left|D^\alpha u; L_q\ (G;\omega)\right| = \left(\int_G |D^\alpha u|^q \omega(x) dx\right)^{1/q},$$

where $1 \leq q < \infty$,

$$D^\alpha u = \frac{\partial^\alpha u}{\partial x^\alpha} = \frac{\partial^{\alpha_1 + ... + \alpha_n} u}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

are mixed derivatives corresponding to the multi-index $\alpha = \alpha_1, ..., \alpha_n$. We denote the class

$$C^\infty W = \{u \in C^\infty \ \ : |u; W_{\bar{p}, p}^{\bar{l}}(G;\, \bar{\rho}, \upsilon)| < \infty\}$$

by $C^\infty W$.

We consider issues related to the description of the conditions under which the embedding takes place:

$$W^{\bar{l}}_{\bar{p},\,p}(G;\,\bar{\rho},\upsilon) \to L^{\alpha}_{q}(G;\omega). \tag{1}$$

The embedding (1) is revealed through the embedding inequality as follows:

$$\left(\int_G |D^{\alpha}u|^q \omega(x)dx\right)^{1/q} \leq C|u; W^{\bar{l}}_{\bar{p},\,p}(G;\,\bar{\rho},\upsilon)|, u \in C^{\infty}W.$$

This article discusses the embeddings of spaces $W^{\bar{l}}_{\bar{p},\,p}\,(G;\,\bar{\rho},\,\upsilon)$ in the case when $\kappa_{\alpha} = \left|(1+\alpha) : \bar{l}\right| > 1$.

Further, by introducing weights satisfying multiplicative boundedness conditions on average on parallelepipeds (Definition 1), we obtain embeddings of multiweighted anisotropic Sobolev type spaces $W^{\bar{l}}_{\bar{p},p}(G,\bar{\rho},\upsilon)$ into the weighted space $L^{\alpha}_{q}(G;\,\omega)$ on a domain G with irregular geometry, and anisotropy is present in the orders of derivatives, in terms of summability and in weight multipliers for these derivatives.

2 Preliminaries

Let $a = (a_1, ..., a_n)$, $a_i > 0$. We denote the parallelepiped

$$Q_a = Q_a(x)\{y \in R^n \ \ : |y_i - x_i| < \frac{a_i}{2},\ i = 1, 2, \ldots, n\},$$

by $Q_a = Q_a(x)$. For $\lambda > 0$ let $\lambda Q = \lambda Q_a = Q_{\lambda a.}$

We will consider a positive vector function $\bar{d}(x) = (d_1(x), ..., \, d_n(x))$, satisfying the conditions:

1) $Q(x) = Q_{\bar{d}(x)}(x) \subset G$;

2) $\sup\limits_{G} d_i(x) < \infty$, $i = 1, ..., n$;

3) There exist the constants $0 < \varepsilon < 1$ and $b_0 > 1$:

$$\forall i = 1, ..., n \quad b_0^{-1} \leq \frac{d_i(y)}{d_i(x)} \leq b_0$$

as soon as

$$Q_{(\varepsilon)}(y) \cap Q_{(\varepsilon)}(x) \neq \emptyset,$$

where $Q_{(\varepsilon)}(x) = (1-\varepsilon)Q(x)$. We call the function $\bar{d}(x)$ by the edge length function. Let $\{Q(x), x \in G\}$ be the parallelepiped family

$$Q(x) = \{y \in R^n \ \ : |y_i - x_i| < \frac{d_i(x)}{2},\ \ i = 1, 2, \ldots, n\},$$

satisfying the conditions 1)–3). Let $\rho_i(x)$, $\omega(x)$, $\upsilon(x)$ be the weights in G, and

$$\sigma_i(x) = \tilde{\rho}_i(x) d_i^{l_i p_i'}(x) \in L(G; loc),$$

$$\tilde{\rho}_i = \rho_i^{1-p_i'} (p_i' = \frac{p_i}{p_i - 1}), 1 < p_i < \infty \ (i = 1, ..., n).$$

Suppose

$$B(x) = \left[\prod_{i=1}^{n} \left(\int\limits_{Q_{(\varepsilon)}(x)} \sigma_i\right)^{1/p_i'}\right]^{1/n} |Q_{\varepsilon}(x)|^{-1},$$

$$N_{(\delta)}(Q)\{e \subset Q \ : \ |e| \leq \delta|Q|\}, \ 0 \leq \delta < 1.$$

Let

$$M_{(\varepsilon,\delta)}(x) = B(x) \inf_{e \in N_{(\delta)}(Q_{(\varepsilon)}(x))} \left(\int\limits_{Q_{(\varepsilon)}(x) \setminus e} \upsilon \right)^{1/p}, \ \varepsilon \in [0,1).$$

Definition 1. [5] We say that the weight pair $(\bar{\rho}, \upsilon)$ satisfies the condition $\Pi_{(\delta,\varepsilon)}$ with respect to $\bar{d}(x) = (d_1(x), ..., d_n(x))$, if there are numbers $\delta \in [0,1)$ and $\varepsilon \in [0,1)$ such that

$$M_{(\varepsilon,\delta)}(x) \geq 1 \text{ for a.a. } x \in G.$$

Short notation:

$$(\bar{\rho}, \upsilon) \in \Pi_{(\delta,\varepsilon)}.$$

At the same time $0 < \varepsilon < 1$, if $G \subset R^n$, $G \neq R^n$.

Lemma 1. [3] Let $1 < p_i, p < q < \infty$, $(i = 1, ..., n)$, $r = \min\{p_1, ..., p_n, p\}$, $\omega \in L_1(\nabla)$, where $\xi \in \nabla = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$, $\omega \geq 0$ and

$$M^{1/q} = \sup_{\substack{t>0 \\ Q_{(t,\bar{l})} \subset \nabla}} t^{1-\kappa} |Q_{(t,\bar{l})}|_{\omega}^{1/q} < \infty.$$

Then

$$\left(\int\limits_{\nabla} |D^{\alpha} f|^q \omega(\xi) d\xi \right)^{1/q} \leq c M^{1/q} \|f\|_{W_r^{\bar{l}}(\nabla)},$$

where c does not depend on $f \in C^{\infty}(\bar{\nabla})$.

Let $\left\{\bar{P}(x), \, x \in E\right\}$ be the closed parallelepiped family

$$\bar{P}(x) = \{y \in R^n \ : \ |y_i - x_i| \leq a_i(x)/2, \ \ i = 1, ..., n\}, \tag{2}$$

where $a(x) = (a_1(x), \ldots, a_n(x))$ is a a positive vector function defined on a bounded set E in R^n.

Theorem 1. Let E be a bounded set in R^n, $\left\{\bar{P}(x), x \in E\right\}$ is the closed parallelepiped family (2), satisfying the conditions:

1) $\sup\limits_{x \in E} a_i(x) < \infty, \ (i = 1, \ldots n)$;

2) there is a number $c > 0$ such that

$$c^{-1} \leq a_i(y)/a_i(x) \leq c \ (i = 1, \ldots n),$$

as soon as

$$\bar{Q}(x) \cap \bar{P}(y) \neq \emptyset.$$

Then it is possible to distinguish from $\{\bar{P}(x), \ x \in E\}$ no more than a countable subfamily $\{\bar{P}^j\}_{j \in J}$, $\bar{P}^j = \bar{P}(x^j)$, such that:

a) $E \subset \bigcup_{j \in J} \bar{P}^j$;

b) $\sum_{j \in J} X_{\bar{P}^j}(x) \leq \kappa_1 = \kappa_1(c,n) < \infty$ for any x in R^n;

c) $\{\bar{P}^j\}_{j \in J}$ is represented as a union of no more than $\kappa_2 = \kappa_1 + 1$ subfamilies $\{\bar{P}^j\}_{j \in J_v}$ of pairwise disjoint parallelepipeds.

The cover $\{\bar{P}^j\}$ of the set E, which has the properties of finite multiplicity and finite separability (respectively, properties b) and c)), in Theorem 1, we will call B-covering.

Let $X_i = X_i(G)$ $(i = 1, 2)$ be spaces of functions defined in G, with the seminorms $\|\cdot\|_{X_i(G)}$, $X_i(G)$ is the space of functions $X_i(G)/G$ with an induced seminorm $\|\cdot\|_{X_i(G)}$.

Lemma 2. [5] Let the spaces X_i $(i = 1, 2)$ meet the following conditions:

i1) $C^\infty(\bar{Q}) \subset X_i$, $Q \in I^n$ such that $\bar{Q} \subset G$;

i2) $\|f\|_{X_i(G_1)} \le \|f\|_{X_i(G_2)}$, if $G_1 \subset G_2 \subset G$;

$$\|f\|_{X_i(G_k)} = \lim_{k\to\infty} \|f\|_{X_i(G_k)},$$

if

$$G_k \subset G_{k+1}\ (k \ge 1),\ G = \bigcup_1^\infty G_k \subset G;$$

i3) $\|f\|_{X_i(G\setminus e)} = \|f\|_{X_i(G)}$, if $|e| = 0$;

i4) There are numbers $s_i \ge 1,\ c_i \ge 1$ $(i = 1, 2)$ such that for any family

$$G = \bigcup_j G_j \subset G$$

$\{G_i\}$ of open sets such that

$$\|f\|^{s_1}_{X_i(G)} \le c_1^{s_1} \sum_j \|f\|^{s_1}_{X_1(G_j)},\ \ f \in X_1(G)\, \|f\|^{s_2}_{X_2(G_j)} \le c_2^{s_2} \|f\|^{s_2}_{X_2(G)},$$

if G_j do not intersect in pairs with $f \in X_2(G)$;

i5) There are such a parallelepiped family $\{P(x),\, x \in G\}$ and a positive function $K(x)$ on G, that

$$\|f\|_{X_1(Q(x))} \le cK(x)\, \|f\|_{X(Q(x))}\ \ \forall f \in C^\infty(\bar{Q}(x));$$

i6) From the family $\{P(x),\, x \in G\}$, we can distinguish B-covers $E = G \cap B(x, r)$, multiplicity κ_1 and coefficients of finite separability κ_2, which do not depend on r. Then we have a true inequality

$$\|f\|_{X_1} \le cK\, \|f\|_{X_2},\ f \in C^\infty(G) \cap X_2,$$

where

$$K = K_{s_1 s_2} = \begin{cases} \sup\limits_{x\in G} K(x), & \text{where } s_2 \le s_1 \\ \sup\left(\sum_{j\in J}\left(K(x^j)\right)^{s_1 s_2/(s_2-s_1)}\right)^{(s_2-s_1)/(s_1 s_2)}, & \text{where } s_2 > s_1 \end{cases}$$

and sup is taken over all at most countable families of $\{Q^j\}_{j\in J}$ pairwise non-intersecting parallelepipeds $Q^j = Q(x^j)$.

3 Localization and embedding theorems

Below we will consider "characteristic parallelepipeds" of the form

$$Q\,(x) = Q^\tau_{\bar{\rho}}\,(x) = \left\{ y \in R^n : |y_i - x_i| < \frac{(\tau\,(x)\,\rho_i\,(x))^{1/l_i}}{2},\ i = 1, 2, ..., n \right\}, \tag{3}$$

where $\tau\,(x)$ such a positive function in G that the functions

$$d_i\,(x) = (\tau\,(x)\ \rho_i\,(x))^{1/l_i} \le 1$$

and satisfy the conditions 1)–3). With respect to the functions ρ_i $(i = 1, 2, ..., n)$ we will assume that

$$c_1 \le \frac{\rho_i(y)}{\rho_i(x)} \le c_2, \quad \text{if} \quad y \in Q^{\tau}_{(\varepsilon)}(x) \ (i = 1, ..., n)$$

and we introduce the following values connecting to the "characteristic parallelepipeds" $Q(x) = Q^{\tau}_{\bar{\rho}}(x)$ of the weight of ρ_i, ω and numeric parameters p_i, p, q, $\alpha = (\alpha_1, ..., \alpha_n)$ and $\bar{l} = (l_1, ..., l_n)$; namely, let us put

$$A^{\tau}_{\bar{p},p,q}(x|\bar{\rho},\omega) = \sup_{t>0}\left\{ t^{1-\kappa}\left(\int_{\bar{p}} \omega\right)^{1/q}, p_t = ((1-\varepsilon)\ t)^{1/\bar{l}}\, Q_{\bar{\rho}^{1/\bar{l}}} \subset (1-\varepsilon)^{1/\bar{l}}\, Q(x) \right\},$$

$$K^{\tau}_{\bar{p},p,q}(x|\bar{\rho},\omega) = \left[\prod_{i=1}^{n}\left(\tau(x)^{1-\frac{1}{p}-\alpha_i-\frac{1}{r}}\right)^{1/l_i}\right] A^{\tau}_{\bar{p},p,q}(x|\bar{\rho},\omega).$$

Lemma 3. [3–5] Let $1 < p_i, p < q < \infty$ $(i = 1, ..., n)$, $r = \min\limits_{1\le i\le n}(p_i, p)$, and the conditions are met:

$$M^{r}_{(\delta,\varepsilon)}(x|\bar{\rho},\tau(\cdot)) \ge 1$$

and

$$\mathbf{K}_{\bar{p},p,q}(x|\bar{\rho},\omega) < \infty.$$

Then

$$\left(\int_{Q} |D^{\alpha}u|^{q}\,\omega\right)^{1/q} \le c\mathbf{K}^{\alpha}_{(\varepsilon),\bar{p},q}(x|\bar{\rho},\omega)\left[\sum_{i=1}^{n}\left(\int_{Q}\left|\rho_i D^{l_i}u\right|^{p_i}\right)^{\frac{1}{p_i}} + \left(\int_{Q}|\upsilon u|^{p}\right)^{\frac{1}{p}}\right],$$

where $Q = Q_{(\varepsilon)}(x) = (1-\varepsilon)^{1/\bar{l}}\, Q^{\tau}_{\bar{\rho}}(x)$.

Proof. Suppose $f(\zeta) = u\left(x + (1-\varepsilon)^{1/\bar{l}}\, d(x)\cdot\zeta\right)$, where

$$\bar{d}(x) = (d_1(x), ..., d_n(x)) \subset d_i(x) = (\tau(x)\,\rho_i(x))^{1/l_i}, d_i = (1-\varepsilon)^{1/l_i}\, d_i(x).$$

Let

$$\hat{\omega}(\zeta) = \omega\left(x + \bar{d}\cdot\zeta\right), \bar{d} = (d_1, ..., d_n), Q = Q_{(\varepsilon)}(x).$$

Then by virtue of Lemma 1

$$\left(\int_{Q_{(\varepsilon)}(x)} |D^{\alpha}u|^{q}\omega\right)^{1/q} = |Q|^{1/q}\left(\prod_{i=1}^{n} d_i^{-\alpha_i}\right)\left(\int_{\nabla}|D^{\alpha}f|^{q}\hat{\omega}(\xi)\,d\xi\right)^{1/q} \le$$

$$\le \left(M\frac{|\alpha|}{\prod_{i=1}^{n} d_i^{q\alpha_i}}\right)^{\frac{1}{q}}\left[\sum_{i=1}^{n}\left(\int_{\nabla}\left|D^{l_i}f\right|^{r}d\xi\right)^{1/r} + \left(\int_{\nabla}|f|^{r}d\xi\right)^{1/r}\right] \le$$

$$\le \left(\frac{M}{\prod_{i=1}^{n} d_i^{q\alpha_i}}\right)^{\frac{1}{q}}|Q|^{\frac{1}{q}-\frac{1}{r}}\left\{\sum_{i=1}^{n}|Q|^{\frac{1}{r}}\left(\int_{Q}\left|D^{l_i}f\right|^{p_i}\right)^{\frac{1}{p_i}} + \left(\int_{Q}|\omega|^{r}dy\right)^{1/r}\right\} \le$$

$$\le \left(\frac{M}{\prod_{i=1}^{n} d_i^{q\alpha_i}}\right)^{\frac{1}{q}}|Q|^{\frac{1}{q}-\frac{1}{r}}\tau(x)\left[\sum_{i=1}^{n}|Q|^{\frac{1}{r}-\frac{1}{p_i}}\zeta_i(x)\left\|D^{l_i}u\right\|_{p_iQ} + \tau(x)^{-1}\left(\int_{Q}|u|^{r}\right)^{1/r}\right] \le$$

$$\leq \left(\frac{M}{\prod_{i=1}^{n} d_i^{q\alpha_i}}\right)^{\frac{1}{q}} |Q|^{\frac{1}{q}-\frac{1}{r}} \left\|u; W^{\hat{l}}_{\hat{p},p}\left(Q;\bar{\rho},\upsilon\right)\right\|, \text{where} \bar{\rho} = \left(\rho_1^{p_1},...,\rho_n^{p_n}\right).$$

It remains to note that

$$\left(\frac{M}{\prod\limits_{i=1}^{n} d_i^{q\alpha_i}}\right)^{\frac{1}{q}} |Q|^{\frac{1}{q}-\frac{1}{r}} \tau\left(x\right) = \tau\left(x\right)\prod_{i=1}^{n} d_i^{-\alpha_i} |Q|^{\frac{1}{q}-\frac{1}{r}} \sup_{Q_{(t,\bar{l})\subset ?}} t^{1-\kappa}\left|Q_{(t,\bar{l})}\right|_{\hat{\omega}}^{1/q} =$$

$$= \tau\left(x\right)\prod_{i=1}^{n} d_i^{-\alpha_i} |Q|^{-\frac{1}{r}} \left(\tau\left(x\right)^{x-1}\sup_{t>0}\left\{t^{1-\kappa}\left(\int\limits_{\bar{p}}\omega\right)^{1/q}, p_t = \left(\left(1-\varepsilon\right)t\right)^{1/\bar{l}} Q_{\bar{\rho}^{1/\bar{l}}} \subset Q\right\}\right) =$$

$$= \prod_{i=1}^{n}\left(\tau\left(x\right)^{1-\frac{1}{r}} \rho_i\left(x\right)^{-\alpha_i-\frac{1}{r}}\right)^{1/l_i} A^{\tau}_{\bar{p},p,q}\left(x|\bar{\rho},\upsilon\right) = \mathbf{K}^{\tau}_{\bar{p},p,q,\alpha}(x|\bar{\rho},\omega).$$

Theorem 2. Let $1 < p_i, p < q < \infty \, (i = 1,...,n)$, $\alpha \in Z_+^n$, $\rho^{\bar{p}} = \left(\rho_1^{\bar{p}_1},...,\rho_n^{\bar{p}_n}\right)$. Let $(\rho^{\bar{p}},\upsilon) \in \prod_{(\delta,\varepsilon)}^r$ and

$$K = \sup_{x\in G} K^{\tau}_{\bar{p},p,q,\alpha}(x|\bar{\rho},r) < \infty.$$

Then on the class $C^\infty W$ the inequality

$$\left(\int_G |D^\alpha u|^q \omega(x)dx\right)^{1/q} \leq C\left|u; W^{\hat{l}}_{\hat{p},p}(G;\bar{\rho},\upsilon)\right|, u \in C^\infty W$$

is valid with an exact constant $C \leq cK$.

Proof. It follows from Lemma 2 and Theorem 1 that a pair of spaces $X_1 = L^\alpha_q(G;\omega)$, $X_2 = (W^{\hat{l}}_{\hat{p},p}(G;\bar{\rho}\upsilon))$ satisfies all the requirements of Lemma 2, from which the statement of the theorem follows.

Corollary 1. Let $1 < p_i, p < q < \infty \, (i = 1,...,n)$, $\kappa = \left|1 : \bar{l}\right| \leq 1$, ω is the weight on R^n, which satisfies the conditions of uniform boundedness on unit cubes $Q = Q_1(x) = x + \nabla$:

$$K = \sup_x \left(\int_{Q_1(x)} \omega\right)^{1/q} < \infty.$$

Then

$$\left(\int_G |u|^q \omega(x)dx\right)^{1/q} \leq cK\|u; W^{\bar{i}}_{\bar{p},p}(R^n)\|, u \in C^\infty W.$$

Let $\overline{\mu} = (m_1,...,\mu_n) - \infty < m_i, v < \infty (i = 1,...n)$,

$$W^{\bar{i}}_{\bar{p},p}(\overline{\mu},v) = W^{\bar{i}}_{\bar{p},p}(R^n;\overline{\rho},v)$$

with

$$\rho_i(x) = (1+|x|)^\mu, v(x) = (1+|x|)^v.$$

Corollary 2. Let $1 < p_i, p < q < \infty\, (i = 1, ..., n)$,$\kappa = \left|1 : \bar{l}\right| \leq 1$, $-\infty < \mu_i, v, \gamma < \infty\, (i = 1, ..., n)$, and let the conditions be met:

$$\frac{v}{p} - \frac{1}{n}\sum_{i=n}^{n}\frac{\mu_i}{p_i} \leq 0, \frac{\gamma}{q} \leq \min_{i<i<n}\frac{\mu_i}{p_i}.$$

Then the inequality is true

$$\left(\int_G |u|^q (1+|x|)^\gamma dx\right)^{1/q} \leq C\|u; W^{\bar{l}}_{\overline{p},p}(\overline{\mu}, v)\|,\ u \in C^\infty W.$$

Embeddings of weighted anisotropic Sobolev type spaces are relevant in applications where it is necessary to consider the heterogeneity of the medium or the complex geometry of the domain, such as in numerical methods for solving differential equations and in the theory of differential operators. Localization methods, in particular, the norms of embeddings on cubes with variable edge length considered in the work, can be applied for embeddings of more complex spaces, including fractional ones. Considering weights that satisfy multiplicative conditions of boundedness on average in parallelepipeds is particularly important for analyzing functions in multidimensional spaces with irregular geometries.

Acknowledgments

This work is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP15473253).

Author Contributions

G.Sh. Iskakova collected and analyzed data, led manuscript preparation, conducted the main research process. M.S. Aitenova and A.K. Sexenbayeva assisted in data collection and analysis. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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Соболев типті көпсалмақты анизотроптық кеңістіктің енгізулері

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Функциялардың әртүрлі интегралды-дифференциалдық сипаттамалары, облыстар мен олардың шекараларының тегістігінің қасиеттері, сондай-ақ салмақтық функцияларының көптеген кластары сияқты параметрлер Соболев типіндегі көп салмақтық анизотропты кеңістіктердің күрделі өзара байланыстары мен енгізу шарттарын анықтайды. Бұл параметрлерді шектемеуге деген ұмтылыс жаңа тәсілдердің дамуына әкеледі, олардағы кеңістіктер мен нормаларды анықтаудың балама нұсқаларын енгізуге немесе оқшаулаудың арнайы әдістеріне негізделген. Мақалада көп салмақтық дифференциалдық индекстерді, қосындылану индекстерін қоса алғанда, кеңістік нормасының барлық айқындаушы сипаттамаларында анизотропиясы бар, сондай-ақ салмақ коэффициенттері бар Соболев типті анизотропты кеңістіктердің енгізілуі зерттелген. Қолданылған локализация әдісі дифференциалдық операторлар теориясында және сандық талдаудың қосымшаларында маңызды болып табылатын кез келген облыс пен жалпы типтегі салмақ үшін енгізгенді алуға мүмкіндік береді.

Кілт сөздер: анизотроптық Соболев кеңістіктері, көп салмақтық кеңістіктер, енгізу теоремалары, локализация әдістері, салмақтық функциялар.

Вложения многовесового анизотропного пространства типа Соболева

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Такие параметры, как различные интегро-дифференциальные характеристики функций, свойства гладкости областей и их границ, а также множества классов весовых функций, обусловливают сложные взаимосвязи и условия вложений многовесовых анизотропных пространств типа Соболева. Стремление не ограничивать эти параметры приводит к развитию новых подходов, основанных на введении альтернативных вариантов определений пространств и норм в них либо на специальных методах локализации. В этой статье исследованы вложения многовесовых анизотропных пространств типа Соболева с анизотропией во всех определяющих характеристиках нормы пространства, включая дифференциальные индексы, индексы суммируемости, а также весовые коэффициенты. Примененный метод локализации позволил получить вложение для случая произвольной области и весов общего типа, что важно в приложениях в теории дифференциальных операторов и численном анализе.

Ключевые слова: анизотропные пространства Соболева, многовесовые пространства, теоремы вложения, методы локализации, весовые функции.

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