

## The problem with the missing Goursat condition at the boundary of the domain for a degenerate hyperbolic equation with a singular coefficient

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The work is devoted to the formulation and study of the solvability for a problem with missing conditions on the characteristic boundary of the domain and an analogue of the Frankl condition on the segment of the degeneracy for a hyperbolic equation. The difference between this problem and known local and nonlocal problems is that, firstly, a hyperbolic equation is taken with arbitrary positive power degeneracy and singular coefficients on the part of the boundary, and secondly, the characteristic boundary of the domain is arbitrarily divided into two pieces and the value of the desired function is set on the first piece, and the second piece is freed from the boundary condition and this missing Goursat condition is replaced by an analogue of the Frankl condition on the degeneracy segment, and the value of an unknown function on another characteristic boundary of the domain is also considered to be known. The conditions for the coefficients of the equation and the data of the formulated problem, ensuring the validity of the uniqueness theorem are found. The theorem of the existence of a solution to the problem is proved by reducing to the problem of solving a non-standard singular integral equation with a non-Fredholm integral operator in the non-characteristic part of the equation, the kernel of which has an isolated first-order singularity. Applying the Carleman regularization method to the received equation, the Wiener-Hopf integral equation is obtained. It is proved that the index of the Wiener-Hopf equation is zero, therefore it is uniquely reduced to the Fredholm integral equation of the second kind, the solvability of which follows from the uniqueness of the problem's solution.

*Keywords:* Hyperbolic equation degenerating at the boundary of the domain, missing Goursat condition, Frankl condition, singular coefficient, complete orthogonal system of functions, singular integral equation, Wiener-Hopf equation, index.

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### Introduction

Many scientific and practical studies conducted in various fields of mathematics in most cases lead to the study of models of gas dynamics problems, the theory of infinitesimal bends of rotation surfaces, the instantaneous theory of shells and mathematical biology. The study of the fundamental laws of gas dynamics by solving boundary value problems for partial differential equations with singularities in coefficients is an urgent problem.

The development of the theory of degenerate hyperbolic and elliptic equations and mixed type equations originates from the well-known fundamental works of G. Darboux (1894), F. Tricomi (1923), E. Holmgren (1927) and S. Gellerstedt (1938).

After these works, the theory of boundary value problems for degenerate hyperbolic and mixed-type equations began to develop rapidly. E. Holmgren, S. Gellerstedt, F.I. Frankl, M. Keldysh, S.G. Mikhlin,

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A.V. Bitsadze, K.I. Babenko, M.S. Salakhiddinov, A.M. Nakhushhev, E.I. Moiseev, and many other scientists made significant contributions to the development of this theory. Degenerate hyperbolic equations with singular coefficients have the peculiarity that the well-posedness of the Cauchy problem does not always take place. The initial problem in the usual formulation may turn out to be unsolvable if the equation degenerates along a line that is both a characteristic or, the coefficients of the hyperbolic equation for the lower terms are singular [1]. Therefore, it is natural to consider a modified Cauchy problem when the initial conditions on the line of parabolic degeneracy are given with weight functions [2, 3].

It is relevant to study the well-posedness of non-standard problems for degenerate hyperbolic equations with singular coefficients. Especially if, in the formulated problems, local and non-local conditions are given in a non-standard form, in particular, the Goursat condition is set on some part of the characteristic; the Bitsadze-Samarsky condition is set on the boundary and parallel internal characteristic, a Frankl type condition is set on the line of degeneracy of the equation, then problems with such non-classical conditions are reduced to previously unexplored new non-standard singular equations, the singular and non-singular parts of the kernel of which are not reduced to each other by means of a fractional linear transformation.

Nowadays, there are many articles and books devoted to the theoretical and applied aspects of degenerate hyperbolic and mixed-type equations [1–14]. It should be noted that the bibliography does not pretend to be complete and mainly concerns issues close to this work.

The work is devoted to the study of the uniqueness and existence of a non-standard problem for a degenerate hyperbolic equation with singular coefficients in a domain bounded by two characteristics of a different family and a segment of the line of degeneration of the equation (characteristic triangle). The peculiarity of this problem is that part of the characteristic boundary of the domain is freed from the Goursat conditions, and Frankl-type conditions are set on the line of the equation's degeneracy.

The purpose of this work is to find conditions for the equation coefficients and the data of the problem, which ensure the validity of the theorems on the existence and uniqueness of the non-standard problem posed.

The work consists of an introduction, three sections and a conclusion.

The first section provides a description of the domain and a restriction on the equation coefficients for a degenerate hyperbolic equation. The statement of the main and auxiliary problems is given.

In the second section, the conditions for the equation coefficients and the data of the problem are found, ensuring the validity of the uniqueness theorem of the problem solution.

In the third section, the existence of a solution to the problem is proved by reducing a non-standard singular integral equation to a solution. Using the Carleman regularization method and the theory of the Wiener-Hopf integral equation, this equation is uniquely reduced to the Fredholm integral equation of the second kind, the solvability of which follows from the uniqueness theorem of the problem solution.

### 1 Problem formulation A

Let  $\Omega^-$  be the characteristic triangle of the half-plane  $y < 0$  bounded by characteristics  $AC_1$  and  $BC_1$  of the equation

$$-(-y)^m u_{xx} + u_{yy} + \alpha_0(-y)^{(m-2)/2} u_x + (\beta_0/y) u_y = 0, \quad y < 0, \quad (1)$$

and segment  $AB$ , where  $A(-1, 0)$ ,  $B(1, 0)$ ,  $C_1\left(0, -((m+2)/2)^{2/(m+2)}\right)$ ,  $m, \alpha_0, \beta_0$  are some constants satisfying conditions  $m > 0$ ,  $-m/2 < \beta_0 < 1$ ,  $-(m+2)/2 < \alpha_0 < (m+2)/2$  [4–7].

Correctness of setting boundary value problems for equation (1) significantly depends on its numerical parameters  $\alpha_0$  and  $\beta_0$  coefficients for the lower terms of the equation, on the parameter plane  $\alpha_0 O \beta_0$

consider a triangle  $A_0^*B_0^*C_0^*$  bounded by straight lines

$$A_0^*C_0^* : \beta_0 + \alpha_0 = -m/2; \quad B_0^*C_0^* : \beta_0 - \alpha_0 = -m/2; \quad A_0^*B_0^* : \beta_0 = 1.$$

Let  $P(\alpha_0, \beta_0) \in \Delta A_0^*B_0^*C_0^*$ , i.e.  $0 < \alpha, \beta < 1$ ,  $\alpha + \beta < 1$ , where  $\alpha = (m + 2(\beta_0 + \alpha_0)) / 2(m + 2)$ ,  $\beta = (m + 2(\beta_0 - \alpha_0)) / 2(m + 2)$ .

Denote by  $A_0$  and  $B_0$  intersection points of the characteristics  $AC_1$  and  $BC_1$  respectively with a characteristic coming from a point  $E(c, 0)$ , where  $c \in J_0 = (-1, 1)$  is an interval of the axis  $y = 0$ .

Let the linear function  $p(x) = \delta - kx$ , where  $k = (1 - c)/(1 + c)$ ,  $\delta = 2c/(1 + c)$  display the set of points of the segment  $[-1, c]$  on the set of points of the segment  $[c, 1]$  and  $p(-1) = 1$ ,  $p(c) = c$ .

In the Goursat problem, the carriers of boundary conditions are boundary characteristics  $AC_1$  and  $BC_1$ .

This work is devoted to the study of the correctness of the problem in the domain  $\Omega^-$ , for hyperbolic equation (1) degenerating at the boundary of the domain, when the boundary characteristic  $AC_1$  of the domain  $\Omega^-$  is arbitrarily divided into two pieces  $AA_0$  and  $A_0C_1$  and on the first piece  $AA_0 \subset AC_1$  the value of the desired function is set, and the second piece  $A_0C_1 \subset AC_1$  it is freed from the boundary condition and this missing Goursat condition is replaced by an analogue of the Frankl condition [8–12] on the degeneration segment  $AB$ .

*Problem A.* In the domain  $\Omega^-$  it is required to find the function  $u(x, y) \in C(\bar{\Omega}^-)$  satisfying the following conditions:

1)  $u(x, y)$  is generalized solution to the equation (1) from the class  $R_1$  [13].

2)

$$u(x, y) |_{BC_1} = \psi_1(x), \quad 0 \leq x \leq 1, \tag{2}$$

3)

$$u(x, y) |_{AA_0} = \psi_2(x), \quad -1 \leq x \leq (c - 1)/2, \tag{3}$$

4)

$$u(x, 0) - \mu u(p(x), 0) = f(x), \quad -1 \leq x \leq c, \tag{4}$$

where  $\mu = const$ ,  $\psi_1(x) \in C[0, 1] \cap C^2(0, 1)$ ,  $\psi_2(x) \in C[-1, (c - 1)/2] \cap C^2(-1, (c - 1)/2)$ ,  $f(x) \in C[-1, c] \cap C^2(-1, c)$ , and  $\psi_1(1) = 0$ ,  $\psi_2(-1) = 0$ ,  $f(c) = 0$ .

Condition (3) is an incomplete condition of the Course, since it is set only on  $AA_0$  part of characteristic  $AC_1$ .

Condition (4) is an analogue of Frankl condition [14] on the degeneracy segment  $AB$ .

By virtue of the designation  $u(x, 0) = \tau(x)$  condition (4) we write in the form

$$\tau(x) - \mu\tau(p(x)) = f(x), \quad x \in [-1, c]. \tag{4*}$$

Let  $\Omega^+$  be a symmetrical domain to the  $\Omega^-$  with respect to the axis  $y = 0$ , lying in a half-plane  $y > 0$  and let  $\Omega = \Omega^- \cup \Omega^+ \cup AB$ . The domain  $\Omega^+$  is bounded with characteristics  $AC_2$  and  $BC_2$  of the equation

$$-y^m u_{xx} + u_{yy} + \alpha_0 y^{(m-2)/2} u_x + (\beta_0/y) u_y = 0, \quad y > 0, \tag{5}$$

where  $C_2(0, (m + 2)/2)^{2/(m+2)}$ .

Note that if  $u(x, y)$  is a solution to equation (1) in a half-plane  $y < 0$ , then  $u(x, -y)$  is a solution to equation (5) in a half-plane  $y > 0$ . Due to this property of solutions to equations (1) and (5) in a symmetrical domain  $\Omega$  we consider an auxiliary problem  $A^*$ .

*Problem formulation  $A^*$ .* It is required to find in the domain  $\Omega$  the function  $u(x, y) \in C(\bar{\Omega})$  satisfying conditions:

1)  $u(x, y)$  is a generalized solution from the class  $R_1$  in domains  $\Omega^-$  and  $\Omega^+$ ;

2)  $u(x, y)$  satisfies the condition

$$u(x, y) |_{BC_2} = \psi_1(x), \quad 0 \leq x \leq 1, \tag{6}$$

and conditions (3) and (4) of Problem A.

3) on the degeneracy segment  $y = 0, -1 < x < 1$ , a conjugation condition takes place

$$\lim_{y \rightarrow -0} (-y)^{\beta_0} \frac{\partial u}{\partial y} = - \lim_{y \rightarrow +0} y^{\beta_0} \frac{\partial u}{\partial y} = \nu(x), \quad x \in J_0, \tag{7}$$

moreover, these limits at  $x \rightarrow \pm 1$  may have features of the order less than  $1 - \alpha - \beta$ , where  $\alpha + \beta = (m + 2\beta_0)/(m + 2) \in (0, 1)$ .

Let  $u(x, y)$  is a solution to the problem  $A^*$ , we show that  $u(x, y) |_{BC_1} = \psi(x)$ . It is obvious from the form equations (1) and (5) that if  $u(x, y)$  is a solution to equation (1) in the half-plane  $y < 0$  ( $y > 0$ ) then  $u(x, -y)$  is a solution to equation (5) in the half-plane  $y > 0$  ( $y < 0$ ). Hence from the design of solutions (see below (9)) of equations (1) and (5) it can be seen that for symmetric with respect to the axis  $y = 0$  points  $(x, y) \in \Omega^-$  and  $(x, -y) \in \Omega^+$  the equality  $u(x, y) = u(x, -y)$  takes place and by virtue of continuity of solutions, this equality is also preserved for points of characteristics  $BC_1$  and  $BC_2$  then by virtue of (2)  $u(x, y) |_{BC_1} = u(x, -y) |_{BC_2} = \psi(x)$ , where  $y < 0$ , that is what needed to be shown.

Hence the solution to the problem  $A^*$ , in domain  $\Omega^-$  will also be a solution to the  $A$  problem in the same domain  $\Omega^-$ . Thus, the study of the problem  $A$  is reduced to solving the problem  $A^*$ .

### 2 Uniqueness of the problem solution $A^*$

The solution to equation (1) in domains  $\Omega^-, \Omega^+$  satisfying modified Cauchy conditions:

$$\lim_{y \rightarrow 0} u(x; y) = \tau(x), \quad x \in \bar{J}; \quad \lim_{y \rightarrow \pm 0} |y|^{\beta_0} \frac{\partial u}{\partial y} = \mp \nu(x), \quad x \in J_0, \tag{8}$$

has the form [14]

$$u(x, y) = \gamma_1 \int_{-1}^1 \tau \left[ x + \frac{2t}{m+2} |y|^{\frac{m+2}{2}} \right] (1+t)^{\beta-1} (1-t)^{\alpha-1} dt + \gamma_2 |y|^{1-\beta_0} \times \int_{-1}^1 \nu \left[ x + \frac{2t}{m+2} |y|^{\frac{m+2}{2}} \right] (1+t)^{-\alpha} (1-t)^{-\beta} dt, \tag{9}$$

where

$$\gamma_1 = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} 2^{1-\alpha-\beta}, \quad \gamma_2 = - \frac{\Gamma(2 - \alpha - \beta)}{(1 - \beta_0)\Gamma(1 - \alpha)\Gamma(1 - \beta)} 2^{\alpha+\beta-1}.$$

By virtue of (9) from boundary condition (6) (taking place in domain  $\Omega^+$ ) we have

$$\gamma_1 \left( \frac{1-x}{2} \right)^{1-\alpha-\beta} \int_x^1 \frac{\tau(s)(1-s)^{\alpha-1}}{(s-x)^{1-\beta}} ds - \gamma_2 \left( \frac{m+2}{2} \right)^{1-\alpha-\beta} \times \int_x^1 \frac{\nu(s)(1-s)^{-\beta}}{(s-x)^\alpha} ds = \Psi_1(x), \quad x \in (-1, 1),$$

or

$$\nu(x) = -\gamma D_{x,1}^{1-\alpha-\beta} \tau(x) + \Psi_1(x), \quad x \in (-1, 1), \tag{10}$$

where  $D_{x,1}^l$  is a fractional differentiation operator.

$$\gamma = \frac{2\Gamma(1-\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(1-\alpha-\beta)} \left(\frac{m+2}{4}\right)^{\alpha+\beta},$$

$$\Psi_1(x) = -\frac{(2/(m+2))^{1-\alpha-\beta}}{\gamma_2\Gamma(1-\alpha)}(1-x)^\beta D_{x,1}^{1-\alpha}\psi_1\left(\frac{1+x}{2}\right).$$

Relation (10) is the first functional relationship between unknown functions  $\tau(x)$  and  $\nu(x)$  brought to the interval  $(-1, 1)$  from the domain  $\Omega^+$ . Note, that relation (10) is valid for the entire interval  $(-1, 1)$ .

Now by virtue of (9) from boundary condition (3)(taking place in the domain  $\Omega^-$ ) we obtain

$$\nu(x) = \gamma D_{-1,x}^{1-\alpha-\beta}\tau(x) + \Psi_2(x), \quad x \in (-1, c), \tag{11}$$

where

$$\Psi_2(x) = \frac{(2/(m+2))^{1-\alpha-\beta}}{\gamma_2\Gamma(1-\beta)}(1+x)^\alpha D_{-1,x}^{1-\beta}\psi_2\left(\frac{x-1}{2}\right).$$

Relation (11) is the second functional relationship between unknown functions  $\tau(x)$  and  $\nu(x)$  brought to the interval  $(-1, c)$  from the domain  $\Omega^-$ .

*Theorem 1.* The problem  $A^*$  when the condition

$$k^{\alpha+\beta} < \mu^2 \tag{12}$$

is met, can have no more than one solution.

*Proof.*  $1^0$ . Using (10) ( $c \Psi_1(x) \equiv 0$ ), we prove that

$$J = \int_{-1}^1 \tau(x)\nu(x)dx \leq 0. \tag{13}$$

We calculate

$$J = -\gamma \int_{-1}^1 \tau(x) \left( D_{x,1}^{1-\alpha-\beta}\tau(x) \right) dx = \frac{\gamma}{\Gamma(\alpha+\beta)} \int_{-1}^1 \tau(x) \times$$

$$\times \left( \frac{d}{dx} \int_x^1 \frac{\tau(t)dt}{(t-x)^{1-\alpha-\beta}} \right) dx. \tag{14}$$

Let

$$\tau(x) = \int_x^1 \frac{\tau_1(s)ds}{(s-x)^{\alpha+\beta}}, \quad x \in (-1, 1), \tag{15}$$

where  $\tau_1(x) \in C(\bar{J}) \cap C^2(J)$ ,  $\tau_1(1) = \tau_1'(1) = 0$ .

By virtue of (15) equality (14) has the form

$$J = \frac{\gamma}{\Gamma(\alpha+\beta)} \int_{-1}^1 \tau(x) \left( \frac{d}{dx} \int_x^1 \frac{dt}{(t-x)^{1-\alpha-\beta}} \int_t^1 \frac{\tau_1(s)ds}{(s-t)^{\alpha+\beta}} \right) dx. \tag{16}$$

It is not difficult to prove that

$$\frac{d}{dx} \int_x^1 \frac{dt}{(t-x)^{1-\alpha-\beta}} \int_t^1 \frac{\tau_1(s)ds}{(s-t)^{\alpha+\beta}} = -\Gamma(\alpha+\beta)\Gamma(1-\alpha-\beta)\tau_1(x). \tag{17}$$

Therefore, taking into account (17) we write equality (16) in the form

$$J = -\gamma\Gamma(1 - \alpha - \beta) \int_{-1}^1 \tau(x)\tau_1(x)dx. \tag{18}$$

Now by virtue (15) we transform (18) to the form

$$J = -\gamma\Gamma(1 - \alpha - \beta) \int_{-1}^1 \tau_1(x)dx \int_x^1 \frac{\tau_1(s)ds}{(s - x)^{\alpha+\beta}}. \tag{19}$$

Here, changing the order of integration, we have

$$J = -\gamma\Gamma(1 - \alpha - \beta) \int_{-1}^1 \tau_1(s)ds \int_{-1}^s \frac{\tau_1(x)dx}{(s - x)^{\alpha+\beta}}. \tag{20}$$

In (19) swapping integration variables  $s$  and  $x$ , we have

$$J = -\gamma\Gamma(1 - \alpha - \beta) \int_{-1}^1 \tau_1(s)ds \int_s^1 \frac{\tau_1(x)dx}{(x - s)^{\alpha+\beta}}. \tag{21}$$

Now summing up (20) and (21), we obtain

$$J = -\frac{\gamma\Gamma(1 - \alpha - \beta)}{2} \int_{-1}^1 \int_{-1}^1 \frac{\tau_1(s)\tau_1(x)}{|s - x|^{\alpha+\beta}} dx ds. \tag{22}$$

Let us now use the following well-known formula for the function  $\Gamma(z)$

$$\int_0^\infty t^{z-1} \cos(kt) dt = \frac{\Gamma(z)}{k^z} \cos\left(\frac{z\pi}{2}\right), \quad k > 0, \quad 0 < z < 1. \tag{23}$$

Let in (23)  $k = |s - x|$ ,  $z = \alpha + \beta$  then from (23) we have

$$\frac{1}{|s - x|^{\alpha+\beta}} = \frac{1}{\Gamma(\alpha + \beta) \cos((\alpha + \beta)\pi/2)} \int_0^\infty \xi^{\alpha+\beta-1} \cos((s - x)\xi) d\xi. \tag{24}$$

By virtue (24) we write equality (22) in the form

$$\begin{aligned} J &= -\frac{\gamma\Gamma(1 - \alpha - \beta)}{2\Gamma(\alpha + \beta) \cos((\alpha + \beta)\pi/2)} \int_0^\infty \xi^{\alpha+\beta-1} d\xi \int_{-1}^1 \int_{-1}^1 \tau_1(s) \cdot \\ &\cdot \tau_1(x) \cos((s - x)\xi) dx ds = -\frac{\gamma\Gamma(1 - \alpha - \beta)}{2\Gamma(\alpha + \beta) \cos((\alpha + \beta)\pi/2)} \int_0^\infty \xi^{\alpha+\beta-1} \cdot \\ &\cdot \left\{ \left[ \int_{-1}^1 \tau_1(t) \cos(t\xi) dt \right]^2 + \left[ \int_{-1}^1 \tau_1(t) \sin(t\xi) dt \right]^2 \right\} d\xi. \end{aligned} \tag{25}$$

Thus, by virtue (25) we obtain inequality (13).

$2^0$ . Now using (11) ( $\Psi_2(x) \equiv 0$ ) and condition (4\*) we show that integral (13) is not negative, i.e.

$$J = \int_{-1}^1 \tau(x)\nu(x)dx \geq 0. \tag{26}$$

Indeed

$$J = \int_{-1}^1 \tau(x)\nu(x)dx = \int_{-1}^c \tau(x)\nu(x)dx + \int_c^1 \tau(x)\nu(x)dx, \tag{27}$$

here we transform the second integral of the right-hand part (27) i.e.

$$J_1 = \int_c^1 \tau(x)\nu(x)dx. \tag{28}$$

In (28) by replacing the variable integration  $x = p(t) = \delta - kt$ , we get

$$J_1 = k \int_{-1}^c \tau(p(t))\nu(p(t))dt. \tag{29}$$

Now we will find  $\nu(p(x))$ , for this purpose, we use relation (10) which is the case for the entire interval  $J_0 = (-1, 1)$  in particular for  $x \in (-1, 1)$ :

$$\nu(x) = -\gamma D_{x,1}^{1-\alpha-\beta} \tau(x) = \frac{\gamma}{\Gamma(\alpha + \beta)} \frac{d}{dx} \int_x^1 \frac{\tau(t)dt}{(t-x)^{1-\alpha-\beta}}, \quad x \in (c, 1).$$

Here, firstly, performing the integration operation in parts, then, performing the differentiation operation, we have

$$\nu(x) = \frac{\gamma}{\Gamma(\alpha + \beta)} \int_x^1 \frac{\tau'(t)dt}{(t-x)^{1-\alpha-\beta}}, \quad x \in (c, 1). \tag{30}$$

In (30) by replacing the variable  $x \in (c, 1)$  to  $p(x)$  (where  $p(x) \in (c, 1)$ , and an argument  $x \in (-1, c)$ ) we obtain

$$\nu(p(x)) = \frac{\gamma}{\Gamma(\alpha + \beta)} \int_{p(x)}^1 \frac{\tau'(t)dt}{(t-p(x))^{1-\alpha-\beta}}, \quad x \in (-1, c). \tag{31}$$

Now, in (31) by replacing the variable integration  $t = p(s)$ , taking into account the condition (4\*) ( $f(x) \equiv 0$ ):  $\tau(x) = \mu\tau(p(x))$ ,  $\tau'(x) = -\mu\tau'(p(x))$ , we calculate

$$\begin{aligned} \nu(p(x)) &= -\frac{\gamma k^{\alpha+\beta-1}}{\mu\Gamma(\alpha + \beta)} \int_{-1}^x \frac{\tau'(t)dt}{(x-t)^{1-\alpha-\beta}} = -\frac{\gamma k^{\alpha+\beta-1}}{\mu\Gamma(\alpha + \beta)} \lim_{\varepsilon \rightarrow 0} \int_{-1}^{x-\varepsilon} \frac{d\tau(t)}{(x-t)^{1-\alpha-\beta}} = \\ &= -\frac{\gamma k^{\alpha+\beta-1}}{\mu\Gamma(\alpha + \beta)} \lim_{\varepsilon \rightarrow 0} \left[ \frac{\tau(t)}{(x-t)^{1-\alpha-\beta}} \Big|_{-1}^{x-\varepsilon} - (1-\alpha-\beta) \int_{-1}^{x-\varepsilon} \frac{\tau(t)dt}{(x-t)^{2-\alpha-\beta}} \right] = \\ &= -\frac{\gamma k^{\alpha+\beta-1}}{\mu\Gamma(\alpha + \beta)} \lim_{\varepsilon \rightarrow 0} \left[ \frac{d}{dx} \int_{-1}^{x-\varepsilon} \frac{\tau(t)dt}{(x-t)^{1-\alpha-\beta}} \right]. \end{aligned} \tag{32}$$

In (32) moving to the limit at  $\varepsilon \rightarrow 0$ , we will have

$$\nu(p(x)) = -\frac{\gamma k^{\alpha+\beta-1}}{\mu} D_{-1,x}^{1-\alpha-\beta} \tau(x), \quad x \in (-1, c),$$

due to this equality, relation (29) is written as

$$J_1 = -\frac{\gamma k^{\alpha+\beta}}{\mu} \int_{-1}^c \tau(p(x)) D_{-1,x}^{1-\alpha-\beta} \tau(x) dx. \tag{33}$$

Now, taking into account (4\*) ( $c f(x) \equiv 0$ ):  $\tau(p(x)) = \tau(x)/\mu$  and relations (33) equality (27) has the form

$$\begin{aligned} J &= \int_{-1}^1 \tau(x)\nu(x)dx = \gamma \left( 1 - \frac{k^{\alpha+\beta}}{\mu^2} \right) \int_{-1}^c \tau(x) \left( D_{-1,x}^{1-\alpha-\beta} \tau(x) \right) dx = \\ &= \frac{\gamma\Gamma(1-\alpha-\beta)a^{2-(\alpha+\beta)}}{2\Gamma(\alpha + \beta)\cos((\alpha + \beta)\pi/2)} \left( 1 - \frac{k^{\alpha+\beta}}{\mu^2} \right) \int_0^\infty \xi^{\alpha+\beta-1} \times \\ &\times \left\{ \left[ \int_{-1}^1 \tau_2(at-b)\cos(t\xi)dt \right]^2 + \left[ \int_{-1}^1 \tau_2(at-b)\sin(t\xi)dt \right]^2 \right\} d\xi, \end{aligned} \tag{34}$$

where

$$\tau(x) = \int_{-1}^x \frac{\tau_2(s)ds}{(x-s)^{\alpha+\beta}}, \quad x \in (-1, c),$$

$\tau_2(x) \in C[-1, c] \cap C^2(-1, c)$ ,  $\tau_2(-1) = \tau_2'(-1) = 0$ , taking into account (12) from (34) it follows (26).

Therefore by virtue of inequalities (13) and (26) we have

$$J = \int_{-1}^1 \tau(x)\nu(x)dx = 0.$$

Thus, the right-hand side of (25) is equal to zero, but both terms of the integral expression in (25) are non-negative, therefore they are also equal to zero:

$$\int_{-1}^1 \tau_1(t)\cos(t\xi)dt \equiv 0, \quad \int_{-1}^1 \tau_1(t)\sin(t\xi)dt \equiv 0, \tag{35}$$

for all  $\xi \in [0, +\infty]$  and in particular for  $\xi = k\pi, k = 0, 1, 2, \dots$ , for such values  $\xi$  trigonometric systems of functions  $\cos(t\xi)$  and  $\sin(t\xi)$  form a complete orthogonal system of functions in  $L_2[-1, 1]$ . Therefore, in (35)  $\tau_1(t) \equiv 0$  almost everywhere on  $[-1, 1]$ , but by virtue of continuity of the function  $\tau_1(x)$  on  $[-1, 1]$  it follows, that  $\tau_1(x) \equiv 0$  everywhere  $\forall x \in [-1, 1]$ , hence by virtue of (15) we conclude that  $\tau(x) \equiv 0, \forall x \in [-1, 1]$ . Hence by virtue of (10) ( $c \Psi_1(x) \equiv 0$ ) and also it follows, that  $\nu(x) \equiv 0, \forall x \in (-1, 1)$ . Now by virtue of (7), restoring the solution of the problem  $A^*$  as solutions of modified Cauchy problem with zero modified initial Cauchy data (8) ( $c \tau(x) \equiv 0, \nu(x) \equiv 0$ ) according to the Darboux formula (9) we obtain  $u(x, y) \equiv 0$  в  $\bar{\Omega}$ . Theorem 1 is proved.

### 3 The existence of a solution to the problem $A^*$

*Theorem 2.* Let for the numerical parameters of problem  $A^*$  inequality (12) be valid

$$\frac{k^{(1-2\theta)/2} \sin(\theta\pi) |\ln k|}{\mu} < 1, \tag{36}$$

where  $2\theta = 1 - \alpha - \beta$ , then the problem  $A^*$  is unambiguously solvable.

Note that the set of numerical parameters of the problem  $A^*$ , satisfying inequalities (12) and (36) is non-empty. Indeed, if we suppose  $c > 0$ , i.e.  $k > 1$  and  $\mu > 1$  then inequality (12) holds.

By virtue of (12) ( $k^{\alpha+\beta} < \mu^2$ ) taking into account  $2\theta = 1 - \alpha - \beta$  from (36), we have

$$\begin{aligned} \frac{k^{(1-2\theta)/2} \sin(\theta\pi) |\ln k|}{\mu} &= \frac{k^{(1-2\theta)/2} \cos(((\alpha + \beta)\pi)/2) |\ln k|}{\mu} < \\ &< \frac{\mu \cos(((\alpha + \beta)\pi)/2) |\ln k|}{\mu} < |\ln k| = \left| \ln \frac{1-c}{1+c} \right| < 1, \end{aligned}$$

from here it is obvious that if  $c \in (0, (e-1)/(1+e))$ , then inequality (36) holds.

Thus, the set of numerical parameters of the problem  $A^*$  is nonempty, since inequalities (12) and (36) holds for the values of numerical parameters  $c \in (0, (e-1)/(1+e))$  and  $\mu > 1$ .

*Proof of Theorem 2.*



3.1 Derivation of the singular integral equation

From functional relations (10) and (11) excluding  $\nu(x)$ , we obtain

$$D_{-1,x}^{1-\alpha-\beta} \tau(x) + D_{x,1}^{1-\alpha-\beta} \tau(x) = \frac{1}{\gamma} (\Psi_1(x) - \Psi_2(x)), \quad x \in (-1, c). \tag{37}$$

Applying the fractional integration operator  $D_{-1,x}^{\alpha+\beta-1}$  to equality (37) taking into account  $\tau(-1) = 0$  and identities

$$D_{-1,x}^{1-\alpha-\beta} D_{-1,x}^{\alpha+\beta-1} \tau(x) = \tau(x),$$

$$D_{-1,x}^{1-\alpha-\beta} D_{x,1}^{\alpha+\beta-1} \tau(x) = \cos((1-\alpha-\beta)\pi) \tau(x) - \frac{\sin((1-\alpha-\beta)\pi)}{\pi} \times$$

$$\times \int_{-1}^1 \left(\frac{1+x}{1+t}\right)^{1-\alpha-\beta} \frac{\tau(t)dt}{t-x},$$

equation (37) is written in the form

$$\tau(x) - \lambda \int_{-1}^1 \left(\frac{1+x}{1+t}\right)^{1-\alpha-\beta} \frac{\tau(t)dt}{t-x} = F(x), \quad x \in (-1, c), \tag{38}$$

where

$$\lambda = \frac{\sin((\alpha+\beta)\pi)}{\pi(1-\cos(\alpha+\beta)\pi)}, \quad F(x) = \frac{1}{\gamma(1-\cos(\alpha+\beta)\pi)} (\Psi_1(x) - \Psi_2(x)).$$

Note that in (38)  $x \in (-1, c)$ , therefore equation (38) has a singular feature only when the integration variable is  $t \in (-1, c)$ . In order to highlight the singular part of equation (38) integration interval  $(-1, 1)$  divide it into two intervals  $(-1, c)$  and  $(c, 1)$  and write (38) in the form

$$\tau(x) - \lambda \int_{-1}^c \left(\frac{1+x}{1+t}\right)^{1-\alpha-\beta} \frac{\tau(t)dt}{t-x} - \lambda \int_c^1 \left(\frac{1+t}{1+x}\right)^{1-\alpha-\beta} \frac{\tau(t)dt}{t-x} = F(x), \quad x \in (-1, c). \tag{39}$$

In the second integral of the left-hand side of (39), by replacing the integration variable  $t = p(s)$ ,  $dt = -kds$ ,  $p(-1) = 1$ ,  $p(c) = c$ , we obtain

$$\tau(x) - \lambda \int_{-1}^c \left(\frac{1+x}{1+t}\right)^{1-\alpha-\beta} \frac{\tau(t)dt}{t-x} - \lambda k \int_{-1}^c \left(\frac{1+p(s)}{1+x}\right)^{1-\alpha-\beta} \times$$

$$\times \frac{\tau(p(s))ds}{p(s)-x} = F(x), \quad x \in (-1, c). \tag{40}$$

By virtue of condition (4\*) equation (40) is written in the form

$$\tau(x) - \lambda \int_{-1}^c \left(\frac{1+x}{1+t}\right)^{1-\alpha-\beta} \frac{\tau(t)dt}{t-x} = \frac{\lambda k}{\mu} \int_{-1}^c \left(\frac{1+x}{1+p(s)}\right)^{1-\alpha-\beta}$$

$$\frac{\tau(s)ds}{p(s)-x} + F_1(x), \quad x \in (-1, c), \tag{41}$$

$$F_1(x) = F(x) - \frac{\lambda k}{\mu} \int_{-1}^c \left(\frac{1+x}{1+p(s)}\right)^{1-\alpha-\beta} \frac{f(s)ds}{p(s)-x}.$$

The singular integral equation (41) is typical that the kernel of the right-hand side of the equation has an isolated first-order singularity for  $s = c, x = c$ , hence the integral operator of the right-hand side of (41) is not a Fredholm operator.

Temporarily considering the right-hand side of the equation (41) as a known function, we write it as

$$\tau(x) - \lambda \int_{-1}^c \left( \frac{1+x}{1+t} \right)^{1-\alpha-\beta} \frac{\tau(t)dt}{t-x} = g_0(x), \quad x \in (-1, c), \tag{42}$$

where

$$g_0(x) = \frac{\lambda k}{\mu} \int_{-1}^c \left( \frac{1+x}{1+p(s)} \right)^{1-\alpha-\beta} \frac{\tau(s)ds}{p(s)-x} + F_1(x). \tag{43}$$

*Theorem 3.* If  $g_0(x)$  satisfies the Helder condition for  $x \in (-1, c)$  and  $g_0(x) \in L_p(-1, c), p > 1$ , then the solution to equation (42) in the class of Helder functions  $H$ , in which the function  $(1+x)^{\alpha+\beta-1}\tau(x)$  can be unlimited at the left end of the interval  $(-1, c)$  and bounded at the right end of the interval  $(-1, c)$  expressed by the formula

$$\tau(x) = \frac{g_0(x)}{1 + \lambda^2\pi^2} + \frac{\lambda}{1 + \lambda^2\pi^2} \int_{-1}^c \left( \frac{(c-x)(1+x)}{(c-t)(1+t)} \right)^\theta \frac{g_0(t)dt}{t-x}. \tag{44}$$

The proof of Theorem 3 is identical to the proof of a similar theorem in work [13].

### 3.2 Derivation and investigation of the Wiener-Hopf integral equation

Substituting the expression for  $g_0(x)$  from (43) into (44), we have

$$\begin{aligned} \tau(x) = & \lambda_1 \int_{-1}^c \left( \frac{1+x}{1+p(s)} \right)^{2\theta} \frac{\tau(s)ds}{p(s)-x} + \lambda\lambda_1 \int_{-1}^c \left( \frac{(c-x)(1+x)}{(c-t)(1+t)} \right)^\theta \frac{dt}{t-x} \times \\ & \times \int_{-1}^c \left( \frac{1+t}{1+p(s)} \right)^{2\theta} \frac{\tau(s)ds}{(p(s)-t)} + F_2(x), \end{aligned} \tag{45}$$

where

$$\begin{aligned} \lambda_1 = & \frac{\lambda k}{\mu(1 + \lambda^2\pi^2)}, \\ F_2(x) = & \frac{F_1(x)}{1 + \lambda^2\mu^2} + \frac{\lambda}{1 + \lambda^2\pi^2} \int_{-1}^c \left( \frac{(c-x)(1+x)}{(c-t)(1+t)} \right)^\theta \frac{F_1(t)dt}{t-x}. \end{aligned}$$

In the double integral of equation (45) changing the order of integration, we have

$$\begin{aligned} \tau(x) = & \lambda_1 \int_{-1}^c \left( \frac{1+x}{1+p(s)} \right)^{2\theta} \frac{\tau(s)ds}{p(s)-x} + \lambda\lambda_1 \times \\ & \times \int_{-1}^c \frac{(c-x)^\theta(1+x)^\theta \tau(s)ds}{(1+p(s))^{2\theta}} \int_{-1}^c \left( \frac{1+t}{c-t} \right)^\theta \frac{dt}{(t-x)(p(s)-t)} + F_2(x), \quad x \in (-1, c). \end{aligned} \tag{46}$$

Calculate the internal integral in (46)

$$A(x, s) = \int_{-1}^c \left( \frac{1+t}{c-t} \right)^\theta \frac{dt}{(t-x)(p(s)-t)}, \tag{47}$$

to do this, we decompose the rational multiplier of the integrand into simple fractions

$$\frac{1}{(t-x)(p(s)-t)} = \left( \frac{1}{t-x} + \frac{1}{p(s)-t} \right) \cdot \frac{1}{p(s)-x},$$

then (47) has the form

$$A(x, s) = \frac{1}{p(s) - x} \left[ \int_{-1}^c \left( \frac{1+t}{c-t} \right)^\theta \frac{dt}{t-x} + \int_{-1}^c \left( \frac{1+t}{c-t} \right)^\theta \frac{dt}{p(s)-t} \right] = \frac{1}{p(s) - x} [A_1(x) + A_2(s)]. \quad (48)$$

We calculate  $A_1(x)$  by the formula

$$\int_a^b \frac{(x-a)^{\alpha-1}(b-x)^{\beta-1}}{x-y} dx = \frac{\pi \operatorname{ctg}(\beta\pi)}{(y-a)^{1-\alpha}(b-y)^{1-\beta}} - (b-a)^{\alpha+\beta-2} B(\alpha, \beta-1) F\left(1, 2-\alpha-\beta, 2-\beta; \frac{b-y}{b-a}\right).$$

Here  $a = -1, b = c; \alpha - 1 = \theta, \alpha = 1 + \theta; \beta - 1 = -\theta, \beta = 1 - \theta$ .

Thus,

$$A_1(x) = \int_{-1}^c \frac{(1+t)^\theta(c-t)^{-\theta}}{t-x} dt = (1+x)^\theta(c-x)^{-\theta} \pi \operatorname{ctg}(1-\theta)\pi - (1+c)^\theta B(1+\theta, -\theta) F\left(1, \theta, 1+\theta; \frac{c-x}{2}\right) = -\pi \operatorname{ctg}(\theta\pi) \left(\frac{1+x}{c-x}\right)^\theta + \frac{\pi}{\sin(\theta\pi)}. \quad (49)$$

Now we calculate

$$A_2(s) = \int_{-1}^c \left( \frac{1+t}{c-t} \right)^\theta \frac{dt}{p(s)-t}.$$

Here we will replace the variable integration  $t = -1 + (1+c)\sigma$  and using the integral representation of the hypergeometric function, we have

$$A_2(s) = \frac{1+c}{1+\delta-ks} \frac{\Gamma(1+\theta)\Gamma(1-\theta)}{\Gamma(2)} F\left(1+\theta, 1, 2; \frac{1+c}{1-\delta-ks}\right).$$

Here, applying the autotransformation formula

$$F(a, b, c; x) = (1-x)^{c-a-b} F(c-a, c-b, c; x),$$

we have

$$A_2(s) = \frac{(1+c)\Gamma(1+\theta)\Gamma(1-\theta)}{1+\delta-ks} \left(\frac{\delta-ks-c}{1+\delta-ks}\right)^{-\theta} \cdot F\left(1-\theta, 1, 2; \frac{1+c}{1-\delta-ks}\right).$$

Next, using the formula

$$F(a, b, c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} z^{-a} F\left(a, a-c+1, a+b-c+1; \frac{z+1}{z}\right) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} z^{a-c} (1-z)^{c-a-b} F\left(c-a, 1-a, c-a-b+1; \frac{z+1}{z}\right),$$

we have

$$A_2(s) = \frac{(1+c)\Gamma(1+\theta)\Gamma(1-\theta)}{1+\delta-ks} \left(\frac{1+\delta-ks}{\delta-ks-c}\right)^\theta \times$$

$$\begin{aligned} & \times \left[ \frac{\Gamma(2)\Gamma(\theta)}{\Gamma(1+\theta)\Gamma(1)} \left( \frac{1+c}{1+\delta-ks} \right)^{\theta-1} F\left(1-\theta, -\theta, 1-\theta; \frac{c-\delta+ks}{1+c}\right) + \right. \\ & \left. + \frac{\Gamma(2)\Gamma(-\theta)}{\Gamma(1-\theta)\Gamma(1)} \left( \frac{1+c}{1+\delta-ks} \right)^{-\theta-1} \left( \frac{\delta-ks-c}{1+\delta-ks} \right)^\theta \cdot F\left(1+\theta, \theta, 1+\theta; \frac{c-\delta+ks}{1+c}\right) \right]. \end{aligned}$$

Here, taking into account the equality

$$F(a, c, c; x) = (1-x)^{-a},$$

we have

$$A_2(s) = \frac{\pi}{\sin(\theta\pi)} \left( \frac{1+\delta-ks}{\delta-ks-c} \right)^\theta - \frac{\pi}{\sin(\theta\pi)}. \tag{50}$$

Now substituting the expressions for  $A_1(x)$  and  $A_2(s)$  from (49) and (50) into (48) respectively, we obtain

$$\begin{aligned} A(x, s) &= \frac{1}{p(s)-x} [A_1(x) + A_2(s)] = \frac{1}{p(s)-x} \times \\ & \times \left[ -\pi \operatorname{ctg}(\theta\pi) \left( \frac{1+x}{c-x} \right)^\theta + \frac{\pi}{\sin(\theta\pi)} \left( \frac{1+\delta-ks}{\delta-ks-c} \right)^\theta \right]. \end{aligned} \tag{51}$$

By virtue (51) the equation (46) is transformed to the form

$$\begin{aligned} \tau(x) &= \lambda_1(1 - \lambda\pi \operatorname{ctg}(\theta\pi)) \int_{-1}^c \left( \frac{1+x}{1+p(s)} \right)^{2\theta} \frac{\tau(s)ds}{p(s)-x} + \\ & + \lambda\lambda_1 \frac{\pi}{\sin(\theta\pi)} \int_{-1}^c \left( \frac{1+x}{1+p(s)} \right)^\theta \left( \frac{c-x}{p(s)-c} \right)^\theta \frac{\tau(s)ds}{p(s)-x} + F_2(x). \end{aligned} \tag{52}$$

By virtue of the identity  $1 - \lambda\pi \operatorname{ctg}(\theta\pi) = 0$ , the equation (52) has the form

$$\tau(x) = \frac{\lambda\lambda_1\pi}{\sin(\theta\pi)} \int_{-1}^c \left( \frac{1+x}{1+p(s)} \right)^\theta \left( \frac{c-x}{p(s)-c} \right)^\theta \frac{\tau(s)ds}{p(s)-x} + F_2(x), \quad x \in (-1, c). \tag{53}$$

Thus, by virtue of the identities

$$p(s) - c = k(c - s), \quad p(s) - x = k(c - s) + c - x,$$

equation (53) is written in the form

$$\tau(x) = \frac{\lambda\lambda_1\pi}{\sin(\theta\pi) \cdot k^\theta} \int_{-1}^c \left( \frac{c-x}{c-s} \right)^\theta \frac{\tau(s)ds}{k(c-s) + c - x} + R_1[\tau(x)] + F_2(x), \quad x \in (-1, c), \tag{54}$$

where  $R_1[\tau(x)] = \frac{\lambda\lambda_1\pi}{\sin(\theta\pi)k^\theta} \int_{-1}^c \left[ \left( \frac{1+x}{1+p(s)} \right)^\theta - 1 \right] \frac{\tau(s)ds}{k(c-s) + c - x}$  is a regular operator.

In equation (54) we make substitutions  $s = c - (1+c)e^{-t}$ ,  $x = c - (1+c)e^{-y}$ , where  $t \in [0, +\infty)$ ,  $y \in [0, +\infty)$  and introducing notations

$$\rho(y) = \tau[c - (1+c)e^{-y}]e^{(\theta-\frac{1}{2})y},$$

we write equation (54) in the form

$$\rho(y) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} K(y-t)\rho(t)dt + R_2[\rho] + F_3(y), \tag{55}$$

where

$$K(x) = \frac{\sqrt{2\pi^3}\lambda\lambda_1}{\sin(\theta\pi)k^\theta(k e^{x/2} + e^{-x/2})}, \quad F_3(y) = F_2[c - (1 + e)e^{-y}]e^{(\theta - \frac{1}{2})y}.$$

$R_2[\rho] = R_1[\tau]e^{(\theta - \frac{1}{2})y}$  is a regular operator. Note that, since  $2\theta = 1 - \alpha - \beta$  then the following inequality holds:  $\theta - 1/2 < 0$ . Equation (55) is the Wiener-Hopf integral equation [15]. Using the Fourier transform, like the well-known characteristic special integral equation with the Cauchy kernel, This equation is reduced to the Riemann boundary value problem and thereby it is solved in quadratures. Fredholm's theorems for integral equations of the convolution type will be valid only in one case, when the index of these equations is equal to zero.

The index  $\chi$  of equation (55) will be the index of the expression  $1 - K^\wedge(x)$  with the reverse sign, i.e.  $\chi = -\text{Ind}(1 - K^\wedge(x))$ , here [15]

$$\begin{aligned} K^\wedge(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} K(t)e^{ixt} dt = \\ &= \frac{\lambda\lambda_1\pi}{k^\theta \sin(\theta\pi)} \int_{-\infty}^{+\infty} \frac{e^{ixt} dt}{k e^{t/2} + e^{-t/2}} = \frac{\lambda\lambda_1\pi}{k^\theta \sin(\theta\pi)} \cdot \frac{\pi e^{-ix} \ln k}{\sqrt{k} \cdot \text{ch}(\pi x)} = \\ &= \frac{\lambda\lambda_1\pi^2 \ln k}{k^{1/2+\theta} \sin(\theta\pi)} \frac{e^{-ix}}{\text{ch}(\pi x)} = A^*(x) - iB^*(x), \end{aligned}$$

where

$$A^*(x) = \frac{\lambda\lambda_1\pi^2 \ln k}{k^{1/2+\theta} \sin(\theta\pi)} \frac{\cos x}{\text{ch}(\pi x)}; \quad B^*(x) = i \frac{\lambda\lambda_1\pi^2 \ln k}{k^{1/2+\theta} \sin(\theta\pi)} \frac{\sin x}{\text{ch}(\pi x)}. \quad (56)$$

From (56) it can be seen that

$$\left. \begin{aligned} |A^*(x)| \\ |B^*(x)| \end{aligned} \right\} \leq \frac{\lambda\lambda_1\pi^2 |\ln k|}{k^{1/2+\theta} \sin(\theta\pi)} \cdot \frac{1}{\text{ch}(\pi x)},$$

and  $A^*(x) = O(1/\text{ch}(\pi x)), B^*(x) = O(1/\text{ch}(\pi x))$  for large enough ones  $|x|$ . Hence, by virtue of condition (36) of Theorem 2 it follows that

$$\left. \begin{aligned} |A^*(x)| \\ |B^*(x)| \end{aligned} \right\} \leq \frac{\lambda\lambda_1\pi^2 |\ln k|}{k^{1/2+\theta} \sin|\theta\pi|} = \frac{k^{1/2-\theta} \sin(\theta\pi) |\ln k|}{\mu} < 1.$$

Hence

$$\text{Re}(1 - K^\wedge(x)) > 0. \quad (57)$$

Changing the argument of a complex-valued function  $1 - K^\wedge(x)$  on the real axis, expressed in full revolutions and taken with the reverse sign [15] taking into account the inequality (57) the index  $\chi$  of equation (55) is equal to

$$\begin{aligned} x &= -\text{Ind}(1 - K^\wedge(x)) = -\frac{1}{2\pi} [\arg(1 - K^\wedge(x))]_{-\infty}^{+\infty} = \\ &= -\frac{1}{2\pi} \left[ \text{arctg} \frac{\text{Im}(1 - K^\wedge(x))}{\text{Re}(1 - K^\wedge(x))} \right]_{-\infty}^{+\infty} = -\frac{1}{2\pi} \left[ \text{arctg} \frac{B^*(x)}{1 - A^*(x)} \right]_{-\infty}^{+\infty} = \\ &= -\frac{1}{2\pi} \left[ \text{arctg} \frac{0}{1} - \text{arctg} \frac{0}{1} \right] = 0, \end{aligned}$$

since  $A^*(\pm\infty) = 0, B^*(\pm\infty) = 0$ . Consequently, equation (55) is uniquely reduced to the Fredholm integral equation of the second kind, the unambiguous solvability of which follows from the uniqueness of the solution of the problem  $A^*$ . Theorem 2 is proved.

### *Conclusion*

The paper investigates the issues of unique solvability for one class of problems in a non-standard formulation for a degenerate hyperbolic equation with singular coefficients (1) in a bounded domain.

For equation (1), when the conditions  $0 < \alpha$ ,  $\beta < 1$ ,  $\alpha + \beta < 1$  are hold, a non-classical problem is formulated with missing Goursat conditions (3) on the characteristic boundary of the domain and an analog of Frankl condition (4) on the boundary of degeneracy.

It is shown that the validity of the theorem on the uniqueness of the solution to problem A (1)–(4) significantly depends on the ratio between the coefficient  $\mu$  in Frankl conditions (4), the location of point  $c$  lying on the line of degeneracy and on the coefficients  $\alpha_0$  and  $\beta_0$  in equation (1).

The theorem on the existence of a solution to problem A (1)–(4) is proved by reducing it to the problem of solving a non-standard singular integral equation with a non-Fredholm integral operator in the non-characteristic part of the equation, the kernel of which has an isolated first-order singularity. Further, using the Carleman regularization method, the theory of Wiener-Hopf equations, the problem is equivalently (in the sense of solvability) reduced to an integral equation of the second kind, the solvability of which follows from the uniqueness of the solution to the problem A.

In conclusion, we note that the constructive properties of solutions to equation (1) significantly depend on the values of the parameters  $m$ ,  $\alpha$ ,  $\beta$ .

Issues of setting and studying the solvability of similar non-standard problems for other parameter values when  $P(\alpha_0, \beta_0) \notin \Delta A_0^* B_0^* C_0^*$  have not been investigated.

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### *Author contributions*

All authors contributed equally to this work.

### *Conflict of Interest*

The authors declare no conflict of interest.

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## Аймақтың шекарасында өзгешеленетін сингуляр коэффициентті гиперболалық теңдеу үшін жетіспейтін Гурса шарты бар есеп

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Жұмыс гиперболалық теңдеу үшін аймақтың характеристикалық шекарасында жетіспейтін шарттары мен өзгешеленетін сегментіндегі Франкль шартының аналогы бар есепті қоюға және оның шешілу мәселелерін зерттеуге арналған. Бұл есептің белгілі локалды және локалды емес есептерден айырмашылығы: біріншіден, гиперболалық типтегі теңдеу ерікті оң дәрежелі өзгешеленетін және шекара бөліктеріндегі сингулярлық коэффициенттермен алынады, екіншіден, аймақтың характеристикалық шекарасы ерікті түрде екі бөлікке бөлінеді және бірінші бөлікте ізделінді функцияның мәні беріледі,

ал екінші бөлік шеттік шарттан босатылған және бұл жетіспейтін Гурса шарты өзгешелену сегментіндегі Франкль шартының аналогымен алмастырылады, сондай-ақ белгісіз функцияның аймақтың басқа характеристикалық шекарасындағы мәні белгілі болып саналады. Қойылған есептің жалғыздық жөніндегі теореманың әділдігін қамтамасыз ететін теңдеудің коэффициенттері мен есептің берілгендері үшін жеткілікті шарттар табылған. Есеп шешімінің бар болу теоремасы ядросы бірінші ретті оқшауланған ерекшелігі бар теңдеудің характеристикалық емес бөлігінде фредгольмдік емес интегралдық операторы бар стандартты емес сингулярлық интегралдық теңдеуді шешу туралы есепке келтіру арқылы дәлелденеді. Алынған теңдеуге Карлеманның регуляризациялау әдісін қолдана отырып, Винер-Хопф интегралдық теңдеуін қосамыз. Винер-Хопф теңдеуінің индексі нөлге тең екендігі дәлелденген, сондықтан ол екінші типтегі Фредгольмнің интегралдық теңдеуіне келтіріледі, ал оның шешімділігі есеп шешімінің жалғыздығынан туындайды.

*Кілт сөздер:* аймақтың шекарасында өзгешеленген гиперболалық теңдеу, жетіспейтін Гурса шарты, Франкль шарты, сингулярлы коэффициент, функциялардың толық ортогональ жүйесі, сингулярлы интегралдық теңдеу, Винер-Хопф теңдеуі, индекс.

## Задача с недостающим условием Гурса для вырождающегося на границе области гиперболического уравнения с сингулярным коэффициентом

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Работа посвящена постановке и изучению вопросов разрешимости задачи с недостающими условиями на характеристической границе области и аналогом условия Франкля на отрезке вырождения для гиперболического уравнения. Отличие данной задачи от известных локальных и нелокальных задач состоит в том, что, во-первых, уравнение гиперболического типа берется с произвольным положительным степенным вырождением и сингулярными коэффициентами на части границы, и, во-вторых, характеристическая граница области произвольным образом разбивается на два куска, и на первом куске задается значение искомой функции, а второй кусок освобожден от краевого условия, и это недостающее условие Гурса заменено аналогом условия Франкля на отрезке вырождения, а также считается известным значение неизвестной функции на другой характеристической границе области. Найдены условия на коэффициенты уравнения и данные сформулированной задачи, обеспечивающие справедливость теоремы единственности. Теорема существования решения задачи доказывается сведением к задаче о решении нестандартного сингулярного интегрального уравнения с нефредгольмовым интегральным оператором в нехарактеристической части уравнения, ядро которого имеет изолированную особенность первого порядка. К полученному уравнению, применяя метод регуляризации Карлемана, получается интегральное уравнение Винера-Хопфа. Доказано, что индекс уравнения Винера-Хопфа равен нулю, следовательно, оно однозначно редуцируется к интегральному уравнению Фредгольма второго рода, разрешимость которого следует из единственности решения задачи.

*Ключевые слова:* вырождающееся на границе области гиперболическое уравнение, недостающее условие Гурса, условие Франкля, сингулярный коэффициент, полная ортогональная система функций, сингулярное интегральное уравнение, уравнение Винера-Хопфа, индекс.



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