DOI 10.31489/2023M4/163-172

UDC 517.9

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Iterated discrete Hardy-type inequalities with three weights for a class of matrix operators

Iterated Hardy-type inequalities are one of the main objects of current research on the theory of Hardy inequalities. These inequalities have become well-known after study boundedness properties of the multi-dimensional Hardy operator acting from the weighted Lebesgue space to the local Morrie-type space. In addition, the results of quasilinear inequalities can be applied to study bilinear Hardy inequalities. In the paper, we discussed weighted discrete Hardy-type inequalities containing some quasilinear operators with a matrix kernel where matrix entries satisfy discrete Oinarov condition. The research of weighted Hardy-type inequalities depends on the relations between parameters p, q and θ, so we considered the cases $1 < p \leq q < \theta < \infty$ and $p \leq q < \theta < \infty$, $0 < p \leq 1$, criteria for the fulfillment of iterated discrete Hardy-type inequalities are obtained. Moreover, an alternative method of proof was shown in the work.

Keywords: Inequality, discrete Lebesgue space, Hardy-type operator, weight, quasilinear operator, matrix operator.

Introduction

The iterated integral Hardy-type inequality has the following form

$$\left(\int_0^\infty w^\theta(x)\left(\int_0^x\left|\varphi(t)\int_0^t f(s)ds\right|^q dt\right)^{\frac{\theta}{q}}dx\right)^{\frac{1}{\theta}} \leq C\left(\int_0^\infty |u(x)f(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall f\in L_{p,u}(0,\infty), \tag{1}$$

where $0 < q, p, \theta < \infty$, $u(\cdot)$, $\varphi(\cdot)$ and $w(\cdot)$ are positive functions and locally integrable on the interval $(0;\infty)$, $L_{p,u}(0,\infty)$ is a weighted Lebesgue space of functions for which the right side of the inequality (1) is finite.

At the beginning the inequality (1) has been studied with various quasilinear operators in the works [1, 2]. The equivalence of inequality (1) to the inequality, which defines the boundedness of the multidimensional Hardy operator from the Lebesgue space to the local Morrey-type space has been shown by V. Burenkov and R. Oinarov [3]. After this work researchers have become interested in an iterated integral Hardy-type inequality, then they began to use it intensively [4, 5]. In the last decade, researchers have studied weighted Hardy-type inequalities for the class of quasilinear operators including the kernel [6, 7].

Characterizations of inequality (1) was studied more deeply than discrete analogue. A discrete version of inequality (1) will be as follows

$$\left(\sum_{n=1}^\infty w_n^\theta\left(\sum_{k=1}^n\left|\varphi_k\sum_{i=1}^k f_i\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \leq C_1\left(\sum_{i=1}^\infty |u_i f_i|^p\right)^{\frac{1}{p}}, \quad \forall f\in l_{p.u}, \tag{2}$$

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where the positive constant C is independent from f, $0 < q, p, \theta < \infty$, and $\varphi = \{\varphi_i\}_{i=1}^{\infty}$ is a non-negative sequence, $u = \{u_i\}_{i=1}^{\infty}$, $w = \{w_i\}_{i=1}^{\infty}$ are positive sequences of real numbers. $l_{p,u}$ is the space of sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers such that

$$\|f\|_{p,u} = \left(\sum_{i=1}^{\infty} |u_i f_i|^p\right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty.$$

Nowadays, inequality (2) is being considered in many works. In the papers [8–10], necessity and sufficient conditions for the fulfillment of iterated discrete Hardy-type inequalities were obtained for the different relations of parameters q, p and θ, namely, for the case $p \le \theta < \infty$, in the sense that q can be any positive number. The most difficult cases $0 < \theta < min\{p, q\} < \infty$ and $0 < q < \theta < p < \infty$ for these inequalities was studied in the papers [11, 12]. Moreover, the paper [13] includes characterization of the following discrete iterated Hardy-type inequality

$$\left(\sum_{n\in Z} w_n \left(\sup_{i\ge n} \varphi_i \sum_{k\le i} f_k\right)^{\theta}\right)^{\frac{1}{\theta}} \le C \left(\sum_{n\in Z} f_n^p u_n\right)^{\frac{1}{p}}.$$

It is obvious to us that by using previously obtained results of iterated Hardy inequalities we can find characteristics of bilinear Hardy inequalities [14–16].

The aim of this paper is to characterize the iterated discrete Hardy-type inequality with matrix kernel defined as follows

$$\left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=1}^{n} \left|\varphi_k \sum_{i=1}^{k} a_{k,i} f_i\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \le C_1 \left(\sum_{i=1}^{\infty} |u_i f_i|^p\right)^{\frac{1}{p}}, \quad \forall f \in l_{p.u} \tag{3}$$

and the dual discrete Hardy-type inequality has the following form

$$\left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=n}^{\infty} \left|\varphi_k \sum_{i=k}^{\infty} a_{i,k} f_i\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \le C_2 \left(\sum_{i=1}^{\infty} |u_i f_i|^p\right)^{\frac{1}{p}}, \quad \forall f \in l_{p.u}, \tag{4}$$

where $(a_{k,i})$, $k \ge i \ge 1$, is a matrix non-negative entries of which satisfy the discrete Oinarov condition: there exists constant $d \ge 1$, entries $a_{k,i}$ are non-decreasing in k and non-increasing in i, such that the inequalities

$$\frac{1}{d}(a_{k,j} + a_{j,i}) \le a_{k,i} \le d(a_{k,j} + a_{j,i}) \tag{5}$$

hold for all $k \ge j \ge i \ge 1$.

The recent papers [17] and [18], where inequalities (3) and (4) are firstly studied for the matrix $(a_{k,i})$, $i \ge k \ge 1$, entries of which satisfy condition (5). The paper [17] contains results for only inequality (3) for the case $0 < q < p \le \theta < \infty$. In work [18], authors have used the localization method and considered the case $0 < p \le \theta < \infty$, $0 < q < \infty$. As we know, we can divide this case into the following three conditions

1) $0 < p \le \theta < q < \infty$;
2) $0 < p \le q < \theta < \infty$;
3) $0 < q \le p \le \theta < \infty$.

In the paper, we obtained necessity and sufficient conditions for the fulfillment of the inequalities (3) and (4) in the case $0 < p \le q < \theta < \infty$ by using an alternative method which is different from the method in [18]. This method requires $q < \theta$ condition since we will use the dual principle in the space l_p. It is important to note that in this paper we present the results for the case $0 < p \le 1$ which is interting because integral Hardy-type inequalities hold in trivial cases only [19].

1 Preliminaries

We need following known statements to obtain the main results. Let's start with reverse Hölder inequalities for weighted sequence l_p spaces and $1 < p < \infty$:

$$\left(\sum_{i=1}^{\infty} d_i^p z_i\right)^{\frac{1}{p}} = \sup_{h\geq 0}\left(\sum_{i=1}^{\infty} d_i h_i\right)\left(\sum_{i=1}^{\infty} h_i^{p'} z_i^{1-p'}\right)^{-\frac{1}{p'}},$$

$$\left(\sum_{i=1}^{\infty} d_i^p z_i\right)^{\frac{1}{p}} = \sup_{h\geq 0}\left(\sum_{i=1}^{\infty} d_i h_i z_i\right)\left(\sum_{i=1}^{\infty} h_i^{p'} z_i\right)^{-\frac{1}{p'}}.$$

We also apply theorems regarding discrete Hardy-type inequality for one class of matrix operators:

$$\left(\sum_{k=1}^{\infty} v_k^q \Big|\sum_{i=1}^{k} a_{k,i} f_i\Big|^q\right)^{\frac{1}{q}} \leq C\left(\sum_{i=1}^{\infty} |u_i f_i|^p\right)^{\frac{1}{p}}, \quad \forall f \in l_{p,u}, \tag{6}$$

where the entries of the matrix $(a_{k,i})$ satisfy discrete Oinarov condition. The boundedness of Hardy-type operators with matrix kernel was considered in the manuscripts [20–22].

Theorem 1. [21] Let $p \leq q < \infty$ and $0 < p \leq 1$. Let the entries of the matrix $(a_{k,i})$ such that $a_{k,i}$ non-increasing in second index. Then inequality (6) holds if and only if $A_2 < \infty$, where

$$A_1 = \sup_{j\geq 1}\left(\sum_{k=j}^{\infty} a_{k,j}^q v_k^q\right)^{\frac{1}{q}} u_j^{-1} < \infty.$$

Moreover, $C \approx A_2$, where C is the best constant in (6).

Theorem 2. [22] Let $1 < p \leq q < \infty$ and the entries of the matrix $(a_{k,i})$ satisfy condition (5). Then the inequality (6) holds if and only if $A = \max\{A_3, A_4\} < \infty$, where

$$A_2 = \sup_{j\geq 1}\left(\sum_{i=1}^{j} u_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{k=j}^{\infty} a_{k,j}^q v_k^q\right)^{\frac{1}{q}},$$

$$A_3 = \sup_{j\geq 1}\left(\sum_{i=1}^{j} a_{j,i}^{p'} u_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{k=j}^{\infty} v_k^q\right)^{\frac{1}{q}}.$$

Moreover, $C \approx A$, where C is the best constant in (6).

2 The main results

Theorem 3. Let $0 < p \leq q < \theta < \infty$. Let the entries of the matrix $(a_{k,i})$ satisfy condition (5). Then inequality (3) holds if and only if

(i) If $0 < p \leq 1$, $B_1 < \infty$, where

$$B_1 = \sup_{j\geq 1}\left(\sum_{n=j}^{\infty} w_n^{\theta}\left(\sum_{i=j}^{n} a_{i,j}^q \varphi_i^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} u_j^{-1}.$$

Moreover, $C_1 \approx B_1$, where C_1 is the best constant in (3).

(ii) If $p > 1$, $B = \max\{B_2, B_3\} < \infty$, where

$$B_2 = \sup_{j\geq 1} \left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{i=j}^{n} a_{i,j}^{q} \varphi_i^{q} \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \left(\sum_{k=1}^{j} u_k^{-p'} \right)^{\frac{1}{p'}},$$

$$B_3 = \sup_{j\geq 1} \left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{i=j}^{n} \varphi_i^{q} \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \left(\sum_{k=1}^{j} a_{j,k}^{p'} u_i^{-p'} \right)^{\frac{1}{p'}}.$$

Moreover, $C_1 \approx B$, where C_1 is the best constant in (3).

Proof. We assume that $0 < p \leq q < \theta < \infty$ and $0 \leq f \in l_{p,u}$. Then from inequality (3) we get that

$$C_1 = \sup_{f\geq 0} \left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=1}^{n} \left| \varphi_k \sum_{i=1}^{k} a_{k,i} f_i \right|^{q} \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \|f\|_{p,u}^{-1} < \infty. \tag{7}$$

By raising both sides of (7) to power q and denoting $S_k = \varphi_k^q \left(\sum\limits_{i=1}^{k} a_{k,i} f_i \right)^q$, we obtain that

$$C_1^q = \sup_{f\geq 0} \left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=1}^{n} S_k \right)^{\frac{\theta}{q}} \right)^{\frac{q}{\theta}} \|f\|_{p,u}^{-q}. \tag{8}$$

As $\frac{\theta}{q} > 1$, we can use of reverse Hölder inequality to (8). Then we find

$$C_1^q = \sup_{f\geq 0} \|f\|_{p,u}^{-q} \sup_{h\geq 0} \left(\sum_{n=1}^{\infty} h_n \sum_{k=1}^{n} S_k \right) \left(\sum_{n=1}^{\infty} h_n^{\frac{\theta}{\theta-q}} w_n^{-\frac{q\theta}{\theta-q}} \right)^{-\frac{\theta-q}{\theta}}$$

$$= \sup_{h\geq 0} \left(\sum_{n=1}^{\infty} \left(h_n w_n^{-q} \right)^{\frac{\theta}{\theta-q}} \right)^{-\frac{\theta-q}{\theta}} \sup_{f\geq 0} \|f\|_{p,u}^{-q} \left(\sum_{n=1}^{\infty} h_n \sum_{k=1}^{n} S_k \right).$$

By replacing S_k we obtain

$$C_1^q = \sup_{h\geq 0} \left(\sum_{n=1}^{\infty} \left(h_n w_n^{-q} \right)^{\frac{\theta}{\theta-q}} \right)^{-\frac{\theta-q}{\theta}} \left[\sup_{f\geq 0} \|f\|_{p,u}^{-1} \left(\sum_{n=1}^{\infty} h_n \sum_{k=1}^{n} \varphi_k^q \left(\sum_{i=1}^{k} a_{k,i} f_i \right)^{q} \right)^{\frac{1}{q}} \right]^{q}.$$

By changing the orders of sums we get that

$$C_1^q = \sup_{h\geq 0} \left(\sum_{n=1}^{\infty} \left(h_n w_n^{-q} \right)^{\frac{\theta}{\theta-q}} \right)^{-\frac{\theta-q}{\theta}} \left[\sup_{f\geq 0} \|f\|_{p,u}^{-1} \left(\sum_{k=1}^{\infty} \varphi_k^q \left(\sum_{i=1}^{k} a_{k,i} f_i \right)^{q} \sum_{n=k}^{\infty} h_n \right)^{\frac{1}{q}} \right]^{q}. \tag{9}$$

Let us define $H_k := \varphi_k^q \sum\limits_{n=k}^{\infty} h_n$. We will investigate separately the supremum which relates to f.

$$I := \sup_{f\geq 0} \frac{\left(\sum\limits_{k=1}^{\infty} H_k \left(\sum\limits_{i=1}^{k} a_{k,i} f_i \right)^{q} \right)^{\frac{1}{q}}}{\left(\sum\limits_{i=1}^{\infty} |u_i f_i|^p \right)^{\frac{1}{p}}}. \tag{10}$$

As you have noticed, we have obtained a discrete Hardy-type inequality for a class of matrix operators. Therefore, we consider two conditions regarding p. At first, if $0 < p < 1$, we use Theorem 1, then we have

$$\sup_{f\geq 0}\frac{\left(\sum\limits_{k=1}^{\infty}H_k\left(\sum\limits_{i=1}^{k}a_{k,i}f_i\right)^q\right)^{\frac{1}{q}}}{\left(\sum\limits_{i=1}^{\infty}|u_if_i|^p\right)^{\frac{1}{p}}}\approx\sup_{j\geq 1}\left(\sum_{k=j}^{\infty}a_{k,j}^qH_k\right)^{\frac{1}{q}}u_j^{-1}. \tag{11}$$

By inserting (11) into (9), we get that

$$C_1^q\approx\sup_{h\geq 0}\left(\sum_{n=1}^{\infty}\left(h_nw_n^{-q}\right)^{\frac{\theta}{\theta-q}}\right)^{-\frac{\theta-q}{\theta}}\sup_{j\geq 1}u_j^{-q}\sum_{k=j}^{\infty}\varphi_k^qa_{k,j}^q\sum_{i=k}^{\infty}h_i=$$

$$=\sup_{j\geq 1}u_j^{-q}\sup_{h\geq 0}\left(\sum_{n=1}^{\infty}\left(h_nw_n^{-q}\right)^{\frac{\theta}{\theta-q}}\right)^{-\frac{\theta-q}{\theta}}\sum_{k=j}^{\infty}\varphi_k^qa_{k,j}^q\sum_{i=k}^{\infty}h_i. \tag{12}$$

We denote $z_{k,j}=\varphi_k^qa_{k,j}^q$, $v_n=w_n^{-q}$ and $p_1=\frac{\theta}{\theta-q}$. Then we rewrite (12) as follows

$$C_1^q\approx\sup_{j\geq 1}u_j^{-q}\sup_{h\geq 0}\frac{\sum\limits_{k=j}^{\infty}z_{k,j}\sum\limits_{i=k}^{\infty}h_i}{\left(\sum\limits_{n=1}^{\infty}(h_nv_n)^{p_1}\right)^{\frac{1}{p_1}}}=\sup_{j\geq 1}u_j^{-q}\sup_{h\geq 0}\frac{\sum\limits_{i=j}^{\infty}h_i\sum\limits_{k=j}^{i}z_{k,j}}{\left(\sum\limits_{n=1}^{\infty}(h_nv_n)^{p_1}\right)^{\frac{1}{p_1}}}=\sup_{j\geq 1}u_j^{-q}\sup_{h\geq 0}\frac{\sum\limits_{i=1}^{\infty}\mathcal{X}_ih_i\sum\limits_{k=j}^{i}z_{k,j}}{\left(\sum\limits_{n=1}^{\infty}(h_nv_n)^{p_1}\right)^{\frac{1}{p_1}}}, \tag{13}$$

where $\mathcal{X}_k=0$ for $1\leq k<j$ and $\mathcal{X}_k=1$ for $k\geq j$. Since $p_1=\frac{\theta}{\theta-q}>1$ by applying reverse Hölder inequalities to (13) we get that

$$C_1^q\approx\sup_{j\geq 1}u_j^{-q}\left[\sum_{n=1}^{\infty}\left(\mathcal{X}_n\sum_{i=j}^{n}z_{i,j}\right)^{p_1'}v_n^{-p_1'}\right]^{\frac{1}{p_1'}}=\sup_{j\geq 1}u_j^{-q}\left[\sum_{n=j}^{\infty}\left(\sum_{i=j}^{n}z_{i,j}\right)^{p_1'}v_n^{-p_1'}\right]^{\frac{1}{p_1'}}.$$

Then we rewrite previously applied designations and obtain

$$C_1^q\approx\sup_{j\geq 1}u_j^{-q}\left[\sum_{n=j}^{\infty}\left(\sum_{i=j}^{n}a_{i,j}^q\varphi_i^q\right)^{\frac{\theta}{q}}w_n^{\theta}\right]^{\frac{q}{\theta}},$$

so that

$$C_1\approx B_1.$$

Therefore, we find that $C_1\approx B_1$ in the case $0<p<1$ and the constant C_1 depends only on the parameters p,q and θ.

Let us start estimating (10) for the case $p>1$. Actually, by using Theorem 2 we obtain $I\approx\max\{B_2^*,B_3^*\}$, where

$$B_2^*=\sup_{j\geq 1}\left(\sum_{i=1}^{j}u_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{k=j}^{\infty}a_{k,j}^qH_k\right)^{\frac{1}{q}},$$

$$B_3^* = \sup_{j\geq 1}\left(\sum_{i=1}^{j} a_{j,i}^{p'} u_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{k=j}^{\infty} H_k\right)^{\frac{1}{q}}.$$

First we estimate (9) with B_2^* and B_3^*. By applying previously used designations and by changing the supremums' order of execution we get that

$$C_1^q \approx \max\left\{\sup_{j\geq 1}\left(\sum_{i=1}^{j} u_i^{-p'}\right)^{\frac{q}{p'}}\sup_{h\geq 0}\left(\sum_{n=1}^{\infty}(h_n v_n)^{p_1}\right)^{-\frac{1}{p_1}}\sum_{k=j}^{\infty} z_{k,j}\sum_{i=k}^{\infty} h_i,\right.$$

$$\left.\sup_{j\geq 1}\left(\sum_{i=1}^{j} a_{j,i}^{p'} u_i^{-p'}\right)^{\frac{q}{p'}}\sup_{h\geq 0}\left(\sum_{n=1}^{\infty}(h_n v_n)^{p_1}\right)^{-\frac{1}{p_1}}\sum_{k=j}^{\infty}\varphi_k^q\sum_{i=k}^{\infty} h_i\right\}.$$

We can estimate the value of the best constant C_1^q in the same way as we calculated before. By changing the order of sums, applying the reverse Hölder inequality for these results and substituting the designations, we have

$$C_1^q \approx \max\left\{\sup_{j\geq 1}\left(\sum_{i=1}^{j} u_i^{-p'}\right)^{\frac{q}{p'}}\left(\sum_{n=j}^{\infty}\left(\sum_{i=j}^{n} a_{i,j}^q\varphi_i^q\right)^{\frac{\theta}{q}} w_n^{\theta}\right)^{\frac{q}{\theta}},\right.$$

$$\left.\sup_{j\geq 1}\left(\sum_{i=1}^{j} a_{j,i}^{p'} u_i^{-p'}\right)^{\frac{q}{p'}}\left(\sum_{n=j}^{\infty}\left(\sum_{i=j}^{n}\varphi_i^q\right)^{\frac{\theta}{q}} w_n^{\theta}\right)^{\frac{q}{\theta}}\right\}$$

then

$$C_1^q \approx \max\left\{B_2^q, B_3^q\right\}.$$

So we find that $C_1 \approx \max\{B_2, B_3\}$. We obtain $C_1 \approx B_1$ in the condition $0 < p < 1$, $p \leq q < \theta < \infty$ and $C_1 \approx \max\{B_2, B_3\}$ in the condition $1 < p \leq q < \theta < \infty$. Moreover, the equivalence constants depend only on p, q and θ. The proof is complete.

Theorem 4. Let $0 < p \leq q < \theta < \infty$. Let the entries of the matrix $(a_{k,i})$ satisfy condition (5). Then inequality (4) holds if and only if

(i) If $0 < p \leq 1$, $D_1 < \infty$, where

$$D_1 = \sup_{j\geq 1}\left(\sum_{n=1}^{j} w_n^{\theta}\left(\sum_{i=n}^{j} a_{j,i}^q\varphi_i^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} u_j^{-1}.$$

Moreover, $C_2 \approx D_1$, where C_2 is the best constant in (4).

(ii) If $p > 1$, $D = \max\{D_2, D_3\} < \infty$, where

$$D_2 = \sup_{j\geq 1}\left(\sum_{n=1}^{j} w_n^{\theta}\left(\sum_{i=n}^{j} a_{j,i}^q\varphi_i^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}\left(\sum_{k=j}^{\infty} u_k^{-p'}\right)^{\frac{1}{p'}},$$

$$D_3 = \sup_{j\geq 1} \left(\sum_{n=1}^{j} w_n^{\theta} \left(\sum_{i=n}^{j} \varphi_i^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \left(\sum_{k=j}^{\infty} a_{k,j}^{p'} u_i^{-p'} \right)^{\frac{1}{p'}} .$$

Moreover, $C_2 \approx D$, where C_2 is the best constant in (4).

Theorem 4 is devoted for inequality (4) and it can be proved similarly as Theorem 3.

Acknowledgments

We thank the referee and Prof. Ryskul Oinarov for some suggestions which improved the final version of this paper. This work was financially supported by the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP09259084).

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Матрицалық операторлар класы үшін үш салмақты итерацияланған дискретті Харди типті теңсіздіктер

Итерацияланған Харди тәріздес теңсіздіктер Харди теңсіздіктері теориясының қазіргі таңдағы зерттеулерінің негізгі объектілерінің бірі. Бұл теңсіздіктер көп өлшемді Харди операторының салмақты Лебег кеңістігінен локальды Морри тәріздес кеңістігінің шенелімділік қасиеттерін зерттегеннен кейін белгілі болды. Сонымен қатар, квазисызықты теңсіздіктердің нәтижелерін қоссызықты Харди теңсіздіктерін зерттеу кезінде қолдануға болады. Мақалада матрицалық ядросы бар кейбір квазисызықты операторлар қатысқан салмақты дискреттік Харди тәріздес теңсіздіктер қарастырылды, мұнда матрица элементтері дискретті Ойнаров шартын қанағаттандырады. Салмақты Харди тәріздес теңсіздіктерді зерттеу p, q және θ параметрлері арасындағы қатынастарға байланысты, сондықтан біз $1 < p \leq q < \theta < \infty$ және $p \leq q < \theta < \infty$, $0 < p \leq 1$ жағдайларын қарастырдық, итерацияланған дискреттік Харди тәріздес теңсіздіктердің орындалу критерийлері алынды. Сонымен қатар бұл жұмыста дәлелдеудің балама әдісі көрсетілген.

Кілт сөздер: теңсіздік, дискретті Лебег кеңістігі, Харди тәріздес оператор, салмақтар, квазисызықты оператор, матрицалық оператор.

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Итерационные дискретные неравенства типа Харди с тремя весами для класса матричных операторов

Итерированные неравенства типа Харди являются одним из основных объектов современных исследований теории неравенств Харди. Эти неравенства стали широко известны после изучения свойств ограниченности многомерного оператора Харди из весового пространства Лебега в локальное пространство типа Морри. Кроме того, результаты квазилинейных неравенств могут быть применены для изучения билинейных неравенств Харди. В статье рассмотрены весовые дискретные неравенства типа Харди, содержащие некоторые квазилинейные операторы с матричным ядром, где элементы матрицы удовлетворяют дискретному условию Ойнарова. Исследование весовых неравенств типа Харди зависит от соотношения параметров p, q и θ, поэтому мы рассмотрели случаи $1 < p \leq q < \theta < \infty$ и $p \leq q < \theta < \infty$, $0 < p \leq 1$; получили критерии выполнения итерационных дискретных неравенств типа Харди в случаях $1 < p \leq q < \theta < \infty$, $p \leq q < \theta < \infty$ и $0 < p \leq 1$. Более того, в работе показан альтернативный метод доказательства.

Ключевые слова: неравенство, дискретное пространство Лебега, оператор типа Харди, вес, квазилинейный оператор, матричный оператор.

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