

On the behaviors of solutions of a nonlinear diffusion system with a source and nonlinear boundary conditions

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We study the global solvability and unsolvability of a nonlinear diffusion system with nonlinear boundary conditions in the case of slow diffusion. We obtain the critical exponent of the Fujita type and the critical global existence exponent, which plays a significant part in analyzing the qualitative characteristics of nonlinear models of reaction-diffusion, heat transfer, filtration, and other physical, chemical, and biological processes. In the global solvability case, the key components of the asymptotic solutions are obtained. Iterative methods, which quickly converge to the exact solution while maintaining the qualitative characteristics of the nonlinear processes under study, are known to require the presence of an appropriate initial approximation. This presents a significant challenge for the numerical solution of nonlinear problems. A successful selection of initial approximations allows for the resolution of this challenge, which depends on the value of the numerical parameters of the equation, which are primarily in the computations recommended using an asymptotic formula. Using the asymptotics of self-similar solutions as the initial approximation for the iterative process, numerical calculations and analysis of the results are carried out. The outcomes of numerical experiments demonstrate that the results are in excellent accord with the physics of the process under consideration of the nonlinear diffusion system.

Keywords: blow-up, nonlinear boundary condition, critical global existence curve, degenerate parabolic systems, critical exponents of Fujita type.

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Introduction

The source for this article’s discussion of the doubly degenerate parabolic equations is as follows:

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x} \left(\left| \frac{\partial u_i^k}{\partial x} \right|^{m-1} \frac{\partial u_i^k}{\partial x} \right) + u_i^{p_i}, \quad x \in R_+, \quad t > 0, \quad i = 1, 2, \quad (1)$$

coupled through nonlinear boundary conditions:

$$-\left| \frac{\partial u_i^k}{\partial x} \right|^{m-1} \frac{\partial u_i^k}{\partial x} \Big|_{x=0} = u_{3-i}^{q_i}(0, t), \quad t > 0, \quad i = 1, 2, \quad (2)$$

where $m > 1$, $k \geq 1$, and $q_i, p_i > 0$ are numerical parameters. The following preliminary information should be considered:

$$u_i|_{t=0} = u_{i0}(x), \quad i = 1, 2. \quad (3)$$

It is expected that the function and its corresponding first- and second-order derivatives conform to a set of criteria. Specifically, these derivatives should exhibit a degree of continuity, non-negativity, and compactness within the domain of R_+ .

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Population dynamics, heat transfer, chemical processes, and other phenomena all use parabolic equations with nonlinearity (1).

The functions $u_1(t, x)$, $u_2(t, x)$ represent the biological two populations' densities during migration, the thickness of two types of chemical reagents during a chemical reaction, and the temperatures of two various sorts of materials during propagation. By incorporating the dependent on the power-law of shear stress and displacement velocity, equation (1) becomes an invaluable tool for analyzing a liquid medium with inconsistent fluxes. This allows for a comprehensive understanding of the complex dynamics and behavior exhibited under polytropic conditions, providing specialized professionals and enthusiasts with the means to look into the details of these systems.

Parabolic equations (1) with nonlinearity have a significant role in several scientific fields, such as population dynamics, heat transfer, chemical reactions, and many others. They are widely employed to investigate a variety of phenomena, such as the biological densities of two populations during migration and the thickness of two different kinds of chemical reagents during a chemical reaction. These equations are also used to determine the temperature of two different types of materials during propagation. In population dynamics, the functions $u_1(t, x)$, $u_2(t, x)$ describe the growth or decline of animal or plant populations. Similarly, in heat transfer, they help to determine the heat flux in a material with varying temperatures. Furthermore, they are used to describe unsteady flows in a liquid media, especially when shear stress and displacement velocity exhibit a power-law relationship.

The local presence of ineffective solutions to problem (1)–(3) in the problem-solving domain has been a topic of much discussion and analysis. The strict testing and experimentation conducted in this field have consistently shown that the usual integration method is a reliable approach for determining this specific phenomenon. This widely acknowledged fact within the community of experts demonstrates the thorough knowledge and expertise that underpins our understanding of complex systems. Moreover, it is worth mentioning that such a local existence can be easily established and understood by applying the comparison principle, which has been extensively reviewed in several studies ([1; 316], [2; 26], [3–11]). Therefore, it is safe to say that the determination of the local existence in this particular problem can be achieved with a high level of accuracy and precision, using the appropriate tools and methods at hand.

The study of nonlinear parabolic systems has piqued the interest of researchers all around the world. With the aim of understanding the global existence and blow-up conditions of such systems, researchers have employed diverse techniques and strategies to investigate this phenomenon. The existing literature in this area is extensive, with several noteworthy contributions from experts in the field (see [1; 176], [2, 3, 7–9, 12] and references therein). The essential for several nonlinear parabolic equations in mathematical physics, the Fujita exponent is one of the major topics of research, which has drawn significant attention from mathematicians. Researchers have delved deep into this area, studying various aspects of critical Fujita exponents in great detail (see [2, 10, 11, 13–16] and references therein). Overall, the understanding of nonlinear parabolic systems' global existence and blow-up circumstances, as well as the critical Fujita exponent, continues to be an area of active research. With further study and investigation, researchers hope to gain deeper insights into these systems, leading to a better understanding of the complex phenomena that underlie them.

Let us now consider and revisit some well-known results. In the research conducted by V.A. Galaktionov, and H.A. Levine mentioned in reference [4], they extensively investigated the situation using a single equation

$$\begin{cases} u_t = (u^k)_{xx}, & x > 0, \quad 0 < t < T, \\ -(u^k)_x(0, t) = u^q(0, t), & 0 < t < T, \\ u(x, 0) = u_0(x), & x > 0, \end{cases} \quad (4)$$

and the gradient diffusion heat conduction equation

$$\begin{cases} u_t = \left(|u_x|^{k-1} u_x \right)_x, & x > 0, 0 < t < T, \\ -|u_x|^{k-1} u_x(0, t) = u^q(0, t), & 0 < t < T, \\ u(x, 0) = u_0(x), & x > 0, \end{cases} \quad (5)$$

with $k \geq 1$, $q > 0$, and u_0 has compact support. It has been established that for the problem (4), $q_0 = \frac{1}{2}(k+1)$ is the critical global exponent, where $q_c = k+1$ is the crucial Fujita exponent, as opposed to (5), the critical Fujita exponent is $q_c = 2k$ as well as the critical global exponent being $q_0 = \frac{2k}{k+1}$.

In [5] authors analyzed the following issue with single equation and gradient diffusion:

$$\begin{cases} \rho(x)u_t = \left(|u_x|^{k-2} u_x \right)_x + \rho(x)u^\beta, & (x, t) \in R_+ \times (0, +\infty), \\ -|u_x|^{k-2} u_x(0, t) = u^m(0, t), & t > 0, \\ u(x, 0) = u_0(x) > 0, & x \in R_+, \end{cases} \quad (6)$$

with $k > 2$, $\beta, m > 0$, $\rho(x) = x^{-n}$, $n \in R$, $u_0(x)$ is a bounded, continuous, nonnegative, and nontrivial initial value. They determined that the problem (6):

- in case of $0 < \beta \leq 1$, and $0 < m \leq \frac{(2-n)(k-1)}{k-n}$ the issue can be resolved globally;
- in case of $\beta < 1$, and $m > \frac{(2-n)(k-1)}{k-n}$ the issue has a blow-up solution.

Consideration of the following problem is the focus of the research conducted by Zhaoyin Xiang, Chunlai Mu, and Yulan Wang in their study published in [12]. The problem under scrutiny has been given thorough attention and analysis by the researchers.

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(\left| \frac{\partial u^{m_1}}{\partial x} \right|^{p_1-2} \frac{\partial u^{m_1}}{\partial x} \right) \\ \frac{\partial v}{\partial t} = \frac{\partial}{\partial x} \left(\left| \frac{\partial v^{m_2}}{\partial x} \right|^{p_2-2} \frac{\partial v^{m_2}}{\partial x} \right) \end{cases}, \quad (x, t) \in R_+ \times (0, T), \quad (7)$$

$$\begin{cases} - \left| \frac{\partial u^{m_1}}{\partial x} \right|^{p_1-2} \frac{\partial u^{m_1}}{\partial x} \Big|_{x=0} = v^{q_1}(0, t) \\ - \left| \frac{\partial v^{m_2}}{\partial x} \right|^{p_2-2} \frac{\partial v^{m_2}}{\partial x} \Big|_{x=0} = u^{q_2}(0, t) \end{cases}, \quad t \in (0, T), \quad (8)$$

$$\begin{cases} u(x, 0) = u_0(x) \\ v(x, 0) = v_0(x) \end{cases}, \quad x \in R_+, \quad (9)$$

where $m_i > 1$, $p_i > 2$, $q_i > 0$, $i = 1, 2$. They determined that:

(i) in case of $q_1 q_2 \leq ((p_1 - 1)(p_2 - 1)(m_1 + 1)(m_2 + 1))/p_1 p_2$ the problem's every nonnegative solutions (7)–(9) are all global in time;

(ii) in case of $q_1 q_2 > ((p_1 - 1)(p_2 - 1)(m_1 + 1)(m_2 + 1))/p_1 p_2$, then the problem (7)–(9) has solutions that blow-up in a limited length of time.

If $q_1 q_2 > ((p_1 - 1)(p_2 - 1)(m_1 + 1)(m_2 + 1))/p_1 p_2$:

- (i) in case of $\min\{\alpha_1 + \beta_1, \alpha_2 + \beta_2\} > 0$, then solution of the problem (7)–(9) is global in time;
- (ii) in case of $\max\{\alpha_1 + \beta_1, \alpha_2 + \beta_2\} < 0$, then the solution of problem (7)–(9) is blow-up.

Many mathematical models of nonlinear cross-diffusion in [17, 18] are described using nonlinearly linked partial differential equation systems. Finding explicit analytical solutions for these systems is difficult, though. To tackle the complexities of these systems, researchers have delved into the realm of numerical methods, employing them to derive approximations. In their pursuit, they have turned to the use of nonlinear parabolic equations, coupled with nonlinear boundary conditions, as a means to accurately describe and analyze these intricate systems. By harnessing the power of these mathematical tools, deeper investigation of the intricacies is possible for researchers and intricacies of these complex phenomena, providing valuable insights that pave the way for advancements in their respective fields. To investigate the qualitative properties of a cross-diffusion system with nonlocal boundary conditions and nonlinearity, self-similar analysis and the standard equation approach have been employed. The results of these studies have helped researchers understand the behavior of these systems under different conditions. Despite the challenges posed by the nonlinearly coupled systems of partial differential equations, and improvements in numerical techniques have paved the way for obtaining accurate approximations, thus making significant contributions to the field of nonlinear cross-diffusion. The quest for further exploration and understanding of these systems continues to fuel research in this area. The situation of slow diffusion, researchers have devised several self-similar solutions to tackle the cross-diffusion problem. The intricate nature of a nonlinear cross-diffusion system, comprised of interconnected parabolic equations, poses a significant challenge in the realm of mathematical analysis. These complex systems often exhibit behavior that defies traditional methods of solution due to the presence of nonlinear boundary conditions. As a result, finding global solutions becomes an arduous task requiring advanced computational techniques and deep understanding of the underlying dynamics at play. Self-similar analysis and the comparison principle were used to identify the critical exponents, namely the global solvability and Fujita type critical exponents. The comparison theorem has further enabled researchers to establish upper and lower limits for global solutions and blow-up solutions, respectively. These findings underscore the importance of carefully considering numerical parameters when dealing with slow-diffusion scenarios.

This article, influenced by the works we have mentioned earlier, serves a twofold purpose. First, it aims to identify the (1)–(3) system’s essential global existence curve, and in order to achieve that, the article emphasizes the importance of constructing self-similar super-solution and sub-solution. Second, the essay presents a theory regarding the critical curve of the Fujita type supported by certain recent findings. As opposed to dealing with a single equation, we are dealing with a system, we need to devise some innovative strategies to tackle the challenges that come with it. In conclusion, this article is a valuable addition to the literature on critical global existence curves, self-similar super- and subsolutions, and the critical curve of the Fujita type.

It is widely accepted in the field of mathematics that degenerate equations often lack classical solutions. When confronted with such equations, mathematicians have to find other solutions that are more general in nature. In conclusion, while degenerate equations may present unique challenges, there are still various ways to approach them and derive meaningful solutions.

Definition. The function $u(x, t)$ is viewed as an insufficient solution to problems (1)–(3) in $\Omega = \{(0, +\infty) \times (0, T)\}$, if $0 \leq u_i(x, t) \in C(\Omega)$, $\left| \frac{\partial u_i^k}{\partial x} \right|^{m-1} \frac{\partial u_i^k}{\partial x} \in C(\Omega)$, $i = 1, 2$, if it complies with (1)–(3) with regard to distribution in Ω , where the longest time period that can be allowed is $T > 0$, see [5].

1 Main results

Solutions to the global existence and nonexistence theorems play a crucial role in understanding complex systems. To further explore this topic, it is necessary to discuss the creation of self-similar sub-

and super-solutions to equations (1)–(3). These solutions provide valuable insights into the behavior of these equations under various conditions. By examining the properties of sub-solutions, we can gain a deeper understanding of how certain factors contribute to the existence of global solutions. As opposed to that, studying super-solutions allows us to analyze situations where nonexistence solutions arise. This comprehensive approach enables researchers and professionals to make informed decisions when dealing with complex systems in their respective fields.

We will use the comparison principle to prove our first theorem, which focuses on determining the conditions necessary for the global solution of problem (1)–(3). By establishing a framework for analyzing self-similar sub-solutions and super-solutions, we gain valuable insights into the intricacies of global existence and nonexistence solutions. This theorem represents a significant advancement in our understanding of complex systems, as it showcases the interplay between comparison principles and the concept of self-similarity. The comprehensive examination of these factors allows us to delve deeper into the realm of global solutions, providing a solid foundation for further research and analysis in this field. Our findings highlight the importance of considering self-similar sub- and super-solutions when studying problems with global implications.

Theorem 1. If $r_1 r_2 \leq \left(\frac{m}{m+1}\right)^2 (k+1-s_1)(k+1-s_2)$, then every nonnegative solution of the problem (1)–(3) is global in time.

Proof. By emphasizing the construction of a self-similar super-solution, one can gain an additional understanding of the theorem and its intricate nuances. This particular super-solution serves as a powerful demonstration of the theorem’s validity and its ability to address complex problems. Through meticulous analysis, it becomes evident that this super-solution possesses certain limitations for any given $t > 0$. As researchers strive towards achieving their objective, their attention has been directed towards the identification and analysis of strict super-solutions that conform to the self-similar form. These endeavors pave the way for a more comprehensive understanding of the intricacies involved in this intricate realm of study,

$$\bar{u}_i(t, x) = e^{h_{2i-1}} \left(N + e^{-K_i x e^{-h_{2i} t}} \right)^{\frac{1}{k}}, \tag{10}$$

where $K_i > 0$, $h_{2i-1, 2i} > 0$, $N = \max \left\{ \|\bar{u}_i\|_{\infty}^k + 1 \right\}$; $i = 1, 2$.

Using comparison principles and the substitution of (10) into (1)-(2), it has been determined:

$$\begin{aligned} \frac{\partial \bar{u}_i}{\partial t} &= h_{2i-1} \cdot e^{h_{2i-1} t} \cdot \left(N + e^{-K_i x e^{-h_{2i} t}} \right)^{\frac{1}{k}} + e^{(h_{2i-1} - h_{2i}) t} \cdot \frac{1}{k} \cdot K_i \cdot x \cdot h_{2i} \left(N + e^{-K_i x e^{-h_{2i} t}} \right)^{\frac{1}{k} - 1} \geq \\ &\geq h_{2i-1} e^{h_{2i-1} t} \left(N + e^{-K_i x e^{-h_{2i} t}} \right)^{\frac{1}{k}} \geq h_{2i-1} e^{h_{2i-1} t} N^{\frac{1}{k}}, \\ \frac{\partial}{\partial x} \left(\left| \frac{\partial \bar{u}_i^k}{\partial x} \right|^{m-1} \frac{\partial \bar{u}_i^k}{\partial x} \right) &= m K_i^{m+1} e^{[h_{2i-1} k m - (m+1) h_{2i}] t} \times e^{-m K_i x e^{-h_{2i} t}} \leq m K_i^{m+1} e^{[h_{2i-1} k m - (m+1) h_{2i}] t}, \\ \bar{u}_i^{s_i} &= e^{s_i h_{2i-1} t} \left(N + e^{-K_i x e^{-h_{2i} t}} \right)^{\frac{s_i}{k}} \leq e^{s_i h_{2i-1} t} (N + 1)^{\frac{s_i}{k}}, \\ - \left| \frac{\partial \bar{u}_i^k}{\partial x} \right|^{m-1} \frac{\partial \bar{u}_i^k}{\partial x} \Big|_{x=0} &= K_i^m e^{(h_{2i-1} k - h_{2i}) m t}, \bar{u}_{3-i}^{r_i} \Big|_{x=0} = e^{r_i h_{5-2i} t} (N + 1)^{\frac{r_i}{k}}. \end{aligned}$$

The solution \bar{u}_i is regarded as global, if inequalities:

$$\frac{\partial \bar{u}_i}{\partial t} \geq \frac{\partial}{\partial x} \left(\left| \frac{\partial \bar{u}_i^k}{\partial x} \right|^{m-1} \frac{\partial \bar{u}_i^k}{\partial x} \right) + \bar{u}_i^{s_i}, \quad i = 1, 2, \tag{11}$$

hold for any $x \in R_+, t > 0$. Eventually, the following expressions have been achieved using the computations above in (11):

$$\begin{aligned} h_{2i-1} e^{h_{2i-1} t} N^{\frac{1}{k}} &\geq m K_i^{m+1} e^{[h_{2i-1} k m - (m+1) h_{2i}] t} + e^{s_i h_{2i-1} t} (N+1)^{\frac{s_i}{k}}, \\ K_i^m e^{(h_{2i-1} k - h_{2i}) m t} &= e^{r_i h_{5-2i} t} (N+1)^{\frac{r_i}{k}}, \\ K_i &= (N+1)^{\frac{r_i}{k m}}, \quad r_i h_{5-2i} = (h_{2i-1} k - h_{2i}) m, \quad i = 1, 2, \\ h_{2i-1} &\geq h_{2i-1} k m - (m+1) h_{2i} + s_i h_{2i-1}, \quad h_1 k - h_2 = \frac{r_1}{m} h_3, \\ h_3 k - h_4 &= \frac{r_2}{m} h_1, \\ h_{2i} &\geq \frac{(k m + s_i - 1) h_{2i-1}}{m+1}, \\ h_1 k - \frac{r_1}{m} h_3 &\geq \frac{(k m + s_1 - 1) h_1}{m+1}, \\ h_3 k - \frac{r_1}{m} h_1 &\geq \frac{(k m + s_2 - 1) h_3}{m+1}. \end{aligned}$$

Thus, it is evident that for the solution of the problem (1)–(3) to be global in time, the last inequality should always hold for any $m > 1, k \geq 1$, as the theorem proves.

Remark. Theorem 1 demonstrates that $r_1 r_2 = \left(\frac{m}{m+1}\right)^2 (k+1-s_1)(k+1-s_2)$ is critical global existence of the problem (1)–(3).

Theorem 2. If $0 < p_i \leq 1$, and $q_i \geq \frac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ or $p_i > 1$, and $r_i \leq \frac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ then, each of the solutions to (1)–(3) blows up.

Proof. To prove the theorem, it was necessary to search for sub-solutions of the problem (1)–(3), and this was achieved by looking for them in the next form:

$$\underline{u}_i(t, x) = t^{\alpha_i} f_i(\xi_i), \quad \xi_i = x t^{-\beta_i}, \tag{12}$$

where $\alpha_i = \frac{1}{1-p_i}, \beta_i = \frac{p_i - k m}{(p_i - 1)(m + 1)}, i = 1, 2$.

By analyzing the super-solutions obtained from equation (12), we can observe the emergence of a self-similar form in the resulting equations (1)–(3). These self-similar inequalities and boundary conditions play a pivotal role in determining whether a solution is deemed as a blow-up solution or not. It is imperative to adhere to these self-similar inequalities and boundary conditions in order to accurately classify and understand the behavior of the system under study. The presence of such intricate relationships highlights the complexity of the problem at hand, requiring a comprehensive and meticulous approach for its exploration. To fully comprehend the underlying dynamics, further research and analysis are warranted to delve deeper into these self-similar forms and their implications on the overall system:

$$\frac{d}{d\xi_i} \left(\left| \frac{df_i^k}{d\xi_i} \right|^{m-1} \frac{df_i^k}{d\xi_i} \right) + \beta_i \xi_i \frac{df}{d\xi_i} - \alpha_i f_i + f_i^{p_i} \geq 0, \tag{13}$$

$$- \left| \frac{\partial \underline{u}_i^k}{\partial x} \right|^{m-1} \frac{\partial \underline{u}_i^k}{\partial x} \Big|_{x=0} \leq \underline{u}_{3-i}^{q_i}(0, t). \tag{14}$$

Let

$$f_i(\xi_i) = A_i \left(a_i^{\frac{m+1}{m}} - \xi_i^{\frac{m+1}{m}} \right)^{\frac{m}{mk-1}}. \tag{15}$$

By substituting equation (15) into inequalities (13) and (14), we can derive the necessary conditions that unequivocally illustrate the occurrence of equation (14) under all circumstances. This crucial step not only solidifies our understanding of the underlying principles, but also provides a robust framework for further analysis and exploration within this complex system:

$$\begin{aligned} & \left(\frac{k(m+1)}{mk-1} \right)^m \left(\frac{m+1}{mk-1} \right) A_i^{mk} \geq \beta_i \frac{m+1}{mk-1} A_i, \\ A_i & \geq \left[\beta_i \left(\frac{mk-1}{k(m+1)} \right)^m \right]^{\frac{1}{mk-1}}, \\ f_i^{p_i} & = A_i^{p_i} \left(a_i^{\frac{m+1}{m}} - \xi_i^{\frac{m+1}{m}} \right)^{\frac{m}{mk-1}} \left(a_i^{\frac{m+1}{m}} - \xi_i^{\frac{m+1}{m}} \right)^{\frac{m}{mk-1}(p_i-1)} \leq \\ & \leq A_i^{p_i} a_i^{\frac{(m+1)(p_i-1)}{mk-1}} \left(a_i^{\frac{m+1}{m}} - \xi_i^{\frac{m+1}{m}} \right)^{\frac{m}{mk-1}}, \\ A_i^{p_i} a_i^{\frac{(m+1)(p_i-1)}{mk-1}} & \geq \alpha_i A_i + A_i^{mk} \left(\frac{k(m+1)}{mk-1} \right)^m. \end{aligned}$$

By taking

$$a_i^{\frac{(m+1)(p_i-1)}{mk-1}} \geq \alpha_i A_i^{1-p_i} + A_i^{mk-p_i} \left(\frac{k(m+1)}{mk-1} \right)^m,$$

$0 < p_i \leq 1$, and $q_i \geq \frac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ can be easily checked and ensure that A_1 , and A_2 can be taken sufficient to prevent inequalities (13) and (14) are valid. Because of this, if the initial data $u_1(x, 0)$, $u_2(x, 0)$ are large enough that $u_{10}(x) \geq \underline{u}_1(x, 0)$, $u_{20}(x) \geq \underline{u}_2(x, 0)$, then $\underline{u}_i(t, x)$, $i = 1, 2$ is a subsolution to (1)–(3). In accordance with the comparison principle, it is established that when dealing with a substantial amount of initial data, the solutions provided in (1)–(3) will eventually blow up within a finite time frame. The comprehensive proof has been successfully concluded, cementing this understanding.

Theorem 3. If $q_1 q_2 < \left(\frac{m(k+1)}{m+1} \right)^2$, and $p_i > \left(1 + \frac{1}{k} \right) m + \frac{1}{k}$, then every solution of problem (1)–(3) blows up in finite time.

Proof. It is vital to comprehend that the delineated by (1)–(3) can be convincingly shown for equations that lack a source. The necessary conditions for this to occur can be satisfied entirely through internal mechanisms. As such, we proceed to build our targeted solution in a subsequent manner.

$$u_{ib}(t, x) = t^{\mu_i} g_i(\xi_i), \quad \xi_i = xt^{-\gamma_i}, \tag{16}$$

where g_i are two compactly supported functions,

$$\begin{aligned} \mu_i & = \frac{m[m(k+1) + (m+1)q_i]}{(m(k+1))^2 - (m+1)^2 q_i q_{3-i}}, \\ \gamma_i & = \frac{m[mk(k+1) + (mk-1)q_i] - (m+1)q_1 q_2}{(m(k+1))^2 - (m+1)^2 q_i q_{3-i}}. \end{aligned}$$

We now insert (16) into (1)–(3) and derive the following result:

$$\frac{d}{d\xi_i} \left(\left| \frac{dg_i^k}{d\xi_i} \right|^{m-1} \frac{dg_i^k}{d\xi_i} \right) + \gamma_i \xi_i \frac{dg_i}{d\xi_i} - \mu_i g_i \geq 0, \tag{17}$$

$$- \left| \frac{dg_i^k}{d\xi_i} \right|^{m-1} \frac{dg_i^k}{d\xi_i} \Big|_{\xi_i=0} \leq g_{3-i}^{q_i}(0). \tag{18}$$

Finding self-similar solutions to the issue (17), (18) is now necessary.

Let

$$\bar{g}_i(\xi_i) = B_i(b_i - \xi_i)^{\frac{m}{mk-1}}, \tag{19}$$

then by inserting (19) into (17), and (18), we obtain

$$\begin{aligned} \frac{d\bar{g}_i}{d\xi_i} &= -\frac{B_i m}{mk-1} (b_i - \xi_i)^{\frac{m}{mk-1}-1}, \\ \gamma_i \xi_i \frac{d\bar{g}_i}{d\xi_i} - \mu_i \bar{g}_i &= -\frac{B_i m}{mk-1} \xi_i (b_i - \xi_i)^{\frac{m}{mk-1}-1} - \mu_i B_i (b_i - \xi_i)^{\frac{m}{mk-1}} = \\ &= -\frac{B_i m}{mk-1} \xi_i (b_i - \xi_i)^{\frac{m}{mk-1}-1} - \mu_i B_i (b_i - \xi_i)^{\frac{m}{mk-1}-1} (b_i - \xi_i) \geq \\ &\geq -\left(\frac{b_i B_i m}{mk-1} - \mu_i b_i B_i \right) (b_i - \xi_i)_+^{\frac{m}{mk-1}-1}, \\ \frac{d}{d\xi_i} \left(\left| \frac{d\bar{g}_i^k}{d\xi_i} \right|^{m-1} \frac{d\bar{g}_i^k}{d\xi_i} \right) &= B_i^{mk} \left(\frac{m}{mk-1} \right)^{m+1} k^m (b_i - \xi_i)_+^{\frac{m}{mk-1}-1} \geq \\ &\geq b_i B_i \left(\mu_i + \frac{m}{mk-1} \right) (b_i - \xi_i)_+^{\frac{m}{mk-1}-1}, \\ B_i^{mk-1} &\geq \frac{b_i}{k^m} \left(\frac{mk-1}{m} \right)^{m+1} \left(\mu_i + \frac{m}{mk-1} \right), \\ - \left| \frac{d\bar{g}_i^k}{d\xi_i} \right|^{m-1} \frac{d\bar{g}_i^k}{d\xi_i} \Big|_{\xi_i=0} &\leq \bar{g}_{3-i}^{q_i}(0). \end{aligned}$$

The following benefits result from applying comparison principles to the aforementioned expressions:

$$\begin{aligned} - \left| \frac{dg_i^k}{d\xi_i} \right|^{m-1} \frac{dg_i^k}{d\xi_i} \Big|_{\xi_i=0} &= \left| B_i^k (b_i - \xi_i)_+^{\frac{m}{mk-1}-1} \right|^{m-1} \cdot \left(B_i^k (b_i - \xi_i)_+^{\frac{m}{mk-1}-1} \right) \Big|_{\xi_i=0} = \\ &= B_i^{mk} (b_i - \xi_i)_+^{\left(\frac{m}{mk-1}-1\right)m} \Big|_{\xi_i=0} = B_i^{mk} b_i^{\frac{m}{mk-1}} \leq B_{3-i}^{q_i} b_{3-i}^{\frac{q_i m}{mk-1}}. \end{aligned}$$

And this illustrates unequivocally that when $p_i > \left(1 + \frac{1}{k}\right)m + \frac{1}{k}$, equations (17) and (18) hold true.

The concept of comparison leads us to conclude that (1)–(3) have solutions that invariably end in blow-up in a finite amount of time.

Theorem 4. If $q_1 q_2 \leq (m(k+1))^2$, and $p_i > 1$, then every solution of the problem (1)–(3) is blow-up in finite time.

Proof. The same approach used in [11, 18] can be used to establish Theorem 4.

Let us demonstrate how self-similar solutions asymptotically behave.

The case $q_1 q_2 > \left(\frac{m(k+1)}{m+1}\right)^2$, and $\frac{1}{m} < p_i \leq 1$. Take into account the following self-similar solution of (1)–(3).

Auxiliary systems of equations are a fundamental aspect of mathematical problem-solving in various fields. The intricacy of these systems can often be overwhelming, but with the right methods and techniques, they can be simplified. Through the application of specific transformations, such as substitution or elimination, the complex nature of these systems can be broken down into more manageable components. These methods have been extensively studied and proven effective in numerous academic research papers. By implementing these strategies, professionals and enthusiasts alike can confidently approach and solve even the most intricate auxiliary systems of equations:

$$u_i(x, t) = (T + t)^{\alpha_i} \varphi_i(\xi_i), \xi_i = x(T + t)^{-\beta_i},$$

where α_i and β_i parameters defined above.

$$\frac{d}{d\xi_i} \left(\left| \frac{d\varphi_i^k}{d\xi_i} \right|^{m-1} \frac{d\varphi_i^k}{d\xi_i} \right) + \beta_i \xi_i \frac{d\varphi_i}{d\xi_i} - \alpha_i \varphi_i + \varphi_i^{\beta_i} = 0, \tag{20}$$

$$- \left| \frac{d\varphi_i^k}{d\xi_i} \right|^{m-1} \frac{d\varphi_i^k}{d\xi_i} \Big|_{\xi_i=0} = \varphi_{3-i}^{q_i}(0). \tag{21}$$

Let us consider the function

$$\bar{\varphi}_i(\xi_i) = \left(d_i - D_i \xi_i^{\frac{m+1}{m}} \right)^{\frac{m}{mk-1}}, \quad d_i > 0, \quad D_i = \frac{\beta_i^{\frac{1}{m}} (mk - 1)}{k(m + 1)}.$$

Theorem 5. The compactly supported solution of problem (20)-(21) has the asymptotic

$$\varphi_i(\xi_i) = \bar{\varphi}_i(\xi_i)(1 + o(1)),$$

when $\xi_i \rightarrow \left(\frac{d_i}{D_i}\right)^{\frac{m}{m+1}} = \xi_{i0}$.

Proof. The function φ_i is looked for in the following form

$$\varphi_i(\xi_i) = \bar{\varphi}_i(\xi_i)\omega_i(\eta_i).$$

It is enough to show that $\omega_i \approx 1$. Let

$$\eta_i = -\ln \left(d_i - D_i \xi_i^{\frac{m+1}{m}} \right), \quad \text{and} \quad \eta_i \xrightarrow{\xi_i \rightarrow \xi_{i0}} +\infty. \tag{22}$$

Upon substituting (22) into (20)-(21) we get the next expressions:

$$\begin{aligned} \bar{\varphi}_i(\xi_i) &= e^{-\frac{m}{mk-1}\eta_i}, \varphi_i(\xi_i) = e^{-\frac{m}{mk-1}\eta_i}\omega_i, \quad \xi_i = (d_i - e^{-\eta_i})^{\frac{m}{m+1}} D_i^{-\frac{m}{m+1}}, \\ \frac{d\xi_i}{d\eta_i} &= \left(\frac{m+1}{m}\right) D_i^{\frac{m}{m+1}} e^{\eta_i} (d_i - e^{-\eta_i})^{\frac{1}{m+1}}, \\ \beta_i \xi_i \frac{d\varphi_i}{d\xi_i} - \alpha_i \varphi_i + \varphi_i^{p_i} &= \beta_i \left(\frac{m+1}{m}\right) e^{(1-\frac{m}{mk-1})\eta_i} (d_i - e^{-\eta_i}) \times \\ &\times \left(\omega_i' - \frac{m}{mk-1}\omega_i\right) - \alpha_i e^{-\frac{m}{mk-1}\eta_i}\omega_i + e^{-\frac{m p_i}{mk-1}\eta_i}\omega_i^{p_i}, \\ \frac{d}{d\xi_i} \left(\left| \frac{d\varphi_i^k}{d\xi_i} \right|^{m-1} \frac{d\varphi_i^k}{d\xi_i} \right) &= \left(\frac{m+1}{m}\right)^{m+1} D_i^m e^{(1-\frac{m}{mk-1})\eta_i} \times \\ &\times (d_i - e^{-\eta_i}) \left[((L_i\omega)^m)' + \left(\frac{e^{-\eta_i}}{d_i - e^{-\eta_i}} - \frac{m}{mk-1}\right) \cdot (L_i\omega)^m \right], \end{aligned}$$

where $L_i\omega = (\omega_i^k)' - \frac{mk}{mk-1}\omega_i^k$.

Now (20) takes a next look:

$$((L_i\omega)^m)' + \left(a_1(\eta_i) - \frac{m}{mk-1}\right) (L_i\omega)^m + a_2(\eta_i)\omega_i^{1-k}L_i\omega - a_3(\eta_i)\omega_i + a_4(\eta_i)\omega_i^{p_i} = 0,$$

where $a_1(\eta_i) = \frac{e^{-\eta_i}}{d_i - e^{-\eta_i}}$, $a_2(\eta_i) = \frac{\beta_i}{k} \left(\frac{m}{D_i(m+1)}\right)^m$,

$$a_3(\eta_i) = d_i \left(\frac{m}{m+1}\right)^{m+1} D_i^{-m} a_1(\eta_i) e^{-\frac{m(p_i-1)}{mk-1}\eta_i}, \eta_i \in [\eta_0; +\infty).$$

In a specific region around $+\infty$, the solutions to the last system fulfill the following inequalities:

$$\omega_i > 0, \quad (\omega_i^k)' - \frac{mk}{mk-1}\omega_i^k \neq 0.$$

Assuming that $\nu_i(\eta_i) = (L_i\omega)^m$, then

$$\nu_i'(\eta_i) = - \left(a_1(\eta_i) - \frac{m}{mk-1}\right) \nu_i - a_2(\eta_i)\omega_i^{1-k}L_i\omega + a_3(\eta_i) - \omega_i a_4(\eta_i)\omega_i^{p_i}. \tag{23}$$

Furthermore, we consider the functions:

$$\theta_i(\eta_i, \mu_i) = - \left(a_1(\eta_i) - \frac{m}{mk-1}\right) \mu_i - a_2(\eta_i)\omega_i^{1-k}L_i\omega + a_3(\eta_i) - \omega_i a_4(\eta_i)\omega_i^{p_i}, \tag{24}$$

where $\mu_i \in R$.

The functions $\theta_i(\eta_i, \mu_i)$ keep the sign for interval $[\eta_{1i}; +\infty) \subset [\eta_{i0}; +\infty)$ regarding each fixed value μ_i . Therefore, the functions $\theta_i(\eta_i, \mu_i)$ satisfies one of the following inequalities, for all $\eta_i \in [\eta_{1i}; +\infty)$,

$$\nu_i' > 0, \quad \text{or} \quad \nu_i' < 0, \tag{25}$$

from what one can conclude that when $\eta_i \in [\eta_{1i}; +\infty)$:

$$\begin{aligned} \lim_{\eta_i \rightarrow +\infty} a_1(\eta_i) &= \lim_{\eta_i \rightarrow +\infty} a_3(\eta_i) = 0, \\ a_2^0 &= \lim_{\eta_i \rightarrow +\infty} a_2(\eta_i) = \frac{\beta_i}{k} \left(\frac{m}{\nu_i(m+1)} \right)^m, \\ a_4^0 &= \lim_{\eta_i \rightarrow +\infty} a_4(\eta_i) = \begin{cases} 0, & \text{if } p_i > 1 - k + \frac{1}{m}, \\ \left(\frac{m}{m+1} \right)^m \frac{D_i^{-m}}{d_i}, & \text{if } p_i = 1 - k + \frac{1}{m}, \\ +\infty, & \text{if } p_i < 1 - k + \frac{1}{m}. \end{cases} \end{aligned}$$

Suppose now that for the functions $\nu_i(\eta_i)$, a limit $\eta_i \rightarrow +\infty$ does not exist. It should be taken into account the situation where one of the inequalities (25) holds. As $\nu_i(\eta_i)$ are oscillating functions around $\bar{\nu}_i = \mu_i$, and in $[\eta_{1i}; +\infty)$, the intersection of this straight line's graph with itself is infinite.

But given that in the interval $[\eta_{1i}; +\infty)$, this is not possible. Since there is only one real inequality (25), it follows from (24) that the graph of the function $\nu_i(\eta_i)$ only crosses the straight line $\bar{\nu}_i = \mu_i$, once over the interval $[\eta_{1i}; +\infty)$. The function $\nu_i(\eta_i)$ therefore has a limit at $\eta \rightarrow +\infty$.

The functions $\nu_i(\eta_i)$ are assumed they have a limit at $\eta \rightarrow +\infty$. Then, $w'_i(\eta_i)$ has a limit at $\eta \rightarrow +\infty$, and this limit is zero. Then

$$\nu_i(\eta_i) = \left(\frac{mk}{mk-1} \right)^m \left(\omega_i^0 \right)^{km} + o(1),$$

at $\eta \rightarrow +\infty$.

Furthermore, by (23) functions $\nu_i(\eta_i)$ derivatives have limits at $\eta \rightarrow +\infty$, which are plainly equal to zero.

As a result, it is required

$$\lim_{\eta_i \rightarrow \infty} \left[\left(a_1(\eta_i) - \frac{m}{mk-1} \right) \nu_i + a_2(\eta_i) \omega_i^{1-k} L_i \omega - a_3(\eta_i) + \omega_i a_4(\eta_i) \omega_i^{p_i} \right] = 0.$$

And the following algebraic equations can be obtained

$$\frac{mk}{mk-1} \left(\frac{mk}{mk-1} \right)^m k^m \left(\omega^0 \right)^{mk} - a_2^0 \frac{mk}{mk-1} \omega_i^0 = 0,$$

or

$$\omega_i^0 = \left(\frac{\beta_i}{k} \left(\frac{mk-1}{kD_i(m+1)} \right)^m \right)^{\frac{1}{mk-1}}. \tag{26}$$

The best case: $\omega_i^0 = 1$. From the last equation (26), it has been achieved that $\omega_i^0 \approx 1$, and thus $\varphi_i(\xi_i) = \bar{\varphi}_i(\xi_i) \omega_i(\eta_i)$.

Theorem 6. If $p_i > 1 - k + \frac{1}{m}$, and $q_1 q_2 < \frac{m^2(k+1-p_1)(k+1-p_2)}{(m+1)^2}$, then

$$u_i(t, x) = c_i(t+T)^{\alpha_i} \bar{g}_i(\xi_i)(1+o(1)),$$

where $c_i = \left(\frac{mk-1}{m} b_i \gamma_i \right)^{\frac{1}{mk-1}} \left(\frac{mk-1}{B_i m} \right)$.

Proof. Theorem 6 is demonstrated in a manner similar to that of Theorem 5.

2 Numerical solution of the problem

Drawing upon the extensive knowledge in the field of numerical analysis, experts have established that the process of selecting an initial approximation is of utmost importance in maintaining the nonlinear characteristics of a system of equations. Through rigorous research and analysis, it has been determined that an ill-suited initial approximation can lead to significant distortions in the accuracy and efficiency of the numerical solution. As such, professionals in this domain are constantly exploring innovative techniques and methodologies to ensure optimal selection of initial approximations for complex systems. Recognizing this significance, a computer experiment was recently undertaken to investigate the qualitative properties of solutions in relation to the global solvability of the system. To ensure utmost accuracy in our calculations, we employed equation (1) as our primary tool. This equation, which takes into account the second order with respect to x and the first order with respect to t , allows us to accurately model complex systems. By leveraging this approximation method, we can gain a deeper understanding of intricate phenomena and make informed decisions based on highly accurate data. The construction of the iterative process for numerical modeling involved employing the Thomas algorithm to calculate the node values during each step of the iteration. This meticulous approach guarantees the precision and reliability of the numerical analysis for the given system of nonlinear equations.

To shed some light on the effectiveness of different approaches, we conducted a series of numerical experiments. Through these numerical experiments, we were able to gain valuable insights into the influence of different initial approximations on both the convergence of the solution and the preservation of the qualitative properties of the intricate nonlinear processes under study. Our findings revealed that even slight variations in the initial approximations could have a significant impact on the final outcome, highlighting the importance of careful consideration and precise initialization in computational simulations. These results underscore the necessity for thorough numerical analysis and further emphasize the intricate nature of these nonlinear systems. Through our experiments, we were able to gather valuable insights into the behavior of the system of nonlinear equations under different numerical parameters and boundary conditions.

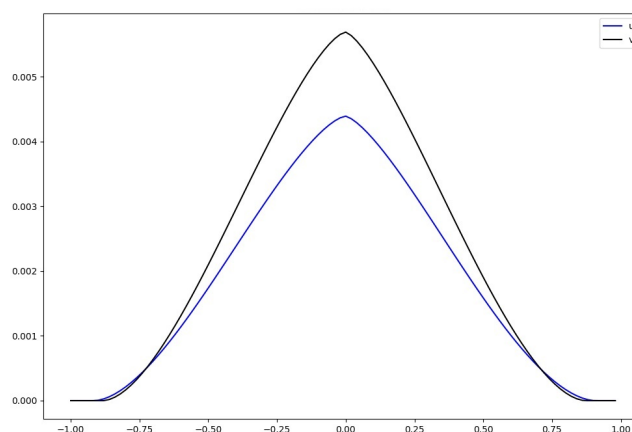


Figure 1. $k = 1.0$, $m = 2.3$, $p_1 = 2.1$, $p_2 = 2.0$, $a_1 = 1$, $a_2 = 1$

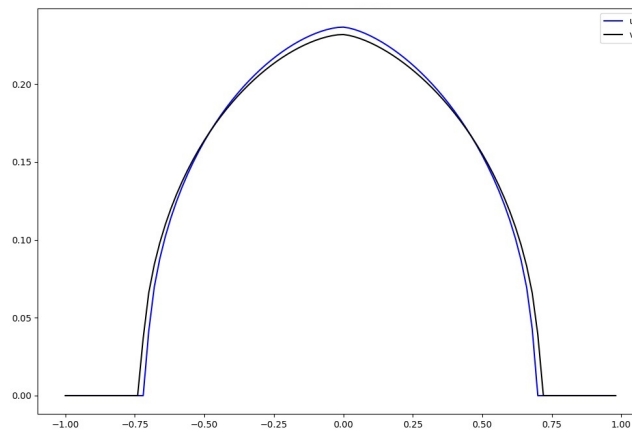


Figure 2. $k = 1.8$, $m = 1.7$, $p_1 = 2.5$, $p_2 = 2.4$, $a_1 = 1$, $a_2 = 1$

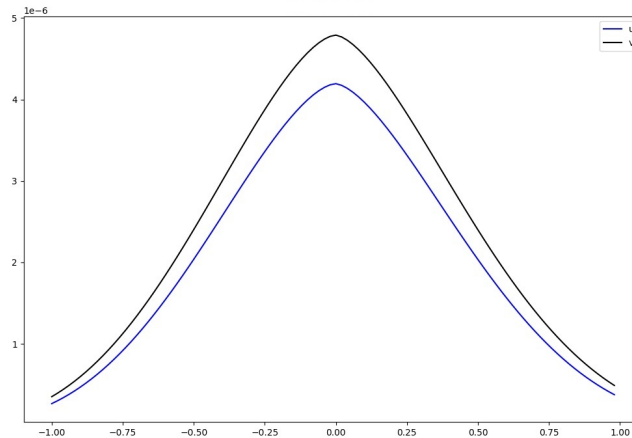


Figure 3. $k = 0.8$, $m = 3.7$, $p_1 = 1.4$, $p_2 = 1.5$, $a_1 = 1$, $a_2 = 1$

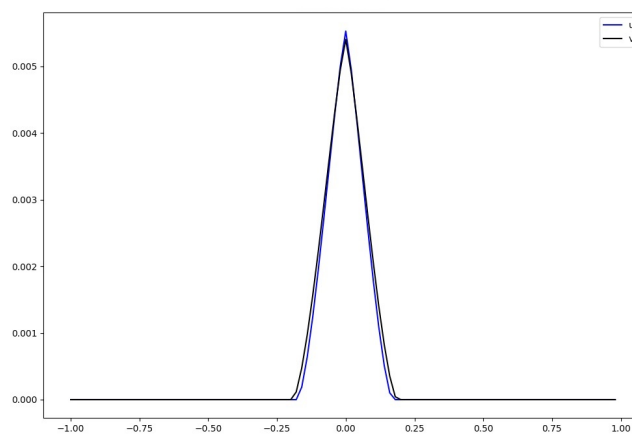


Figure 4. $k = 1.7$, $m = 1.6$, $p_1 = 2.8$, $p_2 = 2.4$, $a_1 = 1$, $a_2 = 1$

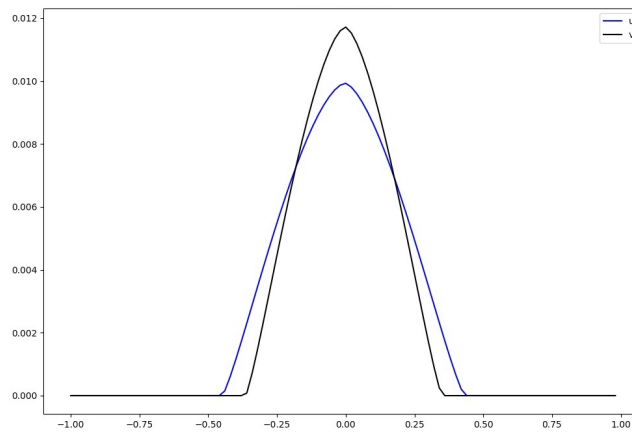


Figure 5. $k = 1.5$, $m = 1.7$, $p_1 = 2.6$, $p_2 = 3.4$, $a_1 = 1$, $a_2 = 1$

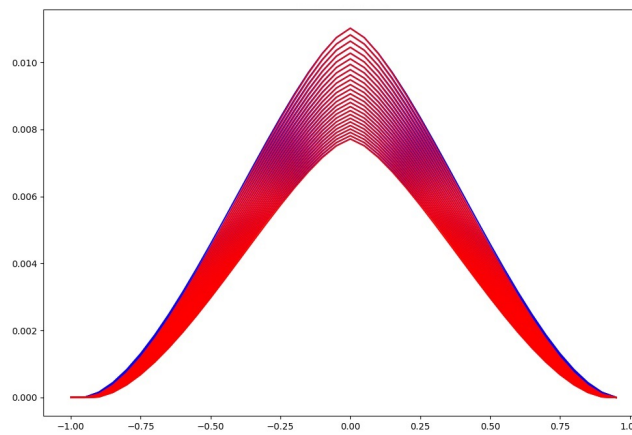


Figure 6. $k = 1.4$, $m = 1.7$, $p_1 = 1.6$, $p_2 = 1.4$, $a_1 = 1$, $a_2 = 1$

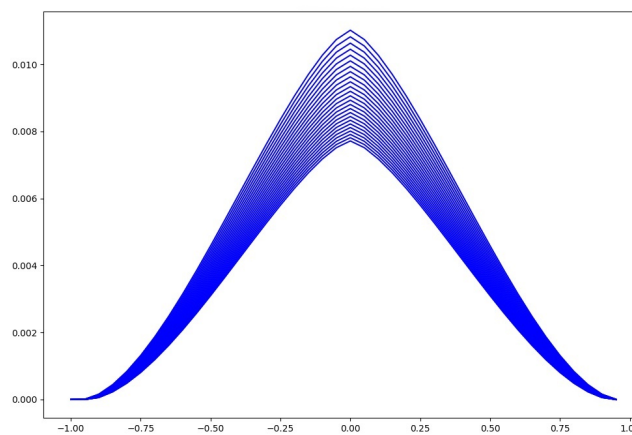


Figure 7. $k = 1.4$, $m = 1.7$, $p_1 = 1.6$, $p_2 = 1.4$, $a_1 = 1$, $a_2 = 1$

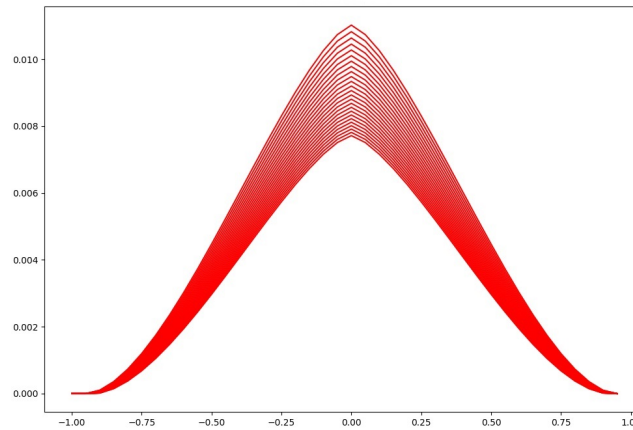


Figure 8. $k = 1.4$, $m = 1.7$, $p_1 = 1.6$, $p_2 = 1.4$, $a_1 = 1$, $a_2 = 1$

Conclusion

It has been established upper and lower estimates for global and unbounded generalized solutions and also Fujita-type critical exponents are obtained for a nonlinear mathematical model of the system of parabolic equations with sources and nonlinear boundary conditions. In the study of a mathematical model of a nonlinear diffusion equation with a double nonlinearity and a source, it has been confirmed that perturbations propagate with finite velocity. This finding sheds light on the behavior of solutions within this complex system, revealing the intricacies of spatial localization. By understanding these properties, researchers can delve deeper into the dynamics of nonlinear diffusion equations, advancing our knowledge in this specialized field of study.

An asymptotic behavior of compactly supported generalized solutions of the nonlinear diffusion problem with a source and with nonlinear damping is proved.

In Figures 1–8, we are presented with a visual representation of the numerical solution to the boundary value problem (1)–(3). These graphs not only provide a comprehensive view of the solution, but also showcase the intricate nature of the problem at hand. By examining these figures, one can discern the complex patterns and behaviors that emerge from this system, further reaffirming the need for rigorous analysis and research in this field. In this case, the process has the property of a finite perturbation propagation velocity. The size of the perturbation propagation region increases with time. The results of numerical experiments provide compelling evidence of the rapid convergence observed in the iterative process. This phenomenon can be attributed to the meticulous selection of the initial approximation, a crucial step that sets the foundation for subsequent computations. Through careful analysis and validation, it becomes evident that this method yields accurate and efficient solutions, making it a valuable tool for tackling complex problems in various domains. All the figures show that the increase in the propagation of a disturbance depends on the numerical parameters of the medium. The numerical experiments conducted in this study have demonstrated the remarkable convergence rate of the iterative process towards the precise solution. This notable result can be attributed to the careful selection of an appropriate initial approximation. Notably, regardless of the variation in numerical parameters, the number of iterations required does not surpass a mere five. Such findings emphasize the efficiency and reliability of our computational methods in solving complex problems.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

- 1 Samarskii, A.A., Galaktionov, V.A., Kurdyumov, S.P., & Mikhailov, A.P. (1995). *Blow-up in Quasilinear Parabolic Equations*. Berlin: Walter de Gruyter.
- 2 Kalashnikov, A.S. (1987). Some problems of the qualitative theory of non-linear degenerate parabolic equations of second-order. *Russian Mathematical Surveys*, 42(2), 169–222. <https://doi.org/10.1070/RM1987v042n02ABEH001309>
- 3 Deng, K., & Levine, H.A. (2000). The role of critical exponents in blow-up theorems. *The sequel J Math Anal Appl.*, 243, 85–126.
- 4 Galaktionov, V.A., & Levine, H.A. (1996). On critical Fujita exponents for heat equations with nonlinear flux conditions on the boundary. *Israel J Math*, 94, 125–146.
- 5 Rakhmonov, Z.R., & Tillaev, A.I. (2018). On the behavior of the solution of a nonlinear polytropic filtration problem with a source and multiple nonlinearities. *Nanosystems: physics, chemistry, mathematics*, 9(3), 323–329. <https://doi.org/10.17586/2220-8054-2018-9-3-323-329>
- 6 Wu, Z.Q., Zhao, J.N., Yin, J.X., & Li, H.L. (2001). *Nonlinear Diffusion Equations*. River Edge, NJ: World Scientific Publishing Co Inc.
- 7 Levine, H.A. (1990). The role of critical exponents in blow up theorems. *SIAM Rev*, 32, 262–288.
- 8 Li, Z.P., Mu, C.L., & Cui, Z.J. (2009). Critical curves for a fast diffusive polytropic filtration system coupled via nonlinear boundary flux. *Math Phys.*, 60, 284–296.
- 9 Chen, B., Mi, Y., & Mu Ch. (2011). Global existence and nonexistence for a doubly degenerate parabolic system coupled via nonlinear boundary flux. *Acta Mathematica Scientia*, 31(2), 681–693.
- 10 Aripov, M.M., Utebaev, D., Kazimbetova, M.M., & Yarlashov, R.Sh. (2023). On convergence of difference schemes of high accuracy for one pseudo-parabolic Sobolev type equation. *Bulletin of the Karaganda University. Mathematics series*, 1(109), 24–37. <https://doi.org/10.31489/2023M1/24-37>
- 11 Aripov, M.M., Utebaev, D., & Nurullaev, Zh.A. (2022). On the convergence of difference schemes of high accuracy for the equation of ion-acoustic waves in a magnetized plasma. *Bulletin of the Karaganda University. Mathematics series*, 4(108), 4–19. <https://doi.org/10.31489/2022M4/4-19>
- 12 Zhaoyin, X., Chunlai, M., & Yulan, W. (2009). Critical curve of the non-Newtonian polytropic filtration equations coupled via nonlinear boundary flux. *Rocky mountain journal of mathematics*, 39(2), 76–89. <https://www.jstor.org/stable/44239573>
- 13 Aripov, M.M., & Matyakubov, A.S. (2017). To the qualitative properties of solution of system equations not in divergence form of polytrophic filtration in variable density. *Nanosystems: Physics, Chemistry, Mathematics*, 8(3), 317–322. <https://doi.org/10.17586/2220-8054-2017-8-1-5-12>
- 14 Rakhmonov, Z., Urunbaev, J., & Alimov, A. (2022). Properties of solutions of a system of nonlinear parabolic equations with nonlinear boundary conditions. *AIP Conference Proceedings*, 2637(1), 040008. <https://doi.org/10.1063/5.0119747>
- 15 Rakhmonov, Z., Parovik, R., & Alimov, A. (2021). Global existence and nonexistence for a multidimensional system of parabolic equations with nonlinear boundary conditions. *AIP Conference Proceedings*, 2365(1), 060022. <https://doi.org/10.1063/5.0057139>

- 16 Aripov, M., Matyakubov, A., & Bobokandov, M. (2023). Cauchy problem for the heat dissipation equation in non-homogeneous medium. *AIP Conference Proceedings*, 2781(1), 020027. <https://doi.org/10.1063/5.0144807>
- 17 Rakhmonov, Z.R., Khaydarov, A.T., & Urumbaev, J.E. (2020). Global Existence and Nonexistence of Solutions to a Cross Diffusion System with Nonlocal Boundary Conditions. *Mathematics and Statistics*, 8(4), 404–409. <https://doi.org/10.13189/ms.2020.080405>
- 18 Rakhmonov, Z. (2016). On the properties of solutions of multidimensional nonlinear filtration problem with variable density and nonlocal boundary condition in the case of fast diffusion. *Journal of Siberian Federal University. Mathematics & Physics*, 9, 236–245. <https://doi.org/10.17516/1997-1397-2016-9-2-225-234>

Бейсыздықты шекаралық шарттары және дереккөзі бар бейсыздықты диффузиялық жүйе шешімдерінің өзгеруі туралы

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Мақалада баяу диффузия жағдайындағы бейсыздықтық шекаралық шарттары бар бейсыздықты диффузиялық жүйенің глобалды шешілетіндігі және шешілмейтіндігі зерттелген. Бейсыздықты модельдерінің реакция-диффузия, жылу алмасу, сүзу және басқа да физикалық, химиялық және биологиялық процестердің сапалық сипаттамаларын талдауда маңызды рөл атқаратын критикалық Фуджита типті көрсеткіші алынды. Глобалды шешімділік жағдайында асимптотикалық шешімдердің негізгі компоненттері алынады. Зерттелетін бейсыздықты процестердің сапалық сипаттамаларын сақтай отырып, нақты шешімге тез жақындайтын итерациялық әдістер сәйкес бастапқы жуықтаудың болуын талап ететіні белгілі. Бұл бейсыздықтық есептерді сандық шешу үшін күрделі мәселе болып табылады. Бастапқы жуықтауларды сәтті таңдау есепті шешуге мүмкіндік береді, ол теңдеудің сандық параметрлерінің мәніне байланысты, олар бірінші кезекте асимптотикалық формуланы қолданатын есептеулерде ұсынылады. Итерациялық процестің бастапқы жуықтауы ретінде өзіндік ұқсас шешімдердің асимптотикасын пайдаланып, сандық есептеулер жүргізілген және нәтижелер талдауы берілген. Сандық тәжірибелерден алынған нәтижелер бейсыздық диффузиялық жүйеде қарастырылатын процестің физикасымен тамаша сәйкес келетінін көрсетеді.

Кілт сөздер: күрделену режимі, бейсыздықты шекаралық шарт, шешімінің бар болуының критикалық глобалды қисығы, өзгешеленген параболалық жүйелер, Фуджита типті критикалық көрсеткіш.

О поведении решений нелинейной диффузионной системы с источником и нелинейными граничными условиями

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Изучены глобальная разрешимость и неразрешимость нелинейной диффузионной системы с нелинейными граничными условиями в случае медленной диффузии. Получены критические показатели типа Фуджита и существования, которые играют существенную роль при анализе качественных характеристик нелинейных моделей реакций—диффузии, теплопереноса, фильтрации и других физических, химических и биологических процессов. В случае глобальной разрешимости получены ключевые компоненты асимптотических решений. Известно, что итерационные методы, быстро сходящиеся к

точному решению при сохранении качественных характеристик изучаемых нелинейных процессов, требуют наличия соответствующего начального приближения. Это представляет собой серьезную проблему для численного решения нелинейных задач. Успешный выбор начальных приближений позволяет решить эту задачу, которая зависит от значения числовых параметров уравнения, которые, в первую очередь, в расчетах рекомендуются с использованием асимптотической формулы. Применяя асимптотику автомодельных решений в качестве начального приближения итерационного процесса, проведены численные расчеты и приведен анализ результатов. Результаты численных экспериментов показывают, что полученные результаты прекрасно согласуются с физикой рассматриваемого процесса в нелинейной диффузионной системе.

Ключевые слова: режим с обострением, нелинейное граничное условие, критическая глобальная кривая существования, вырожденные параболические системы, критические показатели типа Фуджиты.

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