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Mixed inverse problem for a Benney–Luke type integro-differential equation with two redefinition functions and parameters

In this paper, we consider a linear Benney–Luke type partial integro-differential equation of higher order with degenerate kernel and two redefinition functions given at the endpoint of the segment and two parameters. To find these redefinition functions we use two intermediate data. Dirichlet boundary value conditions are used with respect to spatial variable. The Fourier series method of variables separation is applied. The countable system of functional-integral equations is obtained. Theorem on a unique solvability of countable system for functional-integral equations is proved. The method of successive approximations is used in combination with the method of contraction mapping. The triple of solutions of the inverse problem is obtained in the form of Fourier series. Absolutely and uniformly convergences of Fourier series are proved.

Keywords: Inverse problem, two redefinition functions, final conditions, intermediate functions, Fourier method, unique value solvability.

Introduction

Historically, differential equations arose in solving applied problems. Therefore, the development of differential equations at the initial stage was carried out by applied scientists. Gradually, this direction grew into an independent theory — the theory of differential equations. Therefore, it can be said many times that differential and integro-differential equations are great interest from the point of theoretical research and applications in the mathematical physics, engineering, chemistry and in other different fields [1–8]. Recent years, a number of new problems for ordinary and partial differential and integro-differential equations are studied and a large number of research papers are published. Problems with nonlocal conditions for differential and integro-differential equations were considered in [9–29]. In [30–38], integro-differential equations with a degenerate kernel were considered.

In this paper, we study the solvability of the mixed inverse problem for a Benney–Luke type partial integro-differential equation with a degenerate kernel, two parameters, and final conditions at the endpoint of the interval. This paper differs from existing papers in that it requires to find redefinition functions considering at the endpoint of the interval. This inverse problem has features in relation to the direct problem.

In the rectangular domain $\Omega = \left\{0 < t < T,\ 0 < x < l\right\}$ we consider the following partial integro-differential equation of a higher order

$$\frac{\partial^2 U}{\partial t^2} + (-1)^k \frac{\partial^{2k+2} U}{\partial t^2 \partial x^2} + \omega^2 \left[(-1)^k \frac{\partial^{2k} U}{\partial x^{2k}} + \frac{\partial^{4k} U}{\partial x^{4k}}\right] = \alpha(t)\, U(t,x) + \nu \int\limits_0^T K(t,s)\, U(s,x)\, ds, \tag{1}$$

where k is a natural number, $0 < \alpha(t) \in C[0,T]$, T, l are given positive numbers, ω is a positive parameter, ν is a nonzero real parameter, $K(t,s) = \sum\limits_{i=1}^{m} a_i(t)\, b_i(s)$, $a_i(t),\ b_i(s) \in C\,[0,T]$. It is assumed that the systems of functions $\{a_i(t)\}$ and $\{b_i(s)\}$, $i = \overline{1, m}$ are linear independent.

It is known that when applying the method of separation of variables, the Dirichlet condition allows us to reduce partial differential equations to a countable system of ordinary differential equations. So, in solving partial integro-differential equation (1), we use the following Dirichlet boundary value conditions with respect to spatial variable x

$$U(t,0) = U(t,l) = \frac{\partial^2}{\partial x^2}U(t,0) = \frac{\partial^2}{\partial x^2}U(t,l) =$$

$$= \cdots = \frac{\partial^{4k-2}}{\partial x^{4k-2}}U(t,0) = \frac{\partial^{4k-2}}{\partial x^{4k-2}}U(t,l) = 0. \tag{2}$$

We use two conditions at the endpoint of the given segment with respect to time variable t:

$$U(T,x) = \varphi_1(x), \quad U_t(T,x) = \varphi_2(x), \quad 0 \le x \le l, \tag{3}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are redefinition functions and we assume that they are enough smooth on the segment $[0,l]$. For these functions the following conditions will be fulfilled

$$\varphi_i(0) = \varphi_i(l) = \varphi_i''(0) = \varphi_i''(l) = \cdots = \varphi_i^{(4k-2)}(0) = \varphi_i^{(4k-2)}(l) = 0, \quad i = 1,2.$$

In determining the redefinition functions, we use the following two intermediate conditions:

$$U(t_1,x) = \psi_1(x), \quad U_t(t_1,x) = \psi_2(x), \quad 0 \le x \le l, \tag{4}$$

where $\psi_1(x)$ and $\psi_2(x)$ are known functions enough smooth on the segment $[0,l]$, $0 < t_1 < T$. For the functions $\psi_1(x)$ and $\psi_2(x)$ the following conditions will be fulfilled

$$\psi_i(0) = \psi_i(l) = \psi_i''(0) = \psi_i''(l) = \cdots = \psi_i^{(4k-2)}(0) = \psi_i^{(4k-2)}(l) = 0, \quad i = 1,2.$$

The choice of conditions (3) and (4) with the final and intermediate data are important in applications. Indeed, in real practice it is not always possible to determine the initial data for unknown functions. When studying the technological process of aluminum production, before the start of the production cycle, the raw material passes through firing and the state of the raw material by the beginning of the production cycle is not known. And the final expected state of the output will be unknown in reality. We find it from known intermediate conditions. Because after each technological cycle we can determine the quality of the product. So, we have an inverse problem to solve equation (1).

Problem statement. To find triple of functions

$$\left\{U(t,x) \in C\left(\overline{\Omega}\right) \cap C_{t,x}^{2,4k}(\Omega) \cap C_{t,x}^{2+2k}(\Omega),\ \varphi_i(x) \in C[0,l],\ i = 1,2\right\},$$

the first of which satisfies partial integro-differential equation (1) and specified conditions (2)–(4), where $\overline{\Omega} = \left\{0 \le t \le T,\ 0 \le x \le l\right\}$.

Note that problem (1)–(4) is formulated such that direct problem (1)–(3) has a unique solution for all values of the parameter ω, and inverse problem (1)–(4) has a unique solution only for certain values of this parameter ω.

1 Construction of formal solution of the direct problem (1)–(3)

Note that the functions $\vartheta_n(x) = \sqrt{\frac{2}{l}}\sin\lambda_n x$, where $\lambda_n \in \frac{n\pi}{l}$, $n \in \mathbb{N}$, form a complete system of orthonormal eigenfunctions in the space $L_2[0,l]$. Linear equation (1) always has the trivial solution.

Therefore, by virtue of the Dirichlet condition (2), we seek nontrivial solutions to the linear partial integro-differential equation (1) of the higher order in the form of a Fourier series in sines

$$U(t,x)=\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}u_n(t)\,\sin\lambda_n x, \tag{5}$$

where

$$u_n(t)=\sqrt{\frac{2}{l}}\int_0^l U(t,x)\,\sin\lambda_n x\,dx,\quad \lambda_n=\frac{n\,\pi}{l}. \tag{6}$$

Substituting the Fourier series (5) into the given integro-differential equation (1), we obtain a linear second order countable system of ordinary differential equations

$$u''_n(t)+\omega^2\lambda_n^{2k}\,u_n(t)=\frac{\nu}{1+\lambda_n^{2k}}\sum_{i=1}^{m}a_i(t)\,\tau_{n\,i}+\frac{1}{1+\lambda_n^{2k}}\alpha(t)\,u_n(t), \tag{7}$$

where

$$\tau_{i\,n}=\int_0^T b_i(s)\,u_n(s)\,ds. \tag{8}$$

Solving the countable system of differential equations (7) by the variation method of arbitrary constants, we obtain the representation for its solution

$$\begin{aligned} u_n(t)&=A_{1n}\,\cos\lambda_n^k\omega\,t+A_{2n}\,\sin\lambda_n^k\omega\,t+\\ &+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\tau_{i\,n}\int_0^t\sin\lambda_n^k\,\omega\,(t-s)\,a_i(s)\,ds+\\ &+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int_0^t\sin\lambda_n^k\,\omega\,(t-s)\,\alpha(s)\,u_n(s)\,ds, \end{aligned} \tag{9}$$

where A_{1n} and A_{2n} are arbitrary coefficients, which will be determined by the final conditions (3). By differentiating (9) one times on t, we obtain

$$\begin{aligned} u'_n(t)&=-\lambda_n^k\omega A_{1n}\,\sin\lambda_n^k\omega\,t+\lambda_n^k\omega A_{2n}\,\cos\lambda_n^k\omega\,t+\\ &+\frac{\nu}{1+\lambda_n^{2k}}\sum_{i=1}^{m}\tau_{n\,i}\int_0^t\cos\lambda_n^k\,\omega\,(t-s)\,a_i\,(s)\,ds+\\ &+\frac{1}{1+\lambda_n^{2k}}\int_0^t\cos\lambda_n^k\,\omega\,(t-s)\,\alpha(s)\,u_n(s)\,ds. \end{aligned} \tag{10}$$

Now, supposing that the redefinition functions $\varphi_1(x)$ and $\varphi_2(x)$ were expanded into a Fourier series, and using Fourier coefficients (6), from conditions (3) we obtain

$$u_n(T)=\sqrt{\frac{2}{l}}\int_0^l U(T,x)\,\sin\lambda_n x\,dx=\sqrt{\frac{2}{l}}\int_0^l\varphi_1(x)\,\sin\lambda_n x\,dx=\varphi_{1\,n}, \tag{11}$$

$$u'_n(T) = \sqrt{\frac{2}{l}} \int\limits_0^l U_t(T,x) \sin \lambda_n x \, dx = \sqrt{\frac{2}{l}} \int\limits_0^l \varphi_2(x) \sin \lambda_n x \, dx = \varphi_{2n}. \tag{12}$$

To find the unknown coefficients A_{1n} and A_{2n} in presentations (9) and (10), we use final conditions (11) and (12). Then we arrive at a system of algebraic equations (SAE)

$$\begin{cases} A_{1n} \cos \lambda_n^k \omega T + A_{2n} \sin \lambda_n^k \omega T = \gamma_{1n}, \\ -A_{1n} \sin \lambda_n^k \omega T + A_{2n} \cos \lambda_n^k \omega T = \gamma_{2n}, \end{cases} \tag{13}$$

where

$$\gamma_{1n} = \varphi_{1n} - \frac{\nu}{\lambda_n^k (1+\lambda_n^{2k}) \omega} \sum_{i=1}^m \tau_{in} \int\limits_0^T \sin \lambda_n^k \omega (T-s) \, a_i(s) \, ds -$$

$$- \frac{1}{\lambda_n^k (1+\lambda_n^{2k}) \omega} \int\limits_0^T \sin \lambda_n^k \omega (T-s) \, \alpha(s) \, u_n(s) \, ds,$$

$$\gamma_{2n} = \varphi_{2n} - \frac{\nu}{1+\lambda_n^{2k}} \sum_{i=1}^m \tau_{in} \int\limits_0^T \cos \lambda_n^k \omega (T-s) \, a_i(s) \, ds -$$

$$- \frac{1}{1+\lambda_n^{2k}} \int\limits_0^T \cos \lambda_n^k \omega (T-s) \, \alpha(s) \, u_n(s) \, ds.$$

For uniquely solvability of SAE (13), the following condition

$$\delta_{0n} = \begin{vmatrix} \cos \lambda_n^k \omega T & \sin \lambda_n^k \omega T \\ -\sin \lambda_n^k \omega T & \cos \lambda_n^k \omega T \end{vmatrix} \neq 0$$

must be fulfilled. Since $\delta_{0n} = 1$, this condition are fulfilled for all values of the parameter ω. Consequently, SAE (13) has a unique pair of solutions

$$A_{1n} = \delta_{1n} = \begin{vmatrix} \gamma_{1n} & \sin \lambda_n^k \omega T \\ \gamma_{2n} & \cos \lambda_n^k \omega T \end{vmatrix} = \varphi_{1n} \cos \lambda_n^k \omega T - \varphi_{2n} \sin \lambda_n^k \omega T +$$

$$+ \frac{\nu}{\lambda_n^k (1+\lambda_n^{2k}) \omega} \sum_{i=1}^m \tau_{in} \int\limits_0^T \sin \lambda_n^k \omega s \, a_i(s) \, ds + \frac{1}{\lambda_n^k (1+\lambda_n^{2k}) \omega} \int\limits_0^T \sin \lambda_n^k \omega s \, \alpha(s) \, u_n(s) \, ds, \tag{14}$$

$$A_{2n} = \delta_{2n} = \begin{vmatrix} \cos \lambda_n^k \omega T & \gamma_{1n} \\ -\sin \lambda_n^k \omega T & \gamma_{2n} \end{vmatrix} = \varphi_{1n} \sin \lambda_n^k \omega T + \varphi_{2n} \cos \lambda_n^k \omega T +$$

$$+ \frac{\nu}{1+\lambda_n^{2k}} \sum_{i=1}^m \tau_{in} \int\limits_0^T \cos \lambda_n^k \omega s \, a_i(s) \, ds + \frac{1}{1+\lambda_n^{2k}} \int\limits_0^T \cos \lambda_n^k \omega s \, \alpha(s) \, u_n(s) \, ds. \tag{15}$$

Substituting these values of (14) and (15) into presentation (9), we obtain

$$u_n(t,\nu,\omega) = \varphi_{1n} \chi_{1n}(t,\omega) + \varphi_{2n} \chi_{2n}(t,\omega) + \frac{\nu}{\lambda_n^k (1+\lambda_n^{2k}) \omega} \sum_{i=1}^m \tau_{in} \chi_{3in}(t,\omega) +$$

$$+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int\limits_0^T H_n(t,s,\omega)\,\alpha(s)\,u_n(s,\nu,\omega)\,ds, \tag{16}$$

where

$$\chi_{1n}(t,\omega)=\cos\lambda_n^k\omega\,(T-t)-\sin\lambda_n^k\omega\,(T-t),$$

$$\chi_{2n}(t,\omega)=\cos\lambda_n^k\omega\,(T+t)-\sin\lambda_n^k\omega\,(T-t),$$

$$\chi_{3in}(t,\omega)=\int\limits_0^T H_n(t,s,\omega)\,a_i(s)\,ds,$$

$$H_n(t,s,\omega)=\begin{cases}\sin z\,(t+s), \quad z=\lambda_n^k\omega, \quad t<s\le T,\\ \sin z\,(t-s)+\cos z\,t\,\sin z\,s+z\,\sin z\,t\,\sin z\,s, \;\; 0\le s<t.\end{cases}$$

Although functions (16) are Fourier coefficients of the solution to direct problem (1)–(3), it contains extra quantities τ_{in} that are still unknown. To find these quantities, we substitute representation (16) into designation (8) and arrive at a new SAE:

$$\tau_{in}-\frac{\nu}{\bar\lambda}\sum_{j=1}^m\tau_{jn}\,\sigma_{3ijn}(t)=\varphi_{1n}\,\sigma_{1in}+\varphi_{2n}\,\sigma_{2in}+\sigma_{4in}(u_n), \tag{17}$$

where

$$\sigma_{1in}=\int\limits_0^T b_i(s)\,\chi_{1n}(s,\omega)\,ds, \quad \sigma_{2in}=\int\limits_0^T b_i(s)\,\chi_{2n}(s,\omega)\,ds, \quad \bar\lambda=\lambda_n^k\left(1+\lambda_n^{2k}\right)\omega,$$

$$\sigma_{3ijn}=\int\limits_0^T b_i(s)\int\limits_0^T H_n(s,\theta,\omega)\,a_j(\theta)\,d\theta\,ds,$$

$$\sigma_{4in}(u_n)=\frac{1}{\bar\lambda}\int\limits_0^T b_i(s)\int\limits_0^T H_n(s,\theta,\omega)\,\alpha(\theta)\,u_n(\theta)\,d\theta\,ds.$$

To establish the unique solvability of SAE (17), we introduce the following matrix

$$\Theta_{0n}(\nu,\,\omega)=\begin{pmatrix}1-\frac{\nu}{\bar\lambda}\sigma_{311n} & \frac{\nu}{\bar\lambda}\sigma_{312n} & \dots & \frac{\nu}{\bar\lambda}\sigma_{31mn}\\ \frac{\nu}{\bar\lambda}\sigma_{321n} & 1-\frac{\nu}{\bar\lambda}\sigma_{322n} & \dots & \frac{\nu}{\bar\lambda}\sigma_{32mn}\\ \dots & \dots & \dots & \dots\\ \frac{\nu}{\bar\lambda}\sigma_{3m1n} & \frac{\nu}{\bar\lambda}\sigma_{3m2n} & \dots & 1-\frac{\nu}{\bar\lambda}\sigma_{3mmn}\end{pmatrix}$$

and consider the values of the parameter ν, for which the Fredholm determinant is not zero:

$$\Delta_{0n}(\nu,\omega)=\det\Theta_{0n}(\nu,\omega)\neq 0. \tag{18}$$

Determinant $\Delta_{0n}(\nu,\omega)$ in (18) is a polynomial with respect to $\frac{\nu}{\bar\lambda}$ of the degree not higher than m. The countable system of algebraic equations $\Delta_{0n}(\nu,\omega)=0$ has no more than m different real roots for every value of n. We denote them by $\mu_l(l=\overline{1,p},\ 1\le p\le m)$. Then $\nu_n=\nu_{ln}=\bar\lambda\,\mu_l=\lambda_n^k\left(1+\lambda_n^{2k}\right)\omega\,\mu_l$ are called the characteristic (irregular) values of the kernel for integro-differential equation (1). So, we introduce the following two designations

$$\Lambda_1=\left\{(\nu_n,\,\omega):\ \nu_n=\lambda_n^k\left(1+\lambda_n^{2k}\right)\omega\,\mu_l, \quad \omega\in(0,\infty)\right\},$$

$$\Lambda_2=\left\{(\nu_n,\,\omega):\,|\,\Delta_{0n}(\nu,\omega)\,|>0,\ \ \nu_n\neq\lambda_n^k\left(1+\lambda_n^{2k}\right)\omega\,\mu_l,\ \ \omega\in(0,\infty)\right\}.$$

On the number set Λ_2 we consider a matrix

$$\Theta_{ijn}(\nu,\omega)=\begin{pmatrix} 1-\frac{\nu}{\lambda}\sigma_{311n} & \dots & \frac{\nu}{\lambda}\sigma_{31(i-1)n} & \sigma_{j1n} & \frac{\nu}{\lambda}\sigma_{31(i+1)n} & \dots & \frac{\nu}{\lambda}\sigma_{31mn}\\ \frac{\nu}{\lambda}\sigma_{321n} & \dots & \frac{\nu}{\lambda}\sigma_{32(i-1)n} & \sigma_{j2n} & \frac{\nu}{\lambda}\sigma_{32(i+1)n} & \dots & \frac{\nu}{\lambda}\sigma_{32mn}\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \frac{\nu}{\lambda}\sigma_{3m1n} & \dots & \frac{\nu}{\lambda}\sigma_{3m(i-1)n} & \sigma_{j\,mn} & \frac{\nu}{\lambda}\sigma_{3m(i+1)n} & \dots & 1-\frac{\nu}{\lambda}\sigma_{3mmn}\end{pmatrix},$$

$j=1,2,4$. Taking into account the known properties of the matrix $\Theta_{ijn}(\nu,\omega)$, we modified the Cramer method on the set Λ_2 and obtain solutions of SAE (17) in the form

$$\tau_{in}=\varphi_{1n}\frac{\Delta_{1i}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}+\varphi_{2n}\frac{\Delta_{2in}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}+\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)},\quad i=\overline{1,\,m},\ \ (\nu,\omega)\in\Lambda_2, \tag{19}$$

where $\Delta_{ijn}(\nu,\omega)=\det\Theta_{ijn}(\nu,\omega),\ j=1,2,4.$

Substituting solutions (19) into function (16), we obtain

$$u_n(t,\nu,\omega)=\varphi_{1n}\,h_{1n}(t,\nu,\omega)+\varphi_{2n}\,h_{2n}(t,\nu,\omega)+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)}\,\chi_{3in}(t)+$$

$$+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int_0^T H_n(t,s,\omega)\,u_n(s,\nu,\omega)\,ds,\quad(\nu,\omega)\in\Lambda_2, \tag{20}$$

where

$$h_{jn}(t,\nu,\omega)=\chi_{jn}(t,\omega)+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{jn}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}\,\chi_{3in}(t,\omega),\ \ j=1,2,$$

$$\chi_{1n}(t,\omega)=\cos\lambda_n^k\omega\,(T-t)-\sin\lambda_n^k\omega\,(T-t),\quad\chi_{2n}(t,\omega)=\cos\lambda_n^k\omega\,(T+t)-\sin\lambda_n^k\omega\,(T-t),$$

$$\chi_{3in}\,(t,\omega)=\int_0^T H_n(t,s,\omega)\,a_i(s)\,ds,$$

$$H_n(t,s,\omega)=\begin{cases}\sin z\,(t+s), & z=\lambda_n^k\omega,\quad t<s\leq T,\\ \sin z\,(t-s)+\cos z\,t\,\sin z\,s+z\,\sin z\,t\,\sin z\,s, & 0\leq s<t.\end{cases}$$

Representation (20) is a countable system of functional-integral equations. Substituting representation (20) into the Fourier series (5), we obtain a formal solution of direct problem (1)–(3) on the domain Ω

$$U(t,x)=\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\sin\,\lambda_n x\times$$

$$\times\Bigg[\varphi_{1n}\,h_{1n}(t,\nu,\omega)+\varphi_{2n}\,h_{2n}(t,\nu,\omega)+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)}\,\chi_{3in}(t)+$$

$$+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int_0^T H_n(t,s,\omega)\,u_n(s,\nu,\omega)\,ds\Bigg],\quad(\nu,\omega)\in\Lambda_2. \tag{21}$$

But, there are two unknown quantities φ_{1n} and φ_{2n} in (21).

2 Formal solution of the inverse problem (1)–(4)

We will now formally define the redefinition functions $\varphi_1(x)$ and $\varphi_2(x)$. We subordinate function (20) to intermediate conditions (4). For this purpose, we differentiate (21) one times on the time-variable t:

$$U_t(t,x)=\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\sin\lambda_n x\;\Big[\varphi_{1n}\,h'_{1n}(t,\nu,\omega)+\varphi_{2n}\,h'_{2n}(t,\nu,\omega)+$$

$$+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)}\,\chi'_{3in}(t)+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int_0^T H'_n(t,s,\omega)\,u_n(s,\nu,\omega)\,ds\Big],\tag{22}$$

where

$$h'_{jn}(t,\nu,\omega)=\chi'_{jn}(t,\omega)+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{jn}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}\,\chi'_{3in}(t,\omega),\;\; j=1,2,$$

$$\chi'_{1n}(t,\omega)=\lambda_n^k\omega\left(\sin\lambda_n^k\omega\,(T-t)+\cos\lambda_n^k\omega\,(T-t)\right),$$

$$\chi'_{2n}(t,\omega)=-\lambda_n^k\omega\left(\sin\lambda_n^k\omega\,(T+t)+\cos\lambda_n^k\omega\,(T-t)\right),$$

$$\chi'_{3in}(t,\omega)=\int_0^T H'_n(t,s,\omega)\,a_i(s)\,ds,$$

$$H'_n(t,s,\omega)=\begin{cases} z\,\cos z\,(t+s), \quad z=\lambda_n^k\omega, \quad t<s\le T,\\ z\,\cos z\,(t-s)-z\,\sin z\,t\,\sin z\,s+z^2\cos z\,t\,\sin z\,s, \;\; 0\le s<t.\end{cases}$$

Then, applying intermediate conditions (4) to functions (21) and (22), we arrive at the solution of the following SAE:

$$\begin{cases}\varphi_{1n}\,\left[\chi_{1n}(t_1,\omega)+\varepsilon_{11n}\right]+\varphi_{2n}\,\left[\chi_{2n}(t_1,\omega)+\varepsilon_{12n}\right]=\bar{\psi}_{1n},\\ \varphi_{1n}\,\left[\chi'_{1n}(t_1,\omega)+\varepsilon_{21n}\right]+\varphi_{2n}\,\left[\chi'_{2n}(t_1,\omega)+\varepsilon_{22n}\right]=\bar{\psi}_{2n},\end{cases}\tag{23}$$

where

$$\varepsilon_{1jn}=\frac{\nu}{\bar{\lambda}}\sum_{i=1}^{m}\frac{\Delta_{jn}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}\,\chi_{3in}(t_1,\omega),\quad \varepsilon_{2jn}=\frac{\nu}{\bar{\lambda}}\sum_{i=1}^{m}\frac{\Delta_{jn}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}\,\chi'_{3in}(t_1,\omega),\;\; j=1,2,$$

$$\bar{\psi}_{1n}=\psi_{1n}-\frac{\nu}{\bar{\lambda}}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}\,(\nu,\omega)}\,\chi_{3in}(t_1,\omega)+\frac{1}{\bar{\lambda}}\int_0^T H_n(t_1,s,\omega)\,u_n(s,\nu,\omega)\,ds,\tag{24}$$

$$\bar{\psi}_{2n}=\psi_{2n}-\frac{\nu}{\bar{\lambda}}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}\,(\nu,\omega)}\,\chi'_{3i}(t_1,\omega)+\frac{1}{\bar{\lambda}}\int_0^T H'_n(t_1,s,\omega)\,u_n(s,\nu,\omega)\,ds,\tag{25}$$

$$\bar{\lambda}=\lambda_n^k\left(1+\lambda_n^{2k}\right)\omega.$$

The fulfillment of the following condition ensures the unique solvability of SAE (23):

$$V_{0n}(\omega)=\begin{vmatrix}\chi_{1n}(t_1,\omega)+\varepsilon_{11n} & \chi_{2n}(t_1,\omega)+\varepsilon_{12n}\\ \chi'_{1n}(t_1,\omega)+\varepsilon_{21n} & \chi'_{2n}(t_1,\omega)+\varepsilon_{22n}\end{vmatrix}=$$

$$=-z\,\sin 2zT-z\,\cos 2zT+2z\,\sin z(T-t_1)\,\cos z(T-t_1)-z\,\cos 2z(T-t_1)-$$

$$-z\,\varepsilon_{11n}[\sin z(T+t_1)+\cos z(T-t_1)]-z\varepsilon_{12n}[\sin z(T-t_1)+\cos z(T-t_1)]-$$

$$-\varepsilon_{21n}[\cos z(T+t_1) - z\,\sin z(T-t_1)] - \varepsilon_{22n}[\sin z(T-t_1) - z\,\cos z(T-t_1)] +$$
$$+\varepsilon_{11n}\varepsilon_{22n} - \varepsilon_{21n}\varepsilon_{12n} \neq 0. \tag{26}$$

Before proceeding to find the solution of SAE (23), we consider nonzero condition (26). To do this, we suppose the opposite:

$$-z\,\sin 2zT - z\,\cos 2zT + 2z\,\sin z(T-t_1)\,\cos z(T-t_1) - z\,\cos 2z(T-t_1) -$$
$$-z\,\varepsilon_{11n}[\sin z(T+t_1) + \cos z(T-t_1)] - z\varepsilon_{12n}[\sin z(T-t_1) + \cos z(T-t_1)] -$$
$$-\varepsilon_{21n}[\cos z(T+t_1) - z\,\sin z(T-t_1)] - \varepsilon_{22n}[\sin z(T-t_1) - z\,\cos z(T-t_1)] +$$
$$+\varepsilon_{11n}\varepsilon_{22n} - \varepsilon_{21n}\varepsilon_{12n} = 0, \quad z = \lambda_n^k\,\omega. \tag{27}$$

Condition (27) is a transcendental equation, and the set of its solutions with respect to ω is denoted by $\Im$. So, on the set

$$\Lambda_3 = \left\{(\nu_n,\omega) : |\,\Delta_{0n}(\nu,\omega)\,| > 0,\ \ \nu_n \neq \lambda_n^k\left(1+\lambda_n^{2k}\right)\omega\,\mu_l,\ \ \omega \in \Im\right\}$$

SAE (23) is not uniquely solvable. But, on the other set

$$\Lambda_4 = \left\{(\nu_n,\,\omega) : |\,\Delta_{0n}(\nu,\omega)\,| > 0,\ \ |\,V_{0n}(\omega)\,| > 0,\ \ \nu_n \neq \lambda_n^k\left(1+\lambda_n^{2k}\right)\omega\,\mu_l,\ \ \omega \in (0,\infty)\setminus\Im\right\}$$

SAE (23) is uniquely solvable. So, taking into account notations (24) and (25), we obtain

$$\varphi_{jn} = \psi_{1n}w_{j1n}(\omega) + \psi_{2n}w_{j2n}(\omega) + \frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)}\,w_{j3in}(\omega) +$$
$$+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int_0^T W_{j\,n}(s,\omega)\,u_n(s,\nu,\omega)\,ds,\quad j=1,2,\quad (\nu,\omega)\in\Lambda_4, \tag{28}$$

where

$$w_{11n}(\omega) = V_{0n}^{-1}\left(\chi'_{2n}(t_1,\omega) + \varepsilon_{22n}(\omega)\right),\ \ w_{12n}(\omega) = V_{0n}^{-1}\left(-\chi_{2n}(t_1,\omega) + \varepsilon_{12n}(\omega)\right),$$
$$w_{21n}(\omega) = V_{0n}^{-1}\left(\chi'_{1n}(t_1,\omega) + \varepsilon_{21n}(\omega)\right),\ \ w_{22n}(\omega) = V_{0n}^{-1}\left(\chi_{1n}(t_1,\omega) + \varepsilon_{11n}(\omega)\right),$$
$$w_{13n}(\omega) = -\left[\chi_{3i\,n}(t_1,\omega)w_{11n}(\omega) + \chi'_{3in}(t_1,\omega)\,w_{12n}(\omega)\right],$$
$$w_{23n}(\omega) = -\left[\chi_{3in}(t_1,\omega)\,w_{21n}(\omega) + \chi'_{3i\,n}(t_1,\omega)\,w_{22n}(\omega)\right],$$
$$W_{1n}(s,\omega) = H_n(t_1,s)\,w_{11n}(\omega) + H'_n(t_1,s)\,w_{12n}(\omega),$$
$$W_{2n}(s,\omega) = H_n(t_1,s)\,w_{21n}(\omega) + H'_n(t_1,s)\,w_{22n}(\omega).$$

Since φ_{1n} and φ_{2n} are Fourier coefficients, from presentations (28) we obtain the following Fourier series

$$\varphi_j(x) = \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\sin\lambda_n x\,\left[\psi_{1n}w_{j1n} + \psi_{2n}w_{j2n} + \frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)}\,w_{j3in} +\right.$$
$$\left.+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int_0^T W_{jn}(s,\omega)\,u_n(s,\nu,\omega)\,ds\right],\ \ (\nu,\omega)\in\Lambda_4. \tag{29}$$

The functions $u_n(t,\nu,\omega)$ in series (29) are Fourier coefficients of the unknown function $U(t,x,\nu,\omega)$. Therefore, we need to define the Fourier coefficients $u_n(t,\nu,\omega)$ uniquely. Substituting representation (28) into equations (20), we obtain the following countable system of functional-integral equations in the final form

$$u_n(t,\nu,\omega)=S(t,\nu,\omega;u_n)\equiv\psi_{1n}\,g_{1n}(t,\nu,\omega)+\psi_{2n}\,g_{2n}(t,\nu,\omega)+$$
$$+\frac{\nu}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\sum_{i=1}^{m}\frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)}\,g_{3in}(t,\omega)+$$
$$+\frac{1}{\lambda_n^k\,(1+\lambda_n^{2k})\,\omega}\int\limits_0^T G_n(t,s,\nu,\omega)\,u_n(s,\nu,\omega)\,ds,\quad(\nu,\omega)\in\Lambda_4,\tag{30}$$

where

$$g_{1n}(t,\nu,\omega)=w_{11n}(\omega)h_{1n}(t,\nu,\omega)+w_{21n}(\omega)h_{2n}(t,\nu,\omega),$$
$$g_{2n}(t,\nu,\omega)=w_{12n}(\omega)\,h_{1n}(t,\nu,\omega)+w_{22n}(\omega)\,h_{2n}(t,\nu,\omega),$$
$$g_{3in}(t,\omega)=g_{1n}(t,\nu,\omega)\,\chi_{3in}(t_1,\omega)+g_{2n}(t,\nu,\omega)\,\chi'_{3in}(t_1,\omega)+\chi_{3in}(t,\omega),$$
$$G_n(t,s,\nu,\omega)=g_{1n}(t,\nu,\omega)\,H_n(t_1,s,\omega)+g_{2n}(t,\nu,\omega)\,H'(t_1,s,\omega)+H_n(t,s,\omega).$$

Note that this functional-integral equation (30) makes sense only for values of parameters ν, ω from the set Λ_4. In addition, in the countable system of functional-integral equations (30), the unknown function $u_n(t,\nu,\omega)$ is under the sign of the determinant and under the sign of the integral.

3 Solvable of the countable system of functional-integral equations (30)

Let us investigate the system of equations (30) in the sense of the unique solvability. To this, we consider the following well-known Banach spaces, in which we need in our further actions [26,32,33,36]. We consider the space B_2 of function sequences $\{\,u_n(t)\,\}_{n=1}^{\infty}$ on the segment $[0,T]$ with the norm

$$\|\,u(t)\,\|_{B_2}=\sqrt{\sum_{n=1}^{\infty}\left(\max_{t\in[0,T]}|\,u_n(t)\,|\right)^2}<\infty;$$

the space ℓ_2 of number sequences $\{\varphi_n\}_{n=1}^{\infty}$ with the norm

$$\|\,\varphi\,\|_{\ell_2}=\sqrt{\sum_{n=1}^{\infty}|\,\varphi_n\,|^2}<\infty;$$

the space $L_2[0,l]$ of square-integrable functions on an interval $[0,l]$ with norm

$$\|\,\vartheta(x)\,\|_{L_2[0,l]}=\sqrt{\int\limits_0^l|\,\vartheta(x)\,|^2\;dx}<\infty.$$

Smoothness conditions. Let on the segments $[0,l]$ there exist peace-wise continuous derivatives with respect to x up $(4k+2)$-th order for the functions $\psi_i(x)\in C^{4k+1}[0,l]$, $\;i=1,2$. Then, after integration the integrand functions $\psi_{in}=\sqrt{\frac{2}{l}}\int\limits_0^l\psi_i(x)\,\sin\lambda_n x\,dx$, $\;i=1,2$ by part $(4k+2)$ times on the variable x, we obtain the following relation

$$|\,\psi_{i,n}\,|=\left(\frac{l}{\pi}\right)^{4k+2}\frac{\left|\,\psi_{i,n}^{(4\,k+2)}\,\right|}{n^{4k+2}},\;\;i=1,2,\tag{31}$$

where

$$\psi_{i,n}^{(4k+2)} = \int\limits_0^l \frac{\partial^{4k+2} \psi_i(x)}{\partial x^{4k+2}} \vartheta_n(x)\, dx, \;\; i=1,2.$$

Here we note that the Bessel inequality is true

$$\sum_{n=1}^{\infty} \left[\psi_{i,n}^{(4k+2)}\right]^2 \leq \left(\frac{2}{l}\right)^{4k+2} \int\limits_0^l \left[\frac{\partial^{4k+2} \psi_i(x)}{\partial x^{4k+2}}\right]^2 dx, \;\; i=1,2. \tag{32}$$

Theorem 1. Let the smoothness conditions and the following conditions be fulfilled:

$$\max_{t\in[0,T]} \left[|\, g_{1n}(t,\nu,\omega)\,|\, ;\, |\, g_{2n}(t,\nu,\omega)\,|\right] = \delta_{1n} \leq \delta_1 < \infty, \tag{33}$$

$$\rho = |\,\nu\,|\; \delta_2 \left\| \sum_{i=1}^{m} \left| \frac{\bar{\Delta}_{4in}(\nu,\omega)}{\Delta_{0n}(\nu,\omega)} \right| \delta_{0i} \right\|_{\ell_2} + \delta_3 < 1, \tag{34}$$

where δ_2, δ_3 and δ_{0i} will be defined from (38) and (39), while $\bar{\Delta}_{4in}(\nu,\omega)$ is defined from (41). Then the countable systems of functional-integral equations (30) is uniquely solvable in the space B_2. The desired solution can be founded from the following iterative process:

$$\begin{cases} u_n^0(t,\nu,\omega) = \psi_{1n}\, g_{1n}(t,\nu,\omega) + \psi_{2n}\, g_{2n}(t,\nu,\omega), \\ u_n^{r+1}(t,\nu,\omega) = S(t,\nu,\omega; u_n^r), \quad r=0,1,2,... \end{cases} \tag{35}$$

Proof. We use the method of contraction maps in combination with the method of successive approximations in the space B_2. Then, by virtue of smoothness condition (31) and estimate (33), applying the Cauchy–Schwartz inequality and Bessel inequality (32), from approximations (35) we obtain that the following estimate is valid:

$$\sum_{n=1}^{\infty} \max_{t\in[0,T]} \left|\, u_n^0(t)\,\right| \leq \sum_{n=1}^{\infty} \max_{t\in[0,T]} \left[|\,\psi_{1n}\,| \cdot |\, g_{1n}(t,\nu,\omega)\,| + |\,\psi_{2n}\,| \cdot |\, g_{2n}(t,\nu,\omega)\,|\right] \leq$$

$$\leq \delta_1 \left(\frac{l}{\pi}\right)^{4k+2} \left[\sum_{n=1}^{\infty} \frac{\left|\psi_{1,n}^{(4k+2)}\right|}{n^{4k+2}} + \sum_{n=1}^{\infty} \frac{\left|\psi_{2,n}^{(4k+2)}\right|}{n^{4k+2}}\right] \leq$$

$$\leq \delta_1 \left(\frac{\sqrt{2l}}{\pi}\right)^{4k+2} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{8k+4}}} \left[\left\| \frac{\partial^{4k+2} \psi_1(x)}{\partial x^{4k+2}} \right\|_{L_2[0,l]} + \left\| \frac{\partial^{4k+2} \psi_2(x)}{\partial x^{4k+2}} \right\|_{L_2[0,l]}\right] = \delta_0 < \infty. \tag{36}$$

Taking into account estimate (36), applying the Cauchy–Schwartz inequality, for the first difference of approximations (35) we obtain:

$$\sum_{n=1}^{\infty} \max_{t\in[0,T]} \left|\, u_n^1(t) - u_n^0(t)\,\right| \leq |\,\nu\,| \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{3k}\omega} \sum_{i=1}^{m} \left| \frac{\Delta_{4in}\left(\nu,\omega,u_n^0\right)}{\Delta_{0n}(\nu,\omega)} \right| \max_{t\in[0,T]} |g_{3\,i\,n}(t,\omega)| +$$

$$+ \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{3k}\omega} \max_{t\in[0,T]} \left| \int\limits_0^T G_n(t,s,\nu,\omega)\, u_n^0(s,\nu,\omega)\, ds \right| \leq$$

$$\leq |\,\nu\,|\,\delta_2\sqrt{\sum_{n=1}^{\infty}\left[\sum_{i=1}^{m}\left|\,\frac{\Delta_{4in}\,(\nu,\omega,u_n^0)}{\Delta_{0n}(\nu,\omega)}\,\right|\,\delta_{0i}\right]^2}+\delta_3\delta_0<\infty, \tag{37}$$

where

$$\delta_{0i}\geq\delta_{0in}=\max_{t\in[0,T]}|g_{3in}(t,\omega)|\,,\ \ \delta_2=\sqrt{\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{6k}\omega^2}}, \tag{38}$$

$$\delta_3=\sqrt{\sum_{n=1}^{\infty}\max_{t\in[0,T]}\left[\frac{1}{\lambda_n^{3k}\omega}\int_0^T|\,G_n(t,s,\nu,\omega)\,|\,ds\right]^2}\,. \tag{39}$$

Continuing this process, similarly to estimate (37) we obtain

$$\sum_{n=1}^{\infty}\max_{t\in[0,T]}\left|\,u_n^{r+1}(t)-u_n^r(t)\,\right|\leq$$

$$\leq |\,\nu\,|\,\delta_2\sqrt{\sum_{n=1}^{\infty}\left[\sum_{i=1}^{m}\left|\,\frac{\Delta_{4in}\,(\nu,\omega,u_n^r)-\Delta_{4in}\left(\nu,\omega,u_n^{r-1}\right)}{\Delta_{0n}(\nu,\omega)}\,\right|\,\delta_{0i}\right]^2}+$$

$$+\delta_3\sqrt{\sum_{n=1}^{\infty}\max_{t\in[0,T]}\left|\,u_n^r(t,\nu,\omega)-u_n^{r-1}(t,\nu,\omega)\,\right|^2}\leq$$

$$\leq |\,\nu\,|\,\delta_2\sqrt{\sum_{n=1}^{\infty}\left[\sum_{i=1}^{m}\left|\,\frac{\bar{\Delta}_{4in}\,(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}\,\right|\,\delta_{0i}\right]^2}\,\left\|\,u^r(t,\nu,\omega)-u^{r-1}(t,\nu,\omega)\,\right\|_{B_2}+$$

$$+\delta_3\left\|\,u^r(t,\nu,\omega)-u^{r-1}(t,\nu,\omega)\,\right\|_{B_2}\leq\rho\cdot\left\|\,u^r(t,\nu,\omega)-u^{r-1}(t,\nu,\omega)\,\right\|_{B_2}, \tag{40}$$

where

$$\rho=|\,\nu\,|\,\delta_2\left\|\sum_{i=1}^{m}\left|\,\frac{\bar{\Delta}_{4in}\,(\nu,\omega)}{\Delta_{0n}(\nu,\omega)}\,\right|\,\delta_{0i}\right\|_{\ell_2}+\delta_3,$$

$$\bar{\Delta}_{4in}\,(\nu,\omega)=\begin{vmatrix} 1-\frac{\nu}{\lambda}\sigma_{31\,1n} & \dots & \frac{\nu}{\lambda}\sigma_{31\,(i-1)n} & \bar{\sigma}_{41n} & \frac{\nu}{\lambda}\sigma_{31(i+1)n} & \dots & \frac{\nu}{\lambda}\sigma_{31mn} \\ \frac{\nu}{\lambda}\sigma_{321n} & \dots & \frac{\nu}{\lambda}\sigma_{32(i-1)n} & \bar{\sigma}_{42n} & \frac{\nu}{\lambda}\sigma_{32(i+1)n} & \dots & \frac{\nu}{\lambda}\sigma_{32mn} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\nu}{\lambda}\sigma_{3m1n} & \dots & \frac{\nu}{\lambda}\sigma_{3m(i-1)n} & \bar{\sigma}_{4mn} & \frac{\nu}{\lambda}\sigma_{3m(i+1)n} & \dots & 1-\frac{\nu}{\lambda}\sigma_{3mmn} \end{vmatrix}, \tag{41}$$

$$\bar{\sigma}_{4in}=\frac{1}{\bar{\lambda}}\int_0^T|\,b_i(s)\,|\int_0^T|\,H_n(s,\theta,\omega)\,\alpha(\theta)\,|\,d\theta\,ds.$$

According to the last condition (34) of the theorem, we have $\rho<1$. Consequently, it follows from estimate (40) that the operator on the right-hand sides of the countable system of functional-integral equations (30) is contracting. It follows from estimates (36), (37) and (40) that there is a unique fixed point, which is a solution to the countable system of functional-integral equations (30) in the space B_2. Theorem 1 is proved.

4 Uniformly convergence of Fourier series

Theorem 2. Let the conditions of Theorem 1 are fulfilled. Then the series in (29) are convergence in the segment $[0, l]$.

Proof. Let $u_n(t,\nu,\omega) \in B_2$ be a solution of system (30). As in the case of estimates (36) and (40), we obtain

$$| \varphi_j(x) | \leq \sqrt{\frac{2}{l}} \left(\frac{\sqrt{2l}}{\pi} \right)^{4k+2} \delta_1 \delta_2 \left[\left\| \frac{\partial^{4k+2} \psi_1(x)}{\partial x^{4k+2}} \right\|_{L_2(\Omega_l)} + \left\| \frac{\partial^{4k+2} \psi_2(x)}{\partial x^{4k+2}} \right\|_{L_2(\Omega_l)} \right] +$$

$$+ | \nu | \ \delta_2 \sqrt{\sum_{n=1}^{\infty} \left[\sum_{i=1}^{m} \left| \frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)} \right| \delta_{0i} \right]^2} + \delta_3 \, \| u(t,\nu,\omega) \|_{B_2} < \infty, \ \ j = 1, 2. \tag{42}$$

Absolutely and uniformly convergence of the series (29) implies from estimate (42).

Substituting system (30) into Fourier series (5), we obtain

$$U(t,x,\nu,\omega) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \sin \lambda_n x \ [\psi_{1n} \, g_{1n}(t,\nu,\omega) + \psi_{2n} \, g_{2n}(t,\nu,\omega) +$$

$$+ \frac{\nu}{\lambda_n^k \, (1+\lambda_n^{2k}) \, \omega} \sum_{i=1}^{m} \frac{\Delta_{4in}(\nu,\omega,u_n)}{\Delta_{0n}(\nu,\omega)} \, g_{3in}(t,\omega) +$$

$$+ \left. \frac{1}{\lambda_n^k \, (1+\lambda_n^{2k}) \, \omega} \int\limits_0^T G_n(t,s,\nu,\omega) \, u_n(s,\nu,\omega) \, ds \right], \quad (\nu,\omega) \in \Lambda_4. \tag{43}$$

Theorem 3. Let the conditions of Theorem 1 are fulfilled. Then the main unknown function $U(t,x,\nu,\omega)$ of inverse problem (1)–(4) is defined by Fourier series (43) and this series (43) converges absolutely and uniformly in the domain Ω for all $(\nu,\omega) \in \Lambda_4$. Moreover, function (43) belongs to the class $C\,(\overline{\Omega}) \cap C^{2,4k}_{t,x}(\Omega) \cap C^{2+2k}_{t,x}(\Omega)$.

The *proof* of Theorem 3 is similar to the proof of Theorem 2.

Conclusion

In the rectangular domain $\Omega = \big\{ 0 < t < T, \ 0 < x < l \big\}$ we consider a linear Benney–Luke type partial integro-differential equation (1) of a higher order with degenerate kernel and two redefinition functions (3) given at the endpoint of the segment [0,T]. With respect to spatial variable x Dirichlet boundary value conditions (2) is used. To find these redefinition functions intermediate data (4) are used. The Fourier series method of variables separation is applied. The countable system of functional-integral equations (30) is obtained. Theorem 1 on a unique solvability of countable system of functional-integral equations (30) is proved. The method of successive approximations is used in combination with the method of contraction mappings. The triple of solutions for the inverse problem is obtained in the form of Fourier series (29) and (43). The absolutely and uniformly convergence of Fourier series is proved (Theorem 2 and 3).

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Екі қайта анықтау функциясы мен параметрлері бар Бенни-Люк типті интегралдық-дифференциалдық теңдеу үшін аралас кері есеп

Мақалада сегменттің шеттерінде берілген екі қайта анықтау функциясы мен өзгешеленетін ядросы бар Бенни-Люк типті жоғары ретті сызықтық интегралдық-дифференциалды дербес туындылы дифференциалдық теңдеуі қарастырылған. Қайта анықтау функцияларын табу үшін аралық берілгендер пайдаланылған. Кеңістіктік айнымалыға қатысты Дирихле типінің шекаралық шарттары қолданылған. Айнымалыны бөліктеу үшін Фурье әдісі пайдаланылды. Функционалдық интегралдық теңдеулердің есептелетін жүйесі алынды. Функционалдық интегралдық теңдеулердің санаулы жүйесінің бірмәнді шешілетіндігі туралы теорема дәлелденді. Бұл жағдайда біртіндеп жуықтау әдісі сығылған бейнелеу әдісімен бірге қолданылады. Кері есептің шешімі Фурье қатары түрінде құрылады. Алынған Фурье қатарының абсолютті және бірқалыпты жинақтылығы нақтыланды.

Кілт сөздер: кері есеп, екі қайта анықтау функциясы, кейінгі шарттар, аралық функциялар, Фурье әдісі, бірмәнді шешім.

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Смешанная обратная задача для интегро-дифференциального уравнения типа Бенни–Люка с двумя функциями переопределения и параметрами

В статье рассмотрено линейное интегро-дифференциальное уравнение в частных производных типа Бенни–Люка высокого порядка с вырожденным ядром и двумя функциями переопределения, заданными в конце отрезка. Для нахождения этих функций переопределения использованы промежуточные данные. По отношению к пространственной переменной применены краевые условия типа Дирихле. Применяется метод разделения переменных Фурье. Получена счетная система функционально-интегральных уравнений. Доказана теорема об однозначной разрешимости счетной системы функционально-интегральных уравнений. При этом используется метод последовательных приближений в сочетании с методом сжатых отображений. Решение обратной задачи строится в виде ряда Фурье. Доказана абсолютная и равномерная сходимость полученных рядов Фурье.

Ключевые слова: обратная задача, две функции переопределения, финальные условия, промежуточные функции, метод Фурье, однозначная разрешимость.

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