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Roughness in Fuzzy Cayley Graphs

Rough set theory is a worth noticing approach for inexact and uncertain system modelling. When rough set theory accompanies with fuzzy set theory, which both are a complementary generalization of set theory, they will be attended by potency in theoretical discussions. In this paper a definition for fuzzy Cayley subsets is put forward as well as fuzzy Cayley graphs of fuzzy subsets on groups inspired from the definition of Cayley graphs. We introduce rough approximation of a Cayley graph with respect to a fuzzy normal subgroup. We introduce the approximation rough fuzzy Cayley graphs and fuzzy rough fuzzy Cayley graphs. The last approximation is the mixture of the other approximations. Some theorems and properties are investigated and proved.

Keywords: fuzzy subset, rough set, Cayley graph, fuzzy Cayley graph, lower and upper approximations.

1 Introduction and preliminaries

Rough sets have been investigated in many papers. For details we refer to [1–7]. In particular, in [8], rough approximations of Cayley graphs are studied. It has intended to build up a rational connection between rough set theory [7], fuzzy set theory [9] and Cayley graphs. Cayley fuzzy graphs are studied in [10–12]. We present a new definition of fuzzy Cayley sets and so, fuzzy Cayley graphs of generators of the Cayley graph of a group. For a finite group G and a fuzzy subset μ on G, the fuzzy subset μ is called fuzzy Cayley subset, if the subset

$$S_\mu = \{a \in G \mid \mu(a) < 1\}$$

is a Cayley subset of G. It means that $1_G \notin S_\mu$ (where 1_G represents the identity element of G) and if $s \in S_\mu$, then $s^{-1} \in S_\mu$. We define the triple $(G; S_\mu; \mu)$ as a fuzzy Cayley graph. In fact, the fuzzy Cayley graph $(G; S_\mu; \mu)$ is a Cayley graph where the fuzzy Cayley subset μ constructs the Cayley subset of it.

The outline on the paper is as follows. First, we recall some notation and definitions about the simple graph. We also recall the definitions and concepts of the fuzzy subset, fuzzy subgroup, t-level relation and lower approximation operator and upper approximation operator for a fuzzy approximation space that we need for the paper in this section. In Section 2, we present the definitions of fuzzy Cayley subset and fuzzy Cayley graph for fuzzy subsets of groups and some few results for them. In Sections 3 and 4, we deal the concept of fuzzy lower and upper approximations of a Cayley graph and lower and upper approximations of a fuzzy Cayley graph with respect to a fuzzy normal subgroup. Finally, in Section 5, we combine the concept of the lower and upper approximations of a Cayley graph and lower and upper approximations of a fuzzy Cayley graph and present the fuzzy lower and upper approximations of a fuzzy Cayley graph with respect to a fuzzy normal subgroup on a finite group.

For the benefit of the reader, we collect in this section some of the basic concepts and facts that we need in this paper.

Let us introduce some basic notation and definitions about the simple graph. We consider simple graphs, which are undirected, with no loops or multiple edges.

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Now, we recall the definition's fuzzy subset, fuzzy subgroup, fuzzy normal subgroup and some proportion of them [9, 13]. Suppose that X is a universe set. A *fuzzy subset* μ on X is a function $\mu : X \to [0,1]$ mapping all elements x of X into a real number $\mu(x)$ in the closed interval $[0,1]$. Taking fuzzy subsets μ and λ on X. $\mu \subseteq \lambda$ if and only if all $x \in X$ satisfying $\mu(x) \leq \lambda(x)$. Fuzzy subset γ is called the *union* of fuzzy subsets μ and λ, if and only if $\gamma(x) = \max\{\mu(x), \lambda(x)\}$ for all $x \in X$, and γ is denoted by $\mu \cup \lambda$. Fuzzy subset φ is called the *intersection* of fuzzy subsets μ and λ, if and only if $\varphi(x) = \min\{\mu(x), \lambda(x)\}$ for all $x \in X$, and φ is denoted by $\mu \cap \lambda$.

A fuzzy subsets μ on a group G is called a *fuzzy subgroup* of G [13], if the following conditions hold:
1 $\forall a, b \in G,\ \mu(ab) \geq \min\{\mu(a), \mu(b)\}$;
2 $\forall a \in G,\ \mu(a^{-1}) \geq \mu(a)$;
3 $\mu(1_G) = 1$.
For every a in G, $\mu(a^{-1}) = \mu(a)$. This follows at once from part 2. A fuzzy subgroup μ of G, is called a *fuzzy normal subgroup* of G if for any arbitrary elements a and b of G, have to $\mu(ab) = \mu(ba)$.

We recall the t-level relation for fuzzy normal subgroups and some properties and theorems 1 and 2, that we need in the work from [4]. Let μ be a fuzzy normal subgroup of G. For each $t \in [0,1]$, the set

$$\mu_t = \{(a,b) \in G \times G \ \mid \ \mu(ab^{-1}) \geq t\}$$

is called a *t-level relation* of μ. For each t, μ_t is a congruence relation on G. We denote by $[x]_\mu$ the congruence class of μ_t containing the element x of G. Let A be a non-empty subset of G. Then the sets

$$\mu_{t-}(A) = \{x \in G \ \mid \ [x]_\mu \subseteq A\},$$
$$\mu_{t^\wedge}(A) = \{x \in G \ \mid \ [x]_\mu \cap A \neq \emptyset\}$$

are called, respectively, the *lower and upper approximations* of the set A with respect to μ_t. The pair $\mu(A) = (\mu_{t-}(A), \mu_{t^\wedge}(A))$ is called a *rough set* of A in G. A non-empty subset A of a group G is called a *$\mu_{t^\wedge}$-fuzzy rough (normal) subgroup* of G if the upper approximation of A is a (normal) subgroup of G. Similarly, a non-empty subset A of G is called a *$\mu_{t-}(A)$-fuzzy rough (normal) subgroup* of G if lower approximation is a (normal) subgroup of G. Note that, if μ and λ are fuzzy normal subgroups of a group G, then $\mu \cap \lambda$ is also a fuzzy subgroup G.

Theorem 1. Suppose that μ and λ are fuzzy normal subgroups of a group G and $t \in [0,1]$. Let A and B be any non-empty subsets of G. Then
(1) $\mu_{t-}(A) \subseteq A \subseteq \mu_{t^\wedge}(A)$,
(2) $\mu_{t-}(A \cap B) = \mu_{t-}(A) \cap \mu_{t-}(B)$,
(3) $\mu_{t^\wedge}(A \cup B) = \mu_{t^\wedge}(A) \cup \mu_{t^\wedge}(B)$,
(4) $A \subseteq B$ implies $\mu_{t-}(A) \subseteq \mu_{t-}(B)$,
(5) $A \subseteq B$ implies $\mu_{t^\wedge}(A) \subseteq \mu_{t^\wedge}(B)$,
(6) $\mu_{t-}(A \cup B) \supseteq \mu_{t-}(A) \cup \mu_{t-}(B)$,
(7) $\mu_{t^\wedge}(A \cap B) \subseteq \mu_{t^\wedge}(A) \cap \mu_{t^\wedge}(B)$,
(8) $\mu \subseteq \lambda$, implies $\lambda_{t-}(A) \subseteq \mu_{t-}(A)$,
(9) $\mu \subseteq \lambda$, implies $\mu_{t^\wedge}(A) \subseteq \lambda_{t^\wedge}(A)$,
(10) $(\mu \cap \lambda)_t = \mu_t \cap \lambda_t$,
(11) $(\mu \cap \lambda)_{t-}(A) \supseteq \mu_{t-}(A) \cap \lambda_{t-}(A)$,
(12) $(\mu \cap \lambda)_{t^\wedge}(A) \subseteq \mu_{t^\wedge}(A) \cap \lambda_{t^\wedge}(A)$.

Theorem 2. Let μ be a fuzzy normal subgroup of a group G and $t \in [0,1]$. If A is a (normal) subgroup of G, then $\mu_{t^\wedge}(A)$ is a (normal) subgroup of G. Moreover, if the lower approximation of A is non-empty, then it is a (normal) subgroup of G.

Given a continuous triangular norm T on the unit interval $I = [0,1]$. A fuzzy binary relation R on X is called a T-similarity relation if for all $x, y, z \in X$, R satisfies the following conditions:
(1) $R(x,x) = 1$;
(2) $R(x,y) = R(y,x)$;
(3) $R(x,z)TR(x,y) \leq R(z,y)$.
The pair (X,R) is called a fuzzy approximation space (see, for example [14] and [6]). Morsi and Yakout in [6] define the lower approximation operator and upper approximation operator for a fuzzy approximation space (X,R), respectively, for $\mu \in [0,1]^X$, as follows:

$$\underline{A}_R\mu(x) = \inf_{u\in X} \vartheta_T(R(u,x),\mu(u)) \text{ for every } x \in X,$$

$$\overline{A}_R\mu(x) = \sup_{u\in X}(R(u,x)T\mu(u)) \text{ for every } x \in X,$$

when $\vartheta_T(a,b) = \sup\{\theta \in [0,1] \mid aT\theta \leq b\}$, for every $a, b \in [0,1]$. Let G be a group and $C \in I_G$. If C satisfies the following conditions:
(1) $C(xy) \geq C(x)TC(y)$;
(2) $C(x^{-1}) \geq C(x)$;
(3) $C(e) = 1$,
then C is called a T-fuzzy subgroup of G. If $C(xy) = C(yx)$ for every $x, y \in G$, then C is called a T-fuzzy normal subgroup of G. It easily can be verified that the binary relation,

$$B : G \times G \to [0,1],$$

$$B(x,y) = C(xy^{-1}), \text{ for every } x, y \in G$$

is T-similarity relation. Jiashang, Congxin and Degang in [14] define the upper approximation operator $\overline{A}_B$ and the lower approximation operator $\underline{A}_B$ with respect to B on G. In this paper, we limited the triangular norm T, the simplest triangular norm, Min. Let μ be a fuzzy subset and β be a fuzzy normal subgroup on G. We call the fuzzy subsets $\underline{A}_B\mu, \overline{A}_B\mu$ as respectively, the *lower and upper approximations of the fuzzy subset μ on G with respect to the fuzzy normal subgroup B.*

$$\underline{A}_B\mu(x) = \inf_{u\in G} \vartheta_{\min}(B(u,x),\mu(u)), \text{ for every } x \in G,$$

$$\overline{A}_B\mu(x) = \sup_{u\in G}\{\min\{B(u,x),\mu(u)\}\}, \text{ for every } x \in G.$$

The pair$(\underline{A}_B\mu, \overline{A}_B\mu)$ is called a *rough fuzzy set* of μ. The fuzzy subset μ on a group G is called a $\overline{A}_B$ *rough fuzzy (normal) subgroup*, if the upper approximation of μ is a fuzzy (normal) subgroup on G. Similarity, the fuzzy subset μ on a group G is called a $\underline{A}_B$ *rough fuzzy (normal) subgroup*, if the lower approximation of μ is a fuzzy (normal) subgroup on G.

Note that $\vartheta_{\min}(a,b) = 1$, if and only if $a \leq b$, if not it is equal to b.

The next proposition follows at once from [14; Proposition 2.4].

Theorem 3. Let G be a finite group, μ and λ be fuzzy subsets. Let B and C be fuzzy normal subgroups on G. Then
(1) $\underline{A}_B\mu \subseteq \mu \subseteq \overline{A}_B\mu$,
(2) $\overline{A}_B\overline{A}_B\mu = \underline{A}_B\overline{A}_B\mu = \overline{A}_B\mu$,
(3) $\overline{A}_B\underline{A}_B\mu = \underline{A}_B\underline{A}_B\mu = \underline{A}_B\mu$,
(4) $\underline{A}_B\mu = \mu$ if and only if $\overline{A}_B\mu = \mu$,
(5) $\overline{A}_B(\mu \cup \lambda) = \overline{A}_B\mu \cup \overline{A}_B\lambda$,
(6) $\overline{A}_B(\mu \cap \lambda) \subseteq \overline{A}_B\mu \cap \overline{A}_B\lambda$,

(7) $\underline{A}_B(\mu \cup \lambda) \supseteq \underline{A}_B\mu \cup \underline{A}_B\lambda$,
(8) $\underline{A}_B(\mu \cap \lambda) = \underline{A}_B\mu \cap \underline{A}_B\lambda$,
(9) $B \subseteq C$ then $\overline{A}_B\mu \subseteq \overline{A}_C\mu$,
(10) $B \subseteq C$ then $\underline{A}_C\mu \subseteq \underline{A}_B\mu$.

The next corollary easily can be verified based upon the parts (6) and (7) of Theorem 3.

Corollary 1. Let G be a finite group, μ and λ be fuzzy subsets. Let B be a fuzzy normal subgroup on G. If $\mu \subseteq \lambda$, then
(1) $\overline{A}_B\mu \subseteq \overline{A}_B\lambda$,
(2) $\underline{A}_B\mu \subseteq \underline{A}_B\lambda$.

The fuzzy subset $B\text{Min}C$ is defined based on fuzzy subsets B and C as $B\text{Min}C(x) = \min\{B(x), C(x)\}$, $\forall x \in G$. The next theorem follows from [14; Lemma 3.4, Propositions 3.5, 3.6, 4.1 and 4.2].

Theorem 4. Let G be a finite group. Suppose that B and C are fuzzy normal subgroups of G. The following properties hold.
(1) The fuzzy set $B\text{Min}C$ is a fuzzy normal subgroup.
(2) $\underline{A}_B\mu\text{Min}\underline{A}_C\mu \subseteq \underline{A}_{B\text{Min}C}\mu$.
(3) $\overline{A}_{B\text{Min}C}\mu \subseteq \overline{A}_B\mu\text{Min}\overline{A}_C\mu$.
(4) If μ is a fuzzy (normal) subgroup of G, then $\overline{A}_B\mu$ is a fuzzy (normal) subgroup of G.
(5) If μ is a fuzzy (normal) subgroup of G and $B \subseteq \mu$, then $\underline{A}_B\mu$ is a fuzzy (normal) subgroup of G.

Throughout the paper, we will make frequently use of the above mentioned results.

2 *Fuzzy Cayley subsets and graphs*

In this section, we present the definitions of fuzzy Cayley subset and fuzzy Cayley graph for fuzzy subsets on groups.

Let G be a finite group and μ be a fuzzy subset on G. The fuzzy subset μ is called *fuzzy Cayley subset*, if the subset

$$S_\mu = \{a \in G \mid \mu(a) < 1\}$$

is a Cayley subset of G. It follows that $\mu(1_g) = 1$ and if $\mu(a) < 1$, then $\mu(a^{-1}) < 1$. Obviously, every fuzzy group is a fuzzy Cayley subset. Since S_μ is a Cayley set, $(G; S_\mu)$ is a Cayley graph. When μ_S is a fuzzy Cayley subset, we define the triple $(G; S_\mu; \mu)$ and called it *fuzzy Cayley graph*. In fact, the fuzzy Cayley graph $(G; S_\mu; \mu)$ is a Cayley graph where the fuzzy Cayley subset μ constructs the Cayley subset of it.

The next lemma yields that if $\mu(a) \neq \mu(b)$, then $\mu(ab) = \min\{\mu(a), \mu(b)\}$, for some $a, b \in G$, when μ is a fuzzy subgroup on G.

Lemma 1. Suppose that μ is a fuzzy subgroup on G. If $\mu(a) \neq \mu(b)$ then $\mu(ab) = \min\{\mu(a), \mu(b)\}$, for every $a, b \in G$.

Proof. Without less of generality, suppose that $\mu(b) > \mu(a)$. Since μ is a fuzzy subgroup, we get $\mu(a) = \mu(abb^{-1}) \geq \min\{\mu(ab), \mu(b^{-1})\}$. Since $\mu(b^{-1}) = \mu(b)$ and $\mu(b) > \mu(a)$, the last argument yields that $\mu(a) \geq \mu(ab)$. On the other hand, $\mu(ab) \geq \min\{\mu(a), \mu(b)\} = \mu(a)$. Therefore, $\mu(ab) = \mu(a) = \min\{\mu(a), \mu(b)\}$. Similarity, if $\mu(b) < \mu(a)$, then $\mu(ab) = \mu(b)$. Thus, we have $\mu(ab) = \min\{\mu(a), \mu(b)\}$.

Lemma 2. Suppose that μ_1 and μ_2 are fuzzy Cayley subsets on a group G. The following properties hold.
(1) If $\mu_1 \subseteq \mu_2$, then $S_{\mu_2} \subseteq S_{\mu_1}$.
(2) The fuzzy subset $\mu_1 \cup \mu_2$ is a fuzzy Cayley subset and $S_{\mu_1 \cup \mu_2} = S_{\mu_1} \cap S_{\mu_2}$.
(3) The fuzzy subset $\mu_1 \cap \mu_2$ is a fuzzy Cayley subset and $S_{\mu_1 \cap \mu_2} = S_{\mu_1} \cup S_{\mu_2}$.

Proof. (1) If $x \notin S_{\mu_1}$, then $\mu_1(x) = 1$. Since $\mu_1 \leq \mu_2$, we have $\mu_2(x) = 1$, and, thus, $x \notin S_{\mu_2}$. Therefore, $S_{\mu_2} \subseteq S_{\mu_1}$.

(2) It easily can be verified that $S_{\mu_1 \cup \mu_2} = S_{\mu_1} \cap S_{\mu_2}$. Now, suppose that $x \in S_{\mu_1 \cup \mu_2}$. Then $x \in S_{\mu_1} \cap S_{\mu_2}$. Since μ_1 and μ_2 are fuzzy Cayley subsets, we have $x^{-1} \in S_{\mu_1} \cap S_{\mu_2}$ and, thus, $x^{-1} \in S_{\mu_1 \cup \mu_2}$. Similarly, if $1 \in S_{\mu_1 \cup \mu_2}$, then $1 \in S_{\mu_1} \cap S_{\mu_2}$, a contradiction. Therefore, $\mu_1 \cup \mu_2$ is a fuzzy Cayley subset.

(3) In a similar way as last part.

Lemma 3. Let $X_1 = (G; S_1)$ and $X_2 = (G; S_2)$ be Cayley graphs. The following properties hold.
(1) $X_1 \cup X_2 = (G; S_1 \cup S_2)$.
(2) $X_1 \cap X_2 = (G; S_1 \cap S_2)$.
(3) $X_1 \subseteq X_2$ if and only if $S_1 \subseteq S_2$.

Proof. (1) Let e be an edge of $(G; S_1 \cup S_2)$. Then there exist $g \in G$ and $s \in S_1 \cup S_2$ such that e is an edge between two vertices g and gs. Since $s \in S_1 \cup S_2$, we have $s \in S_1$ or $s \in S_2$ and, thus, $e \in E(X_1)$ or $e \in E(X_2)$. Therefore, $e \in E(X_1 \cup X_2)$. Similarly, any edge of $E(X_1 \cup X_2)$ is an edge of $(G; S_1 \cup S_2)$. The result follows.

(2) In a similar way as last part.

(3) Suppose that $S_1 \subseteq S_2$. If $e \in E(X_1)$, then there exist elements $g \in G$ and $s_1 \in S_1$ such that $e = (g, gs_1)$. Since $s_1 \in S_1$ and $S_1 \subseteq S_2$, we obtain $e \in E(X_2)$. Therefore, $X_1 \subseteq X_2$. Now, suppose that $E(X_1) \subseteq E(X_2)$. Let $g \in G$. If $s_1 \in S_1$, then $(g, gs_1) \in E(X_1)$. Therefore, $(g, gs_1) \in E(X_2)$. Then $(g, gs_1) = (g', g's'_1)$ for some $g' \in G$ and $s'_1 \in S_2$. Since $g = g'$, we obtain $s_1 = s'_1$ and, thus, $s_1 \in S_2$. The result follows.

Notice that, if $V(X_1) = V(X_2)$ then $X_1 \cup X_2$ and $X_1 \cap X_2$ are obviously Cayley graphs. The Lemma 2 follows us to define subgraph, union and intersection of fuzzy Cayley graphs.

Definition 1. Suppose that $X = (G; S_\mu; \mu)$ and $Y = (G; S_\lambda; \lambda)$ are fuzzy Cayley graphs. Then
(1) $X \subseteq Y$ if and only if $\lambda \subseteq \mu$;
(2) $X \cup Y = (G; S_\mu \cup S_\lambda; \mu \cap \lambda)$;
(3) $X \cap Y = (G; S_\mu \cap S_\lambda; \mu \cup \lambda)$.

Lemma 4. Suppose that G is a finite group and μ is a fuzzy Cayley subset on G. If μ is a fuzzy subgroup and $S_\mu \neq \emptyset$ then S_μ generates G.

Proof. Suppose that $g \in G$. If $g \notin S_\mu$, then $\mu(g) = 1$. Now, if $a \in S_\mu$, then $\mu(a) < 1$. By Lemma 1, $\mu(ga^{-1}) = \mu(a)$. It follows that $ga^{-1} \in S_\mu$ and, thus, $g = ga^{-1}a \in \langle S_\mu \rangle$. Therefore, $G = \langle S_\mu \rangle$.

The following theorem is easily verified by Lemma 4.

Theorem 5. Suppose that $X = (G; S_\mu; \mu)$ is a fuzzy Cayley graph. If μ is a fuzzy subgroup, then the Cayley graph $(G; S_\mu)$ is connected.

3 *Fuzzy rough Cayley graphs*

Suppose that G is a finite group with identity 1_G, μ is a fuzzy normal subgroup, $0 \leq t \leq 1$, and $X = (G; S)$ is a Cayley graph. Then the following graphs (we will prove these graphs are Cayley graphs)

$$\overline{X}_{\mu_t} = (G; \mu_{t^\wedge}(S)^*) \ \ (\mu_{t^\wedge}(S)^* = \mu_{t^\wedge}(S) \setminus \{1_G\}) \ \text{ and } \ \underline{X}_{\mu_t} = (G; \mu_{t-}(S))$$

are called, respectively, *fuzzy upper and lower approximations* of the Cayley graph X with respect to the fuzzy normal subgroup μ and integer t.

Theorem 6. $\underline{X}_{\mu_t}$ and $\overline{X}_{\mu_t}$ are Cayley graphs.

Proof. By Theorem 1(1), we have $\mu_{t^-}(S) \subseteq S$, and, thus, $1_G \notin \mu_{t^-}(S)$. Suppose that $s \in \mu_{t^-}(S)$. Then $[s]_\mu \subseteq S$. If $x \in [s^{-1}]_\mu$ then $(x, s^{-1}) \in \mu_t$ and, thus, $(x^{-1}, s) \in \mu_t$, because μ is a fuzzy normal subgroup. Thus $x^{-1} \in [s]_\mu \subseteq S$. Since S is a Cayley set, we obtain $x \in S$ and, thus, $[s^{-1}]_\mu \subseteq S$. Hence, $s^{-1} \in \mu_{t^-}(S)$. Therefore, $\mu_{t^-}(S)$ is a Cayley set, and $\underline{X}_{\mu_t}$ is a Cayley graph.

Now, suppose that $s \in \mu_{t^\wedge}(S)^*$. Then $[s]_\mu \cap S \neq \emptyset$ which implies that there exists $a \in [s]_\mu \cap S$. Since $a \in [s]_\mu \cap S$, we obtain $(a, s) \in \mu_t$. As μ is a fuzzy normal subgroup, $(a^{-1}, s^{-1}) \in \mu_t$. Then $a^{-1} \in [s^{-1}]_\mu$. Since S is a Cayley set, we have $[s^{-1}]_\mu \cap S \neq \emptyset$ and, thus, $s^{-1} \in \mu_{t^\wedge}(S)$. Therefore, $\mu_{t^\wedge}(S)^*$ is a Cayley set, and $\overline{X}_{\mu_t}$ is a Cayley graph.

Let G be a group congruence modulo 16 integral number $\mathbb{Z}$. Let B be a fuzzy normal subgroup of G presented in Table, and t be 0.3. Let $X = (G; S)$ be a Cayley graph such that S equals to $\{\underline{1}, \underline{2}, \underline{6}, \underline{10}, \underline{14}, \underline{15}\}$. The congruence relation $B_{0.3}$ partitions G to four classes $\{0, 4, 8, 12\}, \{1, 5, 9, 13\}$, $\{2, 6, 10, 14\}$ and $\{3, 7, 11, 15\}$. Then we have

$$\overline{X}_{B_{0.3}} = (G; \{\underline{1}, \underline{2}, \underline{3}, \underline{5}, \underline{6}, \underline{7}, \underline{9}, \underline{10}, \underline{11}, \underline{13}, \underline{14}, \underline{15}\}) \text{ and } \underline{X}_{B_{0.3}} = (G; \{\underline{2}, \underline{6}, \underline{10}, \underline{14}\}).$$

Table

The fuzzy normal subgroup B

$B(\underline{1}) = 0.1$	$B(\underline{2}) = 0.2$	$B(\underline{3}) = 0.1$	$B(\underline{4}) = 0.4$
$B(\underline{5}) = 0.1$	$B(\underline{6}) = 0.2$	$B(\underline{7}) = 0.1$	$B(\underline{8}) = 0.8$
$B(\underline{9}) = 0.1$	$B(\underline{10}) = 0.2$	$B(\underline{11}) = 0.1$	$B(\underline{12}) = 0.4$
$B(\underline{13}) = 0.1$	$B(\underline{14}) = 0.2$	$B(\underline{15}) = 0.1$	$B(\underline{0}) = 1$

Theorem 7. Suppose that μ and λ are fuzzy normal subgroups of a group G and $t \in [0, 1]$. Let $X = (G; S)$, $X_1 = (G; S_1)$ and $X_2 = (G; S_2)$ be Cayley graphs. The following properties hold.

(1) $\underline{X}_{\mu_t} \subseteq X \subseteq \overline{X}_{\mu_t}$,
(2) $\overline{X_1 \cup X_2}_{\mu_t} = \overline{X_1}_{\mu_t} \cup \overline{X_2}_{\mu_t}$,
(3) $\underline{X_1 \cap X_2}_{\mu_t} = \underline{X_1}_{\mu_t} \cap \underline{X_2}_{\mu_t}$,
(4) $X_1 \subseteq X_2 \Rightarrow \underline{X_1}_{\mu_t} \subseteq \underline{X_2}_{\mu_t}$,
(5) $X_1 \subseteq X_2 \Rightarrow \overline{X_1}_{\mu_t} \subseteq \overline{X_2}_{\mu_t}$,
(6) $\underline{X_1 \cup X_2}_{\mu_t} \supseteq \underline{X_1}_{\mu_t} \cup \underline{X_2}_{\mu_t}$,
(7) $\overline{X_1 \cap X_2}_{\mu_t} \subseteq \overline{X_1}_{\mu_t} \cap \overline{X_2}_{\mu_t}$,
(8) $\mu_t \subseteq \lambda_t \Rightarrow \overline{X}_{\mu_t} \subseteq \overline{X}_{\lambda_t}$,
(9) $\mu_t \subseteq \lambda_t \Rightarrow \underline{X}_{\lambda_t} \subseteq \underline{X}_{\mu_t}$,
(10) $\overline{X}_{(\mu \cap \lambda)_t} \subseteq \overline{X}_{\mu_t} \cap \overline{X}_{\lambda_t}$,
(11) $\underline{X}_{(\mu \cap \lambda)_t} \supseteq \underline{X}_{\mu_t} \cap \underline{X}_{\lambda_t}$.

Proof. (1) By Theorem 1(1), $\mu_{t^-}(S) \subseteq S \subseteq \mu_{t^\wedge}(S)$. Then $\mu_{t^-}(S) \subseteq S \subseteq \mu_{t^\wedge}(S)^*$. It follows that $\underline{X}_{\mu_t} \subseteq X \subseteq \overline{X}_{\mu_t}$.

(2) Based on Lemma 3, $\overline{X_1} \cup \overline{X_2} = (G; \mu_{t^\wedge}(S_1)^* \cup \mu_{t^\wedge}(S_2)^*)$. By Theorem 1(5), we have $\mu_{t^\wedge}(S_1)^*$ and $\mu_{t^\wedge}(S_2)^* \subseteq \mu_{t^\wedge}(S_1 \cup S_2)^*$. Now, by Lemma 3(3), we have $\overline{X_1}_{\mu_t} \cup \overline{X_2}_{\mu_t} \subseteq \overline{X_1 \cup X_2}_{\mu_t}$. Conversely, by Theorem 1(3), $\mu_{t^\wedge}(S_1)^* \cup \mu_{t^\wedge}(S_2)^* = \mu_{t^\wedge}(S_1 \cup S_2)^*$. Suppose that (g, gs) is an edge of $E(\overline{X_1 \cup X_2}_{\mu_t})$. It follows that $s \in \mu_{t^\wedge}(S_1 \cup S_2)^*$. Then $s \in \mu_{t^\wedge}(S_1)^* \cup \mu_{t^\wedge}(S_2)^*$ and, thus, $s \in \mu_{t^\wedge}(S_1)^*$ or $s \in \mu_{t^\wedge}(S_2)^*$. Therefore, (g, gs) is an edge of $\overline{X_1}_{\mu_t}$ or $\overline{X_2}_{\mu_t}$. Finally, we have $\overline{X_1 \cup X_2}_{\mu_t} = \overline{X_1}_{\mu_t} \cup \overline{X_2}_{\mu_t}$.

(3) By Theorem 1(2), the proof is similar to part (2).

(4) Assume that $X_1 \subseteq X_2$. Then $S_1 \subseteq S_2$ and, thus, $\mu_{t-}(S_1) \subseteq \mu_{t-}(S_2)$. Hence, $\underline{X_1}_{\mu_t} \subseteq \underline{X_2}_{\mu_t}$.

(5) By Theorem 1(5), the proof is similar to part (4).

(6) By Theorem 1(6), we have $\mu_{t-}(S_1) \cup \mu_{t-}(S_2) \subseteq \mu_{t-}(S_1 \cup S_2)$. Then $\mu_{t-}(S_1) \subseteq \mu_{t-}(S_1 \cup S_2)$ and $\mu_{t-}(S_2) \subseteq \mu_{t-}(S_1 \cup S_2)$. Therefore, we obtain $\underline{X_1 \cup X_2}_{\mu_t} \supseteq \underline{X_1}_{\mu_t}$ and $\underline{X_1 \cup X_2}_{\mu_t} \supseteq \underline{X_2}_{\mu_t}$. And finally, $\underline{X_1 \cup X_2}_{\mu_t} \supseteq \underline{X_1}_{\mu_t} \cup \underline{X_2}_{\mu_t}$.

(7) By Theorem 1(7), the proof is similar to part (6).

(8) Assume that $\mu_t \subseteq \lambda_t$. Theorem 1(9) yields $\mu_{t^\wedge}(S) \subseteq \lambda_{t^\wedge}(S)$. Then $\mu_{t^\wedge}(S)^* \subseteq \lambda_{t^\wedge}(S)^*$ and, thus, $\overline{X}_{\mu_t} \subseteq \overline{X}_{\lambda_t}$.

(9) By Theorem 1(8), the proof is similar to part (8).

(10) By Theorem 1(12), we have

$$\begin{aligned}\overline{X}_{(\mu\cap\lambda)_t} &= (G; (\mu \cap \lambda)_{t^\wedge}(S)) \\ &\subseteq (G; \mu_{t^\wedge}(S) \cap \lambda_{t^\wedge}(S)) \\ &= (G; \mu_{t^\wedge}(S)) \cap (G; \lambda_{t^\wedge}(S)) \\ &= \overline{X}_{\mu_t} \cap \overline{X}_{\lambda_t}.\end{aligned}$$

(11) By Theorem 1(12), the proof is similar to part (11).

Remark 1. A subset S of G is a minimal Cayley set if it generates G and if $S \setminus \{s, s^{-1}\}$ generates a proper subgroup of G for all $s \in S$.

The pair $(\underline{X}_{\mu_t}, \overline{X}_{\mu_t})$ is called a *fuzzy rough set* of the Cayley graph X. A Cayley graph $X = (G; S)$ is called a $\mu_{t^\wedge}$*-fuzzy rough generating*, if the subset $\mu_{t^\wedge}(S)^*$ is a generating set for G. Similarly, a Cayley graph $X = (G; S)$ is called an μ_{t-}*-fuzzy rough generating*, if the subset $\mu_{t-}(S)$ is a generating set for G. A Cayley graph $X = (G; S)$ is called a $\mu_{t^\wedge}$*-fuzzy rough optimal connected*, if the subset $\mu_{t^\wedge}(S)^*$ is a minimal Cayley set for G. Similarly, a Cayley graph $X = (G; S)$ is called a μ_{t-}*-fuzzy rough optimal connected*, if the subset $\mu_{t-}(S)$ is a minimal Cayley set for G.

Theorem 8. Suppose that $X = (G; S)$ is a Cayley graph.
(1) If X is a $\mu_{t^\wedge}$-fuzzy rough generating, then $\overline{X}_{\mu_t}$ is connected.
(2) If X is a μ_{t-}-fuzzy rough generating, then $\underline{X}_{\mu_t}$ is connected.
(3) If X is a $\mu_{t^\wedge}$-fuzzy rough optimal connected, then $\overline{X}_{\mu_t}$ is optimal connected.
(4) If X is a μ_{t-}-fuzzy rough optimal connected, then $\underline{X}_{\mu_t}$ is optimal connected.

Proof. It is straightforward.

4 Rough fuzzy Cayley graphs

Let G be a finite group with identity 1_G, B a fuzzy normal subgroup on G and $X = (G; S_\mu; \mu)$ be a fuzzy Cayley graph. The following fuzzy Cayley graphs (we will prove these are fuzzy Cayley graphs)

$$\overline{X}_B = (G; S_{\underline{A}_B\mu^\star}; \underline{A}_B\mu^\star) \text{ and } \underline{X}_B = (G; S_{\overline{A}_B\mu}; \overline{A}_B\mu)$$

are called, respectively, *lower and upper approximations* of the fuzzy Cayley graph X with respect to B. In the above definition, $\underline{A}_B\mu^\star(x)$ is similar to $\underline{A}_B\mu(x)$ in all elements, except for 1_G, where $\underline{A}_B\mu^\star(1_G)$ is 1.

Theorem 9. The triples $\underline{X}_B$ and $\overline{X}_B$ are fuzzy Cayley graphs.

Proof. Suppose that $\underline{A}_B\mu(x) = 1$ for some $x \in [0,1]$. Thus

$$\inf_{u\in G} \vartheta_{\min}(B(u,x),\mu(u)) = 1.$$

Therefore, for all elements $u \in G$, $\vartheta_{\min}(B(u,x),\mu(u)) = 1$ and, thus, $B(ux^{-1}) \le \mu(u)$ for every $u \in G$. On the other hand, μ is a fuzzy subgroup, and we have $\mu(u^{-1}) = \mu(u)$. Then $B(ux^{-1}) \le \mu(u^{-1})$. Since B is a fuzzy normal subgroup, we obtain $B(ux^{-1}) = B(x^{-1}u)$ and consequently, are equal to $B(u^{-1}x)$. So $B(u^{-1}x) \le \mu(u^{-1})$. Hence for all u of G, $\vartheta_{\min}(B(u^{-1},x^{-1}),\mu(u^{-1})) = 1$ and, thus,

$$\inf_{u\in G} \vartheta_{\min}(B(u^{-1},x^{-1}),\mu(u^{-1})) = 1.$$

Then

$$\inf_{u\in G} \vartheta_{\min}(B(u,x^{-1}),\mu(u)) = 1.$$

So $\underline{A}_B\mu(x^{-1}) = 1$. Therefore, $\underline{A}_B\mu^\star$ is a fuzzy Cayley subset and $\overline{X}_B$ is a fuzzy Cayley graph.

Theorem 3(1) leads $\mu \subseteq \overline{A}_B\mu$. Since $\mu(1_G) = 1$, we obtain $\overline{A}_B\mu(1_G) = 1$. Now suppose that $\overline{A}_B\mu(x) = 1$. Then

$$\sup_{u\in G}\{\min\{B(ux^{-1}),\mu(u)\}\} = 1.$$

Since G is finite, there exists an element u of G such that $\min\{B(ux^{-1}),\mu(u)\} = 1$. Then $B(ux^{-1}) = \mu(u) = 1$. Since μ is a fuzzy subgroup, we obtain $\mu(u^{-1}) = \mu(u)$. Now as B is a fuzzy normal subgroup, $B(ux^{-1}) = B(x^{-1}u)$ and since B is a fuzzy subgroup, we obtain $B(ux^{-1}) = B(u^{-1}x)$. Thus $\min\{B(u^{-1}x),\mu(u^{-1})\} = 1$, and $\overline{A}_B\mu(x^{-1}) = 1$. Consequently, $\overline{A}_B\mu$ is a fuzzy Cayley subset and, thus, $\underline{X}_B$ is a fuzzy Cayley graph.

Lemma 5. Suppose that G is a finite group and B is a fuzzy normal subgroup of G. If $X = (G; S_\mu; \mu)$ and $Y = (G; S_\lambda; \lambda)$ are fuzzy Cayley graphs, then:

(1) $S_{\underline{A}_B(\mu\cup\lambda)^\star} \subseteq S_{\underline{A}_B\mu^\star} \cap S_{\underline{A}_B\lambda^\star}$,
(2) $S_{\underline{A}_B(\mu\cap\lambda)^\star} = S_{\underline{A}_B\mu^\star} \cup S_{\underline{A}_B\lambda^\star}$,
(3) $S_{\overline{A}_B(\mu\cup\lambda)} = S_{\overline{A}_B\mu} \cap S_{\overline{A}_B\lambda}$,
(4) $S_{\overline{A}_B(\mu\cap\lambda)} \supseteq S_{\overline{A}_B\mu} \cup S_{\overline{A}_B\lambda}$.

Proof. (1) Suppose that $x \in S_{\underline{A}_B(\mu\cup\lambda)^\star}$. Then $\underline{A}_B(\mu\cup\lambda)^\star(x) < 1$ and $x \ne 1_G$. By Theorem 3(7), $\underline{A}_B\mu(x), \underline{A}_B\lambda(x) < 1$. Hence, $x \in S_{\underline{A}_B\mu^\star} \cap S_{\underline{A}_B\lambda^\star}$.

According to Theorem 3, items (2), (3) and (4) are straightforward.

Theorem 10. Suppose that G is a finite group and B and C are fuzzy normal subgroups of G. Let $X = (G; S_\mu; \mu)$ and $Y = (G; S_\lambda; \lambda)$ be fuzzy Cayley graphs. Then

(1) $\underline{X}_B \subseteq X \subseteq \overline{X}_B$,
(2) $\overline{X \cup Y}_B = \overline{X}_B \cup \overline{Y}_B$,
(3) $\overline{X \cap Y}_B \subseteq \overline{X}_B \cap \overline{Y}_B$,
(4) $\underline{X \cup Y}_B \supseteq \underline{X}_B \cup \underline{Y}_B$,
(5) $\underline{X \cap Y}_B = \underline{X}_B \cap \underline{Y}_B$,
(6) $\mu \subseteq \lambda \Rightarrow \underline{Y}_B \subseteq \underline{X}_B$,
(7) $\mu \subseteq \lambda \Rightarrow \overline{Y}_B \subseteq \overline{X}_B$,
(8) $B \subseteq C \Rightarrow \underline{X}_C \subseteq \underline{X}_B$,
(9) $B \subseteq C \Rightarrow \overline{X}_B \subseteq \overline{X}_C$.

Proof. (1) By Theorem 3(1), we have $\underline{A}_B\mu \subseteq \mu \subseteq \overline{A}_B\mu$. Hence $\underline{A}_B\mu^\star \subseteq \mu \subseteq \overline{A}_B\mu$. Lemma 2(1) implies that $\underline{X}_B \subseteq X \subseteq \overline{X}_B$.

(2) By Definition 1(2), $X \cup Y = (G; S_{\mu\cap\lambda}; \mu \cap \lambda)$. Then we have

$$\overline{X \cup Y}_B = (G; S_{\underline{A}_B(\mu\cap\lambda)^\star}; \underline{A}_B(\mu \cap \lambda)^\star).$$

By Theorem 3(8), $\underline{A}_B(\mu \cap \lambda) = \underline{A}_B\mu \cap \underline{A}_B\lambda$ and, thus,

$$\overline{X \cup Y}_B = (G; S_{\underline{A}_B\mu^\star\cap\underline{A}_B\lambda^\star}; \underline{A}_B\mu^\star \cap \underline{A}_B\lambda^\star).$$

Now by 1(2), $\overline{X \cup Y}_B = \overline{X}_B \cup \overline{Y}_B$. The result follows.

(3) By Theorem 3(7), the proof is similar to part (2).

(4) By Theorem 3(6), the proof is similar to part (2).

(5) By Theorem 3(5), the proof is similar to part (2).

(6) If $\mu \subseteq \lambda$, then by Corollary 1(1), $\overline{A}_B\mu \subseteq \overline{A}_B\lambda$. Now, by Definition 1(1), $\underline{Y}_B \subseteq \underline{X}_B$.

(7) By Corollary 1(2), the proof is similar to part (6).

(8) Assume that $B \subseteq C$. By Theorem 3(9), $\overline{A}_B\mu \subseteq \overline{A}_C\mu$. Therefore, we have $\underline{X}_C \subseteq \underline{X}_B$.

(9) According to Theorem 3(10), the proof is similar to part (8).

Theorem 11. Suppose that G is a finite group. If B and C are fuzzy normal subgroups and μ is a fuzzy subset on G, then the following statement hold.

(1) $(G; S_{\underline{A}_{BMinC}\mu}; \underline{A}_{BMinC}\mu) \subseteq (G; S_{\underline{A}_B\mu Min\underline{A}_C\mu}; \underline{A}_B\mu Min\underline{A}_C\mu)$,

(2) $(G; S_{\overline{A}_B\mu Min\overline{A}_C\mu}; \overline{A}_B\mu Min\overline{A}_C\mu) \subseteq (G; S_{\overline{A}_{BMinC}\mu}; \overline{A}_{BMinC}\mu)$.

Proof. According to Theorem 4, the proof of both parts are clear.

The pair $(\underline{X}_B, \overline{X}_B)$ is called a *rough set* of a fuzzy Cayley graph $X = (G; S_\mu; \mu)$. A fuzzy Cayley graph $X = (G; S_\mu; \mu)$ is called an $\overline{A}_B$ *rough generating*, if the subset $S_{\overline{A}_B\mu}$ generates G. Likewise a fuzzy Cayley graph $X = (G; S_\mu; \mu)$ is called an $\underline{A}_B$ *rough generating*, if the subset $S_{\underline{A}_B\mu}$ generates G. A fuzzy Cayley graph $X = (G; S_\mu; \mu)$ is called an $\overline{A}_B$ *rough optimal connected*, if the subset $S_{\overline{A}_B\mu}$ is a minimal Cayley set of G. Similarly a fuzzy Cayley graph $X = (G; S_\mu; \mu)$ is called an $\underline{A}_B$ *rough optimal connected*, if the subset $S_{\underline{A}_B\mu}$ is a minimal Cayley set of G.

Theorem 12. Suppose that G is a finite group, and B is a fuzzy normal subgroup of G. Let $X = (G; S_\mu; \mu)$ be a fuzzy Cayley graph. The following properties hold.

(1) If μ is a fuzzy subgroup of G, then X is a $\overline{A}_B$ rough generating.

(2) If $B \subseteq \mu$ and μ is a fuzzy subgroup of G, then X is a $\underline{A}_B$ rough generating.

Proof. According to Theorems 4 and 4, the proof is straightforward.

5 *Fuzzy rough fuzzy Cayley graphs*

In this section, we get the t-level relation μ_t, for each $t \in [0,1)$, as follows:

$$\mu_t = \{(a,b) \in G \times G \mid \mu(ab^{-1}) > t\}.$$

Similarly, all results related to the t-level relation μ_t are same. Let B be a fuzzy normal subgroup on G and $X = (G; S_\mu; \mu)$ be a fuzzy Cayley graph. The following fuzzy Cayley graphs (we will prove these are fuzzy Cayley graphs)

$$\underline{X}'_B = (G; B_{t_\mu -}(S_\mu); \overline{A}_B\mu^\sharp) \text{ and } \overline{X}'_B = (G; B^\wedge_{t_\mu}(S_\mu)^\star; \underline{A}_B\mu^\sharp)$$

are called, respectively, *fuzzy lower and upper approximations* of the fuzzy Cayley graph X with respect to B. The definitions of t_μ, $\underline{A}_B\mu^\sharp$ and $\overline{A}_B\mu^\sharp$ are as follows:

$$t_\mu = \max\{\mu(x) \mid x \in S_\mu\},$$

$$\text{if } x \in B^\wedge_{t_\mu}(S_\mu)^\star \text{ then } \underline{A}_B\mu^\sharp(x) = \underline{A}_B\mu^\star, \text{ otherwise } \underline{A}_B\mu^\sharp(x) = 1 \text{ and}$$

$$\text{if } x \in B_{t_\mu -}(S_\mu) \text{ then } \overline{A}_B\mu^\sharp(x) = \overline{A}_B\mu(x), \text{ otherwise } \overline{A}_B\mu^\sharp(x) = 1.$$

Theorem 13. The triples $\underline{X}'_B$ and $\overline{X}'_B$ are fuzzy Cayley graphs.

Proof. In the proof Theorem 6, it proved that the subsets $B_{t_\mu -}(S_\mu)$ and $B^\wedge_{t_\mu}(S_\mu)^\star$ are Cayley sets. To prove that the $\underline{X}'_B$ and $\overline{X}'_B$ are fuzzy Cayley graphs, it is sufficient to show that $B_{t_\mu -}(S_\mu) = S_{\overline{A}_B\mu^\sharp}$ and $B^\wedge_{t_\mu}(S_\mu)^\star = S_{\underline{A}_B\mu^\sharp}$.

Suppose that $x \in B_{t_\mu -}(S_\mu)$. Then $\overline{A}_B\mu^\sharp(x) = \overline{A}_B\mu(x)$. If $\overline{A}_B\mu(x) = 1$, then

$$\sup_{u\in G}\{\min\{B(ux^{-1}), \mu(u)\}\} = 1.$$

Since G is finite, there exists an element u in G where $\min\{B(ux^{-1}), \mu(u)\} = 1$ and, thus, $B(ux^{-1}) = \mu(u) = 1$. As $B(ux^{-1}) = 1$, we h obtain $u \in [x]_B$ and, thus, $u \in S_\mu$. Therefore, $\mu(u) < 1$, a contradiction. Then $\overline{A}_B\mu(x) \neq 1$ and, as a result, $\overline{A}_B\mu^\sharp(x) \neq 1$. Now, suppose that $x \notin B_{t_\mu -}(S_\mu)$. Based on the definition, $\overline{A}_B\mu^\sharp(x) = 1$. Therefore, $B_{t_\mu -}(S_\mu) = S_{\overline{A}_B\mu^\sharp}$.

Let x be in $B^\wedge_{t_\mu}(S_\mu)^\star$. Then $\underline{A}_B\mu^\sharp(x) = \underline{A}_B\mu^\star$. Since $x \in B^\wedge_{t_\mu}(S_\mu)^\star$, there exists an element $y \in S_\mu$ such that $y \in [x]_B$. If we have $\underline{A}_B\mu(x) = 1$, then

$$\inf_{u\in G} \vartheta_{\min}(B(ux^{-1}), \mu(u)) = 1.$$

Therefore, we have $B(ux^{-1}) \leq \mu(u)$ for every $u \in G$. Then $B(yx^{-1}) \leq \mu(y)$. Since $\mu(y) \leq t_\mu$, we obtain $B(yx^{-1}) \leq t_\mu$. As $y \in [x]_B$, $B(yx^{-1}) > t_\mu$, a contradiction. Now, suppose that $x \notin B^\wedge_{t_\mu}(S_\mu)^\star$. Based on the definition, $\underline{A}_B\mu^\sharp(x) = 1$. Hence, by above $B^\wedge_{t_\mu}(S_\mu)^\star = S_{\underline{A}_B\mu^\sharp(x)}$.

Lemma 6. Let G be a group and t_1, t_2 and t_3 be integers in the closed interval $[0,1]$. Suppose that μ is fuzzy normal subgroups of G. Let A and B be two non-empty sets. Then

(1) if $t_3 \leq t_1, t_2$, then $\mu_{t_3^\wedge}(A \cup B) \supseteq \mu_{t_1^\wedge}(A) \cup \mu_{t_2^\wedge}(B)$,
(2) if $t_3 \geq t_1, t_2$, then $\mu_{t_3^\wedge}(A \cup B) \subseteq \mu_{t_1^\wedge}(A) \cup \mu_{t_2^\wedge}(B)$,
(3) if $t_3 \geq t_1, t_2$, then $\mu_{t_3^\wedge}(A \cap B) \subseteq \mu_{t_1^\wedge}(A) \cap \mu_{t_2^\wedge}(B)$,
(4) if $t_3 \leq t_1, t_2$, then $\mu_{t_3 -}(A \cap B) \subseteq \mu_{t_1 -}(A) \cap \mu_{t_2 -}(B)$,
(5) if $t_3 \geq t_1, t_2$, then $\mu_{t_3 -}(A \cap B) \supseteq \mu_{t_1 -}(A) \cap \mu_{t_2 -}(B)$,
(6) if $t_3 \geq t_1, t_2$, then $\mu_{t_3 -}(A \cup B) \supseteq \mu_{t_1 -}(A) \cup \mu_{t_2 -}(B)$.

Proof. (1) Let $x \in \mu_{t_1^\wedge}(A) \cup \mu_{t_2^\wedge}(B)$. Then $x \in \mu_{t_1^\wedge}(A)$ or $x \in \mu_{t_2^\wedge}(B)$. Suppose that $x \in \mu_{t_1^\wedge}(A)$. Thus, $[x]_{\mu_{t_1}} \cap A \neq \emptyset$ and consequently, there exists $a \in A$ such that $\mu(xa^{-1}) > t_1$. Since $t_1 \geq t_3$, we have $\mu(xa^{-1}) > t_3$ and, thus, $[x]_{\mu_{t_3}} \cap A \neq \emptyset$. The result gives us that $x \in \mu_{t_3^\wedge}(A \cup B)$. Similarity, if $x \in \mu_{t_2^\wedge}(B)$, the same result can be gained.

(2) Let $x \in \mu_{t_3^\wedge}(A \cup B)$. Thus, $[x]_{\mu_{t_3}} \cap (A \cup B) \neq \emptyset$. Then $[x]_{\mu_{t_3}} \cap A \neq \emptyset$ or $[x]_{\mu_{t_3}} \cap B \neq \emptyset$. Suppose that $[x]_{\mu_{t_3}} \cap A \neq \emptyset$. Hence, there exists $a \in A$ such that $\mu(xa^{-1}) > t_3$. Since $t_3 \geq t_1$, we have $\mu(xa^{-1}) > t_1$ and, thus, $[x]_{\mu_{t_1}} \cap A \neq \emptyset$. The result gives us that $x \in \mu_{t_1^\wedge}(A)$. The result follows.

(3) Let $x \in \mu_{t_3^\wedge}(A \cap B)$. Then, $[x]_{\mu_{t_3}} \cap (A \cap B) \neq \emptyset$ and, thus, $[x]_{\mu_{t_3}} \cap A \neq \emptyset$ and $[x]_{\mu_{t_3}} \cap B \neq \emptyset$. Hence, there exist elements $a \in A$ and $b \in B$ such that $\mu(xa^{-1}) > t_3$ and $\mu(xb^{-1}) > t_3$. Since $t_3 \geq t_1, t_2$, we have $\mu(xa^{-1}) > t_1$ and $\mu(xb^{-1}) > t_2$ and, thus, $[x]_{\mu_{t_1}} \cap A \neq \emptyset$ and $[x]_{\mu_{t_2}} \cap B \neq \emptyset$. The result gives us that $\mu_{t_1^\wedge}(A) \cap \mu_{t_2^\wedge}(B)$. The result follows.

(4) Let $x \in \mu_{t_3-}(A \cap B)$. Then $[x]_{\mu_{t_3}} \subseteq A \cap B$. If $y \in [x]_{\mu_{t_1}}$, then $\mu(yx^{-1}) > t_1$ and, thus, $\mu(yx^{-1}) > t_3$. Then $y \in A$. It follows that $x \in \mu_{t_1-}(A)$. Similarly, we have $x \in \mu_{t_2-}(B)$.

(5) Let $x \in \mu_{t_1-}(A) \cap \mu_{t_2-}(B)$. Then $[x]_{\mu_{t_1}} \subseteq A$ and $[x]_{\mu_{t_2}} \subseteq B$. If $y \in [x]_{\mu_{t_3}}$, then $\mu(yx^{-1}) > t_3$ and, thus, $\mu(yx^{-1}) > t_1$ and $\mu(yx^{-1}) > t_2$. Then $y \in A \cap B$. It follows that $x \in \mu_{t_3-}(A \cap B)$.

(6) Let $x \in \mu_{t_1-}(A) \cup \mu_{t_2-}(B)$. Hence, $x \in \mu_{t_1-}(A)$ or $x \in \mu_{t_2-}(B)$. Suppose that $x \in \mu_{t_1-}(A)$. Hence, $[x]_{\mu_{t_1}} \subseteq A$. If $y \in [x]_{\mu_{t_3}}$, then $\mu(yx^{-1}) > t_3$ and, thus, $\mu(yx^{-1}) > t_1$. Then $y \in A$. It follows that $x \in \mu_{t_3-}(A \cup B)$. The result follows.

Theorem 14. Let G be a finite group. Taking any fuzzy normal subgroups B and C on G. If $X = (G; S_\mu; \mu)$ and $Y = (G; S_\lambda; \lambda)$ are fuzzy Cayley graphs. The following properties hold.

(1) $\underline{X}'_B \subseteq X \subseteq \overline{X}'_B$,
(2) $\overline{X \cap Y}'_B \subseteq \overline{X}'_B \cap \overline{Y}'_B$,
(3) $\overline{X \cup Y}'_B \supseteq \overline{X}'_B \cup \overline{Y}'_B$,
(4) $\underline{X \cap Y}'_B \subseteq \underline{X}'_B \cap \underline{Y}'_B$,
(5) $\mu \subseteq \lambda \Rightarrow \underline{Y}'_B \subseteq \underline{X}'_B$,
(6) $\mu \subseteq \lambda \Rightarrow \overline{Y}'_B \subseteq \overline{X}'_B$,
(7) $B \subseteq C \Rightarrow \underline{X}'_C \subseteq \underline{X}'_B$,
(8) $B \subseteq C \Rightarrow \overline{X}'_B \subseteq \overline{X}'_C$.

Proof. (1) By Theorems 3(1) and 1(1), we have, respectively, $\underline{A}_B\mu \subseteq \mu \subseteq \overline{A}_B\mu$, $B_{t_\mu -}(S_\mu) \subseteq S_\mu \subseteq B^\wedge_{t_\mu}(S_\mu)$. If $x \in B^\wedge_{t_\mu}(S_\mu)^\star$, then $\underline{A}_B\mu^\sharp(x) = \underline{A}_B\mu(x)$ and, thus, $\underline{A}_B\mu^\sharp(x) \leq \mu(x)$. If $x \notin B^\wedge_{t_\mu}(S_\mu)^\star$, then $x \notin S_\mu$ and, thus, $\mu(x) = 1$. So we have $\underline{A}_B\mu^\sharp(x) \leq \mu(x)$. Now by Definition 1, we have $X \subseteq \overline{X}'_B$.

If $x \in B_{t_\mu -}(S_\mu)$, then $\overline{A}_B\mu^\sharp(x) = \overline{A}_B\mu(x)$ and, thus, $\mu(x) \leq \overline{A}_B\mu^\sharp(x)$. If $x \notin B_{t_\mu -}(S_\mu)$, then $\overline{A}_B\mu^\sharp(x) = 1$ and again $\mu(x) \leq \overline{A}_B\mu^\sharp(x)$. Then, we have $\underline{X}'_B \subseteq X$.

(2) We have

$$\begin{aligned}
\overline{X \cap Y}'_B &= \overline{(G; S_{\mu\cup\lambda}; \mu \cup \lambda)}'_B = (G; B^\wedge_{t_{\mu\cup\lambda}}(S_\mu \cap S_\lambda); \underline{A}_B(\mu \cup \lambda)^\sharp),\\
\overline{X}'_B \cap \overline{Y}'_B &= \overline{(G; S_\mu; \mu)}'_B \cap \overline{(G; S_\lambda; \lambda)}'_B\\
&= (G; B^\wedge_{t_\mu}(S_\mu); \underline{A}_B(\mu)^\sharp) \cap (G; B^\wedge_{t_\lambda}(S_\lambda); \underline{A}_B(\lambda)^\sharp)\\
&= (G; B^\wedge_{t_\mu}(S_\mu) \cap B^\wedge_{t_\lambda}(S_\lambda); \underline{A}_B(\mu)^\sharp \cup \underline{A}_B(\lambda)^\sharp).
\end{aligned}$$

Since $t_{\mu\cup\lambda} \geq t_\mu, t_\lambda$, by Theorem 6(3), we have $B^\wedge_{t_{\mu\cup\lambda}}(S_\mu \cap S_\lambda) \subseteq B^\wedge_{t_\mu}(S_\mu) \cap B^\wedge_{t_\lambda}(S_\lambda)$. Also, by Theorem 3(7), we have $\underline{A}_B(\mu \cup \lambda) \supseteq \underline{A}_B\mu \cup \underline{A}_B\lambda$. If $x \in B^\wedge_{t_\mu}(S_\mu) \cap B^\wedge_{t_\lambda}(S_\lambda)$ then

$$\underline{A}_B\mu^\sharp(x) = \underline{A}_B\mu(x), \underline{A}_B\lambda^\sharp(x) = \underline{A}_B\lambda(x)$$

and, thus,

$$\underline{A}_B\mu^\sharp(x) \cup \underline{A}_B\lambda^\sharp(x) \leq \underline{A}_B(\mu\cup\lambda)(x).$$

Since $\underline{A}_B(\mu\cup\lambda)(x) \leq \underline{A}_B(\mu\cup\lambda)^\sharp(x)$, we obtain $\underline{A}_B\mu^\sharp(x)\cup\underline{A}_B\lambda^\sharp(x) \leq \underline{A}_B(\mu\cup\lambda)^\sharp(x)$. If $x \notin B^\wedge_{t_\mu}(S_\mu)\cap B^\wedge_{t_\lambda}(S_\lambda)$, then $x \notin B^\wedge_{t_{\mu\cup\lambda}}(S_\mu\cap S_\lambda)$ and, thus, $\underline{A}_B(\mu\cup\lambda)^\sharp(x) = 1$. Therefore, $\underline{A}_B(\mu\cup\lambda)^\sharp(x) \geq \underline{A}_B\mu^\sharp(x)\cup \underline{A}_B\lambda^\sharp(x)$ and Definition 1 yields $\overline{X\cap Y}'_B \subseteq \overline{X}'_B \cap \overline{Y}'_B$.

(3) We have

$$\begin{aligned}\overline{X\cup Y}'_B &= \overline{(G;S_{\mu\cap\lambda};\mu\cap\lambda)}'_B = (G;B^\wedge_{t_{\mu\cap\lambda}}(S_\mu\cup S_\lambda);\underline{A}_B(\mu\cap\lambda)^\sharp),\\ \overline{X}'_B\cup\overline{Y}'_B &= \overline{(G;S_\mu;\mu)}'_B \cup \overline{(G;S_\lambda;\lambda)}'_B\\ &= (G;B^\wedge_{t_\mu}(S_\mu);\underline{A}_B(\mu)^\sharp)\cup(G;B^\wedge_{t_\lambda}(S_\lambda);\underline{A}_B(\lambda)^\sharp)\\ &= (G;B^\wedge_{t_\mu}(S_\mu)\cup B^\wedge_{t_\lambda}(S_\lambda);\underline{A}_B(\mu)^\sharp\cap\underline{A}_B(\lambda)^\sharp).\end{aligned}$$

Since $t_{\mu\cap\lambda} \leq t_\mu, t_\lambda$, by Theorem 6(1), we have $B^\wedge_{t_{\mu\cap\lambda}}(S_\mu\cup S_\lambda) \supseteq B^\wedge_{t_\mu}(S_\mu)\cup B^\wedge_{t_\lambda}(S_\lambda)$. Also, by Theorem 3(8), we have $\underline{A}_B(\mu\cap\lambda) = \underline{A}_B\mu\cap\underline{A}_B\lambda$. If $x\in B^\wedge_{t_\mu}(S_\mu)\cup B^\wedge_{t_\lambda}(S_\lambda)$ then $x \in B^\wedge_{t_{\mu\cap\lambda}}(S_\mu\cup S_\lambda)$ and, thus,

$$\underline{A}_B(\mu\cap\lambda)^\sharp(x) = \underline{A}_B(\mu\cap\lambda)(x) = \underline{A}_B\mu(x)\cap\underline{A}_B\lambda(x) \leq \underline{A}_B\mu^\sharp(x)\cap\underline{A}_B\lambda^\sharp(x).$$

Now, suppose that $x\notin B^\wedge_{t_\mu}(S_\mu)\cup B^\wedge_{t_\lambda}(S_\lambda)$. Then

$$\underline{A}_B\mu^\sharp(x) = \underline{A}_B\lambda^\sharp(x) = 1$$

and, thus,

$$\underline{A}_B(\mu\cap\lambda)^\sharp(x) \leq \underline{A}_B\mu^\sharp(x)\cap\underline{A}_B\lambda^\sharp(x) = 1.$$

Therefore, $\overline{X\cup Y}'_B \supseteq \overline{X}'_B\cup\overline{Y}'_B$.

(4) We have

$$\begin{aligned}\underline{X\cap Y}'_B &= \underline{(G;S_{\mu\cup\lambda};\mu\cup\lambda)}'_B = (G;B_{t_{\mu\cup\lambda}-}(S_\mu\cap S_\lambda);\overline{A}_B(\mu\cup\lambda)^\sharp),\\ \underline{X}'_B\cap\underline{Y}'_B &= \underline{(G;S_\mu;\mu)}'_B\cap\underline{(G;S_\lambda;\lambda)}'_B\\ &= (G;B_{t_\mu-}(S_\mu);\overline{A}_B(\mu)^\sharp)\cap(G;B_{t_\lambda-}(S_\lambda);\overline{A}_B(\lambda)^\sharp)\\ &= (G;B_{t_\mu-}(S_\mu)\cap B_{t_\lambda-}(S_\lambda);\overline{A}_B(\mu)^\sharp\cup\overline{A}_B(\lambda)^\sharp).\end{aligned}$$

Since $t_{\mu\cup\lambda} \geq t_\mu, t_\lambda$, by Theorem 6(5), we have $B_{t_{\mu\cup\lambda}-}(S_\mu\cap S_\lambda) \supseteq B_{t_\mu-}(S_\mu)\cap B_{t_\lambda-}(S_\lambda)$. Also, by Theorem 3(5), we have $\overline{A}_B(\mu\cup\lambda) = \overline{A}_B(\mu)\cup\overline{A}_B(\lambda)$. If $x\in B_{t_\mu-}(S_\mu)\cap B_{t_\lambda-}(S_\lambda)$ then

$$\overline{A}_B\mu^\sharp(x) = \overline{A}_B\mu(x), \overline{A}_B\lambda^\sharp(x) = \overline{A}_B\lambda(x)$$

and, thus,

$$\overline{A}_B\mu^\sharp(x)\cup\overline{A}_B\lambda^\sharp(x) = \overline{A}_B(\mu\cup\lambda)(x) \leq \overline{A}_B(\mu\cup\lambda)^\sharp(x).$$

If $x\notin B_{t_\mu-}(S_\mu)\cap B_{t_\lambda-}(S_\lambda)$, then $x\notin B_{t_{\mu\cup\lambda}-}(S_\mu\cap S_\lambda)$ and, thus, $\overline{A}_B(\mu\cup\lambda)^\sharp(x) = 1$. Therefore, $\overline{A}_B\mu^\sharp(x)\cup\overline{A}_B\lambda^\sharp(x) \leq \overline{A}_B(\mu\cup\lambda)^\sharp(x)$ and, thus, $\underline{X\cap Y}'_B \subseteq \underline{X}'_B\cap\underline{Y}'_B$.

(5) If $\mu\subseteq\lambda$, then by Corollary 1(1), $\overline{A}_B\mu\subseteq\overline{A}_B\lambda$. In the other hand, by Lemma 1, we have $S_\lambda\subseteq S_\mu$ and, thus, by Theorem 1(4), $B_{t_X-}(S_\lambda)\subseteq B_{t_X-}(S_\mu)$. Then $\underline{Y}'_B\subseteq\underline{X}'_B$.

(6) The proof is similar to part (5).

(7) Since $B\subseteq C$, by Theorem 3(9) we have $\overline{A}_B\mu\subseteq\overline{A}_C\mu$. Also, Theorem 1(8) gives $C_{t_\mu-}(S)\subseteq B_{t_\mu-}(S)$. The result follows.

(8) The proof is similar to part (7).

Conclusion

This paper has intended to build up a rational connection between rough set theory, fuzzy set theory and Cayley graphs. First, formal definitions for fuzzy Cayley sets and fuzzy Cayley graphs have been suggested.

Some illustrative examples have also been presented. Fuzzy Cayley graphs and related approximations might be received attentions in some distributed and networked systems challenges.

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Дәл емес Кейли графтарындағы шамадан ауытқу

Шамамен алынған жиындар теориясы бұл жүйелерді дәл емес және анықталмаған модельдеу үшін лайықты әдіс. Шамамен алынған жиындар теориясы жиындар теориясының одан әрі жалпылауы болып табылатын дәл емес жиындар теориясымен толықтырылғанда, олар теориялық талқылауларда жарамды болады. Мақалада Кэйли графтарының анықтамасынан туындайтын дәл емес Кэйли ішкі жиындарының анықтамасы, демек группалардағы дәл емес ішкі жиындардың дәл емес Кэйли графтары ұсынылған. Авторлар Кэйли графының дәл емес нормаль ішкі группасына қатысты шамамен жуықтауды, сонымен қатар аппроксимацияланатын шамамен алынған дәл емес Кейли графтары және дәл емес шамамен алынған дәл емес Кейли графтарын енгізген. Соңғы жуықтау басқа жуықтаулардың бірігуі болып табылады. Кейбір теоремалар мен қасиеттері зерттелген және дәлелденген.

Кілт сөздер: анық емес жиын, шамамен алынған жиын, Кейли графы, анық емес Кейли графы, төменгі және жоғарғы жуықтаулар.

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Грубость в нечетких графах Кэли

Грубая теория множеств — заслуживающий внимания подход для неточного и неопределенного моделирования систем. Когда грубая теория множеств дополняется теорией нечетких множеств, причем обе являются дополнительным обобщением теории множеств, они будут иметь силу в теоретических дискуссиях. В настоящей статье предложено определение нечетких подмножеств Кэли и, следовательно, нечетких графов Кэли нечетких подмножеств на группах, вдохновленное определением графов Кэли. Авторами введены грубая аппроксимация графа Кэли относительно нечеткой нормальной подгруппы, а также аппроксимационные грубые нечеткие графы Кэли и нечеткие грубые нечеткие графы Кэли. Последнее приближение представляет собой смесь других приближений. Исследованы и доказаны некоторые теоремы и свойства.

Ключевые слова: нечеткое подмножество, грубое множество, граф Кэли, нечеткий граф Кэли, нижняя и верхняя аппроксимации.