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Punctual numberings for families of sets

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This work investigates the structure of punctual numberings for families of punctually enumerable sets with respect to primitive recursively reducibility. We say that a numbering of a certain family is primitive recursively reducible to another numeration of the same family if there exists a primitive recursively procedure (an algorithm not employing unbounded search) mapping the numbers of objects in the first numbering to the numbers of the same objects in the second numbering. This study was motivated by the work of Bazhenov, Mustafa, and Ospichev on punctual Rogers semilattices for families of primitive recursively enumerable functions. The concept of punctually enumerable sets was introduced in the paper, and it was proven that not all recursively enumerable sets are punctually enumerable, but in all m-degrees, recursively enumerable sets include punctually enumerable sets. For two-element families of punctual sets, it was demonstrated that punctual Rogers semilattices can be of at least three types: (1) one-element family, (2) isomorphic to the upper semilattice of recursively enumerable sets with respect to primitive recursively m-reducibility, (3) without the greatest element. It was also proven that the set of all punctually enumerable sets does not have a punctual numbering, and punctual families with a Friedberg numbering do not have the least numbering.

Keywords: primitive recursive functions, punctually enumerable sets, Rogers semilattice, quick functions, punctual numberings.

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Introduction

Theory of computable numberings is one of the actively developing areas in the computability theory. *Numbering* of a countable set S is any surjective mapping $\nu : \omega \to \mathcal{S}$ (Here and further as ω we denote the set of natural numbers). A numbering ν is called computable if the set

$$\{\langle n, x\rangle : n \in \omega, x \in \nu(n)\}$$

is computably enumerable (c.e.) set.

The set of all computable numberings for family $\mathcal{S}$ denotes as $\mathrm{Com}(\mathcal{S})$. Let ν and μ are numberings for family S. Numbering ν is *reducible* to μ if there is computable function f such that $\nu = \mu \circ f$ (denotes $\nu \leq \mu$). This reducibility induces a partially preordered set structure, which factor structure is called *Rogers semilattice for family S* and denoted as $\mathcal{R}(\mathcal{S})$.

There are several interesting results known about Rogers semilattice. For example, if $\mathcal{S}$ is a family of c.e. sets, then either $|\mathcal{R}(\mathcal{S})| = 1$ or $|\mathcal{R}(\mathcal{S})| = \infty$ [1]. In the case when $|\mathcal{R}(\mathcal{S})| = \infty$ the semilattice is not

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a lattice [2]. There is a universal numbering of all partial computable functions [3] and a computable numbering of all c.e. sets without repetitions [4] (these numberings are called friedberg numbering). For more information about the properties of the classical Rogers semilattice, refer to the following articles: [5–8].

In recent years, under the influence of the work [9], interest in primitive recursive (or punctual) properties of algebraic structures has increased. The next articles will help you to find more information about punctual structures [10–27]. In connection with this, studying the punctual properties of numberings is also relevant. Bazhenov, Mustafa, and Ospichev considered punctual numberings of families of functions in their article [28]. The authors established punctual reducibility between numberings, induced by primitive recursive functions, leading to the creation of upper semilattices of degrees known as Rogers pr-semilattices. They demonstrated that any infinite, uniformly primitive recursive family S induces an infinite Rogers pr-semilattice $\mathcal{R}$. It was proven that the semilattice $\mathcal{R}$ is downwards dense, with every nontrivial interval within $\mathcal{R}$ containing an infinite antichain. Additionally, every non-greatest element in $\mathcal{R}$ is a part of an infinite antichain. The authors showed that the Σ_1-fragment of the theory $Th(R)$ is decidable. Several examples were provided to emphasize the contrasts between the punctual framework and the classical theory of computable numberings. Notably, it was demonstrated that some infinite Rogers pr-semilattices $\mathcal{R}$ are lattices, while others are not. The authors obtained a series of results concerning special classes of punctual numberings, including Friedberg numberings and decidable numberings with primitive recursive numeration equivalence. This paper is a logical continuation of the article [28] and aims to investigate punctual numberings for families of sets.

In Chapter 2, we introduce punctual analogs of concepts standard in the theory of computable numberings and define the punctual Rogers semilattice for sets. Chapter 3 is devoted to the structural properties of c.e. degrees induced by the restriction of m-reducibility by primitive recursive functions (called prm-reducibility). Chapters 4 and 5 present some properties of the punctual Rogers semilattice for finite and infinite families, including its connection with the structure of c.e. prm-degrees.

We adhere to the notations and terminology adopted in [29, 30]. We denote by $\{p_e\}_{e\in\omega}$ the computable numbering of all primitive recursive functions. In this article we will consider restricted Church-Turing thesis for primitive recursive functions. We can define this thesis as follows: a function is primitive recursive if and only if it can be described by an algorithm that uses only bounded loops. More about restricted Church-Turing thesis you can find in the work [31].

1 Punctually enumerable sets and numberings

In the paper [28], the numbering ν of a family of primitive recursive functions is called "punctual" if the function $g_\nu(n,x) := (\nu(n))(x)$ is primitive recursive. It seems natural to attempt to extend this definition to a family of c.e. sets, but here we face some difficulties.

The thing is, such a definition of punctual numbering yields the same class of computable numberings for families of c.e. sets because any c.e. set can be represented as the range of a primitive recursive function, which means that a family can be enumerated in a punctual way. On the other hand, even with the presence of a punctual enumeration of a c.e. set, it is not always possible to use primitive recursive constructions, for example, due to the unbounded repetition of elements in the enumeration. In this regard, it makes sense to consider families of sets with stricter enumeration constraints than c.e. sets.

Definition 1. A set A is called *punctually enumerable*, if there is a primitive recursive function p, such that

1) $A = \text{range}(p)$, and
2) If $p(x) = p(y)$ for some $x < y$, then $\text{range}(p) = \{p(0), p(1), \ldots, p(x)\}$.

We will call the function p as a *quick* function for A.

Thus, the quick function p from the definition is injective for an infinite set A, and for a finite set it eventually enumerates all of its elements and then starts repeating them.

Definition 2. A numbering ν of a family $\mathcal{S}$ of punctually enumerable sets is called *punctual*, if there exists a primitive recursive function $g_\nu(n,x)$ such that $\lambda x.g_\nu(n,x)$ is a quick function for $\nu(n)$ for any n. The set of all punctual numberings for family $\mathcal{S}$ we will denote as $\mathrm{Com}_{pr}(\mathcal{S})$.

Reductions on numberings are defined analogously to [28].

Definition 3. We say that numbering ν is *punctually reducible* to numbering μ (denoted as $\nu \leq_{pr} \mu$), if there is primitive recursive function f such that $\nu = \mu \circ f$.

Numberings ν and μ are punctually equivalent and denote as $\nu \equiv_{pr} \mu$, if $\nu \leq_{pr} \mu$ and $\mu \leq_{pr} \nu$.

As in the computable case, the least upper bound of the numberings ν and μ is the numbering $\nu \oplus \mu$, which is defined as

$$(\nu \oplus \mu)(2x) = \nu(x),\ (\nu \oplus \mu)(2x+1) = \mu(x).$$

As punctual Rogers semilattice of the family of punctually enumerable sets $\mathcal{S}$, we will call partially ordered set $\mathcal{R}_{pr}(\mathcal{S}) = (\mathrm{Com}_{pr}(\mathcal{S})_{/\equiv_{pr}}, \leq_{pr}, \oplus)$.

The following theorem demonstrates the independence of the concepts of primitive recursive set (having primitive recursive characteristic function) and punctually enumerable.

Theorem 1. There exist sets A and B such that A is punctually enumerable but not primitive recursive, and B is primitive recursive but not punctually enumerable.

Proof. As a set A we can choose the set $K \oplus \omega$. For this set its quick function we can construct as follows: we fix a primitive recursive approximation of the creative set K, denoted K_i, which at each step enumerates at most one element. Then, we set $f(0) = 1$, and $f(x) = 2s$, where $s \in K_x \setminus K_{x-1}$, if such s exists. If there is no such s, then $f(x)$ is defined as the smallest odd number that has not been used before. It is clear that f is injective primitive recursive function and A is the range of f.

For set B we will construct its primitive recursive characteristic function ϕ such that for B there is no quick function. We fix a computable numbering of all injective primitive recursive functions with the following condition

$$i_e(x)[t]\downarrow= a \Rightarrow a < t.$$

We will define B as a infinite set. So, it is sufficient that there is no injective quick function for B.

At step s we will define $\phi(s)$ as follow: Assume that k is the cardinality of the set $\{t : \ \phi(t) = 1 \ \&\ t < s\}$. If there is more that k elements $x \leq s$ such that $i_k(x)[s]\downarrow$, then define $\phi(s) = 1$. Otherwise, define $\phi(s) = 0$.

It is not hard to see that B is infinite set and any i_k can not enumerate B.

2 *The structure of prm-degrees*

In recent work [32] considered a many-one reductions for computable sets under primitive recursive functions, and have been proven that first-order theory of upper semilattice of degrees of computable sets with respect to primitive recursive many-one reducibility is hereditarily undecidable.

Definition 4. [32] The set A is *prm-reducible* to the set B (written as $A \leq_m^{pr} B$), if there exists a primitive recursive function f such that $A \leq_m B$ via f.

Remark 1. The computable m-degree contains infinitely many prm-degrees.

Theorem 2. For any c.e., but not computable set A, there is c.e. set B such that $A \equiv_m B$ and $A \not\leq_m^{pr} B$.

Proof. Let's define computable majorant for all primitive recursive functions:

$$f(0) = p_0(0) + 1,$$
$$f(x+1) = \max_{i,j \leq x+1} \{p_i(j), f(x)\} + 1.$$

Note that f is not primitive recursive but the set range(f) is primitive recursive.

Let $B = f(A) = \{f(x) : \ x \in A\}$.

It is clear, that $A \leq_m^f B$, since f is strongly increasing, then $x \in B \Leftrightarrow \exists n \leqslant x\,(n \in A \,\&\, f(n) = x)$, which means that reverse reducibility is correct.

Let's show that if $A \leq_m^{pr} B$, then A is computable. We fix primitive recursive function p_e, which reduces A to B, and also step s such that $A \upharpoonright e = A_s \upharpoonright e$ (here by $A \upharpoonright e$ denotes the set $\{x : \ x \in A \,\&\, x \leq e\}$).

Let x be an arbitrary number. If $x \leqslant e$, then $x \in A \Leftrightarrow x \in A_s$. Otherwise, we check the following condition: $p_e(x) \in range(f)$? If it is not, then $x \notin A$. If it is, we effectively find z_o such that $f(z_0) = p_e(x)$.

Repeat for z_0 same procedure as we did for x, and, if $z_0 > e$, then we find number z_1 such that $f(z_1) = p_e(z_0)$ and so on. As a result, we receive sequence $(z_k)_k$. Since $f(z_0) = p_e(x) < f(x)$, by definition of f, then $z_0 < x$, consequently, the sequence $(z_k)_k$ decreases and we find k, such that $p_e(z_k) \notin range(f)$ or $z_k \leqslant e$. Then $x \in A \Leftrightarrow p_e(x) = f(z_0) \Leftrightarrow z_0 \in A \Leftrightarrow p_e(z_0) = f(z_1) \Leftrightarrow \cdots \Leftrightarrow p_e(z_{k-1}) = f(z_k) \Leftrightarrow z_k \in A$. If $p_e(z_k) \notin range(f)$, then $z_k \notin A$, otherwise $z_k \leqslant e$ and $z_k \in A \Leftrightarrow z_k \in A_s$.

Thus, we can effectively define that x belongs to A or not.

Corollary 1. Every non-computable c.e. m-degree contains infinitely many prm-degrees.

Proof. Let A_0 be non-computable c.e. set. By using the previous theorem, we will build c.e. set A_1 such that $A_0 \equiv_m A_1$ and $A_0 \nleq_m^{pr} A_1$, for A_1 similarly build A_2, and for A_2 build A_3 and so on. All sets A_n, $n \in \omega$ are m-equivalent, and for $i < j$ set A_i m-reduces to A_j by $f^{j-i}(x)$ (($(j-i)$-th composition of function f from the previous theorem), consequently, $A_i \nleq_m^{pr} A_j$. Here, note that $B = f(A) \leq_m^{pr} A$. (Proof is similar).

3 Punctual semilattice of two-element families

In the work [28] it was shown that the punctual Rogers semilattice of a finite family of functions always has exactly one element. However, it turns out that this is not the case for families of sets.

In this chapter we assume, that $\mathcal{S} = \{A, B\}$, where A, B are different punctually enumerable sets.

Note that in this case $\mathcal{R}_{pr}(\mathcal{S}) \neq \emptyset$, since the function

$$\alpha(n)(x) = \begin{cases} f(x), & \text{for } n = 2k, \\ g(x), & \text{for } n = 2k+1, \end{cases}$$

where f and g are quick functions for A and B respectively, gives the punctual numbering of the family $\mathcal{S}$.

Proposition 1. Let $\mathcal{S}$ is punctual two-element family such that A or B is finite then $|\mathcal{R}_{pr}(\mathcal{S})| = 1$.

Proof. Let $|A| = N \leqslant |B|$; f and g are quick functions of the sets A and B, respectively.

Let $\nu, \mu \in \mathrm{Com}_{pr}(\mathcal{S})$ are arbitrary and k_ν, k_μ their quick functions. Let's show that $\nu \equiv_{pr} \mu$.

Fix numbers a and b such that $\mu(a) = A$ and $\mu(b) = B$. Then $\nu \leqslant_{pr} \mu$ by primitive recursive function h, which defines as:

$$h(n) = \begin{cases} a, & \text{if } |k_\nu(n,\cdot) \restriction N| \leqslant N, \\ b, & \text{otherwise.} \end{cases}$$

Really, $|k_\nu(n,\cdot) \restriction N| \leqslant N$ means that quick function k_ν on N-th argument starts to repeat the values, consequently, $|\nu(n)| \leqslant N$ and that's why $\nu(n) = A = \mu(a)$.

Reverse reducibility is proved similarly.

Proposition 2. Let $\mathcal{S}$ be a two-element family such that A and B are infinite, $A \cap B$ is finite and one of the sets is primitive recursive, then $|\mathcal{R}_{pr}(\mathcal{S})| = 1$.

Proof. Let $|A \cap B| = N$ and A is primitive recursive.

Let's take two numberings ν and μ of the family $\mathcal{S}$ and fix numbers a and b such that $\mu(a) = A$ and $\mu(b) = B$.

We define

$$h(n) = \begin{cases} a, & \text{if } \exists x \in (k_\nu(n,\cdot) \restriction N) \cap (A \setminus B), \\ b, & \text{otherwise.} \end{cases}$$

Note, that $|k_\nu(n,\cdot) \restriction N| > |A \cap B| \Rightarrow \exists x \in (k_\nu(n,\cdot) \restriction N) \setminus (A \cap B)$. We can check that x belongs to A by primitive recursive procedure, and $\nu(n) = A \Leftrightarrow x \in A$. Consequently, $\nu \leqslant_{pr} \mu$ by function h. It is clear that reverse reducibility is true, then $\nu \equiv_{pr} \mu$.

Theorem 3. There exists family $\mathcal{S} = \{A, B\}$, where $|A \cap B| < \infty$, such that there is no universal numbering for $\mathcal{S}$.

Proof. We will build the sets A, B and numbering α_e for family $\mathcal{S} = \{A, B\}$, satisfying the following requirements:

$$\mathcal{P}_{e,i}: \ \pi_e \in \mathrm{Com}_{pr}(\mathcal{S}) \to \alpha_e \not\leqslant_{pr} \pi_e \text{ by function } p_i,$$

where π_e is computable numbering of all primitive recursive numberings, p_i is computable numbering of all primitive recursive functions. Let k_e be primitive recursive quick function for numbering π_e.

Strategy for $\mathcal{P}_{e,i}$:

1) Pick $w_{e,i}$ – the least number, that we do not use before.
2) Wait until $p_i(w_{e,i}) \downarrow$ and $k_e(p_i(w_{e,i}), 0) \downarrow$ on the step t. While we are waiting, list to $\alpha_e(w_{e,i})$ new numbers.
3) We perform one of the following cases:

Case 1: If $k_e(p_i(w_{e,i}), 0) \in B$, then all elements that we listed to $\alpha_e(w_{e,i})$ until this step, we add to A. Also, we add to $\alpha(w_{e,i})$ all elements from A. After this, we add to $\alpha_e(w_{e,i})$ all elements that we add to A.

Case 2: If $k_e(p_i(w_{e,i}), 0) \in A$, then all elements that we listed to $\alpha_e(w_{e,i})$ until this step, we add to B. Also, we add to $\alpha(w_{e,i})$ all elements from B. After this, we add to $\alpha_e(w_{e,i})$ all elements that we add to B.

Case 3: If $k_e(p_i(w_{e,i}), 0) \notin A \cup B$, then $k_e(p_i(w_{e,i}), 0) \in B$ and return to the Case 1.

Construction. Fix effective linear order of requirements:

$$\mathcal{P}_{0,0} < \mathcal{P}_{1,0} < \mathcal{P}_{0,1} < \mathcal{P}_{2,0} < \mathcal{P}_{1,1} < \mathcal{P}_{0,2} < \ldots$$

On step s of the construction we visit the first s strategies from the list. At every step, fresh numbers are selected and thrown into the sets A or B.

Let $\pi_e \in \mathrm{Com}_{pr}(\mathcal{S})$, by this we build α_e and let there is primitive recursive function p_i, such that $\alpha_e \leq_{pr} \pi_e$ by function p_i. So the following should be performed:

$$\forall x[\alpha_e(x) = \pi_e(p_i(x))].$$

Let's check the witness $w_{e,i}$ in $\mathcal{P}_{e,i}$. By construction, while $p_i(w_{e,i})$ defines on step t, in $\alpha_e(w_{e,i})$ we add new numbers. Suppose that on stage 3 the number $k_e(p_i(w_{e,i}), 0)$ is in the set B, since $A \cap B = \emptyset$, the following performed $\pi_e(p_i(w_{e,i})) = B$. But, by construction $\alpha_e(w_{w,i}) = A$. This contradicts to reducibility α_e to π_e.

For case, when the number $k_e(p_i(w_{e,i}), 0)$ is in the set A, we do the same action. If number $k_e(p_i(w_{e,i}), 0) \notin A \cup B$, then the action is performed as in the first case.

Proposition 3. Let $\mathcal{S}$ be two-elemented family, such that $A \subset B$, then $|\mathcal{R}_{pr}(\mathcal{S})| = \infty$ with universal numbering.

Proof. Since the sets A and B are punctually enumerable, then there are functions p, q – quick functions for A and B, respectively.

We will build the numberings for family $\mathcal{S}$ as follows:

Let W be arbitrary c.e. set, which is not empty and not ω, then numbering ν defines as:

$$\nu_W(x) = \begin{cases} A, \text{if } x \notin W, \\ B, \text{if } x \in W. \end{cases}$$

$W = \cup_s W_s$.

Quick function $\lambda x.h(x, y)$ of the numbering ν will add elements as:

On step 0. $h_\nu(x, 0) = p(0)$.

On step s.

1) $h_\nu(x, s) = p(s)$, if $x \notin W_s$,

2) $h_\nu(x, s) = q(\mu_{z \leq s+1}[q(z) \notin h_\nu(x, \cdot) \upharpoonright s])$, if $x \in W_s$.

It is easy to check, that for an arbitrary W_i, which is not empty and not ω, we can decide that $\nu_{W_i} \in \mathrm{Com}_{pr}(\mathcal{S})$.

W_i reduces to W_j by primitive recursive function if and only if ν_{W_i} reduces by primitive recursive function to ν_{W_j}.

Now, let α be an arbitrary punctual numbering of the family $\mathcal{S}$. Then we can find c.e. set W_i such that $\alpha = \nu_{W_i}$. Since α is numbering of the family $\mathcal{S}$, and W_i is not empty, then there is an element $b \in B \setminus A$, then we can define the function $\varphi(x) = \mu_z[h_\alpha(x, z) = b]$, which is range of c.e. set W_i. Which means that $\mathcal{R}_{pr}(\mathcal{S})$ is isomorphic to $\mathcal{L}^0_{prm}$.

It is known, that the set $K_0 = \{\langle x, y\rangle | x \in W_y\}$ is universal in $\mathcal{L}^0_{prm}$. Consequently, ν_{K_0} is universal punctual numbering of the family $\mathcal{S}$.

Note that $\mathcal{R}_{pr}(\mathcal{S})$ is isomorphic to the upper semilattice of all c.e. sets under *pr*-many-one reducibility. By [32] we can say that first-order theory of $\mathcal{R}_{pr}(\mathcal{S})$ is undecidable.

Proposition 4. There exists the family $\mathcal{S} = \{A, B\}$ such that $|A \cap B| = \infty$ and $|\mathcal{R}_{pr}(\mathcal{S})| = \infty$ without universal numbering.

Proof. Let $A = \omega \setminus \{0\}$ and $B = \omega \setminus \{1\}$. And let μ be an arbitrary numbering of the family $\mathcal{S}$. We will build the numbering $\nu \in \mathrm{Com}_{pr}(\mathcal{S})$ so that $\nu \not\leq_{pr} \mu$.

We will construct quick function q_ν for ν as follows:

$$q_\nu(w, y) = \begin{cases} y + 2, & \{0, 1\} \cap h_\mu(p_e(w) \downarrow, \cdot) \upharpoonright y = \emptyset; \\ 0, & h_\mu(p_e(w), y) = 1; \\ 1, & h_\mu(p_e(w), y) = 0; \\ y + 1, & \{0, 1\} \cap h_\mu(p_e(w) \downarrow, \cdot) \upharpoonright y \neq \emptyset, \end{cases}$$

where h is quick function of the numbering μ and p_e is function that does not reduce ν to μ.

Let numbering ν reduces to numbering μ by primitive recursive function p, then there is w such that $\nu(w) = 0$ and $\mu(p(w)) = 1$ or $\nu(w) = 1$ and $\mu(p(w)) = 0$. Contradiction.

Since μ is an arbitrary numbering from the $\mathrm{Com}_{pr}(\mathcal{S})$, then for any numbering from $\mathrm{Com}_{pr}(\mathcal{S})$ we can construct ν, which does not reduce to μ, which means that there is no universal numbering in this family.

4 Punctual semilattice of the infinite families

Theorem 4. There is no punctual numbering for the family of all punctually enumerable sets.

Proof. Suppose that there is the numbering ν for the family of all punctually enumerable sets. Let $g(x, y)$ be a quick function for ν. We construct punctually enumerable set A such that $\nu(e) \neq A$ for all $e \in \omega$; thus we will come to the contradiction.

Let $q(0) = p(0, 0) + 1$ and $q(n + 1) = \max(\{p(n + 1, z) : \ z \leq n + 1\} \cup \{q(n)\}) + 1$.

It is clear that q is primitive recursive increasing function. Assume $A = \mathrm{range}(q)$.

By contradiction, assume $\nu(n) = A$ for some $n \in \omega$. Since, A is infinite, the function $\lambda y.g(n, y)$ is injective. It is clear that the set $\{p(n, z) : \ z \leq n\}$ has $n+1$ different elements, hence in $\{p(n, z) : \ z \leq n\}$ there is a number greater than $q(n)$. Contradiction.

Corollary 2. There is no punctual numbering for the family of all primitive recursive punctually enumerable sets.

The proof of corollary is the same as the proof of the theorem.

Definition 5. The numbering $\nu \in \mathrm{Com}_{pr}(\mathcal{S})$ is called *friedberg*, if it is injective.

Proposition 5. If the infinite family $\mathcal{S}$ has friedberg numbering, then
1) $\mathcal{R}_{pr}(\mathcal{S})$ does not have the least element,
2) $|\mathcal{R}_{pr}(\mathcal{S})| = \infty$.

Proof. 1) Let ν be friedberg numbering for the infinite family $\mathcal{S}$. Suppose that α is the least numbering of the family $\mathcal{S}$. Then $\alpha \leq_{pr} \nu$ by primitive recursive function g, which means that $alpha$ is punctually decidable, since $\forall n, m\, \alpha(m) = \alpha(n) \Leftrightarrow \nu(g(m)) = \nu(g(n)) \Leftrightarrow g(m) = g(n)$. Consequently ([28], Proposition 3.1(ii)), there is *spd*-numbering $\mu \equiv_{pr} \alpha$. By using the construction from the Theorem 4.1 of the same paper, we can construct the numbering $\mu_0 <_{pr} \mu$, which contradicts to choice of α.

2) Let ν be friedberg numbering of the infinite family $\mathcal{S}$. Consider $\mu = \nu \circ f$, where f is a primitive recursive bijective function such that f^{-1} is not primitive recursive (existence of such function is shown in [33]). It is clear, that $\mu \leq_{pr} \nu$ and μ friedberg: $\mu(m) = \mu(n) \Leftrightarrow \nu(f(m)) = \nu(f(n)) \Leftrightarrow f(m) = f(n) \Leftrightarrow m = n$. Wherein, $\nu = \mu \circ f^{-1}$ and f^{-1} is not primitive recursive, which means that $\nu \not\leq_{pr} \mu$. Thus $\mu <_{pr} \nu$. Continuing the process, you can build an endless-waning chain of friedberg numberings, from where we get required.

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Author Contributions

A. Askarbekkyzy proved Theorem 3. R. Bagaviev worked on Proposition 5. V. Isakov proved Proposition 1. B. Kalmurzayev set the direction of the research and participated in discussions of all the evidence and propositions in this article. D. Nurlanbek proved Proposition 3. F. Rakymzhankyzy proved Theorem 4. A. Slobozhanin proved Theorem 1. All authors participated in discussions of the definitions that were introduced in this article. All authors participated in the revision of the manuscript and approved the final submission.

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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