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M.I. Tleubergenov[1,2], G.K. Vassilina[1,3,*], A.A. Abdrakhmanova[1,2]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *Al-Farabi Kazakh National University, Almaty, Kazakhstan;*
[3] *Almaty University of Power Engineering and Telecommunication named Gumarbek Daukeev, Almaty, Kazakhstan*
(E-mail: marat207@mail.ru, v_gulmira@mail.ru, asilaiaa2001@mail.ru)

Representing a second-order Ito equation as an equation with a given force structure

The problem of constructing equivalent equations with a given structure of forces by the given system of stochastic equations is considered. The equivalence of equations in the sense of almost surely is investigated. The paper determines the conditions under which a given system of second-order Ito stochastic differential equations is represented in the form of stochastic Lagrange equations with non-potential forces of a certain structure. Necessary and sufficient conditions for the representability of stochastic equations in the form of stochastic equations with non-potential forces admitting the Rayleigh function are obtained. The obtained results are illustrated by an example of motion of a symmetric satellite in a circular orbit, assuming a change in pitch under the action of gravitational and aerodynamic forces.

Keywords: Stochastic differential equation, stochastic Lagrange equation, stochastic equations with non-potential forces, equivalence almost surely.

Introduction. Problem statement

In [1], Yerugin constructed a set of ordinary differential equations (ODEs) possessing a given integral curve. This work became seminal in the theory of inverse problems of dynamics. At present, this theory is quite fully developed in the class of ODEs (see for instance [2–10]). In [2, 3], Galiullin presented a classification of the main types of inverse problems of dynamics and developed general methods for their solution in the class of ODEs. Inverse problems of dynamics for Ito stochastic differential equations were studied in [11–18].

In recent decades, the increased interest in the Helmholtz problem [19] has given a new impetus to the study of inverse problems for differential systems (for a literature review, see [20]). The solution of the Helmholtz problem in a wider class of differential equations makes it possible to extend the well-developed mathematical methods of classical mechanics to this class of equations. A special place, in terms of the variety of aspects in the study of the Helmholtz problem, is occupied by the works of Santilli [21, 22], which are devoted to the problem of representing second-order ODEs in the form of the Lagrange, Hamilton, and Birkhoff equations. In [23–26], methods for solving the Helmholtz problem are extended to the class of partial differential equations (PDEs). The Helmholtz problem is considered in [27–29] in a probabilistic formulation. We also note the works [21, 22, 26], which, in addition to the authors' own research, mainly in the class of ODEs and PDEs, present a historical review of literature on the development and generalization of the Helmholtz problem.

Given the second-order stochastic equation

$$d\dot{x}_\nu = F_\nu(x, \dot{x}, t)dt + \sigma_{\nu j}(x, \dot{x}, t)d_0\xi^j, \quad \nu = \overline{1, n},\ j = \overline{1, m}, \tag{1}$$

*Corresponding author.
E-mail: v_gulmira@mail.ru

it is required to construct the equivalent equations of the form

$$d\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k}dt = Q_k(x,\dot{x},t)dt + \sigma'_{kj}(x,\dot{x},t)d_0\xi^j, \quad k=\overline{1,n}, \tag{2}$$

with the given structure of the forces Q_k.

We assume that the functions included in the above equations have the smoothness necessary for further reasoning and satisfy the existence and uniqueness theorem for solutions of the Cauchy problems in the class of Ito stochastic differential equations [30]. In particular, we suppose the following holds for the vector function $F(z,t)$ and the matrix $\sigma(z,t)$ (here$z=(x^T,\dot{x}^T)^T$):

(i) $F(z,t)$ and $\sigma(z,t)$ are continuous in t and satisfy the Lipschitz condition in z, i.e.

$$\left\|\sigma(z',t)-\sigma(z'',t)\right\|^2 + \left\|F(z',t)-F(z'',t)\right\|^2 \le L(1+\left|z'-z''\right|^2 \text{ for all } z',z''\in R^{2n};$$

(ii) the linear growth condition

$$\|\sigma(z,t)\|^2 + \|F(z,t)\|^2 \le L(1+|z|^2)$$

is met for all $z\in R^{2n}$.

Let (Ω,U,P) be a probability space with a flow $\{U_t\}$. Here $\{\xi^1(t),\xi^2(t),...,\xi^m(t)\}$ is a system of Wiener processes with the unit matrix of local variances. The equivalence of solutions of equations (1) and (2) is understood in the sense of the following definition.

Definition 1. The equations

$$d\dot{y} = Y_1(y,\dot{y},t)dt + Y_2(y,\dot{y},t)d\xi \tag{a}$$

and

$$d\dot{z} = Z_1(z,\dot{z},t)dt + Z_2(z,\dot{z},t)d\xi \tag{b}$$

are said to be equivalent almost surely (a. s.) if $y(t_0)=z(t_0),\ \dot{y}(t_0)=\dot{z}(t_0)$ a.s. imply $y(t,t_0,y_0,\dot{y}_0)= z(t,t_0,z_0,\dot{z}_0),\ \dot{y}(t,t_0,y_0,\dot{y}_0)=\dot{z}(t,t_0,z_0,\dot{z}_0)$ a. s., for all $t\ge t_0$.

The problem of construction of equation (2) by the given equation (1) was considered in [31] in the case of the absence of random perturbations $\sigma_{\nu j}\equiv\sigma'_{\nu j}\equiv 0$. The case of the presence of random perturbations and $Q_k\equiv 0$ was studied in [32] by the method of additional variables.

Hereinafter, summation is assumed for the repeated indices of the factors. The indices i, k, and v run from 1 to n, and the index j runs from 1 to m.

In other words, the problem is stated as follows: for given $F_\nu,\sigma_{\nu j}$ it is required to determine the conditions on the functions L and $\sigma'_{\nu j}$, under which equation (2) is equivalent to equation (1) with the given structure of forces Q_k.

Case A. Let Q_k be arbitrary non-potential forces.

Theorem 1. Equation (1) is represented in the form of equation (2) with arbitrary non-potential forces if and only if

$$\frac{\partial^2 L}{\partial \dot{x}_k\partial \dot{x}_\nu} = \delta_k^\nu, \quad \text{where} \quad \delta_k^\nu = \begin{cases} 1, \nu=k \\ 0, \nu\ne k \end{cases}, \tag{3}$$

and

$$\sigma'_{kj}(x,\dot{x},t) = \sigma_{\nu j}(x,\dot{x},t). \tag{4}$$

Proof. By the Ito's rule of stochastic differentiation, we obtain

$$d\left(\frac{\partial L}{\partial \dot{x}_k}\right) = \left[\frac{\partial^2 L}{\partial \dot{x}_k\partial t} + \frac{\partial^2 L}{\partial \dot{x}_k\partial x^\nu}\dot{x}_\nu + \frac{\partial^2 L}{\partial \dot{x}_k\partial \dot{x}_\nu}F_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k\partial \dot{x}_i\partial \dot{x}_\nu}\sigma_{ij}\sigma_{\nu j}\right]dt + \frac{\partial^2 L}{\partial \dot{x}_k\partial \dot{x}_\nu}\sigma_{\nu j}d_0\xi^j.$$

Since $F_\nu dt + \sigma_{\nu j} d_0\xi^j = d\dot{x}_\nu$, we have

$$d\left(\frac{\partial L}{\partial \dot{x}_k}\right) = \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} d\dot{x}_\nu + \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma_{ij}\sigma_{\nu j}\right] dt. \tag{5}$$

Hence, in view of (5), equation (2) takes the form

$$d\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k} dt - \sigma'_{kj}(x,\dot{x},t) d_0\xi^j \equiv$$

$$\equiv \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} d\dot{x}_\nu + \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma_{ij}\sigma_{\nu j} - \frac{\partial L}{\partial x_k} - Q_k\right] dt - \sigma'_{kj}(x,\dot{x},t) d_0\xi^j. \tag{6}$$

Then, taking into account (6) and the original equation (1), we have

$$\frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} d\dot{x}_\nu + \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma_{ij}\sigma_{\nu j} - \frac{\partial L}{\partial x_k} - Q_k\right] dt -$$

$$-\sigma'_{kj}(x,\dot{x},t) d_0\xi^j \equiv d\dot{x}_k - F_k(x,\dot{x},t)dt - \sigma_{kj}(x,\dot{x},t) d_0\xi^j. \tag{7}$$

The above equation implies the fulfillment of condition (3) of Theorem, which in turn implies

$$\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu} \equiv 0. \tag{8}$$

Equating the coefficients of dt and $d\xi^j$ in (7), on the basis of (8) we obtain the fulfillment of condition (4) of Theorem and the following expression

$$Q_k = \frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu - \frac{\partial L}{\partial x_k} + F_k(x,\dot{x},t). \tag{9}$$

Expression (9) determines the non-potential force Q_k. If instead of (2) we consider the equation

$$d\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k} dt = Q_k(x,\dot{x},t)dt + \sigma_{kj}(x,\dot{x},t) d_0\xi^j, \tag{$2'$}$$

we obtain the following corollary of Theorem 1.

Corollary 1. Equation (1) is represented in the form of equation ($2'$) with arbitrary non-potential forces if and only if condition (3) is met.

In particular, for $x \in R^1, \xi \in R^1$, conditions (3) and (4) for the transition from (1) to (2) take the form

$$\frac{\partial^2 L}{\partial \dot{x}^2} = 1, \quad \sigma' = \sigma,$$

and an arbitrary non-potential force is determined as

$$Q = \frac{\partial^2 L}{\partial \dot{x} \partial t} + \frac{\partial^2 L}{\partial \dot{x} \partial x}\dot{x} - \frac{\partial L}{\partial x} + F.$$

Case B. Let Q_k admit the generalized Rayleigh function $R(x,\dot{x})$, that is,

$$Q_k(x,\dot{x}) = -\frac{\partial R}{\partial x_k}. \tag{10}$$

Then equation (2) is represented in the form

$$d\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k} dt = -\frac{\partial R}{\partial x_k} + \sigma'_{kj}(x,\dot{x},t) d_0\xi^j. \tag{11}$$

Theorem 2. Equation (1) is represented in the form of equation (11) with non-potential forces admitting the Rayleigh function if and only if conditions (3), (4) and

$$\frac{\partial R}{\partial x_k} = \frac{\partial L}{\partial x_k} - \frac{\partial^2 L}{\partial \dot{x}_k \partial t} - \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu - F_k(x, \dot{x}, t) \tag{12}$$

are met.

This theorem is proved in the same way as Theorem 1.

Theorem 2 implies the following statement for the equation

$$d(\frac{\partial L}{\partial \dot{x}_k}) - \frac{\partial L}{\partial x_k}dt = -\frac{\partial R}{\partial x_k} + \sigma_{kj}(x, \dot{x}, t)d_0\xi^j. \tag{$11'$}$$

Corollary 2. Equation (1) is represented in the form of equation ($11'$) with non-potential forces admitting the Rayleigh function if and only if conditions (3) and (12) are met.

In particular, for $x \in R^1, \xi \in R^1$, conditions (3), (4) and (12) for the transition from (1) to (11) take the forms

$$\frac{\partial^2 L}{\partial \dot{x}^2} = 1, \quad \sigma' = \sigma, \quad \frac{\partial R}{\partial x} = \frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial \dot{x}\partial t} - \frac{\partial^2 L}{\partial \dot{x}\partial x}\dot{x} - F,$$

respectively.

We now extend the definition introduced by R.M. Santilli [21] to the class of Ito stochastic differential equations.

Definition 2. We say that equation

$$A_{\nu i}(x, \dot{x}, t)d\dot{x}^i + B_\nu(x, \dot{x}, t)dt = \sigma_{\nu j}(x, \dot{x}, t)d_0\xi^j \tag{$1'$}$$

admits the analytic representation in the form of the Lagrange stochastic equation with a given structure of forces, if there exist n^2 functions $h_k^\nu(x, \dot{x}, t)$, $det(h_k^\nu) \neq 0$, such that the following identity holds:

$$d(\frac{\partial L}{\partial \dot{x}_k}) - \frac{\partial L}{\partial x_k}dt - Q_k dt - \sigma'_{kj}(x, \dot{x}, t)d_0\xi^j \equiv h_k^\nu(A_{\nu i}d\dot{x}_i + B_\nu dt - \sigma_{\nu j}d_0\xi^j). \tag{13}$$

Let us consider the problem of analytic representation in the sense of Definition 2. In other words, given the functions $A_{\nu,j}, B_\nu, \sigma_{\nu,j}$ and the forces Q_k in equation ($2'$), it is required to determine the conditions on the functions $h_k^\nu, L, \sigma'_{\nu j}$, under which the relation (13) holds.

Теорема 3. For the indirect representation of the equation with arbitrary non-potential forces Q_k, the necessary and sufficient conditions are

$$\frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_i} = h_k^\nu A_{\nu i}, \tag{14}$$

$$\sigma'_{kj} = h_k^\nu \sigma_{\nu j}. \tag{15}$$

Proof. We set $F_\nu^* = -A_{\nu i}^{-1}B_i$, $\sigma_{\nu j}^* = A_{\nu i}^{-1}\sigma_{ij}$. Then, we have

$$d(\frac{\partial L}{\partial \dot{x}_k}) = \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu}F_\nu^* + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma_{ij}^*\sigma_{\nu j}^*\right]dt + \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu}\sigma_{\nu j}^* d_0\xi^j.$$

Since $F_\nu^* dt + \sigma_{\nu j}^* d_0\xi^j = d\dot{x}_\nu$, we obtain

$$d\left(\frac{\partial L}{\partial \dot{x}_k}\right) = \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu}d\dot{x}_\nu + \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma_{ij}^*\sigma_{\nu j}^*\right]dt. \tag{16}$$

Hence, in view of (16), relation (13) takes the form

$$\frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} d\dot{x}_\nu + \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma^*_{ij}\sigma^*_{\nu j} - \frac{\partial L}{\partial x_k} - Q_k\right] dt -$$

$$-\sigma'_{kj} d_0\xi^j \equiv h^\nu_k \left(A_{\nu i}(x,\dot{x},t)d\dot{x}_i + B_\nu(x,\dot{x},t)dt - \sigma_{\nu j}(x,\dot{x},t)d_0\xi^j\right).$$

Equating the coefficients of $d\dot{x}_i$ and $d\xi^j_0$, we obtain relations (14) and (15) of Theorem 3. The coefficients of dt on the right- and left-hand sides of the equation are equal due to an arbitrary non-potential force Q_k of the form

$$Q_k = \frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma^*_{ij}\sigma^*_{\nu j} - \frac{\partial L}{\partial x_k} - h^\nu_k B_\nu.$$

The following statement holds in the case when non-potential forces admit the generalized Rayleigh function (10).

Theorem 4. Equation $(2')$ has an indirect representation in the form of the Lagrange equation with non-potential forces admitting the generalized Rayleigh function (10) if and only if conditions (14), (15) and

$$\frac{\partial R}{\partial x_k} = \frac{\partial L}{\partial x_k} - \frac{\partial^2 L}{\partial \dot{x}_k \partial t} - \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu - \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma^*_{ij}\sigma^*_{\nu j} + h^\nu_k B_\nu,$$

with $\sigma^*_{\nu j} = A^{-1}_{\nu i}\sigma_{ij}$, hold.

Proof. Let us apply Ito's rule of stochastic differentiation to the expression $d(\frac{\partial L}{\partial \dot{x}_k})$ and plug it into (13). Then (13) takes the following form:

$$\frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_\nu} d\dot{x}_\nu + \left[\frac{\partial^2 L}{\partial \dot{x}_k \partial t} + \frac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu + \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma^*_{ij}\sigma^*_{\nu j} - \frac{\partial L}{\partial x_k} + \frac{\partial R}{\partial x_k}\right] dt -$$

$$-\sigma'_{kj} d_0\xi^j \equiv h^\nu_k \left(A_{\nu i}(x,\dot{x},t)d\dot{x}_i + B_\nu(x,\dot{x},t)dt - \sigma_{\nu j}(x,\dot{x},t)d_0\xi^j\right). \quad (17)$$

Equating the coefficients on the left- and right-hand sides, we obtain the fulfillment of conditions (14), (15) and (17) of Theorem 4.

In particular, for $x \in R^1$, $\sigma \in R^1$, conditions (14), (15) and (17) take the forms

$$\frac{\partial^2 L}{\partial \dot{x}^2} = hA, \quad \sigma' = h\sigma, \quad \frac{\partial R}{\partial x} = \frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial \dot{x} \partial t} - \frac{\partial^2 L}{\partial \dot{x} \partial x}\dot{x} - \frac{1}{2}\frac{\partial^3 L}{\partial \dot{x}^3}\sigma^2 + hB.$$

If the desired Lagrangian is sought, following R.M. Santilli [21], in the form

$$L = K(x,\dot{x},t) + D_\mu(x,t)\dot{x}_\mu + C(x,t), \quad (18)$$

then we obtain the following statement in terms of functions K, D^μ and C.

Theorem 5. Equation $(2')$ has an indirect representation in the form of the Lagrange equation with non-potential forces, admitting the generalized Rayleigh function (10), and the Lagrangian of the form (18) if and only if conditions (15) and

$$\begin{cases} \dfrac{\partial^2 K}{\partial \dot{x}_k \partial \dot{x}_i} = h^\nu_k A_{\nu i}, \quad \dfrac{\partial R}{\partial x_k} = \left(\dfrac{\partial K}{\partial x_k} + \dfrac{\partial C}{\partial x_k}\right) - \dfrac{\partial D_k}{\partial t} - \\ -\dfrac{\partial^2 K}{\partial \dot{x}_k \partial t} - \dfrac{\partial^2 K}{\partial \dot{x}_k \partial x_\nu}\dot{x}_\nu - \dfrac{1}{2}\dfrac{\partial^3 K}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}\sigma^*_{ij}\sigma^*_{\nu j} + h^\nu_k B_\nu \end{cases}$$

are fulfilled.

Proof follows from Theorem 4 and the relations

$$\begin{cases} \dfrac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_i} = \dfrac{\partial^2 K}{\partial \dot{x}_k \partial \dot{x}_i}, \quad \dfrac{\partial^2 L}{\partial \dot{x}_k \partial t} = \dfrac{\partial^2 K}{\partial \dot{x}_k \partial t} + \dfrac{\partial D_k}{\partial t}, \\ \dfrac{\partial^2 L}{\partial \dot{x}_k \partial x_\nu} = \dfrac{\partial^2 K}{\partial \dot{x}_k \partial x_\nu} + \dfrac{\partial D_k}{\partial x_\nu}, \quad \dfrac{\partial^3 L}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu} = \dfrac{\partial^3 K}{\partial \dot{x}_k \partial \dot{x}_i \partial \dot{x}_\nu}, \\ \dfrac{\partial L}{\partial x_k} = \dfrac{\partial K}{\partial x_k} + \dfrac{\partial D_\mu}{\partial x_k} \dot{x}_\mu + \dfrac{\partial C}{\partial x_k}. \end{cases}$$

Example

Let us consider the planar motion of a symmetric satellite in a circular orbit, assuming a change in pitch under the action of gravitational and aerodynamic forces [33]

$$\ddot{\theta} = f(\theta, \dot{\theta}) + \sigma(\theta, \dot{\theta})\dot{\xi}_0, \tag{19}$$

where θ is the pitch angle and

$$f = Ml \sin 2\theta - M[g(\theta) + \eta\dot{\theta}], \sigma = M\delta[g(\theta) + \eta\dot{\theta}]. \tag{20}$$

Case A. Let Q be arbitrary non-potential forces.

A(i). Assume that the Lagrange function is of the form $L = \dfrac{1}{2}\dot{\theta}^2$. Then equation (2) takes the form $\ddot{\theta} = Q + \sigma^{'}\xi_0$. Hence, by Theorem 1, condition (19) for the representation in the form (2) with the given L is written as $\sigma^{'} = \sigma$ for $Q = f$.

A(ii). Let $L = \dfrac{1}{2}\dot{\theta}^2 + \alpha(\theta)\dot{\theta}) + \beta(\theta)$. Then equation (2) takes the form $\ddot{\theta} - \beta_\theta = Q + \sigma^{'}\xi_0$. Taking into account the form (20) of the function f, we determine $\beta_\theta = Ml \sin 2\theta - Mg(\theta)$, or $\beta = -\dfrac{1}{2}Ml \cos 2\theta - MG(\theta)$, where $G = \int g(\theta) d\theta$. Let us now assume $\sigma^{'} = \sigma = M\delta[g(\theta) + \eta\dot{\theta}]$. Then, by Theorem 1, we conclude that the Lagrangian $L = \dfrac{1}{2}\dot{\theta}^2 - M[\dfrac{1}{2}l \cos 2\theta + G(\theta)]$ for $Q = -Mg\dot{\theta}$ provides the representation of equation (19) in the form (2).

Case B. Let Q admit the generalized Rayleigh function R (10).

B(i). By Theorem 2, for $L = \dfrac{1}{2}\dot{\theta}^2$ the function R takes the form $R = -[N(\theta)\dot{\theta} + \dfrac{1}{2}H\dot{\theta}^2]$, where $N(\theta) = Ml \sin 2\theta - Mg(\theta)$ and $H = -M\eta$. Hence, (24) is represented in the form $(11^{'})$.

B(ii). For $L = \dfrac{1}{2}\dot{\theta}^2 + \alpha(\theta)\dot{\theta}) + \beta(\theta)$, as in the case A(ii), we determine β and, by Theorem 2, conclude that for $R = \dfrac{1}{2}M\eta\dot{\theta}^2$ and $L = \dfrac{1}{2}\dot{\theta}^2 - M[\dfrac{1}{2}l \cos 2\theta + G(\theta)]$ equation (19) can be represented in the form $(11^{'})$.

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References

1 Галиуллин А.С. Методы решения обратных задач динамики / А.С. Галиуллин. — М.: РУДН, 1986. — 224 с.

2 Галиуллин А.С. Избранные труды в двух томах. — Т. II / А.С. Галиуллин. — М.: РУДН, 2009. — 462 с.

3 Мухаметзянов И.А. Уравнения программных движений / И.А. Мухаметзянов, Р.Г. Мухарлямов. — М.: РУДН, 1986. — 88 с.

4 Mukharlyamov R.G. Differential-algebraic equations of programmed motions of Lagrangian dynamical systems / R.G. Mukharlyamov // Mechanics of Solids. — 2011. — 46. — No. 4. — P. 534–543.

5 Mukharlyamov R.G. Control of system dynamics and constrains stabilization / R.G. Mukharlyamov, M.I. Tleubergenov // Communications in Computer and Information Science. — 2017. — No. 700. — P. 431–442. https://doi.org/10.1007/978-3-319-66836-936.

6 Llibre J. Inverse Problems in Ordinary Differential Equations and Applications / J. Llibre, R. Ramirez. — Switzerland: Springer International Publishing, 2016. — 266 p.

7 Zhumatov S.S. Asymptotic Stability of Implicit Differential Systems in the Vicinity of Program Manifold / S.S. Zhumatov // Ukrainian Mathematical Journal. — 2014. — 66. — No. 4. — P. 625–632. https://doi.org/10.1007/s11253-014-0959-y.

8 Zhumatov S.S. Exponential Stability of a Program Manifold of Indirect Control Systems / S.S. Zhumatov // Ukrainian Mathematical Journal. — 2010. — 62. — No. 6. — P. 907–915. https://doi.org/10.1007/s11253-010-0399-2

9 Zhumatov S.S. Stability of a Program Manifold of Control Systems with Locally Quadratic Relations / S.S. Zhumatov // Ukrainian Mathematical Journal. — 2009. — 61. — No. 3. — P. 500–509. https://doi.org/10.1007/s11253-009-0224-y.

10 Еругин Н.П. Построение всего множества систем дифференциальных уравнений, имеющих заданную интегральную кривую / Н.П. Еругин // Прикладная математика и механика. — 1952. — 16. — Вып. 6. — С. 659–670.

11 Tleubergenov M.I. Main Inverse Problem for Differential Systems With Degenerate Diffusion / M.I. Tleubergenov, G.T. Ibraeva // Ukrainian Mathematical Journal. — 2013. — 65. — No. 5. — P. 787–792.

12 Tleubergenov M.I. On the inverse stochastic reconstruction problem / M.I. Tleubergenov // Differential equations. — 2014. — 50. – Issue 2. — P. 274–278. https://doi.org/10.1134/S0012266114020165

13 Tleubergenov M.I. Stochastic Inverse Problem with Indirect Control / M.I. Tleubergenov, G.T. Ibraeva // Differential equations. — 2017. — 53. — Issue 10. — P. 1387–1391. https://doi.org/10.1134/S0012266117100172

14 Vassilina G.K. Solution of the Problem of Stochastic Stability of an Integral Manifold by the Second Lyapunov Method / G.K. Vassilina, M.I. Tleubergenov // Ukrainian Mathematical Journal. – 2016. — 68. — No. 1. – P. 14–28. https://doi.org/10.1007/s11253-016-1205-6

15 Tleubergenov M.I. On the Solvability of the Main Inverse Problem for Stochastic Differential Systems / M.I. Tleubergenov, G.T. Ibraeva // Ukrainian Mathematical Journal. — 2019. — 71. — No. 1. — P. 157–165. https://doi.org/10.1007/s11253-019-01631-w

16 Tleubergenov M.I. On the Restoration Problem with Degenerated Diffusion / M.I. Tleubergenov, G.T. Ibraeva // Turkic World Mathematical Society Journal of Pure and Applied Mathematics. — 2015. — 6. — No. 1. — P. 93–99.

17 Tleubergenov M.I. On Stochastic Inverse Problem of Construction of Stable Program Motion / M.I. Tleubergenov, G.K. Vassilina // Open Mathematics. — 2021. — 19. — P. 157–162. https://doi.org/10.1515/math-2021-0005

18 Tleubergenov M.I. On the Closure of Stochastic Differential Equations of Motion / M.I. Tleubergenov, G.T. Ibraeva // Eurasian Mathematical Journal. — 2021. — 12. — No. 2. — P. 82–89.

19 Гельмгольц Г. О физическом значении принципа наименьшего действия / Г. Гельмгольц // Вариационные принципы механики: сб. ст. классиков науки. — 1959. — С. 430–459.

20 Галиуллин А.С. Системы Гельмгольца / А.С. Галиуллин. — М.: РУДН, 1995. — 86 с.
21 Santilli R.M. Foundations of Theoretical Mechanics. 1. The Inverse Problem in Newtonian Mechanics / R.M. Santilli. — New-York: Springer-Verlag, 1978. — 266 p.
22 Santilli R.M. Foundation of Theoretical Mechanics. 2. Birkhoffian Generalization of Hamiltonian Mechanics / R.M. Santilli. — New-York: Springer-Verlag, 1983. — 370 p.
23 Budochkina S.A. An Operator Equation with the Second Time Derivative and Hamilton-admissible Equations / S.A. Budochkina, V.M. Savchin // Doklady Mathematics. — 2016. — 94. — No. 2. — P. 487–489.
24 Savchin V.M. Nonclassical Hamilton's Actions and the Numerical Performance of Variational Mathodssfor Some Dissipative Problems / V.M. Savchin, S.A. Budochkina // Communications in Computer and Information Science. — 2016. — 678. — P. 624–634.
25 Savchin V.M. Invariance of functionals and related Euler-Lagrange equations / V.M. Savchin, S.A. Budochkina // Russian Mathematics. — 2017. — 61. — No. 2. — P. 49–54.
26 Филиппов В.М. Вариационные принципы для непотенциальных операторов / В.М. Филиппов, В.М. Савчин, С.Г. Шорохов // Итоги науки и техники. Сер. Современные проблемы математики. Новейшие достижения / ВИНИТИ. — 1992. — 40. — С. 3–178.
27 Tleubergenov M.I. Stochastical problem of Helmholtz for Birkhoff system / M.I. Tleubergenov, D.T. Azhymbaev // Bulletin of the Karaganda University. Mathematics series. — 2019. — No. 1(93). — P. 78–87.
28 Tleubergenov M.I. Construction of system's differential equations of the program motion in Lagrangian variables in the presence of random perturbations / M.I. Tleubergenov, G.K. Vassilina, D.T. Azhymbaev // Bulletin of the Karaganda University. Mathematics series. — 2022. — No. 1(105). — P. 118–126.
29 Tleubergenov M.I. Construction of stochastic differential equations of motion in canonical variables / M.I. Tleubergenov, G.K. Vassilina, S.R. Seisenbayeva // Bulletin of the Karaganda University. Mathematics series. — 2022. — No. 3(107). — P. 152–162.
30 Ватанабэ С. Стохастические дифференциальные уравнения и диффузионные процессы / С. Ватанабэ, Н. Икэда. — М., 1986. — 445 с.
31 Шорохов С.Г. Представимость систем дифференциальных уравнений в виде уравнений механики с заданной структурой сил / С.Г. Шорохов // Дифференциальные уравнения. — 1988. — 24. — № 10. — С. 1738–1746.
32 Tleubergenov M.I. On the Solvability of Stochastic Helmholtz Problem / M.I. Tleubergenov, D.T. Azhymbaev // Journal of Mathematical Sciences. — 2021. — 253 — No. 2. — P. 297–305.
33 Сагиров П. Стохастические методы в динамике спутников / П. Сагиров // Механика: период. сб. пер. ин. ст. — 1974. — № 5 (147). — С. 28–47.

М.Ы. Тілеубергенов[1,2], Г.Қ. Василина[1,3], А.А. Абдрахманова[1,2]

[1]*Математика және математикалық модельдеу институты, Алматы, Қазақстан;*
[2]*Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан;*
[3]*Ғұмарбек Дәукеев атындағы Алматы энергетика және байланыс университеті, Алматы, Қазақстан*

Екінші ретті Ито теңдеуін берілген құрылымы бар күштердің теңдеуі түрінде құру

Берілген стохастикалық теңдеулер жүйесінен берілген құрылымы бар күштердің эквивалентті теңдеулерді құру есебі қарастырылған. Теңдеулердің эквиваленттілігі шамамен ықтимал мағынада зерттеледі. Екінші ретті Ито стохастикалық дифференциалдық теңдеулер жүйесі белгілі бір құрылымның потенциалды емес күштері бар стохастикалық Лагранж теңдеулері ретінде ұсынылу шарттары анықталған. Стохастикалық теңдеулердің Рэлей функциясын қабылдайтын потенциалды емес күштері бар стохастикалық теңдеулер түріндегі бейнеленуінің қажетті және жеткілікті шарттары алынды. Зерттеу нәтижелері ауырлық күші мен аэродинамикалық күштердің әсерінен тангаждық өзгерістерге ұшыраған дөңгелек орбитадағы симметриялық жерсеріктің қозғалысының мысалында көрсетілген.

Кілт сөздер: стохастикалық дифференциалдық теңдеу, стохастикалық Лагранж теңдеуі, потенциалдық емес күштері бар стохастикалық теңдеулер, шамамен ықтимал эквиваленттілік.

М.И. Тлеубергенов[1,2], Г.К. Василина[1,3], А.А. Абдрахманова[1,2]

[1] *Институт математики и математического моделирования, Алматы, Казахстан;*
[2] *Казахский национальный университет имени аль-Фараби, Алматы, Казахстан;*
[3] *Алматинский университет энергетики и связи имени Гумарбека Даукеева, Алматы, Казахстан*

Представление уравнения Ито второго порядка в виде уравнения с заданной структурой сил

Рассмотрена задача построения по заданной системе стохастических уравнений, эквивалентных уравнений с заданной структурой сил. Исследована эквивалентность уравнений в смысле почти наверное. Определены условия, при которых заданная система стохастических дифференциальных уравнений Ито второго порядка представима в виде стохастических уравнений Лагранжа с непотенциальными силами определенной структуры. Получены необходимые и достаточные условия представимости стохастических уравнений в виде стохастических уравнений с непотенциальными силами, допускающими функцию Рэлея. Результаты исследования проиллюстрированы на примере движения симметричного спутника по круговой орбите в предположении изменения тангажа под действием аэродинамических сил и тяготения.

Ключевые слова: стохастическое дифференциальное уравнение, стохастическое уравнение Лагранжа, стохастические уравнения с непотенциальными силами, эквивалентность почти наверное.

References

1 Galiullin, A.S. (1986). *Metody resheniia obratnykh zadach dinamiki [Methods for solving inverse problems of dynamics]*. Moscow: RUDN [in Russian].

2 Galiullin, A.S. (2009). *Izbrannye trudy v dvukh tomakh. Tom II. [Selected works in two volumes. Vol. II]*. Moscow: RUDN [in Russian].

3 Mukhametzyanov, I.A., & Mukharlyamov, R.G. (1986). *Uravneniia programmnykh dvizhenii [Equations of program motions]*. Moscow: RUDN [in Russian].

4 Mukharlyamov, R.G. (2011). Differential-algebraic equations of programmed motions of Lagrangian dynamical systems. *Mechanics of Solids, 46*(4), 534–543.

5 Mukharlyamov, R.G., & Tleubergenov, M.I. (2017). Control of system dynamics and constrains stabilization. *Communications in Computer and Information Science,* 700, 431–442. https://doi.org/10.1007/978-3-319-66836-936

6 Llibre, J. (2016). *Inverse Problems in Ordinary Differential Equations and Applications.* Switzerland: Springer International Publishing.

7 Zhumatov, S.S. (2014). Asymptotic Stability of Implicit Differential Systems in the Vicinity of Program Manifold. *Ukrainian Mathematical Journal, 66*(4), 625–632. https://doi.org/10.1007/s11253-014-0959-y

8 Zhumatov, S.S. (2010). Exponential Stability of a Program Manifold of Indirect Control Systems. *Ukrainian Mathematical Journal, 62*(6), 907–915. https://doi.org/10.1007/s11253-010-0399-2.

9 Zhumatov, S.S. (2009). Stability of a Program Manifold of Control Systems with Locally Quadratic Relations. *Ukrainian Mathematical Journal, 61*(3), 500–509. https://doi.org/10.1007/s11253-009-0224-y.

10 Erugin, N.P. (1952). Postroenie vsego mnozhestva sistem differentsialnykh uravnenii, imeiushchikh zadannuiu integralnuiu krivuiu [Construction of the entire set of systems of differential equations with a given integral curve]. *Prikladnaia matematika i mekhanika — Applied Mathematics and Mechanics, 16*(6), 659–670 [in Russian].

11 Tleubergenov, M.I. & Ibraeva, G.T. (2013). Main Inverse Problem for Differential Systems With Degenerate Diffusion. *Ukrainian Mathematical Journal, 65*(5), 787–792.

12 Tleubergenov, M.I., (2014). On the inverse stochastic reconstruction problem. *Differential equations, 50*(2), 274–278. https://doi.org/10.1134/S0012266114020165.

13 Tleubergenov, M.I., & Ibraeva, G.T. (2017). Stochastic Inverse Problem with Indirect Control. *Differential equations, 53*(10), 1387–1391. https://doi.org/10.1134/S0012266117100172.

14 Vassilina, G.K., & Tleubergenov, M.I. (2016). Solution of the Problem of Stochastic Stability of an Integral Manifold by the Second Lyapunov Method. *Ukrainian Mathematical Journal, 68*(1), 14–28. https://doi.org/10.1007/s11253-016-1205-6.

15 Tleubergenov, M.I., & Ibraeva, G.T. (2019). On the Solvability of the Main Inverse Problem for Stochastic Differential Systems. *Ukrainian Mathematical Journal, 71*(1), 157–165. https://doi.org/10.1007/s11253-019-01631-w

16 Tleubergenov, M.I., & Ibraeva, G.T. (2015). On the Restoration Problem with Degenerated Diffusion. *Turkic World Mathematical Society Journal of Pure and Applied Mathematics, 6*(1), 93–99.

17 Tleubergenov, M.I., & Vassilina, G.K. (2021). On Stochastic Inverse Problem of Construction of Stable Program Motion. *Open Mathematics, 19*, 157–162. https://doi.org/10.1515/math-2021-0005

18 Tleubergenov, M.I., & Ibraeva, G.T. (2021). On the Closure of Stochastic Differential Equations of Motion. *Eurasian Mathematical Journal, 12*(2), 82–89.

19 Helmholtz, G. (1959). O fizicheskom znachenii printsipa naimenshego deistviia [On the physical meaning of the principle of least action]. *Variatsionnyye printsipy mekhaniki — Variational principles of mechanics,* 430–459 [in Russian].

20 Galiullin, A.S. (1995). *Sistemy Gelmgoltsa [Helmholtz systems].* Moscow: RUDN [in Russian].

21 Santilli, R.M. (1978). *Foundations of Theoretical Mechanics. 1. The Inverse Problem in Newtonian Mechanics.* New-York: Springer-Verlag.

22 Santilli, R.M. (1983).*Foundation of Theoretical Mechanics. 2. Birkhoffian Generalization of Hamiltonian Mechanics.* New-York: Springer-Verlag.

23 Budochkina, S.A., & Savchin, V.M. (2016). An Operator Equation with the Second Time Derivative and Hamilton-admissible Equations. *Doklady Mathematics, 94*(2), 487–489.

24 Savchin, V.M., & Budochkina, S.A. (2016). Nonclassical Hamilton's Actions and the Numerical Performance of Variational Mathodssfor Some Dissipative Problems. *Communications in Computer and Information Science, 678*, 624–634.

25 Savchin, V.M., & Budochkina, S.A. (2017). Invariance of functionals and related Euler-Lagrange equations. *Russian Mathematics, 61*(2), 49–54.

26 Filippov, V.M., Savchin, V.M., & Shorokhov, S.G. (1992). Variatsionnyye printsipy dlia nepotentsialnykh operatorov [Variational Principles for Nonpotential Operators]. *Itogi nauki i tekhniki. Seriia Sovremennyye problemy matematiki. Noveishiie dostizheniia / VINITI — Results of science and technology. Series Modern Problems of Mathematics. Latest achievements / VINITI, 40,*

3–178 [in Russian].
27 Tleubergenov, M.I., & Azhymbaev, D.T. (2019). Stochastical problem of Helmholtz for Birkhoff system. *Bulletin of the Karaganda University. Mathematics series,* 1(93), 78–87.
28 Tleubergenov, M.I., Vassilina, G.K., & Azhymbaev, D.T. (2022). Construction of system's differential equations of the program motion in Lagrangian variables in the presence of random perturbations. *Bulletin of the Karaganda University. Mathematics series,* 1(105), 118–126.
29 Tleubergenov, M.I., Vassilina, G.K., & Seisenbayeva, S.R. (2022). Construction of stochastic differential equations of motion in canonical variables. *Bulletin of the Karaganda University. Mathematics series,* 3(107), 152–162.
30 Watanabe, S., & Ikeda, N. (1986). *Stokhasticheskiie differentsialnye uravneniia i diffuzionnye protsessy [Stochastic differential equations and diffusion processes].* Moscow [in Russian].
31 Shorokhov, S.G. (1988). Predstavimost sistem differentsialnykh uravnenii v vide uravnenii mekhaniki s zadannoi strukturoi sil [Representability of systems of differential equations in the form of equations of mechanics with a given structure of forces]. *Differentsialnye uravnenia — Differential equations, 24*(10), 1738–1746 [in Russian].
32 Tleubergenov, M.I., & Azhymbaev, D.T. (2021). On the Solvability of Stochastic Helmholtz Problem. *Journal of Mathematical Sciences, 253*(2), 297–305.
33 Sagirov, P. (1974). Stokhasticheskie metody v dinamike sputnikov [Stochastic methods in dynamics of satellites]. *Mekhanika: periodicheskii sbornik perevodov inostrannykh statei — Mechanics. Periodical Collection of Translations of Foreign Articles, 5*(147), 28–47 [in Russian].