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On the spectral problem for three-dimesional bi-Laplacian in the unit sphere

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In this work, we introduce a new concept of the stream function and derive the equation for the stream function in the three-dimensional case. To construct a basis in the space of solutions of the Navier-Stokes system, we solve an auxiliary spectral problem for the bi-Laplacian with Dirichlet conditions on the boundary. Then, using the formulas employed for introducing the stream function, we find a system of functions forming a basis in the space of solutions of the Navier-Stokes system. It is worth noting that this basis can be utilized for the approximate solution of direct and inverse problems for the Navier-Stokes system, both in its linearized and nonlinear forms. The main idea of this work can be summarized as follows: instead of changing the boundary conditions (which remain unchanged), we change the differential equations for the stream function with a spectral parameter. As a result, we obtain a spectral problem for the bi-Laplacian in the domain represented by a three-dimensional unit sphere, with Dirichlet conditions on the boundary of the domain. By solving this problem, we find a system of eigenfunctions forming a basis in the space of solutions to the Navier-Stokes equations. Importantly, the boundary conditions are preserved, and the continuity equation for the fluid is satisfied. It is also noteworthy that, for the three-dimensional case of the Navier-Stokes system, an analogue of the stream function was previously unknown.

Keywords: Navier-Stokes system, bi-Laplacian, spectral problem, stream function.

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Introduction

Previously, we solved the spectral problem for the bi-Laplacian in the unit circle with Dirichlet conditions on the boundary. As is known, in the two-dimensional case the linear Navier-Stokes system can be transformed into a single equation for the stream function [1–3]. Note that the spectral problem for the two-dimensional bi-Laplacian in the unit circle was solved in [4–6], and its results were applied to an approximate solution of the inverse problem with final redefinition conditions for the two-dimensional system of Navier-Stokes equations. For the bi-Laplacian, the solvability of two-dimensional spectral problems for square domains was considered in [7–12], and for the $2m-$Laplacian, spectral problems for multidimensional domains with smooth and non-smooth boundaries – in [13–16]. In [8, 10, 11], lower bounds for eigenvalues were obtained by introducing intermediate spectral problems (the main thing was the fact that one of the boundary conditions was replaced by a family of approximate conditions on the boundary, which in the limit tended to original). In [13–16], estimates were given for the number of eigenvalues not exceeding a given number. However, the calculation of eigenvalues and eigenfunctions in the above spectral problems has remained open. This issue is dedicated to submitted work.

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The question of constructing a basis applicable to domains with time-varying boundaries also remains open. For example, problems of this kind in degenerate domains or in domains with time-varying boundaries were considered in papers [17–29]. Note that the results of this work can be used in the construction of this basis.

1 Stream function for a three-dimensional linearized Navier-Stokes system. Statement of the spectral problem

Let $y = (y_1, y_2, y_3)$, $Q_{yt} = \{y, t : |y| < 1,\ 0 < t < T\}$ be a cylindrical domain, and Ω be a section (sphere with unit radius) of the cylinder Q_{yt} for any fixed time $t \in [0, T]$ with boundary $\partial\Omega$, $\Sigma_{yt} = \partial\Omega \times (0, T)$. In the cylindrical domain Q_{yt} we consider the following initial boundary value problem for the linear three-dimensional Navier-Stokes equation of determining the vector function $w(y, t) = \{w_1(y, t), w_2(y, t), w_3(y, t)\}$ and scalar function $P(y, t)$:

$$\partial_t w - \Delta w = f - \nabla P, \quad (y, t) \in Q_{yt}, \tag{1.1}$$

$$\operatorname{div} w = 0, \ \ (y, t) \in Q_{yt}, \tag{1.2}$$

$$w = 0, \ \ (y, t) \in \Sigma_{yt} \text{ is a lateral surface of the cylinder}, \tag{1.3}$$

$$w = 0, \ \ y \in \Omega \text{ is a unit sphere, base of cylinder.} \tag{1.4}$$

Let's introduce the notations of spaces $\mathbf{V}$, $\mathbf{H}$, $\mathbf{L}^2(\Omega)$, $\mathbf{H}^1_0(\Omega)$ and $\mathbf{H}^2(\Omega)$, used in studying the solvability of the initial boundary value problem (1.1)–(1.4), and which we will use in the future:

$$\mathbf{V} = \{v : v \in \mathbf{H}^1_0(\Omega) = \left(H^1_0(\Omega)\right)^3, \operatorname{div} v = 0\},$$

$$\mathbf{H} = \left\{v : v \in \mathbf{L}^2(\Omega), \operatorname{div} v = 0\right\},$$

$$\mathbf{L}^2(\Omega) = \left(L^2(\Omega)\right)^3, \ \ \mathbf{H}^2(\Omega) = \left(H^2(\Omega)\right)^3.$$

The following dense embeddings take place

$$\mathbf{V} \subset \mathbf{H} \equiv \mathbf{H}' \subset \mathbf{V}', \ \ \mathbf{H}^1_0(\Omega) \subset \mathbf{L}^2(\Omega) \equiv \left(\mathbf{L}^2(\Omega)\right)' \subset \mathbf{H}^{-1}(\Omega),$$

and $(\cdot, \cdot)$, $((\cdot, \cdot))$ are scalar products in spaces $\mathbf{H}$, $\mathbf{L}^2(\Omega)$ and $\mathbf{V}$, $\mathbf{H}^1_0(\Omega)$, respectively. The Helmholtz decomposition of space $\mathbf{L}^2(\Omega)$: $\mathbf{L}^2(\Omega) = \mathbf{H} \oplus \mathbf{H}^\perp$, where

$$\mathbf{H}^\perp \text{ is an orthogonal complement to } \mathbf{H} \text{ in the space } \mathbf{L}^2(\Omega),$$

$$\mathbf{H}^\perp = \{v : v \in \mathbf{L}^2(\Omega), v = \nabla u, u \in \mathbf{H}^1(\Omega)\},$$

$$\left(\mathbf{H} \oplus \mathbf{H}^\perp\right)' \equiv \left(\mathbf{L}^2(\Omega)\right)' \equiv \mathbf{L}^2(\Omega) \equiv \mathbf{H} \oplus \mathbf{H}^\perp,$$

and the "prime" symbol denotes a topologically dual space.

So, we will look for a solution of the initial boundary value problem (1.1)–(1.4) in the spaces of the vector functions of liquid velocities $w(y, t) = \{w_1(y, t), w_2(y, t), w_3(y, t)\} \in L^2(0, T; \mathbf{V} \cap \mathbf{H}^2(\Omega)) \cap H^1(0, T; \mathbf{H}(\Omega))$, and scalar liquid pressure function $P(y, t) \in L^2(0, T; \mathbf{H}^1(\Omega))$ for a given vector functions of the acting forces $f(y, t) = \{f_1(y, t), f_2(y, t), f_3(y, t)\} \in L^2(0, T; \mathbf{H}(\Omega))$.

Let us transform boundary value problem (1.1)–(1.4). For this purpose, in the domain Q_{yt} we introduce the scalar stream function $U(y, t)$, defined up to an additive constant, by the equations:

$$w_1 = \partial_{y_2} U - \partial_{y_3} U, \quad w_2 = \partial_{y_3} U - \partial_{y_1} U, \quad w_3 = \partial_{y_1} U - \partial_{y_2} U. \tag{1.5}$$

We will act with the operators $\partial_{y_2}-\partial_{y_3},\ \partial_{y_3}-\partial_{y_1},\ \partial_{y_1}-\partial_{y_2}$ to equations (1.1) respectively and add the obtained results. Then for $U(y,t)$ we obtain the equation

$$(\partial_t-\Delta)\left(\Delta-\partial^2_{y_1y_2}-\partial^2_{y_2y_3}-\partial^2_{y_3y_1}\right)U=G(y,t),\quad \{y,t\}\in Q_{yt}, \tag{1.6}$$

where

$$2G(y,t)\equiv(\partial_{y_2}-\partial_{y_3})\,f_1+(\partial_{y_3}-\partial_{y_1})\,f_2+(\partial_{y_1}-\partial_{y_2})\,f_3.$$

From relations (1.3) and (1.5) we have the identities:

$$(\partial_{y_1}-\partial_{y_2})\,U\equiv(\partial_{y_2}-\partial_{y_3})\,U\equiv(\partial_{y_3}-\partial_{y_1})\,U\equiv 0,\quad (y,t)\in\Sigma_{yt} \tag{1.7}$$

or

$$\partial_{y_1}U\equiv\partial_{y_2}U\equiv\partial_{y_3}U,\ \ (y,t)\in\Sigma_{yt}. \tag{1.8}$$

Note that relations (1.7)–(1.8) do not completely determine the boundary conditions on the lateral surface of the cylinder Q_{yt}. In addition to (1.7)–(1.8) we will require that $\partial_{y_1}U\equiv 0$ on Σ_{yt}, which do not contradict relations (1.7)–(1.8). So, instead of (1.8) we will have:

$$\partial_{y_1}U\equiv\partial_{y_2}U\equiv\partial_{y_3}U\equiv 0,\ \ (y,t)\in\Sigma_{yt}. \tag{1.9}$$

Thus, equalities (1.9) allow us to set the following boundary conditions for equation (1.6)

$$\partial_{\vec n}U=0,\ \ (y,t)\in\Sigma_{yt}, \tag{1.10}$$

$$U=0,\ \ (y,t)\in\Sigma_{yt}, \tag{1.11}$$

where $\vec n$ is the outer unit normal to the sphere $|y|=1$, and from (1.4) (doing the same thing as when establishing conditions (1.10)–(1.11)) we obtain the initial condition

$$U=0,\ \ y\in\Omega\equiv\{|y|<1\},\ \ t=0. \tag{1.12}$$

To numerically solve the initial boundary value problem (1.1)–(1.4) we will need to be able to solve approximately the initial boundary value problem (1.6), (1.10)–(1.12). We will look for a solution to this problem using the method of separation of variables. We have

$$U(y,t)=\sum_{k=1}^{\infty}c_k(t)u_k(y).$$

Then from equation (1.6) we obtain

$$c_k'(t)\left[\triangle u_k(y)-\partial^2_{y_1y_2}u_k(y)-\partial^2_{y_2y_3}u_k(y)-\partial^2_{y_3y_1}u_k(y)\right]=$$

$$=c_k(t)\triangle\left[\triangle u_k(y)-\partial^2_{y_1y_2}u_k(y)-\partial^2_{y_2y_3}u_k(y)-\partial^2_{y_3y_1}u_k(y)\right].$$

Further, we have

$$\frac{c_k'(t)}{c_k(t)}=\frac{\triangle\left(\triangle-\partial^2_{y_1y_2}-\partial^2_{y_2y_3}-\partial^2_{y_3y_1}\right)u_k(y)}{\left(\triangle-\partial^2_{y_1y_2}-\partial^2_{y_2y_3}-\partial^2_{y_3y_1}\right)u_k(y)}=-\lambda_k,\ \ \lambda_k>0\quad\text{for each}\quad k\in\mathbb{N},$$

i.e., we finally come to the need to solve the following spectral problem:

$$\triangle\left(\triangle-\partial^2_{y_1y_2}-\partial^2_{y_2y_3}-\partial^2_{y_3y_1}\right)u(y)=-\lambda\left(\triangle-\partial^2_{y_1y_2}-\partial^2_{y_2y_3}-\partial^2_{y_3y_1}\right)u(y), \tag{1.13}$$

$$u(y)_{|\partial\Omega} = \partial_{\vec{n}} u(y)_{|\partial\Omega} = 0. \tag{1.14}$$

Solving the spectral problem (1.13)–(1.14) poses certain difficulties (details in Appendix A). We actually need to construct a basis in the space solutions of the Navier-Stokes system $\mathbf{V} \cap \mathbf{H}^2(\Omega)$, the elements of which would ensure the fulfillment of equation (1.2) and boundary conditions (1.3). Therefore, it will be enough for us to use the solution to the following spectral problem, also posed on a unit sphere (but with a simplification of the equation in which there are no terms with mixed derivatives of the desired function):

$$(-\Delta)^2 Z(y) = \mu^2(-\Delta\, Z(y)), \ y \in \Omega = \{|y| < 1\},$$

$$\partial_{\vec{n}} Z(y) = 0, \ \text{at } |y| = 1, \tag{1.15}$$

$$Z(y) = 0, \ \text{at } |y| = 1. \tag{1.16}$$

Let us rewrite the equation in the form of a system for unknown functions $\{Z(y), Y(y)\}$:

$$-\Delta\, Z(y) = Y(y), \ \ -\Delta\, Y(y) = \mu^2 Y(y) \quad y \in \Omega. \tag{1.17}$$

So, we got spectral problem (1.17), (1.15) and (1.16).

2 Transition to spherical coordinates in the spectral problem

Let us write spectral problem (1.17), (1.15) and (1.16) in a spherical coordinate system $\{r, \theta, \zeta\} \in \Omega \equiv \{0 \le r < 1, \ \theta \in (0, \pi], \ \zeta \in (0, 2\pi]\}$ using transformation formulas

$$y_1 = r \sin\theta \cos\zeta, \ \ y_2 = r \sin\theta \sin\zeta, \ \ y_3 = r\cos\theta,$$

regarding the functions $Z(r, \theta, \zeta), \ \ Y(r, \theta, \zeta)$ (in this case, for the sake of simplicity, we leave the function designations unchanged):

$$-\frac{1}{r^2}\partial_r\left(r^2 \partial_r Z\right) - \frac{1}{r^2}\Delta_{\theta,\zeta} Z = Y, \quad \{r, \theta, \zeta\} \in \Omega, \tag{2.1}$$

$$\Delta_{\theta,\zeta} Z \equiv \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\, \partial_\theta Z\right) + \frac{1}{\sin^2\theta}\partial_\zeta^2 Z, \quad \{r, \theta, \zeta\} \in \Omega, \tag{2.2}$$

$$-\frac{1}{r^2}\partial_r\left(r^2 \partial_r Y\right) - \frac{1}{r^2}\Delta_{\theta,\zeta} Y = \mu^2 Y, \quad \{r, \theta, \zeta\} \in \Omega, \tag{2.3}$$

$$\Delta_{\theta,\zeta} Y \equiv \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\, \partial_\theta Y\right) + \frac{1}{\sin^2\theta}\partial_\zeta^2 Y, \quad \{r, \theta, \zeta\} \in \Omega, \tag{2.4}$$

$$Z \ \text{ is bounded in the neighborhood of the point } \ r = 0, \tag{2.5}$$

$$\partial_r Z = 0 \ \text{ at } \ r = 1, \tag{2.6}$$

$$Z = 0 \ \text{ at } \ r = 1. \tag{2.7}$$

3 Solution of the spectral problem in spherical coordinates

We will solve problem (2.1)–(2.7) using the method of separation of variables:

$$Z(r,\theta,\zeta)=\sum_{j}R_{Z_j}(r)\Theta_{Z_j}(\theta,\zeta),\ \ Y(r,\theta,\zeta)=\sum_{j}R_{Y_j}(r)\Theta_{Y_j}(\theta,\zeta),\tag{3.1}$$

$$\frac{\left(r^2R'_{Y_j}\right)'+\mu_j^2r^2R_{Y_j}}{R_{Yj}}=-\frac{\Delta_{\theta,\zeta}\Theta_{Y_j}}{\theta_{Y_j}}=\mu^2_{Y_j},\ \ \frac{\left(r^2R'_{Z_j}\right)'+r^2R_{Y_j}}{R_{Z_j}}=-\frac{\Delta_{\theta,\zeta}\Theta_{Z_j}}{\theta_{Z_j}}=\mu^2_{Z_j},\tag{3.2}$$

where the "prime" symbol here and below denotes the derivative with respect to the variable r.

The second relation from (3.2) follows from the fact that the boundary value problems (3.3)–(3.4) and (3.5)–(3.6) for the functions $\Theta_{Z_j}(\theta,\zeta)$ and $\Theta_{Y_j}(\theta,\zeta)$ coincide, then their solutions can be taken equal to each other, i.e. $\Theta_{Z_j}(\theta,\zeta)=\Theta_{Y_j}(\theta,\zeta)$ and $\mu^2_{Z_j}=\mu^2_{Y_j}$.

Substituting (3.1) into (2.1)–(2.7) and taking (3.2) into account, we obtain

$$-\Delta_{\theta,\zeta}\Theta_{Z_j}=\mu^2_{Z_j}\Theta_{Z_j},\ \ \theta\in(0,\pi),\ \ \zeta\in(0,2\pi),\ \ \Theta_{Z_j}(\theta,\zeta)=\Theta_{Z_j}(\theta,\zeta+2\pi),\tag{3.3}$$

$$\text{conditions of boundedness}\ \ \Theta_{Z_j}(\theta,\zeta)\ \ \text{at}\ \ \theta=0,\ \ \theta=\pi,\tag{3.4}$$

$$-\Delta_{\theta,\zeta}\Theta_{Y_j}=\mu^2_{Y_j}\Theta_{Y_j},\ \ \theta\in(0,\pi),\ \ \zeta\in(0,2\pi),\Theta_{Y_j}(\theta,\Theta_{Y_j}(\theta,\zeta+2\pi),\tag{3.5}$$

$$\text{conditions of boundedness}\ \ \Theta_{Z_j}(\theta,\zeta)\ \ \text{at}\ \theta=0,\ \ \theta=\pi,\tag{3.6}$$

$$r^2R''_{Z_j}(r)+2rR'_{Z_j}(r)-\mu^2_{Z_j}R_{Z_j}(r)=-r^2R_{Y_j}(r),\tag{3.7}$$

$$r^2R''_{Y_j}(r)+2rR'_{Y_j}(r)+\left(\mu_j^2r^2-\mu^2_{Y_j}\right)R_{Y_j}(r)=0,\tag{3.8}$$

$$R_{Z_j}(r)\ \ \text{are bounded in the neighborhood of zero,}\ \ R_{Z_j}(1)=0,\ \ R'_{Z_j}(1)=0.\tag{3.9}$$

Let us deal with the solution of boundary value problems (3.3)–(3.4) and (3.5)–(3.6). Let us use the variable separation method:

$$\Theta_{Z_j}(\theta,\zeta)=\sum_{m}P_{Z_{jm}}(\theta)Q_{Z_{jm}}(\zeta),\ \ \Theta_{Y_{jm}}(\theta,\zeta)=\sum_{m}P_{Y_{jm}}(\theta)Q_{Y_{jm}}(\zeta).\tag{3.10}$$

Then (3.3)–(3.4) and (3.5)–(3.6) are reduced to the following systems:

$$Q''_{Z_{jm}}(\zeta)+m^2Q_{Z_{jm}}(\zeta)=0,\ \ \zeta\in[0,2\pi),\ \ m^2\in\{0,1,2,\ldots\},\ \ Q_{Z_{jm}}(\zeta)=Q_{Z_{jm}}(\zeta+2\pi),\tag{3.11}$$

$$\frac{1}{\sin\theta}\left(\sin\theta\,P'_{Z_{jm}}(\theta)\right)'+\left[\mu^2_{Z_j}-\frac{m^2}{\sin^2\theta}\right]P_{Z_{jm}}(\theta)=0,\tag{3.12}$$

$$\text{conditions of boundedness}\ \ P_{Z_{jm}}(\theta)\ \ \text{at points}\ \ \theta=0,\ \ \theta=\pi,\tag{3.13}$$

$$Q''_{Y_{jm}}(\zeta)+m^2Q_{Y_{jm}}(\zeta)=0,\ \ \zeta\in[0,2\pi),\ \ m^2\in\{0,1,2,\ldots\},\ \ Q_{Y_{jm}}(\zeta)=Q_{Y_{jm}}(\zeta+2\pi),\tag{3.14}$$

$$\frac{1}{\sin\theta}\left(\sin\theta\,P'_{Y_{jm}}(\theta)\right)'+\left[\mu^2_{Y_j}-\frac{m^2}{\sin^2\theta}\right]P_{Y_{jm}}(\theta)=0,\tag{3.15}$$

$$\text{conditions of boundedness}\ \ P_{Y_{jm}}(\theta)\ \ \text{at points}\ \ \theta=0,\ \ \theta=\pi,\tag{3.16}$$

where the "prime" symbol denotes the derivative with respect to the variables ζ and θ.

The solutions of boundary value problems (3.11) and (3.14) coincide and are equal:

$$Q_{Z_{jm}}(\zeta)=Q_{Y_{jm}}(\zeta)=\{\cos m\zeta,\ \sin m\zeta\},\ \ \zeta\in[0,2\pi),\ \ m\in\{0,1,2,\ldots\}.\tag{3.17}$$

In addition, it is easy to see that relations (3.12)–(3.13) and (3.15)–(3.16) also coincide, and their solutions were found, for example, in ([30], p. 374–376) with using Legendre polynomials $P_{Z_j}(\theta)$ and $P_{Y_j}(\theta)$.

If in the equation (3.12) we make the substitution $t = \cos\theta$ and denote $X(t)|_{t=\cos\theta} = X(\cos\theta) = P_{Z_j}(\theta)$, so we get the equation

$$\left(\left(1-t^2\right)X'(t)\right)' + \left(\mu_{Zj}^2 - \frac{m^2}{1-t^2}\right)X(t) = 0, \;\; |t| < 1. \tag{3.18}$$

Relation (3.12)–(3.13) admits bounded solutions only if and only if $\mu_{Z_j}^2 = j(j+1)$ (3.20):

$$X(t)_{|t=\cos\theta} = P_j^{(m)}(t)_{|t=\cos} = P_j^{(m)}(\cos\theta) = P_{Z_j}(\theta), \quad \text{where} \quad m = 0, 1, 2, \ldots, j. \tag{3.19}$$

Thus, according to (3.10) and (3.17)–(3.19) we obtain the eigenvalues

$$\mu_{Zj}^2 = \mu_{Yj}^2 = j(j+1), \tag{3.20}$$

each of which corresponds to $2j+1$ spherical functions

$$\Theta_{Z_j}^{(0)}(\theta,\zeta) = P_j(\theta),$$

$$\Theta_{Z_j}^{(-1)}(\theta,\zeta) = P_j^{(1)}(\cos\theta)\cos\zeta, \;\; \Theta_{Z_j}^{(1)}(\theta,\zeta) = P_j^{(1)}(\cos\theta)\sin\zeta,$$

$$\Theta_{Z_j}^{(-2)}(\theta,\zeta) = P_j^{(2)}(\cos\theta)\cos 2\zeta, \;\; \Theta_{Z_j}^{(2)}(\theta,\zeta) = P_j^{(2)}(\cos\theta)\sin 2\zeta,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\Theta_{Z_j}^{(-l)}(\theta,\zeta) = P_j^{(l)}(\cos\theta)\cos l\zeta, \;\; \Theta_{Z_j}^{(l)}(\theta,\zeta) = P_j^{(l)}(\cos\theta)\sin l\zeta,$$

$$l = 1, 2, \ldots, j, \tag{3.21}$$

where $P_j^{(\pm l)}(\cos\theta)$ are Legendre polynomials.

It should be noted that the system of spherical functions $\left\{\Theta_{Z_j}(\theta,\zeta), \;\; j = 0, 1, 2, \ldots\right\}$ is orthogonal with weight $\sin\theta$ and forms an orthogonal basis in $L_2(\Sigma)$, where $\{1, \theta, \zeta\} \in \Sigma$ is the surface of the unit sphere. We can normalize this system of functions using the condition

$$\int_0^{\pi}\int_0^{2\pi} \left|\Theta_{Z_j}^{(\mp l)}(\theta,\zeta)\right|^2 \sin\theta \, d\theta \, d\zeta = 1.$$

Functions $\Theta_{Z_j}^{(0)}(\theta,\zeta) = P_j(\cos\theta)$ do not depend on ζ and called zonal. Since $P_j(t)$ has exactly j zeros inside the interval $(-1, 1)$, the unit sphere is divided into $(j+1)$ latitude zones, inside which the zonal function retains its sign.

Let us consider the behavior of the function on the sphere

$$\Theta_{Z_j}^{(-l)}(\theta,\zeta) = \sin^l\theta \left[\frac{d^l}{dt^l}P_j(t)\right]\Bigg|_{t=\cos\theta} \cos l\zeta, \;\; \Theta_{Z_j}^{(+l)}(\theta,\zeta) = \sin^l\theta \left[\frac{d^l}{dt^l}P_j(t)\right]\Bigg|_{t=\cos\theta} \sin l\zeta.$$

Since $\sin\theta$ becomes zero at the poles and $\sin l\zeta$ or $\cos l\zeta$ becomes zero at $2l$ meridians, and $\frac{d^l}{dt^l}P_j(t)$ at $(j-l)$ latitudes, the entire sphere is divided into cells in which $\Theta_{Z_j}^{(\mp l)}(\theta,\zeta)$ maintains a constant sign. Functions $\Theta_{Z_j}^{(\pm l)}(\theta,\zeta)$ at $l > 0$ are called tesseral.

Similar constructions are valid for boundary value problem (3.15)–(3.16).

Now we transform equations (3.7)–(3.8), by making the following substitutions

$$R_{Y_j}(r) = \frac{\Phi_{Y_j}(r)}{\sqrt{r}}, \;\; R_{Z_j}(r) = \frac{\Phi_{Z_j}(r)}{\sqrt{r}}. \tag{3.22}$$

Then, taking into account (3.20), instead of (3.7)–(3.9), we obtain the following equations with boundary conditions:

$$r^2\Phi''_{Z_j}(r) + r\Phi'_{Z_j}(r) - \nu^2_{Z_j}\Phi_{Z_j}(r) = -r^2\Phi_{Y_j}(r), \;\; \nu^2_{Z_j} = (j+1/2)^2\,, \tag{3.23}$$

$$r^2\Phi''_{Y_j}(r) + r\Phi'_{Y_j}(r) + \left(\mu_j^2 r^2 - \nu^2_{Y_j}\right)\Phi_{Y_j}(r) = 0, \;\; \nu^2_{Y_j} = (j+1/2)^2\,, \tag{3.24}$$

$$r^{-\frac{1}{2}}\Phi_{Z_j}(r) \quad \text{are bounded in the neighborhood of zero,}$$

$$\Phi_{Z_j}(1) = 0, \;\; \Phi'_{Z_j}(1) = 0.$$

If in (3.24) we make the replacement $\rho = \mu_j r$, then by definition the cylindrical function $\Phi_{Y_j}(r) = J_{\nu_{Y_j}}(\mu_j r)$ will satisfy the equation (3.24), here $\nu_{Y_j} = \nu_{Z_j} = j + \frac{1}{2}$, $j = 0, 1, 2, \ldots$

So, according to the definition of cylindrical functions ([31], chapter VII, § 3) for the equation (3.24) the following statement is true.

Lemma 1. Equation (3.24) has a general solution in the form of a cylindrical function $\Phi_{Y_j}(r) = J_{j+\frac{1}{2}}(\mu_j r)$, $j = 0, 1, 2, \ldots$

Substituting this solution into equation (3.23), we will have a boundary value problem for a second-order nonhomogeneous ordinary differential equation:

$$\begin{gathered} r^2\Phi''_{Z_j}(r) + r\Phi'_{Z_j}(r) - \nu^2_{Z_j}\Phi_{Z_j}(r) = -r^2 J_{j+\frac{1}{2}}\left(\mu_j r\right), r \in (0,1), \\ r^{-\frac{1}{2}}\Phi_{Z_j}(r) \quad \text{are bounded in the neighborhood of zero,} \\ \Phi_{Z_j}(1) = 0, \;\; \Phi'_{Z_j}(1) = 0, \end{gathered} \tag{3.25}$$

where $j = 0, 1, 2, \ldots$

For boundary value problem (3.25) we establish the following lemma.

Lemma 2. For each $j \in \{0, 1, 2, \ldots\}$ the boundary value problem (3.25) has a countable family of solutions

$$\left\{\Phi_{Z_j}(r) = \int_0^1 G_j(r,\rho) J_{j+\frac{1}{2}}(\mu_{j+1,k}\,\rho)\,d\,\rho, \;\; \mu^2_{j+1,k}\right\}, \;\; k = 1, 2, \ldots,$$

where $\mu_{j+1,k}$ are the roots of the equations $J_{j+\frac{3}{2}}(\mu) = 0$, and G_j, $j = 0, 1, 2, \ldots$ is the corresponding Green's function.

Proof. We look for fundamental solutions for (3.25) in the form $\Phi_{j\,\mathrm{f.s.}}(r) = r^\sigma$, where σ is whole unknown number. Substituting r^σ into the homogeneous case of equation (3.25), we find: for $j \neq 0$ $\sigma = j + \frac{1}{2}$, $\sigma = -j - \frac{1}{2}$; for $j = 0$ $\sigma = \frac{1}{2}$, $\sigma = -\frac{1}{2}$, i.e. fundamental solutions are equal

$$z_{1j}(r) = r^{j+\frac{1}{2}}, \;\; z_{2j}(r) = r^{-j-\frac{1}{2}} \quad \text{for each} \quad j \neq 0, \;\; z_{10}(r) = r^{\frac{1}{2}}, \;\; z_{20}(r) = r^{-\frac{1}{2}}. \tag{3.26}$$

Thus, the general solution of homogeneous equation (3.25) according to (3.26) is written in the form

$$\Phi_{Z_j\,f.s.}(r) = C_{1j}r^{j+\frac{1}{2}} + C_{2j}r^{-j-\frac{1}{2}}, \;\; j \in \{1, 2, \ldots\}, \;\; \Phi_{Z_0\,f.s.}(r) = C_{10}r^{\frac{1}{2}} + C_{20}r^{-\frac{1}{2}}. \tag{3.27}$$

Thus, general solutions for the equation from (3.25), obtained on the basis of fundamental solutions (3.26)–(3.27) ([30], chapter 1, § 5, Cauchy method), have the form:

$$\Phi_{\text{j, gen.s.}}(r) = \begin{cases} C_{1j} r^{j+\frac{1}{2}} + \Phi_{j \text{ part.s.}}(r), & j \neq 0, \\ C_{10} r^{\frac{1}{2}} + \Phi_{0 \text{ part.s.}}(r), & j = 0, \end{cases} = \int_0^1 G_j(r,\rho) J_{j+\frac{1}{2}}(\mu_j \rho)\, d\rho, \;\; j = 0,1,2,\ldots, \tag{3.28}$$

where

$$G_j(r,\rho) = \begin{cases} -\frac{1}{2j+1} r^{j+\frac{1}{2}} \left[\rho^{-j+\frac{1}{2}} - \rho^{j+\frac{3}{2}}\right], & 0 < r < \rho < 1, \\ -\frac{1}{2j+1} \rho^{j+\frac{3}{2}} \left[r^{-j-\frac{1}{2}} - r^{j+\frac{1}{2}}\right], & 0 < \rho < r < 1, \end{cases} \quad j = 1,2,3,\ldots, \tag{3.29}$$

$$G_0(r,\rho) = \begin{cases} -r^{\frac{1}{2}} \left[\rho^{\frac{1}{2}} - \rho^{\frac{3}{2}}\right], & 0 < r < \rho < 1, \\ -\rho^{-\frac{1}{2}} \left[r^{-\frac{1}{2}} - r^{\frac{1}{2}}\right], & 0 < \rho < r < 1, \end{cases} \quad j = 0 \tag{3.30}$$

$$C_{1j} = \begin{cases} -\frac{1}{2j+1} \int\limits_0^1 \left[\rho^{-j+\frac{1}{2}} - \rho^{-j+\frac{3}{2}}\right] J_{j+1/2}(\mu\rho)\, d\rho, & j = 1,2,3,..., \\ -\int\limits_0^1 \left[\rho^{\frac{1}{2}} - \rho^{\frac{3}{2}}\right] J_{1/2}(\mu\rho)\, d\rho, & j = 0, \end{cases} \tag{3.31}$$

$$C_{2j} = 0, \;\; j = 0,1,2,3,\ldots, \tag{3.32}$$

the equality of the coefficients C_{2j} to zero follow from the conditions of boundedness in the neighborhood of the point $r = 0$ from (3.25).

We have included the details of the calculations contained in (3.27)–(3.32) in Appendix B.

Next, taking into account the solution formulas (3.28)–(3.31) and satisfying their second boundary conditions at $r = 1$ from (3.25), we obtain

$$J_{j+\frac{3}{2}}(\mu_{j+1}) = 0, \text{ for each } j \in \{0,1,2,\ldots\}. \tag{3.33}$$

Really, we have

$$\Phi'_{Z_j}(1) = 0 = -\int_0^1 \rho^{j+\frac{3}{2}} J_{j+\frac{1}{2}}(\mu\rho)\, d\rho, \;\; j = 0,1,2,\ldots$$

According to formula (20) from ([31], chapter VII, § 3) the last relations are equivalent to the equalities (3.33).

Finally, as a solution of spectral problem (3.7)–(3.9) and taking into account formula (3.22) as eigenfunctions $R_{Zjk}(r)$ from (3.28)–(3.30), we obtain:

$$R_{Zjk}(r) = r^{-\frac{1}{2}} \int_0^1 G_j(r,\rho) J_{j+\frac{1}{2}}(\mu_{j+1,k}\rho)\, d\rho, \;\; J_{j+\frac{1}{2}}(\mu_{j+1,k}) = 0, \;\; j,k = 1,2,3,\ldots, \tag{3.34}$$

$$R_{Z0k}(r) = r^{-\frac{1}{2}} \int_0^1 G_0(r,\rho) J_{\frac{1}{2}}(\mu_{1,k}\rho)\, d\rho, \;\; J_{\frac{1}{2}}(\mu_{1,k}) = 0, \;\; k = 1,2,3,\ldots \tag{3.35}$$

As the roots of the equations $J_{j+\frac{1}{2}}(\mu_{j+1}) = 0, \; j = 0,1,2,\ldots,$ (into (3.34)–(3.35)) we find the eigenvalues

$$\mu^2_{j+1,k}, \;\; j = 0,1,2,\ldots, \;\; k = 1,2,3,\ldots \tag{3.36}$$

Thus, from solutions (3.7)–(3.9), problems (3.11)–(3.21), (3.25) and (3.34)–(3.36) we obtain the following system of eigenfunctions and the corresponding its eigenvalues:

$$\begin{gathered}\left\{Z^{(\pm)}_{jkm}(r,\theta,\zeta) = R_{Zjk}(r)\Theta^{(\pm m)}_{Zj}(\theta,\zeta),\ \mu^2_{j+1,k}\right\}\\ j\in\{0,1,2,\ldots\},\ \ m\in\{0,1,2,\ldots,j\},\ \ k\in\{1,2,3,\ldots\}.\end{gathered} \tag{3.37}$$

Note that the system of eigenfunctions (3.37) satisfies the orthogonality conditions with weight $r^2\sin\theta$.

4 Construction of eigenfunctions in Cartesian coordinates. Main result

Now in (3.37) let us move on to the Cartesian coordinate system.

The system of eigenfunctions and eigenvalues has the form

$$\begin{gathered}\left\{u^{(\pm)}_{jkm}(y) \equiv R_{Zjk}(|y|)\Theta^{(\pm m)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right),\ \mu^2_{jk}\right\},\\ j\in\{1,2,\ldots\},\ \ m\in\{0,1,2,3,\ldots,j\},\ \ k\in\{1,2,3,\ldots\},\ \ |y|<1,\end{gathered} \tag{4.1}$$

$$\left\{u_{0k0}(y) \equiv R_{Z0k}(|y|)\Theta^{(0)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right),\ \mu^2_{0k}\right\},$$

$$j=0,\ \ m=0,\ \ k\in\{1,2,3,\ldots\},\ \ |y|<1, \tag{4.2}$$

$$\operatorname{arctg}\frac{y_2}{y_1} = \operatorname{arctg}\zeta, \quad \text{where} \quad \zeta\in\begin{cases}[0,\frac{\pi}{2}), & y_1>0,\ y_2\ge 0;\\ (\frac{3\pi}{2},2\pi), & y_1>0,\ y_2<0;\\ (\frac{\pi}{2},\frac{3\pi}{2}), & y_1<0;\\ \frac{\pi}{2}, & y_1=0,\ y_2>0;\\ \frac{3\pi}{2}, & y_1=0,\ y_2<0.\end{cases} \tag{4.3}$$

Note that under the conditions of orthogonality of the system of eigenfunctions (4.1)–(4.2) there will be missing weight $|y|^2\sin\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}\right)$, since the Jacobian of the transformation when passing from the Cartesian system to the spherical coordinate system is equal to $r^2\sin\theta$.

Thus, we have established the validity of the following theorem.

Theorem 1. From the solution formulas (3.17), (3.20), (3.21), (3.34)–(3.37) for boundary value problems (3.3)–(3.4), (3.5)–(3.6) and (3.7)–(3.9) respectively, we obtain the following system of eigenfunctions and the corresponding eigenvalues:

$$\begin{gathered}\left\{u^{(\pm)}_{jkm}(y)\equiv Z^{(\pm)}_{jkm}(r,\theta,\zeta) = R_{Zjk}(r)\Theta^{(\pm m)}_{Zj}(\theta,\zeta),\ \mu^2_{j+1,k}\right\}\\ j\in\{0,1,2,\ldots\},\ \ m\in\{0,1,2,\ldots,j\},\ \ k\in\{1,2,3,\ldots\}.\end{gathered}$$

In the Cartesian system, accordingly, we obtain the relations (4.1)–(4.3).

Now, according to the formulas (1.5), (4.1)–(4.2) we define the system of eigenfunctions $w(y) = \{w_1(y), w_2(y), w_3(y)\}$ for the spectral problem (1)–(1.16).

Using the statement of Theorem 1, we establish the following result.

Theorem 2 (Main result). For all $j \in \{0,1,2,\ldots\}$, $m \in \{0,1,2,\ldots,j\}$, $k \in \{1,2,3,\ldots\}$, $|y|<1$, we have that each triple of eigenfunctions $\left\{w^{(\pm)}_{1jkm}(y), w^{(\pm)}_{2jkm}(y), w^{(\pm)}_{3jkm}(y)\right\}$:

$$w^{(-)}_{1jkm} = (\partial_{y_2}-\partial_{y_3})\, u^{(-)}_{jkm}(y),\ \ w^{(-)}_{2jkm} = (\partial_{y_3}-\partial_{y_1})\, u^{(-)}_{jkm}(y),\ \ w^{(-)}_{3jkm} = (\partial_{y_1}-\partial_{y_2})\, u^{(-)}_{jkm}(y), \tag{4.4}$$

$$w^{(+)}_{1jkm} = (\partial_{y_2}-\partial_{y_3})\, u^{(+)}_{jkm}(y),\ \ w^{(+)}_{2jkm} = (\partial_{y_3}-\partial_{y_1})\, u^{(+)}_{jkm}(y),\ \ w^{(+)}_{3jkm} = (\partial_{y_1}-\partial_{y_2})\, u^{(+)}_{jkm}(y), \tag{4.5}$$

where for $j \neq 0$:

$$(\partial_{y_2}-\partial_{y_3})\, u^{(\pm)}_{jkm}(y) \equiv (\partial_{y_2}-\partial_{y_3})\, R_{Zjk}(|y|)\Theta^{(\pm m)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right), \tag{4.6}$$

$$(\partial_{y_3}-\partial_{y_1})\, u^{(\pm)}_{jkm}(y) \equiv (\partial_{y_3}-\partial_{y_1})\, R_{Zjk}(|y|)\Theta^{(\pm m)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right), \tag{4.7}$$

$$(\partial_{y_1}-\partial_{y_2})\, u^{(\pm)}_{jkm}(y) \equiv (\partial_{y_1}-\partial_{y_2})\, R_{Zjk}(|y|)\Theta^{(\pm m)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right), \tag{4.8}$$

and for $j=0$:

$$(\partial_{y_2}-\partial_{y_3})\, u_{0k0}(y) \equiv (\partial_{y_2}-\partial_{y_3})\, R_{Z0k}(|y|)\Theta^{(0)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right), \tag{4.9}$$

$$(\partial_{y_3}-\partial_{y_1})\, u_{0k0}(y) \equiv (\partial_{y_3}-\partial_{y_1})\, R_{Z0k}(|y|)\Theta^{(0)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right), \tag{4.10}$$

$$(\partial_{y_2}-\partial_{y_3})\, u_{0k0}(y) \equiv (\partial_{y_1}-\partial_{y_2})\, R_{Z0k}(|y|)\Theta^{(0)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right) \tag{4.11}$$

form an orthogonal basis in the space $\mathbf{V}\cap\mathbf{H}^2(\Omega)$.

Remark 1. From (3.34)–(3.35), (4.1)–(4.2) and (3.25) it follows that the boundary conditions from (3.9) are valid for $r=|y|=1$, and from (1.5), (4.4)–(4.11) we obtain the satisfiability of the equation (1.2), i.e. $\operatorname{div} w = 0$.

It is obvious that each triple of functions from (4.4)–(4.11) satisfies the homogeneous Dirichlet condition on the boundary of the unit sphere, with the possible exception of the following six points on the sphere $\{y_1,y_2,y_3\}$: $\{1,0,0\}$, $\{-1,0,0\}$, $\{0,1,0\}$, $\{0,-1,0\}$, $\{0,0,1\}$ and $\{0,0,-1\}$.

5 *Towards an approximate solution of the initial boundary value problem* (1.1)–(1.4)

We have constructed the orthogonal basis $w^{(\pm)}_{jkm}(y)$, $j=0,1,2,\ldots,m=0,1,2,\ldots,j,k=1,2,3,\ldots$ in the space $\mathbf{V}\cap\mathbf{H}^2(\Omega)$. And based on this basis, we will introduce an approximate solution and given functions for the initial boundary value problem (1.1)–(1.4), formulated in weak form (in terms of the integral identity):

$$w^{(\pm)}_N(y,t) = \sum_{j=-N,k=1}^{N}\sum_{m=0}^{j} c^{(\pm)}_{jkmN}(t) w^{(\pm)}_{jkm}(y), \tag{5.1}$$

$$f^{(\pm)}_N(y,t) = \sum_{j=-N,k=1}^{N}\sum_{m=0}^{j} d^{(\pm)}_{jkmN}(t) w^{(\pm)}_{jkm}(y), \tag{5.2}$$

$$P_N^{(\pm)}(y,t) = \sum_{j=-N,k=1}^{N} \sum_{m=0}^{j} e_{jkmN}^{(\pm)}(t) w_{jkm}^{(\pm)}(y), \tag{5.3}$$

$$\left(\partial_t w_N^{(\pm)}, w_{lnp}^{(\pm)}\right) + \left(\left(w_N^{(\pm)}, w_{lnp}^{(\pm)}\right)\right) = \left(f_N^{(\pm)}, w_{lnp}^{(\pm)}\right), \ 0 \le l \le N, \ n = 1, \dots, N, \ p = 0, \dots, l, \tag{5.4}$$

$$w_N^{(\pm)}(y,0) = 0, \tag{5.5}$$

where the expansion coefficients $c_{jkmN}^{(\pm)}(t)$ (5.1) are to be determined at given coefficients $d_{jkmN}^{(\pm)}(t)$ (5.2) from the Cauchy problem for ordinary differential equations (5.4)–(5.5). And the expansion coefficients $e_{jkmN}^{(\pm)}(t)$ (5.3) are determined from equations (1.1). Thus, it is possible to find an approximate solution to the initial boundary value problem for the linearized system of Navier-Stokes equations (1.1)–(1.4).

Conclusion

In this work, a basis is constructed in the space solutions of the system of Navier-Stokes equations $\mathbf{V} \cap \mathbf{H}^2(\Omega)$, composed of eigenfunctions of the generalized spectral problem for a three-dimensional bi-Laplacian with Dirichlet boundary conditions in the unit sphere $\Omega = \{y = (y_1, y_2, y_3) : \ |y| < 1\}$. It is shown that these eigenfunctions satisfy the boundary conditions for the liquid velocity vector $w(y) = \{w_1(y), w_2(y), w_3(y)\}$ and the continuity equation $\operatorname{div} w(y) = 0, \ y \in \Omega$.

Appendix A. Spectral problem (1.13)–(1.14) *in spherical coordinates*

Let us recall the well-known formulas for gradient and divergence in spherical coordinates (r, θ, ζ):

$$\nabla u(y) = \partial_r u(r,\theta,\zeta) \cdot i_1 + \frac{1}{r}\partial_\theta u(r,\theta,\zeta) \cdot i_2 + \frac{1}{r\sin\theta}\partial_\zeta u(r,\theta,\zeta) \cdot i_3, \tag{A.1}$$

$$\operatorname{div} \vec{D}(y) = \frac{1}{r^2}\partial_r \left(r^2 D_1(r,\theta,\zeta)\right) + \frac{1}{r\sin\theta}\partial_\theta \left(\sin\theta D_2(r,\theta,\zeta)\right) + \frac{1}{r\sin\theta}\partial_\zeta D_3(r,\theta,\zeta), \tag{A.2}$$

where the vector $\vec{D} = \left\{\partial_r u(r,\theta,\zeta), \ \frac{1}{r}\partial_\theta u(r,\theta,\zeta), \ \frac{1}{r\sin\theta}\partial_\zeta u(r,\theta,\zeta)\right\}$ defined by the gradient vector. In addition, it is known that if $u(r,\theta,\zeta) = R(r)\Theta(\theta,\zeta)$, then

$$\Delta u(y) = \operatorname{div} \nabla u(y) = \frac{1}{r^2}\left(r^2 R'(r)\right)' \Theta(\theta,\zeta) + \frac{1}{r^2} R(r) \triangle_{\theta,\zeta} \Theta(\theta,\zeta),$$

where

$$\Delta_{\theta,\zeta} Z \equiv \frac{1}{\sin\theta}\partial_\theta \left(\sin\theta \, \partial_\theta Z\right) + \frac{1}{\sin^2\theta}\partial_\zeta^2 Z.$$

Now, instead of gradient (A.1), we introduce a new vector (modified gradient vector):

$$\widetilde{\nabla} u(y) = \frac{1}{r}\partial_\theta u(r,\theta,\zeta) \cdot i_1 + \frac{1}{r\sin\theta}\partial_\zeta u(r,\theta,\zeta) \cdot i_2 + \partial_r u(r,\theta,\zeta) \cdot i_3, \tag{A.3}$$

where $\frac{1}{r}\partial_\theta u = \tilde{D}_1, \quad \frac{1}{r\sin\theta}\partial_\zeta = \tilde{D}_2, \quad \partial_r u = \tilde{D}_3$.

Then, using (A.2) and (A.3), we have:

$$\operatorname{div} \widetilde{\nabla} u(y) \equiv \left(\partial_{y_1 y_2}^2 + \partial_{y_2 y_3}^2 + \partial_{y_3 y_1}^2\right) u(y) =$$

$$= \frac{1}{r^2}\partial_r \left(r^2 \frac{1}{r}\partial_\theta u(r,\theta,\zeta)\right) + \frac{1}{r\sin\theta}\partial_\theta \left(\sin\theta \frac{1}{r\sin\theta}\partial_\zeta u(r,\theta,\zeta)\right) + \frac{1}{r\sin\theta}\partial_\zeta \left(\partial_r u(r,\theta,\zeta)\right).$$

And finally, we have for the required operator (1.13):

$$\left(\triangle-\partial^2_{y_1y_2}-\partial^2_{y_2y_3}-\partial^2_{y_3y_1}\right)u=\frac{1}{r^2}\partial_r\left(r^2\partial_r u(r,\theta,\zeta)\right)+\frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta u(r,\theta,\zeta)\right)+$$

$$+\frac{1}{r^2\sin^2\theta}\partial^2_\zeta u(r,\theta,\zeta)-\frac{1}{r^2}\partial_r\left(r\partial_\theta u(r,\theta,\zeta)\right)-\frac{1}{r^2\sin\theta}\partial^2_{\theta\zeta}u-\frac{1}{r\sin\theta}\partial^2_{\zeta r}u(r,\theta,\zeta).$$

Having separated the variables $u(r,\theta,\zeta)=R(r)\Theta(\theta,\zeta)$, we obtain

$$\left(\triangle-\partial^2_{y_1y_2}-\partial^2_{y_2y_3}-\partial^2_{y_3y_1}\right)u(y)=\frac{1}{r^2}\left(r^2R'(r)\right)'\Theta\left(\theta,\zeta\right)+\frac{1}{r^2}R(r)\triangle_{\theta,\zeta}\Theta\left(\theta,\zeta\right)-$$

$$-\frac{1}{r}R'(r)\left(\partial_\theta\Theta\left(\theta,\zeta\right)+\frac{1}{\sin\theta}\partial_\zeta\Theta\left(\theta,\zeta\right)\right)-\frac{1}{r^2}R(r)\left(\partial_\theta\Theta\left(\theta,\zeta\right)+\frac{1}{\sin\theta}\partial^2_{\theta\zeta}\Theta\left(\theta,\zeta\right)\right);$$

$$\triangle_{\theta,\zeta}\Theta\left(\theta,\zeta\right)=\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Theta\left(\theta,\zeta\right)\right)+\frac{1}{\sin^2\theta}\partial^2_\zeta\Theta\left(\theta,\zeta\right). \qquad (A.4)$$

Thus, we have obtained the spectral problem (A.4) and (1.14), which (in our opinion) is an unsolvable problem to solve. Naturally, the boundary conditions (1.14) must be written on the surface of the unit sphere and at the center of the sphere (in spherical coordinates):

$$u(r,\theta,\zeta)_{|r=1}=0,\;\;\partial_r u(r,\theta,\zeta)_{|r=1}=0,$$

$u(r,\theta,\zeta)$ is bounded in the neighborhood of the center of sphere.

Appendix B. Cauchy Method

According to [23, chapter 1, § 5] a particular solution to the equation (3.25) has the form

$$\Phi_{j\,ch.s.}(r)=-\int\limits_0^r\eta_j(r,\rho)J_{j+\frac12}(\mu_j\rho)d\rho, \qquad (B.1)$$

where for the Cauchy function $\eta_j(r,\rho)$ we have

$$\eta_j(r,\rho)=C_{1j}(\rho)r^{j+\frac12}+C_{2j}(\rho)r^{-j-\frac12}. \qquad (B.2)$$

Using (B.2), we obtain a system of equations for determining the unknown coefficients $C_{1j}(\rho)$ and $C_{2j}(\rho)$:

$$\begin{cases}\eta_j(\rho,\rho) &= C_{1j}(\rho)\rho^{j+\frac12}+C_{2j}(\rho)\rho^{-j-\frac12} &= 0,\\ \partial_r\eta_j(\rho,\rho) &= \left(j+\frac12\right)\left[C_{1j}(\rho)\rho^{j-\frac12}-C_{2j}(\rho)\rho^{-j-\frac32}\right] &= 1.\end{cases} \qquad (B.3)$$

From (B.3) we have:

$$C_{1j}(\rho)=\frac{1}{2j+1}\rho^{-j+\frac12},\quad C_{2j}(\rho)=-\frac{1}{2j+1}\rho^{j+\frac32}. \qquad (B.4)$$

Thus, from (B.2)–(B.4) for the Cauchy function we obtain

$$\eta_j(r,\rho)=\frac{1}{2j+1}\left[\rho^{-j+\frac12}r^{j+\frac12}-\rho^{j+\frac32}r^{-j-\frac12}\right],$$

respectively for the particular solution $\Phi_{j\,ch.s.}(r)$ (B.1):

$$\Phi_{j\,ch.s.}(r) = -\frac{1}{2j+1}\int\limits_0^r \left[\rho^{-j+\frac{1}{2}} r^{j+\frac{1}{2}} - \rho^{j+\frac{3}{2}} r^{-j-\frac{1}{2}}\right] J_{j+\frac{1}{2}}(\mu_j\rho)d\rho. \qquad (B.5)$$

Now, using (B.5) and (3.22), we write the formulas for general solutions of the nonhomogeneous equations (3.25) and (3.7), respectively. We have

$$\Phi_{j\,gen.s.}(r) = C_{1j} r^{j+\frac{1}{2}} + C_{2j} r^{-j-\frac{1}{2}} - \frac{1}{2j+1}\int\limits_0^r \left[\rho^{-j+\frac{1}{2}} r^{j+\frac{1}{2}} - \rho^{j+\frac{3}{2}} r^{-j-\frac{1}{2}}\right] J_{j+\frac{1}{2}}(\mu_j\rho)d\rho, \qquad (B.6)$$

$$R_{j\,gen.s.}(r) = C_{1j} r^{j} + C_{2j} r^{-j-1} - \frac{1}{2j+1}\int\limits_0^r \left[\rho^{-j+\frac{1}{2}} r^{j} - \rho^{j+\frac{3}{2}} r^{-j-1}\right] J_{j+\frac{1}{2}}(\mu_j\rho)d\rho, \qquad (B.7)$$

where in (B.6)-(B.7) C_{1j} and C_{2j} are the unknown constants that need to be found. To do this, we will use the boundary conditions from (3.25). Due to the boundedness of the solution (B.7) in the neighborhood of zero, it is necessary that the coefficients C_{2j} be equal to zero, i.e., $C_{2j} = 0$. According to the boundary condition $R_j(1) = 0$ from (3.25) from (B.7) we get

$$C_{1j} = \frac{1}{2j+1}\int\limits_0^1 \left[\rho^{-j+\frac{1}{2}} - \rho^{j+\frac{3}{2}}\right] J_{j+\frac{1}{2}}(\mu\rho)d\rho,$$

$$\begin{aligned} R_{j\,gen.s.}(r) &= \tfrac{1}{2j+1}\int\limits_0^r \left[r^{-j-\frac{1}{2}} - r^{j+\frac{1}{2}}\right] \rho^{j+\frac{3}{2}} J_{j+\frac{1}{2}}(\mu_j\rho)d\rho + \\ &+ \tfrac{1}{2j+1}\int\limits_r^1 \left[\rho^{-j+\frac{1}{2}} - \rho^{j+\frac{3}{2}}\right] r^{j+\frac{1}{2}} J_{j+\frac{1}{2}}(\mu_j\rho)d\rho. \end{aligned}$$

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Бірлік шардағы үшөлшемді би-Лапласиан үшін қойылған спектрлік есеп туралы

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Мақалада ток функциясының жаңа түсінігін енгіземіз және үшөлшемді жағдайда ток функциясының теңдеуін шығарамыз. Навье-Стокс жүйесінің шешімдерінің кеңістігінде базис құру үшін шекарада Дирихле шарттары бар би-Лапласиан үшін көмекші спектрлік есепті шешеміз. Әрі қарай, ток функциясын енгізу үшін қолданылған формулаларды пайдалана отырып, Навье-Стокс жүйесінің шешімдерінің кеңістігінде базис болатын функциялар жүйесін табамыз. Бұл базисті Навье-Стокс жүйесі үшін сызықты және сызықты емес тура және кері есептерді жуықтап шешу үшін қолдануға болатынын атап өткен жөн. Ұсынылған жұмыстың негізгі идеясы келесідей: шекаралық шарттарды емес (оларды өзгеріссіз қалдырамыз) спектрлік параметрі бар ток функциясының дифференциалдық теңдеулерін өзгерту. Нәтижесінде біз облыс шекарасында Дирихле шарттарымен үшөлшемді бірлік шармен бейнеленген облыстағы би-Лапласианда спектрлік есеп аламыз, оны шешу кезінде Навье-Стокс теңдеулер жүйесінің шешімдерінің кеңістігінде базис құрайтын меншікті функциялар жүйесін табамыз. Бұл жағдайда шекаралық шарттар сақталып, сұйықтың үзіліссіздігі шартымен берілген теңдеу дің орындалғаны маңызды. Навье-Стокс жүйесінің үшөлшемді жағдайы үшін ток функциясының аналогы белгісіз болғанын да ескереміз.

Кілт сөздер: Навье-Стокс жүйесі, би-Лапласиан, спектрлік есеп, ток функциясы.

О спектральной задаче для трехмерного би-Лапласиана в единичном шаре

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В статье мы вводим новое понятие функции тока и выводим уравнение для функции тока в трехмерном случае. Для построения базиса в пространстве решений системы Навье-Стокса мы решаем вспомогательную спектральную задачу для би-Лапласиана с условиями Дирихле на границе. Далее, с помощью формул, которые использовались для введения функции тока, мы находим систему функций, образующую базис в пространстве решений системы Навье-Стокса. Следует отметить, что этот базис может быть использован для приближенного решения прямых и обратных задач для системы Навье-Стокса, как линеаризованной, так и нелинейной. Основная идея представленной работы заключается в следующем: изменять не граничные условия (их оставляем без изменений), а менять дифференциальные уравнения для функции тока со спектральным параметром. В результате мы получаем спектральную задачу для би-Лапласиана в области, представленной трехмерным единичным шаром, с условиями Дирихле на границе области, решая которую, мы находим систему собственных функций, образующих базис в пространстве решений системы уравнений Навье-Стокса. При этом является важным, что сохраняются граничные условия, и выполняется уравнение, представленное условием неразрывности жидкости. Заметим также, что для трехмерного случая системы Навье-Стокса аналог функции тока был неизвестен.

Ключевые слова: система Навье-Стокса, би-Лапласиан, спектральная задача, функция тока.

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