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Problem for differential-algebraic equations with a significant loads

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In this article, the problem for a differential-algebraic equation with a significant loads is studied. Unlike previously studied problems for differential equations with a significant loads, in the considered equation, there is a matrix in the left part with a derivative that is not invertible. Therefore, the system of equations includes both differential and algebraic equations. To solve the problem, we propose a modification of the Dzhumabaev's parametrization method. The considered problem is reduced to a parametric problem for the differential-algebraic equation with significant loads. We apply the Weierstrass canonical form to this problem. We obtain parametric initial value problem for a differential equations and an algebraic equations with a significant loads. The solvability conditions for the considered problem are established.

Keywords: differential-algebraic equations, equations with significant loads, parameter, parametric initial value problem, solution.

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Introduction

Differential equations with significant loads are equations that describe how a system changes over time, taking into account significant external influences or forces, known as "loads". These external influences could represent various factors such as external forces, environmental conditions, or other external factors that affect the behavior of the system.

In the context of scientific and engineering applications, these equations are often used to model dynamic systems where the behavior is influenced by external factors. For example, in the study of diffusion processes, soil moisture dynamics, or the spread of infections, the differential equations with significant loads would mathematically represent how the system evolves over time, considering the impact of external loads on the system's dynamics.

The solutions to these differential equations provide insights into the behavior of the system under the influence of these significant loads, helping researchers and scientists understand and predict the system's evolution over time. The study of such equations is essential in various fields, including physics, biology, engineering, and environmental science [1–17].

Solving differential equations with significant loads can be challenging, and the specific methods you choose depend on the characteristics of the problem. Here are some general approaches: 1) Analytical methods — they include Separation of variables, Integrating factors, Exact equations; 2) Numerical methods — they include Euler's method, Runge-Kutta methods, Finite difference methods; 3) Series solutions — they include Power series. This method is useful for solving linear differential equations with variable coefficients; 4) Transform methods — they include Laplace transform. The Laplace transform

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can simplify the differential equation into an algebraic equation, making it easier to solve; 5) Numerical simulation — they are Finite element method, Boundary element method; 6) Special functions — they include Bessel Functions, Legendre Polynomials, etc.; 7) Computer algebra systems and software — they include Mathematica, MATLAB, or Python with libraries like SciPy to numerically solve differential equations or perform symbolic computations.

When dealing with significant loads, it's crucial to consider the nature of the load (constant, time-dependent, etc.) and the type of differential equation (ordinary or partial). In many real-world situations, a combination of analytical and numerical methods, possibly with the aid of computational tools, is necessary for obtaining solutions.

Differential-algebraic equations with significant loads refer to a class of mathematical equations that involve a combination of differential equations and algebraic equations, where the system is subjected to significant external forces or loads. These equations are common in various scientific and engineering applications, especially when modeling complex dynamic systems [18–33].

The presence of significant loads implies that external forces or influences play a substantial role in the behavior of the system. These loads can be time-dependent, leading to a more intricate mathematical formulation.

Solving differential-algebraic equations with significant loads may require specialized numerical methods or a combination of analytical and numerical techniques. The choice of method depends on the specific characteristics of the problem, such as the nature of the loads and the structure of the equations. Some common methods for solving differential-algebraic equations include implicit and explicit numerical methods, index reduction techniques, and advanced numerical solvers.

Researchers often study and develop methods tailored to the specific challenges posed by differential-algebraic equations with significant loads to accurately model and simulate the behavior of dynamic systems in various fields, including physics, engineering, and biology.

The present article considers a problem for the differential-algebraic equation with significant loads, where the left-hand side of the equation involves a non-invertible matrix. To study and solve this problem, a modification of the Dzhumabaev's parametrization method [34] is proposed. Considered problem is reduced to a parametric initial-boundary value problem for the differential-algebraic equations with significant loads.

1 Statement of problem and reduction to a parametric problem

On $[0,T]$ the following problem for differential-algebraic equations with significant loads is considered:

$$E\dot{x}(t) = Ax(t) + E_0\dot{x}(\theta) + A_0x(\theta) + f(t), \qquad t \in [0,T], \tag{1}$$

$$Bx(0) + Cx(T) = d, \tag{2}$$

where the matrices $E, A \in \mathbb{C}^{n,n}$, $E_0, A_0 \in \mathbb{C}^{n,n}$, and the function $f(t) \in C([0,T],\mathbb{C}^n)$, $0 < \theta < T$, the matrices $B, C \in \mathbb{C}^{n,n}$, the vector $d \in \mathbb{C}^n$.

We suppose that the matrix pair (E, A) is regular.

A solution to problem (1), (2) is called a function $x(t) \in C([0,T],\mathbb{C}^n)$ having derivative $\dot{x}(t) \in C([0,T],\mathbb{C}^n)$, satisfies to differential-algebraic equations with significant loads (1) and two-point condition (2).

The aim of the paper is to propose a constructive method for solving problem (1), (2).

For solving the problem for differential-algebraic equations with significant loads (1), (2) Dzhumabaev's parametrization method is applied [34].

We introduce a parameter ξ in the following form: $E\xi = Ex(0)$, a.e. as a value of the unknown function at the left endpoint. Then, in the problem (1), (2) we replace $x(t)$ by a new function in the form $x(t) = y(t) + \xi$. The two-point problem for differential-algebraic equations with significant loads (1), (2) transfers to the parametric problem

$$E\dot{y}(t) = Ay(t) + E_0\dot{y}(\theta) + A_0y(\theta) + [A + A_0]\xi + f(t), \qquad t \in [0, T], \tag{3}$$

$$Ey(0) = 0, \tag{4}$$

$$[B + C]\xi + Cy(T) = d. \tag{5}$$

We obtain the parametric for differential-algebraic equations with significant loads and initial condition (3)–(5). Relation (5) can be interpreted as an algebraic equation, containing unknown parameter ξ and value of the unknown function $y(t)$ at the point $t = T$.

A solution to the parametric problem for differential-algebraic equations with significant loads and initial condition (3)–(5) is called a pair $(y(t), \xi)$ with elements $y(t) \in C([0, T], \mathbb{C}^n)$ and $\xi \in \mathbb{C}^n$, satisfies to system (3), initial condition (4) and system of algebraic equations (5).

Subsequently, based on the properties of the obtained parametric problem (3)–(5), we give the solvability conditions to the considered problem (1), (2). For this purpose, in next Section the Weierstrass canonical form is applied to the parametric problem (3)–(5).

2 Weierstrass canonical form and solution to parametric problem

Further, we apply Weierstrass canonical form [18], it is a specific representation of differential-algebraic equations. The Weierstrass canonical form simplifies the analysis and numerical solution of differential-algebraic equations by separating the differential and algebraic components of the system. It provides a structured representation that is easier to work with when applying numerical integration techniques or performing stability analysis.

Let P and Q be nonsingular matrices on dimension n which transform (3) to the Weierstrass canonical form

$$PEQ = \begin{bmatrix} I_{n_1} & O_{n_2} \\ O_{n_1} & N \end{bmatrix}, \qquad PAQ = \begin{bmatrix} J & O_{n_2} \\ O_{n_1} & I_{n_2} \end{bmatrix}, \qquad Pf = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix}, \tag{6}$$

where I_{n_1} and I_{n_2} is a identity matrices on dimension n_1, n_2, respectively, O_{n_1} and O_{n_2} is a null matrices on dimension n_1, n_2, respectively, N is a nilpotent matrix on dimension n_2, J is a matrix in Jordan canonical form on dimension n_1, $n_1 + n_2 = n$. Following [18], we call the index of nilpotency of N in (6) the index of the matrix pair (E, A), denoted by $\nu = \text{ind}(E, A)$.

We suppose that the matrices E_0 and A_0 have the forms

$$PE_0Q = \begin{bmatrix} L_{n_1} & O_{n_2} \\ O_{n_1} & L_{n_2} \end{bmatrix}, \qquad PA_0Q = \begin{bmatrix} M_{n_1} & O_{n_2} \\ O_{n_1} & M_{n_2} \end{bmatrix}, \tag{7}$$

where L_{n_1}, M_{n_1} and L_{n_2}, M_{n_2} are a constant matrices on dimension n_1, n_2, respectively.

Using (6), (7) we reduce parametric problem (3)–(5) to the next form:

$$\dot{\tilde{y}}_1(t) = J(\tilde{y}_1(t) + \tilde{\xi}_1) + L_{n_1}\dot{\tilde{y}}_1(\theta) + M_{n_1}\tilde{y}_1(\theta) + M_{n_1}\tilde{\xi}_1 + \tilde{f}_1(t), \tag{8}$$

$$\tilde{y}_1(0) = 0, \tag{9}$$

$$N\dot{\tilde{y}}_2(t) = \tilde{y}_2(t) + \tilde{\xi}_2 + L_{n_2}\dot{\tilde{y}}_2(\theta) + M_{n_2}\tilde{y}_2(\theta) + M_{n_2}\tilde{\xi}_2 + \tilde{f}_2(t), \tag{10}$$

$$N\tilde{y}_2(0) = 0, \tag{11}$$

$$[\tilde{B} + \tilde{C}]\tilde{\xi} = d - \tilde{C}\tilde{y}(T), \tag{12}$$

where $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)^T = Q^{-1}y$, $\tilde{y}_1(t) \in C([0,T], \mathbb{C}^{n_1})$, $\tilde{y}_2(t) \in C([0,T], \mathbb{C}^{n_2})$, $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2)^T = Q^{-1}\xi$, $\tilde{\xi}_1 \in \mathbb{C}^{n_1}$, $\tilde{\xi}_2 \in \mathbb{C}^{n_2}$, $\tilde{B} = BQ, \tilde{C} = CQ$.

Problems (8), (9) and (10), (11) are initial value problems with parameter for differential equations with significant loads.

A pair $(\tilde{y}(t), \tilde{\xi})$ with $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)^T$ and $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2)^T$ is called a solution to problem (8)–(12), if it satisfies the initial value problems (8), (9) and (10), (11), and the relation (12).

From equations (8) and (10) we determine the values $\dot{\tilde{y}}_1(\theta)$ and $\dot{\tilde{y}}_2(\theta)$. We have

$$\dot{\tilde{y}}_1(\theta) = J(\tilde{y}_1(\theta) + \tilde{\xi}_1) + L_{n_1}\dot{\tilde{y}}_1(\theta) + M_{n_1}\tilde{y}_1(\theta) + M_{n_1}\tilde{\xi}_1 + \tilde{f}_1(\theta), \tag{13}$$

$$N\dot{\tilde{y}}_2(\theta) = \tilde{y}_2(\theta) + \tilde{\xi}_2 + L_{n_2}\dot{\tilde{y}}_2(\theta) + M_{n_2}\tilde{y}_2(\theta) + M_{n_2}\tilde{\xi}_2 + \tilde{f}_2(\theta). \tag{14}$$

From (13) and (14) we obtain

$$[I_{n_1} - L_{n_1}]\dot{\tilde{y}}_1(\theta) = [J + M_{n_1}]\tilde{y}_1(\theta) + [J + M_{n_1}]\tilde{\xi}_1 + \tilde{f}_1(\theta), \tag{15}$$

$$[N - L_{n_2}]\dot{\tilde{y}}_2(\theta) = [I_{n_2} + M_{n_2}]\tilde{y}_2(\theta) + [I_{n_2} + M_{n_2}]\tilde{\xi}_2 + \tilde{f}_2(\theta). \tag{16}$$

Assuming that the matrices $I_{n_1} - L_{n_1}$ and $N - L_{n_2}$ are non-singular in (15), (16), we have the following presentations for $\dot{\tilde{y}}_1(\theta)$ and $\dot{\tilde{y}}_2(\theta)$:

$$\dot{\tilde{y}}_1(\theta) = [I_{n_1} - L_{n_1}]^{-1}[J + M_{n_1}]\tilde{y}_1(\theta) + [I_{n_1} - L_{n_1}]^{-1}[J + M_{n_1}]\tilde{\xi}_1 + [I_{n_1} - L_{n_1}]^{-1}\tilde{f}_1(\theta), \tag{17}$$

$$\dot{\tilde{y}}_2(\theta) = [N - L_{n_2}]^{-1}[I_{n_2} + M_{n_2}]\tilde{y}_2(\theta) + [N - L_{n_2}]^{-1}[I_{n_2} + M_{n_2}]\tilde{\xi}_2 + [N - L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{18}$$

Substituting (17), (18) into the equations (8), (10) instead of the values $\dot{\tilde{y}}_1(\theta)$ and $\dot{\tilde{y}}_2(\theta)$, we obtain

$$\dot{\tilde{y}}_1(t) = J\tilde{y}_1(t) + J\tilde{\xi}_1 + \tilde{L}_{n_1}\tilde{y}_1(\theta) + \tilde{L}_{n_1}\tilde{\xi}_1 + \tilde{f}_1(t) + L_{n_1}[I_{n_1} - L_{n_1}]^{-1}\tilde{f}_1(\theta), \tag{19}$$

$$N\dot{\tilde{y}}_2(t) = \tilde{y}_2(t) + \tilde{\xi}_2 + \tilde{L}_{n_2}\tilde{y}_2(\theta) + \tilde{L}_{n_2}\tilde{\xi}_2 + \tilde{f}_2(t) + L_{n_2}[N - L_{n_2}]^{-1}\tilde{f}_2(\theta), \tag{20}$$

where $\tilde{L}_{n_1} = L_{n_1}[I_{n_1} - L_{n_1}]^{-1}[J + M_{n_1}] + M_{n_1}$, $\tilde{L}_{n_2} = L_{n_2}[N - L_{n_2}]^{-1}[I_{n_2} + M_{n_2}] + M_{n_2}$.

For fixed $\tilde{\xi}_1$ solution to initial value problem (19), (9) has the next representation:

$$\tilde{y}_1(t) = \int_0^t e^{(t-s)J}dsJ\tilde{\xi}_1 + \int_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}\tilde{y}_1(\theta) + \int_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}\tilde{\xi}_1 +$$

$$+ \int_0^t e^{(t-s)J}\tilde{f}_1(s)ds + \int_0^t e^{(t-s)J}dsJL_{n_1}[I_{n_1} - L_{n_1}]^{-1}\tilde{f}_1(\theta), \qquad t \in [0,T]. \tag{21}$$

By Lemma 2.8 [18] and property of matrix N, for fixed $\tilde{\xi}_2$ equation (20) has the unique solution in the form:

$$\tilde{y}_2(t) = -\sum_{j=0}^{\nu-1} N^j \Big[\tilde{\xi}_2 + \tilde{L}_{n_2}\tilde{y}_2(\theta) + \tilde{L}_{n_2}\tilde{\xi}_2 + \tilde{f}_2(t) + L_{n_2}[N - L_{n_2}]^{-1}\tilde{f}_2(\theta)\Big]^{(j)} =$$

$$= -\sum_{j=0}^{\nu-1} N^j \tilde{f}_2^{(j)}(t) - \tilde{L}_{n_2}\tilde{y}_2(\theta) - \tilde{\xi}_2 - \tilde{L}_{n_2}\tilde{\xi}_2 - L_{n_2}[N - L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{22}$$

From expressions (21) and (22) we determine values of functions $\tilde{y}_1(t)$ and $\tilde{y}_2(t)$ at the point $t = \theta$:

$$\tilde{y}_1(\theta) = \int_0^\theta e^{(\theta-s)J}dsJ\tilde{\xi}_1 + \int_0^\theta e^{(\theta-s)J}dsJ\tilde{L}_{n_1}\tilde{y}_1(\theta) + \int_0^\theta e^{(\theta-s)J}dsJ\tilde{L}_{n_1}\tilde{\xi}_1 +$$

$$+\int_0^\theta e^{(\theta-s)J}\tilde{f}_1(s)ds+\int_0^\theta e^{(\theta-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta), \tag{23}$$

$$\tilde{y}_2(\theta)=-\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(\theta)-\tilde{L}_{n_2}\tilde{y}_2(\theta)-\tilde{\xi}_2-\tilde{L}_{n_2}\tilde{\xi}_2-L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{24}$$

From equations (23) and (24) we have

$$\Big[I_{n_1}-\int_0^\theta e^{(\theta-s)J}dsJ\tilde{L}_{n_1}\Big]\tilde{y}_1(\theta)=\int_0^\theta e^{(\theta-s)J}dsJ\tilde{\xi}_1+\int_0^\theta e^{(\theta-s)J}dsJ\tilde{L}_{n_1}\tilde{\xi}_1+$$

$$+\int_0^\theta e^{(\theta-s)J}\tilde{f}_1(s)ds+\int_0^\theta e^{(\theta-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta), \tag{25}$$

$$[I_{n_2}+\tilde{L}_{n_2}]\tilde{y}_2(\theta)=-\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(\theta)-\tilde{\xi}_2-\tilde{L}_{n_2}\tilde{\xi}_2-L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{26}$$

We suppose that the matrices $D_{n_1}=I_{n_1}-\int_0^\theta e^{(\theta-s)J}dsJ\tilde{L}_{n_1}$ and $D_{n_2}=I_{n_2}+\tilde{L}_{n_2}$ are invertible, a.e. non-singular. Then, from algebraic equations (25), (26), we obtain the expressions for $\tilde{y}_1(\theta)$ and $\tilde{y}_2(\theta)$:

$$\tilde{y}_1(\theta)=D_{n_1}^{-1}\int_0^\theta e^{(\theta-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]\tilde{\xi}_1+$$

$$+D_{n_1}^{-1}\int_0^\theta e^{(\theta-s)J}\tilde{f}_1(s)ds+D_{n_1}^{-1}\int_0^\theta e^{(\theta-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta), \tag{27}$$

$$\tilde{y}_2(\theta)=-D_{n_2}^{-1}\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(\theta)-\tilde{\xi}_2-D_{n_2}^{-1}L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{28}$$

Substituting (27) and (28) into the expressions (21), (22) instead of the values $\tilde{y}_1(\theta)$ and $\tilde{y}_2(\theta)$, we obtain

$$\tilde{y}_1(t)=\int_0^t e^{(t-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]\tilde{\xi}_1+\int_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}D_{n_1}^{-1}\int_0^\theta e^{(\theta-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]\tilde{\xi}_1+$$

$$+\int_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}\Big\{D_{n_1}^{-1}\int_0^\theta e^{(\theta-s)J}\tilde{f}_1(s)ds+D_{n_1}^{-1}\int_0^\theta e^{(\theta-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta)\Big\}+$$

$$+\int_0^t e^{(t-s)J}\tilde{f}_1(s)ds+\int_0^t e^{(t-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta),\qquad t\in[0,T], \tag{29}$$

$$\tilde{y}_2(t)=-\tilde{\xi}_2-\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(t)+\tilde{L}_{n_2}D_{n_2}^{-1}\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(\theta)+$$

$$+\tilde{L}_{n_2}D_{n_2}^{-1}L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta)-L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{30}$$

Hence, taking into account initial condition (11), we obtain that the second component of the parameter $\tilde{\xi}$ is uniquely determined and the vector $\tilde{\xi}_2$ has the next form

$$\tilde{\xi}_2=-\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(0)+\tilde{L}_{n_2}D_{n_2}^{-1}\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(\theta)+$$

$$+\tilde{L}_{n_2}D_{n_2}^{-1}L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta)-L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{31}$$

As can be seen from (30) and (31) the second components of $\tilde{y}$ and $\tilde{\xi}$ became known.

Now, we are interested in finding only the first components $\tilde{y}_1$ and $\tilde{\xi}_1$ which are interrelated by (29). Therefore, the appropriate number of imposed boundary conditions must match the number of n_1 differential equations in (1).

A natural question arises: what should be the structure of boundary matrices B and C?

3 The solvability of problem (1), (2)

We assume that the $n\times n$ matrices and the right-hand side n-vector of the relation (2) are of the form

$$BQ=\tilde{B}=\begin{bmatrix}\tilde{B}_1 & O_{n_2}\\ O_{n_1} & O_{n_2}\end{bmatrix},\qquad CQ=\tilde{C}=\begin{bmatrix}\tilde{C}_1 & O_{n_2}\\ O_{n_1} & O_{n_2}\end{bmatrix},\qquad d=\begin{bmatrix}d_1\\ O\end{bmatrix}, \tag{32}$$

where $\tilde{B}_1,\tilde{C}_1\in\mathbb{C}^{n_1,n_1}$ and $d\in\mathbb{C}^{n_1}$.

Now, by substituting (29) into (2), we get the following algebraic equation with respect to $\tilde{\xi}_1$:

$$\Phi\tilde{\xi}_1=\tilde{d}, \tag{33}$$

where

$$\Phi=\tilde{B}_1+\tilde{C}_1+\tilde{C}_1\int_0^T e^{(T-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]+\tilde{C}_1\int_0^T e^{(T-s)J}dsJ\tilde{L}_{n_1}D_{n_1}^{-1}\int_0^\theta e^{(\theta-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]$$

and

$$\tilde{d}=d_1-\tilde{C}_1\int_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}D_{n_1}^{-1}\Big\{\int_0^\theta e^{(\theta-s)J}\tilde{f}_1(s)ds+\int_0^\theta e^{(\theta-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta)\Big\}-$$

$$-\tilde{C}_1\int_0^t e^{(t-s)J}\tilde{f}_1(s)ds-\tilde{C}_1\int_0^t e^{(t-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta).$$

If the matrix Φ is nonsingular, a.e. is invertible, then system of algebraic equations (33) has the unique solution $\tilde{\xi}_1^*=\Phi^{-1}\tilde{d}$. Substituting $\tilde{\xi}_1^*$ into (29), we find $\tilde{y}_1^*$ and hence the first components of the unique solution $(\tilde{y}^*,\tilde{\xi}^*)$ of the parametric problem (8)–(12):

$$\tilde{y}_1^*(t)=\int_0^t e^{(t-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]\Phi^{-1}\tilde{d}+\int_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}D_{n_1}^{-1}\int_0^\theta e^{(\theta-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]\Phi^{-1}\tilde{d}+$$

$$+\int_0^t e^{(t-s)J}dsJ\tilde{L}_{n_1}\Big\{D_{n_1}^{-1}\int_0^\theta e^{(\theta-s)J}\tilde{f}_1(s)ds+D_{n_1}^{-1}\int_0^\theta e^{(\theta-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta)\Big\}+$$

$$+\int_0^t e^{(t-s)J}\tilde{f}_1(s)ds+\int_0^t e^{(t-s)J}dsJL_{n_1}[I_{n_1}-L_{n_1}]^{-1}\tilde{f}_1(\theta),\qquad t\in[0,T], \tag{34}$$

$$\tilde{\xi}_1^*=\Phi^{-1}\tilde{d}. \tag{35}$$

As shown earlier, the second components of $(\tilde{y}^*,\tilde{\xi}^*)$ are determined by:

$$\tilde{y}_2^*(t)=\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(0)-\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(t),\qquad t\in[0,T], \tag{36}$$

$$\tilde{\xi}_2^*=-\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(0)+\tilde{L}_{n_2}D_{n_2}^{-1}\sum_{j=0}^{\nu-1}N^j\tilde{f}_2^{(j)}(\theta)+$$

$$+\tilde{L}_{n_2}D_{n_2}^{-1}L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta)-L_{n_2}[N-L_{n_2}]^{-1}\tilde{f}_2(\theta). \tag{37}$$

Therefore, taking into account the interrelation between the parametric problem (3)–(5) and the initial value problem with parameter (8)-(12), we can summarize our result for this case.

Theorem 1. Let (E,A) be a regular pair of square matrices and let P and Q be nonsingular matrices which transform (3) to Weierstrass canonical form (6). Furthermore, let $\nu=\text{ind}(E,A)$ and $f\in C^\nu([0,T],\mathbb{C}^n)$. Assume that:

i) the matrices E_0 and A_0 have the forms (7) with constant matrices L_{n_1},M_{n_1} and L_{n_2},M_{n_2} on dimension n_1, n_2, respectively;

ii) the matrices $I_{n_1}-L_{n_1}$ and $N-L_{n_2}$ are non-singular;

iii) the matrices $D_{n_1}=I_{n_1}-\int_0^\theta e^{(\theta-s)J}dsJ\tilde{L}_{n_1}$ and $D_{n_2}=I_{n_2}+\tilde{L}_{n_2}$ are non-singular, where

$\tilde{L}_{n_1}=L_{n_1}[I_{n_1}-L_{n_1}]^{-1}[J+M_{n_1}]+M_{n_1}$, $\qquad\tilde{L}_{n_2}=L_{n_2}[N-L_{n_2}]^{-1}[I_{n_2}+M_{n_2}]+M_{n_2}$.

Then the initial value problem with parameter (8)–(12) with the matrices B,C, and vector d of the form (32) has a unique solution $(\tilde{y}^*,\tilde{\xi}^*)$ if and only if the matrix

$$\Phi=\tilde{B}_1+\tilde{C}_1+\tilde{C}_1\int_0^T e^{(T-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]+\tilde{C}_1\int_0^T e^{(T-s)J}dsJ\tilde{L}_{n_1}D_{n_1}^{-1}\int_0^\theta e^{(\theta-s)J}dsJ[I_{n_1}+\tilde{L}_{n_1}]$$

is nonsingular.

Taking into account the interrelation between of the parametric problem (3)–(5) and the initial value problem with parameter (8)–(12), we write the unique solution $(y^*(t),\xi^*)$ of the parametric problem (3)–(5) in the following form

$$(y^*(t),\xi^*)=Q\begin{bmatrix}(\tilde{y}_1^*(t),\tilde{\xi}_1^*)\\(\tilde{y}_2^*(t),\tilde{\xi}_2^*)\end{bmatrix}, \tag{38}$$

where the functions $\tilde{y}_1^*(t)$, $\tilde{y}_2^*(t)$ and the vectors $\tilde{\xi}_1^*$, $\tilde{\xi}_2^*$ are determined by (34), (36) and (35), (37), respectively,

$$Pf(t)=\begin{bmatrix}\tilde{f}_1(t)\\\tilde{f}_2(t)\end{bmatrix},$$

$$\tilde{d} = d_1 - \tilde{C}_1 \int_0^t e^{(t-s)J} ds J \tilde{L}_{n_1} D_{n_1}^{-1} \Big\{ \int_0^\theta e^{(\theta-s)J} \tilde{f}_1(s) ds + \int_0^\theta e^{(\theta-s)J} ds J L_{n_1} [I_{n_1} - L_{n_1}]^{-1} \tilde{f}_1(\theta) \Big\} -$$

$$-\tilde{C}_1 \int_0^t e^{(t-s)J} \tilde{f}_1(s) ds - \tilde{C}_1 \int_0^t e^{(t-s)J} ds J L_{n_1} [I_{n_1} - L_{n_1}]^{-1} \tilde{f}_1(\theta).$$

From the equivalence problems (3)–(5) and (1), (2) it follows that

Theorem 2. Let (E, A) be a regular pair of square matrices and let P and Q be nonsingular matrices which transform (3) to Weierstrass canonical form (6). Furthermore, let $\nu = \text{ind}(E, A)$ and $f \in C^\nu([0, T], \mathbb{C}^n)$. Assume that:

i) the matrices E_0 and A_0 have the forms (7) with constant matrices L_{n_1}, M_{n_1} and L_{n_2}, M_{n_2} on dimension n_1, n_2, respectively;

ii) the matrices $I_{n_1} - L_{n_1}$ and $N - L_{n_2}$ are non-singular;

iii) the matrices $D_{n_1} = I_{n_1} - \int_0^\theta e^{(\theta-s)J} ds J \tilde{L}_{n_1}$ and $D_{n_2} = I_{n_2} + \tilde{L}_{n_2}$ are non-singular, where $\tilde{L}_{n_1} = L_{n_1}[I_{n_1} - L_{n_1}]^{-1}[J + M_{n_1}] + M_{n_1}$, $\quad \tilde{L}_{n_2} = L_{n_2}[N - L_{n_2}]^{-1}[I_{n_2} + M_{n_2}] + M_{n_2}$.

Then the problem (1), (2) with the matrices B, C, and vector d of the form (32) has a unique solution if and only if the matrix

$$\Phi = \tilde{B}_1 + \tilde{C}_1 + \tilde{C}_1 \int_0^T e^{(T-s)J} ds J [I_{n_1} + \tilde{L}_{n_1}] + \tilde{C}_1 \int_0^T e^{(T-s)J} ds J \tilde{L}_{n_1} D_{n_1}^{-1} \int_0^\theta e^{(\theta-s)J} ds J [I_{n_1} + \tilde{L}_{n_1}]$$

is nonsingular. And, the solution $x^*(t)$ of problem (1), (2) is determined by equality

$$x^*(t) = y^*(t) + \xi^*, \qquad t \in [0, T],$$

where the function $y^*(t)$ and the vector ξ^* are defined from (38) with components $\tilde{y}_1^*(t)$, $\tilde{y}_2^*(t)$, $\tilde{\xi}_1^*$, $\tilde{\xi}_2^*$ give the expressions (34)–(37).

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Author Contributions

Z.M. Kadirbayeva and S.T.Mynbayeva collected and analyzed data, and led manuscript preparation. R.A. Medetbekova assisted in data collection and analysis. A.T. Assanova served as the principal investigator of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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Елеулі жүктемелері бар дифференциалдық-алгебралық теңдеулер үшін есеп

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Мақалада елеулі жүктемелері бар дифференциалдық-алгебралық теңдеулер үшін есеп зерттелген. Елеулі жүктемелері бар дифференциалдық теңдеулер үшін бұрын зерттелген есептерден айырмашылығы, қарастырылып отырған теңдеудің сол жағындағы туындының алдында қайтымсыз матрица бар. Ендеше, теңдеулер жүйесі дифференциалдық теңдеулермен қоса, алгебралық теңдеулерді де қамтиды. Қойылған есепті шешу үшін Жұмабаевтың параметрлеу әдісінің модификациясы ұсынылады. Қарастырылып отырған есеп елеулі жүктемелері бар дифференциалдық-алгебралық теңдеулер үшін параметрлік есепке келтірілген. Осы есепке Вейерштрасс канондық формасы қолданылады. Елеулі жүктемелері бар дифференциалдық және алгебралық теңдеулер үшін параметрлік бастапқы есеп алынған. Зерттеліп отырған есептің шешілімділік шарттары анықталған.

Кілт сөздер: дифференциалдық-алгебралық теңдеулер, елеулі жүктемелері бар теңдеулер, параметр, параметрлік бастапқы есеп, шешім.

Задача для дифференциально-алгебраических уравнений с существенными нагрузками

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В статье исследована задача для дифференциально-алгебраического уравнения с существенными нагрузками. В отличие от ранее изученных задач для дифференциальных уравнений с существенной нагрузкой, в рассматриваемом уравнении в левой части при производной имеется необратимая матрица. Следовательно, система уравнений включает в себя как дифференциальные, так и алгебраические уравнения. Для решения поставленной задачи предложена модификация метода параметризации Джумабаева, и задача сведена к параметрической задаче для дифференциально-алгебраического уравнения с существенными нагрузками. К этой задаче применяется каноническая форма Вейерштрасса. Получена параметрическая начальная задача для дифференциальных и алгебраических уравнений с существенными нагрузками. Установлены условия разрешимости исследуемой задачи.

Ключевые слова: дифференциально-алгебраические уравнения, уравнения с существенной нагрузкой, параметр, параметрическая начальная задача, решение.

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