DOI 10.31489/2023M4/130-143

UDC 510.67

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Similarities of Jonsson spectra's classes

The study of syntactic and semantic properties of a first-order language, generally speaking, for incomplete theories, is one of the urgent problems of mathematical logic. In this article we study Jonsson theories, which are satisfied by most classical examples from algebra and which, generally speaking, are not complete. A new and relevant method for studying Jonson theories is to study these theories using the concepts of syntactic and semantic similarities. The most invariant concept is the concept of syntactic similarity of theories, because it preserves all the properties of the theories under consideration. The main result of this article is the fact that any perfect Jonson theory which are complete for existential sentences, is syntactically similar to some polygon theory (S-polygon, where S is a monoid). This result extends to the corresponding classes of Jonsson theories from the Jonsson spectrum of an arbitrary model of an arbitrary signature.

Keywords: Jonsson theory, semantic model, perfectness, cosemanticness, S-act, Jonsson spectrum, syntactic and semantic similarities.

Introduction

In the work [1], was proved the fact that any complete theory is similar in some sense to a certain polygon theory (S-act). Moreover, in that work [1] two types of similarity were precisely defined: syntactic and semantic similarities. The value of this result speaks about the universality in the sense of such an algebra as a polygon (S-act). The subject of studying various model-theoretic properties of polygons (S-act) is sufficiently completely studied in [2, 3]. Considering these properties in itself imagines certain essential task. The considering of these properties in itself imagines certaining essential task.

In this article, we want to show that the fact proved in [1] is also true in the class of Jonsson theories, which, generally speaking, are not complete. On the other hand, the class of Jonsson theories includes in itself such basic classical examples from algebra, such as groups, Abelian groups, modules, fields of fixed characteristic, linear orders, Boolean algebras, various classes of lattices and polygons (S-act). Thus, it becomes clear that the class of Jonsson theories is a fairly wide class of theories and the study of their theoretical-model properties is an interesting and relevant task.

In the well-known monograph by J. Barwise «Handbook of mathematical logic» the specialist in logic H.J. Keisler in the review article «Fundamentals of model theory» conditionally divided the content of model theory into two main priorities: «western» and «eastern» model theory [4]. But at the same time, he emphasizes the unity and integrity of these priorities in the framework of the development of the general model theory.

These names are not accidental and are associated with the geographical place of residence of the founders of model theory in North America. Namely, Alfred Tarski and Abraham Robinson lived respectively on the western and eastern coasts of the United States. The tasks that determined these directions differed from each other in two fundamental ways. The first point related to the syntax is that the theories that A. Tarski's school dealt with were complete theories. The followers of A. Robinson

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were engaged in theories with a prenix length of not more than two and, as a rule, Jonsson theories. The second point is related to semantics, more exactly that are regards restrictions of morphisms between models and kinds of models.

In the «western» way actually one has dealt with complete theories, where elementary morphisms were considered. In the case of Jonsson theories logicians dealt with isomorphic embeddings and homomorphisms. Also, in connection with the semantic aspect, it should be noted that in the «eastern» version of model theory, logicians deal mainly with the class of existentially closed models of some fixed inductive theory. The difference in the development of these two directions at the moment of the state of model theory is such that the technique for studying complete theories is much more developed and multilateral. The main stages of development and differences in these directions can be found in the following works [5–25].

One of the methods for studying Jonsson theories is the method of transfer of first-order properties, which is semantic. A first-order property is called semantic if it is invariant with respect to the semantic similarity of Jonsson theories. Thus, when researching two Jonsson theories using the transfer method, the object under study will be a preimage, and the known object will be the image of some mapping that will play the role of a syntactic similarity of these two Jonsson theories. The object under study is unknown and we will be interested in those first-order properties that are formulaic and are preserved under syntactic similarity.

1 Basic concepts and results concerning Jonsson theories

We give the following necessary definitions concerning Jonsson theories and their semantic models.

Definition 1. [4] A theory T is called Jonsson if:
1) the theory T has an infinite model;
2) the theory T is inductive;
3) the theory T has the joint embedding property (JEP);
4) the theory T has the amalgamation property (AP).

Definition 2. [26] Let $\kappa \geq \omega$. Model $\mathcal{M}$ of theory T is called:
- κ-universal for T, if each model of theory T with the power strictly less κ isomorphically imbedded in $\mathcal{M}$;
- κ-homogeneous for T, if for any two models $\mathcal{A}$ and $\mathcal{A}_1$ of theory T, which are submodels of $\mathcal{M}$ with the power strictly less then κ and for isomorphism $f : \mathcal{A} \to \mathcal{A}_1$ for each extension $\mathcal{B}$ of model $\mathcal{A}$, which is a submodel of $\mathcal{M}$ and is model of T with the power strictly less then κ there exists the extension $\mathcal{B}_1$ of model $\mathcal{A}_1$, which is a submodel of $\mathcal{M}$ and an isomorphism $g : \mathcal{B} \to \mathcal{B}_1$ which extends f.

Definition 3. [26] Model $\mathcal{C}$ of Jonsson theory T is called semantic model, if it is ω^+-homogeneous-universal.

Definition 4. [26] The center of Jonsson theory T is called an elementary theory of its semantic model $\mathcal{C}$ and denoted through T^*, i.e. $T^* = \mathrm{Th}(\mathcal{C})$.

Definition 5. [27] Jonsson theory T is called a perfect theory, if each a semantic model of theory T is saturated model of T^*.

The criterion for the perfectness of the Jonsson theory was obtained by Yeshkeyev A.R. and it is as follows:

Theorem 1. [27] For any Jonsson theory T following conditions are equivalent:
1) T is perfect;
2) T^* is the model companion.

The following Definitions 6–8 were taken from [28], where generalized Jonsson theories were defined.

Definition 6. [28] Let $\Gamma \subset L$. Then:

1) notation $T \in \Gamma C_\Delta$ means, that $T \cap \Gamma \vdash \varphi$ for all $\varphi \in T$;

2) if $B \subseteq |\mathcal{A}|$, then $\mathrm{Th}_\Gamma(\mathcal{A}, B)$ denotes the set of all Γ-sentences of the language L_B, true in $\mathcal{A}$;

3) mapping $f : \mathcal{A} \to \mathcal{B}$ is said to be Γ-embedding, if for any $\overline{a} \in \mathcal{A}$ and $\varphi(\overline{x}) \in \Gamma$ from $\mathcal{A} \models \varphi(\overline{a})$ follows $\mathcal{B} \models \varphi(f(\overline{a}))$;

4) if $\mathcal{A} \subseteq \mathcal{B}$, then notation $\mathcal{A} \subseteq_\Gamma \mathcal{B}$ signify, that $\mathrm{Th}_\Gamma(\mathcal{A}, |\mathcal{A}|) \subseteq \mathrm{Th}_\Gamma(\mathcal{B}, |\mathcal{A}|)$;

5) sequence of models $\mathcal{A}_i$, $i < \beta$ called Γ-chain, if $\mathcal{A}_i \subseteq_\Gamma \mathcal{A}_j$, where $i < j < \beta$.

Definition 7. [28]

1) The theory T is persistent with respect to the union of Π_α-chains (or is α-inductive) if the union of any Π_α-chains of models of T is an again model of T.

2) The theory T has the α-joint embedding property (α-JEP), if for any $\mathcal{A}, \mathcal{B} \models T$ there is $\mathcal{M} \models T$ and Π_α-embeddings $f : \mathcal{A} \to \mathcal{M}$ and $g : \mathcal{B} \to \mathcal{M}$.

3) The theory T has the α-amalgamation property (α-AP) if for any $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \models T$ and Π_α-embeddings $f_1 : \mathcal{A} \to \mathcal{B}_1$ and $f_2 : \mathcal{A} \to \mathcal{B}_2$ there is $\mathcal{M} \models T$ and Π_α-embeddings $g_1 : \mathcal{B}_1 \to \mathcal{M}$ and $g_2 : \mathcal{B}_2 \to \mathcal{M}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

The following definition gives us generalized Jonsson theories or α-Jonsson theories.

Definition 8. [28] A theory T is called α-Jonsson ($0 \leq \alpha \leq \omega$) if:

1) the theory T has an infinite model;

2) the theory T is α-inductive;

3) the theory T has α-JEP;

4) the theory T has α-AP.

If compare Definitions 1 and 8, then can notice, that they differ with precision to α. At that in Definition 8 for $\alpha = 0$ we have Jonsson theories, and for $\alpha = \omega$ we have complete Jonsson theories. Further, when we work with 0-Jonsson theories, we will omit 0. Note that from Definition 1 it follows that Jonsson theories, generally speaking, are not complete.

Mustafin T.G. the following useful suggestions were proved in [28]: Proposition 1 and Proposition 2 actually give for us syntactic equivalents of α-JEP and α-AP notions.

Proposition 1. [28] The following conditions are equivalent:

1) T has α-JEP;

2) T has α-JEP for countable models;

3) if $\overline{x} \cap \overline{y} = \oslash$, $p(\overline{x})$ and $q(\overline{y})$ are arbitrary sets of $\Sigma_{\alpha+1}$-formulas, such that $T \cup p(\overline{x})$ and $T \cup q(\overline{y})$ are consistent, then $T \cup p(\overline{x}) \cup q(\overline{y})$ is consistent.

Proposition 2. [28] The following conditions are equivalent:

1) T has α-AP;

2) T has α-AP for countable models;

3) if $p(\overline{x})$ and $q(\overline{x})$ are such sets of $\Sigma_{\alpha+1}$-formulas, that $T \cup p(\overline{x})$, $T \cup q(\overline{x})$, $T \cup \{\neg\varphi(\overline{x}) : \varphi(\overline{x}) \in \Sigma_{\alpha+1}, \varphi(\overline{x}) \notin p(\overline{x}) \cap q(\overline{x})\}$ are consistent sets, then the set $T \cup p(\overline{x}) \cup q(\overline{x})$ is consistent.

4) for any $\mathcal{A} \models T$ and $\overline{a} \in \mathcal{A}$ set $\mathrm{Th}_{\Sigma_{\alpha+1}}(\mathcal{A}, \overline{a})$ it is contained in a unique maximal consistent with T the set $\Sigma_{\alpha+1}$-sentences of the language $L(\overline{a})$.

2 The concepts of syntactic and semantic similarities of complete theories

The notion of similarity between two complete theories was introduced in [1]. For Jonsson theories the similarity between two Jonsson theories was introduced in [27]. In both works were obtained some results which described syntactic and semantic similarity in both cases. We give a list of the necessary definitions of concepts and their necessary model-theoretical properties.

The following definition belongs to T.G. Mustafin [1].

Let $F_n(T)$, $n < \omega$ be the Boolean algebra of formulas of T with exactly n free variables $v_1, \ldots, v_n$ and $F(T) = \bigcup_n F_n(T)$.

Definition 9. [1] Complete theories T_1 and T_2 are syntactically similar if and only if there exists a bijection $f : F(T_1) \to F(T_2)$ such that

1) $f \restriction F_n(T_1)$ is an isomorphism of the Boolean algebras $F_n(T_1)$ and $F_n(T_2)$, $n < \omega$;
2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in F_{n+1}(T)$, $n < \omega$;
3) $f(v_1 = v_2) = (v_1 = v_2)$.

The following example of syntactic similarity of complete theories was given in [1].

Example 1. The following theories T_1 and T_2 of the signature $\sigma = \langle \varphi, \psi \rangle$ are syntactically similar, where φ, ψ are binary functions:

$$T_1 = \mathrm{Th}(\langle Z; +, \cdot \rangle), \quad T_2 = \mathrm{Th}(\langle Z; \cdot, + \rangle).$$

Definition 10. [1]

1) $\langle A, \Gamma, \mathcal{M} \rangle$ is called the pure triple, where A is not empty, Γ is the permutation group of A and $\mathcal{M}$ is the family of subsets of A such that from $M \in \mathcal{M}$ follows that $g(M) \in \mathcal{M}$ for every $g \in \Gamma$.

2) If $\langle A_1, \Gamma_1, \mathcal{M}_1 \rangle$ and $\langle A_2, \Gamma_2, \mathcal{M}_2 \rangle$ are pure triples and $\psi : A_1 \to A_2$ is a bijection then ψ is an isomorphism if:

(i) $\Gamma_2 = \{\psi g \psi^{-1} : g \in \Gamma_1\}$;
(ii) $\mathcal{M}_2 = \{\psi(E) : E \in \mathcal{M}_1\}$.

Definition 11. [1] The pure triple $\langle C, Aut(\mathcal{C}), Sub(C) \rangle$ is called the semantic triple of complete theory T, where C is carrier of Monster model $\mathcal{C}$ of theory T, $Aut(\mathcal{C})$ is the automorphism group of $\mathcal{C}$, $Sub(C)$ is a class of all subsets of C each of which is a carrier of the corresponding elementary submodel of $\mathcal{C}$.

Definition 12. [1] Complete theories T_1 and T_2 are semantically similar if and only if their semantic triples are isomorphic.

The following example of the semantic similarity of complete theories was given in [1].

Example 2. The following theories T_1 and T_2 are semantically similar, where

$$\begin{gathered} T_1 = Th(\langle \mathcal{M}_1; P_n, n < \omega; a_{nm}, n, m < \omega \rangle), \\ \mathcal{M}_1 = \{a_{nm} : n, m < \omega\}, \\ P_n(\mathcal{M}_1) = \{a_{nm} : m < \omega\}, \end{gathered}$$

and

$$\begin{gathered} T_2 = Th(\langle \mathcal{M}_2; Q_n, n < \omega; Q_{nm}, n, m < \omega; b_{nmk}, n, m, k < \omega \rangle), \\ \mathcal{M}_2 = \{b_{nmk} : n, m, k < \omega\}, \\ Q_n(\mathcal{M}_2) = \{b_{nmk} : m, k < \omega\}, \\ Q_{nm}(\mathcal{M}_2) = \{b_{nmk} : k < \omega\}. \end{gathered}$$

It turned out that the above types of similarity are not equivalent to each other.

Proposition 3. [1] If T_1 and T_2 are syntactically similar, then T_1 and T_2 semantically similar. The converse implication fails.

Let us recall the definition of semantic property.

Definition 13. [1] A property (or a notion) of theories (or models, or elements of models) is called semantic if and only if it is invariant relative to semantic similarity.

For example from [1] it is known that:

Proposition 4. The following properties and notions are semantic:
(1) type;
(2) forking;
(3) λ-stability;
(4) Lascar rank;
(5) Strong type;
(6) Morley sequence;
(7) Orthogonality, regularity of types;
(8) $I(\aleph_\alpha, T)$ – the spectrum function.

In English literature the term polygon over a monoid S usually uses the term S-acts [2, 3, 29, 30]. In this article we follow the terminology of Professor T.G. Mustafin, who first defined and formulated model-theoretical concepts and issues related to polygons topics [26, 31, 32].

Definition 14. [1] By a polygon over a monoid S (or we called as S-acts) we mean a structure with only unary functions $\langle A; f_\alpha : \alpha \in S\rangle$ such that:
1) $f_e(a)$ $\forall a \in A$, where e is the unit of S;
2) $f_{\alpha\beta}(a) = f_\alpha(f_\beta(a))$ $\forall \alpha, \beta \in S$, $\forall a \in A$.

The following results (Theorems 2, 3) show that any complete theory has some syntactic similar theory.

Theorem 2. [1] For every theory T_2 in a finite signature there is a theory T_1 of polygons such that some inessential extension of T_1 is an almost envelope of T_2.

Theorem 3. [1] For every theory T_2 in an infinite signature there is a theory T_1 of polygons such that some inessential extension of T_1 is an envelope of T_2.

3 The concepts of syntactic and semantic similarities of Jonsson theories. Main results

The following definition was introduced in the frame of Jonsson theories study by first author of this current article.

Let T be an arbitrary Jonsson theory, then $E(T) = \bigcup_{n<\omega} E_n(T)$, where $E_n(T)$ is a lattice of $\exists$-formulas with n free variables, T^* is a center of Jonsson theory T, i.e. $T^* = Th(\mathcal{C})$, where $\mathcal{C}$ is semantic model of Jonsson theory T in the sense of [26].

Definition 15. [27] Let T_1 and T_2 are arbitrary Jonsson theories. We say that T_1 and T_2 are Jonsson syntactically similar if exists a bijection $f : E(T_1) \longrightarrow E(T_2)$ such that:
1) restriction f to $E_n(T_1)$ is isomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;
2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;
3) $f(v_1 = v_2) = (v_1 = v_2)$.

We would like to give some examples of syntactic similarity of certain algebraic examples. For this, we recall the basic definitions associated with these examples following denotions from B. Poizat [33].

A Boolean ring is an associative ring with identity, in which $x^2 = x$ for any x is called a Boolean ring; we then have $(x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$, but $(x + y)^2 = x + y$; from which it follows that $xy + yx = 0$ for any x and y. Then $x^2 + x^2 = 0$, and hence $x + x = 0$, for every x, so $x = -x$; a Boolean ring therefore has characteristic 2, and since $xy = -yx = yx$, it is commutative.

To axiomatize this concept, we introduce the language consisting of two constant symbols 0 and 1 and two binary operations + and.

We write down some universal axioms, expressing, that A is the Boolean ring, without forgetting thus $0 \neq 1$. In a Boolean ring we define two binary operations $\wedge$ and $\vee$, and one unary operation $\neg$, in the following way: $x \wedge y = x \cdot y$; $x \vee y = x + y + xy$; $\neg x = 1 + x$.

The reader can check that the following properties are true for all x,y, z:
– (de Morgan's laws or duality laws): $\neg(\neg x) = x$, $\neg(x \wedge y) = \neg x \vee \neg y, \neg(x \vee y) = \neg x \wedge \neg y$;
– (associativity of $\wedge$): $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
– (associativity of $\vee$): $(x \vee y) \vee z = x \vee (y \vee z)$;
– (distributivity of $\wedge$ over $\vee$): $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
– (distributivity of $\vee$ over $\wedge$): $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;
– (commutativity of $\wedge$ and $\vee$): $x \wedge y = y \wedge x$, $x \vee y = y \vee x$;
– $x \wedge \neg x = 0,\ x \vee \neg x = 1$;
– $x \wedge 0 = 0,\ x \vee 0 = x,\ x \wedge 1 = x,\ x \vee 1 = 1$;
– $0 \neq 1, \neg 0 = 1, \neg 1 = 0$.

A structure in the language $(0, 1, \neg, \wedge, \vee)$ that satisfies these universal axioms is called a Boolean algebra.

Boolean algebras and Boolean rings defined in this way are examples of Jonsson theories that are syntactically similar in the sense of definition [29], as a consequence of the following fact:

Fact 1. [33] In each Boolean ring one can interpret a certain Boolean algebra.

It is easy to see that interpretation is a special case of syntactic similarity.

Proof. With the Boolean ring A we have connected some Boolean algebra $b(A)$; the converse is also true: $x \cdot y = x \wedge y$, $x + y = (x \vee y) \wedge (\neg x \vee \neg y)$, then we receive the Boolean ring $a(B)$; and besides $a(b(A)) = A$, $b(a(B)) = B$. Thus we see, that up to a language, the Boolean ring and Boolean algebras have the same structures, the Boolean ring canonically is transformed into a Boolean algebra and vice versa, transformations in both directions are carried out using quantifier free formulas.

As in the case of complete theories (Definition 12), we can define a semantic similarity between two Jonsson theories.

Definition 16. [27] The pure triple $\langle C, Aut(\mathcal{C}), Sub(C)\rangle$ is called the Jonsson semantic triple, where C is carrier of semantic model $\mathcal{C}$ of theory T, $Aut(\mathcal{C})$ is the automorphism group of $\mathcal{C}$, $Sub(C)$ is a class of all subsets of C which are carriers of the corresponding existentially closed submodels of $\mathcal{C}$.

Definition 17. [27] Two Jonsson theories T_1 and T_2 are called Jonsson semantically similar if their Jonsson semantic triples are isomorphic as pure triples.

The correctness of this definition follows from the fact that the perfect Jonsson theory has a unique semantic model up to isomorphism. Otherwise, all semantic models are only elementary equivalent to each other.

For the convenience of further exposition we introduce the following notation. The syntactic and semantic similarities of the complete theories T_1 and T_2 will be denoted $T_1 \overset{S}{\bowtie} T_2$ and $T_1 \underset{S}{\bowtie} T_2$ respectively. In the case when we consider Jonsson theories T_1 and T_2, through $T_1 \overset{S}{\rtimes} T_2$ will be denote the Jonsson syntactic similarity of theories T_1 and T_2, and through $T_1 \underset{S}{\rtimes} T_2$ Jonsson semantic similarity of theories T_1 and T_2.

Theorem 4. [27] Let T_1 and T_2 are $\exists$-complete perfect Jonsson theories, then following conditions are equivalent:

1) $T_1 \overset{S}{\rtimes} T_2$;
2) $T_1^* \overset{S}{\bowtie} T_2^*$.

The following lemma is a Jonsson analogue of Proposition 3.

Lemma 1. If two perfect $\exists$-complete Jonsson theories are Jonsson syntactically similar, then they are Jonsson semantically similar. The converse is, generally speaking, not true.

Proof. Follows from Theorem 4 and Proposition 3.

The following technical lemma is necessary to prove Proposition 5.

Lemma 2. Let T be $\exists$-complete theory and $T \subseteq T'$. Then if $p(\overline{x}) \cup T$ consistent, then $p(\overline{x}) \cup T'$ is also consistent ($p(\overline{x})$ is an arbitrary set $\exists$-formulas).

Proof. It is easy to show that T' will also be $\exists$ -complete, since $T \subseteq T'$.

Proposition 5. Let T be a perfect Jonsson theory, then for every sentence $\varphi \in T^*\backslash T$ the theory $T' = T \cup \{\varphi\}$ is a Jonsson.

Proof. Let us verify the fulfillment of all the conditions for the definition of the Jonsson theory. As T is a perfect Jonsson theory, then T^* is a Jonsson theory. Since $T \subset T' \subset T^*$, then T' is $\forall\exists$-axiomatizable and T' has an infinite model. From Lemma 2 and the syntactic definition of α-JEP (Proposition 1 for $\alpha = 0$) it is easy to see that T' has JEP.

Let us verify the fulfillment of condition 4) of Definition 1. Let $p(\overline{x}) \cup T'$, $q(\overline{x}) \cup T'$, $r(\overline{x}) \cup T'$ are consistent, where $p(\overline{x})$, $q(\overline{x})$, $r(\overline{x})$ the same as in Proposition 2 for $\alpha = 0$. Without loss of generality, we can consider that $\overline{x} = x$. Then by the previous lemma $p(\overline{x}) \cup T^*$ and $q(\overline{x}) \cup T^*$ are consistent. Let $h(x) = \{\varphi(x) : \varphi(x)|\varphi(x) \text{ is existential sentence}, \forall x\varphi(x) \in T^*\}$, $p'(x) = p(x) \cup h(x)$, $q'(x) = q(x) \cup h(x)$. It's obvious that $p'(x) \cup T^*$, $q'(x) \cup T^*$ are consistent. Let $r'(x) = \{\neg\varphi(x) : \varphi(x)|\varphi(x) \text{ is existential sentence}, \varphi(x) \in p'(x) \cap q'(x)\}$. We show that $r'(x) \cup T^*$ is consistent. Suppose the opposite, let $r'(x) \cup T^*$ be inconsistent, then exists $\varphi(x) \in r'(x)$ such that $\varphi(x) \cup T^*$ is inconsistent. Means, $\exists x\varphi(x) \cup T^*$ is inconsistent, then $\forall x\neg\varphi(x) \in T^*$ and $\neg\varphi(x) \in h(x)$. Consequently $\neg\varphi(x) \in p'(x) \cap q'(x)$. Got a contradiction. Thus $r'(x) \cup T^*$ is consistent. We have that $p'(x) \cup T^*$, $q'(x) \cup T^*$, $r'(x) \cup T^*$ are consistent. By virtue of the fact that theory T is Jonsson theory, we obtain, that $p'(x) \cup q'(x) \cup T^*$ is consistent, which means that, $p(x) \cup q(x) \cup T^*$ is also consistent. As $T' \subseteq T^*$ then and $p(x) \cup q(x) \cup T'$ is consistent. So, T' has AP. Thus T' is Jonsson theory.

The following definition was introduced by T.G. Mustafin.

Definition 18. We say that the Jonsson theory T_1 is cosemantic to the Jonsson theory T_2 ($T_1 \bowtie T_2$) if $\mathcal{C}_{T_1} = \mathcal{C}_{T_2}$, where $\mathcal{C}_{T_i}$ are semantic model of T_i, $i = 1, 2$.

This definition easily implies the following lemma.

Lemma 3. Any two cosemantic Jonsson theories are Jonsson semantically similar.

The proof follows from the definition.

Let $\mathcal{A}$ be an arbitrary model of countable language. The set $JSp(\mathcal{A}) = \{T/T \text{ is Jonsson theory in this language and } \mathcal{A} \in \mathrm{Mod}(T)\}$ is said to be the Jonsson spectrum of the model $\mathcal{A}$.

The relation of cosemanticness on a set of theories is an equivalence relation. Then $JSp(\mathcal{A})/_{\bowtie}$ is the factor set of the Jonsson spectrum of the model $\mathcal{A}$ with respect to $\bowtie$.

The concept of the Jonsson spectrum was introduced by the first author of this article in [7]. It is turned out that this notion useful in the following sense. Using the concept of $JSp(A)/_{\bowtie}$ in [7,8], cosemanticity criteria for Abelian groups and R-modules are obtained that refine the well-known theorems on elementary equivalence of Abelian groups [34] and R-modules [35].

We have the following result.

Theorem 5. For any Jonsson perfect $\exists$-complete theory T there is a Jonsson $\exists$-complete theory of the polygon T'_{Π} such that $T \overset{S}{\rtimes} T'_{\Pi}$.

Proof. Let T be perfect $\exists$-complete Jonsson theory. Since T^* is complete, according to Theorem 2 in the case of a finite signature and Theorem 3 in the case of an infinite signature, there is a complete theory of the polygon T_{Π} such that $T^* \overset{S}{\bowtie} T_{\Pi}$. But then, according to Proposition 3, it follows that $T^* \underset{S}{\bowtie} T_{\Pi}$. Since the concept of type is a semantic notion (Proposition 4), the concept of a formula is also semantic. It follows from Propositions 1 and 2 with $\alpha = 0$ that the properties of JEP and AP are equivalent to the consistently of some formulas, i.e. JEP and AP are semantic concepts. It is clear that $\forall\exists$ -axiomatizability is also a semantic property, since all axioms are true in the semantic model.

This means that the property "to be a Jonsson theory" is a semantic concept, and therefore T_Π is also a Jonsson theory.

Since T^* is a perfect Jonsson theory, then semantic model $\mathcal{C}_T$ of theory T is saturated. But $T^* \underset{S}{\bowtie} T_\Pi$ and, by definition, the semantic triples of these theories are isomorphic to each other, then $\mathcal{C}_T \cong \mathcal{C}_{T_\Pi}$, therefore $\mathcal{C}_{T_\Pi}$ is also saturated and therefore T_Π is a perfect Jonsson theory.

Consider $JSp(C_{T_\Pi})$. Since the theory T_Π is perfect then $|JSp(C_{T_\Pi})/_{\bowtie}| = 1$. Let $\Delta \in JSp(C_{T_\Pi})$, i.e. Δ is Jonsson theory and $\Delta^* = T_\Pi$. We show that Δ is perfect $\exists$-complete Jonsson theory. By virtue of $T^* \underset{S}{\bowtie} \Delta^*$, then from the definition of semantic similarity for complete theories it follows that Δ is the perfect Jonsson theory. If Δ is $\exists$-complete, then instead T'_Π we take Δ and then by Theorem 4 it follows that $T \overset{S}{\rtimes} \Delta = T'_\Pi$. If Δ is not $\exists$-complete, then we carry out the following replenishment procedure for this theory. As $\Delta \subset T_\Pi$, then for any existential sentence φ, of the signature language of Δ such that $\Delta \nvdash \varphi$ and $\Delta \nvdash \neg\varphi$, but $\varphi \in T_\Pi$, consider the theory $\Delta' = \Delta \cup \{\varphi\}$. Since $\Delta \subset \Delta' \subset T_\Pi$, and Δ, T_Π are Jonsson theories, it follows from Proposition 5 that Δ' is also a Jonsson theory. If Δ' is not $\exists$-complete, then we continue the procedure of adding existential sentences $\varphi \in T_\Pi$ until Δ' it becomes $\exists$-complete.

Let $\overline{\Delta} = \Delta \cup \{\varphi | \varphi \in \Sigma_1, \varphi \in T_\Pi\}$ is the result of replenishment procedure of the theory Δ, i.e. $\overline{\Delta}$ is $\exists$-complete and at the same time $\overline{\Delta}$ is a Jonsson theory. We show that $\overline{\Delta} \in JSp(C_{T_\Pi})$, hence the perfection of the theory of $\overline{\Delta}$ will follow from here. Suppose the contrary, let $\overline{\Delta} \notin JSp(C_{T_\Pi})$, then $C_{T_\Pi} \notin Mod(\overline{\Delta})$, but this is not true since $C_{T_\Pi} \models \Delta$ and for any sentence $\varphi \in \overline{\Delta} \backslash \Delta$, $\varphi \in T_\Pi$. Consequently, $C_{T_\Pi} \models \varphi$ and $C_{T_\Pi} \in \mathrm{Mod}(\overline{\Delta})$. We obtain a contradiction, i.e. $\overline{\Delta} \in JSp(C_{T_\Pi})$. But C_{T_Π} is saturated, therefore, $\overline{\Delta}$ is a perfect Jonsson theory. Then by Theorem 4 we have $T^* \overset{S}{\bowtie} \overline{\Delta}^* \Leftrightarrow T \overset{S}{\rtimes} \overline{\Delta}$, where $\overline{\Delta} = T'_\Pi$.

We extend the concepts of syntactic and semantic similarity to the spectra of models of arbitrary signature.

Definition 19. Let $\mathcal{A} \in \mathrm{Mod}\sigma_1$, $\mathcal{B} \in \mathrm{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is J-syntactically similar to class $[T]_2$ and denote $[T]_1 \overset{S}{\rtimes} [T]_2$ if for any theory $\Delta \in [T]_1$ there is theory $\Delta' \in [T]_2$ such that $\Delta \overset{S}{\rtimes} \Delta'$.

Definition 20. The pure triple $\langle C, Aut(\mathcal{C}), \overline{E}_{[T]}\rangle$ is called the J-semantic triple for class $[T] \in JSp(A)/_{\bowtie}$, where C is the semantic model of $[T]$, $AutC$ is the group of all automorphisms of C, $\overline{E}_{[T]}$ is the class of isomorphically images of all existentially closed models of $[T]$.

Definition 21. Let $\mathcal{A} \in \mathrm{Mod}\sigma_1$, $\mathcal{B} \in \mathrm{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is J-semantically similar to class $[T]_2$ and denote $[T]_1 \underset{S}{\rtimes} [T]_2$ if their semantically triples are isomorphic as pure triples.

Lemma 3. From syntactic similarity of two classes of Jonsson spectrum follows their semantic similarity. Converse statement does not true.

The proof follows from Lemma 1 and Definition 21.

Lemma 4. Let $\mathcal{A} \in \mathrm{Mod}\sigma_1$, $\mathcal{B} \in \mathrm{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$ are perfect $\exists$-complete classes, then

$$[T]_1 \overset{S}{\rtimes} [T]_2 \Leftrightarrow [T]_1^* \overset{S}{\bowtie} [T]_2^*.$$

Proof. Let $[T]_1 \overset{S}{\rtimes} [T]_2$, then for every theory $\Delta \in [T]_1$ there is $\overline{\Delta} \in [T]_2$ such that $\Delta \overset{S}{\rtimes} \overline{\Delta}$, where Δ and $\overline{\Delta}$ are perfect $\exists$-complete Jonsson theories. Then according to Theorem 4 $\Delta^* \overset{S}{\bowtie} \overline{\Delta}^*$. But $\Delta^* = \mathrm{Th}(C_{[T]_1}) = [T]_1^*$ and $\overline{\Delta}^* = \mathrm{Th}(C_{[T]_2}) = [T]_2^*$, therefore $[T]_1^* \overset{S}{\bowtie} [T]_2^*$.

Conversely, let $[T]_1^* \overset{S}{\bowtie} [T]_2^*$ then by Theorem 4 for any theory $\Delta \in [T]_1$ there is theory $\Delta' \in [T]_2$ such that $\Delta \overset{S}{\rtimes} \overline{\Delta}$, i.e. $[T]_1 \overset{S}{\rtimes} [T]_2$.

The following theorem is a generalization of Theorem 5 to the case of the class of the Jonsson spectrum of an arbitrary model of signature.

Theorem 6. Let $[T] \in JSp(\mathcal{A})/_{\bowtie}$, then for every perfect $\exists$-complete class $[T] \in JSp(\mathcal{A})/_{\bowtie}$ there is a class $[T_\Pi] \in JSp(\mathcal{B})/_{\bowtie}$, where T_Π is $\exists$-complete Jonsson theory of some model $\mathcal{B}$ of a polygon signature such that $[T] \overset{S}{\rtimes} [T_\Pi]$.

Proof. Let $[T] \in JSp(\mathcal{A})/_{\bowtie}$ be a perfect $\exists$-complete class, then by Theorem 5 for each theory $\Delta \in [T]$ there is a Jonsson $\exists$-complete polygon theory T_Π^Δ such that $\Delta \overset{S}{\rtimes} T_\Pi^\Delta$. Then by Theorem 4 $\Delta^* \overset{S}{\bowtie} (T_\Pi^\Delta)^*$, but since $\Delta \in [T]$, then $\Delta^* = [T]^*$. T_Π^Δ is the Jonsson theory of some model of $\mathcal{B}$ signature, then $T_\Pi^\Delta \in JSp(\mathcal{B})$ and $T_\Pi^\Delta \in [T_\Pi] \in JSp(\mathcal{B})/_{\bowtie}$. But then $(T_\Pi^\Delta)^* = [T_\Pi]^*$. Hence, we have $[T]^* \overset{S}{\bowtie} [T_\Pi]^*$. By Lemma 5, it follows that $[T] \overset{S}{\rtimes} [T_\Pi]$.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP09260237).

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Йонсондық спектрлердің кластарының ұқсастықтары

Бірінші ретті тілдің синтаксистік және семантикалық қасиеттерін, жалпы айтқанда, толық емес теорияларды зерттеу математикалық логиканың өзекті мәселелерінің бірі. Мақалада біз йонсондық теорияларды зерттейміз, олар алгебрадағы классикалық мысалдардың көп болуымен қанағаттандырылады және жалпы айтқанда, толық емес. Йонсондық теорияларды зерттеудің жаңа және өзекті әдісі — теорияларды синтаксистік және семантикалық ұқсастық ұғымдары арқылы зерттеу. Ең инвариантты ұғым — теориялардың синтаксистік ұқсастығы ұғымы, өйткені ол қарастырылып отырған теориялардың барлық қасиеттерін сақтайды. Осы мақаланың негізгі нәтижесі келесі факт болып табылады: кез келген толық экзистенциалды сөйлемдер үшін кемел йонсондық теориясының полигон теориясына синтаксистік тұрғыдан ұқсас екендігін көрсету (S-полигон, мұндағы S моноид). Бұл нәтиже кез келген сигнатураның тиісті моделінің йонсондық спектрінен алынған йонсондық теорияның сәйкес кластарына кеңейтіледі.

Кілт сөздер: йонсондық теория, семантикалық модель, кемел йонсондық теория, косемантика, S-полигон, йонсондық спектр, синтаксистік және семантикалық ұқсастық.

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Подобия классов йонсоновских спектров

Исследование синтаксических и семантических свойств языка первого порядка, вообще говоря, неполных теорий, является одной из актуальных задач математической логики. В настоящей статье мы изучаем йонсоновские теории, которым удовлетворяет большинство классических примеров из алгебры, и которые, вообще говоря, не полны. Новым и актуальным методом исследования йонсоновских теорий является изучение этих теорий с помощью понятий синтаксического и семантического подобий. Самым инвариантным понятием представляется понятие синтаксического подобия теорий, так как оно сохраняет все свойства рассматриваемых теорий. Основной результат данной статьи есть тот факт, что любая совершенная йонсоновская теория, полная для экзистенциальных предложений, синтаксически подобна некоторой теории полигонов (S-полигона, где S — моноид). Этот результат переносится на соответствующие классы йонсоновских теорий из йонсоновского спектра произвольной модели произвольной сигнатуры.

Ключевые слова: йонсоновская теория, семантическая модель, совершенность, косемантичность, S-полигон, йонсоновский спектр, синтаксическое и семантическое подобия.

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